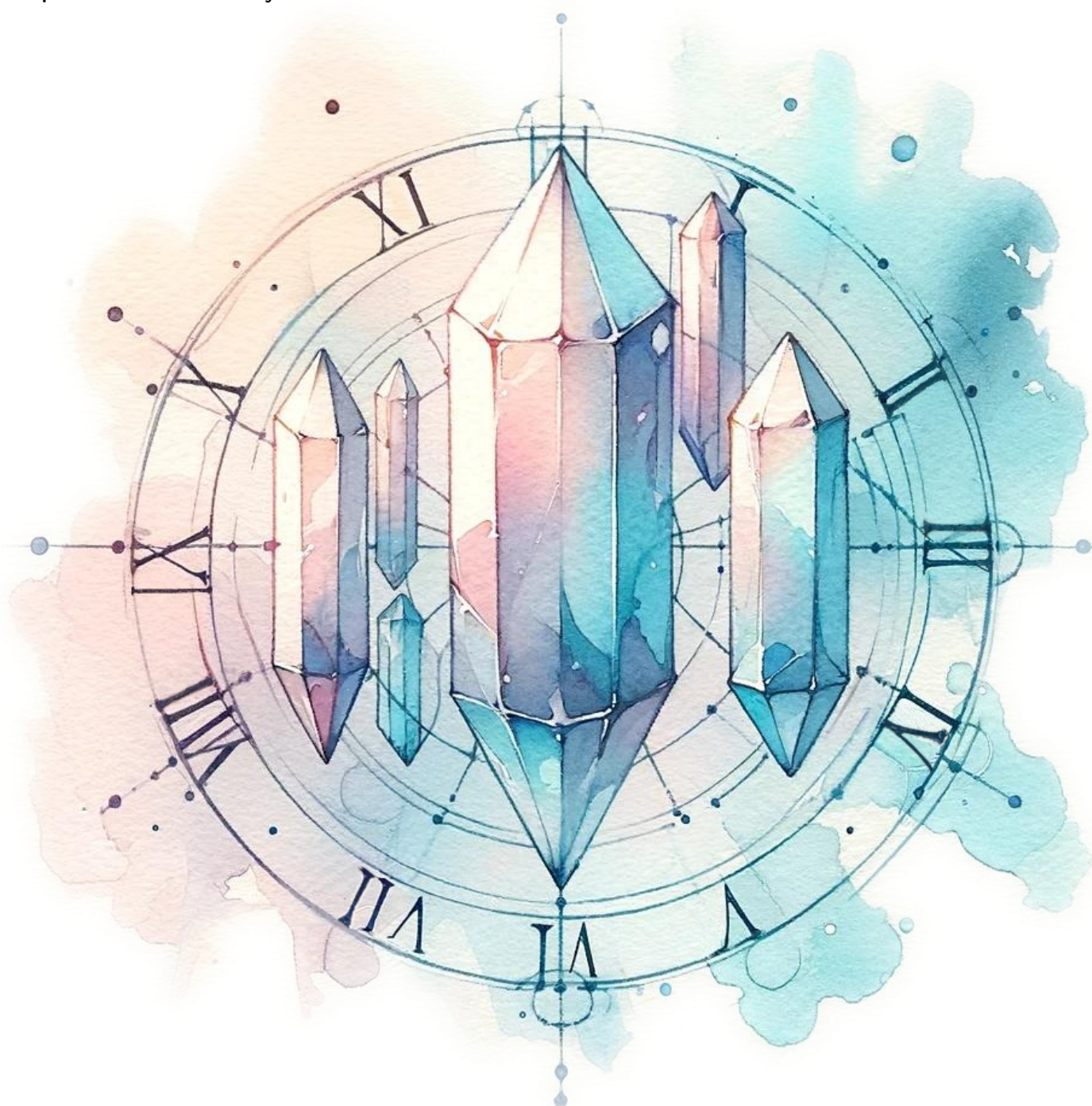




Discrete Time Crystals in Ultra-Cold Atoms

Weronika Golletz || PhD thesis in Theoretical Physics

Prepared under the supervision of
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Abstract

This thesis investigates discrete time crystals (DTCs) in ultra-cold atomic systems, particularly focusing on Bose-Einstein condensate (BECs) bouncing on an oscillating mirror in a gravitational field.

We begin by analyzing the stability of DTCs against experimental imperfections in initial state preparation, demonstrating that sufficiently strong atomic interactions can stabilize the DTC time evolution. Additionally, we explore the impact of quantum many-body fluctuations using the $U(1)$ symmetry-preserving Bogoliubov approach, showing that the depletion of atoms from BEC is negligible compared to the total number of atoms, ensuring the robustness of DTCs.

The analysis is then extended from a single atomic cloud to two clouds interacting via infinite repulsive forces. We first examine the classical dynamics of N impenetrable particles bouncing on a static mirror. For particles with different energies, we introduce new angle-action variables to account for elastic collisions, allowing us to simplify description of the system's dynamics. For particles with equal energies, we use standard angle-action variables. In both cases, we identify the manifolds in phase space on which the particles move. For particles with different energies, the manifold is an N -torus in the $2N$ -dimensional phase space, while for three particles with equal energies, it is a solid torus. We then introduce mirror oscillations and derive the effective Hamiltonian, which describes crystalline structures in time, laying the groundwork for quantum investigations of complex DTCs involving a few atomic clouds.

We then focus on a system of two distinct ultra-cold atomic clouds with infinite repulsive interactions between them and attractive interactions within each cloud. In a one-dimensional space, the lower cloud bounces on an oscillating mirror, driving the upper cloud. We investigate how intra-species interactions can lead to spontaneous time translation symmetry breaking and the formation of complex DTCs. Using a variational Jastrow ansatz, we analyze the system's quasi-energy functional and identify two regimes: a symmetry-preserving phase and a symmetry-broken phase. In the symmetry-broken phase, time translation symmetry is spontaneously broken, resulting in the emergence of a complex discrete time crystal.

Streszczenie

Niniejsza praca została opracowana na podstawie badań dotyczących dyskretnych kryształów czasowych (DTC) w układach ultrazimnych atomów, ze szczególnym uwzględnieniem kondensatów Bosego-Einsteina (BEC) odbijających się od oscylującego lustra w polu grawitacyjnym.

Na początku analizowana jest stabilność DTC w kontekście niedoskonałości eksperymentalnych związanych z przygotowaniem stanu początkowego układu. Wykazano, że wystarczająco silne oddziaływania międzyatomowe mogą stabilizować ewolucję czasową DTC. Ponadto zbadano wpływ fluktuacji kwantowych, stosując podejście Bogoliubova zachowujące symetrię $U(1)$. W ten sposób pokazano, że spadek liczby atomów w stanie kondensatu jest znikomy w porównaniu do całkowitej liczby atomów w układzie, co zapewnia stabilność DTC.

Na kolejnym etapie pracy rozszerzono model DTC, zamieniając pojedynczą chmurę ultrazimnych atomów na dwie chmury, które oddziałują poprzez nieskończoną siłę odpychającą. Pierwszym etapem analizy było zbadanie w ramach mechaniki klasycznej układu N nieprzenikalnych cząstek odbijających się od statycznego lustra. Dla cząstek o różnych energiach wprowadzono nowy rodzaj zmiennych kąta-działanie, które uwzględniają sprężyste zderzenia, co pozwala uprościć opis dynamiki układu. Dla cząstek o równych energiach skorzystano ze standardowej definicji zmiennych kąta-działanie. W obu przypadkach zbadano własności topologiczne rozmaitości, po których poruszają się cząstki w przestrzeni fazowej. W przypadku cząstek o różnych energiach rozmaitość ta jest N -torusem w $2N$ -wymiarowej przestrzeni fazowej, a dla trzech cząstek o równych energiach jest to pełny torus. Ostatecznym krokiem było wprowadzenie do modelu oscylacji lustra i opisanie go poprzez efektywny Hamiltonian, co stanowiło podstawę do dalszych badań kwantowego układu DTC złożonego z kilku chmur atomowych.

Kolejnym etapem badań była analiza układu dwóch różnych chmur atomowych, które oddziałują ze sobą poprzez nieskończoną siłę odpychającą, a wewnątrz każdej z nich występują siły przyciągające. W przestrzeni jednowymiarowej dolna chmura odbija się od oscylującego lustra, wprawiając w ruch górną chmurę. Zbadano, w jaki sposób oddziaływania wewnątrz każdej z chmur mogą prowadzić do spontanicznego łamania dyskretnej symetrii translacyjnej w czasie i powstania złożonych DTC. Z wykorzystaniem ansatzu Jastrowa przeanalizowano funkcjonal kwazienergii układu i wyróżniono dwie fazy modelu: fazę zachowującą symetrię oraz fazę łamiącą symetrię. W fazie ze złamaną symetrią dochodzi do spontanicznego łamania dyskretnej symetrii translacyjnej w czasie, co prowadzi do powstania złożonego dyskretnego kryształu czasowego.

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"And if anyone knows anything about anything," said Bear to himself, "it's Owl who knows something about something," he said, "or my name's not Winnie-the-Pooh," he said.

"Which it is," he added. "So there you are."

I have often found myself relying on Krzysztof for clarity and insight, and for that, I am endlessly grateful.

I would also like to extend my thanks to Arkadiusz Kuroś and Krzysztof Giergiel for introducing me to the fascinating world of time crystals.

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List of abbreviations

DTC	Discrete time crystal
BEC	Bose-Einstein condensate
GPE	Gross-Pitaevskii equation
LMG	Lipkin-Meshkov-Glick

Introduction

Time crystals represent a novel class of quantum mechanical states of matter. Initially proposed by Frank Wilczek in 2012 [1], they remain interesting objects of research. Wilczek’s model pictured an isolated quantum system where continuous time translation symmetry is spontaneously broken into a discrete time translation symmetry in the lowest energy state. Despite the fact that Wilczek’s original concept was not physically realizable, his idea sparked significant interest and motivated further exploration into the phenomenon of time-crystallinity.

Nevertheless, there exist a couple of physically viable models drawing direct inspiration from Wilczek’s insight into broken time translation symmetry. These could be colloquially referred to as time crystals, however they should not be confused with the original idea, as the model-specific details make them into physically sound systems. The first direction to successfully bring time crystals to life was discrete time crystals (DTCs), the theoretical model proposed by Krzysztof Sacha [2], where a higher discrete time translation symmetry is broken into a lower one. His model describes isolated (i.e. conservative) periodically driven ultra-cold atoms that form Bose-Einstein condensate (BEC). In this thesis, we will build on top of that model to study its stability against experimental imperfections and quantum fluctuations. Additionally, we will extend that model into a system of more interacting BECs.

DTCs are not limited to bosonic systems. Shortly after Sacha’s proposal, a DTC realized in a periodically driven spin chain was theoretically proposed [3, 4] and experimentally demonstrated [5, 6]. It has to be noted that an idea akin to Wilczek’s continuous time crystals is possible in the form of dissipative time crystals [7–10]. Time crystals, including DTCs, offer rich insights into time-dependent quantum mechanics and are at the forefront of advancements in studies of exotic quantum states of matter.

In Chapter 1, we introduce the phenomenon of DTCs, focusing on a system of ultra-cold atoms bouncing on an oscillating mirror in the presence of a gravitational field. We provide all the necessary approaches to describe this system, which we will use in subsequent parts of the thesis. In Chapter 2, we discuss the stability of the system in the face of experimental imperfections in the preparation of the system’s state. Additionally, we analyze the system stability against quantum fluctuations. In Chapters 3 and 4, we present the analysis of the extended model of DTCs, specifically a system of N ultra-cold interacting atomic clouds, rather than a single one. Chapter 3 provides a classical analysis of this system, where each ultra-cold atomic cloud is replaced by a classical particle. This analysis includes identifying the manifolds on which the particles move and deriving the effective Hamiltonian. Finally,

in Chapter 4, we examine the extended system from a quantum perspective, analyzing the spontaneous breaking of discrete time translation symmetry.

Chapters 2 and 3 are based on two articles:

1. Weronika Golletz, Andrzej Czarnecki, Krzysztof Sacha, and Arkadiusz Kuroś, Basis for Time Crystal Phenomena in Ultra-Cold Atoms Bouncing on an Oscillating Mirror, New J. Phys. **24** 093002 (2022) [11],
2. Arkadiusz Kuroś, Rick Mukherjee, Weronika Golletz, Frederic Sauvage, Krzysztof Giergiel, Florian Mintert, and Krzysztof Sacha, Phase Diagram and Optimal Control for n-tupling Discrete Time Crystal, New J. Phys. **22** 095001 (2020) [12].

Chapter 4 draws on research for which a manuscript is currently being prepared.

In the course of my studies, I also co-authored the following article, which is not related to the topic of the thesis:

1. Weronika Golletz, Wojciech Górecki, Rafał Ołdziejewski, and Krzysztof Pawłowski, Dark Solitons Revealed in Lieb-Liniger Eigenstates, Phys. Rev. Research **2** 033368 (2020) [13].

Chapter 1

Time crystals

In this chapter, we introduce the concept of time crystals and the essential tools used throughout the thesis. We begin by exploring the formation of space crystals in Section 1.1, providing insights into the spontaneous breaking of continuous translation symmetry into discrete translation symmetry. In Section 1.2, we briefly discuss Wilczek’s pioneering proposal of time crystals [1], followed by an extension of the concept to *discrete* time crystals in Section 1.3.

The subsequent sections introduce the key tools required to describe discrete time crystals in ultra-cold atoms bouncing on an oscillating mirror, the system analyzed in this thesis. First, in Section 1.4, we introduce Floquet theory for describing time-periodic systems. Section 1.5 provides a classical analysis of resonant dynamics of a single particle in this system. Starting with the exact description, we change the reference frame and apply the secular approximation to derive the system’s effective Hamiltonian.

Building on the classical results, Section 1.6 explores phase-space portraits to validate our approach and determine appropriate parameters for resonant driving of the system. This is followed by a quantum description in Section 1.7, where we derive Floquet states and construct the resonant Wannier basis. The Wannier basis will be used to describe the emergence of discrete time crystals from a many-body perspective in Section 1.8.

Finally, Section 1.9 presents the mean-field approach for describing discrete time translation symmetry breaking. The chapter concludes with a brief summary in Section 1.10.

1.1 Spontaneous symmetry breaking phenomena

Spontaneous symmetry breaking occurs when the stable states of a system do not share the same symmetries as the underlying Hamiltonian, even though the equations governing the system possess those symmetries. This means that while the Hamiltonian may commute with a certain symmetry operator, indicating that its eigenfunctions should also be eigenstates of this operator, the stable solutions may break this symmetry. Such symmetry breaking occurs when an eigenstate of the system is highly sensitive to perturbations, causing the system to spontaneously choose a solution that breaks the Hamiltonian’s symmetry. As a result, the

system transitions into a new phase with distinct properties, such as a magnetically ordered state in materials that breaks spin rotation symmetry, the formation of space crystals that break continuous space translation symmetry, or the emergence of a superfluid phase that breaks gauge symmetry. These new phases exhibit unique physical characteristics, such as magnetization in antiferromagnets [14, 15], lattice structures in space crystals [16, 17], or frictionless flow in superfluids [18].

In addition to space symmetries, spontaneous symmetry breaking can also extend to time translation symmetry. Until 2012, when Frank Wilczek proposed the idea [1], it was not considered whether the similar phenomena that lead to the formation of space crystals – where atoms arrange themselves in a repeating pattern in space – could also lead to the formation of *time crystals*, where a system exhibits periodic motion in its ground state, breaking continuous time translation symmetry. This groundbreaking concept opened up new avenues in the study of symmetry breaking, suggesting that time, like space, could exhibit crystalline structures under the right conditions, leading to entirely new phases of matter with unique temporal characteristics.

Before discussing Wilczek’s idea, it is helpful to build intuition about spontaneous symmetry breaking by describing the formation of space crystals. The general Hamiltonian that describes N Coulomb-interacting particles of the mass m_j in a finite volume V with periodic boundary conditions is given by:

$$\hat{H} = \sum_{j=1}^N \frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2} \sum_{j \neq k}^N U_{jk}(\mathbf{r}_j - \mathbf{r}_k), \quad (1.1)$$

and it possesses continuous space translation symmetry. This means that the space translation operator $\hat{T}_{\mathbf{R}}$, which shifts all positions $\mathbf{r}_j$ by the arbitrary vector $\mathbf{R}$, i.e. $\mathbf{r}_j \rightarrow \mathbf{r}_j + \mathbf{R}$, commutes with the Hamiltonian $\hat{H}$ (1.1), and they share common eigenstates $\psi_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$:

$$\psi_n(\mathbf{r}_1 + \mathbf{R}, \dots, \mathbf{r}_2 + \mathbf{R}) = \hat{T}_{\mathbf{R}} \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N) = e^{i\phi} \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N), \quad (1.2)$$

where the modulus of eigenvalues $|e^{i\phi}|^2 = 1$ because $\hat{T}_{\mathbf{R}}$ is unitary. As a result, the probability density of measuring a single particle in space, for the system being in a non-degenerate eigenstate, reveals uniform behavior in space:

$$\begin{aligned} \rho(\mathbf{r}_1 + \mathbf{R}) &= \int d\mathbf{r}_2 \dots d\mathbf{r}_N |\psi_n(\mathbf{r}_1 + \mathbf{R}, \dots, \mathbf{r}_N)|^2 \\ &= \int d\mathbf{r}_1 \dots d\mathbf{r}_N |\psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 = \rho(\mathbf{r}_1). \end{aligned} \quad (1.3)$$

This indicates no preferred location for measuring a particle and no visible crystalline structure, as the density follows the symmetry of the Hamiltonian $\hat{H}$ (1.1), see Fig. 1.1(a). In contrast, crystalline structures can be predicted by the two-point correlation function,

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2) = \int d\mathbf{r}_3 \dots d\mathbf{r}_N |\psi_n(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_N)|^2, \quad (1.4)$$

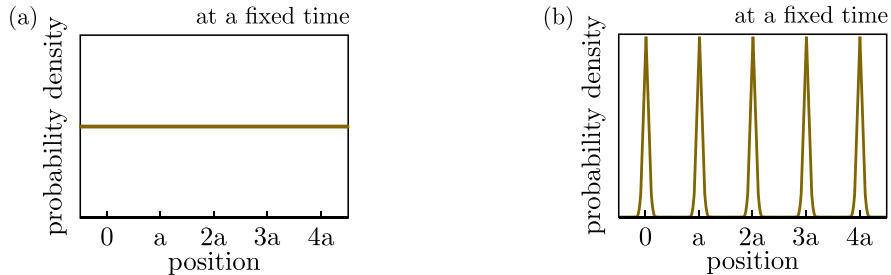


Figure 1.1: The formation of a space crystal through the spontaneous breaking of continuous space translation symmetry into a discrete one. (a) The probability density (1.3) for measuring the first particle in space at a given time is uniform over space. The system is described by the Hamiltonian $\hat{H}$ (1.1) which possesses continuous space translation symmetry. As a result, the probability density (1.3), when calculated for an eigenstate of $\hat{H}$, does not exhibit any crystalline structure. (b) The emergence of a crystalline structure can be revealed through the two-point correlation function (1.4). Physically, this corresponds to introducing a perturbation by measuring the position of one particle at $\mathbf{r}_1$ and then determining the probability of finding a second particle. This manifests as a repeated pattern, which is visible in the probability density, thus indicating the formation of a space crystal. For clarity, the plots are presented for a one-dimensional crystal with a distance between particles equal to a .

which describe the probability density of finding the second particle in space provided that the first one was measured at position $\mathbf{r}_1$. The two-point correlation function ρ_2 may vary only with a change in the relative position of the two particles, $\rho_2(\mathbf{r}_1, \mathbf{r}_2) = \rho_2(\mathbf{r}_1 - \mathbf{r}_2)$, revealing a crystalline structure that reflects the regular distances between particles, see Fig 1.1(b).

The existence of a crystalline structure arises from two-particle interactions in the Hamiltonian, $\hat{H}$ (1.1). However, since both the Hamiltonian and its eigenstates preserve continuous space translation symmetry, the crystal's center of mass is delocalized until a measurement is made. Quantum mechanics ensures that the center of mass behaves like a free massive particle, leading to a fully delocalized ground state. Thus, the probability density for a single particle measurement remains uniform, and the crystal structure remains hidden unless the symmetry is broken. Upon measuring a particle's position, the symmetry is spontaneously broken, selecting one of many possible crystalline configurations. This process results in a non-uniform probability density for subsequent particles. The choice of crystal configuration is random, explaining why this symmetry breaking is spontaneous. Without breaking symmetry, the system's translational invariance conceals the crystal structure. However, measuring relative distances between particles in the ground state would reveal the atomic arrangement without breaking symmetry. In contrast, direct position measurements disrupt this symmetry, revealing the crystalline structure [16, 17, 19–21].

In the next section, we will discuss a similar phenomenon in the time domain, focusing on Wilczek's idea where continuous time translation symmetry was expected to be broken into a discrete one – an idea that has come to be known as time crystals.

1.2 Time crystals

In the previous section, we analyzed how the probability density for detecting particles changes in space at a given time. Now, let us explore how the probability density changes over time at a given position to build intuition about spontaneous time translation symmetry breaking.

Let us start with a time-independent many-body system. The Hamiltonian governing this system commutes with the time translation operator $\hat{T}_t$ (i.e., the evolution operator), meaning they share the same set of eigenstates. This can be described by applying the translation operator to one of the Hamiltonian's eigenstates $\psi_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N; 0)$, with the corresponding eigenvalue E_n :

$$\hat{T}_t \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; 0) = \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; t) = e^{-iE_n t/\hbar} \psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; 0). \quad (1.5)$$

The probability density remains constant over time $|\psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; t)|^2 = |\psi_n(\mathbf{r}_1, \dots, \mathbf{r}_N; 0)|^2$, which is trivial since the system is in a stationary state. This also indicates that the system's Hamiltonian possesses continuous time translation symmetry.

In 2012, Frank Wilczek raised the question of whether this continuous time translation symmetry could be spontaneously broken into a discrete one, analogous to the way continuous space translation symmetry is broken in the formation of space crystals which we presented in Section 1.1. In other words, he asked whether there could exist time-independent systems that, under certain perturbations, could transition into a state of periodic motion, see Fig. 1.2(a)-(b).

Wilczek considered a system of N bosons with unit mass placed on a ring, where the interactions between particles are modeled by an attractive Dirac $\delta(x)$ -potential in the presence of a magnetic flux α . The system, known as the Aharonov-Bohm ring [22], is described by the Hamiltonian:

$$\hat{H} = \sum_{j=1}^N \frac{(p_j - \alpha)^2}{2} + \frac{g_0}{2} \sum_{j \neq k}^N \delta(x_j - x_k), \quad (1.6)$$

where $\hbar = 1$ and ring has a unit length. This system possesses continuous time and space translation symmetry. For $\alpha = 0$, in the mean-field approach, the bosons form a BEC as a ground state, meaning that all particles occupy the same localized wavefunction, which is well approximated by a bright soliton [23–25]. Beyond the mean-field approach, i.e. from a many-body perspective, the ground state possesses continuous space translation symmetry, which is highly vulnerable to perturbation. This means, for example, that a single measurement of a particle's position spontaneously breaks this symmetry, resulting in a localized peak in the probability density [26]. This phenomenon is similar to space crystal formation, as detailed in Section 1.1, where the system spontaneously transitions into a symmetry-broken state with lower symmetry than the governing Hamiltonian.

Wilczek's idea was that by combining the formation of a bright soliton with a properly chosen magnetic flux α , one could achieve the spontaneous breaking of both continuous space and time translation symmetries, resulting in a bright soliton moving periodically along the ring.

In the mean-field limit where the number of particles $N \rightarrow \infty$, interaction strength $g_0 \rightarrow 0$, while keeping $g_0(N-1) = \text{constant}$, such a periodically moving localized wavefunction should last forever.

However, following Wilczek's proposal, subsequent works [27–30] demonstrated that realizing such a system is impossible, culminating in the no-go theorem by Watanabe and Oshikawa [31, 32]. Although the formation of a bright soliton can occur, its motion cannot be observed in the limit $N \rightarrow \infty$.

If we analyze the system from the centre-of-mass coordinate frame, we find that, since the interactions between particles depend only on the relative distances between them, the Hamiltonian (1.6) becomes:

$$H = \frac{(P - N\alpha)^2}{2N} + \text{relative degrees of freedom}, \quad (1.7)$$

where the total momentum of the center of mass P is a conserved quantity and remains a good quantum number. The eigenstates of the system take quantized values $P_n = 2\pi n$ with integer n , and the ground state corresponds to:

$$\frac{\partial H}{\partial P_n} = 2\pi \frac{n}{N} - \alpha \approx 0. \quad (1.8)$$

In the limit $N \rightarrow \infty$, where a bright soliton lives indefinitely, the latter equation can be fulfilled exactly. However, since it is always possible to find such an n that $2\pi/N$ is arbitrarily

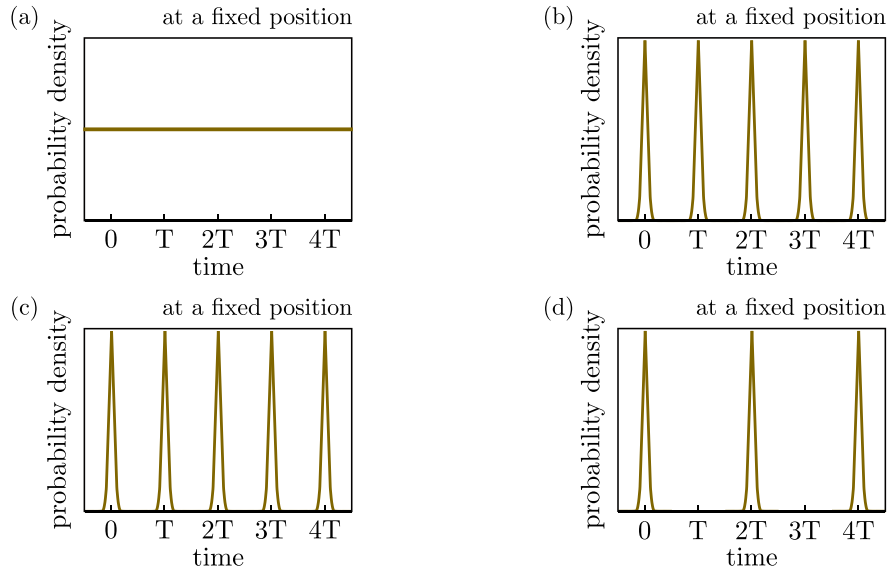


Figure 1.2: The formation of time crystals through (a)-(b) the spontaneous breaking of continuous time translation symmetry into a discrete one (Wilczek's idea [1]), and (c)-(d) the spontaneously breaking of discrete time translation symmetry into another discrete one (an extension of Wilczek's idea by Sacha [2]). Unlike the formation of space crystals, see Fig. 1.1, the probability densities for measuring the particles at a fixed position in space are plotted *over time* at a fixed position.

close to α , the probability flux of the center-of-mass degrees of freedom vanishes. Thus, even if the system can spontaneously form a bright soliton in the ground state for $\alpha \neq 0$, there is no periodic motion [33–36].

For further discussion on the existence of time crystals, we refer to [33, 37–40]. While time crystals – as time-independent systems capable of spontaneously breaking continuous time translation symmetry in the lowest energy state – may not exist, the idea has nevertheless opened up a new avenue for research. In the next section, we briefly explore this emerging field.

1.3 Discrete time crystals

One of the extensions of Wilczek’s idea [1] involves exploring periodically driven systems, where the Hamiltonian $\hat{H}(t + T) = \hat{H}(t)$ is periodic in time with a period $T = 2\pi/\omega$. Some of these systems can spontaneously break the discrete time translation symmetry they already possess, $t \rightarrow t + T$, down to another discrete time translation, $t \rightarrow t + nT$, where n is a rational number. This symmetry breaking results in a self-organization of the system’s motion due to mutual interactions of the particles, causing it to evolve with a period different from that of the driving force, as illustrated in Fig. 1.2(c)-(d). Such systems are known as *discrete time crystals* (or Floquet time crystals).

The first theoretical proposal for discrete time crystals in an isolated (i.e., conservative) periodically driven many-body system was introduced by Krzysztof Sacha in 2015 [2]. This system consists of ultra-cold atoms bouncing on an oscillating atomic mirror. Throughout this thesis, we will focus on this system and analyze its behavior. However, before exploring this specific system, we will briefly review other types of discrete time crystals.

After the first proposal of discrete time crystals, other theoretical models, particularly spin systems, were developed [3, 4, 41], leading to the first experimental observations [5, 6]. Since then, a diverse range of time crystals has been proposed, including clean discrete time crystals [42–46], prethermal discrete time crystals [47], fractional time crystals [46, 48], discrete time quasi-crystals [49–51], dissipative discrete time crystals [52–56] and even continuous dissipative time crystals, where continuous time translation symmetry is broken into a discrete one – something thought impossible for isolated systems [7–10]. There have also been proposals for more exotic concepts such as imaginary time crystals [57] and many others variants [42, 53, 58–73].

Given the rapid development of this field, it is not possible to compile a comprehensive list of all papers on time crystals. However, the key studies mentioned provide a representative overview of the various approaches and discoveries in this area. Along these theoretical developments, various experimental realizations of time crystals have been reported [5, 6, 74–90].

In this thesis, we focus on the first proposal of discrete time crystals [2] involving ultra-cold atoms that forms a BEC, bouncing resonantly on an oscillating mirror. This system is capable of dramatically breaking discrete time translation symmetry, allowing the bouncing

atoms to move with periods up to 100 times longer than the driving period [91, 92]. We will analyze the robustness of these discrete time crystals against external perturbations [12] and quantum fluctuations [12, 93]. Additionally, we will consider extending this model to more complex systems of a few ultra-cold atomic clouds instead of the single one.

Before presenting our results, the remaining sections will introduce all the necessary tools for describing the system of ultra-cold atoms bouncing on an oscillating mirror. We will start with a brief introduction to Floquet theory, which describes periodically driven systems, and then will discuss the mechanism underlying the emergence of discrete time crystals in detail. While other types of time crystals are beyond the scope of this thesis, we encourage readers to consult reviews and a monograph on the topic for further exploration [19, 20, 94–98].

1.4 Floquet theory

Floquet theory is applicable to systems governed by time-periodic Hamiltonians, i.e. $\hat{H}(t) = \hat{H}(t + T)$, where $T = 2\pi/\omega$ corresponds to the period associated with an external driving frequency ω [99, 100]. This theory applies to time-dependent systems that exhibit discrete time translation symmetry and, consequently, where the energy is no longer conserved. Although energy is not conserved, we can still define quasi-stationary states that evolve with the same time-periodicity as the Hamiltonian and span the system's Hilbert space.

According to the Floquet theory, any solution to the time-dependent Schrödinger equation with $\hat{H}(t)$ periodic in time:

$$i\hbar \partial_t |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle, \quad (1.9)$$

can be represented as:

$$|\psi(t)\rangle = \sum_n a_n |\psi_n(t)\rangle = \sum_n a_n e^{-i\epsilon_n t/\hbar} |u_n(t)\rangle. \quad (1.10)$$

Here, ϵ_n are called quasi-energies, and $|u_n(t)\rangle$ are the Floquet states, which are periodic in time with period T , i.e. $|u_n(t)\rangle = |u_n(t + T)\rangle$, and they form a complete basis at any t . The coefficients a_n are time-independent.

Each component $|\psi_n(t)\rangle$ in Eq. (1.10) describes the evolution of a single Floquet state, combining the phase evolution modulated with the periodicity of the Hamiltonian $\hat{H}(t)$. Substituting the single component $|\psi_n(t)\rangle$ into the time-dependent Schrödinger equation (1.9) leads to the eigenequation for Floquet states:

$$\hat{\mathcal{H}}_F |u_n(t)\rangle = \epsilon_n |u_n(t)\rangle. \quad (1.11)$$

with the Floquet Hamiltonian:

$$\hat{\mathcal{H}}_F = \hat{H}(t) - i\hbar \partial_t. \quad (1.12)$$

The quasi-energies ϵ_n are defined modulo the driving frequency ω , and considering only Floquet states corresponding to the spectrum within a single Floquet zone, e.g. $-\omega/2 < \epsilon_n \leq +\omega/2$, is sufficient to fully describe the system. This is because:

$$e^{-i\epsilon_n t/\hbar} |u_n(t)\rangle = e^{-i(\epsilon_n + m\omega)t/\hbar} e^{im\omega t/\hbar} |u_n(t)\rangle = e^{-i\tilde{\epsilon}_n t/\hbar} |\tilde{u}_n(t)\rangle, \quad (1.13)$$

where $|\tilde{u}_n(t)\rangle$ is also a Floquet state with a corresponding eigenvalue $\tilde{\epsilon} = \epsilon + m\omega$, where m is an integer. This shows that the quasi-energy spectrum is periodic, and it is sufficient to restrict it to a single Floquet zone of width ω , as it contains all physically distinct Floquet states.

Floquet theory is analogous to Bloch's theorem, which applies to spatially periodic systems $\hat{H}_B(x+a) = \hat{H}_B(x)$, where a is the lattice constant. The system does not have a well-defined momentum, but we can still define quasi-momentum $\hbar k$ [101, 102]. The solutions to the time-independent Schrödinger equation involve plane waves modulated by spatial periodicity of the Hamiltonian, expressed as $e^{ikx} |u_k(x)\rangle$. Thus, they have a similar structure to the Floquet components $|\psi_n(t)\rangle$ in Eq. (1.10). Just as the Floquet zone maintains information about the time-periodic system, focusing on the Brillouin zone for quasi-momenta within the range $-\pi/a < k \leq \pi/a$ provides a complete picture of space-periodic systems.

Before we switch to a detailed description of the driven system of ultra-cold atoms, it is important to note that for a periodically driven system possessing discrete time translation symmetry, the Hamiltonian $\hat{H}(t) = \hat{H}(t+T)$ commutes with the time translation operator $\hat{\mathcal{T}}_T$, which governs the system's evolution over the period T . Therefore, the Floquet eigenstates are also eigenstates of this operator, cf. Eq. (1.5):

$$\hat{\mathcal{T}}_T |u_n(t=0)\rangle = e^{-i\epsilon_n T/\hbar} |u_n(t=T)\rangle = e^{-i\epsilon_n T/\hbar} |u_n(t=0)\rangle, \quad (1.14)$$

and the probability density of detecting particles at some point in the configuration space will be T -periodic, provided the system is prepared in a single Floquet eigenstate.

Throughout the thesis, we focus on many-body periodically driven systems of ultra-cold atoms bouncing on an oscillating mirror. These systems can spontaneously break discrete time translation symmetry and start evolving with a period different from the driving period T .

In the next section, we will start with a classical analysis of a single atom bouncing on an oscillating mirror. Following this in subsequent sections, we will discuss the stability of the system based on the phase portraits. Finally, we will switch to a quantum many-body description, where we discuss the emergence of discrete time crystals.

1.5 Classical description: periodically driven particle

We begin with the classical analysis of a single particle bouncing on an oscillating atomic mirror under the influence of a gravitational field in a one-dimensional (1D) space, see Fig. 1.3(a). To simplify our calculations, we use gravitational units for length l_0 , energy E_0 , and time t_0 , defined as:

$$l_0 = (\hbar^2/m^2g)^{1/3}, \quad E_0 = mgl_0, \quad t_0 = \hbar/mgl_0, \quad (1.15)$$

where m is the particle's mass, which we assume to be of unit mass, g is the gravitational acceleration, and $\hbar$ is the reduced Planck's constant.

The system's Hamiltonian in the laboratory frame of reference is expressed as:

$$H(x, p, t) = \frac{p^2}{2} + x + F(x + f(t)), \quad (1.16)$$

where $F(y)$ models the atomic mirror as an oscillating hard-wall repulsive potential, defined as $F(y) = \infty$ for $y < 0$, and $F(y) = 0$ for $y \geq 0$. Here, the time-periodic function $f(t) = f(t + T)$ oscillates with frequency ω , which is related to the Hamiltonian period $T = 2\pi/\omega$. Figure 1.3(b) schematically illustrates how the boundary of the linear gravitational potential changes in time due to the presence of the oscillating mirror. Throughout the chapter, we set

$$f(t) = \frac{\lambda}{\omega^2} \cos(\omega t), \quad (1.17)$$

where λ/ω determines the amplitude of the mirror's oscillations in the laboratory frame.

To simplify the analysis and eliminate the time-dependent boundary conditions imposed by the oscillating mirror, we switch from the laboratory frame to a frame oscillating with the mirror using the canonical transformation

$$\begin{aligned} x' &= x + (\lambda/\omega^2) \cos(\omega t), \\ p' &= p - (\lambda/\omega) \sin(\omega t). \end{aligned} \quad (1.18)$$

This transformation fixes the mirror's position and introduces a time-periodically varying effective gravitational field. The transformed Hamiltonian becomes $H'(x', y', t) = \frac{p'^2}{2} + x' + \lambda x' \cos(\omega t) + F(x')$, where we neglected constant terms that do not affect the system's dynamics. Modeling the mirror as a static hard-wall at $x' = 0$ confines the particle's motion to $x' \geq 0$, see Fig. 1.3(c). Consequently, we simplify notation by dropping the primes on x and p , leading to the Hamiltonian [2, 103–105]:

$$H(x, p, t) = \frac{p^2}{2} + x + \lambda x \cos(\omega t), \quad x \geq 0. \quad (1.19)$$

The static mirror case

To provide insights into our system and establish the analytical framework, we start by considering the static mirror case by setting $\lambda = 0$. This case is described by the unperturbed part of the Hamiltonian (1.19) as:

$$H_0(x, p) = \frac{p^2}{2} + x, \quad x \geq 0. \quad (1.20)$$

In this integrable system, the particle follows a periodic orbit in phase space.

To analyze the periodic motion, we perform a canonical transformation from Cartesian coordinates $\{p, x\}$ to action-angle variables $\{I, \theta\}$ [105, 106]. The action variable $I \geq 0$, a constant of motion, represents the area enclosed by the periodic orbit in phase space, divided by 2π , for a given energy E

$$I = \frac{1}{2\pi} \oint p dx = I(E), \quad (1.21)$$

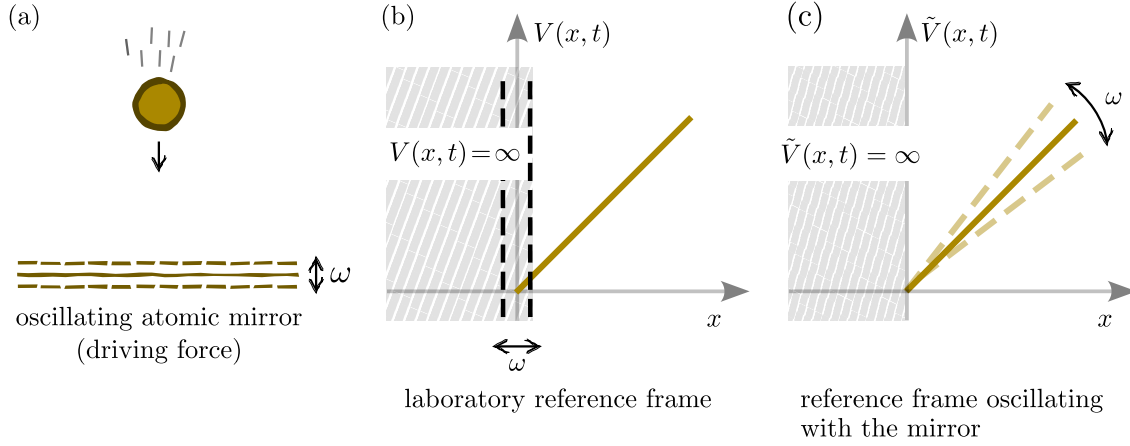


Figure 1.3: (a) Schematic plot of a single particle bouncing on an oscillating atomic mirror in the presence of a gravitational field in one-dimensional space. The mirror oscillates with frequency ω . (b) In the laboratory reference frame, the system is described by the Hamiltonian (1.16) with a time-dependent boundary condition represented by a moving hard-wall potential due to an oscillating mirror. To eliminate this time-dependent boundary, we switch to a reference frame oscillating with the mirror using the canonical transformation (1.18). This results in the Hamiltonian (1.19) confining the particle's motion to $x \geq 0$ (c).

where the momentum $p = \pm\sqrt{2(E - x)}$. Using this relation, we obtain the unperturbed Hamiltonian (1.20) which depends only on the action [2, 20, 105],

$$H_0(I) = \frac{(3\pi I)^{2/3}}{2}, \quad (1.22)$$

leading to Hamilton's equations of motion,

$$\begin{aligned} \frac{\partial I}{\partial t} &= -\frac{\partial H_0(I)}{\partial \theta} = 0, \\ \frac{\partial \theta}{\partial t} &= \frac{\partial H_0(I)}{\partial I} = \Omega(I) = \left(\frac{\pi^2}{3I}\right)^{1/3}, \end{aligned} \quad (1.23)$$

where $\Omega(I)$ is the frequency of the unperturbed motion of the particle.

The conjugate angle variable $\theta \in [0, 2\pi)$ evolves linearly with time:

$$\theta(t) = \Omega(I)t + \theta_0 \pmod{2\pi}, \quad (1.24)$$

mapping the particle's position on its periodic orbit, with θ_0 as the initial position. The action variable I determines the specific orbit on which the particle moves, while θ parametrizes the particle's position along this orbit.

The transformation between the Cartesian variables $\{p, x\}$ and action-angle variables $\{I, \theta\}$

reads:

$$\begin{aligned} x(I, \theta) &= \frac{1}{2}(3\pi I)^{2/3} \left(1 - \frac{\theta^2}{\pi^2}\right), & \theta(p, x) &= \pi \left(1 - \frac{p}{\sqrt{p^2 + 2x}}\right), \\ p(I, \theta) &= \left(\frac{3I}{\pi^2}\right)^{1/3} (\pi - \theta), & I(p, x) &= \frac{1}{3\pi} (p^2 + 2x)^{3/2}. \end{aligned} \quad (1.25)$$

The weakly oscillating mirror case

We now shift our analysis to cases where the mirror oscillates weakly $\lambda \ll 1$. While analytical solutions for a static mirror are known, introducing oscillations complicates the analysis, rendering it analytically unsolvable [106–108]. Turning on the mirror oscillations may lead to irregular classical motion, though certain periodic orbits can remain stable. These stable resonant orbits persist in regular regions of the classical phase space. In the next Section 1.6, we will explore how the phase space for a particle bouncing on an oscillating mirror is structured.

Our analysis focuses on particle motion near resonant orbits, where the driving frequency ω aligns with the s -th harmonic of the particle's unperturbed motion with frequency $\Omega(I_s)$, fulfilling the $s : 1$ resonance condition $\omega = s\Omega(I_s)$. The resonant action $I_s = \pi^2 s^3 / (3\omega^3)$ specifies the orbit where the particle's period is s times longer than the driving period. For instance, $1 : 1$ resonance results in particle motion synchronized with the driving period T . Higher resonances, such as $2 : 1$, lead to the particle bouncing with a period twice as long as that of the driving, along with other higher-order $s : 1$ resonances.

The total Hamiltonian (1.19), expressed in action-angle variables, is represented as:

$$H(I, \theta) = H_0(I) + \lambda \cos(\omega t) x(I, \theta) \quad (1.26a)$$

$$= H_0(I) + \lambda \cos(\omega t) \left(\sum_{n=-\infty}^{\infty} h_n(I) e^{in\theta} \right). \quad (1.26b)$$

Here, we expand $x(I, \theta) = \sum_{n=-\infty}^{\infty} h_n(I) e^{in\theta}$ into a Fourier series due to the 2π -periodicity of the angle variable θ where $h_0 = (\frac{\pi I}{\sqrt{3}})^{2/3}$ and $h_{n \neq 0} = \frac{(-1)^{n+1}}{n^2} (\frac{3I}{\pi^2})^{2/3}$.

To analyze the particle's motion near a resonant orbit, where $I \approx I_s$ and $\omega \approx s\Omega(I)$, we apply the classical secular approximation [106]. This method simplifies the analysis by focusing on the long-term, slower dynamics, averaging out the fast oscillating terms while keeping other variables as constant. It involves a canonical transformation that introduces the slowly varying variable $\theta - \omega t/s$, effectively changing the reference frame to align with the resonant frequency of a particle, $\Omega(I_s) = \omega/s$. The resulting Hamiltonian is then averaged over the rapidly changing dynamical variables.

Before applying the secular approximation, it is necessary to address the time dependency of the system. We extend the phase space to include time t as an additional variable along with its canonically conjugate momentum p_t , which represents the system's energy. The Hamiltonian in this extended phase space is defined as:

$$\mathcal{H} = p_t + H(I, \theta), \quad (1.27)$$

and as we will see in Section 1.7 it is a classical analog of the Floquet Hamiltonian.

Evolution time is now represented by a fictitious time ξ , related to t through Hamilton's equations $\frac{\partial \mathcal{H}}{\partial t} = \frac{dt}{d\xi} = 1$, implying that t and ξ are identical, $t = \xi + t_0$. Consequently, the Hamiltonian (1.27) is independent of ξ , making it a constant of motion as ξ evolves. We now apply the classical secular approximation. To do so, we switch to a reference frame moving with the resonant frequency $\Omega(I_s)$ by implementing the following transformations:

$$I' = I, \quad (1.28a)$$

$$\Theta = \theta - \frac{\omega}{s}t, \quad (1.28b)$$

$$p'_t = p_t + \frac{\omega}{s}I, \quad (1.28c)$$

$$t' = t. \quad (1.28d)$$

These transformations yield the Hamiltonian:

$$\mathcal{H} = H_0(I) - \frac{\omega}{s}I + \lambda \cos(\omega t) \left(\sum_{n=-\infty}^{\infty} h_n(I) e^{in(\Theta + \omega t/s)} \right) + p_t, \quad (1.29)$$

where we simplify notation by dropping the primes. Note that Hamiltonian (1.29) is now sT -periodic, maintaining the periodicity of the internal motion.

For particle motion near the resonant orbit $I \approx I_s$, and if the mirror's oscillations are weak $\lambda \ll 1$, all variables except time evolve slowly:

$$\begin{aligned} dI/dt &= \mathcal{O}(\lambda), \\ d\Theta/dt &= (\Omega(I) - \omega/s + \mathcal{O}(\lambda))|_{I=I_s} = \mathcal{O}(\lambda), \\ dp_t/dt &= \mathcal{O}(\lambda). \end{aligned} \quad (1.30)$$

Therefore, by averaging the Hamiltonian (1.29) over one period sT , and maintaining other variables constant, we derive the effective secular Hamiltonian:

$$\mathcal{H}_{\text{sec}} \approx \frac{P^2}{2m_{\text{eff}}} + V_0 \cos(s\Theta) + p_t. \quad (1.31)$$

Here, $H_0(I)$ is expanded around I_s up to second order, and constant terms are omitted. The negative effective mass $m_{\text{eff}}^{-1} = \frac{d^2 H_0(I_s)}{dI_s^2} = -\frac{\omega^4}{\pi^2 s^4}$, the conjugate momenta $P = I - I_s$, and potential amplitude $V_0 = \frac{\lambda}{\omega^2}(-1)^{s+1}$.

In Section 1.7, we will further explore the implications of effective Hamiltonian (1.31) from a quantum mechanical perspective. However, before we proceed to that discussion, we will first explore the phase-space portraits to gain a better understanding of the system's classical dynamics.

1.6 Phase-space portraits

Secular Hamiltonian (1.31) is independent of time, so we can keep p_t constant and confine the analysis of the system to a two-dimensional (2D) phase space spanned by (I, Θ) . Given that the derivation of the secular Hamiltonian assumes a first-order approximation in λ (1.30), comparing phase-space portraits generated by both the exact Hamiltonian (1.26a) and its approximated, secular form (1.31) can validate this approach. This comparison also helps determine an appropriate value for amplitude λ of the mirror oscillations used in subsequent quantum analysis.

For the static mirror case $\lambda = 0$, the system dynamics is governed by Hamiltonian $H_0(x, p)$ (1.20), which conserves energy and exhibits periodic motion, tracing closed loops in the (x, p)

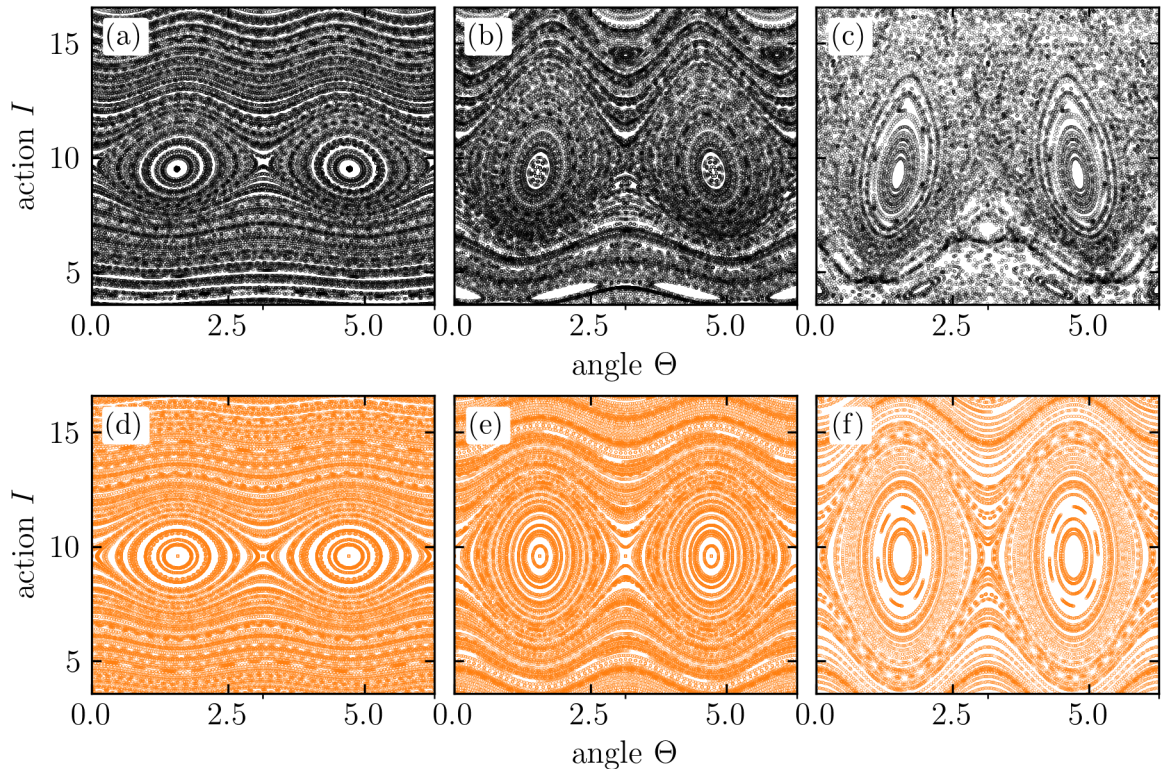


Figure 1.4: Comparison of phase-space portraits from the exact (1.26a) and the secular (1.31) Hamiltonian for the $s = 2$ resonance, illustrating the accuracy of the secular approach. Top panels display results from the exact Hamiltonian, while bottom panels show those from the secular Hamiltonian, across varying mirror amplitude values: (a)-(d) $\lambda = 0.05$, (b)-(e) $\lambda = 0.15$, and (c)-(f) $\lambda = 0.3$. Even with a small amplitude of the mirror oscillations, the resonant islands begin to form. As the perturbation increases, the exact Hamiltonian portraits reveal chaotic motion near the resonant islands (c), a phenomenon that the secular Hamiltonian does not capture (f), indicating its limitations in describing the dynamics under large perturbations. The driving frequency is $\omega = 1.4$, with resonant action $I_s = 9.59$ for $s = 2$.

phase space. As a result, the (I, θ) phase space shows straight lines, each corresponding to a distinct constant I_s value.

When the mirror oscillates with a small amplitude $\lambda \ll 1$, the perturbation manifests as resonant islands around the resonant action I_s , see Fig. 1.4 where the phase space portraits obtained in the exact dynamics (1.26a) and within the secular approach (1.31) are presented. The resonant condition $s : 1$ reveals s resonant islands located around I_s , with centers at fixed points that correspond to a stable resonant trajectory with a period sT . We can divide the dynamics of a particle into librational motion within a resonant island and unbounded rotational motion outside it, separated by a separatrix with unstable points. The area of the resonant island $A_s \propto \sqrt{|\lambda|}$ is determined by amplitude λ of the mirror oscillations [105], indicating that under the resonant driving condition $\Omega(I_s) = \omega/s$ even small perturbations modify the phase space structure.

Figure 1.4 presents phase-space portraits derived from both the exact (1.26a) and the approximated secular Hamiltonian (1.31) for the $s = 2$ resonance. These portraits are stroboscopic snapshots at times that are multiples of sT , represented by $(I(t), \Theta(t))$. The snapshots result from integrating the classical equations of motion generated by each Hamiltonian. Even for a weak perturbation $\lambda = 0.05$, resonant islands around the resonant action I_s are observable (Fig. 1.4(a) and 1.4(d)). With a stronger perturbation $\lambda = 0.15$, the resonant islands expand, yet the phase-space portraits around the islands do not exhibit chaotic motion (Fig. 1.4(b) and 1.4(e)). This implies that the secular Hamiltonian adequately describes the particle's motion near the resonant action I_s . As the strength of the perturbation increases, chaotic motion begins to emerge in exact portraits, while the secular Hamiltonian remains unaffected, indicating that the perturbation is too significant to be accurately described by the secular Hamiltonian (Fig. 1.4(c), 1.4(f)).

1.7 Quantum description: periodically driven particle

In Section 1.5, we derived the effective secular Hamiltonian (1.31). Here, we switch to its quantum counterpart. The transition involves quantization by substitution $P \rightarrow -i\partial_\Theta$ and $p_t \rightarrow -i\partial_t$ valid for $I_s \gg 1$ [20, 91]:

$$\hat{\mathcal{H}}_{\text{sec}} \approx \frac{\hat{P}^2}{2m_{\text{eff}}} + V_0 \cos(s\Theta) - i\partial_t. \quad (1.32)$$

Alternatively, the quantum secular Hamiltonian can be obtained by initiating the analysis from a quantum perspective of the system [20, 105, 109]. The transition to the oscillating mirror frame is accomplished through the unitary transformation $\hat{U}_{\text{mirror}} = e^{i\hat{x}(\lambda/\omega) \sin(\omega t)} e^{i\hat{p}(\lambda/\omega^2) \cos(\omega t)}$, corresponding to its classical equivalent (1.18). For the static mirror case, the solutions to the unperturbed Hamiltonian $\hat{H}_0 |n\rangle = E_n |n\rangle$ are represented by Airy functions, where the zeros of these functions correspond to the eigenvalues $2^{1/3} E_n$ [107, 108]. Once mirror oscillations are introduced, the quantum secular approximation can be applied. This involves switching to a frame moving with the particle's resonant frequency via the canonical transformation

$\hat{U} = e^{i\hat{n}\omega t/s}$, where $\hat{n}|n\rangle = n|n\rangle$, mirroring the classical approach (1.28b). The resulting quantum secular (Floquet) Hamiltonian, for the mirror oscillations described by (1.17), is expressed as:

$$\langle n' | \hat{\mathcal{H}}_{\text{sec}} | n \rangle \approx \left(E_n - n \frac{\omega}{s} \right) \delta_{n,n'} + \frac{\lambda}{2} \langle n' | \hat{x} | n \rangle (\delta_{n',n+s} + \delta_{n',n-s}). \quad (1.33)$$

To complete the quantum analysis of a particle resonantly bouncing on an oscillating mirror, we need to identify the resonant Floquet eigenstates. Before presenting the numerical results, let us discuss the properties for the resonant Floquet eigenstates. In the quantum description, a single localized wave packet moving along a classical $s : 1$ resonant orbit cannot form a Floquet eigenstate because its period is s times longer than the driving period T . Consequently, the wave packet will not align with the Hamiltonian's periodicity, thus failing to meet the criteria for Floquet eigenstates. As a result, true Floquet states are expected to involve states in all s -resonant islands. This configuration includes s localized wave packets, each positioned on a different island. After each driving period T , the wave packets exchange their locations, creating a globally T -periodic behavior [2, 105].

Figure 1.5(a) illustrates the time evolution of probability density of Floquet states $|u_1(x, t)|^2$, for $s = 2$ resonance case. The state consists of two localized wave packets which are $2T$ -periodic, and propagate along the same periodic orbit. The localized wave packets move with a delay of T with respect to each other.

Initially, at time $t = 0$, an interference pattern emerges as the two localized wave packets move in opposite directions, as indicated by the blue arrows. One wave packet (bright blue arrow) travels towards the mirror located at $x = 0$, bounces off at $t = 0.5T$, and returns to the initial position. Simultaneously, the second wave packet (dark blue arrow) heads towards the classical turning point at $x \approx 10$, reaches it at $t = 0.5T$, and then also returns to its initial position. By the end of one period T , the initial probability density is restored, but the roles of the two wave packets switch: the first now moves towards the classical turning point, while the second heads towards the mirror. The original configuration is replicated after two periods $2T$. However, since the two localized wave packets forming the Floquet state are identical, the Floquet state $u_1(x, t)$ itself is T -periodic, aligning with the periodicity of the underlying Floquet Hamiltonian (1.33). There exists a second Floquet state $u_2(x, t)$, which evolves similarly to $u_1(x, t)$ but corresponds to a quasi-energy nearly degenerate (modulo ω/s) $E_2 - E_1 = \omega/2 + J$, differing by a small value J associated with the tunneling process, see Fig. 1.5(b).

The localized wave packets constituting the Floquet states can be extracted by forming appropriate superpositions. At $t = 0$, a superposition of Floquet states can be prepared as:

$$\Phi^\pm(x, t = 0) = (u_1(x, 0) \pm e^{-i\omega t/2} u_2(x, 0)) / \sqrt{2}, \quad (1.34)$$

resulting in two time-dependent basis vectors. Each corresponds to a localized wave packet, referred to as a Wannier state, analogous to those derived from Bloch states in solid-state physics [101]. Wannier states, shown in Fig. 1.5(c)-(d), are $2T$ -periodic and evolve as [2, 20]:

$$\hat{U}(t)\Phi^\pm(x, t) = \frac{e^{-iE_1 t}}{\sqrt{2}} (u_1(x, t) \pm e^{-i(\omega/2 + J)t} u_2(x, t)), \quad (1.35)$$

where the evolution operator $\hat{U}(t)$ takes into account the tunneling between the localized wave packets. If the system is prepared in one of the Wannier states $\Phi^\pm(x, t)$, it follows a 2 : 1 resonant trajectory with a period $2T$, gradually tunneling to the other wave packet. After a time $\pi/J \gg T$, the tunneling cycle completes, and as the evolution continues, the tunneling process reverses, resulting in periodic oscillations between the Wannier states.

The above description of the Floquet states and corresponding Wannier states can be generalized into any $s : 1$ resonance case. In such scenarios, we have s Floquet states from which s localised Wannier states can be defined. For large $s \gg 1$, we expect to reveal a

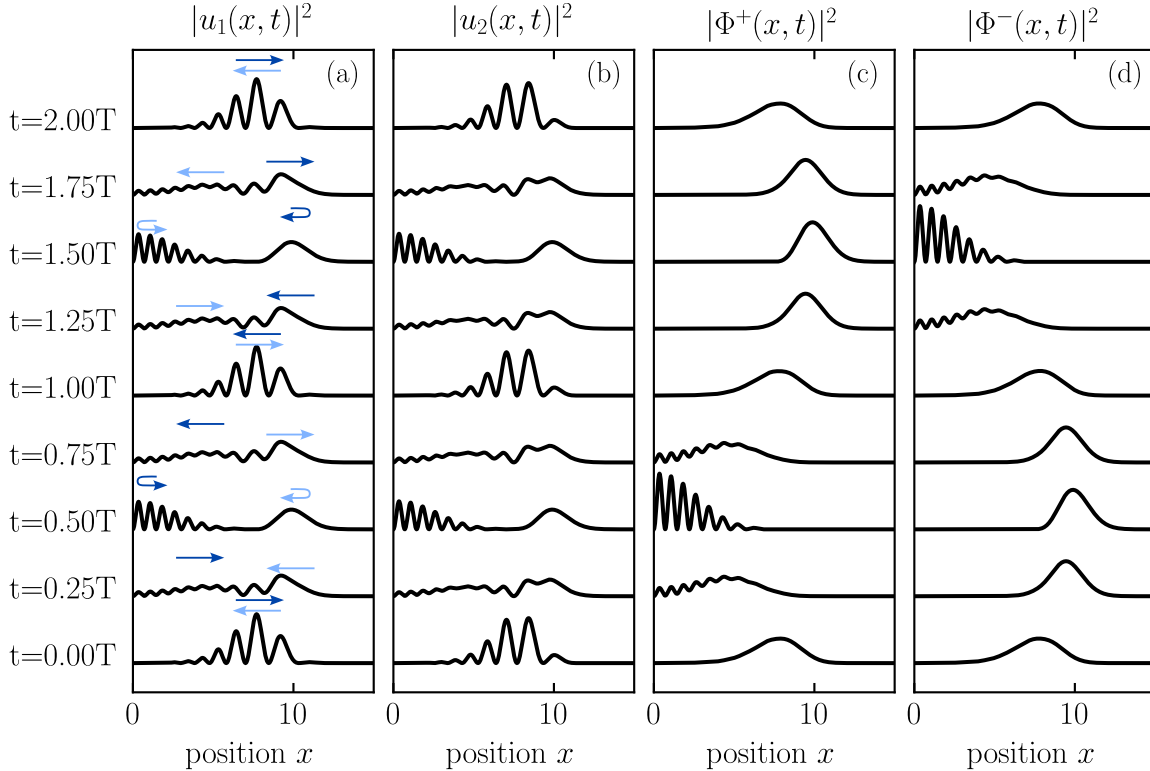


Figure 1.5: (a) Time evolution of the probability density of the resonant Floquet state $u_1(x, t)$, illustrating the quantum dynamics of a single particle bouncing resonantly on an oscillating mirror in the $s = 2$ resonance case. The Floquet state consists of two localized wave packets that move with a delay of T with respect to each other. The arrows indicate their motion, with bright and dark blue highlighting each packet for clarity. After time T , the initial probability density is restored, but the wave packets switch roles (compare the arrows at $t = 0$ and $t = T$), resulting in a T -periodic state, aligning the periodicity of the underlying Hamiltonian (1.33). A second Floquet state $u_2(x, t)$ evolves similarly to $u_1(x, t)$ but with a quasi-energy shifted by approximately $\omega/2$ (b). The localized wave packets in the Floquet states can be extracted and used as time-dependent basis vectors $\Phi^\pm(x, t)$, cf. (1.34). These extracted wave packets exhibit a $2T$ -periodicity (c)-(d). Parameters are consistent with those in Fig. 1.4, the driving frequency is $\omega = 1.4$, amplitude of the mirror's oscillations $\lambda = 0.15$.

band structure in a spectrum of the secular Hamiltonian (1.33). Figure 1.6 presents two first bands for $s = 25$ resonance case.

We can further explore the system using the Wannier basis, restricting the description to the first energy band by properly choosing system parameters, such as the oscillating mirror's amplitude λ and frequency ω . These parameters should be selected to ensure that the tunneling J between the two Wannier states remains smaller than the energy gap between the first and second energy bands. Then, the resonant Hilbert subspace is spanned by s localized wave packets $w_j(x, t)$, and the particle's wavefunction can be expressed as:

$$\psi(x, t) = \sum_j^s w_j(x, t) a_j, \quad (1.36)$$

where $\sum_{j=1}^s |a_j|^2 = 1$. Consequently, the Hamiltonian (1.32) takes the form [2, 20]:

$$\hat{\mathcal{H}}_{\text{sec}} \approx -\frac{J}{2} \sum_j a_{j+1}^* a_j + c.c., \quad (1.37)$$

where $a_{s+1} = a_1$ due to periodic boundary condition arising from periodic behavior of the system. Here, J represents the tunneling amplitude between neighboring Wannier states $w_{j+1}(x, t)$ and $w_j(x, t)$, with negligible longer-range tunneling. Equation (1.37) corresponds to a standard tight-binding model [101], but formulated in terms of the time-periodic Wannier basis.

One may wonder why the crystalline structure is not observed by simply examining the probability density of the Floquet state shown in Fig. 1.5(a)-(b) for $s = 2$ resonance case.

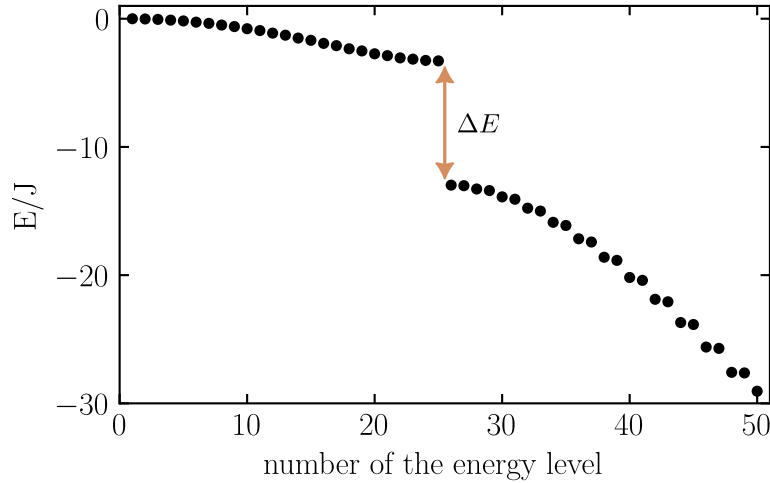


Figure 1.6: Spectrum of the secular Hamiltonian (1.33) for the $s = 25$ resonance case. Energies are given per the tunneling amplitude J , and ΔE is the energy gap between the first and second energy bands. Since the effective mass is negative, the first band corresponds to the highest one on the energy scale. The Hamiltonian parameters: $\omega = 4.3$, $\lambda = 0.18$, $s = 25$.

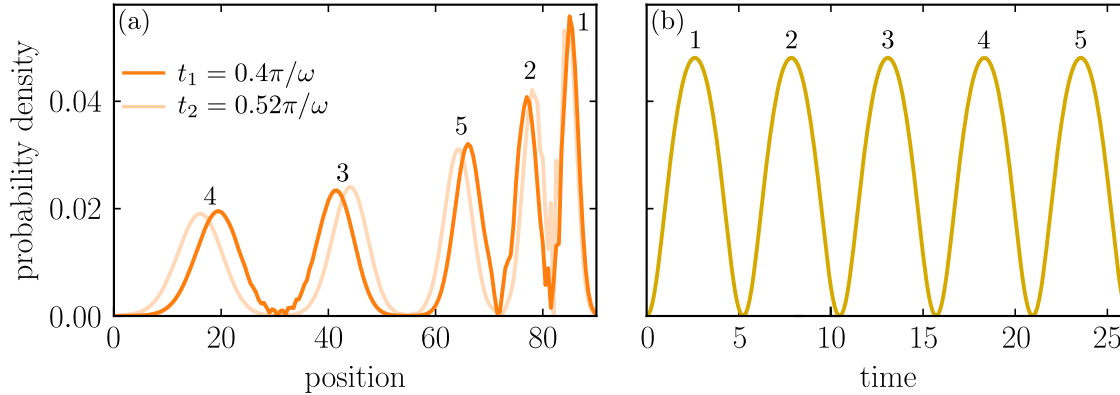


Figure 1.7: The probability density for measuring a single particle in the Floquet state for the $s = 5$ resonance case is presented as follows: (a) the density is plotted over space at two times, $t = 0.4\pi/\omega$ (bright orange) and $t = 0.52\pi/\omega$ (dark orange); (b) the density is plotted over time at position $x_0 = 86$. The periodic behavior of the system is observed in the time domain rather than in the spatial domain. The number of localized wave packets that constitute the Floquet state indicates the order in which they appear at the position x_0 . The Hamiltonian (1.33) parameters: $\omega = 1.2$, $\lambda = 0.1$, $s = 5$.

The reason is that these figures represent the probability densities in configuration space at a given time, while the crystalline structure manifests itself in the time domain. In Figure 1.7, we plot the probability density of a single particle in the Floquet state over space at two different times (Fig. 1.7(a)) and over time while keeping the position coordinate constant at $x_0 = 86$ (Fig. 1.7(b)). Here, we consider the $s = 5$ resonance case, where the Floquet state consists of five localized wave packets. In general, the canonical transformation between action-angle variables and Cartesian coordinates is nonlinear, and there is no guarantee to observe crystalline structure in the time and the space coordinates simultaneously [19].

In the next section, we will introduce the many-body description and discuss spontaneous discrete time translation breaking.

1.8 Periodically driven ultra-cold atoms

In this section, we move to the many-body system composed of N ultra-cold identical bosons bouncing resonantly on an oscillating atomic mirror within a gravitational field. The system is confined in a transverse harmonic potential with a frequency $\omega_\perp$, ensuring that the transverse excitation energy is higher than the energy per particle, and confining the bosons to the ground state in the transverse directions. Thus, the system is effectively 1D. Interactions between particles are modeled by a contact potential $g_0\delta(x)$, with the interaction strength $g_0 = 2\omega_\perp a_s$, where a_s is the s-wave scattering length. We consider attractive interactions by setting $g_0 < 0$.

Our analysis is restricted to the resonant Hilbert subspace spanned by a single particle sT -periodic Wannier states $w_j(x, t)$ introduced in the previous section (cf. (1.34) for the $s = 2$

resonance case). To describe the many-body system, we expand the bosonic field operator using the Wannier basis as:

$$\hat{\Psi}(x, t) \approx \sum_{j=1}^s w_j(x, t) \hat{a}_j, \quad (1.38)$$

where $\hat{a}_j$ are the bosonic annihilation operators. The resulting many-body Floquet Hamiltonian is given by [2, 20, 110]

$$\begin{aligned} \hat{H}_F &= \frac{1}{sT} \int_0^{sT} dt \int_0^\infty dx \hat{\psi}^\dagger \left(\hat{H}(t) - i\partial_t + \frac{g_0}{2} \hat{\psi}^\dagger \hat{\psi} \right) \hat{\psi} \\ &\approx -\frac{J}{2} \sum_{k=1}^s \left(\hat{a}_{k+1}^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_{k+1} \right) + \frac{1}{2} \sum_{k,l=1}^s U_{kl} \hat{a}_k^\dagger \hat{a}_l^\dagger \hat{a}_l \hat{a}_k \end{aligned} \quad (1.39)$$

where $\hat{H}(t)$ is the single-particle Hamiltonian (1.19) and the constant terms are omitted. The tunneling amplitude J is defined by:

$$J = -\frac{2}{sT} \int_0^{sT} dt \int_0^\infty dx w_k^*(x, t) \left(\hat{H}(t) - i\partial_t \right) w_{k+1}(x, t). \quad (1.40)$$

The long-range interactions U_{kl} and the on-site interactions U_{kk} are given as:

$$\begin{aligned} U_{kl} &= \frac{2g_0}{sT} \int_0^{sT} dt \int_0^\infty dx |w_k(x, t)|^2 |w_l(x, t)|^2 \quad \text{for } k \neq l, \\ U_{kk} &= \frac{g_0}{sT} \int_0^{sT} dt \int_0^\infty dx |w_k(x, t)|^4. \end{aligned} \quad (1.41)$$

The many-body Floquet Hamiltonian (1.39) is analogous to the Bose-Hubbard Hamiltonian but is adapted to a time-periodic basis. This approach is valid as long as the interaction energy per particle remains lower than the energy gap between the first and second energy bands of the single-particle Hamiltonian (1.33), as shown in Fig. 1.6.

Formation of discrete time crystals

In the regime of weak interactions $g_0 \rightarrow 0$, the ground state of many-body Floquet Hamiltonian (1.39), i.e. the lowest quasi-energy state within resonant subspace, forms a BEC:

$$\Psi(x_1, x_2, \dots, x_N, t) = \Pi_{j=1}^N \phi(x_j, t), \quad (1.42)$$

where all atoms occupy a state that is a uniform superposition of the Wannier functions:

$$\phi(x, t) \propto \sum_{j=1}^s w_j(x, t). \quad (1.43)$$

Conversely, under sufficiently strong and attractive interactions, it becomes energetically favorable for bosons to cluster in a single localised Wannier state to minimize the system's energy. In this scenario, the ground state is expected to take the form:

$$\Psi^{\text{GS}}(x_1, x_2, \dots, x_N, t) \propto \Pi_{j=1}^N w_k(x_j, t), \quad (1.44)$$

where k is arbitrary. However, the state (1.44) does not quantify as a Floquet state of the Hamiltonian (1.39), as it evolves with a period sT rather than the driving period T , thus violating the time translation symmetry of the Hamiltonian. To align both energy and symmetry requirements, the ground state takes a form that preserves the period T :

$$\Psi^{\text{GS}}(x_1, x_2, \dots, x_N, t) \propto \sum_{k=1}^s \left(\prod_{j=1}^N w_k(x_j, t) \right), \quad (1.45)$$

and in the Fock states basis, it is expressed as:

$$|\Psi^{\text{GS}}\rangle \propto |N, 0, 0, \dots\rangle + |0, N, 0, \dots\rangle + \dots + |0, \dots, 0, N\rangle. \quad (1.46)$$

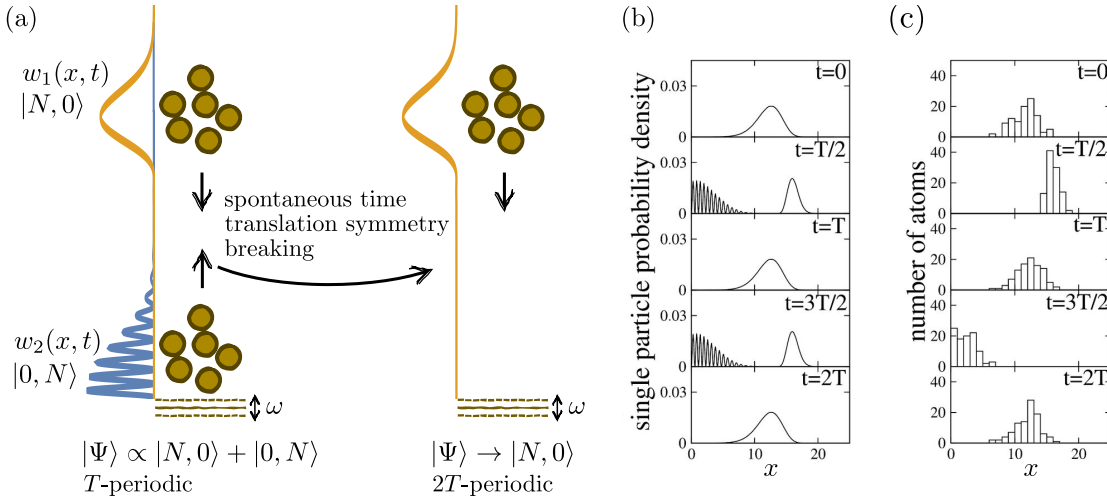


Figure 1.8: (a) The system of N ultra-cold atoms bouncing on an oscillating mirror in the presence of a gravitation field. It is prepared in a many-body Floquet state corresponding to the $s = 2$ resonance case, described either in the position representation (1.45) or in the Fock basis (1.46). To satisfy both energy and symmetry requirements, the many-body state of the system is represented by a uniform superposition of Bose-Einstein condensates, each moving with a period $2T$ but delayed with respect to each other by T (left panel). This configuration results in a T -periodic Schrödinger cat-like state, which is highly vulnerable to perturbations. For instance, measuring the position of a single atom will collapse the state into one of the atomic clouds, leading to spontaneous discrete time translation symmetry breaking. After that, the system evolves with a period twice as long as the original (right panel). (b) Snapshots at different times show the single-particle probability density calculated in the many-body Floquet state (1.46), i.e. before spontaneous discrete time translation symmetry breaking. (c) Snapshots at different times depict the distributions of 100 particles whose positions are measured at $t = 0$, followed by subsequent measurements of next 100 particles at $t = T/2$, and so on. Between measurements, the system's dynamics is governed by the many-body Floquet Hamiltonian (1.39). These snapshots show that when the discrete time translation symmetry is broken, the $2T$ -periodic evolution is stable and observable experimentally. Panels (b)-(c) are adapted from [2]. System parameters: $N = 10^4$, $g_0 N = -0.5$, $\omega = 1.1$, $\lambda = 0.06$, $s = 2$.

This state is a Schrödinger cat-like state, a superposition of macroscopic states. It is extremely sensitive to perturbations. For instance, measuring the position of a single atom causes the state to collapse into one of the macroscopic superpositions:

$$\Psi^{\text{GS}}(x_1, x_2, \dots, x_N) \rightarrow \Psi(x_1, x_2, \dots, x_N) \propto \prod_{j=1}^N w_k(x_j, t), \quad (1.47)$$

and in the Fock representation, it corresponds to:

$$|\Psi\rangle \rightarrow |0, 0, \dots, 0, \underbrace{N}_{k^{\text{th}} \text{ state}}, 0, \dots, 0\rangle. \quad (1.48)$$

The specific Wannier state $w_k(x, t)$ chosen in (1.47) depends on the measurement outcome. This phenomenon demonstrates the spontaneous breaking of discrete time translation symmetry into another discrete time translation symmetry [2], which is schematically represented in Fig. 1.8(a). In other words, while Floquet eigenstates must maintain the symmetry of Hamiltonian (1.39), i.e. evolve with the period of the mirror oscillations T , the system for sufficiently strong interactions chooses the symmetry-broken states that evolve over the period s time longer. This phenomenon is termed discrete time crystal formation [2–4], analogous to space crystal formation, where due to mutual interactions, atoms spontaneously organize into a periodic arrangement in space breaking the continuous space translation symmetry of solid state Hamiltonians [101]. To observe the formation of an s -tupling DTC, the $s : 1$ resonance condition must be fulfilled, where the driving frequency ω is s times greater than the frequency of the unperturbed motion of the particle (1.23). The resulting symmetry-broken state (1.47) is robust against further perturbation, i.e. multiple measurements can be performed while the period of time evolution remains sT , as shown in Fig. 1.8(b). In the limit $N \rightarrow \infty$, $g_0 \rightarrow 0$, with $g_0 N = \text{constant}$, the tunneling time between two states, e.g. $|0, 0, N, 0, \dots, 0\rangle$ and $|0, N, 0, \dots, 0\rangle$, increases as $e^{\alpha N}/N$ with a positive constant α , rendering it unmeasurable [2, 19].

The system of ultra-cold atoms bouncing on an oscillating mirror forms a BEC after spontaneous discrete time translation symmetry breaking. Thus, each atom occupies the same single-particle state, and the system can be accurately described by a mean-field approach [20]. In the next section, we briefly introduce the mean-field description, which allows the calculation of critical value of the interaction strength, $g_0 N$, for the transition to the DTC regime.

1.9 Mean-field approximation

Both, the ground state for weak interactions (1.42) and the symmetry-broken state (1.47) of the Hamiltonian (1.39) are represented as BECs and can be effectively described using the mean-field approximation [23]. Within this framework, the spontaneous breaking of discrete time translation symmetry is demonstrated through bifurcation [111]. This bifurcation leads to the loss of stability in the symmetry-preserving ground state (1.42), resulting in the emergence of new stable solutions that break the discrete time translation symmetry of the Hamiltonian (1.39).

Switching from the full many-body description to the mean-field approximation involves replacing the bosonic field operator $\hat{\Psi}(x, t)$ in (1.39) with the classical field $\psi(x, t)$. This transforms the Hamiltonian (1.39) into a quasi-energy functional, where $\psi(x, t)$ serves as the condensate wavefunction [2, 20]:

$$E_F = \frac{1}{sT} \int_0^{sT} dt \int_0^\infty dx \psi^*(x, t) \left(\hat{H}(t) + \frac{g_0 N}{2} |\psi(x, t)|^2 - i\partial_t \right) \psi(x, t). \quad (1.49)$$

This equation represents the average energy per particle over a time period sT , where s is an integer that reflects the expected periodicity of the system. Specifically, the solution is expected to be sT -periodic. Here, we assume large particle number N .

To obtain the condensate wavefunction, we find the extreme values of the quasi-energy functional E_F by minimizing the functional with the normalized $\psi(x, t)$. This leads to the Gross-Pitaevski equation (GPE) [23, 112]:

$$[\hat{H}(t) + g_0 N |\psi(x, t)|^2 - i\partial_t] \psi(x, t) = \mu \psi(x, t), \quad (1.50)$$

where μ is a Lagrange multiplier corresponding to the chemical potential of the system.

The mean-field approach allows us to find the condensate wavefunction $\psi(x, t)$, which describes a particle experiencing both the external potential and an additional potential related to the mean particle density. The many-body state of the system is then approximated by a product state $\psi(x_1, t)\psi(x_2, t) \dots \psi(x_N, t)$, where we assume weak interaction between particles. The system's evolution is described by the time-dependent GPE:

$$i\partial_t \psi(x, t) = [\hat{H}(t) + g_0 N |\psi(x, t)|^2] \psi(x, t). \quad (1.51)$$

Restricting further analysis to the resonant Hilbert subspace, spanned by s Wannier states as introduced in Section 1.7, we expand the solutions of GPE as: $\psi(x, t) = \sum_{j=1}^s a_j w_j(x, t)$ with the normalization $\sum_{j=1}^s |a_j|^2 = 1$. This results in the following quasi-energy functional:

$$E_F \approx -\frac{J}{2} \sum_{k=1}^s (a_k^* a_{k+1} + a_{k+1}^* a_k) + \frac{N}{2} \sum_{k,l=1}^s U_{kl} |a_k|^2 |a_l|^2, \quad (1.52)$$

where the tunneling amplitude J and the interaction coefficients U_{kl} are defined in Eq. (1.40) and Eq. (1.41), respectively.

By minimizing the quasi-energy functional (1.52) within the resonant Hilbert subspace, we derive condensate wavefunction $\psi(x, t)$, which evolves periodically in time. This is done by solving the following set of equations:

$$\frac{\partial}{\partial a_j^*} \left(E_F - \mu \sum_{k=1}^s |a_k|^2 \right) = 0, \quad (1.53)$$

where μ is a Lagrange multiplier. For weak interactions, the solution is a uniform superposition over all Wannier states (1.43), whereas for sufficiently strong interactions, s degenerate mean-field ground state solutions $\psi(x, t) \approx w_k(x, t)$ emerge [2].

1.10 Summary

In this chapter, we begun by discussing the spontaneous breaking of continuous space translation symmetry into a discrete one, which leads to the formation of space crystals. Using this analogy, we introduced Wilczek’s proposal of time crystals based on the Aharonov-Bohm ring system, briefly discussing both the concept and the challenges that led to the conclusion that such isolated systems cannot exist.

Given the impossibility of realizing Wilczek’s original idea, we then explored the concept of discrete time crystals, focusing on the system of ultra-cold atoms bouncing on an oscillating mirror, where the particles interact via a contact potential. To build intuition and identify suitable parameters for stable system dynamics, we first analyzed the classical behavior of a single particle in the system, employing action-angle variables to simplify the description and derive the secular Hamiltonian that describes the particle’s motion near the resonant orbit.

We then presented phase-space portraits revealing resonant islands emerging due to the oscillating mirror’s perturbation. These resonant islands represent stable resonant trajectories. If the perturbation becomes too strong, the phase-space portraits become covered by chaotic regions. Using the phase-space portraits, we validated the effective Hamiltonian and determined the optimal parameters for the oscillating mirror such as amplitude and frequency.

Afterward, we moved to the quantum description of a single particle bouncing on an oscillating mirror, where we derive the quantum secular Hamiltonian and identified the Floquet states, which act as the system’s *quasi-eigenstates*. These T -periodic states reflect the periodic nature of the underlying Hamiltonian. For a given resonance number s , we obtained s Floquet states within each energy band. Each Floquet state consists of s localized wave packets, which can be isolated by forming appropriate superpositions of Floquet states. These localized wavefunctions, referred to as Wannier states, form the resonance Hilbert subspace used for subsequent many-body analysis.

Using the single-particle Wannier basis, we constructed the many-body Floquet Hamiltonian and discussed the stability of the lowest quasi-energy state. To satisfy both energy and symmetry requirements, the many-body Floquet state is T -periodic but highly vulnerable to perturbations. For instance, measuring a single particle’s position can induce spontaneous discrete time translation symmetry breaking, causing the system to evolve with a period s times longer than the initial one. This marks the emergence of the discrete time crystal.

As the final tool used to describe ultra-cold atoms bouncing on an oscillating mirror in the thesis, we introduced the mean-field description, where the spontaneous time translation symmetry breaking is characterized by bifurcation.

This chapter provided a detailed description of ultra-cold atoms bouncing on an oscillating mirror; however, the approach outlined here can be applied to any periodically driven system exhibiting nonlinear resonances [20, 104–107, 109].

It is worth noting that the system under consideration serves not only as a platform for realizing discrete time crystals but also for investigating a range of condensed matter phenomena in the time domain, including Anderson localization, topological time crystals, many-body

localization, Mott insulator phases, and time lattices with exotic interactions [[113–116](#)].

Chapter 2

Stability of discrete time crystals

In this chapter, we analyze the stability of discrete time crystals against experimental imperfections and quantum many-body fluctuations. We begin by discussing the phenomenon of spontaneous discrete time translation symmetry breaking, drawing connections to $\mathcal{Z}_2$ symmetry breaking (Section 2.1).

Next, we examine the effects of experimental imperfections on discrete time crystal stability in Section 2.2, starting with the Gaussian approximation of the symmetry-broken state (Section 2.2.1) and validating this approach in both weak and sufficiently strong interaction regimes (Section 2.2.2). We introduce the Bayesian optimization method (Section 2.2.2) and propose a method to restore the discrete time crystal phase in experiments (Section 2.2.4).

Finally, we analyze the impact of quantum fluctuations on discrete time crystal stability in Section 2.3. We start by introducing the $U(1)$ symmetry-preserving Bogoliubov approach (Section 2.3.1), then employ this approach and present the results of this analysis (Section 2.3.2).

The chapter concludes with a brief summary in Section 2.4. Sections 2.2 and 2.3 are based on the paper by *A. Kuroś, R. Mukherjee, W. Golletz, F. Sauvage, K. Giergiel, F. Mintert and K. Sacha* [12].

2.1 Spontaneous symmetry breaking and DTC lifetime

In the previous chapter, we introduced the concept of a DTC. Specifically, in Section 1.8, we discussed the formation of DTCs using the Bose-Hubbard Hamiltonian $\hat{H}_F$ in a time-periodic basis (1.39). This model effectively describes ultra-cold atoms bouncing on an oscillating mirror for $s : 1$ resonance, where the spontaneous breaking of discrete time translation symmetry causes the system to evolve with a period sT , rather than the driving period T . In this section, we further explore spontaneous discrete time translation symmetry breaking. Additionally, we discuss the lifetime of DTCs.

Double-well potential: Bose-Hubbard Hamiltonian

The Hamiltonian $\hat{H}_F$ (1.39) also describes bosons trapped in a symmetric s -well potential [118], where the symmetry breaking manifests as self-trapping of all the atoms in one of the s wells when the attractive interactions are sufficiently strong. Let us focus on spontaneous symmetry breaking in the Bose-Hubbard model with a double-well potential, i.e. $s = 2$, possessing $\mathbb{Z}_2$ symmetry and corresponding to the DTC evolving with period $2T$. We consider sufficiently strong attractive interactions, where the interaction coefficients satisfy $|U_{11}| \ll |U_{12}|$ (1.41). In Figure 2.1, we show the spectrum of Hamiltonian (1.39) as a function of the parameter γ , defined as:

$$\gamma = \frac{(N-1)(U_{11} - 2U_{12})}{J}, \quad (2.1)$$

where tunneling amplitude J is defined in (1.40). For $\gamma < -1$, the low quasi-energy spectrum consists of nearly degenerate pairs of the eigenvalues of Hamiltonian (1.39), and the point $\gamma = -1$ corresponds to a critical point where a quantum phase transition occurs [117]. In other words, in the limit of an infinite number of particles $N \rightarrow \infty$ and the interaction strength $g_0 \rightarrow 0$, but keeping $g_0 N = \text{const.}$ and $\gamma = \text{const.} < -1$, the ground level of the system is doubly degenerate with the symmetry-preserving eigenstates that resemble Schrödinger cat-like states (1.45):

$$|\psi_{\pm}\rangle \approx (|N, 0\rangle \pm |0, N\rangle)/\sqrt{2}. \quad (2.2)$$

These states are extremely sensitive to perturbations such as measurement of particle position, which causes the state to collapse to either $|N, 0\rangle$ or $|0, N\rangle$ depending on which well the particle is detected. This self-trapping phenomenon, a manifestation of $\mathbb{Z}_2$ symmetry

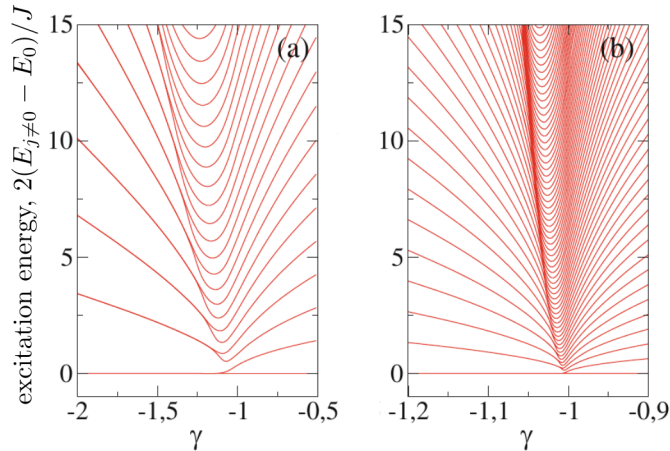


Figure 2.1: Spectrum of the Bose-Hubbard Hamiltonian (1.39) for a double-well potential, as a function of the parameter γ (2.1), presented in terms of the energy difference $2(E_{j \neq 0} - E_0)/J$ for (a) $N = 200$ and (b) $N = 10^4$ particles. In the limit $N \rightarrow \infty$ and $g_0 \rightarrow 0$ while keeping $g_0 N = \text{const.}$, a quantum phase transition occurs at $\gamma = -1$, leading to a near-degeneracy of the two lowest eigenvalues for $\gamma < -1$. The figure is adapted from [117].

breaking [118] which corresponds to spontaneous time translation symmetry breaking that leads to DTC formation.

The lifetime τ of the symmetry-broken state, either $|N, 0\rangle$ or $|0, N\rangle$, is determined by the energy difference between the eigenvalues E_+ and E_- of the eigenstates $|\psi_{\pm}\rangle$. As $N \rightarrow \infty$ while keeping $\gamma = \text{const.} < -1$, the lifetime τ grows exponentially with the number of particles:

$$\tau \propto \frac{1}{E_- - E_+} \propto e^{\alpha N}/N, \quad (2.3)$$

where $\alpha > 0$ [2, 20]. This indicates that the symmetry-broken states, either in a double-well potential or as a DTC, live forever within the description given by Hamiltonian (1.39) [2, 19, 20].

Space crystals: lifetime

On the other hand, the discussion of DTC's lifetime parallels the lifetime analysis of an ordinary space crystal, where spatial crystalline structures emerge due to the breaking of continuous space translation symmetry into a discrete one, as shown in Fig. 1.1. In the limit $N \rightarrow \infty$, and volume $V \rightarrow \infty$ while keeping the density $N/V = \text{const.}$, the energy of the symmetry-broken state becomes infinitely close to the ground state energy leading to an infinite lifetime [19, 119].

Double-well potential: Lipkin-Meshkov-Glick model

Furthermore, the Bose-Hubbard Hamiltonian for a double-well system (1.39) can be mapped to the Lipkin-Meshkov-Glick model (LMG) written in terms of the spin operators [120]:

$$\hat{H}_{\text{LMG}} \approx J \left(-\hat{S}_x + \frac{\gamma}{N} \hat{S}_z^2 \right) = J \left(-\sum_{j=1}^N \hat{s}_j^x + \frac{\gamma}{N} \sum_{j,j'=1}^N \hat{s}_j^z \hat{s}_{j'}^z \right). \quad (2.4)$$

where:

$$\begin{aligned} \hat{S}_x &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) = \sum_{j=1}^N \hat{s}_j^x = \frac{1}{2} \sum_{j=1}^N \hat{\sigma}^x, \\ \hat{S}_z &= \frac{1}{2}(\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1) = \sum_{j=1}^N \hat{s}_j^z = \frac{1}{2} \sum_{j=1}^N \hat{\sigma}^z, \end{aligned} \quad (2.5)$$

Here, $\hat{\sigma}^{x,z}$ are the spin-1/2 Pauli matrices. This model describes 1D chain of spins-1/2 with an infinite-range coupling in the presence of an external magnetic field along the x -axis.

The LMG Hamiltonian (2.4) exhibits $\mathbb{Z}_2$ symmetry, commuting with the spin flip operator $e^{i\pi \hat{S}_x}$. The mean-field approach predicts spontaneous breaking of the $\mathbb{Z}_2$ symmetry in the limit $N \rightarrow \infty$ with $\gamma = \text{const.} < -1$, where all eigenstates below the so-called critical energy threshold $E^* = -JN/2$ correspond to symmetry-broken states [12, 20, 44]. This is analogous to the spontaneous discrete time translation symmetry breaking phenomenon that occurs in ultra-cold atoms bouncing on an oscillating mirror, cf. Section 1.9.

In the subsequent section, we move to analyze the stability of DTCs against experimental imperfections and quantum fluctuations. The presented results are based on reference [12].

2.2 DTC stability against experimental imperfections

In the experimental realization of a DTC, direct observation of the spontaneous breaking of time-translation symmetry is not possible. The symmetry-preserving state (1.45), resembling a Schrödinger cat-like state, is extremely vulnerable to any perturbation and collapses to one of the symmetry-broken states (1.47) before experiment begins. However, we can still detect the formation of a DTC by analyzing the dynamics of the symmetry-broken states over time. For sufficiently strong interactions between particles, the system recurrently returns to its initial state after each period sT , where s denotes the resonance orbit on which the system moves and such an evolution is stable [2, 10].

The symmetry-broken state (1.47) represents the state in which all particles occupy the same Wannier state $w_j(x, t)$, consequently forming a BEC. To experimentally prepare the system in this state, a BEC has to be created in a harmonic trap with frequency ω_0 located at the classical turning point h_0 above the oscillating mirror. This point corresponds to the maximum potential energy in the classical description of a single particle (1.22), and places the system on $s : 1$ resonant orbit. For the ideal case, perfect $s : 1$ resonant driving and an optimal initial position of the atomic mirror are assumed, ensuring that at the moment the atoms bounce, the mirror reaches its highest position. When the trap is turned off and the bosons interact sufficiently strong, we expect the system to return to its initial position after each period sT , as schematically shown in Fig. 2.2(a). In the weak interaction case, we expect to observe sequential tunneling of all atoms to the nearest neighbor unoccupied Wannier state $w_{j\pm 1}(x, t)$ over the time $t = 2\pi/J$, where J is the tunneling coefficient (1.40), which then progresses to subsequent states $w_{j\pm 2}(x, t)$, and so forth, continuing cyclically back to $w_j(x, t)$ [20]. Conversely, with sufficiently strong attractive interactions, the system is expected to remain in the initial state $w_j(x, t)$ throughout the evolution. This characterization of DTC in a bouncing ultra-cold atom system is unique compared to DTCs realized in spin systems, where detuning from perfect conditions is required to facilitate the emergence of DTC [3, 4, 41].

In the following subsections, we analyze the robustness of the DTC against variations in the initial preparation of the BEC, exploring whether introduced imperfections can disrupt the evolution of DTC. We approximate the initial symmetry-broken state $w_j(x, t)$ with the Gaussian function $\phi(x, t = 0)$, which allows us to introduce further imperfections such as variations in the initial average position h and momentum p of the particles:

$$\phi(x, t = 0) = \left(\frac{\omega_0}{\pi}\right)^{1/4} e^{-(\omega_0(x-h)^2/2) + ipx}. \quad (2.6)$$

Such a Gaussian state approximates the initial BEC in the harmonic trap very well. The harmonic trap frequency ω_0 is chosen such that the width of the atomic cloud matches the width of the Wannier state $w_j(x, t)$ at the classical turning point h_0

$$h_0 = 2\pi^2/\Omega^2, \quad (2.7)$$

where Ω is the frequency of unperturbed particle motion (1.23), see Fig. 2.2(b). To analyze the stability of the symmetry broken state under different imperfections, we define the quantum

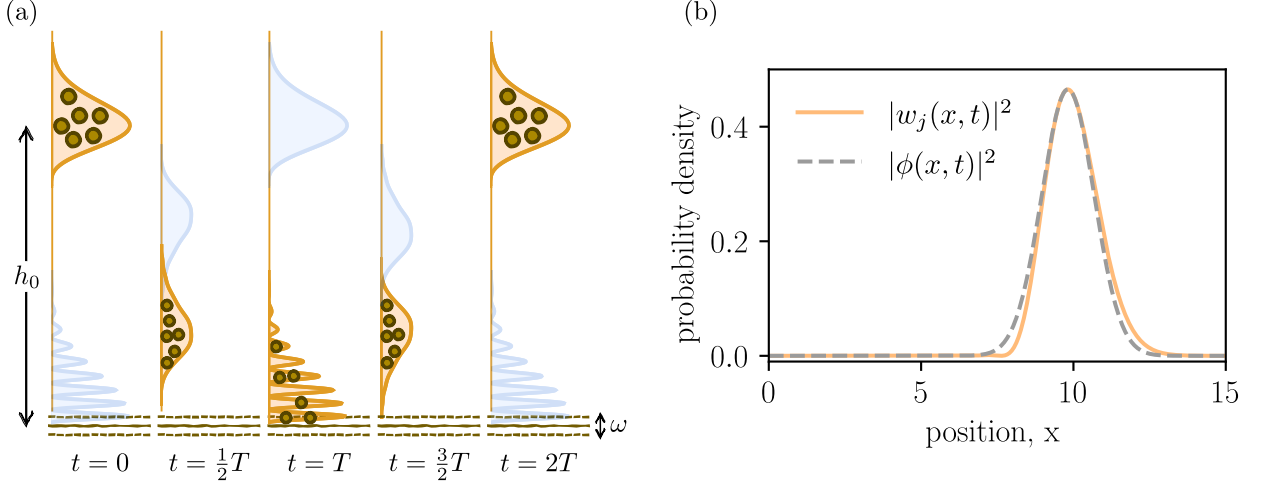


Figure 2.2: (a) Schematic evolution of the symmetry-broken state (orange wave packet) for $s = 2$ resonance case, where all atoms occupy the same Wannier state $w_j(x, t)$ with $j = 1$ or $j = 2$ (1.47). This state results from spontaneous discrete time translation symmetry breaking; for more details see Section 1.8. The system evolution starts from the classical turning point h_0 towards the oscillating mirror. Then, the particles bounce off the mirror at $t = T$, as indicated by interference patterns, and return to the initial position by $t = 2T$. With sufficiently strong interactions, the symmetry-broken state remains robust against perturbations, maintaining the discrete time crystal phase. Conversely, under weak interactions, atoms tunnel to a second, initially unoccupied Wannier state (blue wave packet), after a tunneling time $t_{\text{tun}} = \pi/J$, where J being the tunneling coefficient (1.40), then cyclically return to the initial state, repeating the process throughout their evolution. (b) An initial single-particle state approximated by a Gaussian function $\phi(x, t)$ (2.6), which models a Wannier state $w_j(x, t)$ at time $t = 0$ under optimal conditions; average momentum $p = 0$, and the average position at the classical turning point $h = h_0$. The parameters for the Gaussian state are $\omega_0 = 0.68$, $h_0 = 9.82$. The Hamiltonian (1.19) parameters include a mirror oscillation frequency $\omega = 1.4$, amplitude $\lambda = 0.12$, and the $s = 2$ resonance case.

fidelity function:

$$F(t) = |\langle \phi(x, t) | \phi(x, t = 0) \rangle|^2, \quad (2.8)$$

which quantifies the overlap between the evolved state $\phi(x, t)$ and the initial state.

Before, we move to analyze the DTC's stability against imperfections, we analyze the stability of DTC for optimal Gaussian state to get the insight into the system behaviour in the weak and sufficiently strong interactions regimes.

2.2.1 Stability of DTC for optimal Gaussian state

We focus on the 2 : 1 resonance case where, after spontaneous breaking of discrete time-translation symmetry, the state of N ultra-cold atoms is a product of the same Wannier state $w_j(x, t)$ with $j = 1$ or $j = 2$. To verify the validity of approximating the state $w_j(x, t)$ by Gaussian function $\phi(x, t)$ (2.6), we prepare $\phi(x, t)$ at $t = 0$ under optimal conditions,

with zero average momentum $p = 0$ and centered at the classical turning point $h = h_0$, as depicted in Fig. 2.2(b). The overlap between the Wannier state and the approximated Gaussian state is quantified by $|\langle \phi(x, t = 0) | w_j(x, t = 0) \rangle|^2 = 0.97$, indicating that the exact $2T$ -periodic solution of Eq. (1.50) is well reproduced. We then evolve the system according to the GPE (1.51) and calculate the fidelity function (2.8) at stroboscopic times $t = 0, T, 2T, 3T, \dots$, where $T = 2\pi/\omega$ is the period of the driving force. Given the $2T$ -periodicity of the symmetry-broken state, we expect $F(t) = 0$ at odd multiples of T and $F(t) = 1$ at even multiples of T , provided the interactions are sufficiently strong.

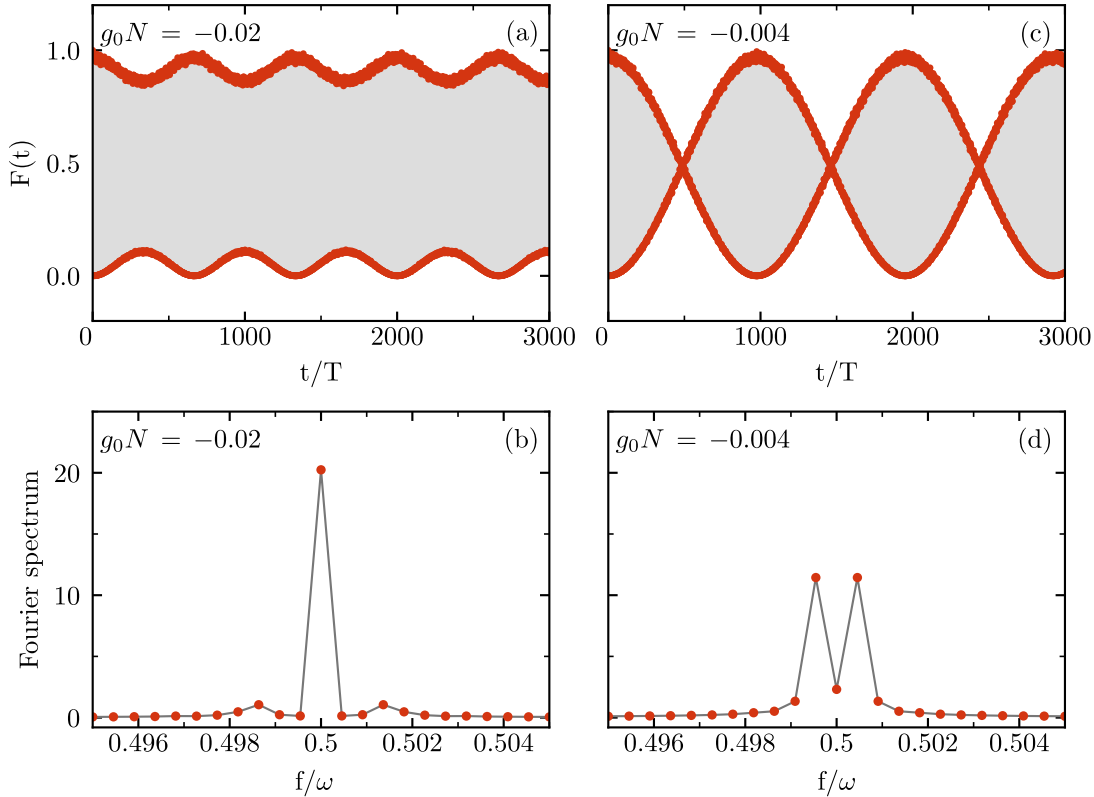


Figure 2.3: The quantum fidelity function $F(t)$ (2.8) between an initial symmetry-broken state, modeled by the Gaussian function $\phi(x, t = 0)$ (2.6), and evolved state according to the Gross-Pitaevski equation (1.51) at stroboscopic times $t = 0, T, 2T, 3T, \dots$. The system exhibits the discrete time crystal phase for sufficiently strong interaction $g_0N = -0.02$ (a). In that regime, the particles tunneling to the initially unoccupied Wannier state is suppressed, as detailed in the main text. Consequently, the Fourier transform of the fidelity function shows a single sharp peak at half the driving frequency $\omega/2$ (b). In contrast, the fidelity function for weak interaction $g_0N = -0.004$ exhibits a beating phenomenon indicating slow tunneling between two Wannier states within the $2 : 1$ resonance Hilbert subspace (c). Consequently, its Fourier transform shows two peaks around $\omega/2$ (d). The parameters for the Gaussian function (2.6) and the Hamiltonian (1.19) are consistent with those in Fig. 2.2(b). The figure is adapted from [12].

Figure 2.3(a) presents the quantum fidelity function (2.8) over stroboscopic times. It is oscillating between zero and one, as expected. We do not observe any signature of tunneling to the second, initially unoccupied symmetry-broken state, cf. Fig. 2.2(a). The lack of tunneling is a consequence of the system being in a sufficiently strong interaction/DTC regime, i.e. $|g_0 N| > |g_{\text{cr}} N| \approx 0.006$ [2], where tunneling to another state is suppressed. Slight fluctuations around $F(t) = 0$ and $F(t) = 1$ arise from the imperfect approximation of the Wannier state $w_j(x, t)$ by the Gaussian function $\phi(x, t)$ (2.6). The stable $2T$ -periodic evolution of the system is reflected in the Fourier transform of the fidelity function $F(t)$, which displays a single peak at half the driving frequency $\omega/2$, see Fig. 2.3(b). This subharmonic response is a signature of the DTC phase, indicating that the system's evolution period is twice the driving period.

In the weakly interacting regime $|g_0 N| < 0.006$, the quantum fidelity function (2.8) exhibits a beating phenomenon, as illustrated in Fig. 2.3(c). This occurs within the $s = 2$ resonant Hilbert subspace, spanned by two Wannier states $w_j(x, t)$. Initially, all atoms occupy nearly perfectly the same Wannier state, while the other remains unoccupied. The beating indicates a slow transfer of atoms from the occupied Wannier state to the unoccupied one, cf. Fig. 2.2(a). After a tunneling period of $t_{\text{tun}} = \pi/J \approx 1000T$, where $J = 7.26 \times 10^{-4}$, all atoms tunnel from the initial Wannier state to the second one, indicating the absence of a DTC phase. The process reverses after another t_{tun} , and the cycle repeats. The dynamics continues with a period of $2\pi/J$, resulting in the Fourier transform of the fidelity function displaying two peaks around $\omega/2$.

Next, we simulate imperfections in the initial state preparation to analyze the stability of DTCs. Before proceeding, we introduce the Bayesian optimization method, which is used to determine the boundary between DTC and symmetry-preserving regimes.

2.2.2 Bayesian optimization

Bayesian optimization is a method used for finding a global maximum of an objective function $G(\mathbf{x})$ with relatively few evaluations of $G(\mathbf{x})$. While it can also be applied to finding minima, this discussion focuses on the maximization aspect. This method is particularly useful when each evaluation of $G(\mathbf{x})$ is costly, time-consuming, or when direct access to the function's gradient is not available [121–128]. The method also handles uncertainty in its model predictions, such as noisy data [129] or when the objective function $G(\mathbf{x})$ behaves unpredictably [130].

Optimization of $G(\mathbf{x})$ involves searching for an argument $\mathbf{x}$ that results in the maximum of $G(\mathbf{x})$:

$$\mathbf{x}^{\text{opt}} = \underset{\mathbf{x}}{\text{argmax}} G(\mathbf{x}), \quad (2.9)$$

where $\mathbf{x}^{\text{opt}}$ represents the optimal set of parameters for the function $G(\mathbf{x})$. Bayesian optimization consists of:

- A probabilistic model $p(g)$ for $G(\mathbf{x})$ which provides a mean and variance describing the expected value and uncertainty of $G(\mathbf{x})$ predictions.

- An acquisition function $\alpha(\mathbf{x})$ that guides the selection of the next points to evaluate by considering both the current model's predictions and the uncertainty associated with the predictions of the probabilistic model. This function converts the probabilistic model into a strategy for selecting the next evaluation points. Common types of acquisition functions include Expectation Improvement, Probability of Improvement, and Upper Confidence Bound. In this work, we use the latter:

$$\alpha_{\text{UCB}}(\mathbf{x}) = \mu_g(\mathbf{x}) + r\sigma_g(\mathbf{x}), \quad (2.10)$$

where r is a user-defined parameter that balances exploration and exploitation. The mean μ_g and the standard deviation σ_g are parameters of the probabilistic model $p(g)$.

- Iterative updating where the probabilistic model and acquisition function are updated at each iteration.

The essence of Bayesian optimization lies in its ability to strategically select evaluation points to efficiently reconstruct the objective function and reduce computational overhead. Unlike manual search, random search [131], or grid search [132], where past performance is not utilized in future iterations, Bayesian optimization leverages knowledge from past iterations to enhance the speed and efficiency of the algorithm.

The implementation steps are as follows:

1. Select a probabilistic model to approximate $G(\mathbf{x})$ with a well-behaved function g that has a prior distribution, and choose an appropriate acquisition function $\alpha(\mathbf{x})$.
2. Begin with an initial set of points $\mathbf{x} = [x_1, x_2, \dots, x_n]$ and evaluate $G(\mathbf{x})$ at these points.
3. Update the probabilistic model as a predictive landscape of $G(\mathbf{x})$ using Bayes' rule:

$$p(g|\mathbf{y}) = \frac{p(g)p(\mathbf{y}|g)}{p(\mathbf{y})}. \quad (2.11)$$

with $\mathbf{y} = [G(x_1), G(x_2), \dots, G(x_n)]$, and use the acquisition function to find the next points $\mathbf{x}$ for evaluation, then evaluate $G(\mathbf{x})$.

4. Continue updating the model and selecting new evaluation points until the optimal value, $\mathbf{x}^{\text{opt}}$, is found.

2.2.3 Stability of DTC against experimental imperfections

In the experimental realization of DTC on the driving ultra-cold atoms setup, numerous imperfections can arise. For example, the initial preparation of the BEC in a harmonic oscillator trap may not perfectly align with the Wannier state $w_j(x, t)$, reflecting an imperfect trap frequency ω_0 , or inaccuracies in positioning the BEC at the classical turning point h_0 . Additionally, the system may suffer from non-zero initial average momentum. These

imperfections can cause detuning of the atoms from the resonant driving by the oscillating mirror and leads to the loss of stability in the system, and thus the DTC. In this subsection, we evaluate whether the symmetry-broken state remains stable despite these initial imperfections.

Introducing imperfections to the initial state affects the Fourier spectrum of the quantum fidelity function (2.8). Under ideal conditions in a sufficiently strong interaction regime, we observe a single sharp peak in the spectrum, confirming the system's stable evolution and indicating the DTC phase, see Fig. 2.3(a)-(b). Conversely, in the weak interaction regime, the spectrum shows two peaks, indicating continuous tunneling between two Wannier states and characterizing a symmetry-unbroken phase without a stable DTC, see Fig. 2.3(c)-(d). When imperfections are introduced to the initial state in the DTC regime, we expect to find a critical point beyond which particle interactions can no longer stabilize the system's evolution. This critical point is expected to manifest as maximal fluctuations in the peak height at $\omega/2$, i.e. between a single peak and two peaks around $\omega/2$, visible in the Fourier spectrum of the fidelity function in Fig. 2.3(b) and Fig. 2.3(d), marking the boundary between the DTC and symmetry-unbroken phases. The boundary can be interpreted as the system's struggle to maintain a clear DTC identity despite imperfections.

To analyze the system stability, we introduce a detuning ϵ from the 2 : 1 resonance orbit, i.e. perfect resonant driving, by slightly varying the initial position of the Gaussian relative to the classical turning point $h = h_0 + \epsilon$. Additionally, we introduce fluctuations in the initial average momentum of the atoms and in the frequency of the harmonic trap ω_0 by sampling them from uniform distributions over the intervals $p = [-\delta p, \delta p]$ and $\omega_0 = [\tilde{\omega}_0 - \delta\omega_0, \tilde{\omega}_0 + \delta\omega_0]$, respectively. The frequency $\tilde{\omega}_0$ is chosen so that the width of the atomic cloud optimally matches the width of the Wannier state $w_j(x, t)$ at the classical turning point. We encode all the imperfections in the initial Gaussian state (2.6).

We then investigate whether particle interactions can stabilize the DTC evolution in the parameters space spanned by the displacement ϵ and interaction strength g_0N , which results in the phase diagram. We consider m realizations of an initial Gaussian state (2.6) with randomly chosen parameters p , and ω_0 for the given values of ϵ and g_0N . Next, we evolve each state according to the GPE (1.51), calculate the quantum fidelity function (2.8) and its Fourier transform. In the end, we calculate the variance of the Fourier peak at $\omega/2$ over all m realizations of the initial Gaussian state. To locate the critical displacement value ϵ , we apply Bayesian optimization (see Section 2.2.2 for details).

In Figure 2.4(a), we show the variance of the Fourier peak amplitude as a function of the displacement parameter ϵ for various values of g_0N . The maximum value of each variance peak indicates the boundary between the DTC and symmetry-unbroken phases, see Fig. 2.4(b). We extend our analysis to the $s = 4$ resonance, as illustrated in Fig. 2.4(c)-(d). In this case, where the symmetry-broken state evolves with period $4T$, we examine how the peak in the Fourier spectrum changes at $\omega/4$. The boundary between the DTC and symmetry-unbroken phases for both resonance cases, $s = 2$ and $s = 4$, becomes sharper as we move deeper into the strong interaction regime, which is visible through the narrowing of the variance peaks. Additionally, the phase diagrams are not symmetric with respect to the perfect initial posi-

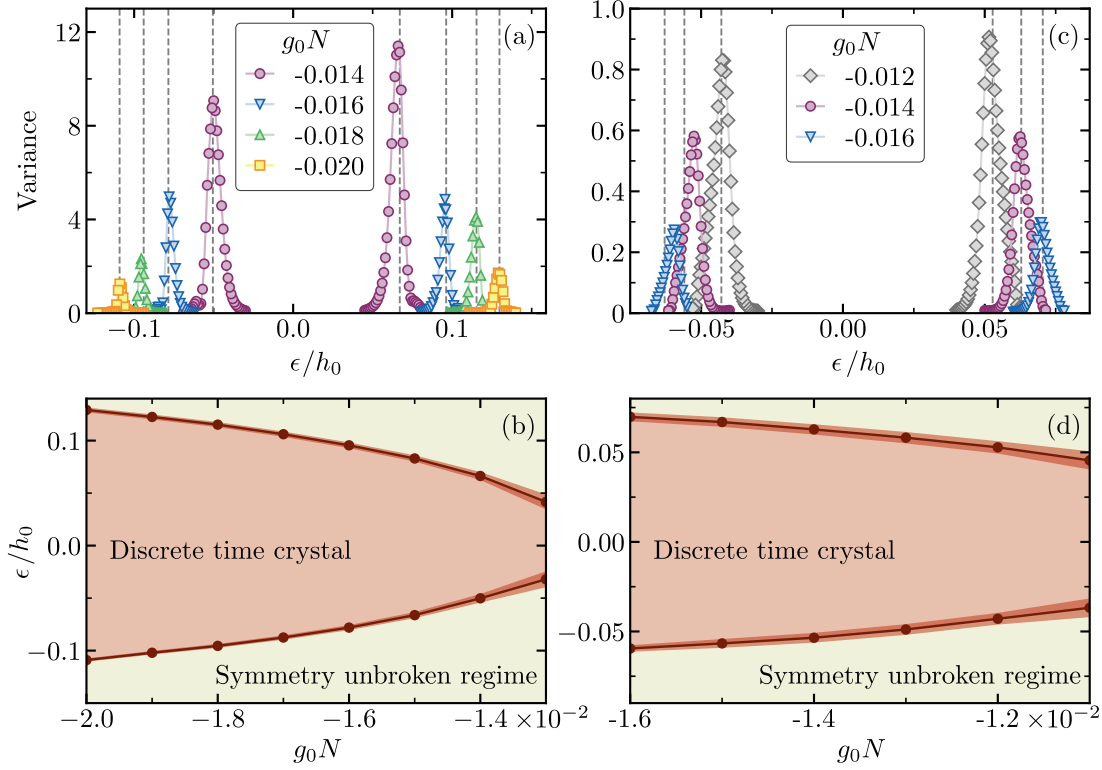


Figure 2.4: Phase diagrams of the discrete time crystal for resonances $s = 2$ (b) and $s = 4$ (d), realized in ultra-cold atoms bouncing on an oscillating mirror setup, depicted as a function of the attractive interaction strength g_0N and the displacement parameter ϵ . The parameter ϵ reflects the imperfect initial positioning of the system on the resonant orbit relative to the optimal one h_0 . The boundaries between the discrete time crystal and symmetry-unbroken phases correspond to the maximum variance of the amplitude peak in the Fourier spectrum of the quantum fidelity function (2.8) at $\omega/2$ for $s = 2$, and $\omega/4$ for $s = 4$, cf. Fig. 2.3. The boundaries are obtained for different realizations of the initial Gaussian state (2.6) with randomly chosen initial average momentum p , and frequency ω_0 of the harmonic trap in which the system is initially prepared, from the uniform distributions $p \in [-\delta p, \delta p]$, and $\omega_0 \in [\tilde{\omega}_0 - \delta\omega_0, \tilde{\omega}_0 + \delta\omega_0]$, respectively. The variance is plotted for resonances $s = 2$ (a), and $s = 4$ (b) as a function of the parameter ϵ for different interaction strengths g_0N . The red shaded areas around the boundaries in the phase diagrams (b, d) represent the standard deviations of the variance shown in panels (a, c). The dashed lines visible in panels (a, c) are predictions of the maximum variance obtained using the Bayesian optimization method. For $s = 2$ ($s = 4$), the parameters are $\omega = 1.40$ (1.79), $\lambda = 0.12$ (0.12), $h_0 = 9.82$ (24.44), $\tilde{\omega}_0 = 0.68$ (0.50), $\delta p = 0.10$ (0.05), and $\delta\omega_0 = 0.02\tilde{\omega}_0$ ($0.02\tilde{\omega}_0$). The figure is adapted from [12].

tion, i.e. $\epsilon = 0$, due to the presence of the gravitational field in the system. As a result, the formation of DTC is more favorable for $h > h_0$.

In analyzing the DTC's robustness against experimental imperfections in the initial state preparation, we assume that the symmetry-broken state consists of a single wave packet

$\phi(x, 0) \approx w_j(x, 0)$. However, this assumption breaks down near the critical interaction strength $g_{\text{cr}}N$ (for $\epsilon = 0$). Specifically, for $|g_{\text{cr}}N| < |g_0N| < 0.013$ ($s = 2$), and $|g_{\text{cr}}N| < |g_0N| < 0.011$ ($s = 4$), the system enters the phase transition regime and the state is no longer represented by a single Wannier state $w_j(x, t)$. This is illustrated in Fig. 2.5 for $s = 40$ resonance case, where the Hilbert subspace is spanned by 40 Wannier states. Within this subspace, the general solution of the GPE (1.50) can be expressed as:

$$\Phi^{\text{GPE}}(x, t) \propto \sum_{j=1}^s a_j w_j(x, t), \quad (2.12)$$

where the occupation amplitudes satisfy the normalization condition $\sum_{j=1}^s |a_j|^2 = 1$.

In the weak interaction regime, the system's state is a uniform superposition of all Wannier states (1.45). However, if the interaction strength increases beyond the critical value, the system transitions through a non-uniform superposition phase, before collapsing into a state where a single coefficient a_j dominates, i.e. all atoms occupy a single Wannier state (1.47).

Therefore, near critical interaction strength $g_{\text{cr}}N$, where the system is represented by non-uniform superposition of Wannier states (2.12), the approach used to analyze system stability against experimental imperfections is not longer valid. A single displacement parameter ϵ is insufficient to capture the full dynamics of the system under these conditions. To address this, a more comprehensive analysis should consider both the relative displacement between the Wannier states and deviations from their optimal superposition.

For larger resonances $s > 4$, the width of the DTC area in the phase diagram can be analyzed numerically using the approach presented in this section, or estimated based on the size of the

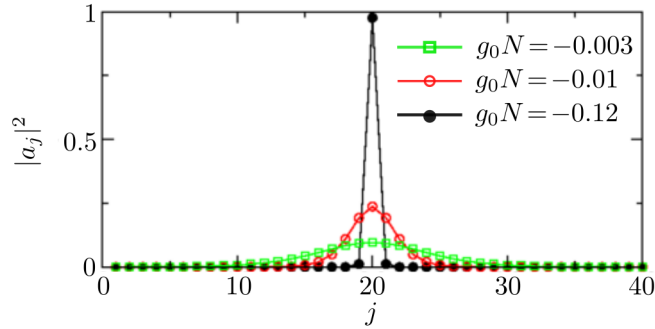


Figure 2.5: Occupation probabilities $|a_j|^2$ corresponding to the lowest quasi-energy solutions of energy functional (1.52), represented by the Wannier states $w_j(x, t)$ (2.12), for different interaction strengths g_0N . For weak attractive interactions $|g_0N| < 1.64 \times 10^{-3}$, the system is in the symmetry unbroken regime, where the state is a uniform superposition of all Wannier states within the resonant Hilbert subspace. As the interaction strength increases, the system transitions through non-uniform superposition phase (green and red lines), reaching a state dominated by a single coefficient a_j at sufficiently strong interactions (black line). The energy functional parameters are as follows: the resonance $s = 40$, $\omega = 4.9$, $\lambda = 0.2$. The figure is adapted from [91].

classical resonant islands for a single particle bouncing on an oscillating mirror, see Section 1.6 for the details. The DTC area scales proportionally to the amplitude λ of the driving force, provided that $\lambda \leq 0.2$, to keep the system in the vicinity of the resonant orbit,

$$\frac{\Delta\epsilon}{h_0} \approx \frac{8\sqrt{\lambda}}{s\pi}, \quad (2.13)$$

where $\Delta\epsilon$ is a range of displacement from the classical turning point h_0 .

In the next section, we analyze the possibility of adjusting the mirror's oscillation frequency ω to accommodate larger derivations in the initial positioning of the BEC, thereby maintaining the system within the DTC phase.

2.2.4 Mirror frequency optimization for DTC

In the previous section, we analyzed the system's behavior under various realizations of the initial Gaussian state (2.6), including imperfections such as fluctuations in the initial average momentum p and frequency ω_0 of the harmonic trap. We explored the space spanned by interaction strength g_0N and the displacement ϵ of the initial position to identify the boundary between the DTC and symmetry-unbroken regimes. However, in experimental realizations of the DTC, the initial position of the atoms with respect to the location of the atomic mirror can be not exactly known, causing the system being beyond the DTC phase, see the phase diagrams in Fig. 2.4(b) and Fig. 2.4(d). Despite this, we demonstrate in this section that adjusting the mirror's oscillation frequency can restore the DTC phase. Adjusting the mirror's oscillation frequency according to the initial position of the system not only changes its periodicity but also adjusts the resonant orbit on which the system evolves. To determine the optimal frequency ω_{opt} of the mirror oscillations, we employ the quantum fidelity function (2.8) as the objective function $F(\omega)$ in Bayesian optimization, as detailed in Section 2.2.2:

$$F(\omega) = |\langle\phi(t=0)|\phi(t=sT)\rangle|^2, \quad (2.14)$$

where s is the resonance number, and $T = 2\pi/\omega$ is the initial period of the mirror oscillations. The function $F(\omega)$ compares an initial Gaussian state (2.6) with the state after sT periods. We modify the frequency ω using the Bayesian optimization, adjusting it to the frequency of the unperturbed motion, i.e. $\omega \approx s\Omega(h)$, where $\Omega(h)$ is determined by the initial location of the atomic cloud, which is not precisely known and can fluctuate between experimental realizations.

Each optimization step involves generating $m = 20$ realizations of the initial state with the position h , initial average momentum p , and the frequency of the harmonic trap ω_0 randomly chosen from the uniform distributions $h \in [\bar{h} - \delta h, \bar{h} + \delta h]$, $p \in [-\delta p, \delta p]$, and $\omega_0 \in [\tilde{\omega}_0 - \delta\omega_0, \tilde{\omega}_0 + \delta\omega_0]$, respectively. The frequency $\tilde{\omega}_0$ is chosen based on the assumed classical turning point h_0 . For each realization, we compute the fidelity $F(\omega)$, average these values, and use the result as input to the optimizer. We restrict the displacement h to ensure that the initial state is positioned above the classical turning point h_0 , $\bar{h} - h_0 \approx 0.3h_0 \gg$

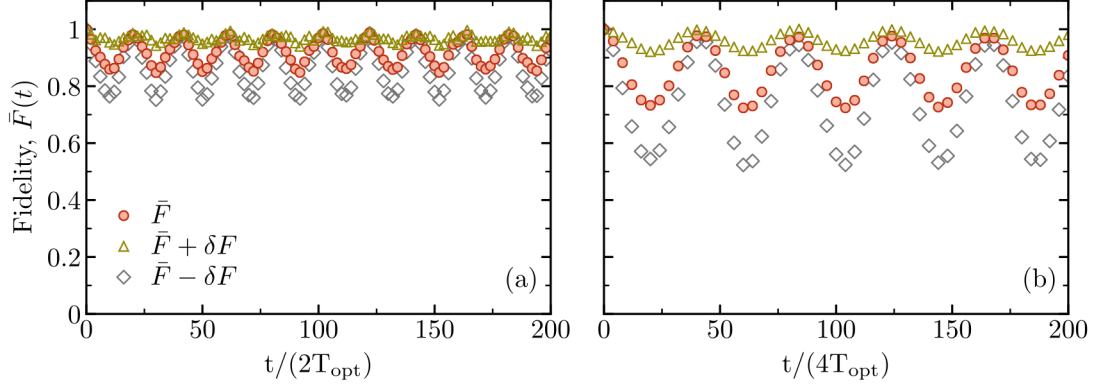


Figure 2.6: Averaged fidelity function $\bar{F}(t)$ (2.8) over $m = 20$ realizations of the initial state (2.6) for stroboscopic times $t = n(sT_{\text{opt}})$ where $n = 0, 1, \dots, 200$ (red dots), for resonances $s = 2$ (a), and $s = 4$ (b). The parameters of the initial Gaussian state (2.6) for each realization, are chosen randomly such an initial position h , frequency of the trap ω_0 , initial average momentum p from uniform distributions $h \in [\bar{h} - \delta h, \bar{h} + \delta h]$, $\omega_0 \in [\bar{\omega}_0 - \delta\omega_0, \bar{\omega}_0 + \delta\omega_0]$, and $p \in [-\delta p, \delta p]$, respectively. The optimal period of the mirror oscillations $T_{\text{opt}} = \frac{2\pi}{\omega_{\text{opt}}}$ corresponds to the optimized frequency ω_{opt} determined after fifteen iterations of Bayesian optimization. Yellow triangles correspond to $\bar{F} + \delta F$, and gray squares denote $\bar{F} - \delta F$, where δF is the standard deviation. The averaged fidelity $\bar{F}$ in both panels remains above 70%, indicating stable evolution of the symmetry-broken state within the discrete time crystal phase. For $s = 2$ ($s = 4$), the parameters are $\bar{h} = 12.78$ (31.77), $\delta h = 0.05\bar{h}$ (0.03 $\bar{h}$). Additional parameters are consistent with those in Fig. 2.4. The figure is adapted from [12].

δh , preventing the initial Gaussian state $\phi(x, t = 0)$ from overlapping with more than one Wannier state, which occurs when $\bar{h} - h_0 < -0.1h_0$.

As a result, we determine the optimal frequency of the mirror oscillations ω_{opt} . In Figure 2.6, we show the fidelity function (2.8) averaged over $m = 20$ realizations of the initial state, each with a randomly chosen initial position h . The evolution of the system maintains an average fidelity $\bar{F}(t)$ above 70% for both considered resonances $s = 2$, and $s = 4$. Without adjusting the mirror oscillation frequency to compensate for imperfections in the initial state's position, the fidelity decreases to zero after a few periods. Consequently, by knowing the average value of $\bar{h}$, which depends on the experimental setup, we can adjust ω_{opt} to ensure the stable evolution of DTCs, thus remaining in the DTC phase.

In the remaining sections, we analyze the DTC's stability against quantum fluctuations.

2.3 Stability of DTC against quantum fluctuations

In Section 2.2, we analyzed the stability of the symmetry-broken state against various perturbations that may occur during the experimental realization of a DTC. We performed the analysis within a many-body Hilbert subspace of product states, assuming that the atomic

cloud forms a perfect BEC and governed the system's dynamics using the GPE (1.51).

We now shift our focus to many-body effects beyond the mean-field approximation. Interactions between bosons may couple the subspace of product states with its complement, potentially leading to quantum depletion of the condensate. To this end, we employ the $U(1)$ symmetry-preserving Bogoliubov approach [112, 133–135], a method for describing weakly interacting bosonic systems, which conserved the total number of particles. By introducing a small perturbation consisting of non-condensed modes and assuming that the number of particles in the condensate exceeds those in the non-condensed modes, the Hamiltonian (2.15) can be expanded to include second-order corrections. These corrections account for the transfer of bosons between condensate and non-condensate modes. The modified Hamiltonian is then diagonalized using the Bogoliubov transformation, which introduces a quasi-particle description. Analyzing these quasi-particle modes allows us to quantify particle depletion from the condensate over time.

The Bogoliubov approach provides a framework for understanding how interactions between bosons can lead to the depletion of the condensate mode and helps assess whether these interactions destabilize the DTC phase. We begin by introducing the $U(1)$ symmetry-preserving Bogoliubov approach and then proceed to analyze the stability of DTCs against quantum fluctuations.

2.3.1 $U(1)$ symmetry-preserving Bogoliubov approach

The general many-body Hamiltonian for N ultra-cold particles interacting via a contact Dirac-delta potential $V(x_k - x_l) = g_0 \delta(x_k - x_l)$, within a trapping time-independent potential $U(x)$, in terms of bosonic field operators $\hat{\Psi}(x)$ is given by:

$$\hat{H} = \int dx \hat{\Psi}^\dagger(x) \left[-\frac{1}{2} \partial_x^2 + U(x) + \frac{g_0}{2} \hat{\Psi}^\dagger(x) \hat{\Psi}(x) - \mu \right] \hat{\Psi}(x). \quad (2.15)$$

where $-\mu \int dx \hat{\Psi}^\dagger(x) \hat{\Psi}(x) = -\mu N$ shifts the energy scale, assuming conservation of the total particle number.

In the weak interaction regime, the many-body ground state of (2.15) is approximated by a product state $\Phi(x_1, x_2, \dots, x_N) \propto \prod_{k=1}^N \phi_0(x_k)$, representing an ideal condensate where all particles occupy the same single-particle (condensate) mode $\phi_0(x)$. This state, in second quantization, is described as:

$$|N\rangle = \frac{1}{\sqrt{N!}} \left(\hat{a}_0^\dagger \right)^N |0\rangle, \quad (2.16)$$

where $\hat{a}_0^\dagger$ ($\hat{a}_0$) creates (annihilates) a boson in the condensate mode $\phi_0(x)$.

However, the product state $|N\rangle$ does not fully capture the exact ground state of a weakly interacting system, as particle correlations are still significant. Interactions between particles can perturb the system during its evolution, potentially causing some bosons to transition out of the condensate mode and into non-condensate states. Our focus is on analyzing the system's stability over time, examining how these interactions influence the persistence of

the system near the product state $|N\rangle$ and whether they lead to a gradual depletion of the condensate mode.

Assumptions of the analysis

Our starting point is the assumption that most particles occupy the condensate mode in the many-body state denoted by $|\Psi\rangle$:

$$\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = \langle \Psi | \hat{a}_0^\dagger \hat{a}_0 | \Psi \rangle = N_0 \simeq N, \quad (2.17)$$

where $N_0 = N$ for a perfect BEC. Formally, the presence of a condensate is indicated when the reduced single-particle density operator $\hat{\rho}_1(x', x) \propto \langle \Psi | \hat{\Psi}^\dagger(x') \hat{\Psi}(x) | \Psi \rangle$ has an eigenvector corresponding to the condensate mode $\phi_0(x)$ with an eigenvalue $N_0 \approx N$ [112]:

$$\int dx \hat{\rho}_1(x', x) \phi_0^*(x) = N_0 \phi_0^*(x'). \quad (2.18)$$

To incorporate the non-condensed components in the analysis, we decompose the bosonic field operator into the condensate part and the non-condensate part:

$$\hat{\Psi}(x) = \phi_0(x) \hat{a}_0 + \delta\hat{\psi}(x), \quad (2.19)$$

where $\delta\hat{\psi}(x) = \sum_{j>0} \phi_j(x) \hat{a}_j$ represents the non-condensate part. It is defined using the projection operator $\hat{Q}$, which projects the field operator $\hat{\Psi}(x)$ onto the space orthogonal to $\phi_0(x)$:

$$\delta\hat{\psi}(x) = \hat{Q} \hat{\Psi}(x) = [1 - |\phi_0(x)\rangle \langle \phi_0(x)|] \hat{\Psi}(x), \quad (2.20)$$

ensuring that $\delta\hat{\psi}$ is orthogonal to $\phi_0(x)$, i.e. its projection on $\phi_0(x)$ is the zero operator:

$$\int dx \phi_0^*(x) \delta\hat{\psi}(x) = 0. \quad (2.21)$$

The commutation relations for $\delta\hat{\psi}(x)$ involve the matrix elements of $\hat{Q}$ that are orthogonal to $\phi_0(x)$:

$$[\delta\hat{\psi}(x_j), \delta\hat{\psi}^\dagger(x_k)] = \delta(x_j - x_k) - \phi_0(x_j) \phi_0^*(x_k). \quad (2.22)$$

The condition (2.18) ensures no coherence in the reduced single-particle density matrix $\hat{\rho}_1(x, x')$ between the condensate and the non-condensed modes:

$$\langle \hat{a}_0^\dagger \delta\hat{\psi} \rangle = 0. \quad (2.23)$$

In our perturbative analysis, $\delta\hat{\psi}(x)$ is treated as a small perturbation compared to dominant condensate term $\phi_0(x) \hat{a}_0$, indicating that the mean number of non-condensed particles is much smaller than that of the condensed particles:

$$\langle \delta\hat{N} \rangle = \int dx \langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle \ll N. \quad (2.24)$$

and the density of the non-condensed particles is significantly lower than the density of the condensate across the entire space:

$$\langle \delta\hat{\psi}^\dagger(x) \delta\hat{\psi}(x) \rangle \ll N_0 |\phi_0(x)|^2. \quad (2.25)$$

Perturbation theory

Substituting the field operator $\hat{\Psi}(x)$ in the form of (2.19) into the many-body Hamiltonian (2.15) results in terms of different orders of $\delta\hat{\psi}(x)$. At zeroth order, where $\delta\hat{\psi}(x)$ is neglected, the system is considered to be an ideal condensate $|N\rangle$. Calculating the average value of the corresponding Hamiltonian in this state $\langle N | \hat{H}^{(0)} | N \rangle$ yields the time-independent GPE:

$$\left[-\frac{1}{2} \partial_x^2 + U(x) + g_0(N-1) |\phi_0(x)|^2 \right] \phi_0(x) = \mu \phi_0(x), \quad (2.26)$$

indicating that $\phi_0(x)$ is the optimal approximation for a single-particle state occupied by a macroscopic number of particles in the presence of interactions.

At the first order, the terms of $\hat{H}^{(1)}$ are grouped as follows:

$$\begin{aligned} & \phi_0(x) \left[-\frac{1}{2} \partial_x^2 + U(x) + g_0(\hat{a}_0^\dagger \hat{a}_0 - 1) |\phi_0(x)|^2 - \mu \right] \hat{a}_0^\dagger \delta\hat{\psi}(x) \\ & + \delta\hat{\psi}^\dagger(x) \hat{a}_0 \left[-\frac{1}{2} \partial_x^2 + U(x) + g_0(\hat{a}_0^\dagger \hat{a}_0 - 1) |\phi_0(x)|^2 - \mu \right] \phi_0(x). \end{aligned} \quad (2.27)$$

Assuming no correlations between the condensate and the non-condensate modes (2.23), this contribution to the Hamiltonian (2.15) vanishes.

At the second order, we consider the direct contributions from non-condensed particles by reformulating terms in $\hat{H}^{(2)}$ to conserve the total number of particles. This involves introducing the field operator [112, 134]:

$$\hat{\Lambda}(x) = \frac{\hat{a}_0^\dagger}{\sqrt{N}} \delta\hat{\psi}(x), \quad (2.28)$$

which commutes with the total number of particles operator $\hat{N}$, and describes the transfer of one non-condensate particle into the condensate. The second-order correction to the many-body Hamiltonian is then expressed as:

$$\hat{H}^{(2)} = f(N) + \frac{1}{2} \int dx \left[\hat{\Lambda}^\dagger(x), \hat{\Lambda}(x) \right] \hat{\mathcal{L}} \left[\begin{matrix} \hat{\Lambda}(x) \\ \hat{\Lambda}^\dagger(x) \end{matrix} \right], \quad (2.29)$$

where the Bogoliubov-de Gennes Hamiltonian $\hat{\mathcal{L}}$ has the form:

$$\hat{\mathcal{L}} = \begin{bmatrix} \hat{Q}(-\frac{1}{2} \partial_x^2 + U(x) + 2g_0 N |\phi_0(x)|^2) \hat{Q} & g_0 N \hat{Q} \phi_0^2(x) \hat{Q}^* \\ -g_0 N \hat{Q}^* \phi_0^{*2}(x) \hat{Q} & -\hat{Q}^*(-\frac{1}{2} \partial_x^2 + U(x) + 2g_0 N |\phi_0(x)|^2) \hat{Q}^* \end{bmatrix} \quad (2.30)$$

The function $f(N)$ depends only on the total number of particles N , and it is a remainder after writing the second-order correction $\hat{H}^{(2)}$ in the compact form (2.29).

Bogoliubov Hamiltonian The second-order Hamiltonian $\hat{H}^{(2)}$ is known as the Bogoliubov Hamiltonian [112, 133] and is diagonalized by means of the Bogoliubov transformation, where the operators $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$:

$$\begin{bmatrix} \hat{\Lambda}(x) \\ \hat{\Lambda}^\dagger(x) \end{bmatrix} = \sum_n \left(\begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix} \hat{b}_n + \begin{bmatrix} v_n^*(x) \\ u_n^*(x) \end{bmatrix} \hat{b}_n^\dagger \right), \quad (2.31)$$

are expanded in the eigenstates of the operator $\hat{\mathcal{L}}$:

$$\begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix} \text{ and } \begin{bmatrix} v_n^*(x) \\ u_n^*(x) \end{bmatrix}, \quad (2.32)$$

which are termed Bogoliubov modes. These modes correspond to eigenvalues $-E_n$ and E_n , respectively [112, 133]. The Bogoliubov modes $u_n(x)$ and $v_n(x)$ satisfy normalization condition:

$$\langle u_n(x) | u_m(x) \rangle - \langle v_n(x) | v_m(x) \rangle = \delta_{nm}. \quad (2.33)$$

and the creation $\hat{b}_n^\dagger$ and annihilation $\hat{b}_n$ operators of quasi-particles follow the bosonic commutation relations:

$$[\hat{b}_n, \hat{b}_m^\dagger] = \delta_{nm}, \quad [\hat{b}_n, \hat{b}_m] = 0. \quad (2.34)$$

Once diagonalized, the Bogoliubov Hamiltonian (2.29) takes the form:

$$\hat{H}^{(2)} = \sum_n E_n \hat{b}_n^\dagger \hat{b}_n + \text{const.} \quad (2.35)$$

The Hamiltonian (2.35) describes a system of non-interacting quasi-particles. The state with no excitations is represented as the Bogoliubov vacuum $|0_B\rangle$, which is annihilated by all $\hat{b}_n$ operators:

$$\forall_n \hat{b}_n |0_B\rangle = 0. \quad (2.36)$$

This analysis concludes with an estimate of the depletion in the Bogoliubov vacuum, quantified by:

$$\begin{aligned} \langle \delta \hat{N} \rangle &= \int dx \langle 0_B | \delta \hat{\psi}^\dagger(x) \delta \hat{\psi}(x) | 0_B \rangle \\ &\simeq \int dx \langle 0_B | \hat{\Lambda}^\dagger(x) \hat{\Lambda}(x) | 0_B \rangle \\ &= \sum_n \langle v_n | v_n \rangle. \end{aligned} \quad (2.37)$$

It describes the impact of particles' interactions on the stability of the condensate, by counting how many particles transition out of the condensate mode. In an ideal condensate, all $v_n(x)$ vanish, indicating no depletion.

Dynamics of the weakly interacting system

Having established the foundations for the Bogoliubov approach, let's summarize our progress so far. We started with a stationary solution $\phi_0(x)$ of the GPE and applied perturbation theory to find corrections due to weak interactions, revealing deviations from the ideal condensate state.

Now, we transition to analyzing the dynamics within the Bogoliubov framework. We assume the system is initially in a product state:

$$\Phi(x_1, x_2, \dots, x_n; t = 0) \propto \prod_{k=1}^N \phi_0(x_k; t = 0), \quad (2.38)$$

indicating no initial depletion $\langle \delta \hat{N} \rangle$ at $t = 0$, where $\phi_0(x, t = 0)$ is an initial condensate wavefunction, which is not necessarily a stationary solution of GPE. The goal is to analyze how the system evolves and how condensate depletion changes over time due to interactions.

The condensate mode $\phi_0(x, t)$ evolves according to GPE:

$$i\partial_t \phi_0(x, t) = \left[-\frac{1}{2} \partial_x^2 + U(x, t) + g_0 N |\phi_0(x, t)|^2 \right] \phi_0(x, t), \quad (2.39)$$

where we consider a general time-dependent potential $U(x, t)$. The Bogoliubov modes follow the Bogoliubov-de Gennes equations [112]:

$$i\partial_t \begin{bmatrix} u_n(x, t) \\ v_n(x, t) \end{bmatrix} = \hat{\mathcal{L}} \begin{bmatrix} u_n(x, t) \\ v_n(x, t) \end{bmatrix}. \quad (2.40)$$

We can initially choose different sets of the Bogoliubov modes. The choice determines properties of the initial Bogoliubov vacuum state [12]. Starting with a Bogoliubov vacuum $|0_B\rangle$, where all components v_n vanish

$$\begin{bmatrix} u_n(x, t = 0) \\ v_n(x, t = 0) \end{bmatrix} = \begin{bmatrix} \chi_n(x) \\ 0 \end{bmatrix}, \quad (2.41)$$

where the set of functions $\{\chi_n(x)\}$ is orthogonal to $\phi_0(x, t = 0)$, and normalized [135]:

$$\langle \phi_0(x, t = 0) | \chi_n(x) \rangle = 0, \quad \langle \chi_n(x) | \chi_m(x) \rangle = \delta_{nm}, \quad (2.42)$$

we choose the initial many-body state (Bogoliubov vacuum $|0_B\rangle$) which corresponds to a perfect BEC, i.e. the depletion $\langle \delta N \rangle = \sum_n \langle v_n | v_n \rangle = 0$. The functions $\chi_n(x)$ should be chosen suitably for the problem under consideration, allowing the evolution of a small number of Bogoliubov modes to achieve convergence in calculating the reduced single-particle density matrix $\hat{\rho}_1(x', x; t)$ and subsequently the condensate depletion $\langle \delta \hat{N} \rangle$.

With the time evolution of both the condensate mode $\phi_0(x, t)$ and the Bogoliubov modes $u_n(x, t)$ and $v_n(x, t)$, we can compute and diagonalize the reduced single-particle density matrix at any given time t [135]:

$$\begin{aligned} \hat{\rho}_1(x', x; t) &\propto \langle \Phi | \hat{\Psi}^\dagger(x', t) \hat{\Psi}(x, t) | \Phi \rangle \\ &\approx N_0(t) \phi_0^*(x', t) \phi_0(x, t) + \sum_n v_n(x', t) v_n^*(x, t) \\ &= N_0(t) \phi_0^*(x', t) \phi_0(x, t) + \sum_j \delta N_j(t) \varphi_j^*(x', t) \varphi_j(x, t), \end{aligned} \quad (2.43)$$

where $\varphi_j(x, t)$ are the eigenstates of $\hat{\rho}_1(x', x; t)$ with eigenvalues δN_j related to the average number of bosons occupying modes $\varphi_j(x, t)$ orthogonal to the condensate mode $\phi_0(x, t)$. The total number of bosons depleted from the condensate at any time is:

$$\langle \delta \hat{N} \rangle(t) = \sum_j \delta N_j(t). \quad (2.44)$$

If BEC is not stable, then the density of non-condensed particles $\langle \delta \hat{N} \rangle(t)$ will diverge over time, until the Bogoliubov approximation fails.

In the next section, we will analyze the stability of DTCs against such quantum fluctuations using the $U(1)$ symmetry-preserving Bogoliubov approach to quantify quantum depletion.

2.3.2 Stability of DTC against quantum fluctuation

We now extend our study of the stability of DTC to include the many-body effects of interactions, using the Bogoliubov framework [112, 133] introduced in the previous section. We focus on the case of a resonance $s = 2$ and DTC initially in a product state:

$$\Phi(x_1, x_2, \dots, x_N; t = 0) \propto \prod_{k=1}^N \phi(x_k, t = 0), \quad (2.45)$$

where $\phi(x, t = 0)$ is the Gaussian function (2.6), prepared under optimal conditions, i.e. with zero average momentum $p = 0$, centered at the classical turning point $h = h_0$, and shaped by a harmonic trap potential with frequency $\tilde{\omega}_0$ which optimally reproduces the Wannier state $w_j(x, t)$, as shown in Fig. 2.2(b).

Within the Bogoliubov approach the initial state (2.45) corresponds to the Bogoliubov vacuum with no depletion. Thus, we have to initialize the Bogoliubov modes $v_n(x, t = 0)$ and $u_n(x, t = 0)$ as in (2.41). We choose $\chi_n(x)$ as the eigenstates of the Hartree-Fock Hamiltonian:

$$\hat{H}_{\text{HF}} = -\frac{1}{2}\partial_x^2 + x + \lambda x + 2g_0 N |\phi(x, t = 0)|^2, \quad (2.46)$$

adjusted such that contributions parallel to the condensate are subtracted $\langle \chi_n(x) | \phi(x, t = 0) \rangle = 0$, and the resulting states are ortho-normalized $\langle \chi_n(x) | \chi_m(x) \rangle = \delta_{nm}$. The choice of the eigenstates of the Hartree-Fock Hamiltonian is adapted to our problem, reflecting the small number of Bogoliubov modes necessary to include for obtaining converged results for the total depletion $\langle \delta \hat{N} \rangle(t)$. The interaction strength g_0 is selected to be in the DTC regime. Then, we simulate the system's evolution using the GPE (2.39):

$$i\partial_t \phi(x, t) = \left[-\frac{1}{2}\partial_x^2 + x + \lambda x \cos(\omega t) + g_0 N |\phi(x, t)|^2 \right] \phi(x, t), \quad (2.47)$$

and Bogoliubov-de Gennes operator, cf. (2.40):

$$\hat{\mathcal{L}} = \begin{bmatrix} \hat{Q}(\hat{H}_0 + 2g_0 N |\phi(x)|^2) \hat{Q} & g_0 N \hat{Q} \phi^2(x) \hat{Q}^* \\ -g_0 N \hat{Q}^* \phi^{*2}(x) \hat{Q} & -\hat{Q}^*(\hat{H}_0 + 2g_0 N |\phi(x)|^2) \hat{Q}^* \end{bmatrix}. \quad (2.48)$$

with the Hamiltonian $\hat{H}_0 = -\frac{1}{2}\partial_x^2 + x + \lambda x \cos(\omega t)$. We expect to observe an increasing quantum depletion (2.44) over time due to presence of interactions. The amount of depleted particles is encoded in the eigenvalues of the reduced single-particle density matrix $\hat{\rho}_1(x', x; t)$ (2.43).

In Figure 2.7(a), we present how the depletion evolves over time. We diagonalize the reduced single-particle matrix $\hat{\rho}_1(x', x; t)$ at each timestep within two tunneling periods $t_f = 2\pi/J$ with $J = 7.26 \times 10^{-4}$. We consider $s = 2$ resonance case, thus the tunneling occurs between two Wannier states $w_1(x, t)$ and $w_2(x, t)$. It turns out that only one eigenmode $\varphi_1(x, t)$, with eigenvalue $\delta N_1(t)$, significantly contributes to the condensate depletion $\langle \delta \hat{N} \rangle(t)$. Throughout the evolution, other modes contribute minimally, with eigenvalues $\delta N_{j \neq 1}(t) \leq 6 \times 10^{-3}$. After a duration equivalent to 1000 bounces off the mirror, or $t_f = 1999T$, where $T = 2\pi/\omega$ is the period of the driving force, the total depletion is:

$$\langle \delta \hat{N} \rangle(t_f) = \sum_j \delta N_j(t_f) \approx \delta N_1(t_f) \approx 1.6, \quad (2.49)$$

indicating that fewer than two atoms are depleted from the condensate and begin to occupy the mode $\varphi_1(x, t)$. Given that a typical experimental setup contains on the order of 10^4 atoms, this quantum depletion is negligible, suggesting that DTC remains stable and robust against quantum fluctuations for the considered duration, which exceeds experimental timescales [91, 92]. Therefore, the mean-field approach and the GPE (2.47) sufficiently accurately describe this system.

In Figure 2.7(b), we present the probability density of the condensate mode $|\phi(x, t_f)|^2$ and the dominant mode $|\varphi_1(x, t_f)|^2$ of the reduced single-particle density matrix $\rho_1(x', x; t_f)$ (2.43). The overlap between the dominant mode $\varphi(x, t_f)$ and the condensate shifted by one period T , $\langle \phi(x, t_f \pm T) | \varphi(x, t_f) \rangle \approx 0.87 \pm 0.02$ implies that particles depleted from the condensate begin occupying the mode that travels along the same resonant orbit as the condensate but is delayed by the period of the mirror oscillations T . Consequently, the evolution of DTC is restricted to two modes: the condensate and the depletion mode, which correspond to the Wannier states $w_1(x, t)$ and $w_2(x, t)$, respectively, which span the resonant Hilbert subspace in the 2 : 1 resonance case, cf. Fig. 2.2(a).

The preceding analysis demonstrated that the mean-field description and the restriction to the resonance Hilbert subspace, spanned by Wannier states, are sufficient to describe the behavior of DTC. We now extend this discussion by comparing results obtained within the Bogoliubov approach with those derived using a two-mode Hamiltonian (1.39) for the 2 : 1 resonance case [118], as follows:

$$\hat{H}_F^{(2:1)} \approx -\frac{J}{2} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U_{11} - 2U_{12}}{2} (\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2). \quad (2.50)$$

We initiate the system in the state $|\psi(t=0)\rangle = |N, 0\rangle$, where all N atoms occupy the Wannier state $w_1(x, t=0)$, and the $w_2(x, t=0)$ remains unoccupied. Given $|\psi(t=0)\rangle$ is not an exact eigenstate of $\hat{H}_F^{(2:1)}$, the quantum depletion $\langle \delta \hat{N} \rangle(t)$ is expected to increase over time, starting from $\langle \delta \hat{N} \rangle(t=0) = 0$. We evolve $|\psi(t=0)\rangle$ according to the Hamiltonian $\hat{H}_F^{(2:1)}$, and at

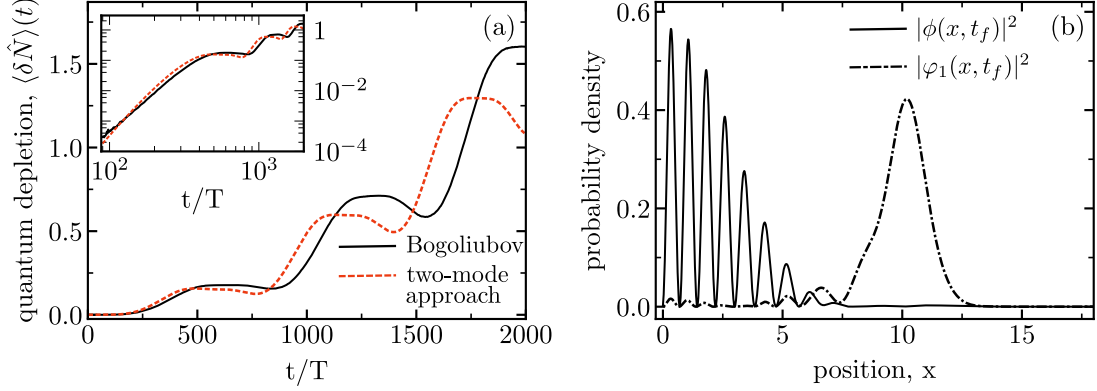


Figure 2.7: (a) Time evolution of quantum depletion $\langle \delta \hat{N} \rangle(t)$ from the condensate over time (black line), starting from an initial perfect condensate state (2.45) with zero depletion $\langle \delta \hat{N} \rangle(t=0) = 0$. Using the $U(1)$ symmetry-preserving Bogoliubov approach [112, 133, 135], the condensate mode follows the Gross-Pitaevski equation (2.47), while the orthogonal modes evolve via the Bogoliubov-de Gennes equations (2.48). The total depletion reaches $\langle \delta \hat{N} \rangle(t_f) \approx 1.6$ at $t_f = 1999T$, which is a negligible fraction compared to the 10^4 particles typically involved in experiments. Depletion is dominated by the mode $\varphi_1(x, t)$ with other modes contributing insignificantly $\delta N_{j \neq 1} \leq 6 \times 10^{-3}$. The red dashed line shows quantum depletion using a two-mode approximation (2.50) for $N = 600$. The differences between the Bogoliubov and two-mode approaches result from approximating the exact condensate mode with a Gaussian function (2.6) in the Bogoliubov case while in the two-mode approach the exact Wannier states $w_{1,2}(x, t)$ are used. Inset display log-log scale of initial algebraic increase of depletion. (b) Probability densities of the condensate mode $|\phi(x, t_f)|^2$ (solid line) and the dominant depletion mode $|\varphi_1(x, t_f)|^2$ (dotted-dashed line), with overlap $|\langle \phi(x, t_f \pm T) | \varphi_1(x, t_f) \rangle|^2 \approx 0.87 \pm 0.02$, indicating that $\varphi_1(x, t)$ closely tracks the condensate mode but is time-delayed by T on the same resonant orbit. Parameters of the mirror oscillations: $\lambda = 0.12$, $\omega = 1.4$, and for the Hamiltonian (2.50) are: $U_{11}/g_0 = 0.23$, $U_{12}/g_0 = 0.10$, $g_0 N = -0.02$, $J = 7.26 \times 10^{-4}$. The figure is adapted from [12]

each timestep within two tunneling periods, we diagonalize the reduced single-particle density matrix:

$$\hat{\rho}_1^{H_F}(t) = \begin{bmatrix} \langle \hat{a}_1^\dagger \hat{a}_1 \rangle_t & \langle \hat{a}_1^\dagger \hat{a}_2 \rangle_t \\ \langle \hat{a}_2^\dagger \hat{a}_1 \rangle_t & \langle \hat{a}_2^\dagger \hat{a}_2 \rangle_t \end{bmatrix}, \quad (2.51)$$

where $\langle \cdot \rangle_t$ represents the expectation value calculated in time-evolved state $|\psi(t)\rangle$. As shown in Figure 2.7(a), the quantum depletion over time reflects the findings of the Bogoliubov approach, although differences arise because the Bogoliubov approach approximates the exact Wannier state $w_1(x, t=0)$ by a Gaussian function $\phi(x, t=0)$ (2.6).

We show the results for $N = 600$ but they remain the same for any $N \geq 600$ provided that $g_0 N = -0.02$. Two-mode approximation also enables exploration of DTC stability against quantum fluctuations over longer timescales than two tunneling periods t_f . Analyzing the system's behavior across various particle numbers $N = 600, 1000, 2000, 4000$, we find that after $t \approx 500T\sqrt{N}$, the quantum depletion saturates at $\langle \delta \hat{N} \rangle \approx 0.02N$. Subsequently, the

depleted particles start to reoccupy the condensate mode, bringing the system close to the ideal BEC state.

An analogous analysis of the stability of DTC realized in ultra-cold atoms bouncing on an oscillating mirror setup was later performed using the truncated-Wigner approximation by *J. Wang, P. Hannaford and B. J. Dalton* [93]. The results presented in the paper are consistent with those we showed here. Additionally, the authors also analyzed how the mean energy per particle changes over time. The energy does not change significantly during this period, indicating that the system reaches a steady state. In this state, no net energy is pumped from the oscillating mirror, and there is no evidence of thermalization over the considered timescale, which is much longer than the duration of the experiment.

2.4 Summary

In this chapter, we explored the phenomenon of spontaneous symmetry breaking, focusing on its relevance to discrete time crystals. We began by examining the Bose-Hubbard model which describes bosons in a double-well potential, and mapped it to the Lipkin-Meshkov Glick model. This allowed us to highlight the similarities between the phase transitions in these models and the discrete time crystal system involving ultra-cold atoms bouncing on an oscillating mirror. Specifically, we discussed the correspondence between the spontaneous discrete time translation symmetry breaking and the spontaneous breaking of $\mathcal{Z}_2$ symmetry. We also analyzed the lifetime of the discrete time crystals in the mean-field limit, where the number of particles $N \rightarrow \infty$, while keeping $\gamma = \text{const.} < -1$ and $g_0 N = \text{const.}$ In this limit, the lifetime of discrete time crystals grows exponentially with the number of particles, indicating that discrete time crystals can live forever within the description reduced to the Bose-Hubbard model in time-periodic Wannier basis.

Next, we analyzed the stability of discrete time crystals against imperfections in the initial state preparation, which are common in experimental realizations. These imperfections included non-zero average momentum, displacement from the classical turning point (i.e. from the resonant orbit), and frequency deviations in the harmonic trap in which the system is initially prepared. To model these imperfections, we used a Gaussian approximation of the symmetry-broken state under optimal conditions (zero average momentum and centered at the classical turning point), and evolved it using the Gross-Pitaevski equation. We calculated the fidelity between the initial and evolved state at stroboscopic times, i.e. integer multiples of the mirror's oscillation period, and analyzed the system's stability by computing the Fourier transform of the fidelity function.

In the weak interaction regime, we observed beating behavior in the fidelity function, indicating the tunneling of atoms into the initially unoccupied states. This was reflected in two peaks around the frequency $\omega/2$ ($s = 2$) and $\omega/4$ ($s = 4$) in the Fourier transform, corresponding to the symmetry-preserving phase, where the discrete time crystal does not exist. However, in the strong interaction regime, we observed stable evolution of the symmetry-broken state in the fidelity function, indicating that interactions between atoms suppress tunneling. This

was characterized by a single sharp peak in the Fourier transform at $\omega/2$ ($s = 2$) and $\omega/4$ ($s = 4$), corresponding to the symmetry-broken phase, where the discrete time crystal exists. We then performed simulations with randomly chosen initial conditions, such as non-zero average momentum and a non-optimal frequency of the harmonic trap. Using the same analysis – evolving the initial state, calculating fidelity, and performing a Fourier transform – we assessed the system’s stability by analyzing the amplitude of the peak at $\omega/2$ ($s = 2$) and $\omega/4$ ($s = 4$) in the Fourier spectrum. As we approached the critical point between the discrete time crystal and symmetry-preserving phases, in the Fourier transform we observed the fluctuations in the peak amplitude across different initial conditions, signaling the boundary between symmetry-preserving and symmetry-broken phases. We identified the boundary corresponding the maximal fluctuations of the peak amplitude.

As a result, we obtained phase diagrams showing the boundary between the discrete time crystal and symmetry-preserving phases. Our analysis demonstrated that sufficiently strong attractive interactions between atoms can compensate for small imperfections in initial conditions, preserving the discrete time crystal phase. Moreover, we proposed a method to restore the discrete time crystal phase when larger imperfections occur. By adjusting the oscillation frequency of the mirror based on the expected displacement from the classical turning point, we can effectively return the system to the resonant orbit and restore the discrete time crystal phase.

We also examined the stability of discrete time crystals against quantum many-body fluctuations using the $U(1)$ symmetry-preserving Bogoliubov approach. This method allows us to extend our analysis beyond the Hilbert space of product states and quantify the influence of interactions between bosons. These interactions can couple the space of product states with its complement, potentially leading to quantum depletion in the discrete time crystal that forms the Bose-Einstein condensate.

Our results indicate that the number of atoms depleted from the condensate is negligible relative to the total number of atoms, especially over the timescale considered, which exceeds the typical duration of the experiment. The depleted atoms predominantly occupy a single mode that travels along the same resonant orbit as the condensate but with a delay corresponding to the period of the mirror oscillations. Therefore, the evolution of discrete time crystals is governed by two modes: the condensate mode and the depletion mode. Thus, a mean-field description within the resonant Hilbert subspace, spanned by the corresponding Wannier states, is sufficient to effectively analyze the properties of discrete time crystals in ultra-cold atoms bouncing on an oscillating mirror.

Chapter 3

Classical basis for complex DTC

In this chapter, we provide a classical basis for complex discrete time crystals. We analyze the system of N impenetrable particles arranged in a one-dimensional stack. The bottom particle bounces on an oscillating mirror, while each subsequent particle bounces on the one directly beneath it. Our focus is on the identification of the manifold on which the particles move in the static mirror case and the derivation of the effective description of the system in the oscillating mirror case. While the results presented here are classical in nature, they provide a foundation for a fully quantum approach to investigate novel discrete time crystals, which we will analyze in the next chapter. In this quantum framework, the N classical particles are replaced with N atomic clouds, each composed of different atomic species and interacting weakly within its cloud but exhibiting infinite repulsion between particles from different clouds.

In Section 3.1, we extend the description of a single particle bouncing on a static mirror, as discussed in Section 1.5, to the system of N independent particles. We then explore the case of N impenetrable particles, with impenetrability modeled by elastic collisions (Section 3.2). This analysis is divided into two cases: particles with different energies (Section 3.2.1) and particles with equal energies (Section 3.2.2). In both cases, we focus on identifying the manifold on which the particles move. For impenetrable particles with equal energies, the manifold needs to be analyzed separately for different numbers of particles. We first examine the case of $N = 2$ in Section 3.2.3, followed by the case of $N = 3$ in Section 3.2.4. In Section 3.2.5 we provide brief remarks on systems with $N = 2$ and $N = 3$ impenetrable particles with equal energies.

In Section 3.3, we derive an effective Hamiltonian to describe the system's behavior near a resonant trajectory when the mirror oscillations are turned on. We separately consider the cases of particles with different energies (Section 3.3.1) and particles with equal energies (Section 3.3.2).

This chapter concludes with a brief summary in Section 3.4. The results presented in the chapter are based on the paper by *W. Golletz, A. Czarnecki, K. Sacha, and A. Kuroś* [11].

3.1 N independent particles: static mirror case

We begin by extending the classical analysis of a single particle bouncing on a static mirror, presented in Section 1.5, to N independent (uncoupled) particles, see Fig. 3.1(a) for $N = 2$. We assume the particles are point-like (hard balls) with zero radius and unit mass. Our focus is now on identifying the manifold on which these particles move in phase space.

The general Hamiltonian for N independent particles bouncing on a static mirror in 1D space, in the presence of the gravitational field, is given by:

$$\begin{aligned} H_0(\mathbf{x}, \mathbf{p}) &= H_0^{(1)}(x_1, p_1) + H_0^{(2)}(x_2, p_2) + \cdots + H_0^{(N)}(x_N, p_N) \\ &= \sum_{j=1}^N \left(\frac{p_j^2}{2} + x_j \right), \quad x_j \geq 0, \end{aligned} \quad (3.1)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_N)$ and $\mathbf{p} = (p_1, p_2, \dots, p_N)$ represent the positions and momenta of all particles, respectively. Each single-particle Hamiltonian conserves the energy related of the j -th particle:

$$H_0^{(j)}(x_j, p_j) = \text{constant} = E_j \quad (3.2)$$

and is integrable. We assume a static mirror (hard wall) at $x_j = 0$ and use gravitational units (1.15) for consistency with the assumptions introduced in Section 1.5.

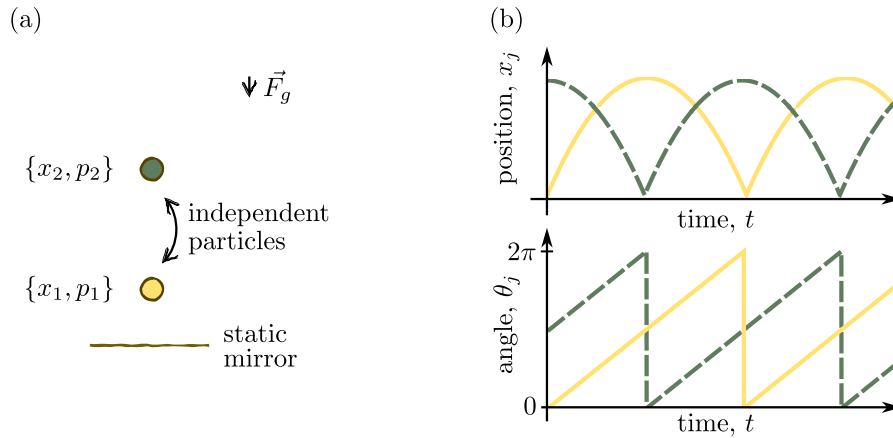


Figure 3.1: (a) The cartoon of the system illustrates $N = 2$ independent particles bouncing on a static mirror in the presence of a gravitational field $\vec{F}_g$. Each particle is characterized by its position x_j and momentum p_j . (b) Following the canonical transformation (3.4) to the angle-action variables $\{\theta, I\}$, the Hamiltonian depends only on action variables $H_0^{(j)}(I_j)$ (3.5), governs the dynamics according to Hamilton's equations (3.7). Consequently, the system is characterized by the 2π -periodic angle variables. As the particles do not interact, their trajectories may intersect during the time evolution. The top panel displays how the particles' positions $x_j(t)$ changes over time, while the bottom panel shows the same particles' evolution but in terms of the angle variables $\theta_j(t)$.

By rearranging equation (3.2) with respect to momentum $p_j = p_j(x_j, E_j)$, we define a new variable, *action*, for each particle separately:

$$I_j(E_j) = \frac{1}{2\pi} \oint_{C(E_j)} p_j(x_j, E_j) dx_j, \quad (3.3)$$

which represents the area enclosed by the energy contour $C(E_j)$. This contour also represents a trajectory in the phase space spanned by position x_j and momentum p_j variables. When we invert the function $I_j(E_j)$, we obtain the energy as a function of the action variable $E_j(I_j)$, transforming $H_0^{(j)}(x_j, p_j)$ to $H_0^{(j)}(I_j)$. This new form of Hamiltonian depends only on one phase-space variable, the new "momentum" I_j , and is independent of the conjugate angle θ_j . Generally, such a canonical transformation transitions the system from the original position-momentum variables to a new set of angle-action variables $\{\mathbf{x}, \mathbf{p}\} \rightarrow \{\boldsymbol{\theta}, \mathbf{I}\}$, such that the Hamiltonian $H_0^{(j)}(x_j, p_j)$ for each particle individually transforms to $H_0^{(j)}(I_j)$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_N)$, and $\mathbf{I} = (I_1, I_2, \dots, I_N)$.

In the system of independent particles bouncing on a static mirror, the canonical transformation is expressed as follows, cf. (1.25):

$$\begin{aligned} x_j(I_j, \theta_j) &= \frac{1}{2}(3\pi I_j)^{2/3} \left(1 - \frac{\theta_j^2}{\pi^2}\right), & \theta_j(p_j, x_j) &= \pi - \frac{\pi}{\sqrt{2E_j}} p_j, \\ p_j(I_j, \theta_j) &= \left(\frac{3I_j}{\pi^2}\right)^{1/3} (\pi - \theta_j), & I_j(p_j, x_j) &= \frac{(2E_j)^{3/2}}{3\pi}. \end{aligned} \quad (3.4)$$

Accordingly, the Hamiltonian for each particle depends only on the action variable:

$$H_0^{(j)}(x_j, p_j) \rightarrow H_0^{(j)}(I_j) = \frac{(3\pi I_j)^{2/3}}{2}. \quad (3.5)$$

Hamilton's canonical equations for the angle-action variables are given by:

$$\begin{aligned} \frac{\partial \theta_j}{\partial t} &= \frac{\partial H_0^{(j)}(I_j)}{\partial I_j} = \Omega_j(I_j) = \left(\frac{\pi^2}{3I_j}\right)^{1/3}, \\ \frac{\partial I_j}{\partial t} &= -\frac{\partial H_0^{(j)}(I_j)}{\partial \theta_j} = 0. \end{aligned} \quad (3.6)$$

The system's dynamics is thus described by the solutions of Eq. (3.6):

$$I_j(t) = \text{constant}, \quad \theta_j(t) = \Omega_j(I_j)t + \theta_{0,j} \pmod{2\pi}, \quad (3.7)$$

where $\Omega_j(I_j)$ is the frequency of the j -th particle motion, and $\theta_{0,j}$ is the initial angle variable of j -th particle. The example of the system evolution is shown in Fig. 3.1(b).

Identification of the manifold: a single-particle system

To identify the manifold on which the system moves, we analyze its phase-space trajectory for different initial conditions. We begin with illustrating our approach with the simplest case: a

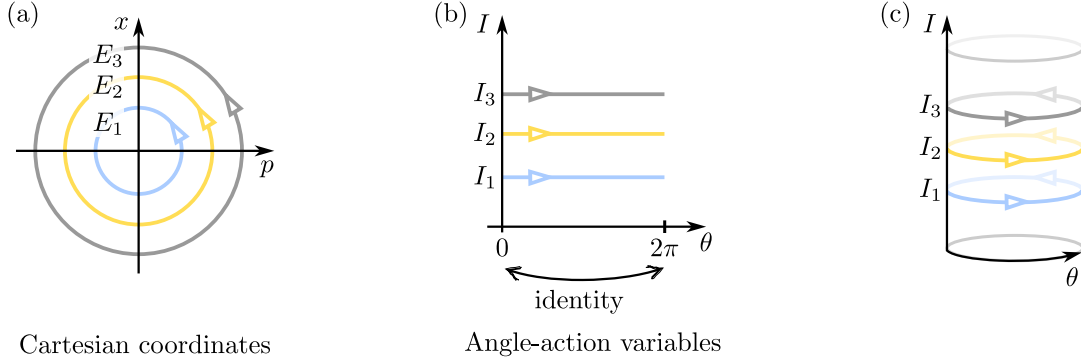


Figure 3.2: The phase space for a single-particle system bouncing on a static mirror in the presence of a gravitational field. (a) The phase space is spanned by Cartesian position-momentum coordinates $\{x, p\}$. The particle follows the periodic trajectory with given energy E_k , where $k = 1, 2, 3$, during the evolution. As the energy of the system increases, so does the period of the system. In the panel three distinct trajectories with different constant energies $E_1 < E_2 < E_3$ (blue, yellow, and gray lines, respectively) are shown. (b) Following the canonical transformation (3.4) to angle-action variables $\{\theta, I\}$, the dynamics of a particle is simplified. The action variable $I(t)$ remains constant of the motion, while the angle variable $\theta(t)$ evolves linearly over time (3.7). Due to the 2π -periodicity of θ , the points at $\theta_j = 0$ and $\theta_j = 2\pi$ for a given action I_j are equivalent. By connecting these points, the system's trajectories form a cylinder (c). Depending on the initial condition I_0 , the particle follows a specific circular path on this manifold.

single-particle system, as shown in Fig. 3.2. Starting with the original Cartesian coordinates $\{x, p\}$, the phase space consists of periodic trajectories, where the period increases with the particle's energy E (Fig. 3.2(a)). We then apply a canonical transformation (3.4) to switch to angle-action variables $\{\theta, I\}$, which simplify the dynamics and reduce the system's complexity (Fig. 3.2(b)). In these new coordinates, the action $I(t)$ remains constant, representing the system's energy, while all dynamic changes are described by the angle variable $\theta(t)$. The angle variable is 2π -periodic, meaning that the points at $\theta = 0$ and $\theta = 2\pi$ are equivalent. This periodicity allows us to conceptually "wrap" the trajectories corresponding to different initial conditions to form a cylinder (Fig. 3.2(c)). As a result, the phase space becomes topologically equivalent to a cylinder $\mathbb{S}^1 \times \mathbb{R}$, where θ is an element of the circle $\theta \in \mathbb{S}^1$ and I is a non-negative real number $I \in \mathbb{R}$. Throughout this chapter, we apply this approach to analyze the system's dynamics under various initial conditions, enabling us to identify the manifold on which the system evolves.

Identification of the manifold: a two-particle system

In the system with two independent particles, the phase space is spanned by four variables $\{x_1, x_2, p_1, p_2\}$. Transitioning to angle-action variables introduces two angle variables $\{\theta_1, \theta_2\}$ which evolve linearly in time, and two constant action variables $\{I_1, I_2\}$. This simplifies the visualization of the manifold in the angle space for a given action values, since attempting to plot a four-dimensional phase space directly is not feasible, see Fig 3.3. Each point in the action space corresponds to a two-dimensional torus (2-torus) $\mathbb{S}^1 \times \mathbb{S}^1$ in angle space. This

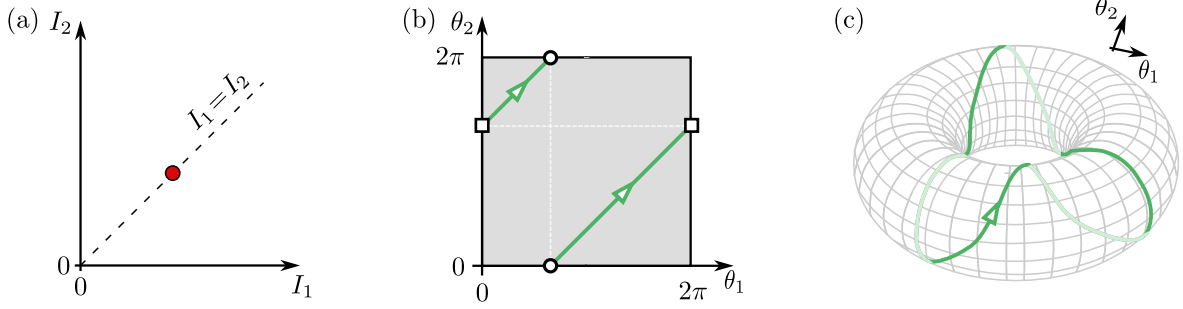


Figure 3.3: The phase space for two independent particles bouncing on a static mirror in the presence of a gravitation field. (a) The action variables $\{I_1, I_2\}$, which are constants of motion, are determined by the initial conditions. Each point in the action space corresponds to a trajectory in the angle space (θ_1, θ_2) , which the system follows. In this example, we choose $I_1 = I_2$ (red dot) which corresponds to the same unperturbed frequency of the particles' motion $\Omega(I_1) = \Omega(I_2)$ (3.7). (b) The trajectory of the particles (green line with arrows indicating the direction of motion) consists of two separate fragments. To identify the manifold on which the particles move, these fragments are connected to form a continuous line. (c) The complete trajectory is obtained by "gluing" the circles (squares) which results in the motion occurring on a 2-torus. Consequently, the four-dimensional phase space is filled with a 2-parameter family $\{I_1, I_2\}$ of 2-tori.

2-torus emerges by appropriately connecting the particles' trajectories into a continuous one. The dynamical behavior on the 2-torus depends on the ratio of the particles' periods Ω_1/Ω_2 . If the ratio is rational, the trajectory is periodic (closed); if irrational, the trajectory never repeats, eventually filling the torus densely over time.

Throughout this thesis, we focus on the periodic motion of the system, which implies all frequencies are rational multiples of each other [136], indicated as:

$$\Omega_1/\Omega_2 = l_1/l_2 = \text{a rational number.} \quad (3.8)$$

Identification of the manifold: a N -particle system

Lastly, the phase space for a system of N independent particles is $2N$ -dimensional. For given constant values of the action variables $\mathbf{I}$ each trajectory is restricted to an N -torus, represented as $\underbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{N \text{ times}}$.

Here, each pair of angle-action variables $\{I_j, \theta_j\}$ is always associated with one particle. In the next section, we will see that this is not the case with impenetrable particles.

3.2 Impenetrable particles: static mirror case

In the previous section, we extended the classical analysis of a single particle bouncing on a static mirror to N independent particles. Here, we focus on the case where these particles are impenetrable, meaning that the lowest particle bounces elastically on a static mirror, while

each subsequent particle bounces elastically on the one directly below it. This results in the exchange of both momentum $p_j \rightleftharpoons p_{j+1}$ and energy $E_j \rightleftharpoons E_{j+1}$ between adjacent particles. Upon collision, the lowest particle reverses its momentum $p_1 \rightarrow -p_1$ due to collision with a static mirror. The particles are ordered in ascending order as $0 \leq x_1 \leq x_2 \leq \dots x_N$, see Fig. 3.4(a) for $N = 2$. Between collisions, the particles follow trajectories similar to those of independent particles, as discussed in Section 3.1.

Our goal is to find the new angle-action variables $\{\vartheta_j, \mathcal{I}_j\}$, which encode the impenetrability of the particles. In these variables, the Hamiltonian depends only on the action variables $H_0(\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_N) = \sum_j H_0^{(j)}(\mathcal{I}_j)$, and the system's dynamics is given solutions of Hamilton's equations, in the form similar to Eq. (3.7):

$$\mathcal{I}_j = \text{constant}, \quad \vartheta_j(t) = \frac{\partial H_0^{(j)}(\mathcal{I}_j)}{\partial \mathcal{I}_j} t + \vartheta_{0,j} \pmod{2\pi}, \quad (3.9)$$

where $\Omega(\mathcal{I}_j) = \frac{\partial H_0^{(j)}(\mathcal{I}_j)}{\partial \mathcal{I}_j}$ is the frequency of the j -th unperturbed particle motion and $\vartheta_{0,j}$ is the initial angle variable of the j -th particle. The new angle-action variables $\{\vartheta, \mathcal{I}\}$, where $\vartheta = \{\vartheta_1, \vartheta_2, \dots, \vartheta_N\}$ and $\mathcal{I} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_N\}$, allows us in further analysis identify the manifold on which the particles move and effectively describe the system's dynamics Hamiltonian near the resonant orbit in the presence of the mirror oscillations.

We begin the analysis by the particles with different energies, and then proceed to examine the case where all particles have equal energies.

3.2.1 Impenetrable particles with all different energies

We consider the system of N impenetrable particles with the same masses, where each has a different energy and initially:

$$E_1 < E_2 < \dots < E_N. \quad (3.10)$$

Over time, due to collisions, the colliding particles exchange their trajectories, see Fig. 3.4(b). This affects both their momenta $p_j \rightleftharpoons p_{j+1}$ and energies $E_j \rightleftharpoons E_{j+1}$, such that the initial relation (3.10) no longer holds. However, we can sort in ascending order the set of different energies $\{E_j\}$ at any time during the system's evolution, obtaining the relation:

$$\mathcal{E}_1 < \mathcal{E}_2 < \dots < \mathcal{E}_{j'} < \dots < \mathcal{E}_N, \quad (3.11)$$

which preserves the order over the time. In Eq. (3.11), $\mathcal{E}_1$ represents the lowest energy, $\mathcal{E}_2$ the next lowest, and so forth, up to the highest energy $\mathcal{E}_N$. It is important to note that, in general, $E_{j'}$ does not equal $\mathcal{E}_{j'}$, as the index j' now corresponds to a specific energy value, rather than a particular particle, see Fig. 3.4(c).

Then, a canonical transformation to novel angle-action variables $\{\vartheta, \mathcal{I}\}$ using the set of

ordered energies $\{\mathcal{E}_{j'}\}$ is follows as:

$$\begin{aligned}\mathcal{X}_{j'} &= \frac{1}{2} \left(\frac{3\mathcal{I}_{j'}}{\pi^2} \right)^{2/3} (2\pi - \vartheta_{j'}) \vartheta_{j'}, & \vartheta_{j'} &= \pi - \frac{\pi}{\sqrt{2\mathcal{E}_{j'}}} \sum_{j=1}^N p_j \delta_{E_j, \mathcal{E}_{j'}}, \\ \mathcal{P}_{j'} &= \left(\frac{3\mathcal{I}_{j'}}{\pi^2} \right)^{1/3} (\pi - \vartheta_{j'}), & \mathcal{I}_{j'} &= \frac{(2\mathcal{E}_{j'})^{3/2}}{3\pi}.\end{aligned}\tag{3.12}$$

where $\delta_{E_j, \mathcal{E}_{j'}} = \begin{cases} 1, & \text{if } E_j = \mathcal{E}_{j'}, \\ 0, & \text{otherwise.} \end{cases}$

To revert to the original coordinates, we need to compute all pairs $\{\mathcal{X}_{j'}, \mathcal{P}_{j'}\}$, sort them by ascending $\mathcal{X}_{j'}$ values, and assign $x_1 = \mathcal{X}_{j'} < x_2 = \mathcal{X}_{k'} < \dots < x_N = \mathcal{X}_{l'}$ and $p_1 = \mathcal{P}_{j'}, p_2 =$

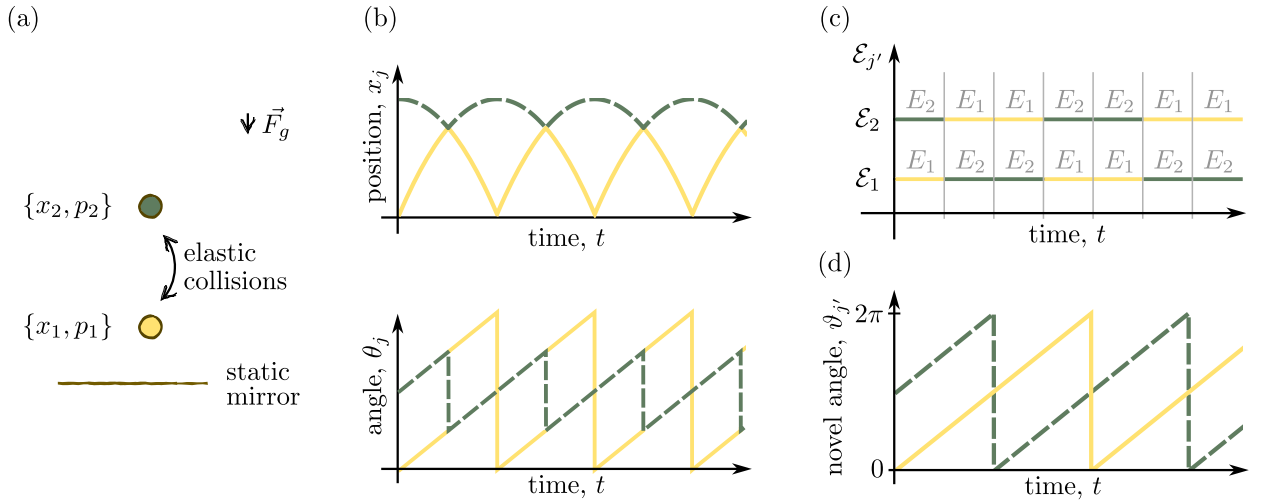


Figure 3.4: (a) The cartoon of the system illustrates $N = 2$ impenetrable particles in the presence of a gravitational field $\vec{F}_g$, cf. Fig. 3.1. The particles are ordered ascendingly $0 \leq x_1 \leq x_2 \leq \dots \leq x_N$. The lowest particle bounces on a static mirror, while subsequent particles bounce on the one directly beneath them. Elastic collisions between particles with the same masses lead to the exchange of momenta $p_j \leftrightarrow p_{j+1}$ and energies $E_j \leftrightarrow E_{j+1}$ between adjacent particles. The lowest particle reverses its momentum $p_1 \rightarrow -p_1$ due to a collision with a static mirror. (b) The top panel shows how the particles' positions $x_j(t)$ evolve over time, while the bottom panel presents this motion in terms of the angle variables $\theta_j(t)$ (3.4). Elastic collisions cause an exchange of paths that colliding particles follow at $x_j = x_{j+1}$, cf. independent particles' evolution in Fig. 3.1(b). (c) Initially, the condition for individual particles' energies is set as $0 \leq E_1 \leq E_2 \leq \dots \leq E_N$. However, these relations only hold until the first collision, after which energies and momenta are exchanged. For particles with all different energies, a set of constant energies $\{\mathcal{E}_{j'}\}$ is introduced, following specific initial energies from the set $\{E_j\}$ rather than individual particles. This allows the condition $0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \leq \dots \leq \mathcal{E}_N$ to remain valid over time, enabling the definition of a canonical transformation to novel angle-action variables $\{\vartheta, \mathcal{I}\}$ (3.12). (d) Consequently, in terms of the novel angle-action variables, the system effectively behaves like independent particles.

$\mathcal{P}_{k'}, \dots, p_N = \mathcal{P}_{l'}$ to each particle.

After applying the canonical transformation (3.12), the Hamiltonian of the system is given by:

$$H_0(\mathcal{I}) = \sum_{j'=1}^N \frac{(3\pi\mathcal{I}_{j'})^{2/3}}{2}, \quad (3.13)$$

where action variables $\mathcal{I}$ are determined by the initial condition (3.10):

$$\mathcal{I}_1 < \mathcal{I}_2 < \dots < \mathcal{I}_{j'} < \dots < \mathcal{I}_N. \quad (3.14)$$

The dynamics of the system is governed by Hamilton's equations with the solutions presented in Eq. (3.9).

These novel angle-action variables $\{\mathfrak{g}, \mathcal{I}\}$ provide a framework for identifying the manifold on which the impenetrable particles move within $2N$ -dimensional phase space. The angle-action variables correspond now to specific energy value $\mathcal{E}_{j'}$ rather than individual particles and as a result, there is no longer direct influence of collisions, i.e. there is no exchange of energies $\mathcal{E}_{j'}$, thus the corresponding angles $\mathfrak{g}_{j'}$ and actions $\mathcal{I}_{j'}$, on the trajectories in the phase space. Consequently, we effectively end up with a system that behaves like independent particles and the impenetrable particles with all different energies move on an N -torus for given values of $\mathcal{I}$.

3.2.2 Impenetrable particles with all equal energies

In the previous subsection, we proposed new angle-action variables $\{\mathfrak{g}, \mathcal{I}\}$ which simplifies the description of the evolution of impenetrable particles with all different energies (3.9) and identification of the manifold on which this system moves in phase space.

However, in a situation where all particles possess identical energies,

$$E := E_1 = E_2 = \dots = E_N \quad (3.15)$$

the application of the novel angle-action variables $\{\mathfrak{g}, \mathcal{I}\}$ becomes unfeasible. This limitation arises from the inability to sort the set of particles energies $\{E_j\}$ in ascending order, as all energies are now the same, and as a result the canonical transformation (3.12) is not well-defined.

Despite this limitation, we can still analyze the system's dynamics by integrating Hamilton's equations using Cartesian variables or the old angle-action variables $\{\theta, \mathbf{I}\}$ introduced in Section 3.1 for independent particles. The latter, are canonically conjugate variables but are not angle-action variables adapted to the system where the actions should be constant and the angles should evolve linearly in time, i.e. it does not allow us to describe the evolution of the system in the form of Eq. (3.9). In the further analysis, we will use the old angle-action variables $\{\theta, \mathbf{I}\}$.

The dynamics of the system then follow Hamilton's equations (3.7), with an additional condition for particle collisions. When two particles collide $x_j = x_{j+1}$, they swap momenta

$p_j \rightleftharpoons p_{j+1}$. This condition, formulated in angle-action variables $\{\theta, I\}$ (3.4), is can be expressed as:

$$\text{if } (2\pi - \theta_j)\theta_j = (2\pi - \theta_{j+1})\theta_{j+1} \xrightarrow{\text{then}} \theta_j \rightleftharpoons \theta_{j+1}, \quad (3.16)$$

where the equal energies constraint is encoded in the action variables $I := I_1 = I_2 = \dots = I_N$. To maintain consistency with previous sections, we preserve the ascending order of the particles $0 \leq x_1 \leq x_2 \leq \dots \leq x_N$, and equivalently:

$$0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2 \leq \dots \leq (2\pi - \theta_N)\theta_N. \quad (3.17)$$

This relation restricts the available space for the particles' motion and reflects on the topology of the manifold we aim to identify.

To identify the manifold, we analyze the system's dynamics and the resulting trajectory in the phase space. This trajectory is fragmented due to collisions between particles, and to identify the manifold, we must "conceptually" join these fragments together. This approach is not easily generalizable, as each number of particles N requires a separate analysis.

In the next subsection, we analyze the case of two impenetrable particles with equal energies to illustrate this approach, and then extend it to a system of three particles.

3.2.3 Two impenetrable particles: equal energies

Let us consider a system of $N = 2$ impenetrable particles with equal energies $E := E_1 = E_2$, or equivalently $I := I_1 = I_2$, bouncing on a static mirror in the presence of a gravitational field. The four-dimensional phase space is spanned by angle-action variables $\{\theta_1, \theta_2, I_1, I_2\}$. Given the equal action variables $I := I_1 = I_2$, the available angle space $\{\theta_1, \theta_2\}$ is constrained to two triangular regions, according to the relation (3.17):

$$0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2, \quad (3.18)$$

as illustrated in Fig. 3.5(a)-(b). The available angle space corresponds to half of the space available for two independent particles, cf. Fig. 3.3(b).

We identify the manifold on which these particles move by analyzing the solutions of the collision condition:

$$(2\pi - \theta_1)\theta_1 = (2\pi - \theta_2)\theta_2. \quad (3.19)$$

The first solution, $\theta_1 = \theta_2$, describes both particles moving along the same trajectory with equal momenta $p_1 = p_2$. The second solution, $\theta_2 = 2\pi - \theta_1$, indicates an exchange of the particles' momenta $p_1 \rightleftharpoons p_2$ and is related to the fragmentation of the system's trajectory in phase space. In Fig. 3.5(a)-(c), we graphically present these solutions and demonstrate how to "glue" the trajectory fragments to form a continuous path, thereby identifying the manifold on which the system moves. As a result, we determine that the phase space for the given action values $I = I_1 = I_2$ forms a Möbius strip.

These results align with those presented by *K. Giergiel, A. Kuroś, A. Kosior, and K. Sacha* in Ref. [137]. The authors analyze the dynamics of a single particle bouncing between two

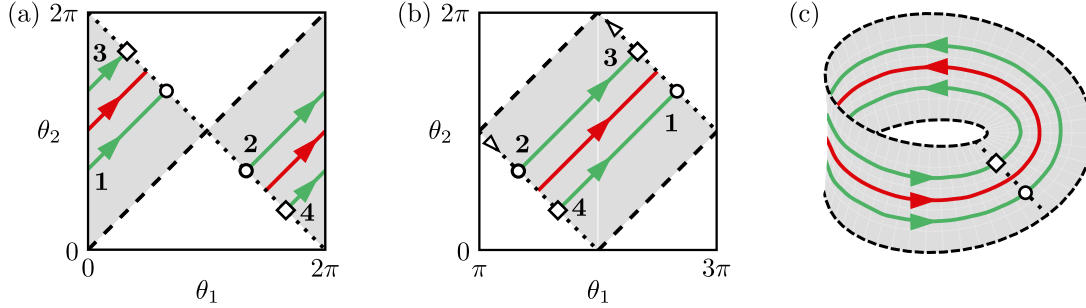


Figure 3.5: The phase space for two impenetrable particles with equal energies. (a) The available space (gray area) is half that of two independent particles, cf. Fig. 3.3(b). This reduction results from the elastic collisions between the particles, described by the condition $0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2$. The typical trajectory of the particles (green line) is divided into four parts numbered (1)-(4). The identification of the manifold on which the system moves is performed by properly gluing of these segments to ensure the trajectory is continuous and consistent with the system dynamics. (b) The first step involves connecting the edges at $\theta_1 = 0$ and $\theta_1 = 2\pi$, which are identical due to the 2π -periodicity of the angle variable θ_1 . This connection is achieved by shifting the left gray triangle to the right. Subsequent gluing focuses on connecting the ends of green segments – circle to circle and square to square. (c) For clarity, only one typical trajectory is depicted (green line). For all possible trajectories, this method of joining the path segments results in connecting two edges (dotted line) so that the direction of arrows (black with white filling) align. This final step introduces a twist of the edge corresponding to one of the solutions to Eq. (3.19), $\theta_2 = 2\pi - \theta_1$. Physically, this represents the points in phase space where the two particles collide and exchange momenta. As a result, the angle space for the given action values $I := I_1 = I_2$ forms a Möbius strip. Among all the trajectories, there is a unique one (red line) with a period twice shorter than the others. The figure is adapted from [11].

static mirrors, characterized by angle β between them. When $\beta = \pi/2$, corresponding to orthogonal mirrors, the dynamics simplifies to independent 1D motions along x and y axes:

$$H^{2D} = \frac{p_x^2 + p_y^2}{2} + x + y. \quad (3.20)$$

This behavior is analogous to that of two independent particles bouncing on a static mirror in 1D space, as discussed in Section 3.1.

For an angle $\beta = \pi/4$, where one of the mirrors aligns with the gravitational field, the available space for particle motion is reduced by half compared to the setup with orthogonal mirrors. This configuration introduces an additional constraint in the Hamiltonian (3.20): $0 \leq x \leq y$. When the particle bounces off the vertical mirror, its momentum swaps $p_x \rightleftharpoons p_y$. On bounces off the other mirror, the x component of its momentum reverses $p_x \rightarrow -p_x$, while the p_y component remains unchanged. This scenario is analogous to the problem considered in this section, involving two impenetrable particles with equal energies bouncing on a static mirror.

The authors of Ref. [137] identify the manifolds on which the particles move. For orthogonal

mirrors $\beta = \pi/2$, the manifold is a 2-torus, and for mirrors forming a wedge of $\beta = \pi/4$, it manifests as a Möbius strip.

In the next section, we take a step further by considering the dynamics of three impenetrable particles with equal energies, bouncing on a static mirror in a gravitational field. Our goal is to explore and identify the manifold on which these particles move.

3.2.4 Three impenetrable particles: equal energies

To identify the manifold on which three impenetrable particles with equal energies move, we use the original action-angle variables $\{\boldsymbol{\theta}, \mathbf{I}\}$, introduced for independent particles as detailed in Section 3.1. As in the previous section, we introduce additional constraints to Hamilton's equations (3.7) related to all possible collisions between particles:

$$\begin{aligned} (2\pi - \theta_1)\theta_1 &= (2\pi - \theta_2)\theta_2, \\ (2\pi - \theta_2)\theta_2 &= (2\pi - \theta_3)\theta_3, \\ (2\pi - \theta_1)\theta_1 &= (2\pi - \theta_2)\theta_2 = (2\pi - \theta_3)\theta_3, \end{aligned} \tag{3.21}$$

where we assume the ascending order of particles:

$$0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2 \leq (2\pi - \theta_3)\theta_3, \tag{3.22}$$

and equal energies $I := I_1 = I_2 = I_3$, or equivalently $0 \leq x_1 \leq x_2 \leq x_3$ and $E := E_1 = E_2 = E_3$. The first two equations (3.21) describe the collisions between the first and second particle, and second and third particle, respectively. The third equation (3.21) describes three-particle collisions.

We identify the manifold on which the system moves by analyzing the dynamics of the system in angle space $\{\theta_1, \theta_2, \theta_3\}$ for various initial conditions $\{\theta_{0,1}, \theta_{0,2}, \theta_{0,3}\}$, based on solutions of Hamilton's equations (3.7) with constraints (3.21), and given the action values $\mathbf{I}$. Due to the collisions, the available space for the particles' motion is restricted to 1/6 of the space explored by independent particles Eq. (3.22). This restricted space consists of two tetrahedrons joined by a common edge, as illustrated in Fig. 3.6(a)-(c). The orange and yellow faces of the tetrahedrons represent points where two particles collide, and together with the tetrahedrons' interiors, they form the interior of a three-dimensional manifold with a two-dimensional surface represented by four white tetrahedron faces. A typical system's trajectory (green line) is divided into six fragments, labeled (1)-(6), indicating the sequence in time. Identification of the manifold involves properly gluing these fragments to ensure that the trajectories are continuous and consistent with the system dynamics. We also present two examples of such trajectories, plotted as the particles' positions over time in terms of Cartesian coordinates and angles variables in Fig. 3.6(d)-(e).

We divide the process of identification of the manifold into three steps. First, we continuously deform two tetrahedrons into two cuboids to simplify the subsequent steps, as presented in Fig. 3.7. In the second step, we focus on identifying the surface (boundary) of the manifold by analyzing the system's motion near the edges of the white faces, see Fig. 3.8. By gluing

the white faces together such that the trajectory fragments align chronologically, we reveal a 2-torus, $\mathbb{S}^1 \times \mathbb{S}^1$.

The third step involves identifying the interior of the manifold by determining its orientability. To do this, we examine the order of trajectories near the central trajectory (red line), as illustrated in Fig. 3.6(c, f). This region surrounding the central trajectory can be then expanded to fully encompass the interior of the manifold. Mathematically, the geometry of tetrahedrons' interiors corresponds to a product of the segment $[0, 1]$ and the interior of the 2-simplex $\text{int } \Delta^2$, with endpoints $\{1\} \times \text{int } \Delta^2$ and $\{0\} \times \text{int } \Delta^2$ identified by a $2\pi/3$ clockwise rotation. This rotation ensures the continuity of the system's trajectory. The interior of the 2-simplex and the open 2-discs, used in the visualization in Fig. 3.9, are homeomorphic. Consequently, the interior of the manifold is a product of an open disc and a circle, $\mathbb{D}^2 \times \mathbb{S}^1$, resembling a solid torus without boundary.

As the last step, we merge the above results. Given that the boundary of the manifold is 2-torus and the interior is a solid torus without a boundary, the manifold itself is homeomorphic to a solid torus, in accordance with the Loop Theorem about 3-manifolds [138]. For a complete argument, refer to Proposition 3.4 in Ref. [139].

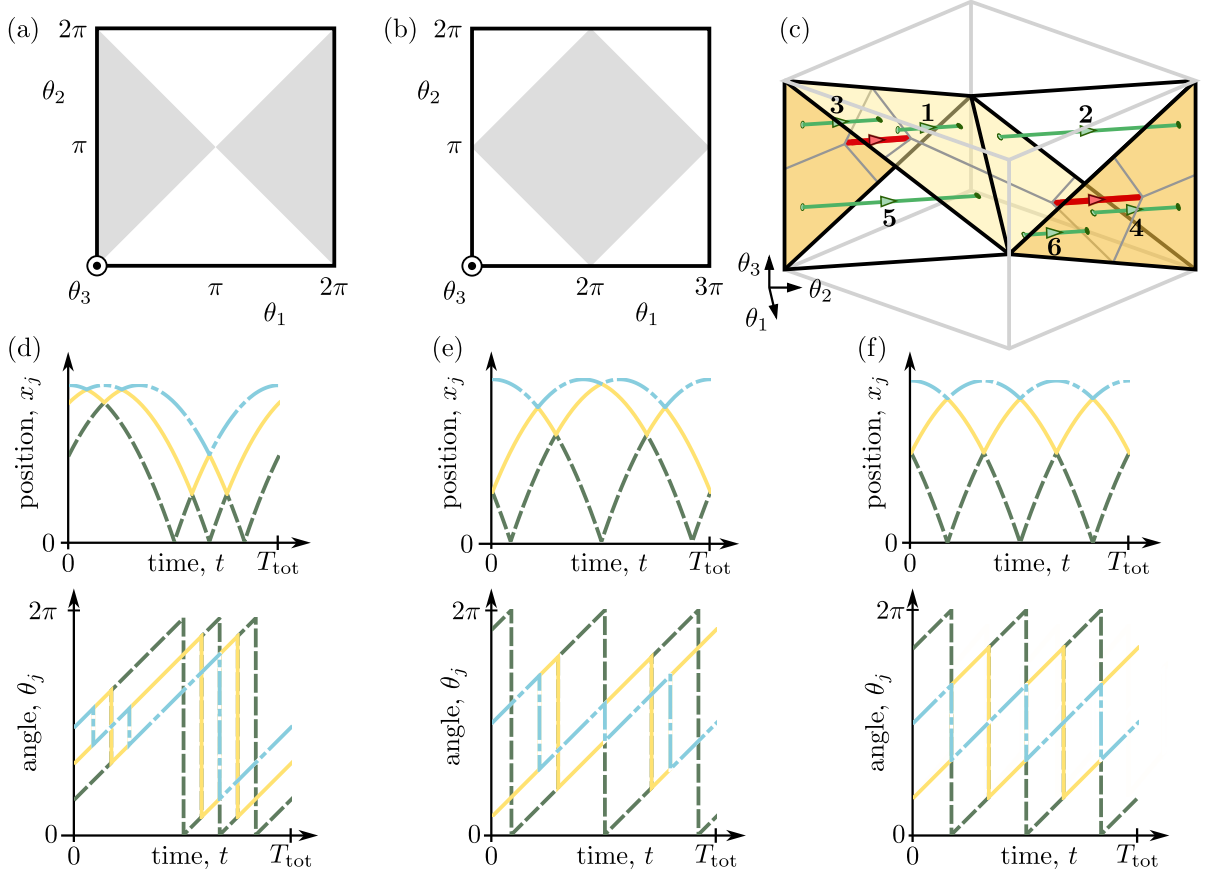


Figure 3.6: The phase space for a three impenetrable particles with equal energies. The available space is $1/6$ that of three independent particles, constrained by the condition $0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2 \leq (2\pi - \theta_3)\theta_3$ (3.22). (a) The original space is shown as a projection of the available space (gray area) onto the (θ_1, θ_2) plane for clarity. (b) To simplify the further identification of the manifold, we shift the cuboid defined by $(\theta_1 \in (0, \pi]) \times (\theta_2, \theta_3 \in (0, 2\pi])$ by 2π in accordance with 2π -periodicity of θ_1 , cf. Fig. 3.5(a)-(b). (c) The resulting space explored by the system consists of two tetrahedrons joined by a common edge. The yellow and orange faces of the tetrahedrons represent areas where two particles collide, and together with the tetrahedrons' interiors, they form the interior of a three-dimensional manifold with a two-dimensional surface comprised of by the four white faces. A typical trajectory (green line) is divided into six parts, labeled (1)-(6), indicating the sequence in time. Each segment corresponds to the motion of particles between collisions. Proper gluing these segments ensures the trajectory is continuous and consistent with the system dynamics, and this leads to the identification of the manifold. Among such trajectories, there is a unique (central) one with a period three times shorter than the others (red line). It runs through the orthocentre of the colored faces. (d)-(e) Two examples of the evolution of particles corresponding to typical trajectories, shown as changes in the positions $x_j(t)$ (top panels) and angles $\theta_j(t)$ (bottom panels) over one period of the system evolution T_{tot} . (f) The evolution of the system corresponding to the unique trajectory (red fragments in (c)), represented as changes in positions $x_j(t)$ (top panel) and angles $\theta_j(t)$ (bottom panel) with the period $T_{\text{tot}}/3$. The panels (a)-(c) are adapted from [11].

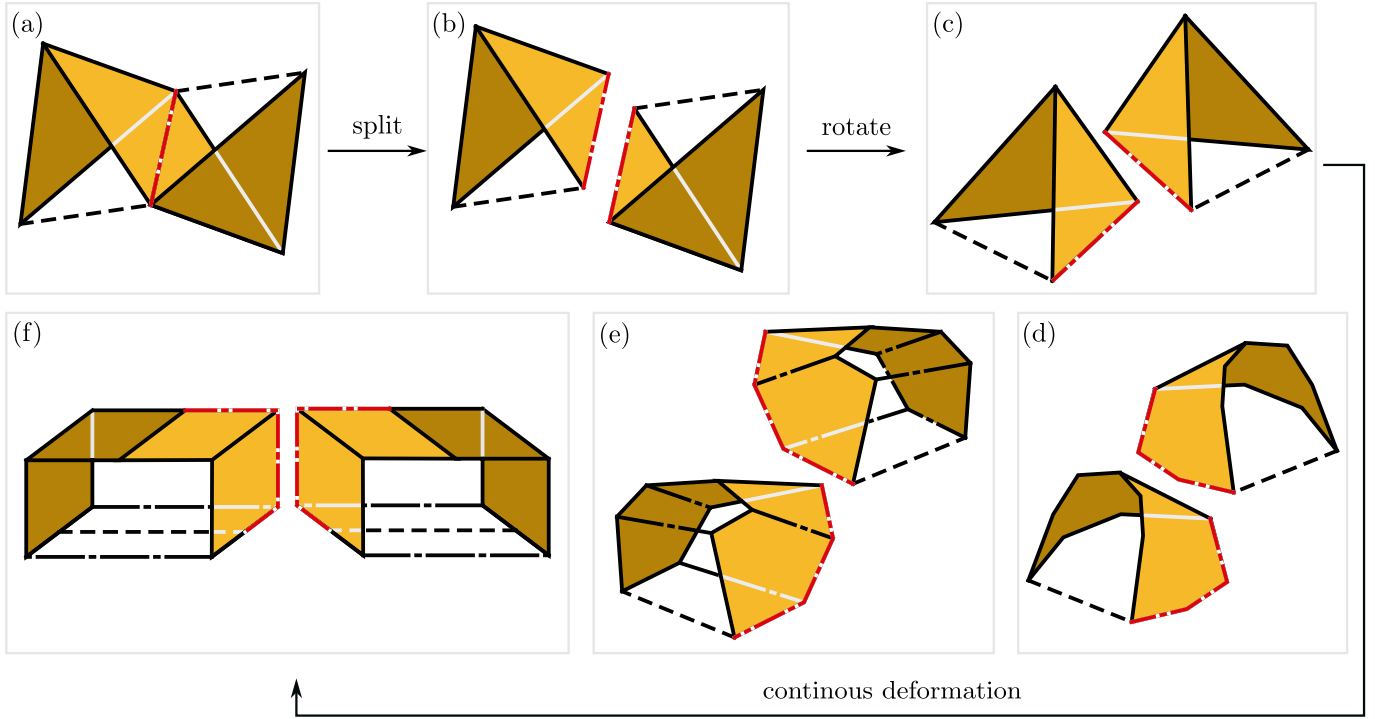


Figure 3.7: Topological deformation of two tetrahedrons, initially joined by a common edge (red dot-dashed line), into two cuboids. (a) We start with the original phase space configuration, cf. Fig. 3.6(c). (b) For simplicity and without loss of generality, we split the manifold into two separate tetrahedrons. (c) Each tetrahedron is then rotated to position one of its white face downwards, aligning them for the subsequent transformation. (d)-(f) We proceed by molding the tetrahedrons into two cuboids. This transformation maintains topological equivalence if the resulting cuboids are connected along a common edge (red dot-dashed line); they are depicted as separate for visual clarity. The white faces of cuboids form the two-dimensional surface of the manifold. The colored faces, along with the interiors of the cuboids, comprise the manifold's interior the particles with equal energies move on, cf. Fig. 3.6(c).

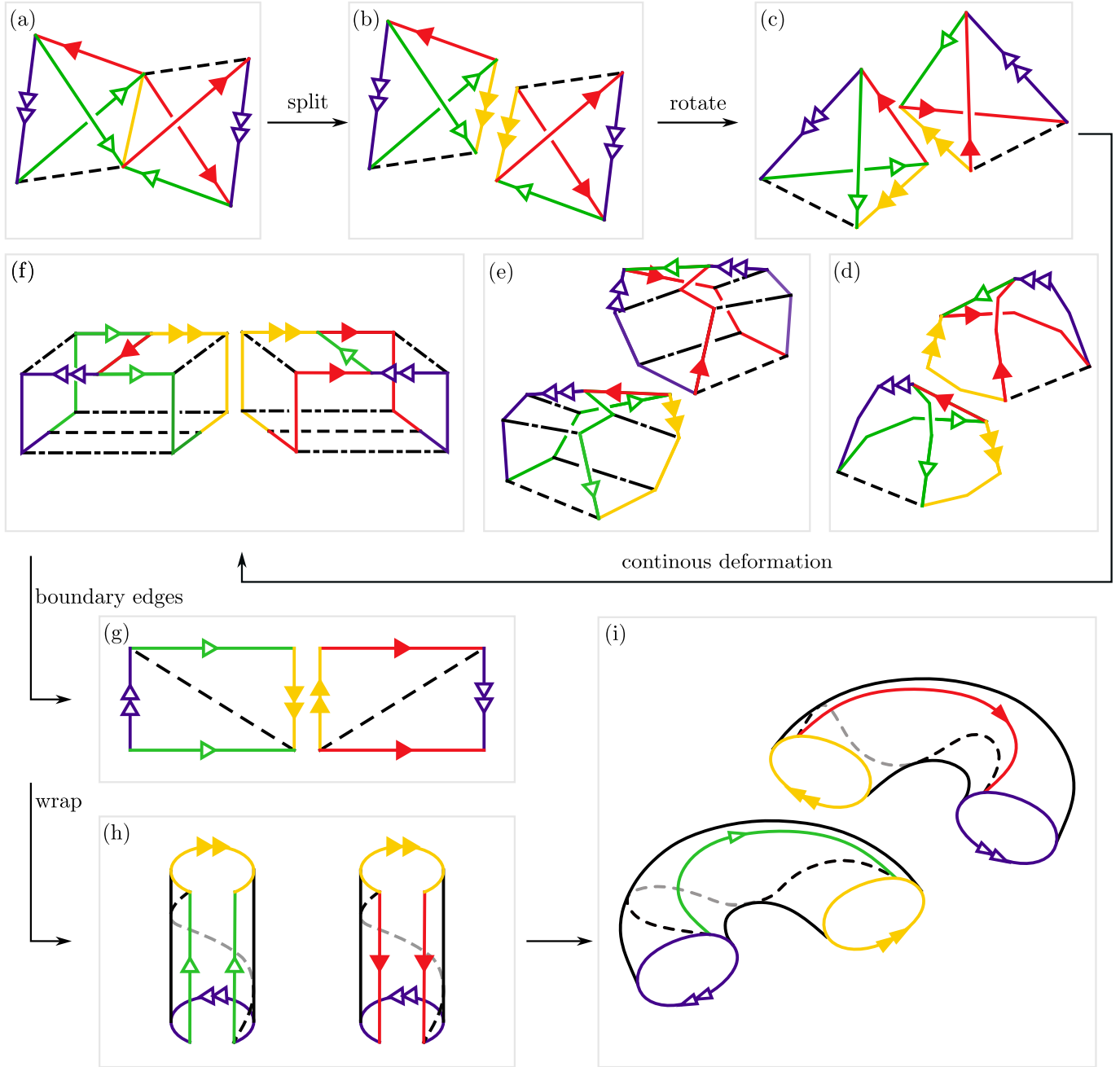


Figure 3.8: While deforming two tetrahedrons as depicted in Fig. 3.7, we keep track of the system's motion near the edges of the faces (a)-(f). To ensure the continuity of the trajectories and their chronological order – thus maintaining consistency with the system dynamics – the edges should be joined color-wise, such that the orientation of the arrows aligns. The tetrahedrons are continuously transformed into two cuboids, plotted separately for clarity. (g) To identify the two-dimensional surface of the manifold, we focus on the four white faces from Fig. 3.7(f). (h) The green and red edges of the white faces are glued together, forming two cylinders. (i) As the last step, the cylinders are then curved and subsequently joined along yellow and violet edges, forming a 2-torus.

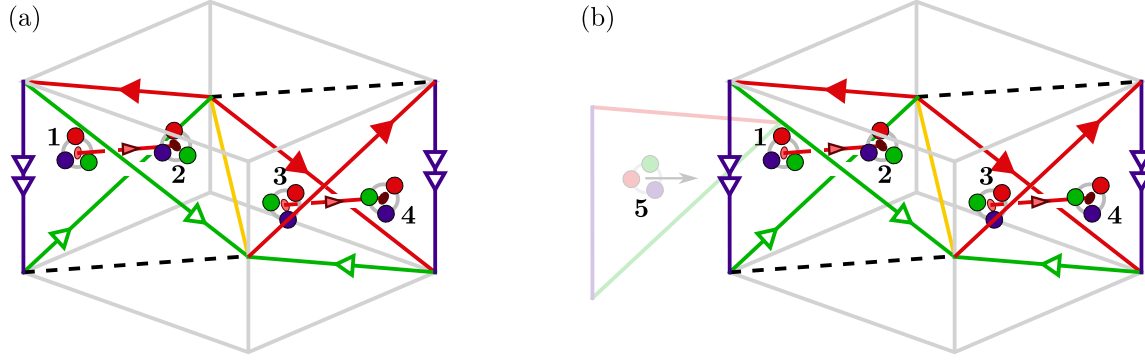


Figure 3.9: Identification of the interior of the manifold presented in Fig. 3.6(c). We analyze a fibre bundle of open 2-discs centered on the central trajectory (red line). (a) For visual clarity, we represent three trajectories by dots, each marking where a trajectory intersects a face of the tetrahedrons. Physically, these points correspond to particle collisions. Within the tetrahedrons (segments (1)-(2) and (3)-(4)), trajectories maintain their order along the gray circle. We analyze three selected trajectories (red, green and blue dots), starting from left at point (1) and progressing to point (4), before returning to the initial position (5). The colors of the trajectories (dots) correspond to the color of the nearest edges of the starting face (1). The color of the edges are known from the system dynamics, and provide the information about proper gluing that the trajectories are continuous, cf. Fig. 3.8. Crossing faces (2)-(3), (4)-(5) results in switches in the order of two trajectories, reflecting the exchange of the particles' momenta. The first collision alters the positions of the blue and green dots, while the second collision switches the red and blue dots. (b) Finally, the order of the trajectories remains consistent with the initial arrangement but undergoes a clockwise rotation of $2\pi/3$.

3.2.5 General remarks

In Section 3.2.3, we analyzed the manifold on which the two impenetrable particles with equal energies move. This manifold corresponds to a Möbius strip, which is a non-orientable fibre bundle over a circle. Physically, this can be understood as collisions between particles introducing a twist of the axes, faces, etc. where the collisions occur. For a two-particle system, there is only one type of collision between particles, which involves an axis twist and forms a Möbius strip.

In the case of three impenetrable particles presented in Section 3.2.4, there are generally two types of collisions: between the first and second particles, and the second and third particles. These collisions can be represented as a topological fibred product of two Möbius strips, resulting in an orientable solid torus. We suspect that this phenomenon will persist for $N > 3$, where the phase space would manifest as a fibred product of $N - 1$ Möbius strips. However, this hypothesis requires further analysis, which is beyond the scope of this work.

In the next section, we will explore the dynamics of impenetrable particles in the presence of the mirror oscillations by deriving the effective Hamiltonians for N impenetrable particles with different energies and for $N = 3$ impenetrable particles with equal energies.

3.3 Impenetrable particles: oscillating mirror

In this section, we consider the oscillating mirror and analyze the resonant dynamics of impenetrable particles. This analysis extends the single-particle bouncing on an oscillating mirror problem presented in Chapter 1. To maintain consistency with the assumptions from the previous sections, we keep the configuration where the particles are ordered in ascending order in 1D space, with the lowest particle bouncing on the oscillating mirror. The system Hamiltonian is given by:

$$H(\mathbf{x}, \mathbf{p}; t) = \sum_{j=1}^N \left(\frac{p_j^2}{2} + x_j \right) + F(x_1 - \lambda f(t)), \quad (3.23)$$

where $f(t) \leq x_1 \leq x_2 \leq \dots \leq x_N$ and λ is the amplitude of the perturbation, indicating how strongly the system is driven by the oscillating mirror. $F(x)$ represents the mirror potential. The periodic function $f(t) = f(t + T)$ with period $T = 2\pi/\omega$ and frequency ω , describes the oscillations of the mirror.

To simplify the analysis, we apply a canonical transformation from the laboratory frame to a frame oscillating with the mirror:

$$\begin{aligned} x'_j &= x_j - \lambda f(t), \\ p'_j &= p_j - \lambda \partial_t f(t). \end{aligned} \quad (3.24)$$

This transformation fixes the mirror position and introduces a time-periodically varying effective gravitational field, as shown in Fig. 1.3. The resulting Hamiltonian is:

$$H(\mathbf{x}, \mathbf{p}; t) = \underbrace{\sum_{j=1}^N \left(\frac{p_j^2}{2} + x_j \right)}_{=H_0(\mathbf{x}, \mathbf{p})} + \underbrace{\sum_{j=1}^N \lambda \partial_t^2 f(t) x_j}_{H_1(\mathbf{x}; t)} + F(x_1), \quad (3.25)$$

where we simplify the notation by dropping the primes and $F(x_1)$ term, assuming that the mirror is placed at $x_1 = 0$ in the oscillating mirror frame. Thus, the motion of the particles is confined to $0 \leq x_1 \leq x_2 \leq \dots \leq x_N$.

By expanding the time periodic function $\partial_t^2 f(t)$ into a Fourier series:

$$\partial_t^2 f(t) = \sum_{k \neq 0} f_k e^{ik\omega t} \quad (3.26)$$

where f_k are the Fourier components, the Hamiltonian takes the form:

$$H(\mathbf{x}, \mathbf{p}; t) = H_0(\mathbf{x}, \mathbf{p}) + \sum_{j=1}^N \left(\lambda x_j \sum_{k \neq 0} f_k e^{ik\omega t} \right), \quad (3.27)$$

In the subsequent sections, we will analyze the resonant behavior of impenetrable particles, considering cases with both different and equal energies. For this, we will use the novel angle-action variable $\{\boldsymbol{\theta}, \boldsymbol{\mathcal{I}}\}$ introduced in Section 3.2 for particles with different energies, and the

original action-angle variables $\{\boldsymbol{\theta}, \mathbf{I}\}$ introduced for independent particle in Section 3.1 for particles with equal energies.

3.3.1 Particles with different energies

The Hamiltonian for N impenetrable particles, each with a different energy, expressed in terms of the novel action-angle variables $\{\boldsymbol{\vartheta}, \mathbf{I}\}$ (3.12), is given by:

$$H(\boldsymbol{\vartheta}, \mathbf{I}; t) = \sum_{j=1}^N \left(\frac{(3\pi\mathcal{I}_j)^{2/3}}{2} + \frac{1}{2}\lambda \sum_{n_j} h_{n_j}(\mathcal{I}_j) e^{in_j\vartheta_j} \sum_{k \neq 0} f_k e^{ik\omega t} \right), \quad (3.28)$$

where the spatial part of the perturbation $H_1(\mathbf{x}; t) \rightarrow H_1(\boldsymbol{\vartheta}; t)$, expanded into a Fourier series with coefficients $h_{n_j}(\mathcal{I}_j)$:

$$h_{n_j}(\mathcal{I}_j) = \begin{cases} (\pi\mathcal{I}_j/\sqrt{3})^{2/3}, & \text{for } n_j = 0, \\ \frac{(-1)^{n_j+1}}{n_j^2} \left(\frac{3\mathcal{I}_j}{\pi^2} \right)^{2/3}, & \text{for } n_j \neq 0, \end{cases} \quad (3.29)$$

reflecting the 2π -periodicity of the angle variables ϑ_j .

We analyze the particles' motion near resonant orbit, keeping the action values $\mathcal{I}_j$ close to the unperturbed motion $\mathcal{I}_j \approx \mathcal{I}_{0,j}$, characterized by the resonant actions $\mathcal{I}_{0,j}$. The resonant condition is formulated as, cf. Eq. (3.8):

$$\omega = s_1\Omega(\mathcal{I}_{0,1}) = s_2\Omega(\mathcal{I}_{0,2}) = \dots = s_N\Omega(\mathcal{I}_{0,N}), \quad (3.30)$$

where integers $s_1 < s_2 < \dots < s_N$. To focus on the effective dynamics and eliminate fast oscillating terms, we apply the classical secular approximation [106, 140]. Firstly, performing a canonical transformation to a reference frame moving along a resonant trajectory, cf. the canonical transformation for a single-particle case (1.28):

$$\begin{aligned} \mathcal{I}'_j &= \mathcal{I}_j, \\ \vartheta'_j &= \vartheta_j - \frac{\omega}{s_j}t. \end{aligned} \quad (3.31)$$

In the new reference frame, the Hamiltonian is:

$$H(\boldsymbol{\vartheta}', \mathbf{I}'; t) = \sum_{j=1}^N \left(\frac{(3\pi\mathcal{I}'_j)^{2/3}}{2} - \frac{\omega}{s_j}\mathcal{I}_j + \lambda \sum_{k \neq 0} f_k e^{ik\omega t} \sum_{n_j} h_{n_j}(\mathcal{I}'_j) e^{in_j(\vartheta'_j + \omega t/s_j)} \right). \quad (3.32)$$

For particles close to resonant orbit $\mathcal{I}'_j \approx \mathcal{I}_{0,j}$, all dynamical variables $\mathcal{I}'_j, \vartheta'_j$ except time evolve slowly, cf. equations (1.30). Therefore, to eliminate fast oscillating terms, we average the Hamiltonian (3.32) over time to obtain the effective Hamiltonian:

$$H_{\text{eff}}(\boldsymbol{\vartheta}, \mathbf{I}) \approx \sum_{j=1}^N \left(\frac{P_j^2}{2m_j} + \lambda V_{\text{eff}}^{(j)}(\vartheta'_j) \right), \quad (3.33)$$

where the potential $V_{\text{eff}}^{(j)}(\vartheta'_j) = \sum_{n_j} f_{n_j} h_{n_j s_j}(\mathcal{I}_{0,j}) e^{in_j s_j \vartheta'_j}$, effective mass $m_j = -\frac{1}{2} \left(\frac{9\mathcal{I}_{0,j}^2}{\pi} \right)^{2/3}$, and conjugate momenta $P_j = \mathcal{I}'_j - \mathcal{I}_{0,j}$.

The effective Hamiltonian $H_{\text{eff}}(\boldsymbol{\vartheta}, \boldsymbol{\mathcal{I}})$ describes the system of N impenetrable particles with different energies near a resonant trajectory [106, 109]. This Hamiltonian can also be interpreted as that of a single particle moving in an N -dimensional space within a time-independent effective potential $\sum_{j=1}^N V_{\text{eff}}^{(j)}(\vartheta'_j)$. By selecting specific Fourier components f_k (3.26) of the mirror oscillations, an arbitrary separable effective potential can be engineered to affect the particle dynamics [91].

The simplest example is the creation of an N -dimensional potential with s_j wells along each j -th direction. By selecting a harmonic driving force such that $\partial_t^2 f(t) = \cos(\omega t)$, the effective Hamiltonian (3.33) simplifies to:

$$H_{\text{eff}}(\boldsymbol{\vartheta}, \boldsymbol{\mathcal{I}}) \approx \sum_{j=1}^N \left(\frac{P_j^2}{2m_j} + \lambda h_{s_j}(\mathcal{I}_{0,j}) \cos(s_j \vartheta'_j) \right). \quad (3.34)$$

Similar to the analysis of a single particle bouncing on an oscillating mirror in a gravitational field, which transitions from classical to quantum analysis (see Section 1.7), we can extend the classical analysis of N impenetrable particles to a quantum framework. By maintaining large resonances $s_j \gg 1$ and engineering the effective potential $\sum_{j=1}^N V_{\text{eff}}^{(j)}(\vartheta'_j)$, various condensed matter phenomena in N -dimensional space can be explored, such as disordered crystalline structures or crystals with nontrivial topological properties [20, 114, 116, 141, 142].

In the next section, we will derive the effective Hamiltonian for a system of $N = 3$ impenetrable particles with equal energies. For a detailed analysis of the $N = 2$ system, refer to Ref. [137].

3.3.2 Particles with equal energies

To derive the effective Hamiltonian for a system of $N = 3$ impenetrable particles, we employ the original angle-action variables $\{\boldsymbol{\theta}, \boldsymbol{I}\}$ introduced for the independent particle system in Section 3.1. The novel angle-action variables $\{\boldsymbol{\vartheta}, \boldsymbol{\mathcal{I}}\}$ are not suitable for particles with equal energies, since a canonical transformation (3.12) is not well-defined in this case, as discussed in Section 3.2.

Following the introductory part of Section 3.3, the Hamiltonian for the system in terms of the original angle-action variables $\{\boldsymbol{\vartheta}, \boldsymbol{\mathcal{I}}\}$ is given by:

$$H(\boldsymbol{\theta}, \boldsymbol{I}; t) = \sum_{j=1}^3 \left(\frac{(3\pi I_j)^{2/3}}{2} + \frac{1}{2} \lambda \left(\frac{3I_j}{\pi^2} \right)^{2/3} (2\pi - \theta_j) \theta_j \partial_t^2 f(t) \right), \quad (3.35)$$

which is invariant under the exchange of both angle $\theta_j \rightleftharpoons \theta_{j+1}$ and action $I_j \rightleftharpoons I_{j+1}$ variables between any two colliding particles.

We derive the effective Hamiltonian of the system close to the resonant trajectory following the approach presented in Section 3.3.1 and Section 1.5. The resonant condition (3.30)

simplifies to $\omega = s\Omega(I_0)$, where I_0 is the resonant action. Initially, we switch to a frame moving along a resonant trajectory through $\theta'_j = \theta_j - \omega t/s$ and $I'_j = I_j$, cf. (3.31), and then average the resulting Hamiltonian over time while keeping I'_j and θ'_j fixed, as their slowly fluctuate in the vicinity of the resonant trajectory, provided the perturbation is weak $\lambda \ll 1$. The effective Hamiltonian then becomes:

$$H(\boldsymbol{\theta}', \boldsymbol{I}') \approx \sum_{j=1}^3 \left(\frac{P_j^2}{2m_j} + \lambda V_{\text{eff}}(\theta'_j) \right), \quad (3.36)$$

where $P_j = I'_j - I_0$, and $m_j = -\frac{1}{2} \left(\frac{9I_0^2}{\pi} \right)^{2/3}$ is the effective mass, and V_{eff} is the effective potential shaped by mirror oscillations.

The Hamiltonian effectively describes the system of $N = 3$ impenetrable particles with equal energies near a resonant trajectory. Given that it is expressed using the original angle-action variables $\{\boldsymbol{\theta}, \boldsymbol{I}\}$ designed for independent particles, as detailed in Section 3.1, additional constraints related to particle collisions are necessary. The system dynamics is described by Hamilton's equations, governed by the effective Hamiltonian (3.36). Monitoring the evolution of particle collisions $I_j(t) = I'_j(t)$ and $\theta_j(t) = \theta'_j(t) + \omega t/s$ in the oscillating mirror reference frame is needed to fully capture the system behavior. If any of the inequalities:

$$0 \leq (2\pi - \theta_1)\theta_1 \leq (2\pi - \theta_2)\theta_2 \leq (2\pi - \theta_3)\theta_3, \quad (3.37)$$

is saturated, indicating a collision, the colliding particles' angles $\theta'_j \rightleftharpoons \theta'_{j+1}$ and actions $I'_j \rightleftharpoons I'_{j+1}$ must be swapped.

3.4 Summary

In this chapter, we discussed the extension of a single-particle system bouncing on an oscillating mirror in a gravitational field to a system of N impenetrable particles in a one-dimensional space. We began by extending the classical analysis of a single particle bouncing on a static mirror to a system of N independent particles. Since the particles are independent, we used angle-action variables $\{\boldsymbol{\theta}, \boldsymbol{I}\}$ to describe the system's dynamics, simplifying the identification of the manifold in phase space. These variables have a one-to-one correspondence with Cartesian coordinates: for each particle $x_j \rightleftharpoons \theta_j$ and $p_j \rightleftharpoons I_j$. The action variable I_j is proportional to the particle's energy E_j and remains constant over time, while the angle variable θ_j describes the particle's position on periodic orbits. The system's dynamics are encoded in the angle variables, which evolve linearly in time and are 2π -periodic. Since the particles are independent, each particle moves separately along a circular trajectory in phase space. This results in the manifold being an N -torus in the $2N$ -dimensional phase space for the system of N independent particles.

Next, we explored the system of N impenetrable particles bouncing on a static mirror, where the impenetrability is modeled by elastic collisions between particles. These collisions result in the exchange of momenta and energies between the colliding particles. As a consequence, the

angle-action variables $\{\theta_j, I_j\}$ introduced for independent particles are no longer feasible to describe the dynamics of impenetrable particles, as we have to account for the constraints imposed by elastic collisions. To address this, we proposed new angle-action variables $\{\vartheta_{j'}, \mathcal{I}_{j'}\}$, which encode the effects of collisions. These variables are effective for particles with different energies, where we can sort the set of particle energies $\{E_j\}$ in ascending order at any time of the system's evolution. This ordered set $\{\mathcal{E}_{j'}\}$ remains preserved over time. An important distinction is that the index j' no longer corresponds to a specific particle but instead to a particular energy value. The new angle-action variables $\{\vartheta_{j'}, \mathcal{I}_{j'}\}$ capture the dynamics of particle collisions, allowing us to describe the system as though it consists of independent particles. Using these new variables, we identified the manifold on which the system of N impenetrable particles with different energies moves. It is an N -torus in the $2N$ -dimensional phase space.

However, the new angle-action variables cannot be well-defined for impenetrable particles with equal energies, as we cannot distinguish between their energies. To identify the manifold on which this system moves, we returned to the angle-action variables $\{\theta_j, I_j\}$ introduced for independent particles. To account for elastic collisions in the system's dynamics, we introduced additional constraints to Hamilton's equations. Each collision results in the exchange of momentum and energy between the colliding particles, causing discontinuities in the trajectory in phase space. To identify the manifold, we needed to conceptually "join" these fragmented trajectories into a continuous one that is consistent with the system's dynamics. This requires an analysis of trajectories in the phase space for different initial conditions. The analysis has to be performed separately for each number of particles. In this chapter, we focused on the case of $N = 2$ and $N = 3$, and find that the manifolds are a Möbius strip and a solid torus, respectively.

Initially, we analyzed the system with a static mirror to identify the manifold on which the particles move. After introducing mirror oscillations, we derived the effective Hamiltonian to describe the system near the resonance trajectory. Using both types of angle-action variables, we distinguished between cases with particles having different and equal energies. The resulting Hamiltonians describe not only the systems under consideration but also a fictitious particle moving in an N -dimensional effective potential shaped by the Fourier components of the function modeling mirror oscillations. This Fourier dependence allows the engineering of various crystalline structures and opens up opportunities for exploring N -dimensional condensed matter phenomena. The derived Hamiltonian, with a time-independent potential, describes crystalline structures in time domain when viewed from the laboratory reference frame.

The results presented in this chapter are classical in nature but form a basis for a fully quantum approach to investigate novel discrete time crystals. One might replace N classical particles with N atomic clouds, each composed of different atomic species and interacting weakly within its cloud but exhibiting infinite repulsion between particles from different clouds. The presence of intra-cloud interactions could lead to spontaneous symmetry breaking, forming various types of discrete time crystals, which will be further explored in the following chapter.

Chapter 4

Quantum analysis of complex DTC

In this chapter, we extend the analysis from Chapter 3 by replacing the N classical particles with N atomic clouds, each composed of different atomic species. Specifically, we focus on $N = 2$ case. The particles within each cloud interact attractively, while particles from different clouds interact with infinite repulsive strength. Our goal is to analyze the overall symmetry of the system in the presence of intra-species interactions, which can lead to the spontaneous time translation symmetry breaking and thus form a complex discrete time crystal.

We begin with a brief description of the system under consideration (Section 4.1). Next, we explore the case of two independent clouds to introduce a single-particle basis for each subsystem and revisit the concept of spontaneous discrete time translation symmetry, which leads to the formation of discrete time crystals (Section 4.2).

In Section 4.3, we focus on the case of two infinitely interacting clouds and propose an ansatz for the many-body wavefunction within the two-particle basis. Sections 4.4-4.5 cover the key building blocks of our approach, including the antisymmetric two-particle basis, as well as the numerical challenges and approximations used to address them.

Sections 4.6-4.7 present the results obtained from minimizing the quasi-energy functional for the proposed ansatz. Finally, Section 4.8 provides a brief summary of the chapter.

4.1 Introduction to complex DTC

We analyze a system composed of $N = 2$ distinct atomic clouds, A and B , as presented in Fig. 4.1. These clouds are arranged in a 1D stack: atoms in the lower cloud A bounce on an oscillating mirror, modeled by the cosine function $f(t) = \frac{\lambda}{\omega^2} \cos(\omega t)$ where λ/ω^2 determines the amplitude of the oscillations. The atoms in the upper cloud B bounce on those from cloud A .

The atoms within each cloud are identical, but the atoms in cloud A are of a different type than those in cloud B . Both types of atoms exhibit weak, attractive intra-species interactions characterized by the strength $g_\sigma \leq 0$, where $\sigma \in \{A, B\}$. In contrast, the inter-species

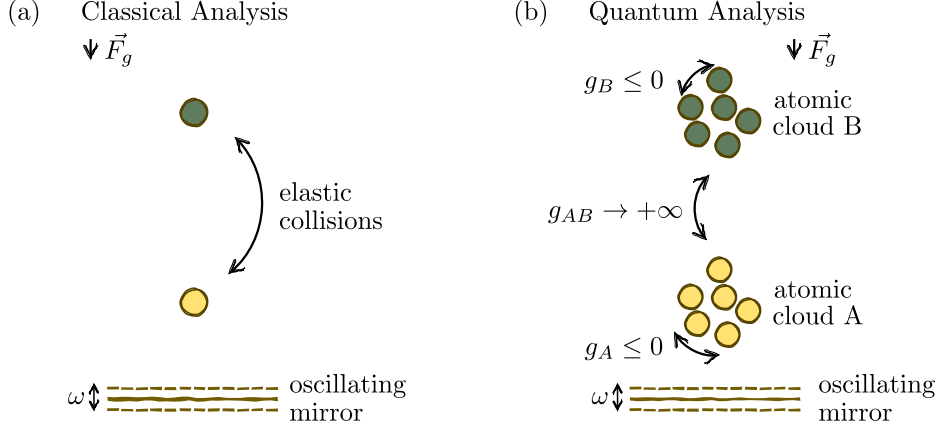


Figure 4.1: Illustration of the system under study. (a) The classical analysis of $N = 2$ classical impenetrable particles, presented in Chapter 3, is extended to two distinct ultra-cold atomic clouds, A and B (b). The system is arranged in a one-dimensional stack: where the lower cloud A bounces on an oscillating mirror, and the upper cloud B bounces on cloud A . The entire system is subjected to a gravitational field $\vec{F}_g$. Classical impenetrability, simulated by elastic collisions, is now modeled by infinitely repulsive contact interactions $g_{AB} \rightarrow +\infty$ between the clouds. Within each cloud, the ultra-cold atoms interact weakly and attractively, characterized by the interaction strength $g_\sigma \leq 0$, where $\sigma \in \{A, B\}$. The mirror's oscillations are defined by a frequency ω and an amplitude λ/ω^2 .

interactions between atoms from different clouds are infinitely repulsive, $g_{AB} \rightarrow +\infty$. The entire system is subjected to a gravitational field.

This quantum system extends the model discussed in Chapter 3, which considered a 1D stack of classical impenetrable particles. In contrast, our current model replaces these classical particles with ultra-cold atomic clouds that interact via an infinitely repulsive contact potential. If the system consisted of only one atom per cloud, it would correspond to the Tonks-Girardeau gas in the limit of infinite repulsive interactions [143]. Conversely, reducing the number of clouds to one aligns the system with the first theoretical proposal for discrete time crystals [2], which is described in detail in Chapter 1.

The dynamics of the system, composed of $\tilde{N} = \tilde{N}_A + \tilde{N}_B$ bosonic particles, where $\tilde{N}_\sigma$ is the number of atoms in cloud σ , is governed by the many-body Floquet Hamiltonian:

$$\hat{\mathcal{H}} = \hat{H}_A + \hat{H}_B + \hat{H}_{AB} - i\partial_t. \quad (4.1)$$

$\hat{H}_\sigma$ represents the Hamiltonian for the attractively interacting atoms within the cloud σ , while $\hat{H}_{AB}$ describes the interactions between the two distinct clouds:

$$\hat{H}_A = \sum_{j=1}^{\tilde{N}_A} \left(-\frac{1}{2} \partial_{x_j}^2 + x_j + \lambda x_j \cos(\omega t) \right) + \frac{1}{2} g_A \sum_{j,k=1}^{\tilde{N}_A} \delta(x_j - x_k), \quad (4.2)$$

$$\hat{H}_B = \sum_{j=1}^{\tilde{N}_B} \left(-\frac{1}{2} \partial_{y_j}^2 + y_j + \lambda y_j \cos(\omega t) \right) + \frac{1}{2} g_B \sum_{j,k=1}^{\tilde{N}_B} \delta(y_j - y_k), \quad (4.3)$$

$$\hat{H}_{AB} = \frac{1}{2} g_{AB} \sum_{j=1}^{\tilde{N}_A} \sum_{k=1}^{\tilde{N}_B} \delta(x_j - y_k). \quad (4.4)$$

All interactions between atoms are modeled by a Dirac $\delta(x)$ contact potential, with the intra- and inter-species coupling constants $g_\sigma \leq 0$ and $g_{AB} \rightarrow +\infty$, respectively. Additionally, we assume identical masses of particles $m_A = m_B$, which could correspond the same kinds of atoms but in different internal states of the same species for each cloud. Throughout the thesis, we use the gravitational units as defined in Eq. (1.15).

In the next section, we analyze independent ultra-cold atomic clouds to introduce a single-particle basis for the further analysis of the system.

4.2 Two independent ultra-cold atomic clouds

In the absence of inter-species interactions $g_{AB} = 0$, the system simplifies into two independent subsystems σ . Each subsystem is governed by its respective many-body Floquet Hamiltonian:

$$\hat{H}_\sigma \rightarrow \hat{\mathcal{H}}_\sigma = \hat{H}_\sigma - i\partial_t. \quad (4.5)$$

The Hamiltonian $\hat{\mathcal{H}}_\sigma$ describes a single cloud of ultra-cold atoms bouncing on an oscillating mirror, as discussed in Section 1.7. Here, we briefly summarize the approaches presented in that section to introduce the one-particle basis that spans the resonant Hilbert subspace for each subsystem σ .

The analysis of these independent systems begins with a classical single-particle problem near the resonance orbit, effectively described using a classical secular approximation. This approach transitions to its quantum counterpart through quantization. Alternatively, one can start with a quantum single-particle description from the beginning and apply a quantum secular approximation to describe the particle near the resonant orbit [20], which leads to the following Hamiltonian, cf. Eq. (1.33):

$$\langle n' | \hat{\mathcal{H}}_\sigma^{\text{sec}} | n \rangle \approx \left(E_n^\sigma - n \frac{\omega}{s_\sigma} \right) \delta_{n,n'} + \frac{\lambda}{2} \langle n' | \hat{x} | n \rangle (\delta_{n',n+s_\sigma} + \delta_{n',n-s_\sigma}). \quad (4.6)$$

Here, E_n^σ are the eigenvalues of the unperturbed Hamiltonian with $\lambda = 0$, $\hat{H}_\sigma^{\lambda=0} | n \rangle = E_n^\sigma | n \rangle$ within the basis of Airy functions $\{| n \rangle\}$.

The eigenstates of the Hamiltonian $\hat{\mathcal{H}}_\sigma^{\text{sec}}$ are Floquet states $u_j^\sigma(x, t)$, characterizing the behavior of a single particle near the resonant trajectory. For a given resonance $s_\sigma : 1$ and restricting the analysis to the first energy band, see Fig. 1.6 as a reference, we obtain s_σ

Floquet states. Throughout this chapter, we choose the following resonances for the subsystems: $s_A = 2$, and $s_B = 3$. The corresponding Floquet states are presented in Fig. 4.2(a)-(c) (Panel 1 and Panel 2). Each Floquet state consists of s_σ localized wave packets that move with a period $s_\sigma T$, where $T = 2\pi/\omega$ is the period of the mirror's oscillations, cf. Fig. 1.5.

As discussed in Section 1.7, we can extract the $s_\sigma T$ -periodic localized wavepackets $w_j^\sigma(x, t)$, termed Wannier states, by forming a proper superposition of the Floquet states $u_j^\sigma(x, t)$. These Wannier states form a single-particle basis that spans a resonant Hilbert subspace for each independent subsystem σ . Within this subspace, the particle's wavefunction can be expanded as:

$$\psi_\sigma(x, t) = \sum_{j=1}^{s_\sigma} w_j^\sigma(x, t) c_j^\sigma, \quad (4.7)$$

where $\sum_{j=1}^{s_\sigma} |c_j^\sigma|^2 = 1$. The Wannier states are shown in Fig. 4.2(d)-(f) (Panel 1 and Panel 2). Having the resonant basis, we move to the many-body interacting Hamiltonian for each subsystem σ within the framework of second quantization, keeping the inter-species interaction strength $g_{AB} = 0$ and weak intra-species interaction strength $g_\sigma \leq 0$. The bosonic field operator, expressed in terms of the Wannier states $w_j^\sigma(x, t)$, is [20, 110]:

$$\hat{\Psi}_\sigma(x, t) \approx \sum_{j=1}^{s_\sigma} w_j^\sigma(x, t) \hat{c}_j^\sigma, \quad (4.8)$$

where $\hat{c}_j^\sigma$ is the bosonic annihilation operator. The many-body Hamiltonian is then formu-

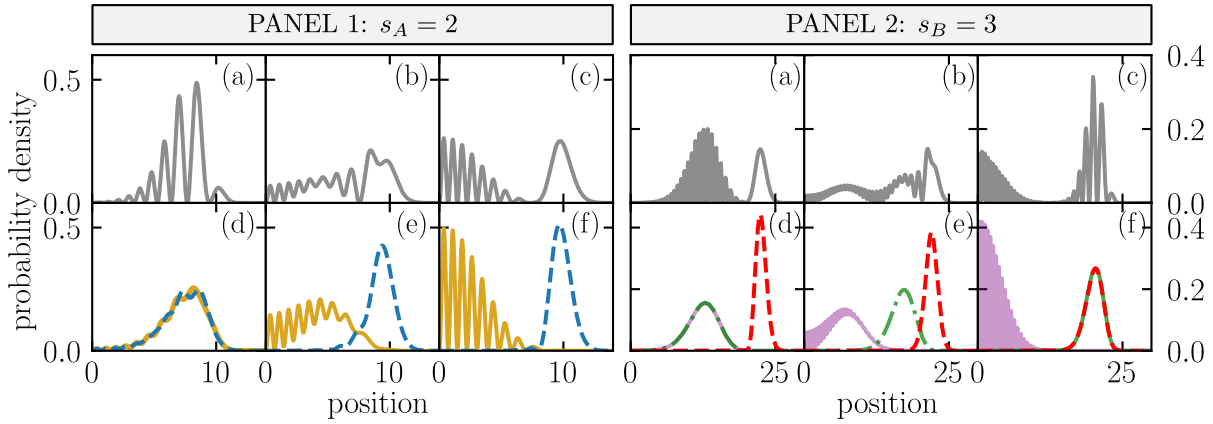


Figure 4.2: (a)-(c) The evolution of the T -periodic Floquet states governed by the single-particle Floquet Hamiltonian (4.6) for the resonances $s_A = 2$ (Panel 1) and $s_B = 3$ (Panel 2). Snapshots are taken at times $t = 0, 0.5\pi/\omega, \pi/\omega$, from left to right. Each Floquet state consists of s_σ localized wave packets, with $\sigma \in \{A, B\}$. From a proper superposition of these Floquet states, $s_\sigma T$ -periodic Wannier states $w_j^\sigma(x, t)$ are extracted (d)-(f), cf. Fig. 1.5. The Hamiltonian parameters are $\omega = 1.4$, $\lambda = 0.07$.

lated as, cf. Hamiltonian (1.39):

$$\begin{aligned}\hat{\mathcal{H}}_\sigma &= \frac{1}{s_\sigma T} \int_0^{s_\sigma T} dt \int_0^\infty dx \hat{\Psi}_\sigma^\dagger \left(\hat{H}_\sigma(t) - i\partial_t + \frac{g_\sigma}{2} \hat{\Psi}_\sigma^\dagger \hat{\Psi}_\sigma \right) \hat{\Psi}_\sigma \\ &\approx -\frac{J^\sigma}{2} \sum_{j=1}^{s_\sigma} \left(\hat{c}_{j+1}^\sigma \hat{c}_j^\sigma + \hat{c}_j^\sigma \hat{c}_{j+1}^\sigma \right) + \frac{1}{2} \sum_{j,k=1}^{s_\sigma} U_{jk}^\sigma \hat{c}_j^\sigma \hat{c}_k^\sigma \hat{c}_k^\sigma \hat{c}_j^\sigma,\end{aligned}\tag{4.9}$$

with the tunneling amplitude:

$$J^\sigma = -\frac{2}{s_\sigma T} \int_0^{s_\sigma T} dt \int_0^\infty dx w_j^{\sigma*}(x, t) (\hat{H}_\sigma(t) - i\partial_t) w_{j+1}^\sigma(x, t),\tag{4.10}$$

and the interaction coefficients:

$$\begin{aligned}U_{jk}^\sigma &= \frac{2g_\sigma}{s_\sigma T} \int_0^{s_\sigma T} dt \int_0^\infty dx |w_j^\sigma(x, t)|^2 |w_k^\sigma(x, t)|^2, \quad \text{for } j \neq k, \\ U_{jj}^\sigma &= \frac{g_\sigma}{s_\sigma T} \int_0^{s_\sigma T} dt \int_0^\infty dx |w_j^\sigma(x, t)|^4.\end{aligned}\tag{4.11}$$

The final form of the Hamiltonian $\hat{\mathcal{H}}_\sigma$ is analogous to the Bose-Hubbard Hamiltonian written in the time-periodic basis of Wannier functions. For sufficiently strong attractive intra-species interactions g_σ , the ground state of $\hat{\mathcal{H}}_\sigma$ takes the form of a Schrödinger cat-like state:

$$\Psi_\sigma^{\text{GS}}(\mathbf{x}; t) \propto \sum_{k=1}^{s_\sigma} \Pi_{j=1}^{\tilde{N}_\sigma} w_k^\sigma(x_j, t),\tag{4.12}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{\tilde{N}_\sigma})$ represents the positions of the atoms (see Section 1.8). The state preserves the discrete time translation symmetry of the Hamiltonian $\hat{\mathcal{H}}_\sigma$ (4.9) and evolves with the same period as the mirror's oscillations period T . It is extremely vulnerable to perturbations; for instance measuring the position of a single atom can induce a collapse into one of the product states:

$$\Psi_\sigma^{\text{GS}}(\mathbf{x}; t) \rightarrow \Psi_\sigma(\mathbf{x}; t) \propto \Pi_{j=1}^{\tilde{N}_\sigma} w_k^\sigma(x_j, t).\tag{4.13}$$

The resulting symmetry-broken state $\Psi_\sigma(\mathbf{x}; t)$ evolves with a period $s_\sigma T$, which is s_σ times longer than that of the state $\Psi_\sigma^{\text{GS}}(\mathbf{x}; t)$. This phenomenon corresponds to spontaneous discrete time translation symmetry breaking, leading to the formation of DTC characterized by a s_σ -tupling of the oscillating mirror's period, as presented in Fig. 1.8 for $s_\sigma = 2$. Given that the two subsystems σ are independent, each form its own DTC for a sufficiently strong intra-species interactions g_σ . Here, we analyze the resonances such that $s_A = 2$ results in a doubling (2-tupling) DTC, while $s_B = 3$ leads to a tripling (3-tupling) DTC. For a comprehensive review, see Chapter 1.

In the next section, we introduce the approach we will use to extend the analysis by incorporating infinitely repulsive inter-species interactions $g_{AB} \rightarrow +\infty$ between the two subsystems σ .

4.3 Two interacting ultra-cold atomic clouds

In the presence of infinitely repulsive inter-species interactions, $g_{AB} \rightarrow +\infty$, non-trivial correlations emerge between particles from different subsystems σ , making the system non-separable. We model these interactions as those of hard-core bosons by applying the Bose-Fermi mapping, which ensures that the many-particle wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$ vanishes whenever atoms of different types σ meet, i.e. $\Psi(\mathbf{x}, \mathbf{y}; t) = 0|_{x_j=y_k}$ [143]. This mapping effectively eliminates the interaction term $\hat{H}_{AB}$ from the Floquet Hamiltonian $\hat{\mathcal{H}}$ (4.1), resulting in:

$$\hat{\mathcal{H}} = \hat{H}_A + \hat{H}_B - i\partial_t, \quad (4.14)$$

where $\hat{H}_\sigma$ is defined by Eqs. (4.2)-(4.3). The infinitely repulsive interactions are thus encoded through the partial antisymmetrization of the many-particle wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$ with respect to the exchange of any pair of particle's coordinates $x_j \rightleftharpoons y_{j'}$, analogous to the approach used for the Tonks-Girardeau gas [143–146]. For clarity, we keep the different notations for the atomic positions: $\mathbf{x} = (x_1, \dots, x_{\tilde{N}_A})$ for subsystem A and $\mathbf{y} = (y_1, \dots, y_{\tilde{N}_B})$ for subsystem B .

To incorporate the infinitely repulsive inter-species interactions g_{AB} into the many-body wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$, we first introduce an antisymmetric two-particle wavefunction using the Wannier basis $\{w_j^\sigma(z, t)\}$ with $z \in \{x, y\}$:

$$\begin{aligned} \Phi(x, y; t) &= \mathcal{A}[\psi_A(x, t)\psi_B(y, t)] \\ &= \mathcal{N} \sum_{j=1}^{s_A} \sum_{k=1}^{s_B} c_j^A c_k^B (w_j^A(x, t)w_k^B(y, t) - w_j^A(y, t)w_k^B(x, t)), \end{aligned} \quad (4.15)$$

where the single-particle state $\psi_\sigma(z, t)$ is defined by Eq. (4.7), and $\mathcal{A}[\cdot]$ denotes antisymmetrization to ensure that the wavefunction vanishes when particles from different clouds meet. $\mathcal{N}$ is the normalization constant.

This two-particle state $\Phi(x, y; t)$ is then used to construct the many-particle Floquet state $\Psi(\mathbf{x}, \mathbf{y}; t)$, which takes the form of a Jastrow ansatz [147, 148]:

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{y}; t) &= \Pi_{m=1}^{\tilde{N}_A} \Pi_{m'=1}^{\tilde{N}_B} \Phi(x_m, y_{m'}; t) \\ &= \tilde{\mathcal{N}} \Pi_{m=1}^{\tilde{N}_A} \Pi_{m'=1}^{\tilde{N}_B} \left[\sum_{j=1}^{s_A} \sum_{k=1}^{s_B} c_j^A c_k^B (w_j^A(x_m, t)w_k^B(y_{m'}, t) - w_j^A(y_{m'}, t)w_k^B(x_m, t)) \right], \end{aligned} \quad (4.16)$$

and serves as the variational ansatz. The ansatz (4.16) incorporates the symmetrization of the many-particle wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$ with respect to the exchange of any pair of coordinates for particles of the same type σ , i.e. $x_j \rightleftharpoons x_{j'}$ and $y_j \rightleftharpoons y_{j'}$, accounting for the bosonic nature of the atoms.

In general, the Jastrow wavefunction is a variational ansatz used to incorporate correlations between particles in many-body quantum systems. The central idea of the Jastrow ansatz

is to start with a simple reference wavefunction $\Phi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$, e.g. a Slater determinant for fermions or a Hartree product for bosons, and then multiply it by a correlation factor $e^U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ that accounts for the interactions between particles. The general form of the Jastrow wavefunction is:

$$\Psi^J(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \Phi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) e^{U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)}. \quad (4.17)$$

The Jastrow factor $U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ typically has the form of:

$$U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \sum_{j < k} u(\mathbf{r}_j, \mathbf{r}_k), \quad (4.18)$$

where the pairwise correlation function $u(\mathbf{r}_j, \mathbf{r}_k)$ often depends on the distance between particles and may be chosen to reflect the nature of the interaction. The main motivation behind the Jastrow ansatz is to efficiently capture short-range correlations between particles, which are often poorly described by mean-field approaches. It allows for a compact representation of many-body effects, improving the accuracy of quantum Monte Carlo simulations and variational calculations [149]. On the other hand, the Jastrow ansatz alone does not effectively model long-range correlations unless specifically adapted for such interactions, which can complicate its form. Additionally, as the number of particles increases, the computational cost of evaluating and optimizing the Jastrow factor can become prohibitively expensive.

Returning to the problem under consideration, to find the lowest quasi-energy state of the system in the resonant Hilbert subspace, we minimize the average value of the Floquet Hamiltonian (4.14) in the state (4.16) over the parameter space spanned by variational coefficients $\{\mathbf{c}_A, \mathbf{c}_B\} = \{c_1^A, c_2^A, \dots, c_{s_A}^A; c_1^B, c_2^B, \dots, c_{s_B}^B\}$, as they appear in the many-particle ansatz (4.16), for the given intra-species interaction strengths g_σ :

$$E(g_a, g_b) := \min [E[\Psi(\mathbf{c}_A, \mathbf{c}_B)]] = \min \left[\frac{\int_0^{T_{\text{tot}}} dt \int_0^\infty d\mathbf{x} d\mathbf{y} \Psi^*(\mathbf{x}, \mathbf{y}; t) \hat{H} \Psi(\mathbf{x}, \mathbf{y}; t)}{\int_0^{T_{\text{tot}}} dt \int_0^\infty d\mathbf{x} d\mathbf{y} \Psi^*(\mathbf{x}, \mathbf{y}; t) \Psi(\mathbf{x}, \mathbf{y}; t)} \right] \quad (4.19)$$

where $T_{\text{tot}} = s_A s_B T$ is the total period of the system. Anticipating the emergence of solutions with period T_{tot} , since this is generally the period of the Jastrow ansatz (4.16), we average the quasi-energy functional (4.19) over a period $s_A s_B$ times longer than the driving period T . Before detailing the minimization procedure, we will first discuss our general approach for analyzing the system's properties in the next section.

4.4 Antisymmetric basis for two atomic clouds

To discuss the implications of using the antisymmetric two-particle basis (4.15) for constructing the Jastrow ansatz (4.16), we begin by exploring the basis itself, followed by an analysis of the many-body wavefunction.

Antisymmetric two-particle basis

In this chapter, we consider a system of two ultra-cold atomic clouds, where the lower cloud bounces on the oscillating mirror and, through infinite interactions, $g_{AB} \rightarrow +\infty$, effectively acts as a driving force for the upper cloud. To handle these infinite interactions between particles from different clouds, we introduce the antisymmetric two-particle basis $\Phi(x, y; t)$ (4.15), which is antisymmetric with respect to the exchange of particle coordinates $x \rightleftharpoons y$. This basis precisely captures the effect of the infinite interactions and simplifies the system's Hamiltonian which is then reduced to (4.14).

The two-particle basis encodes the impenetrability of two particles of different types but does not distinguish between the cases where $x > y$ and $x < y$, as it accounts for both scenarios simultaneously. The two-particle states describing the scenarios are orthogonal, and this orthogonality is preserved throughout the system's evolution. In other words, we can express the two-particle basis (4.15):

$$\Phi(x, y; t) = \Phi^{x>y}(x, y; t) + \Phi^{x<y}(x, y; t), \quad (4.20)$$

with

$$\Phi^{x>y}(x, y; t) \propto \sum_{j=1}^{s_A} \sum_{k=1}^{s_B} c_j^A c_k^B (w_j^A(x, t) w_k^B(y, t) - w_j^A(y, t) w_k^B(x, t)) \Theta(x - y) \quad (4.21)$$

where $\Phi^{x<y}(x, y; t)$ has a similar form but with $\Theta(y - x)$. At any time t , the states $\Phi^{x>y}(x, y; t)$ and $\Phi^{x<y}(x, y; t)$ form an orthogonal basis $\langle \Phi^{x<y}(t) | \Phi^{x>y}(t) \rangle$ and they are not coupled by the Floquet Hamiltonian (4.14):

$$\langle \Phi^{x<y}(t) | \hat{\mathcal{H}} | \Phi^{x>y}(t) \rangle = 0. \quad (4.22)$$

To simulate the experiment where one cloud of atoms is prepared below the other, we will perform particle measurements and then sort the results based on the specific configuration of the particles, see Section 4.7.

Many-body wavefunction: Jastrow ansatz

We use the antisymmetric two-particle states $\Phi(x, y; t)$ (4.15) as the basis for constructing the many-particle state $\Psi(\mathbf{x}, \mathbf{y}; t)$ in the form of Jastrow ansatz (4.16). The Jastrow ansatz accounts the indistinguishable nature of bosons, ensuring that the resulting state is symmetric with respect to exchange of any pair of coordinates for particles of the same type σ . This approach inherently includes all possible fragmentations of the system, ranging from the scenario where all atoms of type A are above those of type B , $\forall_{m, m'} x_m > y_{m'}$, through mixed cases, e.g. $x_1, x_2 > y_1 > x_3, \dots, x_{\tilde{N}_A} > y_2, \dots, y_{\tilde{N}_B}$, to the scenario where all atoms of type B are above those of type A , $\forall_{m, m'} x_m < y_{m'}$.

For example, the Jastrow ansatz for the subsystems with $\tilde{N}_A = 1$ and $\tilde{N}_B = 2$ particles can be written as:

$$\Psi(\mathbf{x}, \mathbf{y}; t) \propto \Phi^{x_1>y_1}(x_1, y_1; t) \Phi^{x_1>y_2}(x_1, y_2; t) \quad (4.23)$$

$$+ \Phi^{x_1>y_1}(x_1, y_1; t) \Phi^{x_1<y_2}(x_1, y_2; t) \quad (4.24)$$

$$+ \Phi^{x_1<y_1}(x_1, y_1; t) \Phi^{x_1>y_2}(x_1, y_2; t) \quad (4.25)$$

$$+ \Phi^{x_1<y_1}(x_1, y_1; t) \Phi^{x_1<y_2}(x_1, y_2; t), \quad (4.26)$$

where we identify three sectors: $x_1 > y_1, y_2$ (Eq. (4.23)), $x_1 < y_1, y_2$ (Eq. (4.26)), and mixed sector, which includes two cases due to the indistinguishability of particles of the same type: $y_1 < x_1 < y_2$ and $y_2 < x_1 < y_1$ (Eqs. (4.24)-(4.25)).

For the subsystem with more particles, the Jastrow ansatz becomes more complex, encompassing a wide variety of configurations. To manage this complexity, one could perform calculations separately for each case, as the many-body Floquet Hamiltonian (4.14) does not couple states from different sectors, cf. Eq. (4.22). However, this approach is not easily tackled numerically, as it requires integrating the Heaviside theta functions appearing in the energy functional (4.19). Alternatively, one can calculate the energy functional using the initial form of Jastrow ansatz (4.16), which includes all sectors together, and then in post-analysis extract the results corresponding to different particle configurations through measurements performed within the framework of the Monte Carlo method. We will use this latter approach. It is important to note that, while the wave function $\Psi(\mathbf{x}, \mathbf{y}; t)$ encompassing all sectors is fully valid within a quantum mechanical framework, experimentally it may be challenging to prepare such a state, as typical setups often involve initializing one cloud of atoms positioned above the other.

In the next section, we discuss the numerical complexity of calculating the energy functional (4.19) and the approach to address it.

4.5 Numerical evaluation of quasi-energy functional

In general, we use a basis consisting of $\tilde{N}_A \tilde{N}_B$ two-particle states $\Phi(x, y; t)$ (4.15) to construct the many-body wavefunction in the form of a Jastrow ansatz $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.16). However, this ansatz quickly becomes computationally intractable due to its exponential complexity, necessitating additional approximations to find the lowest-energy states of the system, which correspond to the minima of the quasi-energy functional (4.19). Expanding the product form of the many-particle state $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.16) results in a sum of $(2s_A s_B)^{\tilde{N}_A \tilde{N}_B}$ terms. Calculating the energy functional (4.19) involves squaring this number of terms, as it requires evaluating the expectation value of the Floquet Hamiltonian (4.14) by considering both the many-particle wavefunction (4.16) and its complex conjugate.

Given the complexity and computational demands of calculating the quasi-energy functional, we focus on a system consisting of two ultra-cold clouds, each with $\tilde{N}_A = \tilde{N}_B = 2$ particles, and resonances $s_A = 2$ and $s_B = 3$. Without considering the Hamiltonian structure, this configuration results in 429 981 696 integrals over time and space, cf. (4.19).

To address these computational challenges, we apply a Gaussian approximation to the Wannier states, replacing $w_j^\sigma(z, t)$ with $g_j^\sigma(z; \mu_j^\sigma(t), \nu_j^\sigma(t), k_j^\sigma(t))$:

$$g_j^\sigma(z; \chi_j^\sigma(t)) := g_j^\sigma(z; \mu_j^\sigma(t), \nu_j^\sigma(t), k_j^\sigma(t)) = \left(\frac{1}{2\pi(\nu_j^\sigma(t))^2} \right)^{1/4} \exp \left(-\frac{(z - \mu_j^\sigma(t))^2}{4(\nu_j^\sigma(t))^2} + ik_j^\sigma(t)z \right), \quad (4.27)$$

where each Gaussian function is characterized by its mean $\mu_j^\sigma(t)$, standard deviation $\nu_j^\sigma(t)$,

and momentum $k_j^\sigma(t)$ at time t . This approach enables us to find analytical solutions for the integrals that appear in the energy functional (4.19).

After computing these integrals analytically, we express the results as functions dependent on the set of Gaussian parameters $\chi_j^\sigma(t) = \{\mu_j^\sigma(t), \nu_j^\sigma(t), k_j^\sigma(t)\}$ for $\sigma \in \{A, B\}$, with $j = 1, 2$ for subsystem A and $j = 1, 2, 3$ for subsystem B . To determine the final form of the quasi-energy functional (4.19), we evaluate the solutions for discrete time steps over the period $T_{\text{tot}} = s_A s_B T$, and then average the results over time. This process leads to the quasi-energy functional as a function of the variational coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$.

Before moving on to the minimization procedure for the quasi-energy functional, we will first discuss the process of selecting the system parameters, such as the amplitude and frequency of the oscillating mirror, in the next section. We will also present the results obtained from the Gaussian approximation of the Wannier states.

4.5.1 System parameters and Gaussian approximation

We select the system parameters, such as the amplitude λ and frequency ω of the oscillating mirror, based on single-particle classical analysis. For more details, see Section 1.5 for a detailed description of the resonant behavior of a particle bouncing on the oscillating mirror, and Section 1.6 for the classical phase space portraits, which we use to validate the approach employed, i.e. the classical secular approximation. For $\lambda \leq 0.2$, the phase space portraits obtained by integrating the exact Hamiltonian (1.26a) align with those governed by the secular Hamiltonian (1.20) [91, 92], as illustrated in Fig. 1.4.

With the upper bound on λ established, we select the frequency of the mirror oscillations to ensure that the tunneling amplitude J^σ (4.10) between two neighboring Wannier states remains smaller than the energy gap ΔE between the first and second energy bands of the quantum secular Hamiltonian (4.6), cf. Fig. 1.6. We choose the frequency ω so that the energy gap ΔE is at least an order of magnitude greater than the tunneling amplitude J^σ . The parameters selected for system consisting of two ultra-cold clouds are presented in Table 4.1.

Following the selection of the system's parameters, we compute the T -periodic Floquet states, which are the eigenstates of the single-particle Floquet Hamiltonian $\hat{\mathcal{H}}_\sigma^{\text{sec}}$ (4.6). We then extract the $s_\sigma T$ -periodic Wannier states $w_j^\sigma(z, t)$ using the method described in Section 1.7.

Table 4.1: The system parameters: amplitude λ and frequency ω of the oscillating mirror, the resulting tunneling amplitude J^σ (4.10) between two neighboring Wannier states, and the energy gap ΔE between the first and second energy bands of the secular Hamiltonian (4.6), are presented for each subsystem $\sigma \in \{A, B\}$ in resonance $s_\sigma : 1$.

σ	s_σ	λ	ω	J^σ	ΔE
A	2	0.07	1.4	$2.158 \cdot 10^{-3}$	$3.511 \cdot 10^{-2}$
B	3			$2.560 \cdot 10^{-5}$	$3.096 \cdot 10^{-2}$

These Wannier states form a single-particle basis, which we use to construct a variational ansatz for the many-body state of the system (4.16), and we then optimize its variational parameters to minimize the quasi-energy functional (4.19).

As discussed in the previous section, the Jastrow ansatz (4.16) becomes quickly computationally intractable due to its exponential complexity. To address this, we approximate the Wannier states $w_j^\sigma(z, t)$ by Gaussian functions $g_j^\sigma(z; \chi_j^\sigma(t))$ (4.27). To find the optimal parameters $\chi_j^\sigma(t)$ of the Gaussian functions, we maximize the squared overlap:

$$F_j^\sigma(t) = \max_{\mu, \nu, k} |\langle g_j^\sigma(z; \chi_j^\sigma(t)) | w_j^\sigma(z, t) \rangle|^2. \quad (4.28)$$

The average squared overlap over a period $T_{\text{tot}} = s_A s_B T$:

$$F_{j, \text{avr}}^\sigma = \frac{1}{T_{\text{tot}}} \int_0^{T_{\text{tot}}} F_j^\sigma(t) dt, \quad (4.29)$$

quantifies the approximation's effectiveness. In further analysis, we use the optimized Gaussian functions $g_j^\sigma(z; \chi_j^\sigma(t))$ as a basis for the single-particle resonant Hilbert subspace instead of the Wannier states. This simplifies the computation of the energy functional (4.19) into analytical calculations of integrals. In Fig. 4.3, we present the densities of the Wannier states $w_1^\sigma(z, t)$ and optimized Gaussian functions $g_1^\sigma(z; \chi_j^\sigma(t))$ for times $t = 0, 0.5\pi/\omega, \pi/\omega, 1.5\pi/\omega$ for each subsystem σ .

The average squared overlap (4.29) between the Wannier state and its Gaussian approximation (4.27) is equal to $F_{\text{avr}}^A = 0.71$ for subsystem A and $F_{\text{avr}}^B = 0.70$ for subsystem B, as detailed in Tab. 4.2. This table also includes additional values that characterize the subsystems, such as the tunneling amplitude between two states:

$$J_{jk}^\sigma = -\frac{2}{s_\sigma T} \int_0^{s_\sigma T} dt \int_0^\infty dx w_j^{\sigma*}(z, t) (\hat{H}_\sigma(t) - i\partial_t) w_k^\sigma(z, t), \quad (4.30)$$

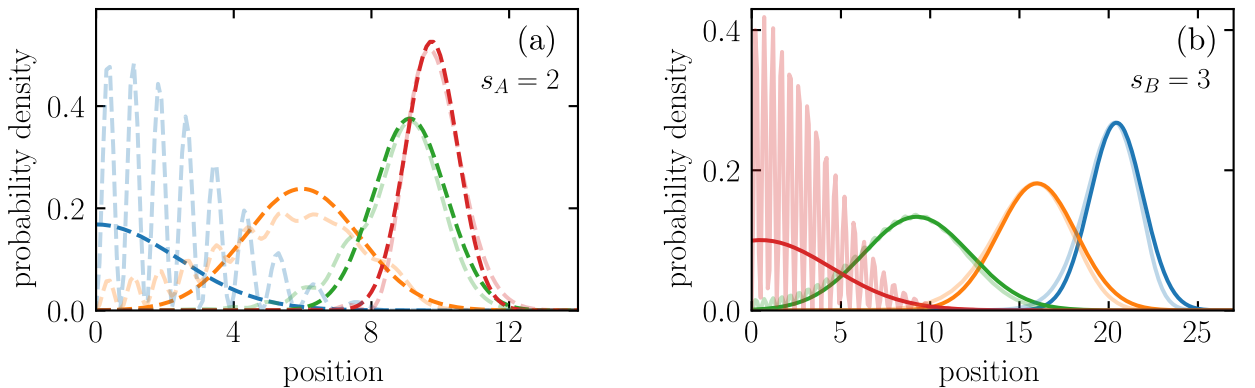


Figure 4.3: The densities of the Wannier states $w_1^\sigma(z, t)$ (light colors) and the optimized Gaussian functions $g_1^\sigma(z; \chi_j^\sigma(t))$ (darker colors) are shown for times $t = 0$ (blue), $t = 0.5\pi/\omega$ (orange), $t = \pi/\omega$ (green), and $t = 1.5\pi/\omega$ (red) for the subsystem A (a) and subsystem B (b). The Hamiltonian parameters are: $\omega = 1.4$, $\lambda = 0.07$, $s_A = 2$, $s_B = 3$.

Table 4.2: The tunneling amplitude J_{jk}^σ (4.30) and interaction coefficient $U_{j,k}^\sigma$ (4.11) calculated within the Wannier basis and Gaussian basis for subsystem $\sigma \in \{A, B\}$ in resonance $s_\sigma : 1$. The average overlap over time $F_{j,\text{avr}}^\sigma$ (4.29) quantifies the effectiveness of the approximation. The Hamiltonian parameters are: $\omega = 1.4$, $\lambda = 0.7$.

	σ	s_σ	$ J_{j,j+1}^\sigma $	$ J_{jj}^\sigma $	$U_{j,j+1}^\sigma$	U_{jj}^σ
Wannier	A	2	$2.158 \cdot 10^{-3}$	3.192	0.05323	0.2251
	B	3	$5.140 \cdot 10^{-5}$	7.454	0.02539	0.1755
Gauss	A	2	$5.351 \cdot 10^{-6}$	3.519	0.05454	0.2271
	B	3	$3.041 \cdot 10^{-8}$	7.731	0.02596	0.1645
Overlap $F_{j,\text{avr}}^\sigma$	A	0.71				
	B	0.70				

and interactions coefficients (4.11), calculated for both the Wannier states $w_j^\sigma(z, t)$ and the Gaussian functions $g_j^\sigma(z; \chi_j^\sigma(t))$. The presented quantities are the same for all Wannier (Gaussian) state pairs within a given resonance s_σ . Since each Wannier state is $s_\sigma T$ -periodic and alternates with its neighboring Wannier state, i.e. $w_j^\sigma(z, t) \rightleftharpoons w_{j+1}^\sigma(z, t)$, after each period T , the whole system effectively maintains a period equal to that of the oscillating mirror.

Nearly all calculated quantities using the optimized Gaussian functions (4.27) align with those obtained from the Wannier states, with over 90 % agreement, except for the tunneling amplitude $J_{j,j+1}^\sigma$. The smaller tunneling amplitude obtained using Gaussian functions is due to the distinct nature of the Gaussian tails when compared to those of the Wannier states. Specifically, the exponential decay of the Wannier states' tails increases the overlap between neighboring states $w_j^\sigma(z, t)$ and $w_{j\pm 1}^\sigma(z, t)$. In contrast, the Gaussian tails decay like $e^{-(z^2/4\gamma^2)}$, leading to a smaller overlap between neighboring Gaussian states $g_j^\sigma(z; \chi_j^\sigma(t))$ and $g_{j\pm 1}^\sigma(z; \chi_j^\sigma(t))$, which results in a lower tunneling amplitude.

To correctly reproduce the complex part of the tunneling amplitude, we need to adjust the phases of the Gaussian functions. The phase difference between the Wannier state and its Gaussian counterpart is determined by the phase of the overlap at a given time t :

$$\arg(\langle g_j^\sigma(z; \chi_j^\sigma(t)) | w_j^\sigma(z, t) \rangle) = \varphi_j^\sigma(t). \quad (4.31)$$

In the subsequent calculations of the quasi-energy functional (4.19), we use the two-particle states (4.15), consisting of the Gaussian functions together with their phases $\varphi_j^\sigma(t)$, i.e. $e^{i\varphi_j^\sigma(t)} g_j^\sigma(z; \chi_j^\sigma(t))$ instead of the Wannier states $w_j^\sigma(z, t)$.

In the next section, we present the results of minimizing the quasi-energy functional (4.19) for different values of the intra-species interaction strengths g_σ .

4.6 Minimization of quasi-energy functional

To find the lowest energy state within the resonant Hilbert subspace of the system, consisting of two ultra-cold atomic clouds governed by the Floquet Hamiltonian (4.14), we minimize the quasi-energy functional $E[\Psi(\mathbf{c}_A, \mathbf{c}_B)]$ (4.19). The functional is defined based on the Jastrow ansatz $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.16) and depends on the variational complex-valued parameters $\{\mathbf{c}_A, \mathbf{c}_B\}$, as well as the intra-species interaction strengths g_σ .

To minimize the quasi-energy functional, we optimize the variational parameters for a given intra-species interaction. The normalization conditions are incorporated into the optimization process using Lagrange multipliers λ_k where $k = 1, 2, 3$. This leads to the following set of equations that must be solved:

$$\frac{\partial}{\partial c_j^{\sigma*}} \left(E[\mathbf{c}_A, \mathbf{c}_B] - \lambda_1 \langle \Psi | \Psi \rangle - \lambda_2 \left(\sum_{j=1}^{s_A} |c_j^A|^2 \right) - \lambda_3 \left(\sum_{j=1}^{s_B} |c_j^B|^2 \right) \right) = 0. \quad (4.32)$$

The normalization of the Jastrow ansatz (4.16) is defined as:

$$\langle \Psi | \Psi \rangle = \frac{1}{T_{\text{tot}}} \int_0^{T_{\text{tot}}} dt \int_0^\infty d\mathbf{x} d\mathbf{y} |\Psi(\mathbf{x}, \mathbf{y}; t)|^2. \quad (4.33)$$

In Figure 4.4(a)-(b), we present optimized coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$ for different values of the intra-species interaction strength g_A while keeping $g_B = 0$. The results correspond to the resonance cases $s_A = 2$ and $s_B = 3$, with the number of atoms $\tilde{N}_A = \tilde{N}_B = 2$. For $|g_A| \lesssim 400$, the quasi-energy functional (4.19) is minimized by the uniform solution: $|c_j^A|^2 = 1/2$ and $|c_k^B|^2 = 1/3$ where $j = 1, 2$ and $k = 1, 2, 3$. This results in the many-body state $\Psi(\mathbf{x}, \mathbf{y}; t)$ evolving with the same period T as the period of the oscillating mirror, without any breaking of discrete time translation symmetry. However, for $|g_A| \gtrsim 400$, the uniform solution becomes unstable, and symmetry-broken states emerge, evolving with the period $s_A s_B T$ times longer than the period of the oscillating mirror, i.e. $T_{\text{tot}} = s_A s_B T$. The discrete time translation symmetry breaking occurs simultaneously for both resonances, $s_A = 2$ and $s_B = 3$. In the range $|g_A| \in [400, 680]$ the lowest quasi-energy solution exhibits non-uniform distributions of the coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$. For even larger $|g_A| \gtrsim 680$, the symmetry-broken state is described by a single Wannier state from each resonance $s_\sigma : 1$, meaning only one of the coefficients $\mathbf{c}_\sigma$ is non-zero for each $\sigma \in \{A, B\}$.

The intra-species interaction strength g_A , required to observe the discrete time translation symmetry breaking, as indicated by non-uniform distributions of the coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$, is large compared to the interaction strength g_0 needed for symmetry breaking in the case of a single cloud bouncing on the oscillating mirror, as discussed in Chapter 1. To understand why this is the case, let us revisit the discussion from Section (4.4). There, we analyzed the implications of using a two-particle basis $\Phi(x, y; t)$ (4.15), which incorporates infinite inter-species interactions, $g_{AB} \rightarrow +\infty$, through antisymmetrization. This approach adds complexity: the antisymmetrized state $\Phi(x, y; t)$ (4.20) encompasses two independent configurations of the particles, i.e. $x > y$ and $x < y$. These two configuration sectors are not coupled by the

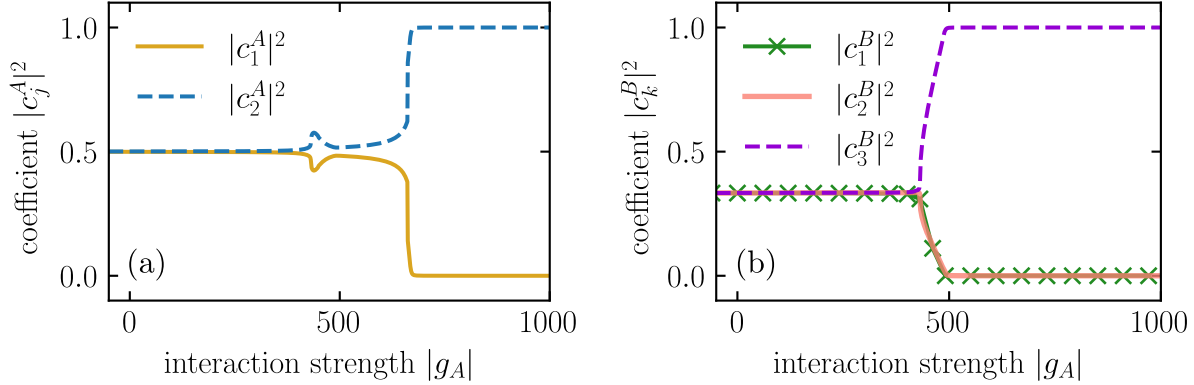


Figure 4.4: The lowest quasi-energy solutions of the functional (4.19) for different intra-species interaction strength $|g_A|$ while keeping $g_B = 0$. For clarity, the optimized coefficients $\{\mathbf{c}_A, \mathbf{c}_B\} = \{c_1^A, c_2^A; c_1^B, c_2^B, c_3^B\}$ are plotted separately for resonance $s_A = 2$ (blue and yellow lines) in the Panel (a) and $s_B = 3$ (violet, pink and green lines) in the Panel (b). For $|g_a| < 400$, the discrete time translation symmetry of the system is preserved, indicated by the uniform solution within each resonance $|c_j^A|^2 = 1/2$ and $|c_k^B|^2 = 1/3$ where $j = 1, 2$ and $k = 1, 2, 3$. That means that the system evolves with the same period as the external driving period T . In contrast, the system begins to evolve with a period $s_A s_B T$ longer than the external driving period, i.e. $T_{\text{tot}} = s_A s_B T$ for $|g_a| > 400$. This is indicated by the non-uniform distributions of the coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$. The Floquet Hamiltonian (4.14) parameters are: $\omega = 1.4$, $\lambda = 0.07$, $s_A = 2$, $s_B = 3$.

Floquet Hamiltonian (4.14), cf. Eq. (4.22). Thus, if the system starts in a state belonging to one sector, it remains confined to that sector throughout its time evolution.

While it is possible to restrict to one of the configuration sectors by introducing the Heaviside theta function (4.21) at the initial stage, this approach complicates the evaluation of integrals within the quasi-energy functional (4.19). Thus, in our analysis, we use the full antisymmetric two-particle basis (4.15) and perform the minimization of the quasi-energy functional corresponding to the Jastrow ansatz $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.19). This results in the minimal value of the quasi-energy averaged over all configuration sectors. Once the Jastrow state is determined, we can simulate experiments with the system prepared in different sectors. The details of this final step will be provided in the next section. Before that, we further explore the implications of averaging the quasi-energy over all configuration sectors.

In general, to induce time translation symmetry breaking, the intra-species interaction energy terms in the quasi-energy functional (4.19) must dominate over the other terms. However, the contributions to these interaction terms are not uniform across the different particle configuration sectors. Sectors where all atoms of the same type are above (or below) those of the other type, i.e. when $x_j > y_k$ (or $x_j < y_k$) for all pairs, contribute most to the interaction terms in the quasi-energy functional. In contrast, sectors where particles of different types alternate contribute less, e.g. $x_1 > y_1 > y_2 > x_2 > \dots$, or in some cases, contribute nothing, e.g. $x_1 > y_1 > x_2 > y_2 > \dots$.

In particular, for the system considered here with $\tilde{N}_\sigma = 2$ particles, the Jastrow wavefunc-

tion $\Psi(\mathbf{x}, \mathbf{y}; t)$ can be expressed as the following sum over independent configuration sectors:

$$\Psi(\mathbf{x}, \mathbf{y}; t) \propto 2 \Phi^{x_1 > y_1}(x_1, y_1; t) \Phi^{x_2 > y_1}(x_2, y_1; t) \Phi^{x_1 > y_2}(x_1, y_2; t) \Phi^{x_2 > y_2}(x_2, y_2; t) \quad (4.34)$$

$$+ 2 \Phi^{x_1 < y_1}(x_1, y_1; t) \Phi^{x_2 < y_1}(x_2, y_1; t) \Phi^{x_1 < y_2}(x_1, y_2; t) \Phi^{x_2 < y_2}(x_2, y_2; t) \quad (4.35)$$

$$+ 4 \Phi^{x_1 > y_1}(x_1, y_1; t) \Phi^{x_2 < y_1}(x_2, y_1; t) \Phi^{x_1 > y_2}(x_1, y_2; t) \Phi^{x_2 < y_2}(x_2, y_2; t) \quad (4.36)$$

$$+ 4 \Phi^{x_1 < y_1}(x_1, y_1; t) \Phi^{x_2 > y_1}(x_2, y_1; t) \Phi^{x_1 < y_2}(x_1, y_2; t) \Phi^{x_2 > y_2}(x_2, y_2; t) \quad (4.37)$$

$$+ 6 \Phi^{x_1 > y_1}(x_1, y_1; t) \Phi^{x_2 < y_1}(x_2, y_1; t) \Phi^{x_1 > y_2}(x_1, y_2; t) \Phi^{x_2 > y_2}(x_2, y_2; t) \quad (4.38)$$

$$+ 6 \Phi^{x_1 < y_1}(x_1, y_1; t) \Phi^{x_2 > y_1}(x_2, y_1; t) \Phi^{x_1 < y_2}(x_1, y_2; t) \Phi^{x_2 < y_2}(x_2, y_2; t), \quad (4.39)$$

where the terms like $\Phi^{x > y}(x, y; t)$ are defined in Eq. (4.21). We schematically present different configuration sectors in Fig. 4.5. The sectors described by (4.34)-(4.35) contribute the most to the interaction terms in the quasi-energy functional, as they represent situations where all particles of the same type are either below or above those of the other type, resulting in all possible intra-species interactions (Fig. 4.5(a)-(b)). Sectors described by (4.36)-(4.37) contribute two times less because the atoms of one type are surrounded by those of the other type (Fig. 4.5(c)-(d)). Finally, the last two sectors (4.38)-(4.39) describe configurations where atoms of different types alternate, resulting in no contribution to the intra-species interaction terms (Fig. 4.5(e)-(f)). The different weights of the terms in (4.34)-(4.39) arise from indistinguishable nature of bosons. For example, configurations such as $y_1 < x_1 < x_2 < y_2$ and $y_1 < x_2 < x_1 < y_2$ are treated equivalently, and their contributions are combined. This symmetry leads to different weights depending on the number of such equivalent configurations.

To observe spontaneous breaking of time translation symmetry, the attractive interaction terms (which are negative) must overcome the kinetic energy terms (which are positive and

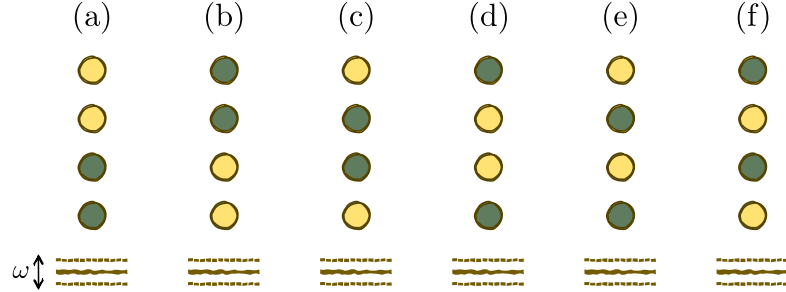


Figure 4.5: Different configuration sectors for two distinct ultra-cold atomic clouds, A and B, bouncing on an oscillating mirror with a frequency ω . The atoms within each cloud are identical, but the atoms in cloud A (yellow circles) are of a different type than those in cloud B (green circles). Each cloud consists of two particles. The sectors are not coupled by the Floquet Hamiltonian (4.14), so the many-body wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.16) can be expressed as a sum of states belonging to independent sectors, cf. Eq. (4.34)-(4.39). Since particles of the same type are indistinguishable, we do not label them by their position coordinates, i.e. $\mathbf{x} = \{x_1, x_2\}$ and $\mathbf{y} = \{y_1, y_2\}$. Consequently, we count them as belonging to the same sectors.

increase in the symmetry-broken phase). As we minimize the quasi-energy functional averaged over sectors, including those with no intra-species interaction terms, the critical interaction strength required is larger than it would be if we minimized the functional solely in sector (4.34) or (4.35).

The results presented in Fig. 4.4 show the system's behaviour for different values of the intra-species interaction strength g_A , while keeping $g_B = 0$. One might ask what happens for different values of g_B while keeping $g_A = 0$, or for both $g_A < 0$ and $g_B < 0$. Since we consider an equal number of particles $\tilde{N}_\sigma = 2$ in each subsystem σ , the corresponding intra-species interaction terms in the Floquet Hamiltonian (4.14) always contribute equally to the quasi-energy functional. It is sufficient to keep $g_A + g_B = \text{constant} > 400$ to observe the emergence of symmetry-broken states. This equal contribution arises because the antisymmetric two-particle basis (4.15) incorporates the sectors of different particle configurations, and the Jastrow ansatz (4.16) ensures that the resulting state is symmetric with respect to the exchange of any pair of coordinates for particles of the same type σ . As long as the number of particles in each cloud $\tilde{N}_\sigma$ remains the same, the contributions to the intra-species interaction terms are equal, cf. (4.2)-(4.3). Thus, we obtain the same results for different values of g_B while keeping $g_A = 0$ (or $g_A < 0$ and $g_B < 0$), as shown in Fig. 4.4. The key point is that, to spontaneously break time translation symmetry, the attractive interaction terms must overcome the kinetic energy terms in the quasi-energy functional.

In the next section, we present the post-processing procedure used to extract the contributions from different particle configuration sectors in the many-body wavefunction $\Psi(\mathbf{x}, \mathbf{y}; t)$ (4.16) corresponding to symmetry-preserving and symmetry-broken regimes.

4.7 Sampling of many-particle probability density

To simulate the experiment where one cloud of atoms is prepared above the other one with the help of the Jastrow state (4.16), we perform a series of particle measurements using the Monte Carlo algorithm [150] based on a Markovian walk that explores the configuration space [151] and next apply post-selection. This approach corresponds to sampling the many-particle probability density $|\Psi(\mathbf{x}, \mathbf{y}; t)|^2$ to find the desired probability distribution.

The steps of the Monte Carlo algorithm used are as follows:

1. **Initial configuration:** Start with an initial particle configuration $R = \{x_1, x_2, y_1, y_1\}$ randomly chosen from a uniform distribution P_R . The initial configuration forms the zero element of the Markov chain $\{\mathcal{R}_0\}$, where $\mathcal{R}_0 = R$.
2. **Propose new configuration:** Using the same uniform distribution P_R , generate the next configuration $R' = \{x'_1, x'_2, y'_1, y'_2\}$.
3. **Acceptance criterion:** In each iteration of the algorithm, apply the following acceptance criterion to decide whether to accept R' :

- (a) Calculate the ratio $q = |\Psi(R'; t)|^2 / |\Psi(R; t)|^2$.

- (b) If $q \geq 1$, accept R' and append it to the Markov chain, i.e. $\{\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{n-1}\} \rightarrow \{\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{n-1}, \mathcal{R}_n\}$ where $\mathcal{R}_n = R'$, and replace the previous element, $R = R'$.
- (c) Otherwise, if $q < 1$, choose a random number τ from a uniform distribution $P_\tau \in [0, 1]$. If $q > \tau$, accept the candidate R' and append it to the Markov chain as in point (b).

To accurately reproduce the desired probability distribution $|\Psi(\mathbf{x}, \mathbf{y}; t)|^2$, the steps (2)-(3) should be repeated many times.

We focus on finding the probability density for the configuration sector where cloud A is positioned below the cloud B (4.35), see Fig. 4.5(b). To achieve this, we perform a Markovian walk following the steps outline above. We run for 50 000 iterations and then select configurations from the resulting Markovian chain that meet the condition $x_j < y_k$ for any pair of particles of different types. In Figure 4.6, we represent histograms (bottom panels) corresponding to the probability density $|\Psi(\mathbf{x}, \mathbf{y}; t)|^2$ (4.16), restricted to the sector presented in Fig. 4.5(b). The upper panels display the corresponding density profiles showing distributions of the $\{x\}$ and $\{y\}$ particle positions (upper panels). Each histogram reflects the densities of the many-body wavefunction with optimized coefficients $\{\mathbf{c}_A, \mathbf{c}_B\} = \{c_1^A, c_2^A; c_1^B, c_2^B, c_3^B\}$ that minimize the quasi-energy functional (4.19) for a given intra-species interaction strength g_A . We select the values of g_A based on the Fig. 4.4. For the symmetry-preserving regime, we choose $g_A = 0$, which corresponds to $\{|\mathbf{c}_A|^2, |\mathbf{c}_B|^2\} = \{0.5, 0.5; 0.33, 0.33, 0.33\}$ (Panels (a)-(b)). For symmetry-broken regime, we choose two representatives: $|g_A| = 610$ with $\{|\mathbf{c}_A|^2, |\mathbf{c}_B|^2\} = \{0.46, 0.53; 1, 0, 0\}$ (Panels (c)-(d)), and $|g_A| > 680$ with $\{|\mathbf{c}_A|^2, |\mathbf{c}_B|^2\} = \{1, 0; 1, 0, 0\}$ (Panels (e)-(h)). In the latter case, we show two different times: when two particles of different types are close to each other in configuration space (Panels (e)-(f)), and when they are far apart (Panels (g)-(h)).

Let us summarize the results presented in this section and the previous one, which analyze two distinct ultra-cold atomic clouds. The lower cloud bounces on the oscillating mirror and effectively acts as a driving force for the upper cloud due to the infinite repulsive interactions between them. The particles within each cloud interact attractively, and by adjusting the strength of this intra-species interaction, the system transitions to a symmetry-broken regime, where the period of the many-body state (4.16) differs from that given by the Floquet Hamiltonian (4.14). The probability densities shown in Panels (e)-(h) correspond to the many-body state (4.16) where the breaking of the discrete time translation symmetry is most pronounced, i.e. particles of the same type σ occupy a single wave packet. Combined with the results presented in Fig. 4.4, this shows that sufficiently strong intra-species interaction lead to simultaneous breaking of discrete time translation symmetry in both clouds, resulting in the formation of a complex discrete time crystal.

Weaker intra-species interaction strengths $|g_A| \in [400, 680]$ also correspond to a symmetry-broken regime, but the corresponding many-body state (4.16) is characterized by a non-uniform distribution of the coefficients $\{\mathbf{c}_A, \mathbf{c}_B\}$, see Fig. 4.6(c)-(d). In contrast, in the symmetry-preserving regime, $|g_A| < 400$, this state is characterized by a uniform distribution within each resonance s_σ , i.e. $|c_j^A|^2 = 1/2$ and $|c_j^B|^2 = 1/3$.

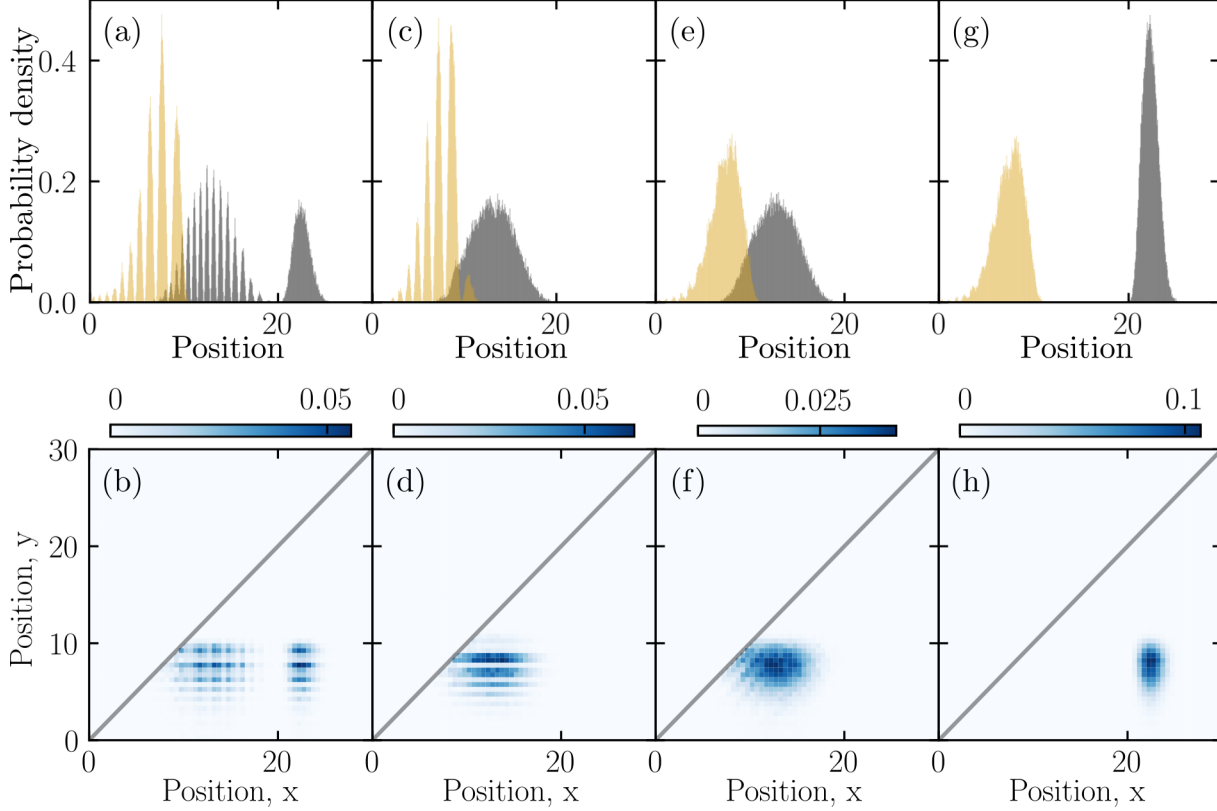


Figure 4.6: The histograms represent the probabilities densities given by the many-body wavefunction (4.16) for the optimized coefficients $\{c_A, c_B\} = \{c_1^A, c_2^A; c_1^B, c_2^B, c_3^B\}$, which minimize the quasi-energy functional (4.19) for a given intra-species interaction strength g_A . The histograms are shown as density profiles of the $\{x\}$ (yellow bars) and $\{y\}$ (gray bars) particles positions (upper panels), and as 2D plots (bottom panels). The probabilities are obtained using a Markovian walk with 50 000 iterations (details in the main text), selecting configurations where $x_j < y_k$ for all pair of particles of different types, corresponding to the case where all atoms from cloud A are below those from cloud B . Panels (a)-(b) show the symmetry-preserving regime with coefficients $\{|c_A|^2, |c_B|^2\} = \{0.5, 0.5; 0.33, 0.33, 0.33\}$ for $g_A = 0$. Panels (c)-(d) represent the symmetry-broken regime with coefficients $\{|c_A|^2, |c_B|^2\} = \{0.46, 0.53; 1, 0, 0\}$ for $|g_A| = 610$. Panels (e)-(h) show the symmetry-broken regime with coefficients $\{|c_A|^2, |c_B|^2\} = \{1, 0; 1, 0, 0\}$ at two different times: where the particles of different types are close to each other in configuration space (e)-(f), and when they are far apart (g)-(h), both for $|g_A| > 680$. The Floquet Hamiltonian (4.14) parameters are: $\omega = 1.4$, $\lambda = 0.07$, $s_A = 2$, $s_B = 3$, $g_B = 0$.

4.8 Summary

In this chapter, we analyzed a system consisting of two distinct ultra-cold atomic clouds. Each composed of the same kind of atoms, but differing by their internal state between the clouds. The atoms from different clouds interact with infinite repulsive strength, while the atoms within each cloud interact attractively. The system is one-dimensional and placed in a

gravitational field, where the lower cloud bounces on an oscillating mirror. Due to the infinite repulsive inter-species interaction, the lower cloud effectively acts as a driving force for the upper cloud. Our goal was to investigate how sufficiently strong intra-species interactions can induce discrete time translation symmetry breaking, leading to the formation of a complex discrete time crystal.

We began by considering case of two independent clouds to introduce a single-particle basis for each subsystem, which was later used to construct the antisymmetric two-particle basis. This provided a foundation for revisiting the concept of spontaneous discrete time translation symmetry breaking in a single cloud bouncing on an oscillating mirror, which leads to the formation of discrete time crystals.

Next, we turned to the case of two interacting ultra-cold atomic clouds with infinite repulsive inter-species interactions. To find the low-lying quasi-energy state of the system within the resonant Hilbert subspace, we proposed a variational ansatz in the form of a Jastrow state. This ansatz respects the bosonic nature of the particles by ensuring that the wavefunction is symmetric under the exchange of two particles of the same type. The antisymmetric two-particle basis, which accounts for the infinite repulsive interaction between atoms from different clouds, simplified the Floquet Hamiltonian by eliminating of the infinite interaction terms.

However, constructing the Jastrow ansatz using the antisymmetric two-particle basis incorporates different configuration sectors. These sectors encoded scenarios where all atoms of one type are either below or above the other type, as well as mixed cases where atoms of both types are interspersed. While such a state is theoretically valid in quantum description, it is very difficult to prepare experimentally, as typical setups involve one type of ultra-cold atomic cloud being placed above the other. Since the different configuration sectors are not coupled by the Floquet Hamiltonian, i.e., meaning the system remains confined to a sector throughout its evolution, we analyzed these sectors separately through post-processing after the main analysis.

To calculate the quasi-energy functional for the proposed Jastrow ansatz, we applied a Gaussian approximation to the single-particle Wannier basis of each subsystem. This approximation was necessary due to the exponential growth of computational complexity as the number of particles increased. The Gaussian approximation enabled us to find analytical solutions for the integrals in the quasi-energy functional. We then minimized the quasi-energy functional by optimizing the variational parameters that characterize the Jastrow ansatz for different intra-species interaction strengths.

Our analysis revealed two regimes: a symmetry-preserving regime and a symmetry-broken regime. In the symmetry-preserving regime, the Jastrow ansatz is characterized by a uniform distribution of the variational coefficients for each resonance. In the symmetry-broken regime, the uniform state became unstable, and a new solution emerged, characterized by a non-uniform distribution of the variational coefficients. The optimized variational coefficients revealed the spontaneous breaking of time translation symmetry, corresponding to the emergence of a complex discrete time crystal.

Since the antisymmetric two-particle basis consists of the sectors of different particle config-

urations, we needed a post-processing procedure to extract the contributions from different configuration sectors to describe a typical experiment. Specifically, we focused on the sector where all particles from one cloud were above those from the other cloud. After minimizing the quasi-energy functional for the proposed Jastrow ansatz, we sampled the optimized many-body wavefunction using the Monte Carlo method. From the collected configurations, we selected those corresponding to the case where one cloud was above the other. We then presented the probability densities for this sector, showing results for different intra-species interaction strengths in both the symmetry-preserving and symmetry-broken phases.

The main finding of this chapter is that when the intra-species interactions are sufficiently strong, discrete time translation symmetry is simultaneously broken in both clouds. Consequently, the system evolves with a period different from that of the oscillating mirror, resulting in the emergence of a complex discrete time crystal.

Conclusions

In Chapter 1, we introduce the concept of discrete time crystals, beginning with a discussion of spontaneous symmetry breaking in space crystals and extending this analogy to time crystals. We present Wilczek’s original proposal and address the challenges that prevent its realization. Focusing on a system of ultra-cold atoms bouncing on an oscillating mirror, we provide the foundational tools used in subsequent chapters to analyze the properties of discrete time crystals. Although this chapter does not present any new results, it offers a comprehensive exploration of the system’s classical and quantum dynamics. We explore both the classical and quantum dynamics of a single particle, employing action-angle variables to derive the secular Hamiltonian and examine phase-space portraits, identifying resonant trajectories. In the quantum case, we derive the quantum secular Hamiltonian and identify Floquet states as the system’s quasi-eigenstates. Using the single-particle Wannier basis, we construct the many-body Floquet Hamiltonian and discuss the stability of the lowest quasi-energy state within the resonance Hilbert subspace. We show that measuring particle positions could induce discrete time translation symmetry breaking, leading to the formation of a discrete time crystal. Lastly, we introduce a mean-field description, characterizing spontaneous time translation symmetry breaking via bifurcation.

In Chapter 2, we investigate the stability of DTCs against experimental imperfections in the initial state preparation such as non-zero average momentum, variations in the initial average position and deviations in trap frequency. Using the Gross-Pitaevski equation, we track the system’s evolution and observe that sufficiently strong interactions can maintain DTC phase. Phase diagrams reveal how attractive interactions preserve DTCs despite these imperfections. Additionally, we examine the impact of quantum many-body fluctuations using the $U(1)$ symmetry-preserving Bogoliubov approach. Our results show that quantum depletion of the Bose-Einstein condensate is negligible, confirming that a mean-field description is sufficient to capture the DTC dynamics in ultra-cold atomic systems.

In Chapter 3, we extend the classical analysis of a single particle bouncing on a static mirror to a system of N impenetrable particles in a one-dimensional space. We first examine the dynamics of independent particles using standard angle-action variables, which simplify the system’s description and allow us to identify the manifold as an N -torus in the $2N$ -dimensional phase space. For impenetrable particles, we introduce new angle-action variables to account for elastic collisions, capturing the exchange of momentum and energy between particles. This approach works well for particles with different energies, where the manifold remains an N -torus in the $2N$ -dimensional phase space. However, for impenetrable particles with equal energies, the standard angle-action variables are used in combination with additional constraints to reflect the effects of collisions. In the case of $N = 3$ particles with equal energies, we identify the manifold as a solid torus. When we introduce oscillations to the mirror, we derive an effective Hamiltonian that describes the system near resonance. This Hamiltonian captures the dynamics of the system and opens up possibilities for engineering crystalline structures in N -dimensional space, with potential applications to condensed matter phenomena. Though the results are classical, they lay the foundation for quantum analysis. By

replacing the particles with atomic clouds composed of different atomic species that interact repulsively between clouds, the system can exhibit spontaneous symmetry breaking, leading to the formation of complex discrete time crystals.

In chapter 4, we analyze a system of two distinct ultra-cold atomic clouds, composed of atoms in different internal states, interacting via infinite repulsive forces between the clouds and attractive forces within each cloud. The system is placed in a one-dimensional gravitational field where the lower cloud bounces on an oscillating mirror, driving the upper cloud. Our focus is on understanding how intra-species interactions can lead to discrete time translation symmetry breaking, resulting in the formation of a complex discrete time crystal. We begin by introducing a single-particle basis for independent clouds, later extending it to an anti-symmetric two-particle basis to account for the infinite repulsive interaction between clouds. A variational Jastrow ansatz is proposed to find the low-energy states within the resonant Hilbert subspace, ensuring the wavefunction remains symmetric for particles of the same type. We apply a Gaussian approximation to handle the computational complexity of the many-body system and minimize the quasi-energy functional for various intra-species interaction strengths. Our analysis reveals two regimes: a symmetry-preserving regime with uniform distributions of variational coefficients and a symmetry-broken regime with non-uniform distributions, corresponding to the spontaneous breaking of time translation symmetry. The later demonstrate the emergence of a complex discrete time crystal.

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