



JAGIELLONIAN UNIVERSITY

DOCTORAL THESIS

Cosmological models with running cosmological constant

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Oświadczenie

Ja niżej podpisany Aleksander Stachowski (nr indeksu: 1016516) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „*Cosmological models with running cosmological constant*” jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr. hab. Marka Szydłowskiego. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia

Abstract

The thesis undertakes an attempt to solve the problems of cosmological constant as well as of coincidence in the models in which dark energy is described by a running cosmological constant. Three reasons to possibly underlie the constant's changeability are considered: dark energy's decay, diffusion between dark energy and dark matter, and modified gravity. It aims also to provide a parametric form of density of dark energy for the models that involve a running cosmological constant, which would describe inflation in the early Universe.

The principal method used in my investigations was the dynamical analysis. The cosmological equations were accordingly recast as a dynamical system, which enabled me to draw up the relevant phase portrait, much useful in considering the possible evolutionary paths of the Universe.

The cosmological models were estimated by taking into account a number of astronomical data, such as observations of type Ia supernovae, cosmic microwave background, baryon acoustic oscillations, measurements of the Hubble function for galaxies, and the Alcock-Paczynski test.

The results of my investigations have been published in eleven papers.

Keywords: cosmology, dark energy, dark matter

Streszczenie

Rozprawa podejmuje próbę rozwiązania problemu stałej kosmologicznej i problemu koincydencji w modelach kosmologicznych, gdzie ciemna energia jest opisywana zmienną stałą kosmologiczną. Są rozważane trzy możliwe przyczyny zmienności stałej kosmologicznej: rozpadająca się ciemna energia, dyfuzja pomiędzy ciemną materią a ciemną energią oraz zmodyfikowana gravitacja. Celem jest także wprowadzenie parametryzacji gęstości ciemnej energii w podejściu ze zmienną stałą kosmologiczną, która opisuje inflację we wczesnym Wszechświecie.

Najważniejszą metodą użytą w tych badaniach była analiza dynamiczna. Równania kosmologiczne były przepisane do postaci układu dynamicznego, który umożliwiał narysowanie odpowiedniego portretu fazowego przydatnego przy badaniu możliwych ścieżek ewolucji Wszechświata.

Modele kosmologiczne były estymowane z uwzględnieniem obserwacji astronomicznych takich jak: obserwacje supernowych typu Ia, mikrofalowego promieniowania tła, barionowych oscylacji akustycznych, pomiarów wartości funkcji Hubble'a dla galaktyk i testu Alcocka-Paczyńskiego.

Wyniki badań zostały zamieszczone w jedenastu opublikowanych pracach.

Słowa kluczowe: kosmologia, ciemna energia, ciemna materia

This dissertation has been based on the scientific results previously reported in the following articles:

- Marek Szydlowski, **Aleksander Stachowski**, *Cosmology with decaying cosmological constant – exact solutions and model testing*, JCAP **1510** (2015) no. 10, 066,
doi:10.1088/1475-7516/2015/10/066.
- Marek Szydlowski, **Aleksander Stachowski**, *Cosmological models with running cosmological term and decaying dark matter*, Phys. Dark Univ. **15** (2017) 96–104,
doi:10.1016/j.dark.2017.01.002.
- **Aleksander Stachowski**, Marek Szydlowski, Krzysztof Urbanowski, *Cosmological implications of the transition from the false vacuum to the true vacuum state*, Eur. Phys. J. **C77** (2017) no. 6, 357,
doi:10.1140/epjc/s10052-017-4934-2.
- Marek Szydlowski, **Aleksander Stachowski**, Krzysztof Urbanowski, *Quantum mechanical look at the radioactive-like decay of metastable dark energy*, Eur. Phys. J. **C77** (2017) no. 12, 902,
doi:10.1140/epjc/s10052-017-5471-8.
- Zbigniew Haba, **Aleksander Stachowski**, Marek Szydlowski, *Dynamics of the diffusive DM-DE interaction – Dynamical system approach*, JCAP **1607** (2016) no. 07, 024,
doi:10.1088/1475-7516/2016/07/024.
- Marek Szydlowski, **Aleksander Stachowski**, *Does the diffusion dark matter-dark energy interaction model solve cosmological puzzles?*, Phys. Rev. **D94** (2016) no. 4, 043521,
doi:10.1103/PhysRevD.94.043521.
- **Aleksander Stachowski**, Marek Szydlowski, *Dynamical system approach to running Λ cosmological models*, Eur. Phys. J. **C76** (2016) no. 11, 606,
doi:10.1140/epjc/s10052-016-4439-4.

- **Aleksander Stachowski**, Marek Szydlowski, Andrzej Borowiec, *Starobinsky cosmological model in Palatini formalism*, Eur. Phys. J. **C77** (2017) no. 6, 406,
[doi:10.1140/epjc/s10052-017-4981-8](https://doi.org/10.1140/epjc/s10052-017-4981-8).
- Marek Szydlowski, **Aleksander Stachowski**, Andrzej Borowiec, *Emergence of running dark energy from polynomial $f(R)$ theory in Palatini formalism*, Eur. Phys. J. **C77** (2017) no. 9, 603,
[doi:10.1140/epjc/s10052-017-5181-2](https://doi.org/10.1140/epjc/s10052-017-5181-2).
- Marek Szydlowski, **Aleksander Stachowski**, *Simple cosmological model with inflation and late times acceleration*, Eur. Phys. J. **C78** (2018) no. 3, 249,
[doi:10.1140/epjc/s10052-018-5722-3](https://doi.org/10.1140/epjc/s10052-018-5722-3).
- Marek Szydlowski, **Aleksander Stachowski**, *Polynomial $f(R)$ Palatini cosmology: Dynamical system approach*, Phys. Rev. **D97** (2018) 103524,
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Chapter 1

Introduction

The main aim of the thesis is to address the following question: **can the cosmological constant problem and the coincidence problem be solved?** In the investigations, it is assumed that the cosmological constant evolves in time.

The cosmological constant problem is a consequence of interpreting dark energy as a vacuum energy. The presently observed value of the constant is by 120 orders of magnitude smaller than the value resulting from quantum physics [1].

The coincidence problem [2] consists in finding an explanation why the cosmological constant has the same order of magnitude as the energy density of matter today.

Another goal of the dissertation is **to provide a description of inflation in the early Universe involving an assumption of the running cosmological constant.**

For the purpose of proving the main thesis of my dissertation that **the running cosmological constant is actually able to explain the cosmological problems**, three hypotheses concerning the cosmological constant's changeability are put forward:

- it is a consequence of the decay of metastable dark energy,
- it is a result of diffusion interaction of dark energy with dark matter,
- it is an intrinsic attribute of dark energy in the Starobinsky model using the Palatini formalism.

We neglected the influence of radiation on the evolution of the Universe. The baryonic matter is treated as dust (the equation of state is $p_m = 0$, where p_m is the pressure of matter) in our papers. Throughout the thesis, dark matter is generally treated as dust too. It was assumed that dark energy has the equation of state $p_{de} = -\rho_{de}$, where p_{de} is the pressure and ρ_{de} is the energy density. In most cases, we do not take into account the curvature effect in the cosmological equations. In the thesis, the following convention is used: $c = 8\pi G = 1$.

One of the models investigated is one with a decaying meta-stable dark energy. The vacuum energy decay was considered in the following papers: [3, 4, 5, 6, 7, 8]. It is assumed that the process of dark energy decay is a transition the false vacuum state to the true vacuum. The Fock-Krylov theory of quantum unstable states is applied here [5]. Next, the Breit-Wigner energy distribution function is used for the model of the quantum unstable systems [9]. In this context, we examined the radioactive-like model of the decay of the false vacuum. The late-time approximation of this model are also considered ($\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$). It assumes an interaction in the dark sector (i.e. dark matter and dark energy). The models are analysed also by the statistical analysis methods.

Another model investigated in the thesis is one with a diffusion between dark matter and dark energy. The natural result of this interaction is a modification of the standard scale law of the dark matter energy density. This model is also statistically analysed using astronomical data.

The third one is the Starobinsky cosmological model in the Palatini formalism, which we examined in both the Jordan and Einstein frames, looking for differences between them by the dynamical methods. In the case of the Einstein frame, the model belongs to the class involving an interaction between dark matter and dark energy. The special point of these investigations is to search for singularities within the models, while the models' fitting is done through statistical analysis.

The important method of investigation of the evolution of the Universe consists in making conclusions on the basis of phase portraits. Accordingly, we recast cosmological equations into the form of the dynamical system, which allows for drawing the phase portrait. The analysis of trajectories that represent the particular paths of the evolution of the Universe as well as critical points gives us the most interesting scenarios of the evolution of the Universe. Such a method can be helpful in solving of the problems of the cosmological constant and of coincidence.

The methods of dynamical analysis are used in most of my papers, while my main article on dynamical systems in cosmology is *Eur.Phys.J. C76 (2016) no. 11, 606* [10], dealing with the dynamics of cosmological models in the different parametrization of the running cosmological constant. In this paper, five classes of models are investigated:

- $\Lambda(H)$ CDM, where H is the Hubble constant,
- $\Lambda(a)$, where a is the scale factor,
- $\Lambda(R)$ CDM, where R is the Ricci scalar,

- $\Lambda(\phi)$ with diffusion, where ϕ is a scalar field,
- $\Lambda(X)$, where $X = \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$ is a kinetic part of density of the scalar field.

The structure of the thesis is as follows. The statistical analysis used in my papers is described in Chap. 2. In Chap. 3, models with decaying dark energy and my papers pertaining to this subject are contemplated. In Chap. 4, the model with diffusion in the dark sector and my papers about this model are considered. In Chap. 5, my paper *Eur.Phys.J. C76 (2016) no. 11, 606* [10] is discussed. The Starobinsky cosmological model in the Palatini formalism and my papers on this model are discussed in Chap. 6. The conclusions of the thesis are provided in Chap. 7.

Chapter 2

Statistical analysis of cosmological models

The cosmological models considered in the thesis are analysed by the statistical methods in order to find the best fit for the values of parameters and their errors. In the statistical analysis, I used my own `CosmoDarkBox` script for estimating model parameters. In order to find errors of the best fit, this code uses the Monte-Carlo methods — the Metropolis-Hastings algorithm [11, 12].

For the purpose of statistical analysis in my papers, I used the following astronomical data:

- supernovae of type Ia (SNIa; [Union 2.1](#)¹ dataset [13]),
- Baryon Acoustic Oscillations (BAO) data from:
 - [Sloan Digital Sky Survey Release 7 \(SDSS DR7\)](#)² dataset at $z = 0.275$ [14],
 - [6dF Galaxy Redshift Survey](#)³ measurements at redshift $z = 0.1$ [15],
 - [WiggleZ](#)⁴ measurements at redshift $z = 0.44, 0.60, 0.73$ [16],
- measurements of the Hubble parameter $H(z)$ of galaxies [17, 18, 19],
- the Alcock-Paczynski test [20, 21] (AP; data from [22, 23, 24, 25, 26, 27, 28, 29, 30])
- measurements of Cosmic Microwave Background (CMB) and lensing by [Planck satellite](#)⁵ [31] and low ℓ polarization from WMAP.

The overall likelihood function is expressed by the following formula:

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB+lensing}}, \quad (2.1)$$

¹ <http://supernova.lbl.gov/union/>

² <https://classic.sdss.org/dr7/>

³ <http://www.6dfgs.net>

⁴ <http://wigglez.swin.edu.au/site/>

⁵ https://www.esa.int/Our_Activities/Space_Science/Planck

where the likelihood functions $L_{\text{SN Ia}}$, L_{BAO} , L_{AP} , $L_{H(z)}$, $L_{\text{CMB+lensing}}$ are for SN Ia, BAO, AP, measurements of $H(z)$ and CMB respectively, which are defined in the following way:

$$L_{\text{SN Ia}} = \exp \left[-\frac{1}{2} [A - B^2/C + \log(C/(2\pi))] \right], \quad (2.2)$$

where $A = (\mu^{\text{obs}} - \mu^{\text{th}}) \mathbb{C}^{-1} (\mu^{\text{obs}} - \mu^{\text{th}})$, $B = \mathbb{C}^{-1} (\mu^{\text{obs}} - \mu^{\text{th}})$, $C = \text{Tr } \mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for SN Ia, μ^{obs} is the observed distance modulus and μ^{th} is the theoretical distance modulus,

$$L_{\text{BAO}} = \exp \left[-\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(z)} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(z)} \right) \right], \quad (2.3)$$

where $r_s(z_d)$ is the sound horizon in the drag epoch [32, 33], $1/\mathbf{d}^{\text{obs}}$ is the observed value of the acoustic-scale distance ratio, $D_V = \left((1+z^2) D_A^2(z) \frac{cz}{H(z)} \right)^{1/3}$, where D_A is the angular diameter distance,

$$L_{H(z)} = \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2 \right], \quad (2.4)$$

where σ is the measurement error,

$$L_{\text{AP}} = \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{AP(z_i)^{\text{obs}} - AP(z_i)^{\text{th}}}{\sigma_i} \right)^2 \right], \quad (2.5)$$

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data and

$$L_{\text{CMB+lensing}} = \exp \left[-\frac{1}{2} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \mathbb{C}^{-1} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \right], \quad (2.6)$$

where $\mathbb{C}$ is the covariance matrix with the errors, $\mathbf{x}$ is a vector of the acoustic scale l_A , the shift parameter R and $\Omega_{\text{b},0}$, where $l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')}$, $R = \sqrt{\Omega_{\text{m},0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}$, and $\Omega_{\text{b},0} = \frac{\rho_{\text{b},0}}{3H_0^2}$, where z^* is the redshift in the recombination epoch [32], r_s is the sound horizon, $\rho_{\text{b},0}$ is the present value of the energy density of baryonic matter, H_0 is the present value of the Hubble function, and $\Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{3H_0^2}$, where $\rho_{\text{m},0}$ is the present value of the energy density of matter.

In my paper *Phys. Dark Univ. 15 (2017) 96-104*, the likelihood function for CMB is given by

$$L_{\text{CMB}} = \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{D_{\ell, \text{th}}^{TT}(\ell_i) - D_{\ell, \text{obs}}^{TT}(\ell_i)}{\sigma_i} \right)^2 \right], \quad (2.7)$$

where $D_\ell^{TT}(\ell)$ is the value of the temperature power spectrum of CMB and ℓ is a multipole. The temperature power spectrum is determined for ℓ in the interval of $\langle 30, 2508 \rangle$.

In my analysis of cosmological models, I used the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) [34, 35]:

$$\text{AIC} = -2 \ln L + 2d, \quad (2.8)$$

$$\text{BIC} = -2 \ln L + d \ln n, \quad (2.9)$$

where L is the value of the likelihood function in the best fit, d is the number of model parameters, and n is the number of data points involved in the estimation.

Chapter 3

Cosmology with decay of metastable dark energy

3.1 Decay of metastable dark energy from quantum vacuum

This section is based on Eur.Phys.J. C77 (2017) no. 6, 357 [36] and Eur.Phys.J. C77 (2017) no. 12, 902 [37].

The quantum unstable systems are characterized by their survival probability (decay law). The survival probability $\mathcal{P}(t)$ of a state $|\phi\rangle$ of vacuum equals $\mathcal{P}(t) = |A(t)|^2$, where $A(t)$ is the probability amplitude ($A(t) = \langle\phi|\phi(t)\rangle$) and $|\phi(t)\rangle$ is the solution of the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \mathfrak{H} |\phi(t)\rangle, \quad (3.1)$$

where $\mathfrak{H}$ is the Hamiltonian. The amplitude $A(t)$ can be expressed as the following Fourier integral:

$$A(t) \equiv A(t - t_0) = \int_{E_{\min}}^{\infty} \omega(E) e^{-\frac{i}{\hbar} E (t - t_0)} dE, \quad (3.2)$$

where $\omega(E) > 0$ (see: [5, 6, 7]).

From the Schrödinger equation (3.1), we can obtain that:

$$i\hbar \frac{\partial}{\partial t} \langle\phi|\phi(t)\rangle = \langle\phi|\mathfrak{H}|\phi(t)\rangle. \quad (3.3)$$

This relation gives us the amplitude $A(t)$ as a solution of the following equation:

$$i\hbar \frac{\partial A(t)}{\partial t} = h(t) A(t), \quad (3.4)$$

where

$$h(t) = \frac{\langle\phi|\mathfrak{H}|\phi(t)\rangle}{A(t)} \equiv \frac{i\hbar}{A(t)} \frac{\partial A(t)}{\partial t} \quad (3.5)$$

and $h(t)$ is the effective Hamiltonian. In result, we get:

$$h(t) = E_\phi(t) - \frac{i}{2} \Gamma_\phi(t), \quad (3.6)$$

where $E_\phi(t) = \Re[h(t)]$, $\Gamma_\phi(t) = -2\Im[h(t)]$ are the instantaneous energy (mass) $E_\phi(t)$ and the instantaneous decay rate $\Gamma_\phi(t)$ [38, 39, 40]. We interpret the expression $\Gamma_\phi(t) = -2\Im[h(t)]$ as the decay rate, because it satisfies the definition of the decay rate used in quantum theory: $\frac{\Gamma_\phi(t)}{\hbar} \stackrel{\text{def}}{=} -\frac{1}{\mathcal{P}(t)} \frac{\partial \mathcal{P}(t)}{\partial t}$.

From the form of the effective Hamiltonian $h(t)$, we get the following solutions of Eq. (3.4):

$$A(t) = e^{-i \frac{t}{\hbar} \overline{h(t)}} \equiv e^{-i \frac{t}{\hbar} (\overline{E_\phi(t)} - \frac{i}{2} \overline{\Gamma_\phi(t)})}, \quad (3.7)$$

where $\overline{h(t)}$ is the average effective Hamiltonian $h(t)$ for the time interval $[0, t]$: $\overline{h(t)} \stackrel{\text{def}}{=} \frac{1}{t} \int_0^t h(x) dx$ (averages $\overline{E_\phi(t)}$, $\overline{\Gamma_\phi(t)}$ are defined analogously).

We assume that $\omega(E)$ is given in the Breit-Wigner (BW) form: $\omega(E) \equiv \omega_{BW}(E) \stackrel{\text{def}}{=} \frac{N}{2\pi} \Theta(E - E_{\min}) \frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}$, where N is a normalization constant and $\Theta(E) = 1$ for $E \geq 0$ and $\Theta(E) = 0$ for $E < 0$. E_0 is the energy of the system, Γ_0 is a decay rate, while $E_{\min}$ is the minimal energy of the system. Inserting $\omega_{BW}(E)$ into formula (3.2), we get:

$$A(t) = A(t - t_0) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta \left(\frac{\Gamma_0(t - t_0)}{\hbar} \right), \quad (3.8)$$

where

$$I_\beta(\tau) \stackrel{\text{def}}{=} \int_{-\beta}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta. \quad (3.9)$$

Here $\tau = \frac{\Gamma_0(t - t_0)}{\hbar}$ and $\beta = \frac{E_0 - E_{\min}}{\Gamma_0} > 0$. It is assumed that $t_0 = 0$.

Using $A(t)$, as given by Eqs (3.8), and the effective Hamiltonian (3.5), we find the Breit-Wigner model as:

$$h(t) = E_0 + \Gamma_0 \frac{J_\beta(\frac{\Gamma_0 t}{\hbar})}{I_\beta(\frac{\Gamma_0 t}{\hbar})}, \quad (3.10)$$

where

$$J_\beta(\tau) = \int_{-\beta}^{\infty} \frac{x}{x^2 + \frac{1}{4}} e^{-ix\tau} dx. \quad (3.11)$$

In result, the instantaneous energy $E_\phi(t)$ has the following form:

$$E_\phi(t) = \Re[h(t)] = E_0 + \Gamma_0 \Re \left[\frac{J_\beta(\frac{\Gamma_0 t}{\hbar})}{I_\beta(\frac{\Gamma_0 t}{\hbar})} \right]. \quad (3.12)$$

The simplest way to extend the classical model of the decay:

$$\rho_{\text{de}}(t) = \rho_{\text{de}}(t_0) \times \exp[-\Gamma(t - t_0)] \equiv \rho_{\text{de}}(t - t_0) \quad (3.13)$$

is to replace the classical decay rate Γ by the decay rate $\Gamma_\phi(t)/\hbar$ appearing in quantum theoretical considerations. In consequence, we get:

$$\dot{\rho}_{\text{de}}(t) = -\frac{1}{\hbar} \Gamma_\phi(t) \rho_{\text{de}}(t) \quad (3.14)$$

instead of the classical fundamental equation of the radioactive decay theory.

Ultimately, the formula for the decay is:

$$\begin{aligned} \rho_{\text{de}}(t) &= \rho_{\text{de}}(t_0) \times \exp\left[-\frac{t}{\hbar} \overline{\Gamma_\phi(t)}\right] \\ &\equiv \rho_{\text{de}}(t_0) \times \exp\left[-\frac{1}{\hbar} \int_{t_0}^t \Gamma_\phi(x) dx\right]. \end{aligned} \quad (3.15)$$

This relation, superseding the classic decay formula, contains quantum corrections resulting from the use of the quantum theory decay rate. Using (3.8), we can rewrite the relation (3.15) as:

$$\rho_{\text{de}}(t) \equiv \frac{N^2}{4\pi^2} \rho_{\text{de}}(t_0) \left| I_\beta \left(\frac{\Gamma_0(t - t_0)}{\hbar} \right) \right|^2. \quad (3.16)$$

The model can be expressed in a more general form of the energy density:

$$\tilde{\rho}_{\text{de}}(t) = \rho_{\text{de}}(t) - \rho_{\text{bare}}, \quad (3.17)$$

where $\rho_{\text{bare}} = \text{const}$ is the minimal value of the dark energy density. When $t \rightarrow \infty$, the density $\rho_{\text{de}}(t)$ tends to ρ_{bare} .

3.2 Late-time approximation of decaying metastable dark energy

This section is based on JCAP 1510 (2015) no. 10, 066 [41] and Phys.Dark Univ. 15 (2017) 96-104 [42].

We investigated the late-time approximation of the model with decaying dark energy as the $\Lambda(t)$ CDM model, where $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$, where α is a model parameter. We assume that $\alpha^2 > 0$ or $\alpha^2 < 0$. α can be imaginary. This parametrization of dark energy is a late-time approximation of Eq. (3.16). This model is an example of the model involving interaction between dark matter and dark energy.

In result, we get the following Friedmann equation:

$$3H(t)^2 = \rho_m(t) + \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}. \quad (3.18)$$

Due to the assumption that the energy-momentum tensor for all fluids satisfies the conservation condition:

$$T_{;\alpha}^{\alpha\beta} = 0, \quad (3.19)$$

we get the conservation equation:

$$\dot{\rho}_m + 3H\rho_m = -\frac{d\Lambda}{dt}. \quad (3.20)$$

This form of the conservation equation guarantees that the interaction in the dark sector is actual.

From Eqs (3.18) and (3.20), we obtain that:

$$\dot{H} = -\frac{1}{2}\rho_m. \quad (3.21)$$

This equation can be rewritten as:

$$\dot{H} = \frac{1}{2} \left(\Lambda_{\text{bare}} + \frac{\alpha^2}{t^2} - 3H(t)^2 \right). \quad (3.22)$$

The above formula has the following solution:

$$h(t) = \frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 t}{2} \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 t}{2} \right)}, \quad (3.23)$$

where $h = \frac{H}{H_0}$, H_0 is the present value of the Hubble constant, $\Omega_{\Lambda,0} = \frac{\Lambda_{\text{bare}}}{3H_0^2}$, I_n is the modified Bessel function of the first kind, and $n = \frac{1}{2} \sqrt{1 + 9\Omega_{\alpha^2,0} T_0^2 H_0^2}$, where $\Omega_{\alpha^2,0} = \frac{\alpha^2}{3H_0^2 T_0^2}$ and T_0 is the present age of the Universe.

The formula for the scale factor a can be derived from the Eq. (3.23) and after the calculations we get:

$$a(t) = C \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 t}{2} \right) \right) \right]^{\frac{2}{3}}. \quad (3.24)$$

Constant C is equal to $\left[\sqrt{T_0} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 T_0}{2} \right) \right) \right]^{-\frac{2}{3}}$, as it is assumed that $a(T_0) = 1$. Since $a(t)$ function is monotonic, one can

obtain formula for $t(a)$ function from Eq. (3.24):

$$t(a) = \frac{2}{3i\sqrt{\Omega_{\Lambda,0}}H_0} S_{n-\frac{1}{2}}^{-1} \left(\frac{\sqrt{3\pi\sqrt{\Omega_{\Lambda,0}}H_0} i^{n+1/2}}{2} \left(\frac{a}{C} \right)^{\frac{3}{2}} \right), \quad (3.25)$$

where $S_n(x)$ is a Riccati-Bessel function $S_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x)$. J_n is the Bessel function of the first kind.

As the formula for $H(t)$ is known, the equation for ρ_m can be derived from Eqs (3.18) and (3.23):

$$\rho_m(t) = -3H_0^2 \left[\Omega_{\Lambda,0} + \frac{\Omega_{\alpha^2,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)} \right)^2 \right]. \quad (3.26)$$

Since we assume the interaction is between dark matter and dark energy only, the energy density of baryonic matter $\rho_b(t)$ scales as a^{-3} . In result, we get

$$\rho_m(t) = \rho_{dm}(t) + \rho_b(t) = \rho_{dm}(t) + \rho_{b,0}a(t)^{-3}, \quad (3.27)$$

where $\rho_{b,0}$ is the present value the energy density of baryonic matter.

From Eqs (3.24), (3.26) and (3.27), we can obtain a formula for the energy density of dark matter $\rho_{dm}(t)$:

$$\rho_{dm}(t) = -3H_0^2 \left[\Omega_{\Lambda,0} + \frac{\Omega_{\alpha^2,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)} \right)^2 \right] - \rho_{b,0}C^{-3} \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right]^{-2}. \quad (3.28)$$

3.3 Model testing

This section is based on *JCAP 1510 (2015) no. 10, 066* [41].

The paper [41] concerns the cosmological model with the following parametrization of the dark energy: $\rho_{\text{de}} = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$. In particular, we investigated the behaviour of the jerk using Sahni et al. [43, 44, 45] $Om(z)$ diagnostic test. We also performed the dynamical and statistical analysis of the model.

From the Eqs (3.23) and (3.24), we find that the jerk function is given by the following equation:

$$j = \frac{1}{H(t)^3 a(t)} \left[\frac{d^3 a(t)}{dt^3} \right] = 1 - \frac{3\Omega_{\alpha,0} T_0^2}{H_0 t^3} \left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} t \right)} \right)^{-3}. \quad (3.29)$$

In the present epoch, the jerk function is given by

$$j_0 = 1 - \frac{3\Omega_{\alpha,0}}{H_0 T_0}, \quad (3.30)$$

where T_0 is the present age of the Universe.

The evolution of the jerk function is shown in Fig. 5 in *JCAP 1510 (2015) no. 10, 066* [41].

The $Om(z)$ diagnostic test measures the deviation from the Λ CDM model ($Om(z) = \Omega_m$ for Λ CDM model). The function $Om(z)$ is $Om(z) = \frac{h^2(x)-1}{x^3-1}$, where $x = 1+z$. For our model, it has the following form:

$$Om(t) = \frac{\left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} t \right)} \right)^2 - 1}{\left(\left[\sqrt{T_0} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} T_0 \right) \right) \right]^2 \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0}{2} t \right) \right) \right]^{-2} \right) - 1}. \quad (3.31)$$

The evolution of the $Om(z)$ function is shown in Fig. 6 in *JCAP 1510 (2015) no. 10, 066* [41].

The behaviour of the jerk and $Om(z)$ function provides a test for the deviation from the Λ CDM model. These tests tools commonly

used to indicate variability of dark energy in time.

After recasting the cosmological equations (3.18) and (3.20) to the form of the dynamical system we have:

$$X' = -3X + 3X^2 + 2\sqrt{3}\alpha^2 Z^3, \quad (3.32)$$

$$Y' = \frac{3}{2}XY, \quad (3.33)$$

$$Z' = -\sqrt{3}Z^2 + \frac{3}{2}ZX, \quad (3.34)$$

where

$$X = \frac{\rho_m}{3H^2} \quad (3.35)$$

and the squares of Y and Z are equal to:

$$Y^2 = \frac{\Lambda_{\text{bare}}}{3H^2}, \quad Z^2 = \frac{1}{3H^2 t^2} \quad (3.36)$$

and $' \equiv \frac{d}{d \ln a}$. The critical points of the system (3.32)-(3.35) are collected in Table 1 in *JCAP 1510 (2015) no. 10, 066* [41].

In the statistical analysis of the model parameters, we have used the SNIa [13], BAO (SDSS DR7 data) [14], CMB and lensing observations [31], measurements of $H(z)$ [17, 18, 19] and the Alcock-Paczyński test [22, 23, 24, 25, 26, 27, 28, 29, 30]. The value of the best fit and errors are given in Table 2 and 3 in *JCAP 1510 (2015) no. 10, 066* [41]. The analysis shows that the model with negative values of the α^2 parameter is more favoured than one with positive values.

3.4 Modified scaling law of matter density

This section is based on Phys.Dark Univ. 15 (2017) 96-104 [42].

In the paper [42], we consider the cosmological model with the parametrization of the dark energy $\rho_{\text{de}} = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$. We check how this parametrization modified the scaling law of the energy density of matter and dark matter. The cosmological equations (3.18) and (3.20) give us the formula for the energy density of matter (see Eq. (3.26)). We can rewrite Eq. (3.26) as:

$$\rho_m = \rho_{m,0} a^{-3+\bar{\delta}(t)}, \quad (3.37)$$

where $\bar{\delta} = \frac{1}{\log a} \int \delta(t) d \log a$, where $\delta(t) = \frac{2\alpha^2}{t^3 H(t) \rho_m(t)}$. The evolution of $\bar{\delta}(t)$ function is presented in Fig. 5 in *Phys.Dark Univ. 15 (2017) 96-104* [42]. If $\delta(t)$ is constant, then we get that $\bar{\delta}(t)$ is constant too. In

this case, Eq. (3.37) is given by:

$$\rho_m = \rho_{m,0} a^{-3+\delta}. \quad (3.38)$$

When $\delta(t)$ is a constant, then also:

$$a = a_0 t^{\frac{2}{3-\delta}} \quad (3.39)$$

and

$$\rho_m = \rho_{m,0} a_0^{-3+\delta} t^{-2}. \quad (3.40)$$

For the early Universe, $\delta(t)$ function can be approximated as:

$$\delta(t) = \frac{9\alpha^2}{(\sqrt{1+3\alpha^2}+1)^2}. \quad (3.41)$$

We can use the same approach in the case of dark matter ρ_{dm} rewriting Eq. (3.28) as:

$$\rho_{dm} = \rho_{dm,0} a^{-3+\lambda(t)}, \quad (3.42)$$

where $\lambda(t) = \frac{1}{\log a(t)} \log \frac{\Omega_{m,0} a^{\delta(t)} - \Omega_{b,0}}{\Omega_{m,0} - \Omega_{b,0}}$. For the early Universe, $\lambda(t) = \text{const}$. In result, $\rho_{dm} = \rho_{dm,0} a^{-3+\lambda}$. The evolution of $\lambda(t)$ function is presented in Fig. 6 in *Phys.Dark Univ. 15 (2017) 96-104* [42].

The statistical analysis in this paper is based on the astronomical observations, such as SNIa [13], BAO [14, 15, 16], observations of the temperature power spectrum of CMB [31], measurements of $H(z)$ [17, 18, 19] and the Alcock-Paczyński test [22, 23, 24, 25, 26, 27, 28, 29, 30]. The value of the best fit and errors are given in Table 1 in *Phys.Dark Univ. 15 (2017) 96-104* [42]. We obtain the decay of particles of dark matter rather than their creation. The AIC criterion favours this model just very weakly in comparison to the Λ CDM model, while the BIC criterion supports positively the Λ CDM model. However, this is not sufficient for rejecting it.

3.5 Cosmological implications of transition from false to true vacuum state

This section is based on Eur.Phys.J. C77 (2017) no. 6, 357 [36].

In the paper [36], we investigate a cosmological model with decaying metastable dark energy. Here, the model of the decaying metastable dark energy is provided by quantum mechanics. The

parametrization of dark energy is given by Eq. (3.12). Replacing energy by the density of energy in Eq. (3.12), we obtain:

$$\rho_{\text{de}} = \Lambda_{\text{bare}} + E_R \left[1 + \frac{\alpha}{1-\alpha} \Re \left(\frac{J(t)}{I(t)} \right) \right], \quad (3.43)$$

where $E_R = E_0 - \Lambda_{\text{bare}}$ and α is a model parameter, which belongs to the interval $(0, 1)$. The functions $I(t)$ and $J(t)$ are:

$$\begin{aligned} J(t) &= \int_{-\frac{1-\alpha}{\alpha}}^{\infty} \frac{\eta}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta \\ &= \frac{1}{2} e^{-\tau/2} \left(-2i\pi + e^{\tau} E_1 \left(\left[\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right. \\ &\quad \left. + E_1 \left(\left[-\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right) \end{aligned} \quad (3.44)$$

and $I(t)$ can be expressed as:

$$\begin{aligned} I(t) &= \int_{-\frac{1-\alpha}{\alpha}}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta \\ &= 2\pi e^{-\tau/2} \left(1 + \frac{i}{2\pi} \left(-e^{\tau} E_1 \left(\left[\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right. \right. \\ &\quad \left. \left. + E_1 \left(\left[-\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right) \right), \end{aligned} \quad (3.45)$$

where $\tau = \frac{\alpha(E_0 - \Lambda_{\text{bare}})}{h(1-\alpha)} V_0 t$ and V_0 is the volume of sphere of radius, which is equal to the Planck length. The function E_1 is the exponential integral $E_1(z) = \int_z^{\infty} \frac{e^{-x}}{x} dx$.

As this model involves interactions between dark matter and dark energy, we have the following cosmological equations:

$$3H^2 = 3 \left(\frac{\dot{a}}{a} \right)^2 = \rho_{\text{tot}} = \rho_{\text{b}} + \rho_{\text{dm}} + \rho_{\text{de}}, \quad (3.46)$$

$$\dot{\rho}_{\text{b}} = -3H\rho_{\text{b}}, \quad (3.47)$$

$$\dot{\rho}_{\text{dm}} = -3H\rho_{\text{dm}} + Q \quad (3.48)$$

and

$$\dot{\rho}_{\text{de}} = -Q, \quad (3.49)$$

where ρ_{b} is the density of baryonic matter and $Q = -\frac{d\rho_{\text{de}}}{dt}$ is the interaction between dark matter and dark energy, which actually consists energy transfer. If $Q > 0$, then energy flows from dark energy to dark matter, while if $Q < 0$, then energy flows from dark matter to dark energy.

In this model, there occurs an intermediate phase of oscillations of the dark energy density between the phases of constant dark energy. We found also a mechanism to cause jumping of the value of energy density of dark energy from the initial value of E_0 ($E_0 = 10^{120}$) to present value of the cosmological constant.

The oscillations appear when $0 < \alpha < 0.4$. Their number, period, and amplitude, as well as the duration of this intermediate phase, decrease when α parameter grows. For $\alpha > 0.4$, the oscillations disappear altogether.

In the statistical analysis, we use the astronomical observations such as the supernovae of type Ia (SNIa) [13], BAO [14, 15, 16], measurements of $H(z)$ for galaxies [17, 18, 19], the Alcock-Paczyński test [22, 23, 24, 25, 26, 27, 28, 29, 30] and the measurements CMB [31]. The analysis showed us that independently of the values of the parameters α and E_0 , we obtain the present value of the energy density of the dark energy. The value of the best fit and errors are given in Table 1 in *Eur.Phys.J. C77 (2017) no. 6, 357* [36].

3.6 Radioactive-like decay of metastable dark energy

This section is based on Eur.Phys.J. C77 (2017) no. 12, 902 [37].

In the paper [37], we consider the model with the radioactive-like decay of metastable dark energy. The cosmological equations are:

$$3H^2 = \rho_m + \rho_{de}, \quad (3.50)$$

$$\dot{\rho}_m = -3H\rho_m - \dot{\rho}_{de}, \quad (3.51)$$

where the density of dark energy ρ_{de} is parametrized as follows:

$$\rho_{de}(t) = \rho_{bare} + \epsilon \left| I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right) \right|^2, \quad (3.52)$$

where $I_\beta(\tau)$ is defined as

$$I_\beta(\tau) = \int_{-\beta}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta, \quad (3.53)$$

where $\tau = \frac{\Gamma_0 t}{\hbar}$. The parameter $\epsilon \equiv \epsilon(\beta) = \frac{\rho_{de}(0) - \rho_{bare}}{|I_\beta(0)|^2}$ measures the deviation from the Λ CDM model ($I_\beta(0) \equiv \frac{2\pi}{N} = \pi + 2 \arctan(2\beta)$ and $\beta > 0$), β is equal to $\frac{E_0 - E_{min}}{\Gamma_0} > 0$, while the parameters E_0 and Γ_0

correspond to the energy of the system in the unstable state and its decay rate at the exponential (or canonical) regime of the decay process.

For $t > t_L = \frac{\hbar}{\Gamma_0} \frac{2\beta}{\beta^2 + \frac{1}{4}}$ [46], the approximation of (3.52) is given in the following form:

$$\rho_{\text{de}}(t) \approx \rho_{\text{bare}} + \epsilon \left(4\pi^2 e^{-\frac{\Gamma_0}{\hbar} t} + \frac{4\pi e^{-\frac{\Gamma_0}{2\hbar} t} \sin\left(\beta \frac{\Gamma_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar} t} + \frac{1}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar} t\right)^2} \right). \quad (3.54)$$

For the late time, Eq. (3.54) can be approximated as:

$$\rho_{\text{de}}(t) \approx \rho_{\text{bare}} + \frac{\epsilon}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar}\right)^2 t^2}. \quad (3.55)$$

If we use formula (3.54), the Friedmann equation (3.50) is:

$$3H^2 = \rho_{\text{tot}} = \rho_{\text{B}} + \rho_{\text{DM}} + \rho_{\text{bare}} + \rho_{\text{rad.dec}} + \rho_{\text{dam.osc}} + \rho_{\text{pow.law}}, \quad (3.56)$$

where $\rho_{\text{rad.dec}} = 4\pi^2 \epsilon e^{-\frac{\Gamma_0}{\hbar} t}$ is the radioactive-like decay dark energy, $\rho_{\text{dam.osc}} = \frac{4\pi \epsilon e^{-\frac{\Gamma_0}{2\hbar} t} \sin\left(\beta \frac{\Gamma_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar} t}$ is the damping oscillating dark energy and $\rho_{\text{pow.law}} = \frac{\epsilon}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar} t\right)^2}$ is the power-law dark energy. The radioactive type of decay dominates up to $2.2 \times 10^4 T_0$.

We performed also statistical analysis using the following astronomical observations: supernovae of type Ia (SN Ia, Union 2.1 dataset [13]), BAO data (Sloan Digital Sky Survey Release 7 (SDSS DR7)) dataset at $z = 0.275$ [14], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [15], WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [16]), measurements of the Hubble parameter $H(z)$ of galaxies [17, 18, 19], the Alcock-Paczynski test [20, 21] (data from [22, 23, 24, 25, 26, 27, 28, 29, 30]) and measurements of CMB and lensing [31]. The value of the best fit and errors are given in Table 1 in *Eur.Phys.J. C77 (2017) no. 12, 902* [37].

We found that the decay half-life time $T_{1/2}$ of dark energy is $8503 \text{ Gyr} \approx 616 \times T_0$ and the radioactive type of decay is the most effective mechanism of decaying metastable dark energy.

3.7 Main results

The model with decaying dark energy belongs to the class involving interaction in the dark sector. For the late-time approximation of the model (α^2/t^2), the deviation from the standard scale law of

the energy density of dark matter is noticeable. However, the production of dark matter is no longer an effective process. Note that this modification for the early Universe is independent on time. From the statistical analysis, we get for α^2/t^2 model the decay of particles of dark matter instead of the creation of one.

The analysis indicates also that the present value of dark energy is not sensitive to the value of α and E_0 parameters.

This model can solve the cosmological constant problem, because it involves the mechanism of jumping from the initial value of dark energy $E_0 = 10^{120}$ to the present value of the cosmological constant.

The characteristic feature of the model are oscillations of the density of dark energy occurring for $0 < \alpha < 0.4$.

The radioactive-like decaying model of dark energy for the late-time Universe ($t = 2T_0$) has three different forms of decay of dark energy: radioactive, damping oscillating, and power-law. In the beginning, the radioactive type of decay dominates up to $2.2 \times 10^4 T_0$. After the radioactive type of decay, damping oscillating type of decay appears, which is later superseded by a power-law type of decay ($1/t^2$).

Chapter 4

Diffusion dark matter-dark energy interaction model

4.1 Relativistic diffusion interacting of dark matter with dark energy

This section is based on JCAP 1607 (2016) no. 07, 024 [47] and Phys.Rev. D94 (2016) no. 4, 043521 [48].

We consider a particular model of energy-momentum exchange between dark matter and dark energy, where baryonic matter is preserved. In this approach, it is assumed that the total number of particles is conserved and the relativistic version of the energy-momentum tensor:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - g^{\mu\nu}p. \quad (4.1)$$

In this model, the energy-momentum tensor consists of two parts:

$$T^{\mu\nu} = T_{\text{de}}^{\mu\nu} + T_{\text{m}}^{\mu\nu}, \quad (4.2)$$

where $T_{\text{de}}^{\mu\nu}$ is the energy-momentum tensor for dark energy and $T_{\text{m}}^{\mu\nu}$ is the energy-momentum tensor for matter.

We assume the conservation of the total energy momentum in the following form:

$$-\nabla_\mu T_{\text{de}}^{\mu\nu} = \nabla_\mu T_{\text{m}}^{\mu\nu} \equiv 3\kappa^2 J^\nu, \quad (4.3)$$

where κ^2 is the diffusion constant and J^ν is the current which describes a flow of particles.

This model provides that the dark matter is transferred by a diffusion mechanism in an environment corresponding to the perfect fluid, while predicting a unique diffusion which is relativistically invariant and preserves the mass m of a particle [49].

The Friedmann equation is given here as:

$$3H^2 = \rho_b + \rho_{dm} + \rho_{de}, \quad (4.4)$$

where ρ_b is the density of baryonic matter, ρ_{dm} is the density of dark matter, ρ_{de} is the density of dark energy, and $\rho_m = \rho_b + \rho_{dm}$. The densities ρ_m and ρ_{de} are given by:

$$\rho_m = \rho_{b,0}a^{-3} + \rho_{dm,0}a^{-3} + \gamma(t - t_0)a^{-3}, \quad (4.5)$$

$$\rho_{de} = \rho_{de}(0) - \gamma \int^t a^{-3} dt, \quad (4.6)$$

where γ is a positive model parameter.

If we choose t_0 as zero, then we get a modified scale law for the energy density of dark matter:

$$\rho_{dm} = \rho_{dm,0}a^{-3} + \gamma ta^{-3}. \quad (4.7)$$

The current J^μ is conserved [50, 51, 52]. In result, we get:

$$\nabla_\mu J^\mu = 0. \quad (4.8)$$

For the FRW metric from the above equation, we obtain:

$$J^0 = \gamma/3\kappa^2 a^{-3}. \quad (4.9)$$

From Eq. (4.3), we get the following conservation equations:

$$\dot{\rho}_m = -3H\rho_m + \gamma a^{-3}, \quad (4.10)$$

$$\dot{\rho}_{de} = -\gamma a^{-3}, \quad (4.11)$$

where we assume that the equation of state for dark energy is $p_{de} = -\rho_{de}$ and for matter is $p_m = 0$. Here, $' \equiv \frac{d}{dt}$.

This model of diffusion interaction in the dark sector is free from the difficulties afflicting Alho et al.'s models with diffusion [53]. It involves no non-physical trajectories crossing the boundary set $\rho_m = 0$.

4.2 Diffusive DM-DE interaction: coincidence problem

This section is based on *Phys.Rev. D94 (2016) no. 4, 043521* [48].

In the paper [48], we recast cosmological equations of the diffusion cosmological model as a dynamical system. By inserting Eqs (4.5) and (4.6) into the Friedman equation (4.4), we get:

$$3H^2 = \rho_{b,0}a^{-3} + \rho_{dm,0}a^{-3} + \gamma(t - t_0)a^{-3} + \rho_{de}(0) - \gamma \int^t a^{-3} dt. \quad (4.12)$$

Now let $x = \Omega_m$, $y = \Omega_{de}$, $\delta = \frac{\gamma a^{-3}}{H\rho_m}$ and $' \equiv \frac{d}{d \ln a}$ is a differentiation with respect to the reparametrized time $\ln a(t)$. Equations (4.10), (4.11) and (4.12) can be rewritten as the dynamical system in variables x , y and z with respect to time $\ln a(t)$. Thus we get the following dynamical system:

$$x' = x(-3 + \delta + 3x), \quad (4.13)$$

$$y' = x(-\delta + 3y), \quad (4.14)$$

$$\delta' = \delta(-\delta + \frac{3}{2}x). \quad (4.15)$$

From Eq. (4.12), we have that $\frac{\rho_m}{3H^2} + \frac{\rho_{de}}{3H^2} = 1$. In result, we get that $x + y = 1$. Accordingly, dynamical system (4.13)-(4.15) is reduced to a two-dimension dynamical system.

In order to analyse this system in the infinity, we use the rewritten forms of Eqs (4.13) and (4.15) in variables

$$X = \frac{x}{\sqrt{x^2 + \delta^2}}, \quad \Delta = \frac{\delta}{\sqrt{x^2 + \delta^2}}. \quad (4.16)$$

Ultimately, we get the following dynamical system:

$$X' = X \left[-\Delta^2 \left(\frac{3}{2}X - \Delta \right) + (1 - X^2)(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (4.17)$$

$$\Delta' = \Delta \left[(1 - \Delta^2) \left(\frac{3}{2}X - \Delta \right) - X^2(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (4.18)$$

where $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{d \ln a(t)}$. The critical points of the system (4.17) and (4.18) are collected in Table I in *Phys.Rev. D94 (2016) no. 4, 043521* [48].

We considered also the case when the equations of state for baryonic and dark matter and dark energy are in a generalized form:

$$p_{\text{de}} = w\rho_{\text{de}}, \quad (4.19)$$

$$p_{\text{dm}} = \tilde{w}\rho_{\text{dm}}, \quad (4.20)$$

$$p_{\text{b}} = 0, \quad (4.21)$$

where w and $\tilde{w}$ are constant coefficients for dark energy and matter respectively.

Now the continuity equations are:

$$\dot{\rho}_{\text{dm}} = -3(1 + \tilde{w})H\rho_{\text{dm}} + \gamma a^{-3}, \quad (4.22)$$

$$\dot{\rho}_{\text{de}} = -3(1 + w)H\rho_{\text{de}} - \gamma a^{-3}, \quad (4.23)$$

$$\dot{\rho}_{\text{b}} = -3H\rho_{\text{b}}. \quad (4.24)$$

From the above equations and Eq. (4.10), we get the following dynamical system in the analogous way like (4.13)-(4.15):

$$\frac{dx}{d \ln a} = 3x \left[(1 + \tilde{w})(x - 1) + (1 + w)y + \frac{z}{3} \right], \quad (4.25)$$

$$\frac{dy}{d \ln a} = 3y \left[(1 + w)(y - 1) + (1 + \tilde{w})x - xz \right], \quad (4.26)$$

$$\frac{dz}{d \ln a} = z \left[3\tilde{w} - z + \frac{3}{2}[(1 + \tilde{w})x + (1 + w)y] \right]. \quad (4.27)$$

As $x + y = 1$, the above system is reduced to a two-dimensional one. The critical points of this model are collected in Table II in *Phys.Rev. D94 (2016) no. 4, 043521* [48]. The critical point $\{x_0 = -\frac{1+3w}{3(\tilde{w}-w)}, z_0 = 1 + 3\tilde{w}\}$ represents a scaling solution $\rho_{\text{dm}} = \rho_{\text{de}}$, thus providing a mechanism to solve the coincidence problem.

We considered the special case of Eqs (4.25) and (4.27) when dark matter is relativistic ($\tilde{w} = 1/3$) and $w = -1$. Then they simplify to the following form:

$$x' = x(-4 + z + 4x), \quad (4.28)$$

$$z' = z(1 - z + 2x). \quad (4.29)$$

For the purpose of examining Eqs (4.28) and (4.29) in the infinity, we choose variables $X = \frac{x}{\sqrt{x^2 + \delta^2}}$, $\Delta = \frac{\delta}{\sqrt{x^2 + \delta^2}}$. Thus we get:

$$X' = X \left[-\Delta^2 \left(\sqrt{1 - X^2 - \Delta^2} + \frac{3}{2}X - \Delta \right) + (1 - X^2)(3X + \Delta - 4\sqrt{1 - X^2 - \Delta^2}) \right], \quad (4.30)$$

$$\Delta' = \Delta \left[(1 - \Delta^2) \left(\sqrt{1 - X^2 - \Delta^2} + \frac{3}{2}X - \Delta \right) - X^2(3X + \Delta - 4\sqrt{1 - X^2 - \Delta^2}) \right], \quad (4.31)$$

where $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{d\tau}$.

The critical points of system (4.30)-(4.31) are collected in Table III in *Phys.Rev. D94 (2016) no. 4, 043521* [48].

4.3 Diffusive DM-DE interaction: non-relativistic case and statistical analysis

This section is based on JCAP 1607 (2016) no. 07, 024 [47].

In the paper [47], we examine two cases of the diffusion interaction in the dark sector: relativistic and non-relativistic. The relativistic case was considered in the previous sections. The other one uses the non-relativistic limit of the above energy-momentum tensor:

$$\begin{aligned} \rho_{\text{dm}} = \tilde{T}^{00} &= \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} p^0 \Omega = g^{-\frac{1}{2}} Z m + \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \frac{a^2 \mathbf{p}^2}{2m} \Omega \\ &\equiv Z m a^{-3} + a^{-2} \rho_{\text{nr}}, \end{aligned} \quad (4.32)$$

where

$$\rho_{\text{nr}} = \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \Omega a^4 \frac{\mathbf{p}^2}{2m}, \quad (4.33)$$

where Ω is the concentration of mass, $\mathbf{p}$ is the momentum and m is the mass of the particle of dark matter. The constant Z is given by:

$$Z \equiv \frac{\gamma}{3\kappa^2} = g \int \frac{d\mathbf{p}}{(2\pi)^3} \Omega, \quad (4.34)$$

where κ^2 is the diffusion constant.

In this case, the conservation equation for dark matter is:

$$\dot{\rho}_{\text{dm}} + 5H\rho_{\text{dm}} = 3Z\kappa^2 a^{-3} + 2ZmHa^{-3}. \quad (4.35)$$

Let

$$x = \frac{\rho_{\text{dm}}}{3H^2}, \quad y = \frac{\rho_{\text{de}}}{3H^2}, \quad u = \frac{(2Zm)a^{-3}}{\rho_{\text{dm}}}, \quad \delta = \frac{\gamma a^{-3}}{H\rho_{\text{dm}}} \quad (4.36)$$

and $\tau = \ln a$ is a reparametrized time. Then we get the following dynamical system:

$$x' = x \left(-5 + \delta + u - 2\frac{\dot{H}}{H^2} \right), \quad (4.37)$$

$$y' = -x(\delta + u) - 2y\frac{\dot{H}}{H^2}, \quad (4.38)$$

$$u' = u(2 - \delta - u), \quad (4.39)$$

$$\delta' = \delta \left(2 - \delta - u - \frac{\dot{H}}{H^2} \right), \quad (4.40)$$

where $' \equiv \frac{d}{d\tau}$ and $\frac{\dot{H}}{H^2} = -\frac{1}{2}x(5 - u)$.

Since $\Omega_{\text{dm}} + \Omega_{\text{de}} = 1$ we have $x + y = 1$. In effect, the above dynamical system reduces to a three-dimensional dynamical system. The system (4.37)-(4.40) has the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$ determined by the equations $x = 0$ or $u = 5$. Its other submanifold is $\delta = 0$. For this invariant submanifold, the system reduces to

$$x' = x(u + 5(x - 1) - xu), \quad (4.41)$$

$$u' = u(2 - u). \quad (4.42)$$

In the statistical analysis of the model parameters of relativistic and non-relativistic case, we used the following astronomical observations: supernovae of type Ia (SNIa, Union 2.1 dataset [13]), BAO data [14, 15, 16], measurements of the Hubble parameter $H(z)$ of galaxies [17, 18, 19], the Alcock-Paczynski test [20, 21] (data from [22, 23, 24, 25, 26, 27, 28, 29, 30]) and measurements of CMB and lensing [31]. The value of the best fit and errors are given in Table 3 and 4 in *JCAP 1607 (2016) no. 07, 024* [47]. The BIC criterion gives a strong evidence in favour of the Λ CDM model in comparison to these models. However, this is not sufficient for rejecting of the diffusion models.

Chapter 5

Dynamical system approach to running Λ cosmological models

This chapter is based on Eur.Phys.J. C76 (2016) no. 11, 606 [10].

In the paper [10], we investigate cosmological models in which the cosmological constant term is a time-dependent function, examining the following parametrization of cosmological parameter Λ : $\Lambda(H)$, $\Lambda(a)$ as well as three covariant ones: $\Lambda(R)$, $\Lambda(\phi)$ – cosmologies with diffusion, and $\Lambda(X)$, where $X = \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$ is the kinetic part of density of the scalar field. We also considered an emergent relation $\Lambda(a)$ obtained from the behaviour of trajectories in a neighbourhood of an invariant submanifold. In the thesis, we limit to $\Lambda(H)$, $\Lambda(R)$, $\Lambda(X)$, and $\Lambda(a)$.

5.1 $\Lambda(H)$ CDM cosmologies

We take the parametrization $\Lambda(H)$ in the form of the Taylor series:

$$\Lambda(H) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dH^n} \Lambda(H)|_0 H^n = \sum_{n=1}^{\infty} \alpha_n H^n. \quad (5.1)$$

Here, a reflection symmetry $H \rightarrow -H$ is additionally assumed. Only terms of type H^{2n} in the above expansion series have this symmetry, thus [54]:

$$\Lambda(H) = \Lambda_{\text{bare}} + \alpha_2 H^2 + \alpha_4 H^4 + \dots. \quad (5.2)$$

The cosmological equations with the parametrization $\Lambda(H)$ are:

$$\dot{H} = -H^2 - \frac{1}{6}\rho_m + \frac{1}{3}\Lambda(H), \quad (5.3)$$

$$\dot{\rho}_m = -3H\rho_m - \Lambda'(H) \left(-H^2 - \frac{1}{6}\rho_m + \frac{\Lambda(H)}{3} \right). \quad (5.4)$$

Let $x = H^2$, $y = \rho_m$ and $\tau = \ln a$ is a new parametrization of time. Then system (5.3)-(5.4) gives the following dynamical system:

$$x' \equiv \frac{dx}{d \ln a} = 2 \left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) \right], \quad (5.5)$$

$$y' \equiv \frac{dy}{d \ln a} = -3y - \frac{1}{3}(\alpha_2 + 2\alpha_4 x + \dots) \times \left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) \right]. \quad (5.6)$$

As $3H^2 = \rho_m + \Lambda(H)$, we get an additional equation:

$$y - 3x = -(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots). \quad (5.7)$$

This equation lets us reduce the system (5.5)-(5.6) to one dimension.

After cutting the second term out of the series (5.2) we get the following equations:

$$\frac{dx}{d\tau} = x(\alpha_2 - 3) + \Lambda, \quad (5.8)$$

$$y = (3 - \alpha_2)x - \Lambda. \quad (5.9)$$

The system has the critical point:

$$x_0 = \frac{\Lambda}{3 - \alpha_2}, \quad y = 0. \quad (5.10)$$

Now we introduce a new variable $x \rightarrow X = x - x_0$, obtaining

$$\frac{dX}{d\tau} = (\alpha_2 - 3)X. \quad (5.11)$$

The above equation has an exact solution in the form:

$$X = X_0 e^{\tau(\alpha_2 - 3)} = X_0 a^{-3 + \alpha_2}, \quad (5.12)$$

which can be interpreted as the Alcaniz-Lima solution [55]:

$$x = H^2 = \frac{\tilde{\rho}_{m,0}}{3} a^{-3 + \alpha_2} + \frac{\rho_{\Lambda,0}}{3}, \quad (5.13)$$

where $\tilde{\rho}_{m,0} = \frac{3}{3-\alpha_2} \rho_{m,0}$. This constitutes the scaling solution $\rho_\Lambda(a) \sim \rho_m(a)$, which provides a way to solve the coincidence problem.

5.2 $\Lambda(R)$ CDM cosmologies

We investigate the parametrization of $\Lambda(R)$ in the form $\rho_\Lambda = -\frac{\alpha}{2}R = 3\alpha(\dot{H} + 2H^2 + \frac{k}{a^2})$ [56], where $k = -1, 0, +1$, getting the following cosmological equations:

$$\dot{H} = -H^2 - \frac{1}{6}(\rho_m + \rho_\Lambda), \quad (5.14)$$

$$\dot{\rho} = -3H\rho_m \quad (5.15)$$

with the first integral of the form

$$H^2 = \frac{1}{3} \left(-\frac{3k}{a^2} + \frac{2}{2-\alpha} \rho_{m,0} a^{-3} + f_0 a^{2\frac{1-2\alpha}{\alpha}} \right), \quad (5.16)$$

where f_0 is an integration constant.

We can rewrite the above equations as a dynamical system in the variables $a, x = \dot{a}$

$$\dot{a} = x, \quad (5.17)$$

$$\begin{aligned} \dot{x} = & -\Omega_{m,0} \frac{1}{2-\alpha} a^{-2} + \\ & + \left(\frac{1}{\alpha} - 1 \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) a^{\frac{2}{\alpha}-3}. \end{aligned} \quad (5.18)$$

We can analyse the system (5.17)-(5.18) in the infinity, using variables $A = \frac{1}{a}, X = \frac{x}{a}$. Then we get the following dynamical system:

$$\dot{A} = -XA, \quad (5.19)$$

$$\begin{aligned} \dot{X} = & A^3 \left[-\Omega_{m,0} \frac{1}{2-\alpha} \right. \\ & \left. + \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) A^{\frac{\alpha-2}{\alpha}} \right] - X^2. \end{aligned} \quad (5.20)$$

We can use also the Poincaré sphere to investigate critical points at the infinity. If we take $B = \frac{a}{\sqrt{1+a^2+x^2}}, Y = \frac{x}{\sqrt{1+a^2+x^2}}$, then

we obtain a dynamical system in the following form:

$$\begin{aligned}
 B' &= YB^2(1 - B^2) \\
 &\quad - BY \left[-\Omega_{m,0} \frac{1}{2-\alpha} (1 - B^2 - Y^2)^{3/2} \right. \\
 &\quad \left. + \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) B^{-1+2/\alpha} \right. \\
 &\quad \left. \times (1 - B^2 - Y^2)^{2-1/\alpha} \right], \tag{5.21}
 \end{aligned}$$

$$\begin{aligned}
 Y' &= \left[-\Omega_{m,0} \frac{1}{2-\alpha} (1 - B^2 - Y^2)^{3/2} \right. \\
 &\quad \left. + \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) B^{-1+2/\alpha} \right. \\
 &\quad \left. (1 - B^2 - Y^2)^{2-1/\alpha} \right] (1 - Y^2) - Y^2 B^3, \tag{5.22}
 \end{aligned}$$

where $' \equiv B^2 \frac{d}{dt}$.

5.3 Non-canonical scalar field cosmology

The dark energy can be also parameterized by a non-canonical scalar field ϕ [57]. In the canonical scalar field approach, the pressure p_ϕ is given by the formula $p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$, where $\dot{\phi} \equiv \frac{d\phi}{dt}$ and $V(\phi)$ is the potential of the scalar field. In the non-canonical scalar field, the pressure is described by the formula $p_\phi = \left(\frac{\dot{\phi}^2}{2} \right)^\alpha - V(\phi)$, where α is a parameter. Note that when α is equal to 1, then the pressure of the non-canonical scalar field corresponds to the canonical case.

The cosmological equations for this model are the Friedmann equation:

$$3H^2 = \rho_m + (2\alpha - 1) \left(\frac{\dot{\phi}^2}{2} \right)^\alpha + V(\phi) - \frac{3k}{a^2}, \tag{5.23}$$

where $k = -1, 0, +1$ and the Klein-Gordon equation:

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} + \left(\frac{V'(\phi)}{\alpha(2\alpha - 1)} \right) \left(\frac{2}{\dot{\phi}^2} \right)^{\alpha-1} = 0. \tag{5.24}$$

The above equations can be rewritten as a dynamical system. We choose a and $x = \dot{a}$ as variables, obtaining from Eqs (5.23) and

(5.24):

$$a' = xa^2, \quad (5.25)$$

$$x' = -\frac{\rho_{m,0}}{6} - \frac{\alpha+1}{3}a^{\frac{3}{1-2\alpha}} + \frac{\Lambda}{3}a^3, \quad (5.26)$$

where $' \equiv a^2 \frac{d}{dt}$.

For the purpose of analysing critical points in the infinity, we choose the coordinates: $A = \frac{1}{a}$, $X = \frac{x}{a}$ and $B = \frac{a}{x}$, $Y = \frac{1}{x}$.

The dynamical system for variables A and X is:

$$A' = -XA^2, \quad (5.27)$$

$$X' = A^4 \left(-\frac{\rho_{m,0}}{6} - \frac{\alpha+1}{3}A^{\frac{3}{2\alpha-1}} \right) + A \left(\frac{\Lambda}{3} - X^2 \right), \quad (5.28)$$

where $' \equiv A \frac{d}{dt}$. We can then obtain the following dynamical system based on variables B and Y :

$$\dot{B} = BY \left[B + \left(\frac{\rho}{6}Y^3 + \frac{\alpha+1}{3}B^{\frac{3}{1-2\alpha}}Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3}B^3 \right) \right], \quad (5.29)$$

$$\dot{Y} = Y^2 \left(\frac{\rho}{6}Y^3 + \frac{\alpha+1}{3}B^{\frac{3}{1-2\alpha}}Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3}B^3 \right), \quad (5.30)$$

where $\dot{} \equiv B^2Y \frac{d}{dt}$.

5.4 Cosmology with emergent $\Lambda(a)$ relation

We consider cosmology with a scalar field which is non-minimally coupled to gravity. In this case, the cosmological equations are:

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V'(\phi) = 0, \quad (5.31)$$

where $' \equiv \frac{d}{d\phi}$, ϕ is a scalar field, $V(\phi)$ is a potential of the scalar field and

$$3H^2 = \rho_m + \frac{1}{2}\dot{\phi}^2 + 3\xi H^2\phi^2 + 6\xi H\phi\dot{\phi} + V(\phi) \quad (5.32)$$

and

$$\dot{\rho}_m = -3H\rho_m. \quad (5.33)$$

We introduce the following variables [58]:

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \quad z \equiv \frac{\phi}{\sqrt{6}}. \quad (5.34)$$

In result, we get:

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\phi + \Omega_m = (1 - 6\xi)x^2 + y^2 + 6\xi(x + z)^2 + \Omega_{m,0}a^{-3}. \quad (5.35)$$

Our aim is to generalize the Λ CDM model by including a contribution beyond Λ_{bare} in the above equation. In our further analysis we will call it 'the emergent Λ term'. Thus,

$$\Omega_{\Lambda,\text{emergent}} = (1 - 6\xi)x^2 + y^2 + 6\xi(x + z)^2. \quad (5.36)$$

The dynamical system which describes the evolution in phase space has the form:

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x - 12\xi z + \frac{1}{2}\lambda y^2 - (x + 6\xi z)\frac{\dot{H}}{H^2}, \quad (5.37)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = -\frac{1}{2}\lambda xy - y\frac{\dot{H}}{H^2}, \quad (5.38)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (5.39)$$

$$\frac{d\lambda}{d(\ln a)} = \frac{d\lambda}{d\tau} = -\lambda^2(\Gamma(\lambda) - 1)x, \quad (5.40)$$

where $\Gamma = \frac{V''(\phi)V(\phi)}{V'^2(\phi)}$, $\lambda \equiv -\sqrt{6}\frac{V'(\phi)}{V(\phi)}$ and

$$\begin{aligned} \frac{\dot{H}}{H^2} &= \frac{1}{H^2} \left[-\frac{1}{2}(\rho_\phi + p_\phi) - \frac{1}{2}\rho_{m,0}a^{-3} \right] \\ &= \frac{1}{6\xi z^2(1 - 6\xi) - 1} \left[-12\xi(1 - 6\xi)z^2 - 3\xi\lambda y^2 z \right. \\ &\quad \left. + \frac{3}{2}(1 - 6\xi)x^2 + 3\xi(x + z)^2 + \frac{3}{2} - \frac{3}{2}y^2 \right]. \end{aligned} \quad (5.41)$$

For the sake of illustrating the emergent $\Lambda(a)$ relation, we consider two cosmologies for which we derive $\Lambda = \Lambda(a)$: $V = \text{const}$ or $\lambda = 0$, if $\xi = 0$ (minimal coupling), and $V = \text{const}$, if $\xi = \frac{1}{6}$ (conformal coupling). For the above cases, the system (5.37)-(5.40) reduces to

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x - x\frac{\dot{H}}{H^2}, \quad (5.42)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = -y\frac{\dot{H}}{H^2}, \quad (5.43)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (5.44)$$

where

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}x^2 - \frac{3}{2} + \frac{3}{2}y^2 \quad (5.45)$$

and

$$\frac{dx}{d\tau} = -3x - 2z - \frac{\dot{H}}{H^2}(x + z), \quad (5.46)$$

$$\frac{dy}{d\tau} = -y \frac{\dot{H}}{H^2}, \quad (5.47)$$

$$\frac{dz}{d\tau} = x, \quad (5.48)$$

where

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}(x + z)^2 - \frac{3}{2} + \frac{3}{2}y^2. \quad (5.49)$$

For the minimal coupling case ($\xi = 0, V = \text{const}$), the dynamical system (5.42)-(5.44) is expressed by:

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x, \quad (5.50)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = 0, \quad (5.51)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (5.52)$$

with the condition

$$0 = x^2 - y^2 + 1. \quad (5.53)$$

The solution of the above system is $x = C_1 a^{-3}$, $y = \text{const}$ and $z = -\frac{1}{3}C_1 a^{-3} + C_2$.

Accordingly, $\Omega_{\Lambda, \text{emergent}}$ for this case is:

$$\Omega_{\Lambda, \text{emergent}} = \Omega_{\Lambda, \text{emergent}, 0} a^{-6} + \Omega_{\Lambda, 0}. \quad (5.54)$$

For the conformal coupling case, the system (5.42)-(5.44) is:

$$\frac{dx}{d\tau} = -3x - 2z, \quad (5.55)$$

$$\frac{dy}{d\tau} = 0 \quad \Rightarrow \quad y = \text{const}, \quad (5.56)$$

$$\frac{dz}{d\tau} = x \quad (5.57)$$

with the condition

$$0 = (x + z)^2 - 3y^2 + 3. \quad (5.58)$$

The solution of the above dynamical system is $x = -2C_1a^{-2} - C_2a^{-1}$, $y = \text{const}$ and $z = C_1a^{-2} + C_2a^{-1}$.

In consequence, $\Omega_{\Lambda,\text{emergent}}$ is:

$$\Omega_{\Lambda,\text{emergent}} = \Omega_{\Lambda,0} + \Omega_{\Lambda,\text{emergent},0}a^{-4}. \quad (5.59)$$

The model with $\xi = 1/6$ (conformal coupling) and $V = \text{const}$ involves the early constant ratio dark energy $\Omega_{\text{de}} = \text{const}$ during the radiation epoch. In this case, we can use the fractional early dark energy parameter $\Omega_{\text{d}}^{\text{e}} = 1 - \frac{\Omega_{\text{m}}}{\Omega_{\text{tot}}}$, where Ω_{tot} is the sum of the densities of both matter and dark energy [59, 60]. For the fractional early dark energy parameter, there is a strong observational upper limit ($\Omega_{\text{d}}^{\text{e}} < 0.0036$) [31]. Accordingly, we obtain the following limit on the running Λ parameter in the present epoch: $\Omega_{\text{em},0} < 3.19 \times 10^{-7}$.

Chapter 6

Starobinsky cosmological model in Palatini formalism

6.1 Palatini formalism in Jordan frame

This section is based on Eur.Phys.J. C77 (2017) no. 6, 406 [61], Eur.Phys.J. C77 (2017) no. 9, 603 [62], Eur.Phys.J. C78 (2018) no. 3, 249 [63], and Phys.Rev. D97 (2018) 103524 [64].

In this section, we consider the Starobinsky cosmological model ($f(R) = R + \gamma R^2$) in the Palatini formalism. This model can be formulated either in the Jordan frame or in the Einstein frame.

First, the model will be considered in the Jordan frame. Then its action has the following form:

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \quad (6.1)$$

where $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\Gamma)$ is the generalized Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ [65, 66]. Since we assume that the equation of state for matter is given in the form $p = p(\rho)$, the action for matter S_m is [67]:

$$S_m = \int -\sqrt{-g} \rho \left(1 + \int \frac{p(\rho)}{\rho^2} d\rho \right) d^4x. \quad (6.2)$$

After varying Eq. (6.1) with respect to the metric $g_{\mu\nu}$ and the connection Γ , we get the equations of motion:

$$f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \quad (6.3)$$

$$\hat{\nabla}_\alpha (\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \quad (6.4)$$

where $\hat{\nabla}_\alpha$ is the covariant derivative obtained with respect to the connection Γ and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor for which $\nabla^\mu T_{\mu\nu} = 0$.

The structural equation is obtained from the trace of Eq. (6.3) as:

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T. \quad (6.5)$$

For the Starobinsky model's case, Eq. (6.5) simplifies to

$$-\hat{R} = T. \quad (6.6)$$

As we consider the perfect fluid, the energy-momentum tensor is given by:

$$T_{\nu}^{\mu} = \text{diag}(-\rho, p, p, p), \quad (6.7)$$

where p is the pressure of matter. In this case, the equation of state has the form $p = w\rho$, where w is a constant, which equals zero for dust, $1/3$ for radiation and -1 for dark energy. The trace of the energy-momentum tensor is:

$$T = \sum_i \rho_{i,0} (3w_i - 1) a(t)^{-3(1+w_i)}. \quad (6.8)$$

Since $\nabla^{\mu} T_{\mu\nu} = 0$, the density of matter ρ is equal to $\rho_{m,0} a^{-3(1+w)}$. For the case of dust, we get $\rho = \rho_{m,0} a^{-3}$ and for the case of radiation $\rho = \rho_{m,0} a^{-4}$.

We assume that matter has the form of dust and dark energy is described by the cosmological constant Λ , so the trace of the tensor energy-momentum T is $\rho_{m,0} a^{-3} + \Lambda$. In consequence, Eq. (6.6) gives the relation between the Ricci scalar $\hat{R}$ and the scale factor a :

$$\hat{R} = \rho_{m,0} a^{-3} + 4\Lambda. \quad (6.9)$$

In the Palatini formalism in the Jordan frame for the FRW metric, we get the Friedmann equation from Eq. (6.3):

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left[\Omega_{\gamma} (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0})^2 \frac{(K-3)(K+1)}{2b} + (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}) + \frac{\Omega_{r,0} a^{-4}}{b} + \Omega_k \right], \quad (6.10)$$

where $\Omega_k = -\frac{k}{H_0^2 a^2}$, $\Omega_{r,0} = \frac{\rho_{r,0}}{3H_0^2}$, $\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}$, $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$, $\Omega_{\gamma} = 3\gamma H_0^2$, $K = \frac{3\Omega_{\Lambda,0}}{(\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0})}$, $b = f'(\hat{R}) = 1 + 2\Omega_{\gamma} (\Omega_{m,0} a^{-3} + 4\Omega_{\Lambda,0})$, $d = \frac{1}{H} \frac{db}{dt} = -2\Omega_{\gamma} (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}) (3 - K)$, H_0 is the Hubble constant, $\rho_{r,0}$ is the present value of the energy density of radiation and $\rho_{m,0}$ is the present value of the density of matter.

As $\nabla^{\mu} T_{\mu\nu} = 0$, we get the following continuity equation:

$$\dot{\rho}_m = -3H\rho_m. \quad (6.11)$$

When γ is zero, then the model is equivalent to the Λ CDM model.

6.2 Palatini formalism in Einstein frame

This section is based on Eur.Phys.J. C77 (2017) no. 6, 406 [61], Eur.Phys.J. C77 (2017) no. 9, 603 [62], Eur.Phys.J. C78 (2018) no. 3, 249 [63], and Phys.Rev. D97 (2018) 103524 [64].

In the Einstein frame, if $f''(\hat{R}) \neq 0$ the action (6.1) is equivalent to the Palatini gravitational action [68]:

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) = \frac{1}{2} \int d^4x \sqrt{-g} \left(f'(\chi)(\hat{R} - \chi) + f(\chi) \right) + S_m(g_{\mu\nu}, \psi). \quad (6.12)$$

Now we can introduce a new scalar field $\Phi = f'(\chi)$, where $\chi = \hat{R}$. In this case action (6.12) is given by:

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) = \frac{1}{2} \int d^4x \sqrt{-g} \left(\Phi \hat{R} - U(\Phi) \right) + S_m(g_{\mu\nu}, \psi). \quad (6.13)$$

Here, the function $U(\Phi)$ is a potential of the form:

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)), \quad (6.14)$$

where $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

After varying the action (6.13) with respect of the metric $g_{\mu\nu}$ and the connection Γ , we get

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - T_{\mu\nu} = 0, \quad (6.15)$$

$$\hat{\nabla}_\alpha (\sqrt{-g} \Phi g^{\mu\nu}) = 0. \quad (6.16)$$

From Eq. (6.16), we get the connection $\hat{\Gamma}$ for the new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$. A new structural equation can be obtained from the trace of Eq. (6.15):

$$2U(\Phi) - U'(\Phi)\Phi = T. \quad (6.17)$$

Let $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $\bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. Then Eq. (6.15) is:

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{U}(\Phi), \quad (6.18)$$

where $\bar{U}(\Phi) = U(\Phi)/\Phi^2$ and $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$. Because $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$ then

$$\Phi \bar{R} - (\Phi^2 \bar{U}(\Phi))' = 0. \quad (6.19)$$

From Eq. (6.18), we get a new structural equation:

$$\Phi \bar{U}'(\Phi) + \bar{T} = 0. \quad (6.20)$$

In this parametrization, the action (6.13) has the following form:

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2\kappa} \int d^4x \sqrt{-\bar{g}} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1} \bar{g}_{\mu\nu}, \psi), \quad (6.21)$$

where

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} S_m = (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} \bar{g}^{\mu\nu} = \Phi^{-3} T^{\mu\nu}, \quad (6.22)$$

and $\bar{u}^\mu = \Phi^{-\frac{1}{2}} u^\mu$, $\bar{\rho} = \Phi^{-2} \rho$, $\bar{p} = \Phi^{-2} p$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$ [69, 70].

As we use a new metric $\bar{g}_{\mu\nu}$, the FRW line element has a new form:

$$d\bar{s}^2 = d\bar{t}^2 - \bar{a}^2(t) [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (6.23)$$

where the new cosmological time $d\bar{t} = \Phi(t)^{\frac{1}{2}} dt$ and the new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$.

We assume the barotropic matter ($p = w\rho$). Accordingly, the cosmological equations are:

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \quad (6.24)$$

$$6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_m(1 + 3w), \quad (6.25)$$

where $\bar{\rho}_\Phi = \frac{1}{2} \bar{U}(\Phi)$, $\bar{\rho}_m = \rho_0 \bar{a}^{-3(1+w)} \Phi^{\frac{1}{2}(3w-1)}$ and $w = \bar{p}_m/\bar{\rho}_m$. In this case, the conservation equation is:

$$\dot{\bar{\rho}}_m + 3\bar{H}\bar{\rho}_m(1 + w) = -\dot{\bar{\rho}}_\Phi. \quad (6.26)$$

The Starobinsky model ($f(\hat{R}) = \hat{R} + \gamma \hat{R}$) in cosmology yields the potential $\bar{U}$ in the form:

$$\bar{U}(\Phi) = \left(\frac{1}{4\gamma} + 2\Lambda \right) \frac{1}{\Phi^2} - \frac{1}{2\gamma} \frac{1}{\Phi} + \frac{1}{4\gamma}. \quad (6.27)$$

From Eq. (6.20), we can obtain the scalar field $\Phi(\bar{a})$ as:

$$\Phi(a) = 1 - 8\gamma\Lambda + 2\gamma\rho_m + 8\gamma p_m \quad (6.28)$$

or

$$\Phi(\bar{a}) = 1 - 8\gamma\Lambda + (2\gamma\bar{\rho}_m + 8\gamma\bar{p}_m)\Phi^2(\bar{a}). \quad (6.29)$$

Ultimately, Φ is dependent on $\bar{\rho}_m$:

$$\Phi(\bar{a}) = \frac{1 + \sqrt{1 - 8\gamma(\bar{\rho}_m + 4\bar{p}_m)(1 - 8\gamma\Lambda)}}{4\gamma(\bar{\rho}_m + 4\bar{p}_m)} \quad (6.30)$$

or

$$\Phi(\bar{a}) = \frac{1 - \sqrt{1 - 8\gamma(\bar{\rho}_m + 4\bar{p}_m)(1 - 8\gamma\Lambda)}}{4\gamma(\bar{\rho}_m + 4\bar{p}_m)}. \quad (6.31)$$

We can obtain the Friedmann equation in form $3\bar{H}(\hat{R})^2$ from Eqs (6.20) and (6.24), getting:

$$3\bar{H}(\hat{R})^2 = \bar{\rho}_m(\hat{R}) + \frac{\bar{U}(\hat{R})}{2} + \Lambda = \frac{\hat{R}(2 + \gamma\hat{R})}{2(1 + 2\gamma\hat{R})^2} - 3\Lambda. \quad (6.32)$$

6.3 Starobinsky cosmological model in Palatini formalism: dynamical system approach

This section is based on Eur.Phys.J. C77 (2017) no. 6, 406 [61].

In the paper [61], we consider singularities that can appear in the Starobinsky cosmological model in the Palatini formalism. Investigating it in the Einstein frame, we found inflation in the model when matter is negligible in comparison to $\bar{\rho}_\Phi = \frac{\bar{U}}{2}$ and the value of γ parameter is close to zero. Moreover, when number of e-folds is equal to 60, then the value of γ parameter is $1.16 \times 10^{-69} \text{ s}^2$.

We investigate also singularities in the Jordan frame, introducing the classification of singularities in FRW cosmology and reducing dynamics to the dynamical system of the Newtonian type. This classification is given in terms of the geometry of a potential $V(a)$ if this potential has a pole.

In the standard cosmology, the potential $V(a)$ is expressed by the following equation:

$$\dot{a}^2 = -2V(a), \quad (6.33)$$

where $V(a) = -\frac{\rho(a)a^2}{6}$. In consequence, we obtain that:

$$\ddot{a} = -\frac{\partial V(a)}{\partial a}. \quad (6.34)$$

This leads to the following dynamical system:

$$\dot{a} = x, \quad (6.35)$$

$$\dot{x} = -\frac{\partial V(a)}{\partial a}. \quad (6.36)$$

In our model, Eq. (6.10) can be rewritten analogically as a dynamical system (6.35)-(6.36):

$$a' = x, \quad (6.37)$$

$$x' = -\frac{\partial V(a)}{\partial a}, \quad (6.38)$$

where $V = -\frac{a^2}{2} \left(\Omega_\gamma \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \Omega_k \right)$ and $' \equiv \frac{d}{d\sigma} = \frac{b+\frac{d}{2}}{H_0 b} \frac{d}{dt}$ is a new parametrization of time.

We treat the above dynamical system as a sewn dynamical system [71, 72]. Accordingly, we consider two cases. The first one is for $a < a_{\text{sing}}$ and the second one is for $a > a_{\text{sing}}$, where a_{sing} is the value of the scale factor in the singularity.

For $a < a_{\text{sing}}$, the dynamical system (6.37)-(6.38) can be rewritten as:

$$a' = x, \quad (6.39)$$

$$x' = -\frac{\partial V_1(a)}{\partial a}, \quad (6.40)$$

where $V_1 = V(-\eta(a - a_{\text{sing}}) + 1)$ and $\eta(a)$ denotes the Heaviside function.

For $a > a_{\text{sing}}$, we get:

$$a' = x, \quad (6.41)$$

$$x' = -\frac{\partial V_2(a)}{\partial a}, \quad (6.42)$$

where $V_2 = V\eta(a - a_{\text{sing}})$.

In the Starobinsky cosmological model in Palatini formalism in the Jordan frame, we found two new types of singularities of a finite scale factor. The first type is the sewn freeze singularity, for which the Hubble function H , pressure p and energy density ρ are divergent. It appears when γ parameter has a positive value. The second type is the sewn typical singularity, for which the Hubble function and energy density ρ are finite and $\dot{H}$ and pressure p are divergent. It appears when γ parameter has a negative value. At the sewn singularity which is of a finite scale factor type, the singularity in the past meets the singularity in the future. In the Jordan frame, the phase portrait is topologically equivalent to the phase portrait of the Λ CDM model for the positive γ parameter.

In order to estimate this model through statistical analysis, we used 580 supernovae of type Ia [13], BAO [14, 15, 16], measurements of $H(z)$ for galaxies [17, 18, 19], Alcock-Paczyński test [22, 23, 24, 25, 26, 27, 28, 29, 30], measurements of CMB and lensing by Planck, and low ℓ by WMAP [31] finding that the best fit value of $\Omega_\gamma = 3\gamma H_0^2$ is 9.70×10^{-11} . The BIC criterion gives a strong evidence in favour of the Λ CDM model in comparison to this model. However, we are not able to reject it.

6.4 Extended Starobinsky cosmological model in Palatini formalism

This section is based on Eur.Phys.J. C77 (2017) no. 9, 603 [62].

In the paper [62], we consider the FRW cosmological model for $f(R) = R + \gamma R^2 + \delta R^3$ gravity within the Jordan and Einstein frame in the Palatini formalism. We investigate singularities in this model and demonstrate how the Starobinsky model is modified by adding a new term in $f(R)$ formula.

By adding of $\delta \hat{R}^3$ in $f(\hat{R})$ expression in the Jordan frame case, the Friedmann formula (6.10) is modified as follows:

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \times \left[\frac{\Omega_R}{2b} \left[\Omega_\gamma (\Omega_R - 4\Omega_{\text{tot}}) + 2\Omega_\delta \Omega_R (\Omega_R - 3\Omega_{\text{tot}}) \right] + \Omega_{\text{tot}} + \Omega_k \right], \quad (6.43)$$

where

$$\Omega_{\text{tot}} = \Omega_{\text{m},0} a^{-3} + \Omega_{\Lambda,0}, \quad (6.44)$$

$$b = f'(\hat{R}) = 1 + \Omega_R [2\Omega_\gamma + 3\Omega_\delta \Omega_R], \quad (6.45)$$

$$d = \frac{1}{H} \frac{db}{dt} = 6 \frac{\Omega_\gamma + 3\Omega_\delta \Omega_R}{3\Omega_\delta \Omega_R^2 - 1} [\Omega_R (1 - \Omega_\delta \Omega_R^2) - 4\Omega_{\Lambda,0}], \quad (6.46)$$

$$\Omega_\gamma = 3\gamma H_0^2, \quad (6.47)$$

$$\Omega_\delta = 9\delta H_0^4, \quad (6.48)$$

$$\Omega_R = \frac{\hat{R}}{3H_0^2}. \quad (6.49)$$

In the case of the Einstein frame when we insert $\delta\hat{R}^3$ in $f(\hat{R})$ formula, the potential function (6.27) is substituted by:

$$\bar{U}(\hat{R}) = \frac{\hat{R}^2(\gamma + 2\delta\hat{R})}{\left(1 + 2\gamma\hat{R} + 3\delta\hat{R}^2\right)^2}. \quad (6.50)$$

In consequence, the Friedmann equation is modified to the form:

$$3\bar{H}^2 = \bar{\rho}_m(\hat{R}) + \frac{\bar{U}(\hat{R})}{2} + \Lambda = \frac{\hat{R}(2 + \gamma\hat{R})}{2\left(1 + 2\gamma\hat{R} + 3\delta\hat{R}^2\right)^2} - 3\Lambda. \quad (6.51)$$

A major qualitative change in the model occurs after inserting $\delta\hat{R}^3$ into the $f(\hat{R})$ formula in the Jordan frame. In this case, some additional singularities appear in the model. For example, in the case when γ parameter is positive and δ parameter is negative, an additional sewn freeze singularity and a typical sudden singularity appear during the evolution of the Universe.

6.5 Inflation in Starobinsky cosmological model in Palatini formalism

This section is based on [Eur.Phys.J. C78 \(2018\) no. 3, 249 \[63\]](#).

The main aim of the paper [63] is the analysis of inflation in the Starobinsky cosmological model in Palatini formalism within the Einstein frame. We found that inflation appears when matter is negligible with comparison to the $\bar{\rho}_\Phi = \frac{\bar{U}}{2}$. The evolution of the Universe during inflation in this model consists of four phases:

- In the first phase, matter is negligible and the density of matter grows due to the interaction between matter and the dark energy. In the inflation process, the production of matter disturbs inflation beginning from the point when matter can no longer be neglected. In consequence, in the first phase inflation becomes unstable and the second phase sets in.
- During the second phase, the effects of matter are not negligible and the density of matter grows further.
- In the third phase, the density of matter decreases but is still not negligible. During the second and third phases the process of inflationary behaviour of the Universe is terminated.

- In the fourth phase, the effects of matter become negligible and so inflation reappears. During that phase, the Universe follows the Λ CDM model.

In the first and fourth phase ρ_Φ has constant values:

$$\bar{\rho}_\Phi = \frac{1 - 16\gamma\Lambda + \sqrt{1 - 32\gamma\Lambda}}{16\gamma} \quad (6.52)$$

in the first phase and

$$\bar{\rho}_\Phi = \frac{1 - 16\gamma\Lambda - \sqrt{1 - 32\gamma\Lambda}}{16\gamma} \quad (6.53)$$

in the last phase.

If we assume that $N \in \langle 50, 60 \rangle$ [73], then γ parameter belongs to the interval $\langle 1.16 \times 10^{-69} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}, 1.67 \times 10^{-69} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \rangle$.

6.6 Einstein frame vs Jordan frame

This section is based on Phys.Rev. D97 (2018) 103524 [64].

In the paper [64], we consider differences in the Einstein and Jordan frames as applied to the Starobinsky cosmological model in Palatini formalism, finding that the topological structures of the phase space depend on the choice of the frame.

In the case of the Einstein frame, $\bar{H}$ and $\hat{R}$ were chosen as variables of the dynamical system. Eqs (6.24) and (6.26) can be then rewritten as a dynamical system:

$$\begin{aligned} \dot{\bar{H}}(\bar{t}) &= \frac{1}{6(1 + 2\gamma\hat{R}(\bar{t}))^2} \\ &\left(6\Lambda - 6\bar{H}(\bar{t})^2(1 + 2\gamma\hat{R}(\bar{t}))^2 + \hat{R}(\bar{t})(-1 + 24\gamma\Lambda + \gamma(1 + 24\gamma\Lambda)\hat{R}(\bar{t})) \right), \end{aligned} \quad (6.54)$$

$$\begin{aligned} \dot{\hat{R}}(\bar{t}) &= -\frac{3\bar{H}(\bar{t})(1 + 2\gamma\hat{R}(\bar{t}))}{(1 - \gamma\hat{R}(\bar{t}))} \\ &\left(4\Lambda + \hat{R}(\bar{t}) \left(-1 + 16\gamma\Lambda + 16\gamma^2\Lambda\hat{R}(\bar{t}) \right) \right), \end{aligned} \quad (6.55)$$

where $\dot{} \equiv \frac{d}{dt}$.

In the Jordan frame case, Eqs (6.10) and (6.11) can be rewritten as

$$\dot{H}(t) = -\frac{1}{6} \left[6(2\Lambda + H(t)^2) + \hat{R}(t) + \frac{18(1 + 8\gamma\Lambda)(\Lambda - H(t)^2)}{-1 - 12\gamma\Lambda + \gamma\hat{R}(t)} - \frac{18(1 + 8\gamma\Lambda)H(t)^2}{1 + 2\gamma\hat{R}(t)} \right], \quad (6.56)$$

$$\dot{\hat{R}}(t) = -3H(t)(\hat{R}(t) - 4\Lambda), \quad (6.57)$$

where $\dot{} \equiv \frac{d}{dt}$.

In this paper, we consider the behaviour of the trajectories in the phase portrait for both the frames when γ parameter has a positive value and find that the different types of singularities appear in the models. In the Jordan frame, the sewn freeze singularity appears beyond the Big Bang, while in the Einstein frame, the freeze singularity is substituted by the generalized sudden singularity and there is a bounce instead of the Big Bang. In consequence, the models in both the frames are not qualitatively equivalent to the Λ CDM model. The main result of this paper consists in showing that the models in the Jordan and Einstein frame are not equivalent due to the different types of singularities included in them.

Chapter 7

Conclusions

The main aim of the thesis is to point out that the running cosmological constant can solve two major problems of present cosmology. We consider many types of cosmological models, such as models with $f(R)$ gravity, diffusion in the dark sector, decaying dark energy, and with the different parametrization of the density of dark energy, in an attempt to address the cosmological constant problem and the coincidence problem.

The conclusions related to **decay of metastable dark energy** are:

- From statistical analysis, we get that the present value of density dark energy is independent of the model parameters α and E_0 .
- This model provides the mechanism of jumping from the initial value of dark energy $E_0 = 10^{120}$ to the present value of the cosmological constant.
- The oscillation of dark energy density occurs for $0 < \alpha < 0.4$.
- In the radioactive-like decay model of dark energy, for the late-time Universe ($t = 2T_0$), there are three different forms of decaying: radioactive, damping oscillating, and power-law type.
- In the radioactive-like decay model of dark energy, this type of decay dominates to $2.2 \times 10^4 T_0$.
- In the radioactive-like decay model of dark energy, after the radioactive type of decay the damping oscillating type sets in, which later is replaced by the power-law decay ($1/t^2$).
- In the model of $1/t^2$ type of decay of dark energy, there is interaction in the dark sector, which modifies the standard scale law of dark matter: $\rho_{\text{dm}} = \rho_{\text{dm},0} a(t)^{-3+\lambda(t)}$.
- For the early Universe, the $\lambda(t)$ function can be regarded as a constant.

- In the model of $1/t^2$ type of decay of dark energy, statistical analysis favours the negative value of α^2 parameter. In result, we get the decay of particles of dark matter.

The conclusions related to **the diffusion dark matter-dark energy interaction model** are:

- In this model, the standard scale law of energy density of dark matter is modified to $\rho_{\text{dm}} = \rho_{\text{dm},0} a(t)^{-3} + \gamma t a(t)^{-3}$.
- This model is free from the difficulties present in Alho et al.'s models with diffusion [53], since there are not any non-physical trajectories crossing the boundary set $\rho_{\text{m}} = 0$.
- This model involves a mechanism solving the coincidence problem.

The conclusions of this thesis related to **the dynamical system approach to the running Λ** are:

- In $\Lambda(H)$ cosmology, the Alcaniz and Lima's solution represents the scaling type $\rho_{\Lambda}(a) \sim \rho_{\text{m}}(a)$ [55].
- Trajectories within the phase space for which $\rho_{\Lambda}(a) \sim \rho_{\text{m}}(a)$ represent scaling solutions, which could solve the cosmic coincidence problem.
- The non-covariant $\Lambda(a)$ parametrization can be obtained from the covariant action for the scalar field as an emergent parametrization.
- We have found a strong evidence for tuning Λ term in $\Lambda(a)$ cosmology: $\Omega_{\Lambda,0} < 3.19 \times 10^{-7}$.

The conclusions related to **the Starobinsky cosmological model in the Palatini formalism** are:

- In the Einstein frame, there occurs interaction between dark matter and dark energy, as contrary to the Jordan frame.
- In the Jordan frame, new types of singularities appear, such as a sewn freeze singularity for the positive value of γ and a sewn typical sudden singularity for the negative value of γ .
- In the Jordan frame, the phase portrait is topologically equivalent to the phase portrait of the Λ CDM model for the positive γ parameter.
- In the Einstein frame, in the case when matter is negligible as compared to dark energy, inflation sets in when model parameter γ is close to zero ($\gamma \approx 1.16 \times 10^{-69} \text{ s}^2$).

- In the Einstein frame, for the positive value of γ , there is a generalized sudden singularity instead of the Big Bang.
- The phase portraits in the Einstein frame and the Jordan frame are not equivalent, which leads to the lack of physical equivalence of the model considered within these frames.
- The extension of Starobinsky model $f(R) = R + \gamma R^2 + \delta R^3$ in the Jordan frame can generate an additional sewn freeze singularity and a typical sudden singularity instead of the Big Bang.

One interesting phenomenon to appear often in the models considered in the thesis, is the interaction between dark energy and dark matter, which can be treated as an energy transfer in the dark sector. Its obvious effect is a modification of the standard scaling law of the energy density of dark matter ($\rho_{\text{dm}}(t) = \rho_{\text{dm},0}a(t)^{-3}$). From observations, in the model of $1/t^2$ type of decay of dark energy, we have a transfer energy from the dark matter to dark energy sector. Such a mechanism is at work in the Starobinsky model in the Palatini formalism within the Einstein frame, the diffusion dark matter-dark energy interaction model as well as in the decaying dark energy model.

A plausible solution of the problem of the cosmological constant is the model with decaying metastable dark energy, which offers the mechanism decreasing the value of dark energy in the early Universe ($\rho_{\text{de}} = 10^{120}$) to its present value. In this case, the transition consists in an onset of oscillation behaviour of the density of dark energy.

The phenomenon of inflation in the evolution of the Universe is given by the Starobinsky model in the Palatini formalism within the Einstein frame, which determines inflation when matter is negligible as compared to the density of dark energy. Its characteristic feature is the creation of matter throughout process of inflation.

The statistical analysis indicates that none of the three hypotheses put forward in the Introduction cannot be rejected. Accordingly, while abiding by the validity of the main thesis of this dissertation (see [Introduction](#)), we are not in a position to tell decisively which of the mechanisms considered here actually underlies changeability of the cosmological constant.

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Appendix

The Appendix contains all the eleven journal articles that have contributed to this dissertation in the respective formats of the journals in which they originally appeared.

Cosmology with decaying cosmological constant — exact solutions and model testing

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Abstract. We study dynamics of $\Lambda(t)$ cosmological models which are a natural generalization of the standard cosmological model (the Λ CDM model). We consider a class of models: the ones with a prescribed form of $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$. This type of a $\Lambda(t)$ parametrization is motivated by different cosmological approaches. We interpret the model with running Lambda ($\Lambda(t)$) as a special model of an interacting cosmology with the interaction term $-d\Lambda(t)/dt$ in which energy transfer is between dark matter and dark energy sectors. For the $\Lambda(t)$ cosmology with a prescribed form of $\Lambda(t)$ we have found the exact solution in the form of Bessel functions. Our model shows that fractional density of dark energy Ω_e is constant and close to zero during the early evolution of the universe.

We have also constrained the model parameters for this class of models using the astronomical data such as SNIa data, BAO, CMB, measurements of $H(z)$ and the Alcock-Paczyński test. In this context we formulate a simple criterion of variability of Λ with respect to t in terms of variability of the jerk or sign of estimator $(1 - \Omega_{m,0} - \Omega_{\Lambda,0})$. The case study of our model enable us to find an upper limit $\alpha^2 < 0.012$ (2σ C.L.) describing the variation from the cosmological constant while the Λ CDM model seems to be consistent with various data.

Keywords: modified gravity, dark matter theory, dark energy theory

ArXiv ePrint: [1507.02114](https://arxiv.org/abs/1507.02114)

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1 Introduction

The standard cosmological model describes the matter content of the Universe comprising the cold dust matter (baryonic matter and dark matter) which satisfies the equation of state for dust $p = 0$. In turn, dark energy is described in terms of an effective parameter (the cosmological constant) which should be treated as the best ‘economical’ (as only one parameter used to describe the whole dark sector) description of the cause that the Universe expansion accelerates in the current epoch.

The natural interpretation of the cosmological constant arises as an effect of quantum vacuum energy. Since this form of energy should be independent of the reference frame it must be proportional to the only ‘invariant’ second order metric tensor $g_{\mu\nu}$, i.e. $T_{\mu\nu} = \rho_{\text{vac}} g_{\mu\nu}$. If we include the conservation condition which for the cosmological model with the Robertson-Walker (R-W) symmetry assumes the form

$$\dot{\rho} = -3H(\rho + p), \quad (1.1)$$

then we obtain that $\rho_{\text{vac}} = \text{const} = \Lambda$ and $p_{\text{vac}} = -\Lambda$, $H = \frac{d}{dt}(\ln a)$ is the Hubble parameter, where ρ is total energy density, p is total pressure, an overdot denotes differentiation with respect to the cosmological time t ; we use a natural system of units in which $8\pi G = c = 1$.

If we interpret the cosmological constant Λ as a vacuum energy, then there is a difference between its today value required to explain observations of type Ia supernovae (SNIa) and the value of ρ_{vac} estimated from effective field theory. The former is smaller by a factor of 10^{-120} . This discrepancy is called the cosmological constant problem.

To achieve the conservation of energy-momentum tensor (divergence of energy-momentum tensor $T_{\mu\nu}$ is vanishing) different descriptions of dark energy sector have been proposed. In the simplest case the time cosmological term $\Lambda(t)$ is shifted to the right-hand side and treated as a source of gravity. Such an approach is called a $\Lambda(t)$ CDM cosmology.

In this paper we assume $\rho = \rho_{\text{m}} + \rho_{\text{de}}$ where ρ_{m} is a density of matter and ρ_{de} is the density of dark energy. We also assume $\rho_{\text{m}} = \rho_{\text{b}} + \rho_{\text{dm}}$, and $p_{\text{m}} = p_{\text{b}} + p_{\text{dm}}$, where $\rho_{\text{b}} = \rho_{\text{b},0}a(t)^{-3}$ and $p_{\text{b}} = 0$ are a density and a pressure of baryonic matter, ρ_{dm} and $p_{\text{dm}} = 0$ are a density and a pressure of dark matter, $a(t)$ is the scale factor. The state equation for dark energy is assumed as $p_{\text{de}} = -\rho_{\text{de}}$. In this case the conservation condition has the following form

$$\dot{\rho}_{\text{dm}} + 3H\rho_{\text{dm}} = Q, \quad (1.2)$$

$$\dot{\rho}_{\text{b}} + 3H\rho_{\text{b}} = 0 \quad (1.3)$$

and

$$\dot{\rho}_{\text{de}} = -Q, \quad (1.4)$$

where Q describes an interaction between dark matter and dark energy and this case is expressed by $Q = -\dot{\Lambda}$. The conservation condition can be rewritten in the form

$$\dot{\rho}_{\text{m}} = -3H\rho_{\text{m}} - \dot{\Lambda}, \quad (1.5)$$

where ρ_{m} and p_{m} are energy density and pressure of matter.

Pani et al. considered the energy-momentum tensor which ensures the covariantness of general relativity [1]. An alternative approach is to postulate the scalar field ϕ with the potential $V(\phi)$ for this model [2] which guarantee that model is covariance.

We consider a model with a parametrization of Λ following the rule

$$\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}, \quad (1.6)$$

where α^2 is a real constant; Λ_{bare} is a constant and $\rho_{\text{vac}} = \Lambda$. This model belongs to a larger class of cosmological models with interaction. In this case the interaction term is $Q = -d\Lambda/dt$. In this model the interaction is between dark matter and dark energy. This model belongs to a class of models so-called early constant dark energy during the matter dominating stage.

If we replace the cosmological time t by the Hubble scale time in eq. (1.6), then we obtain the $\Lambda(H)$ parametrization which is based on Lima et al. [3–5].

We estimate the value of the parameter α^2 as well as the other models parameters from available astronomical data. This class of models is compared with the standard cosmological model (the Λ CDM model).

Let us enumerate motivations for introducing form (1.6) of parametrization of dark energy.

1. The parametrization of dark energy can be derived from the quantum mechanics which describes how decaying false vacuum states changes in time. It can be shown that at the late time it can be identified as the cosmological constant which is time dependent and changes following the rule (1.6) and parameter α^2 is small and constitutes a leading term for long-term behaviour in power series of energy density of decaying vacuum [2, 6–8].
2. A new model of agegraphic dark energy [9, 10] based on some quantum arguments that the energy density of metric fluctuation of the Minkowski spacetime is proportional to $\frac{1}{t^2}$ and it also motivated Károlyházy uncertainty relation [11]. If we identify the time scale as the age of the Universe T , then we obtain that the agegraphic dark energy is $\rho_q \propto \frac{1}{T^2}$.
3. In the de Sitter universe there is a possibility to define in the framework of general relativity length and time scales $\Lambda(t) = \frac{3}{r_{\Lambda}^2(t)} = \frac{3}{c^2 t_{\Lambda}^2(t)}$ [12]. Otherwise, any cosmological length scale or time scale can determined the relation $\Lambda(t)$. Chen et al. [12] demonstrated how holographic [13, 14] and agegraphic dark energy conceptions can be unified in the framework of interacting cosmology in which the interacting term is $Q = -\dot{\rho}_{\Lambda}$. The variational approach to an interacting quintessence model was recently considered by Böhmer et al. [15].

4. Ringermacher and Mead [16] considered dark matter as a perfect fluid satisfying the equation of state $p = -\frac{1}{3}\rho$. The energy density of such fluid mimicking dark matter effects varies like $\frac{1}{t^2}$ rather than $\frac{\Omega_{\text{dark}}}{a^3}$ as in the standard cosmological model.
5. Haba has discussed recently cosmological models of general relativity in which a source of gravity (right-hand sides of the Einstein equations) is a sum of the energy-momentum of particles and the cosmological term describing a dissipation of energy-momentum. He obtained a cosmological model with the cosmological term decaying as $1/t^2$ [17, 18].

2 Exact solutions for $\Lambda(t)$ CDM cosmology with $\Lambda(t) = \Lambda + \frac{\alpha^2}{t^2}$

For the parametrization of $\Lambda(t)$ (1.6) it is possible to obtain exact solutions and discuss cosmological implications of this generalized standard cosmological model. We show that a deviation of this model from the Λ CDM model can be probed by a measurement of a jerk.

We start from the Friedmann first integral in the FRW cosmology with $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$, where t is the cosmological time and α^2 is either positive or negative,

$$3H(t)^2 = \rho_{\text{m}}(t) + \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2} \quad (2.1)$$

and the conservation condition

$$\dot{\rho}_{\text{m}}(t) = -3H(t)\rho_{\text{m}}(t) - \frac{d(\Lambda_{\text{bare}} + \frac{\alpha^2}{t^2})}{dt}. \quad (2.2)$$

Equation (2.1) can be rewritten in the dimensionless parameters

$$\Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{3H_0^2}, \quad \Omega_{\Lambda,0} = \frac{\Lambda_{\text{bare}}}{3H_0^2}, \quad \Omega_{\alpha,0} = \frac{\alpha^2}{3H_0^2 T_0^2}, \quad (2.3)$$

where T_0 is the present age of the Universe, i.e. $T_0 = \int_0^{T_0} dt = \int_0^{a_0} \frac{da}{Ha}$, and quantities labeled by index ‘0’ are defined at the present epoch for which $a_0 = 1$. Then equation (2.1) has the following form

$$\frac{H(t)^2}{H_0^2} = \Omega_{\text{m}}(t) + \Omega_{\Lambda,0} + \Omega_{\alpha,0} \frac{T_0^2}{t^2}, \quad (2.4)$$

where $\Omega_{\text{m}}(t) = \Omega_{\text{b},0}a(t)^{-3} + \Omega_{\text{dm},0}f(t)$ and $f(T_0) = 1$. At present, equation (2.4) is expressed by

$$1 = \Omega_{\text{m},0} + \Omega_{\Lambda,0} + \Omega_{\alpha,0}. \quad (2.5)$$

After differentiation of both sides of (2.1) with respect to t we obtain

$$6H(t)\dot{H}(t) = \dot{\rho}_{\text{m}}(t) + \frac{d(\Lambda_{\text{bare}} + \frac{\alpha^2}{t^2})}{dt}. \quad (2.6)$$

Equation (2.6) can be simplified with the help of (2.2). Then we obtain

$$\dot{H}(t) = -\frac{\rho_{\text{m}}(t)}{2}. \quad (2.7)$$

After substitution of (2.1) to (2.7) we obtain

$$\dot{H}(t) = \frac{1}{2} \left(\Lambda_{\text{bare}} + \frac{\alpha^2}{t^2} - 3H(t)^2 \right). \quad (2.8)$$

Equation (2.8) can be rewritten in the dimensionless parameters. Then we obtain

$$\dot{h}(t) = \frac{3H_0}{2} \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - h(t)^2 \right), \quad (2.9)$$

where $h(t) = \frac{H(t)}{H_0}$.

The general solution of equation (2.9) has the following form

$$h(t) = \frac{2}{3H_0} \frac{d}{dt} \log \left[\sqrt{t} \left(C_1 Y_n \left(\frac{3\sqrt{-\Omega_{\Lambda,0}}H_0}{2} t \right) + J_n \left(\frac{3\sqrt{-\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right], \quad (2.10)$$

where C_1 is a constant, an $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind, the index n of these functions is given in terms of $\Omega_{\alpha,0}$, H_0 and T_0 , $n = \frac{1}{2}\sqrt{1 + 9\Omega_{\alpha,0}T_0^2H_0^2}$. We can rewrite (2.10) to the form

$$h(t) = \frac{2}{3H_0} \frac{d}{dt} \log \left[\sqrt{t} \left(D_1 Y_n \left(\frac{3\sqrt{-\Omega_{\Lambda,0}}H_0}{2} t \right) + D_2 J_n \left(\frac{3\sqrt{-\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right]. \quad (2.11)$$

For the correspondence with the Λ CDM model ($\alpha^2 = 0$) we choose $D_1 = 0$. Then solution (2.11) is given by the formula

$$h(t) = \frac{2}{3H_0} \frac{d}{dt} \log \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right], \quad (2.12)$$

where $I_n(x)$ is the modified Bessel function. Solution (2.12) can be rewritten to the following form

$$h(t) = \frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)}. \quad (2.13)$$

Because $H(t) = \frac{d}{dt} \ln a$, then it is easy to obtain the scale factor from (2.12) in the form

$$a(t) = C_2 \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right]^{\frac{2}{3}}. \quad (2.14)$$

The diagram of $a(t)$ is presented in figure 1.

We obtain a formula for $\rho_m(t)$ from (2.7), (2.9) and (2.12)

$$\rho_m = -3H_0^2 \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - \left(\frac{2}{3H_0} \frac{d}{dt} \log \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right] \right)^2 \right) \quad (2.15)$$

or in an equivalent form

$$\rho_m = -3H_0^2 \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right)} \right)^2 \right). \quad (2.16)$$

The diagrams of $\rho_m(t)$ and $\rho_m(a)$ are presented in figure 2 and 3. In comparison, the diagram of $\rho_{de}(t)$ is demonstrated in figure 4. The dark matter is expressed by

$$\rho_{dm} = \rho_m - \rho_{b,0}a^{-3}. \quad (2.17)$$

If we use formulas (2.14) and (2.16) in equation (2.17) then we get

$$\rho_{dm} = -3H_0^2 \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2}t\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2}t\right)} \right)^2 \right) - \rho_{b,0}C_2^{-3} \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2}t \right) \right) \right]^{-2}. \quad (2.18)$$

The Λ CDM model can be obtained in the limit $\Omega_{\alpha,0} = 0$. Then index $n = \frac{1}{2}$ and $I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x)$. Finally the solution (2.12) reduces to

$$h(t) = \frac{2}{3H_0} \frac{d}{dt} \log \left[\sinh \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2}t \right) \right]. \quad (2.19)$$

Equation (2.19) can be rewritten to the equivalent form

$$h(t) = \sqrt{\Omega_{\Lambda,0}} \coth \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2}t \right). \quad (2.20)$$

In the special case the solution of (2.9) for $\Omega_{\Lambda,0} = 0$ has the following form

$$h(t) = \frac{1}{3Ht} \left[1 + \sqrt{1 + 9H_0^2\Omega_{\alpha,0}T_0^2} \left(1 - \frac{C_1}{t\sqrt{1+9H_0^2\Omega_{\alpha,0}T_0^2} + C_1} \right) \right]. \quad (2.21)$$

For the correspondence with the CDM model we choose $C_1 = 0$. Then equation (2.21) is simplified to

$$h(t) = \frac{1}{3H_0t} \left[1 + \sqrt{1 + 9H_0^2\Omega_{\alpha,0}T_0^2} \right]. \quad (2.22)$$

From equation (2.22) we can obtain an expression for the scale factor

$$a(t) = C_2 t^{\frac{1}{3}} \left(1 + \sqrt{1 + 9H_0^2\Omega_{\alpha,0}T_0^2} \right). \quad (2.23)$$

If we know an exact solution for the scale factor $a(t)$ it will be possible to calculate a dimensionless parameter called a jerk related with a third order time derivative of the scale factor

$$j = \frac{1}{H(t)^3 a(t)} \left[\frac{d^3 a(t)}{dt^3} \right]. \quad (2.24)$$

After some calculations we obtain the third order time derivative of the scale factor in the form

$$\ddot{a} = \frac{3H_0^2\dot{a}}{2} \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - h^2 \right) + \frac{3H_0^2a}{2} \left(-\frac{2\Omega_{\alpha,0}T_0^2}{t^3} - 2h\dot{h} \right) + H_0^2\dot{a}h^2 + 2H_0^2ah\dot{h}. \quad (2.25)$$

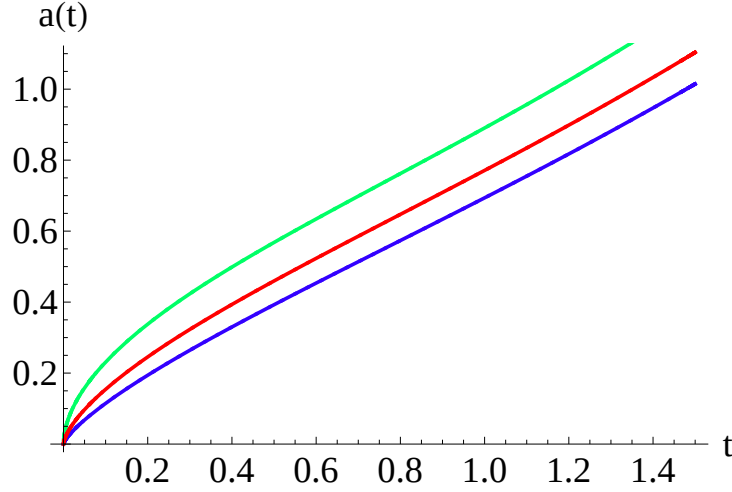


Figure 1. Diagram of the scale factor $a(t)$ for three cases. The top function is for $\alpha^2 = -0.2$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = 0.2$. We assume that $H_0 = 68.27 \text{ km}/(\text{s Mpc})$ and $\Omega_m = 0.35$. Time t is expressed in a unit $(100 \text{ s Mpc}/\text{km})$.

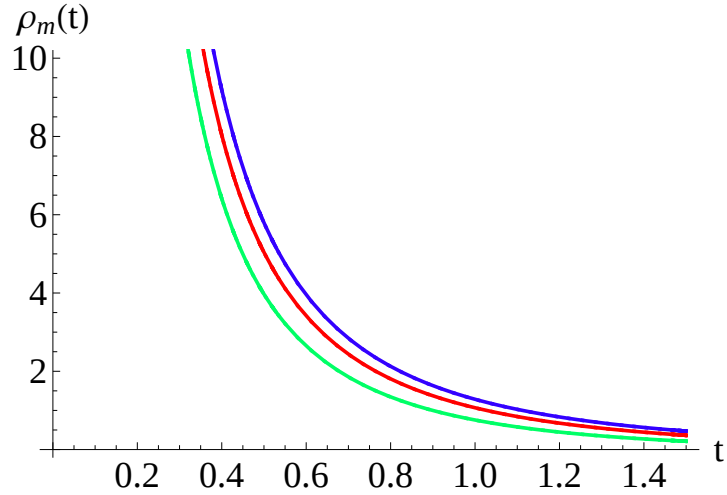


Figure 2. Diagram of the energy density $\rho_m(t)$ for three cases. The top function is for $\alpha^2 = 0.2$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = -0.2$. We assume that $H_0 = 68.27 \text{ km}/(\text{s Mpc})$ and $\Omega_{m,0} = 0.35$. Time t is expressed in unit $(100 \text{ s Mpc}/\text{km})$. We assume $8\pi G = 1$ and we choose for ρ_m a unit $(\text{km}/(100 \text{ s Mpc}))^2$.

A substitution of the expression $h(t)$ from (2.9) gives us the exact formula for the jerk as a function of the cosmological time t

$$j(t) = 1 - \frac{3\Omega_{\alpha,0}T_0^2}{H_0t^3} \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)} \right)^{-3}. \quad (2.26)$$

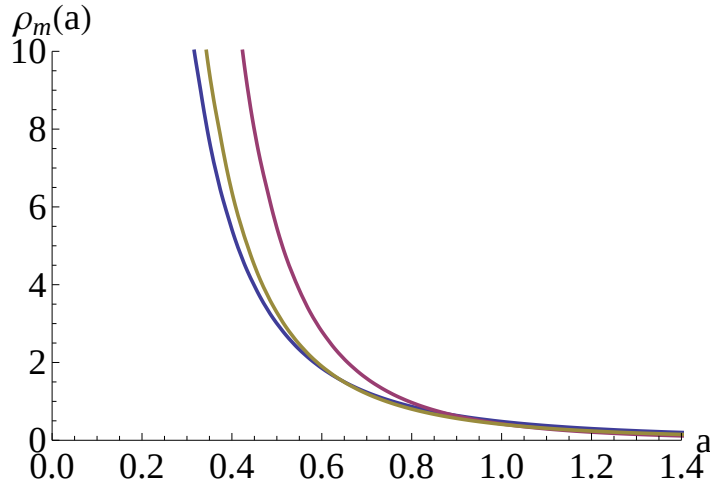


Figure 3. Diagram of the energy density $\rho_m(a)$ for three cases. The top function is for $\alpha^2 = -0.2$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = 0.2$. We assume that $H_0 = 68.27 \text{ km}/(\text{s Mpc})$ and $\Omega_{m,0} = 0.35$. We assume $8\pi G = 1$ and we choose for ρ_m a unit $(\text{km}/(100 \text{ s Mpc}))^2$.

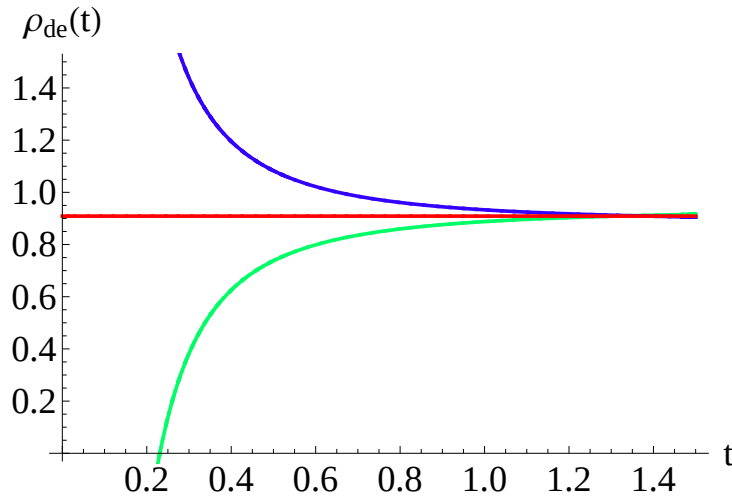


Figure 4. Diagram of the energy density $\rho_{de}(t)$ for three cases. The top function is for $\alpha^2 = 0.05$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = -0.05$. We assume that $H_0 = 68.27 \text{ km}/(\text{s Mpc})$ and $\Omega_{m,0} = 0.35$. Time t is expressed in unit $(100 \text{ s Mpc}/\text{km})$. We assume $8\pi G = 1$ and we choose for ρ_{de} a unit $(\text{km}/(100 \text{ s Mpc}))^2$.

The jerk calculated for $t = T_0$, i.e. for the present epoch is given by formula

$$j_0 = 1 - \frac{3\Omega_{\alpha,0}}{H_0 T_0}. \quad (2.27)$$

The diagram of $j(z)$ is presented in figure 5. One can see the jerk can be treated as a tool for detection the variability of dark energy.

From the exact solution (2.26) one can see that the deviation of the generalized model from the Λ CDM model is given by time dependent contribution to the jerk because for the

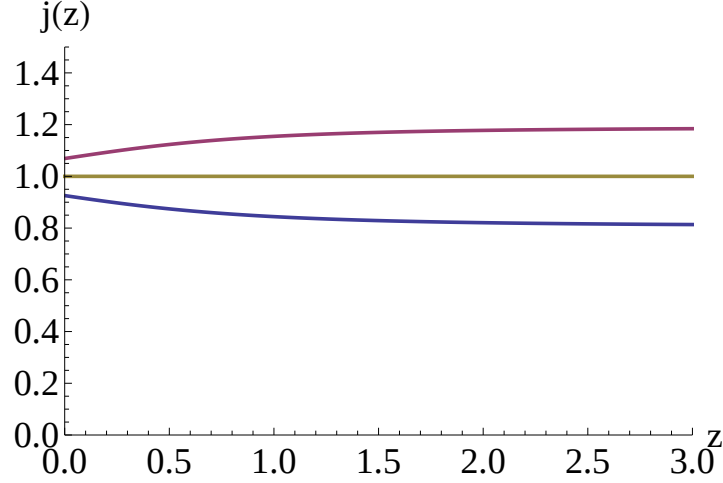


Figure 5. Diagram of $j(z)$ for $H_0 = 68.27$ km/(s Mpc) and $\Omega_{m,0} = 0.35$. The top function is for $\alpha^2 = -0.05$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = 0.05$.

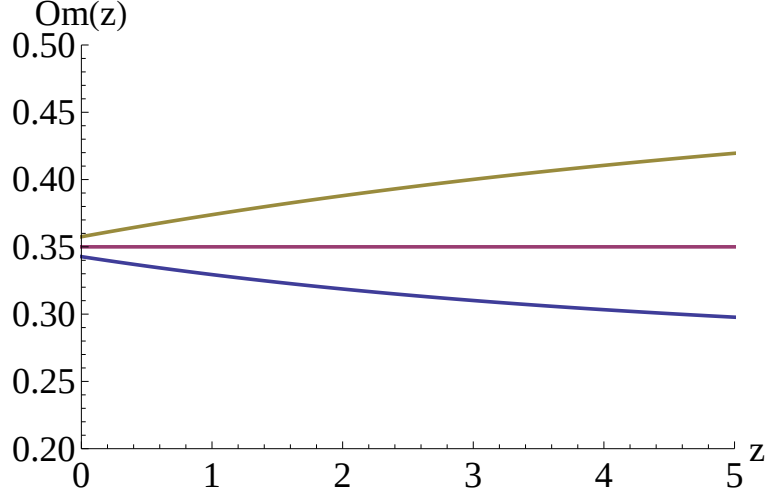


Figure 6. Diagram of $Om(z)$ for $H_0 = 68.27$ km/(s Mpc) and $\Omega_{m,0} = 0.35$. The top function is for $\alpha^2 = -0.05$, the middle function represents the Λ CDM model and the bottom function is for $\alpha^2 = 0.05$.

Λ CDM model it is equal one. Therefore if we can detect from the astronomical observations the time variability of the jerk it will be a simple diagnostic of decaying vacuum. If $\Omega_{\alpha,0}$ is non-zero this means that $\Omega_{m,0} + \Omega_{\Lambda,0} < 1$. Note that

$$\Omega_{\alpha,0} = 1 - \Omega_{m,0} - \Omega_{\Lambda,0} = \frac{\alpha^2}{3H_0^2 T_0^2}. \quad (2.28)$$

Because $T_0 \leq \frac{1}{H_0}$, i.e. $H_0^2 T_0^2 \leq 1$ and $\alpha^2 = 3H_0^2 T_0^2 \Omega_{\alpha,0}$, i.e.

$$\frac{\alpha^2}{3} \leq \Omega_{\alpha,0} \quad (2.29)$$

and from the estimation of $\Omega_{\alpha,0}$ one can obtain an upper limit on $\frac{\alpha^2}{3}$.

From the value of the jerk for the current epoch (see formula (2.27)) there comes also the limit values of the jerk

$$1 - 3\Omega_{\alpha,0} \leq j_0 \leq 1, \quad \text{for } \alpha^2 > 0 \quad (2.30)$$

and

$$1 \leq j_0 \leq 1 - 3\Omega_{\alpha,0}, \quad \text{for } \alpha^2 < 0. \quad (2.31)$$

Sahni et al. [19–21] proposed in the context of testing and comparison of alternatives for the Λ CDM model $Om(z)$ diagnostic test

$$Om(z) = \frac{h^2(x) - 1}{x^3 - 1}, \quad (2.32)$$

where $x = 1 + z$. While this parameter is constant for the Λ CDM model, $Om(x) = \Omega_{m,0}$ for any deviation from zero would discard the Λ CDM model for the description of the cosmic evolution of the current Universe for low z . But $Om(z)$ diagnostic test is not constant for high z because Λ CDM model should respect radiation for high z . Note that if the radiation density is included then the behavior of $Om(z)$ will be different for the case of matter and cosmological constant. For high redshift the contribution from radiation density will dominate. In our paper matter and energy density is present at very beginning and effect of radiation density is not included because of complexity of analytical calculations. Therefore our comparison of a jerk and $Om(z)$ is not valid for high redshift. Let us note that in our case $Om(x)$ is not constant and evolves with the cosmological time as

$$Om(t) = \frac{\left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0 t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0 t}{2}\right)} \right)^2 - 1}{\left(\left[\sqrt{T_0} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} T_0 \right) \right) \right]^2 \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0}{2} t \right) \right) \right]^{-2} \right) - 1}. \quad (2.33)$$

The diagram of $Om(z)$ is presented in figure 6. From comparison of figures 5 and 6 one can observe two alternative ways of the detection of the variability of dark energy with respect to time.

3 Dynamics of the generalized Λ CDM model

For a deeper analysis of dynamics it is useful to investigate how exact solutions (trajectories) depend on initial conditions. The natural language for such a discussion is the phase space which a space of all solutions for all admissible initial conditions.

Let us consider now the dynamics of the model under consideration as a dynamical system. In this paper we consider the case of a positive cosmological constant $\Lambda_{\text{bare}} > 0$ and strictly positive energy density of matter $\rho_m > 0$. The first step in a formulation of the dynamics in terms of a dynamical system is a choice of the state variables. Assume state variables are as follows

$$x^2 = \frac{\rho_m}{3H^2}, \quad y^2 = \frac{\Lambda_{\text{bare}}}{3H^2}, \quad z^2 = \frac{1}{3H^2 t^2}. \quad (3.1)$$

We also choose a new time variable τ : $\tau = \ln a$; let a prime denotes the differentiation with respect to the Hubble time τ . Then, we differentiate with respect to τ the expressions for x^2 , y^2 and z^2 in (3.1) and obtain

$$2xx' = \frac{2x\dot{x}}{H} = \frac{\dot{\rho}_m}{3H^3} - \frac{2\rho_m\dot{H}}{3H^4}, \quad (3.2)$$

$$2yy' = \frac{2y\dot{y}}{H} = -\frac{2\Lambda_{\text{bare}}\dot{H}}{3H^4}, \quad (3.3)$$

$$2zz' = \frac{2z\dot{z}}{H} = -\frac{2}{3H^3t^3} - \frac{2\dot{H}}{3H^4t^2}. \quad (3.4)$$

Due to relation (2.2) the expression for $\dot{\rho}_m$ can be replaced by $-3H\rho_m + \frac{2\alpha^2}{t^3}$ and then with the help of (2.7) $\dot{H}$ can be replaced by $-\frac{\rho_m}{2}$. As a consequence we obtain the set of equations

$$2xx' = -\frac{\rho_m}{H^2} + \frac{2\alpha^2}{3H^3t^3} + \frac{2\rho_m^2}{6H^4}, \quad (3.5)$$

$$2yy' = \frac{2\Lambda_{\text{bare}}\rho_m}{6H^4}, \quad (3.6)$$

$$2zz' = -\frac{2}{3H^3t^3} + \frac{2\rho_m}{6H^4t^2}. \quad (3.7)$$

After returning to the original variables x , y , z we obtain the system

$$x' = -\frac{3}{2}x + \sqrt{3}\alpha^2\frac{z^3}{x} + \frac{3}{2}x^3, \quad (3.8)$$

$$y' = \frac{3}{2}x^2y, \quad (3.9)$$

$$z' = -\sqrt{3}z^2 + \frac{3}{2}x^2z. \quad (3.10)$$

Note that the right-hand side of (3.8) is not defined on the plane $x = 0$. All state variables are constrained by the condition $x^2 + y^2 + z^2 = 1$, i.e. phase space is a surface of a three-dimensional sphere.

We regularize system (3.8)–(3.10) in such a way that its right-hand sides are in a polynomial form. For this purpose we introduce new state variables X, Y, Z : $X = x^2, Y = y, Z = z$. Note that transformation $x \rightarrow X$ is not a diffeomorphism on the line $x = 0$. Then system (3.8)–(3.10) represents the dynamical system with smooth right-hand side functions, namely

$$X' = -3X + 3X^2 + 2\sqrt{3}\alpha^2Z^3, \quad (3.11)$$

$$Y' = \frac{3}{2}XY, \quad (3.12)$$

$$Z' = -\sqrt{3}Z^2 + \frac{3}{2}ZX, \quad (3.13)$$

where the phase space is restricted by the condition

$$X + Y^2 + \alpha^2Z^2 = 1. \quad (3.14)$$

The critical points of the system (3.11)–(3.13), their type and dominant contribution in the energy constraint $X + Y^2 + \alpha^2Z^2 = 1$ are presented in table 1.

No	position of critical point	type	dominant contribution in (3.14)
1	$X_0 = 0, Y_0 = 1, Z_0 = 0$	stable node	Λ dominant state in the future (de Sitter)
2	$X_0 = 1, Y_0 = 0, Z_0 = 0$	unstable node	matter dominant state in the past (Einstein-de Sitter)
3	$X_0 = \frac{2}{3\alpha^2}(-1 + \sqrt{1 + 3\alpha^2})$ $Y_0 = 0$ $Z_0 = \frac{1}{\sqrt{3}\alpha^2}(-1 + \sqrt{1 + 3\alpha^2})$	saddle	both decaying vacuum effects and matter effects are dominating in the past

Table 1. The critical points of the system (3.11)–(3.13), their type and dominant contribution in the energy constraint $X + Y^2 + \alpha^2 Z^2 = 1$.

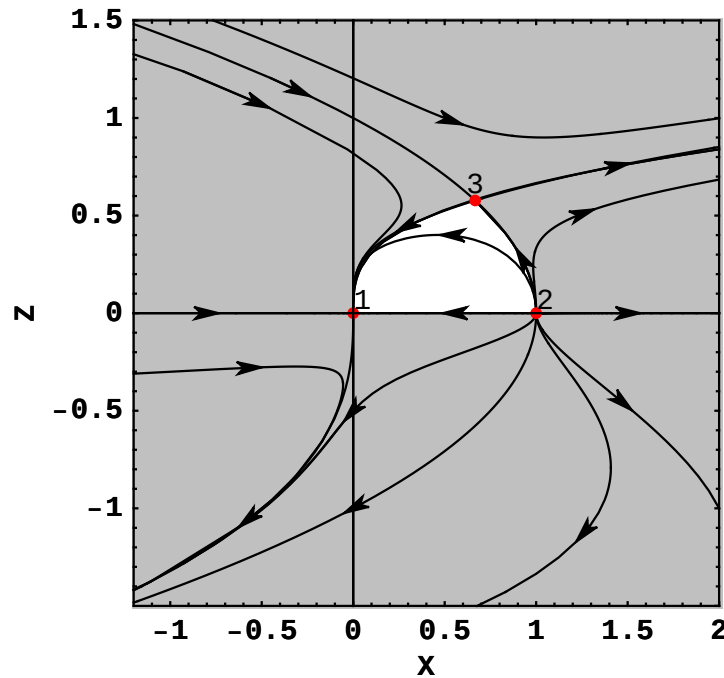


Figure 7. The phase portrait for dynamical system (3.11)–(3.13) for real α and $H > 0$. The grey domain represents non-physical solutions. The phase portrait is organized by three critical points: the de Sitter universe represented by a stable node (point 1), the Einstein-de Sitter universe represented by an unstable node (point 2) and the generalization of Einstein-de Sitter represented by a saddle (point 3).

System (3.11)–(3.13) is three-dimensional but it has invariant submanifolds $Y = 0$ and $Z = 0$. The behavior of trajectories on the invariant submanifold $Y = 0$ describes fully the global dynamic. The phase portraits on the plane (X, Z) are presented in figures 7 and 8. Because of the constraint $Y^2 = 1 - X - \alpha^2 Z^2$ the physical trajectories lie in the region $Y^2 \geq 0$. Beyond this region is situated a non-physical region (the shaded region in figures 7 and 8). The boundary of the physical region is determined by a parabola $X = 1 - \alpha^2 Z^2$ and a line $X = 0$.

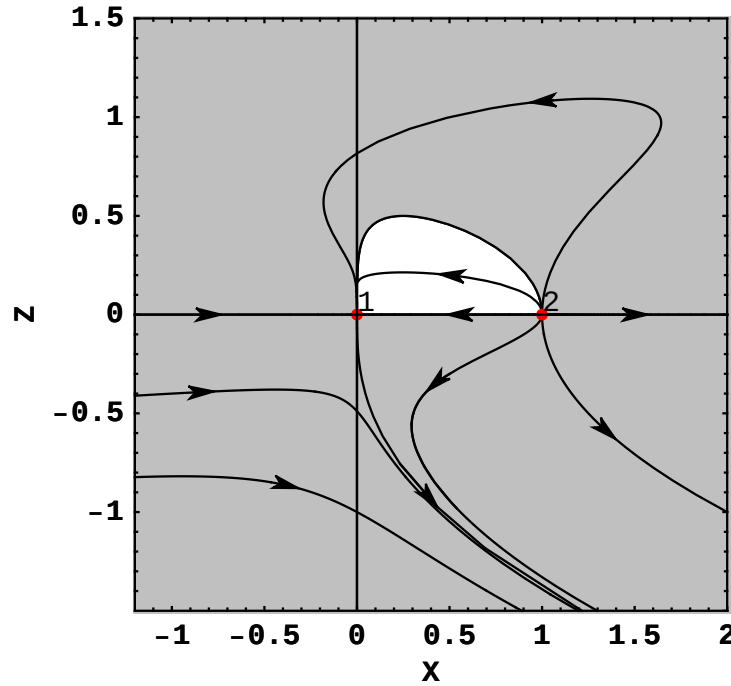


Figure 8. The phase portrait for dynamical system (3.11)–(3.13) for imaginary α and $H > 0$. The grey domain represents non-physical solutions. The phase portrait is organized by two critical points: the de Sitter universe represented by a stable node (point 1) and the Einstein-de Sitter universe represented by an unstable node (point 2).

All critical points lie on this boundary. From the physical point of view they represent asymptotic states of system (3.11)–(3.13) which are started at $\tau \rightarrow -\infty$ and reach the critical points at $\tau = +\infty$. The critical point marked as (1) represents the de Sitter universe and is a stable node. It is a global attractor for trajectories from its neighborhood. The critical point (2) is an unstable node and it represents the CDM model.

The novelty on the phase portrait is the presence of critical point (3). It is of saddle type. At this critical point $\rho_m = 2 \frac{\sqrt{1+3\alpha^2}-1}{\alpha^2} H^2$ and $Ht = \frac{\alpha^2}{\sqrt{1+3\alpha^2}-1}$ i.e., it represents a universe dominated by both decaying vacuum and matter.

The de Sitter state (critical point 1) is connected by an outcoming separatrix with the saddle (point 3). The second separatrix gets in the saddle (point 3) and gets out from the Einstein-de Sitter state (point 2). The other trajectories in a non-shaded region start from the Einstein-de Sitter state and finish in the de Sitter state. In all cases the time flows from $\tau = -\infty$ ($a = 0$) to $\tau = +\infty$ ($a = +\infty$). At the critical point (3) the decaying Λ and matter play important role and cannot be neglected. At this critical point scale factor a and ρ_m behaves like: $a \propto t^{\frac{\alpha^2}{\sqrt{1+3\alpha^2}-1}}$, $H = \frac{\alpha^2}{\sqrt{1+3\alpha^2}-1} t^{-1}$ and $\rho_m = \frac{2\alpha^2}{\sqrt{1+3\alpha^2}-1} t^{-2}$.

This critical point exist only if $\alpha \neq 0$. If $\alpha \rightarrow 0$ then it coincides with the CDM universe. This critical point represents a new generalized CDM model in which $\rho_m = \frac{4}{3} t^{-2}$ and $a(t) \propto t^{\frac{2}{3}}$ in the early universe.

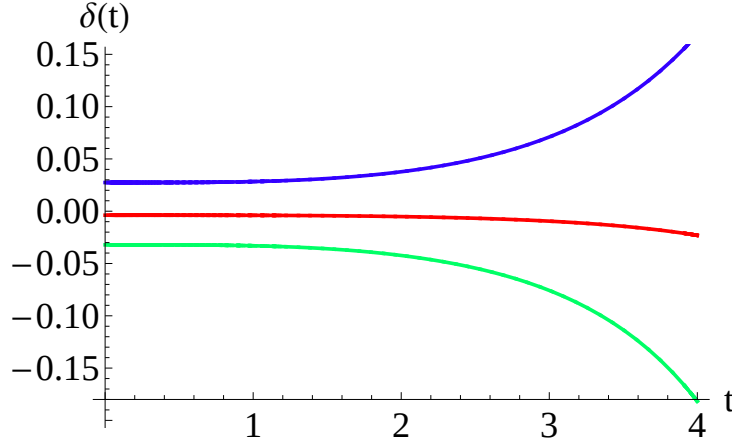


Figure 9. Diagram of $\delta(t)$. The top function represents the evolution of δ assuming the best fit of $H_0 = 68.27$ km/(s Mpc) and right boundary values of 95% C.L. for $\Omega_{m,0} = 0.2416$, $\Omega_\alpha = 0.0039$, the middle function represents the best fit for all parameters. The top function represents the evolution of δ assuming the best of $H_0 = 68.27$ km/(s Mpc) and the left boundary values of 95% C.L. for $\Omega_{m,0} = 0.3542$, $\Omega_{\alpha,0} = -0.0056$. On the t -axis we use a unit times 100 Mpc s/km. See also table 2.

If the function

$$\delta = -\frac{d\Lambda(t)}{H\rho_m} = \frac{2\alpha^2}{H\rho_m} = -\frac{\alpha^2}{\frac{3}{2}t^3H_0^3 \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)} \right)} \times \left(\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}}H_0t}{2}\right)} \right)^2 \right)^{-1} \quad (3.15)$$

is slowly changing then

$$\rho_m = \rho_{m,0}a^{-3+\delta(t)}. \quad (3.16)$$

Let $\delta(t) = \delta = \text{const}$ then $\rho_m = \rho_{m,0}a^{-3+\delta}$. If $t \rightarrow \infty$ then $\delta(t) \rightarrow \text{const}$. At the critical point (3) $\delta(t) = \delta = (\sqrt{1+3\alpha^2}-1)^2/\alpha^2$. The diagram of $\delta(t)$ is presented in figure 9.

Some interesting interpretation of our postulated $\Lambda(t)$ relation can be derive if we apply Starobinsky's argument [22] that ρ_ϕ after some averaging over time in the interval $\Delta t \gg m^{-1}$ assumes the following form in the quintessence epoch

$$\rho_\phi = V_0 + Aa^{-3}. \quad (3.17)$$

Therefore in the matter dominating phase we obtain the $\Lambda(t)$ parametrization (1.6). Finally the model involved belongs to the class of models with so called early dark energy constant in which $\Omega_{de} = \text{const} \equiv \Omega_e$ during the matter dominated stage (the same refers to the radiation dominated stage, too, but with a different value of Ω_e).

If $\delta \ll 1$ for the fractional density of dark energy Ω_e [23, 24] it assumes in the intermediate domain of the universe the following form

$$1 - \Omega_e(a(t)) = \frac{\Omega_{m,0}T_0^2t^{-2}}{\Omega_{m,0}T_0^2t^{-2} + \Omega_{\Lambda,0} + \Omega_\alpha T_0^2t^{-2}}. \quad (3.18)$$

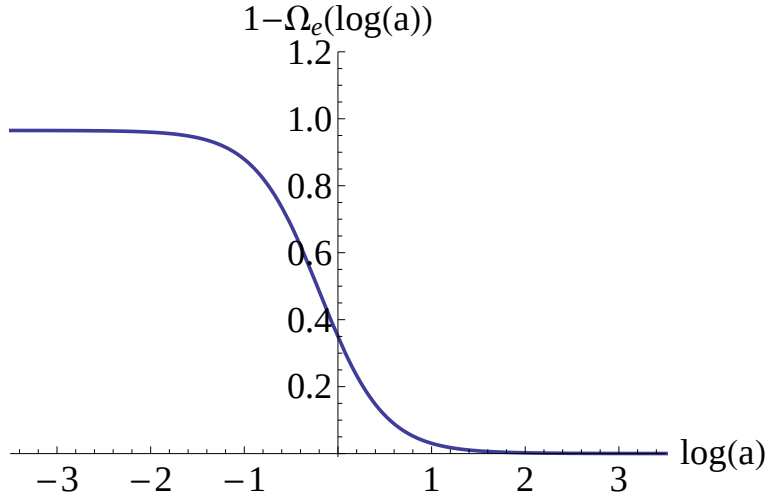


Figure 10. Diagram of $1 - \Omega_e(\log(a))$ which is a share of energy density of matter in the total energy density. The function represents the evolution of $1 - \Omega_e$ for $H_0 = 68.27 \text{ km}/(\text{s Mpc})$, $\Omega_{m,0} = 0.35$ and $\Omega_{\alpha,0} = 0.05$. The present epoch is at $\log(a) = 0$. For negative values of $\log(a)$ we have past evolution of $1 - \Omega_e(\log(a))$ with a constant phase of the fractional density of dark energy Ω_e at the early universe (Ω_e is small and close to zero). For positive values of $\log(a)$ we have future evolution of $1 - \Omega_e(\log(a))$ with a constant phase of the fractional density of dark energy Ω_e at the late universe (Ω_e is big and close to one). Between these two constant phases there is an intermediate phase of changing $1 - \Omega_e(\log(a))$ in which we are living.

In the early universe this value is constant

$$\Omega_e = \frac{\Omega_{\alpha,0}}{\Omega_{m,0} + \Omega_{\alpha,0}}. \quad (3.19)$$

Therefore for a small value of $\Omega_{\alpha,0}$, Ω_e is obtained as $\Omega_e = \frac{\Omega_{\alpha,0}}{\Omega_{m,0}}$.

Hojjati et al. [25] found the fraction in total density contributed by early dark energy which is approximately equivalent to Ω_e .

In our model the exact form of $\Omega_e(t)$ is

$$\Omega_e(t) = \frac{\Omega_{\Lambda,0} + \frac{\Omega_{\alpha,0} T_0^2}{t^2}}{\left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}} H_0 t}{2}\right)} \right)^2}. \quad (3.20)$$

The evolution of fractional density of dark energy Ω_e has a shape of the “logistic” curve. For convenience, the diagram $1 - \Omega_e(\log(a))$ is presented in figure 10.

There is another interesting approach to running cosmologies, proposed by Starobinsky [26]. In this approach (the bottom up), the universe in the quintessence epoch is described by a scalar field minimally coupled to gravity with some self-interacting potential. He proposed the reconstruction of this potential from the evolution of the scalar perturbation (or also luminosity function) in dust like matter component.

Masso et al. [27] discussed some aspects of contribution to the dark energy density of coherent scalar field oscillation in the potential. They obtained using the analytical method

of adiabatic invariance that for a quadratic potential the energy density ρ_ϕ evolves as a^{-3} and a quartic potential $V(\phi) \approx \phi^4$ evolves like for the radiation matter a^{-4} . Therefore if we add Λ_{bare} to the potential in a matter (or radiation) dominating universe $\rho_\phi = \Lambda(t)$.

4 Statistical analysis of the model

In this section we present a statistical analysis of the model parameters using the SNIa, BAO, CMB observations, measurements of $H(z)$ and the Alcock-Paczyński test.

First, we use the Union 2.1 sample of 580 supernovae [28]. For the SNIa data we have the following likelihood function

$$\ln L_{\text{SNIa}} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{\mu_i^{\text{obs}} - \mu_i^{\text{th}}}{\sigma_i} \right)^2, \quad (4.1)$$

where the summing is over the SNIa sample; the distance modulus $\mu^{\text{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of SNIa stars) and $\mu^{\text{th}} = 5 \log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$ and σ is the uncertainties.

We use the BAO (baryon acoustic oscillation) data which were taken from the Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset which consists of 893 319 galaxies [29]. The likelihood function is given by

$$\ln L_{\text{BAO}} = -\frac{1}{2} \frac{\left(\frac{r_s(z_d)}{D_V(z)} - d(z) \right)^2}{\sigma^2}, \quad (4.2)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch and $z = 0.275$, $d(z) = 0.1390$, $\sigma = 0.0037$ [30].

The next likelihood function encompasses the Planck observations of cosmic microwave background (CMB) radiation [31], the information on lensing from the Planck and low- ℓ polarization from the WMAP and has the form

$$\ln L_{\text{CMB+lensing+WP}} = -\frac{1}{2} \sum_{ij} (x_i^{\text{th}} - x_i^{\text{obs}}) \mathbb{C}^{-1} (x^{\text{th}} - x^{\text{obs}}), \quad (4.3)$$

where $\mathbb{C}$ is the covariance matrix with the errors, x is a vector of the acoustic scale l_A , the shift parameter R and $\Omega_b h^2$ where

$$l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')}, \quad (4.4)$$

$$R = \sqrt{\Omega_{m,0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}, \quad (4.5)$$

where z^* is the recombination redshift and r_s is the sound horizon.

The idea of the Alcock-Paczyński test is the comparison of the radial and tangential size of an object, which is isotropic in the correct choice of model [32, 33]. The likelihood function is independent of the parameter H_0 and has the following form

$$\ln L_{\text{AP}} = -\frac{1}{2} \sum_i \frac{(AP^{\text{th}}(z_i) - AP^{\text{obs}}(z_i))^2}{\sigma^2}, \quad (4.6)$$

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data [34–42].

At the end it is also valuable to add the constraints on the Hubble parameter, i.e. $H(z=0) \equiv H_0$.

Data of $H(z)$ for samples of different galaxies were also used [43–45].

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2. \quad (4.7)$$

The final likelihood function for the observational Hubble function is

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{CMB+lensing+WP}} L_{\text{AP}} L_{H(z)}. \quad (4.8)$$

To estimate the model parameters we use our own code CosmoDarkBox implementing the Metropolis-Hastings algorithm [46, 47]. We use the dynamical system formulation of model to obtain the likelihood function [30, 48].

We use observation data of 580 supernovae of type Ia, selected subsets of the data points of Hubble function, the measurements of BAO from SDSS DR7. We also use data for the application of the Alcock-Paczyński test 18 observational points. At last, we estimated model parameters with CMB data from Planck, low- ℓ polarization from WMAP and lensing from Planck. To estimate the model parameters we chose interval (64.00, 74.00) for H_0 and (0.21, 0.37) for $\Omega_{\text{m},0}$. The values of estimated parameters for α^2 from the interval $(-0.05, 0.05)$ are shown in table 2 and for positive α^2 from the interval $(0.00, 0.05)$ are shown in table 3. The best fit for model with α^2 from the interval $(-0.05, 0.05)$ is in the part of likelihood function where α^2 is negative.

If it is chosen the lower limit of the interval of α^2 larger than the value of the best fit then the best fit of the model for the new interval is equal the value of the lower limit of this interval. A consequence is the lower limit of the error is equal zero for α^2 . So the specific values of the best fit and errors of $\Omega_{\alpha,0}$, α^2 and j_0 in table 3 are a result of the choice of limits of the interval of α^2 .

To illustrate the results of statistical analysis the diagrams of PDF are shown in figures 11, 12 and 13. In turn figures 14 and 15 shown the likelihood function with 68% and 95% confidence level projection on the $(\Omega_{\alpha,0}, \Omega_{\text{m},0})$ plane and the $(\Omega_{\alpha,0}, H_0)$ plane, respectively.

5 Conclusion

The aim of the paper was to study the dynamics of the emerging $\Lambda(t)$ CDM cosmological models. In the study of dynamics we find exact solutions and use dynamical system methods for the analysis of dependence of solutions on initial conditions. In the latter evolutionary paths of cosmological model are represented by trajectories in the phase space. Due to geometrical visualization of dynamics we have the space of all solutions and can discuss their stability. We are looking for such trajectories for which the Λ CDM model is a global attractor in the phase space.

We study in details dynamics of cosmological model with the prescribed form of $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$, where Λ_{bare} is a positive constant and α^2 is either positive or negative. We calculate exact solutions for the scale factor and subsequently calculate the jerk. It is demonstrated that this parameter is time dependent if and only if the effects of time contribution to $\Lambda(t)$ are non-zero. We propose the measurement of the jerk as a diagnostic of decaying Λ , i.e. $\dot{\Lambda} < 0$. Due to analysis of dynamics in the phase space we have found an interesting solution

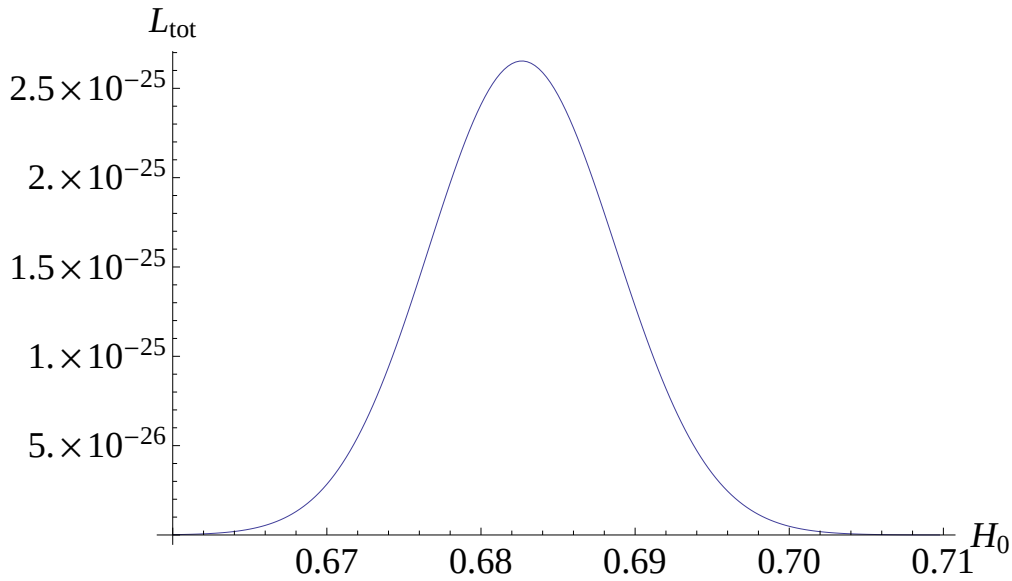


Figure 11. Diagram of PDF for parameter H_0 in units $\text{km}/(100 \text{ s Mpc})$ obtained as an intersection of a likelihood function. Two planes of intersection likelihood function are $\Omega_{\text{m},0} = 0.2938$ and $\Omega_{\alpha,0} = -0.0006$.

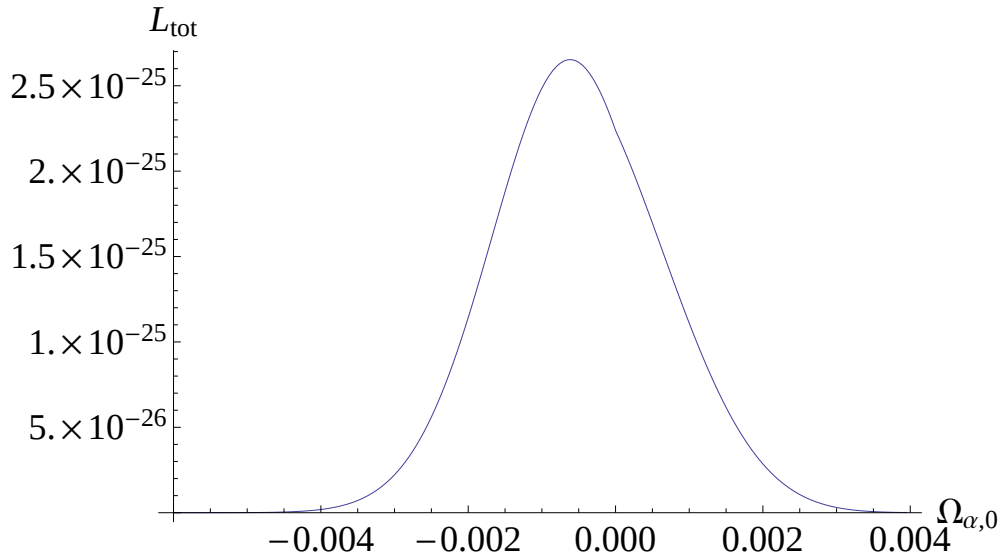


Figure 12. Diagram of PDF for parameter $\Omega_{\alpha,0}$ obtained as an intersection of a likelihood function. Two planes of intersection likelihood function are $\Omega_{\text{m},0} = 0.2938$ and $H_0 = 68.27 \text{ km}/(\text{s Mpc})$.

in the phase space, a saddle critical point, at which $\rho_{\text{m}}(t)$ scales like t^{-2} . This solution was recently proposed by Ringermacher and Mead [16] as a description characteristic for the dark matter evolution.

From the phase portrait we derive the generic scenario for an evolution of cosmological models with such a form of the dark energy parametrization. Trajectories start from initial singularity (the Einstein-de Sitter model) and then go in vicinity of the saddle point where they spend a lot of time and then go to the de Sitter state. It is a typical behavior for all

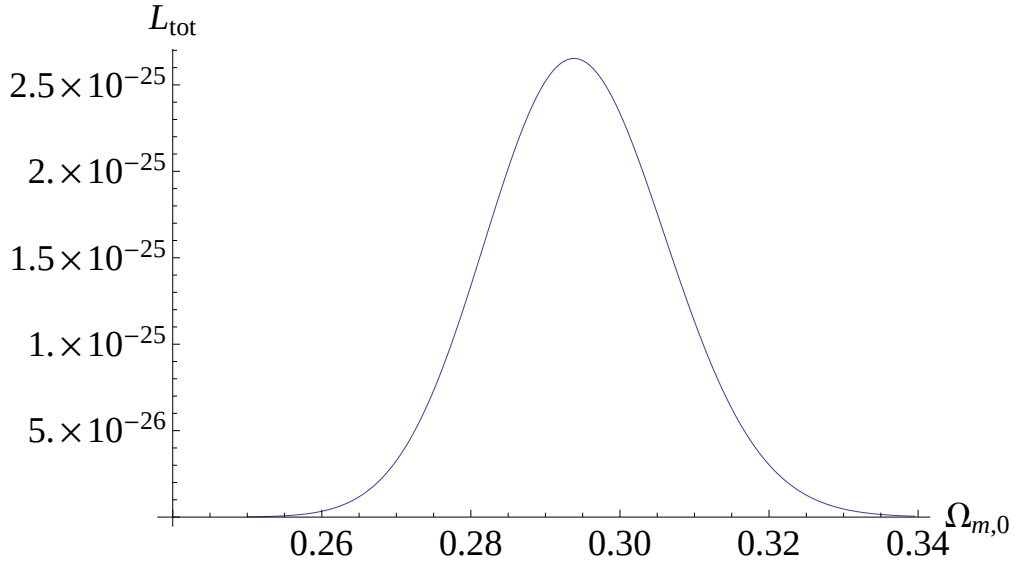


Figure 13. Diagram of PDF for parameter $\Omega_{m,0}$ obtained as an intersection of a likelihood function. Two planes of intersection likelihood function are $\Omega_{\alpha,0} = -0.0006$ and $H_0 = 68.27$ km/(s Mpc).

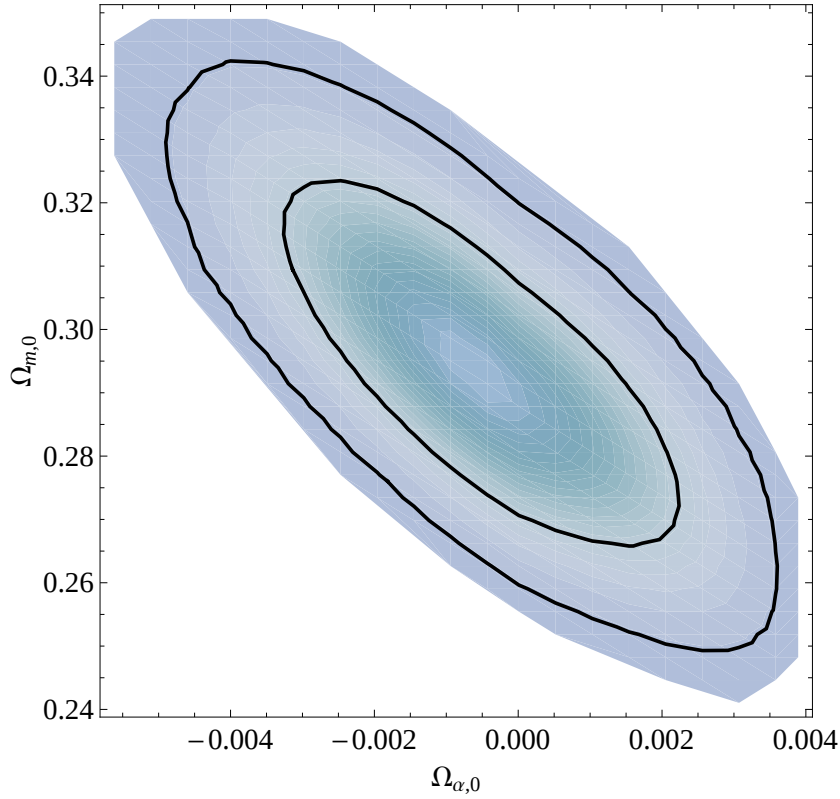


Figure 14. The likelihood function of two model parameters $(\Omega_{\alpha,0}, \Omega_{m,0})$ with the marked 68% and 95% confidence levels. The value of Hubble constant is estimated from the data as best fit value $H_0 = 68.27$ km/(s Mpc) and then the diagram of likelihood function is obtained for this value.

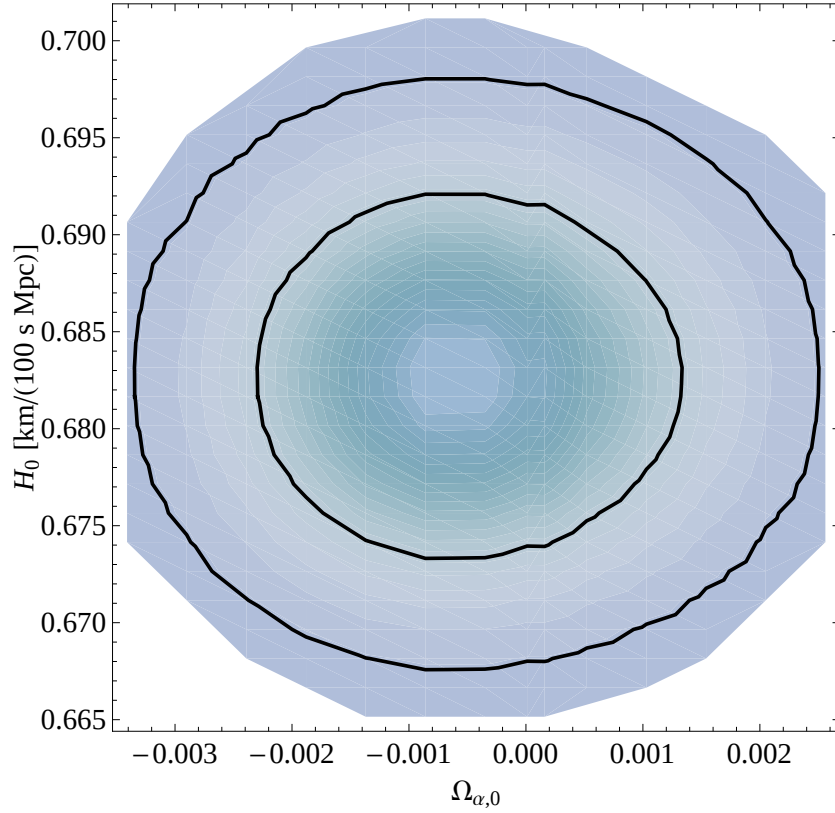


Figure 15. The likelihood function of two model parameters ($\Omega_{\alpha,0}, H_0$) with the marked 68% and 95% confidence levels. The value of $\Omega_{m,0}$ constant is estimated from the data as best fit value $\Omega_{m,0} = 0.2938$ and then the diagram of likelihood function is obtained for this value.

parameter	best fit	68% CL	95% CL
H_0	68.27 km/(s Mpc)	+0.96 −1.07	+1.67 −1.68
$\Omega_{m,0}$	0.2938	+0.0355 −0.0325	+0.0604 −0.0522
$\Omega_{\alpha,0}$	−0.0006	+0.0031 −0.0030	+0.0045 −0.0050
j_0	1.002	+0.010 −0.008	+0.016 −0.013
α^2	−0.002	+0.010 −0.007	+0.014 −0.012

Table 2. The best fit and errors for the estimated model with α^2 from the interval $(-0.05, 0.05)$.

generic trajectories in the phase space. The new critical point is emerging in the phase space due to the effect of a time dependence of the cosmological constant. If $\alpha^2 \leq 0$ then this point is absent (it is gluing with the critical point representing the Einstein-de Sitter model).

parameter	best fit	68% CL	95% CL
H_0	68.38 km/(s Mpc)	+0.97 −0.97	+1.16 −1.16
$\Omega_{m,0}$	0.2877	+0.0198 −0.0242	+0.0331 −0.0442
$\Omega_{\alpha,0}$	0.0000	+0.0025 −0.0000	+0.0038 −0.0000
j_0	1.000	+0.000 −0.008	+0.000 −0.011
α^2	0.000	+0.008 −0.000	+0.012 −0.000

Table 3. The best fit and errors for the estimated model with positive α^2 from the interval (0.00, 0.05).

We also tested this model using astronomical data. Statistical estimations show that the model fits to data as well as the standard cosmological model (the Λ CDM model). In any case the value of $\alpha^2 = 0$ belongs to the confidence interval for the estimated parameter α^2 we cannot reject that $\alpha^2 \neq 0$. Only if we find the best fit value α^2 with the error of the one order less than this value the problem of $\alpha_{\text{estimated}}^2 \neq \alpha_0^2 \neq 0$ could be solvable. We can obtain the limits on the value of parameter α^2 and $-0.009 < \alpha^2 = 3H_0^2 T_0^2 (1 - \Omega_{m,0} - \Omega_{\Lambda,0}) < 0.008$ (for 68% C.L.) and $-0.014 < \alpha^2 < 0.012$ (for 95% C.L.).

In papers of Doran and Robbers [23] and Pettorino et al. [24] there are limits on fractional dark energy at early time. Recently Ade et al. [49] have found $\Omega_e < 0.0036$. It is interesting that they have obtained a similar limit to our limit on $\Omega_\alpha < 0.0038$ in the other parametrization of dark energy.

Note that if we apply Starobinsky's idea the parameter α^2 can be constrained through Ω_e measurement. This parameter measures amount of dark energy at the early evolution of the Universe. If Ω_e is different from zero then we obtain value information about this alternative evolutionary scenarios which are consistent with the present epoch.

In our case $\Omega_e = \frac{\Omega_{\alpha,0}}{\Omega_{m,0}} = \frac{\alpha^2}{3H_0^2 T_0^2} < 0.0036$ and therefore $\alpha^2 < 3H_0^2 T_0^2 \Omega_{m,0} \Omega_e$. If we put $\Omega_{m,0} = 0.25$ and $H_0^2 T_0^2 = 1$ then we obtain $\alpha^2 < \frac{3}{4} \Omega_e = 0.0027$.

Finally we obtain a stronger limit for α^2 then in table 3. However, note that this estimation is model dependent (it is assumed Starobinsky's argument). note that the case study of our model fully confirm existence of phase during the early universe at which fractional energy density of dark energy is constant (see figure 10 and eq. (3.20)).

Acknowledgments

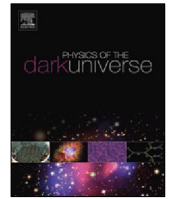
The work was supported by the grant NCN DEC-2013/09/B/ST2/03455. We are very grateful of prof. A. Borowiec, Z. Haba, A. Krawiec and K. Urbanowski for stimulating discussion and remarks. Especially I would like to thank S. Odintsov and V. Oikonomou for discussion of the problem of a covariance of the vacuum. We also thank referees for their remarks, especially for indicating the possibility of estimation α^2 from measurement of the fractional dark energy density Ω_e .

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Cosmological models with running cosmological term and decaying dark matter

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ABSTRACT

We investigate the dynamics of the generalized Λ CDM model, which the Λ term is running with the cosmological time. On the example of the model $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$ we show the existence of a mechanism of the modification of the scaling law for energy density of dark matter: $\rho_{\text{dm}} \propto a^{-3+\lambda(t)}$. We use an approach developed by Urbanowski in which properties of unstable vacuum states are analyzed from the point of view of the quantum theory of unstable states. We discuss the evolution of $\Lambda(t)$ term and pointed out that during the cosmic evolution there is a long phase in which this term is approximately constant. We also present the statistical analysis of both the $\Lambda(t)$ CDM model with dark energy and decaying dark matter and the Λ CDM standard cosmological model. We use data such as Planck, SNIa, BAO, $H(z)$ and AP test. While for the former we find the best fit value of the parameter $\Omega_{\alpha^2,0}$ is negative (energy transfer is from the dark matter to dark energy sector) and the parameter $\Omega_{\alpha^2,0}$ belongs to the interval $(-0.000040, -0.000383)$ at $2\text{-}\sigma$ level. The decaying dark matter causes to lowering a mass of dark matter particles which are lighter than CDM particles and remain relativistic. The rate of the process of decaying matter is estimated. Our model is consistent with the decaying mechanism producing unstable particles (e.g. sterile neutrinos) for which α^2 is negative.

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1. Introduction

In cosmology, the standard cosmological model (Λ CDM model) is an effective theory which well describes the current Universe in the accelerating phase of the expansion. All the astronomical observations of supernovae SNIa and measurements of CMB favor this model over the alternatives but we are still looking for theoretical models to dethrone the Λ CDM model.

On the other hand the Λ CDM model has serious problems like the cosmological constant problem or the coincidence problem which are open and waiting for a solution. Among different propositions, it is an idea of introducing the running cosmological term [1]. The most popular way of introducing a dynamical form of the cosmological term is a parametrization by the scalar field, i.e. $\Lambda \equiv \Lambda(\phi)$ or the Ricci scalar, i.e. $\Lambda \equiv \Lambda(R)$, where R is the Ricci scalar. Recently an interesting approach toward a unified description of both dark matter and dark energy was developed by consideration non-canonical Lagrangian for the scalar field $L = X^\alpha - \Lambda$, where $X = \dot{\phi}^2/2$ is a kinetic part of the scalar field energy [2] (see also [3]). In the both mentioned cases, the covariance

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of field equation is not violated and $\Lambda \equiv \Lambda(t)$ relation emerges from covariant theories.

Two elements appear in the Λ CDM model, namely dark matter and dark energy. The main aim of observational cosmology is to constrain the density parameters for dark energy as well as dark matter. In the testing of the Λ CDM model, the idea of dark energy is usually separated from the dark matter problem. The latter is considered as the explanation of flat galactic curves. Of course the conception of dark matter is also needed for the consistency of the model of cosmological structures but the hypothesis of dark energy and dark matter should be tested not as a isolated hypothesis.

In this paper, we explore the $\Lambda(t)$ CDM model with $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$, where t is the cosmological time for which we know an exact solution [1]. It is interesting that this type of a $\Lambda(t)$ relation is supported by the non-critical string theory consideration [4]. This enables us to show the nontrivial interactions between the sectors of dark matter and dark energy. It would be demonstrated that the model, which is under consideration, constitutes the special case of models with the interaction [1] term $Q = -\frac{d\Lambda(t)}{dt}$. We will be demonstrated that the time dependence of the Λ term is responsible for the modification of the standard scaling law of dark matter $\rho_{\text{dm}} = \rho_{\text{dm},0}a^{-3}$, where a is the scale factor [1]. Wang and Meng [5] developed a phenomenological approach which is based on the modified matter scaling relation $\rho_{\text{m}} = \rho_{\text{m},0}a^{-3+\delta}$, where δ is

the parameter which measures a deviation from the standard case of cold dark matter (CDM).

The both effect of the decaying Λ term and the modification of the scaling relation are strictly related in our model. One can obtain that CDM particles dilute more slowly in comparison to the standard relation $\rho_m \propto a^{-3}$ in this model. The coupling parameter δ is also a subject of testing using astronomical data [6–8].

Parametrization of dark energy in the form $1/t^2$ was used by many authors in different contexts. From the dimensional considerations, it is always possible to write Λ in terms of the Planck energy density as a dimensionless quantity [9,10]: $\Lambda \approx \rho_{\text{Pl}}(t_{\text{Pl}}/t_{\text{H}})^\alpha$, where $t_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}}$ and $t_{\text{H}} = H^{-1}$ are the Planck time and Hubble time, respectively, and $\rho_{\text{Pl}} = \frac{c^5}{\hbar G^2}$ is the Planck energy density. For the case of $\alpha = 2$, which gives the right value of Λ at the present epoch, we get $\Lambda = H^2$ [11]. He noted that such a parametrization of Λ is invoked to solve the cosmological constant problem, and is consistent with Mach's idea.

Vishwakarma also studied the magnitude–redshift relation for the type Ia supernovae data and the angular size–redshift relation for the updated compact radio sources data [12].

Note that for a power law type of the scale factor $a(t) = t^\alpha$ both parametrizations of $\Lambda \sim H^2$ and $\Lambda \sim t^{-2}$ correspond. The scaling evolution of the cosmological constant was investigated by Shapiro and Sola [13].

Lopez and Nanopoulos noted that this ansatz, which is similar to $\Lambda = \frac{\Lambda_{\text{Pl}}}{(a/\ell_{\text{Pl}})^{-2}} \propto a^{-2}$, where ℓ_{Pl} is the Planck length, gives to $\Lambda \propto 1/t^2$ [4].

In this paper, due to it is known the exact solutions of our model it is possible to check how it works the model and one can strictly constrain the model parameters [1].

We estimate the value of $\lambda(t)$: $\rho_{\text{dm}} = \rho_{\text{dm},0} a^{-3+\lambda(t)}$ where ρ_{dm} is energy density of dark matter. We use the astronomical data which is consisted of SNIa, BAO, $H(z)$, the AP test, Planck data.

We also analyze the model under considerations in details. In this analysis the model with $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$ is our case study. For this model we show the terms $\lambda(t)$, $\delta(t)$ are slow-changing with respect to the cosmological time and it justified to treat them as constants.

The organization of the text is following. In Section 2, we present the model with $\Lambda(t) = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$ and its interpretation in the perfect fluid cosmology. In Section 3, it is demonstrated how $\Lambda(t)$ CDM cosmologies can be interpolated as interacting cosmologies with the interacting term $Q = -\frac{d\Lambda(t)}{dt}$ [14]. In Section 4, we present some results of the statistical estimations of the model parameters obtained from some astronomical data. Finally the conclusion are presented in Section 5.

2. $\Lambda(t)$ CDM cosmology with $\Lambda = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$

Let us consider about the flat cosmological model with homogeneity and isotropy (the Robertson–Walker symmetry). The source of gravity is in the time dependent cosmological term and matter is in the form of a perfect fluid with energy density $\rho_m = \rho_m(t)$, where t is the cosmological time. The cosmic evolution is determined by the Einstein equations which admit the Friedmann first integral in the form

$$3H^2(t) = \rho_m(t) + \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}, \quad (1)$$

where $H(t) = \frac{d \log a}{dt}$ is the Hubble function and $a(t)$ is the scale factor and $\alpha^2 \in \mathbb{R}$ is a real dimensionless parameter. The sign of α^2 depends of the type of particle and the distribution of their energy. In the generic case the Breit–Wigner distribution gives rise the negative sign of α^2 [15–18]. Note that this parametrization is

distinguished by a dimensional analysis because a dimension of H^2 should coincide with a dimension of a time dependent part of $\Lambda(t)$.

It is assumed that the energy–momentum tensor for all fluids in the form of perfect fluid satisfies the conservation condition

$$T_{;\alpha}^{\alpha\beta} = 0, \quad (2)$$

where $T^{\alpha\beta} = T_m^{\alpha\beta} + \Lambda(t)g^{\alpha\beta}$. The consequence of this relation is that

$$\dot{\rho}_m + 3H\rho_m = -\frac{d\Lambda}{dt}. \quad (3)$$

The cosmic evolution is governed by the second order acceleration equation

$$\dot{H} = -H^2 - \frac{1}{6}(\rho_{\text{eff}} + 3p_{\text{eff}}), \quad (4)$$

where ρ_{eff} and p_{eff} are effective energy density of all fluids and pressure respectively. In the model under the consideration we have

$$\rho_{\text{eff}} = \rho_m + \rho_\Lambda, \quad (5)$$

$$p_{\text{eff}} = p_m - \rho_\Lambda, \quad (6)$$

where $p_m = 0$, $\rho_\Lambda = \Lambda_{\text{bare}} + \frac{\alpha^2}{t^2}$ and α^2 is a real number.

For this case the exact solution of (1) and (3) for the Hubble parameter $h \equiv \frac{H}{H_0}$ can be obtained in terms of modified Bessel functions of the first kind

$$h(t) = \frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}H_0}t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}H_0}t}{2}\right)} \quad (7)$$

where H_0 is the present value of the Hubble constant, $\Omega_{\Lambda,0} = \frac{\Lambda_{\text{bare}}}{3H_0^2}$,

$\Omega_{\alpha^2,0} = \frac{\alpha^2}{3H_0^2 T_0^2}$, T_0 is the present age of the Universe $T_0 = \int_0^{T_0} dt$ and

$n = \frac{1}{2}\sqrt{1 + 9\Omega_{\alpha^2,0}T_0^2H_0^2}$ is the index of the Bessel function. From (7), the expression for the scale factor can be obtained in the simple form

$$a(t) = C_2 \left[\sqrt{t} \left(I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0}H_0}t}{2} \right) \right) \right]^{\frac{2}{3}}. \quad (8)$$

The inverse formula for $t(a)$ is given by

$$t(a) = \frac{2}{3i\sqrt{\Omega_{\Lambda,0}H_0}} S_{n-\frac{1}{2}}^{-1} \left(\frac{\sqrt{3\pi\sqrt{\Omega_{\Lambda,0}H_0}} t^{n+1/2}}{2} \left(\frac{a}{C_2} \right)^{\frac{3}{2}} \right), \quad (9)$$

where $S_n(x)$ is a Riccati–Bessel function $S_n(x) = \sqrt{\frac{\pi x}{2}} J_{n+\frac{1}{2}}(x)$.

Finally the exact formula for total mass $\rho_m(t) = \rho_{\text{dm}}(t) + \rho_b(t)$ is given in the form

$$\rho_m = -3H_0^2 \left[\Omega_{\Lambda,0} + \frac{\Omega_{\alpha^2,0}T_0^2}{t^2} - \left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1}\left(\frac{3\sqrt{\Omega_{\Lambda,0}H_0}t}{2}\right)}{I_n\left(\frac{3\sqrt{\Omega_{\Lambda,0}H_0}t}{2}\right)} \right)^2 \right]. \quad (10)$$

The diagram of ρ_{dm} , ρ_{de} and $\frac{\rho_{\text{dm}}(\log(a))}{\rho_{\text{de}}(\log(a))}$ as a function of $\log a$ obtained for low z data is presented in Figs. 1–3. Note that at the

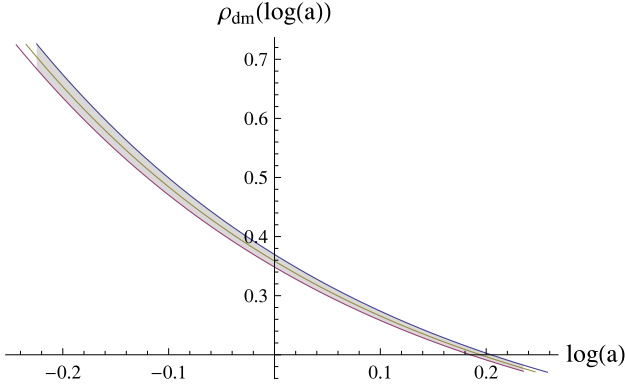


Fig. 1. A diagram of the evolution of $\rho_{dm}(\log(a))$. The bottom thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 67.83$ km/(s Mpc), $\Omega_{m,0} = 0.2875$ and $\Omega_{\alpha^2,0} = -0.000040$. The top thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 68.94$ km/(s Mpc), $\Omega_{m,0} = 0.2922$ and $\Omega_{\alpha^2,0} = -0.000383$. The medium line represents the best fit (see Table 1). We assumed $8\pi G = 1$ and we choose $100^2 \text{ km}^2 / (\text{Mpc}^2 \text{ s}^2)$ as a unit of $\rho_{dm}(\log(a))$.

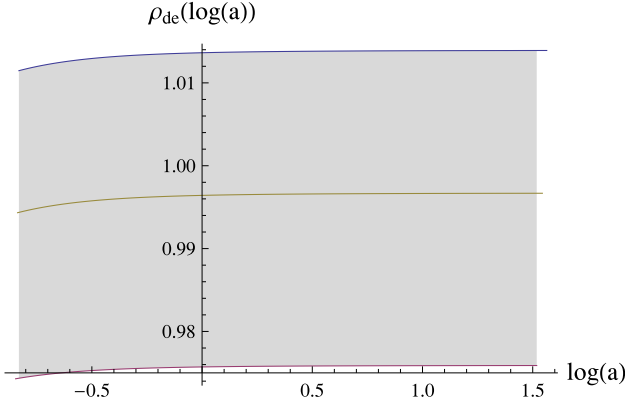


Fig. 2. A diagram of the evolution of $\rho_{de}(\log(a))$. The top thick line represents the evolution of $\rho_{de}(\log(a))$ for $H_0 = 67.62$ km/(s Mpc), $\Omega_{m,0} = 0.2888$ and $\Omega_{\alpha^2,0} = -0.000143$. The bottom thick line represents the evolution of $\rho_{de}(\log(a))$ for $H_0 = 68.97$ km/(s Mpc), $\Omega_{m,0} = 0.2896$ and $\Omega_{\alpha^2,0} = -0.000218$. The medium line represents the best fit (see Table 1). The gray region is the 2σ uncertainties. We assumed $8\pi G = 1$ and we choose $100^2 \text{ km}^2 / (\text{Mpc}^2 \text{ s}^2)$ as a unit of $\rho_{de}(\log(a))$.

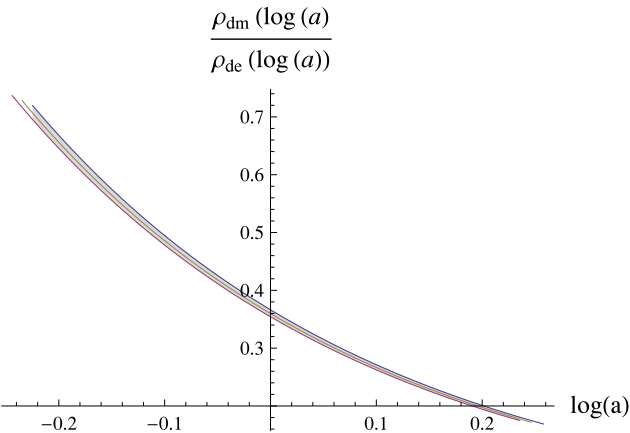


Fig. 3. A diagram of the evolution of $\frac{\rho_{dm}(\log(a))}{\rho_{de}(\log(a))}$. The bottom thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 67.83$ km/(s Mpc), $\Omega_{m,0} = 0.2875$ and $\Omega_{\alpha^2,0} = -0.000040$. The top thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 68.94$ km/(s Mpc), $\Omega_{m,0} = 0.2922$ and $\Omega_{\alpha^2,0} = -0.000383$. The medium line represents the best fit (see Table 1). The gray region is the 2σ uncertainties.

present epoch ($\log(a) = 0$) both energy densities of dark matter and dark energy are of the same order (Fig. 2).

While the relation $\Lambda = \Lambda(t)$ violates the covariance of the general relativity Lagrangian, it can be simply demonstrated that such a relation can emerge from the covariant theory of the perfect fluid.

The action of general relativity for a perfect fluid has the following form

$$S = \int \sqrt{-g}(R + \mathcal{L}_m) d^4x, \quad (11)$$

where R is the Ricci scalar, $\mathcal{L}_m = -\rho \left(1 + \int \frac{p(\rho)}{\rho^2} d\rho\right)$ [19] and $g_{\mu\nu}$ is the metric tensor. The signature of $g_{\mu\nu}$ is chosen as $(+, -, -, -)$.

For the Friedmann–Lemaître–Robertson–Walker (FLRW) metric without the curvature the Ricci scalar is expressed by $R = -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$. The Einstein equations are consequence of the variation of the Lagrangian $\mathcal{L}$ with respect to the metric tensor $g_{\mu\nu}$. The Einstein equations for dust and a minimal coupling scalar field are the following

$$3H^2 = \rho \quad (12)$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p). \quad (13)$$

We rewrite ρ as $\rho = \rho_m + \rho_{de}$ and we assume the equation of state for ρ_m as $p_m = 0$ and for ρ_{de} as $p_{de} = -\rho_{de}$. In consequence $p = p_{de} = -\rho_{de}$.

The conservation equation is in the form

$$\dot{\rho} + 3H(\rho + p) = \dot{\rho}_m + \dot{\rho}_{de} + 3H\rho_m = 0. \quad (14)$$

Because in our case $\frac{d\rho_{de}}{dt} = -\frac{2\alpha^2}{t^3}$ then the conservation equation can be rewritten as two equations

$$\dot{\rho}_m + 3H\rho_m = Q \quad (15)$$

and

$$\dot{\rho}_{de} = -Q, \quad (16)$$

where the expression $Q = \frac{2\alpha^2}{t^3}$ describes an interaction between ρ_m and ρ_{de} .

From Eq. (10) we can obtain a formula for $\rho = \rho_m + \rho_{de}$ as

$$\rho = 3H_0^2 \left(\frac{1-2n}{3H_0 t} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0} H_0}}{2} t \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0} H_0}}{2} t \right)} \right)^2. \quad (17)$$

Formula (17) guarantees that $H^2 \geq 0$ for every value of the cosmological time t .

From $p = p_{de} = -\rho_{de} = -\Lambda_{\text{bare}} - \frac{\alpha^2}{t^2}$ we can obtain the formula $t(p)$

$$t = \sqrt{-\frac{\alpha^2}{p + 3H_0^2 \Omega_{\Lambda}}}. \quad (18)$$

From (17) and (18) we have a formula for $\rho(p)$

$$\rho = 3H_0^2 \left(\frac{1-2n}{3H_0 \sqrt{-\frac{\alpha^2}{p + 3H_0^2 \Omega_{\Lambda}}}} + \sqrt{\Omega_{\Lambda,0}} \frac{I_{n-1} \left(\frac{3\sqrt{\Omega_{\Lambda,0} H_0}}{2} \sqrt{-\frac{\alpha^2}{p + 3H_0^2 \Omega_{\Lambda}}} \right)}{I_n \left(\frac{3\sqrt{\Omega_{\Lambda,0} H_0}}{2} \sqrt{-\frac{\alpha^2}{p + 3H_0^2 \Omega_{\Lambda}}} \right)} \right)^2. \quad (19)$$

Because the above function is strictly monotonic we have the specific equation of state in the form $p(\rho)$. We can use this formula for the equation of state in $\mathcal{L}_m$. In consequence our theory is equivalent to the covariant theory with the perfect fluid, which is described by the equation of state (19). Note that formally one can always find Lagrangian if ρ depends on t because if $a(t)$ is reversible function it is possible from inverse relation obtain $t = t(a)$ and consequently $\rho(t(a))$ and $p = p(\rho)$ which we put into Lagrangian.

However, more natural, covariant interpretation of our model than the perfect fluid interpretation is a model with a diffuse dark matter–dark energy interaction [20–22]. In these models the Einstein equations and equations of current density J^μ are the following

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda(t)g_{\mu\nu} = T_{\mu\nu}, \quad (20)$$

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu, \quad (21)$$

$$\nabla_\mu J^\mu = 0, \quad (22)$$

where $R_{\mu\nu}$ is the Riemann tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the energy–momentum tensor and σ is a positive parameter. From the Bianchi identity $\nabla^\mu (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$, Eqs. (20) and (21) we have the following formula for $\Lambda(t)$

$$\nabla_\mu \Lambda(t) = \sigma J_\mu. \quad (23)$$

We assume that the matter is a perfect fluid. Then the energy–momentum tensor is described by the following formula

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu), \quad (24)$$

where ρ is density of the matter, p is pressure of the matter and u_μ is the 4-velocity. We assume also the form of J^μ as

$$J^\mu = Qu^\mu = -\frac{d\Lambda(t)}{dt}u^\mu, \quad (25)$$

where Q has the interpretation as the interaction between dark matter and dark energy.

Under above considerations Eq. (21) is expressed by the following formula

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = -\sigma \frac{d\Lambda(t)}{dt}u^\mu. \quad (26)$$

We assume that they are symmetric forces in the fluid so $u^\mu = (1, 0, 0, 0)$. Then $J^0 = Q$.

For the FLRW metric equations (20), (21), (26) and (23) for the flat universe are reduced to

$$3H^2 = \rho + \Lambda(t), \quad (27)$$

$$\dot{\rho} + 3H(\rho + p) = -\sigma \frac{d\Lambda(t)}{dt}. \quad (28)$$

Because we assume that matter is the dust, and the parameter σ is equal one then the conservation equation has the following form

$$\dot{\rho} + 3H\rho = -\frac{d\Lambda(t)}{dt}. \quad (29)$$

3. How $\Lambda(t)$ CDM model modifies the scaling relation for dark matter

The existence of dark matter in the Universe is motivated by modern astrophysical observations as well as cosmological observations. From observations of rotation curves of spiral galaxies, masses of infraluster gas, gravitational lensing of clusters of galaxies to cosmological observations of the cosmic microwave background anisotropy and large scale structures we obtain strong evidences of dark matter.

Because models of nucleosynthesis in the early Universe are strongly restricted by the fraction of baryons, we conclude that the nature of dark matter cannot be baryonic matter. On the other hand we imagine that particles of dark matter form a part of standard model (SM) of particles physics. There are many candidates for particles of dark matter, e.g. WIMPs. Lately sterile neutrinos have been also postulated in this context [23,24]. The interesting approach is a search of photon emission from the decay or the annihilation of dark matter particles through the astrophysical observations of X-ray regions [25–27]. For example the radiatively decaying dark matter particles as sterile neutrinos have been searched using X-ray observations [28].

Let us consider the Λ CDM model which describes a homogeneous and isotropic universe which consists of baryonic and dark matter and dark energy. Let us assume an interaction in the dark sectors. Then the conservation equations have the following form

$$\dot{\rho}_b + 3H\rho_b = 0, \quad (30)$$

$$\dot{\rho}_{dm} + 3H\rho_{dm} = Q, \quad (31)$$

$$\dot{\rho}_{de} + 3H\rho_{de} = -Q, \quad (32)$$

where ρ_b is baryonic matter density, ρ_{dm} is dark matter density and ρ_{de} is dark energy density [14]. Q describes the interaction in the dark sector.

Let $\rho_m = \rho_b + \rho_{dm}$ then (30) and (31) give

$$\dot{\rho}_m + 3H\rho_m = Q. \quad (33)$$

For model with $\Lambda(t) = \Lambda_{bare} + \frac{\alpha^2}{t^2}$ the conservation equation has the form $\dot{\rho}_m + 3H\rho_m = -\frac{d\Lambda(t)}{dt}$. So this model can be interpreted as the special case of model with the interacting in the dark sectors. In this model $Q = -\frac{d\Lambda(t)}{dt} = \frac{2\alpha^2}{t^3}$.

Eq. (33) for $Q = \frac{2\alpha^2}{t^3}$ can be rewritten as $\dot{\rho}_m + 3H\rho_m = H\rho_m \frac{2\alpha^2}{t^3 H \rho_m}$ or

$$\frac{d\rho_m}{\rho_m} = \frac{da}{a} \left(-3 + \frac{2\alpha^2}{t^3 H \rho_m} \right). \quad (34)$$

The solution of Eq. (34) is

$$\rho_m = \rho_{m,0} a^{-3+\bar{\delta}(t)}, \quad (35)$$

where $\bar{\delta} = \frac{1}{\log a} \int \delta(t) d \log a$, where $\delta(t) = \frac{2\alpha^2}{t^3 H(t) \rho_m(t)}$. Q can be written as $Q = \delta(t) H \rho_m$. The evolution of $\delta(\log(a))$ and $\bar{\delta}(\log(a))$, which is obtained for low z data, is presented in Figs. 4 and 5. One can observe that $\bar{\delta}(t)$ and $\delta(t)$ is constant since the initial singularity to the present epoch.

If $\delta(t)$ is a slowly changing function than $\bar{\delta}(t) = \delta(t) = \delta$ and (35) has the following form

$$\rho_m = \rho_{m,0} a^{-3+\delta}. \quad (36)$$

In this case $Q = \delta H \rho_m$.

The early time approximation for $\delta(t)$ is

$$\delta(t) = \frac{9\alpha^2}{(\sqrt{1+3\alpha^2}+1)^2}. \quad (37)$$

If $\delta(t) = \delta = \text{const}$ we can easily find that

$$a = a_0 t^{\frac{2}{3-\delta}}, \quad (38)$$

and

$$\rho_m = \rho_{m,0} a_0^{-3+\delta} t^{-2}. \quad (39)$$

Because for the early time universe, Λ_{bare} is neglected, we get the following relation for the early time universe

$$\frac{\rho_m}{\rho_{de}} = \frac{\rho_{m,0} a_0^{-3+\delta}}{\alpha^2}. \quad (40)$$

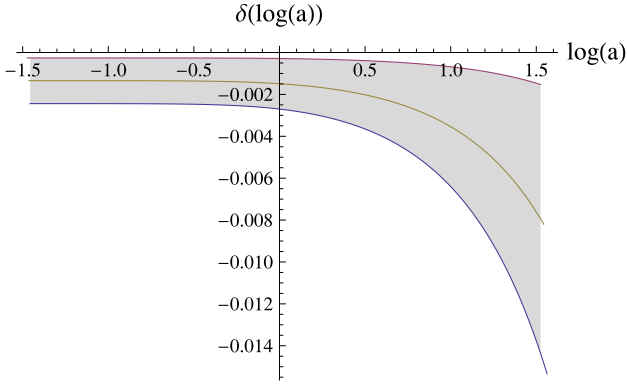


Fig. 4. A diagram of the evolution of $\delta(\log(a))$. The top thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 67.83$ km/(s Mpc), $\Omega_{m,0} = 0.2875$ and $\Omega_{a^2,0} = -0.000040$. The bottom thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 68.94$ km/(s Mpc), $\Omega_{m,0} = 0.2922$ and $\Omega_{a^2,0} = -0.000383$. The medium line represents the best fit (see Table 1). The gray region is the 2σ uncertainties.

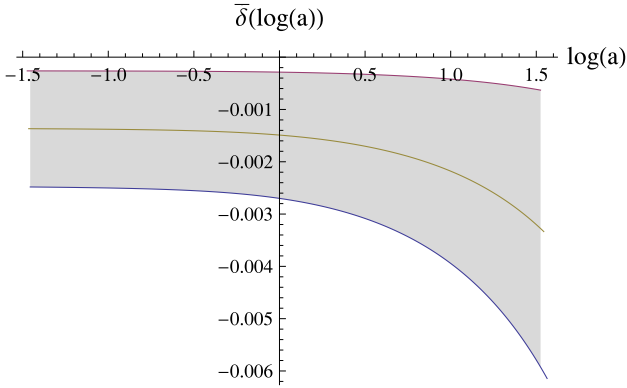


Fig. 5. A diagram of the evolution of $\bar{\delta}(\log(a))$. The top thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 67.83$ km/(s Mpc), $\Omega_{m,0} = 0.2875$ and $\Omega_{a^2,0} = -0.000040$. The bottom thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 68.94$ km/(s Mpc), $\Omega_{m,0} = 0.2922$ and $\Omega_{a^2,0} = -0.000383$. The medium line represents the best fit (see Table 1). The gray region is the 2σ uncertainties. Note that if $\rho_{dm} = 0$ for $a^2 < 0$, i.e. whole dark matter decays then we have the Λ CDM model with baryonic matter.

We can rewrite ρ_{dm} as

$$\rho_{dm} = \rho_{dm,0} a^{-3+\lambda(a)}, \quad (41)$$

where $\lambda(t) = \frac{1}{\log a} \log \frac{\Omega_{m,0} a^{\bar{\delta}(t)} - \Omega_{b,0}}{\Omega_{m,0} - \Omega_{b,0}}$. For the present epoch we can approximate $\lambda(t)$ as $\lambda(t) = \lambda = \text{const}$. So in the present epoch $\rho_{dm} = \rho_{dm,0} a^{-3+\lambda}$. In the consequence, the Friedmann equation can be written as $3H^2 = \rho_{b,0} a^{-3} + \rho_{dm,0} a^{-3+\lambda} + \Lambda_{bare} + \frac{a^2}{t^2}$. The evolution of $\lambda(\log(a))$, which is obtained for low z data, is presented in Fig. 6. One can observe that $\lambda(t)$ is constant since the initial singularity to the present epoch.

Following Amendola and others [29–31] the mass of dark particles can be parametrized by the scale factor as

$$m(a) = m_0 \exp \int^a \kappa(a') d(\log a'), \quad (42)$$

where m_0 is representing of mass of dark matter, $\kappa = \frac{d \log m}{d \log a}$. We consider the mass $m(a)$ as an effective mass of particles in a comoving volume.

In Amendola et al. [29] the parameter $\kappa(a)$ is assumed as a constant. This simplifying assumption has a physical justification as it will be demonstrated in the further dynamical analysis of the model with decaying dark matter. Eq. (42) can be simply

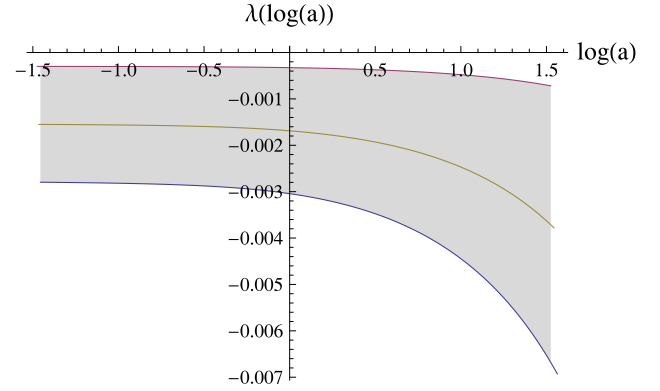


Fig. 6. A diagram of the evolution of $\lambda(\log(a))$. The top thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 67.83$ km/(s Mpc), $\Omega_{m,0} = 0.2875$ and $\Omega_{a^2,0} = -0.000040$. The bottom thick line represents the evolution of $\lambda(\log(a))$ for $H_0 = 68.94$ km/(s Mpc), $\Omega_{m,0} = 0.2922$ and $\Omega_{a^2,0} = -0.000383$. The medium line represents the best fit (see Table 1). The gray region is the 2σ uncertainties.

obtained from (41) because $e = a^{\frac{1}{\log a}}$ and $m(a) = a^3 \rho(a)$. Then $\lambda(a) = \frac{1}{\log(a)} \int^a \kappa(a') d(\log a')$. For illustration the rate of dark matter decaying process it would be useful to define the parameter β

$$\beta = 2^{\frac{\delta-3}{2\lambda}}. \quad (43)$$

If $\lambda(a) = \text{const}$ then Eq. (42) has the equivalent form $m(t) = m_0 a_0^\lambda \exp\left(-\frac{\log 2 \log t}{\log \beta}\right)$, where $\beta = 2^{\frac{\delta-3}{2\lambda}}$.

Let consider the number of dark matter particles $N(t)$ where t is the cosmological time. Than a half of the number of these particle $N(t)/2$ is reached at the moment of time βt .

In Fig. 3 one can see that a quotient ρ_{dm}/ρ_{de} decreases with the scale factor and remains of the same order for today ($\log a = 0$).

Note that the effect of the modification of the scaling law is $\rho_{dm} \propto a^{-3+\lambda}$, then this effect of the nonconservative energy momentum tensor is mimicking the effect of the conservative energy momentum tensor with the perfect fluid with the energy density ρ_{eff} and the pressure

$$p_{eff} = -\frac{\lambda}{3} \rho_{eff}. \quad (44)$$

Let $D(t)$ be the first order perturbation of the density of the matter ρ_m . The equation for evolution of $D(t)$ has the following form [32]

$$\frac{d^2 D(t)}{dt^2} + \left(2H(t) + \frac{Q(t)}{\rho_m(t)}\right) \frac{dD(t)}{dt} - \left(\frac{\rho_m(t)}{2} - 2H(t) \frac{Q(t)}{\rho_m(t)} - \frac{d}{dt} \left[\frac{Q(t)}{\rho_m(t)}\right]\right) D(t) = 0. \quad (45)$$

Because $\delta(t)$ for the early time universe is a constant, we can use Eq. (36) as an approximation of the behavior of ρ_m in the early time universe. In this case Eq. (45) has the following form

$$\frac{d^2 D(t)}{dt^2} + \frac{d_1}{t} \frac{dD(t)}{dt} + \frac{d_2}{t^2} D(t) = 0, \quad (46)$$

where $d_1 = \frac{2\Omega_{a^2,0}}{\Omega_{m,0}} + \frac{4}{3-\delta}$ and $d_2 = \frac{2(1+\delta)\Omega_{a^2,0}}{3-\delta} - \frac{3}{2}H^2 T_0^2 \Omega_{m,0}$. The solution of Eq. (46) is given by the formula

$$D(t) = C_1 t^{\frac{1}{2} \left(1-d_1 - \sqrt{1-4d_2-2d_1+d_1^2}\right)} + C_2 t^{\frac{1}{2} \left(1-d_1 + \sqrt{1-4d_2-2d_1+d_1^2}\right)}. \quad (47)$$

If we put the best fit as values of parameters in (47) then we get

$$D(t) = C_1 t^{-0.83} + C_2 t^{0.50}. \quad (48)$$

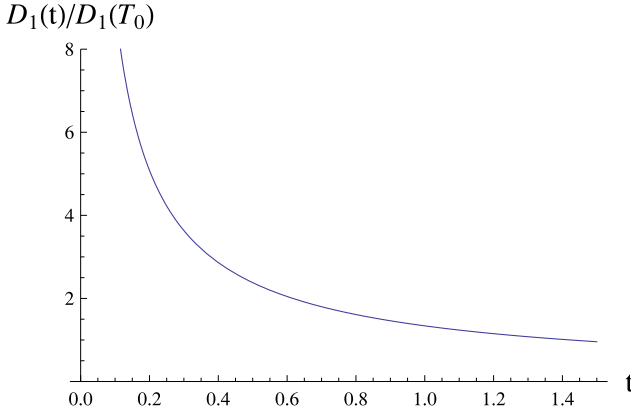


Fig. 7. A diagram of the evolution of the decreasing mode $D_1(t)/D_1(T_0)$ of the function $D(t)$ with the best fit values of parameters. The time is expressed by $\text{Mpc s}/(100 \text{ km})$ unit.

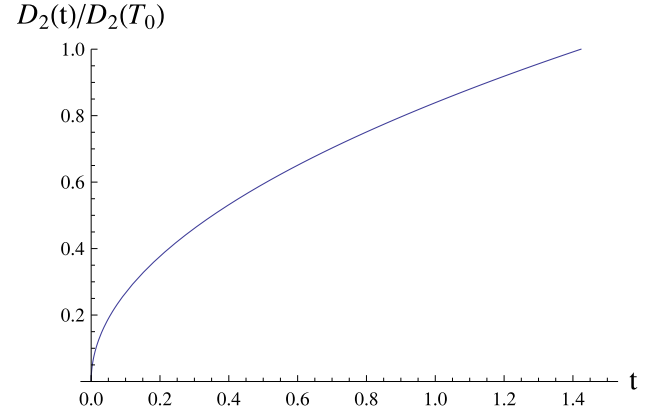


Fig. 8. A diagram of the evolution of the growing mode $D_2(t)/D_2(T_0)$ of the function $D(t)$ with the best fit values of parameters. The time is expressed by $\text{Mpc s}/(100 \text{ km})$ unit.

The first term of the right-hand side of Eq. (48) represents the decreasing mode $D_1(t)$ and its evolution is presented in Fig. 7. The second term represents the growing mode $D_2(t)$ and the evolution of this mode is presented in Fig. 8.

4. Statistical analysis of the model

In this section, we present a statistical analysis of the model parameters such as H_0 , $\Omega_{\text{dm},0}$ and λ_0 . We are using the SNIa, BAO, CMB observations, measurements of $H(z)$ for galaxies and the Alcock–Paczynski test.

We use the data from Union 2.1 which is the sample of 580 supernovae [33]. The likelihood function for SNIa is

$$\log L_{\text{SNIa}} = -\frac{1}{2} [A - B^2/C + \log(C/(2\pi))], \quad (49)$$

where $A = (\mathbf{\mu}^{\text{obs}} - \mathbf{\mu}^{\text{th}}) \mathbb{C}^{-1} (\mathbf{\mu}^{\text{obs}} - \mathbf{\mu}^{\text{th}})$, $B = \mathbb{C}^{-1} (\mathbf{\mu}^{\text{obs}} - \mathbf{\mu}^{\text{th}})$, $C = \text{tr } \mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for SNIa. The distance modulus is $\mu^{\text{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of SNIa) and $\mu^{\text{th}} = 5 \log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$).

We use Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [34], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [35], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [36]. The likelihood function is given by

$$\log L_{\text{BAO}} = -\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(z)} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(z)} \right), \quad (50)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch [37,38].

The likelihood function for the Planck observations of cosmic microwave background (CMB) radiation [39] has the form

$$\log L_{\text{CMB}} = -\frac{1}{2} \sum_i \frac{(D_{\ell, \text{th}}^{\text{TT}}(\ell_i) - D_{\ell, \text{obs}}^{\text{TT}}(\ell_i))^2}{\sigma^2}, \quad (51)$$

where $D_{\ell}^{\text{TT}}(\ell)$ is the value of the temperature power spectrum of CMB, ℓ is multipole. In this statistical analysis, the temperature power spectrum is for ℓ in the interval (30, 2508).

The likelihood function for the Alcock–Paczynski test [40,41] has the following form

$$\log L_{\text{AP}} = -\frac{1}{2} \sum_i \frac{(AP^{\text{th}}(z_i) - AP^{\text{obs}}(z_i))^2}{\sigma^2}. \quad (52)$$

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data [42–50].

Table 1

The best fit and errors for the estimated model for Planck + SNIa + BAO + $H(z)$ + AP test with H_0 from the interval (65.0, 71.0) $\text{km}/(\text{Mpc s})$, $\Omega_{\text{dm},0}$ from the interval (0.20, 0.36), $\Omega_{\alpha^2,0}$ from the interval (−0.005, 0.005) and λ_0 from the interval (−0.025, 0.010). $\Omega_{\text{b},0}$ is assumed as 0.048468. The value of χ^2 for the best fit is equal 2332.25, the value of AIC is equal 2338.25 and BIC is equal 2356.37. In comparison with this model, the χ^2 of the best fit of the Λ CDM model is equal 2335.18, AIC is equal 2339.18 and BIC is equal 2351.26.

Parameter	Best fit	68% CL	95% CL
H_0	68.38	+0.37 −0.42	+0.59 −0.76
$\Omega_{\text{dm},0}$	0.2420	+0.0020 −0.0018	+0.0030 −0.0029
$\Omega_{\alpha^2,0}$	−0.000210	+0.000100 −0.000107	+0.000170 −0.000173
λ_0	−0.00169	+0.00080 −0.00084	+0.00136 −0.00135

We are using some data of $H(z)$ of different galaxies from [51–53] and the likelihood function is

$$\log L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2. \quad (53)$$

The final likelihood function is

$$L_{\text{tot}} = L_{\text{CMB}} L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)}. \quad (54)$$

We use our own code CosmoDarkBox in estimation of the model parameters. The code uses the Metropolis–Hastings algorithm [54,55] and the dynamical system to obtain the likelihood function.

The results of statistical analysis are represented in Table 1. Figs. 9 and 10 where it is shown the likelihood function with 68% and 95% confidence level projection on the $(\Omega_{\text{dm},0}, \lambda_0)$ plane and the (H_0, λ_0) plane, respectively. Diagram of the temperature power spectrum for the best fit values is presented in Fig. 11.

We can use some information criteria in scientific practice to choose the best model. One of information criteria is the Akaike information criterion (AIC), which is given by

$$\text{AIC} = -2 \ln L + 2d, \quad (55)$$

where L is the maximum of the likelihood function and d is the number of model parameters. For our model the parameter d is equal three because we estimate three parameters such as H_0 , Ω_{dm} and $\Omega_{\alpha^2,0}$. It is one more parameter than for the Λ CDM model. Model which is the best approximation to the truth from the set under consideration has the smallest value of the AIC quantity. It is

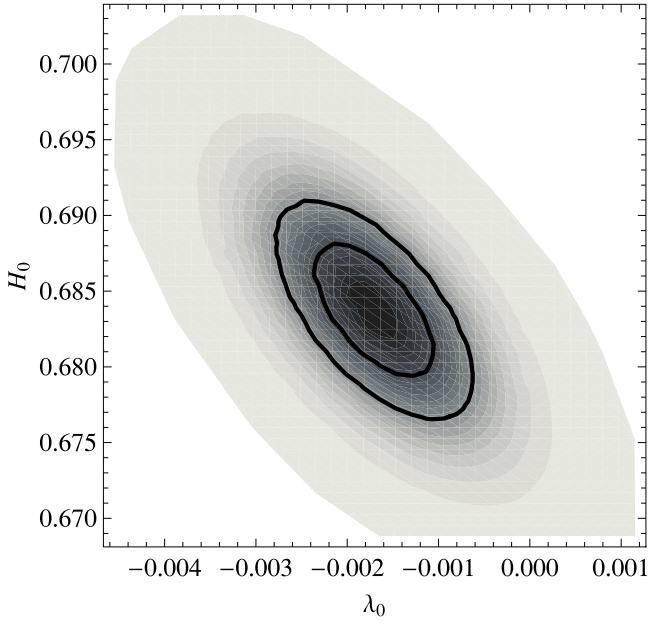


Fig. 9. The intersection of the likelihood function of two model parameters (H_0 , λ_0) with the marked 68% and 95% confidence levels for Planck + SNIa + BAO + $H(z)$ + AP test. The value of Hubble constant is estimated from the data as the best fit of value $\Omega_{\text{dm},0} = 0.2420$ and then the diagram of likelihood function is obtained for this value. We choose 100 km/s Mpc as a unit of H_0 .

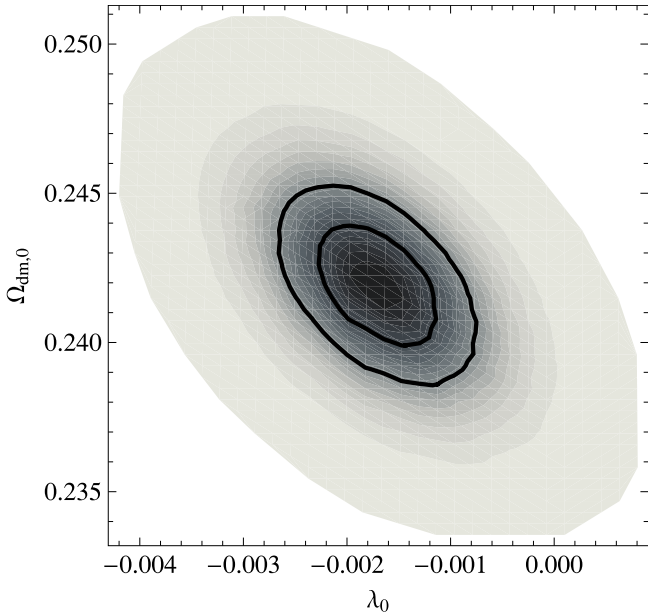


Fig. 10. The intersection of the likelihood function of two model parameters ($\Omega_{\text{dm},0}$, λ_0) with the marked 68% and 95% confidence levels for Planck + SNIa + BAO + $H(z)$ + AP test. The value of the Hubble constant is estimated from the data as the best fit of value $H_0 = 68.38$ km/(s Mpc) and then the diagram of likelihood function is obtained for this value.

convenient to evaluate the differences between the AIC quantities computed for the rest of models from our set and the AIC for the best one. Those differences ΔAIC

$$\Delta\text{AIC}_i = \text{AIC}_i - \text{AIC}_{\min} \quad (56)$$

are easy to interpret and allow a quick “strength of evidence” for the model considered with respect to the best one. In our case the

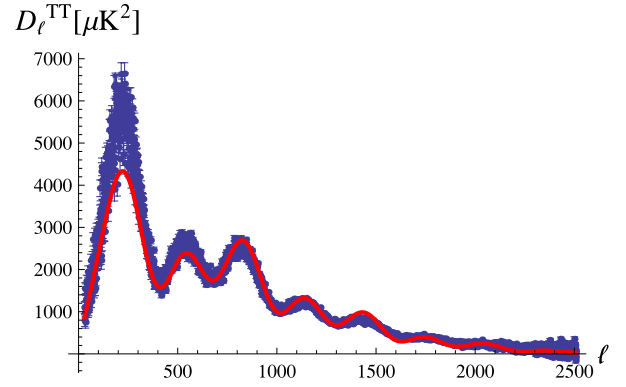


Fig. 11. Diagram of the temperature power spectrum of CMB for the best fit values (red line). The error bars from the Planck data are presented by blue color. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

value of ΔAIC_i is equal 0.93. The AIC favors very weakly our model in comparison to the ΛCDM model.

We also use BIC (Bayesian information criterion) which is defined as

$$\text{BIC} = -2 \ln L + d \ln n, \quad (57)$$

where n is the number of data points [56]. In our case $n = 3101$. For the ΛCDM model we obtain $\text{BIC}_0 = 2351.26$ and for our model $\text{BIC}_1 = 2356.37$. Given a simple relation between the Bayes factor and the BIC

$$2 \ln B_{01} = -(\text{BIC}_0 - \text{BIC}_1) = \Delta\text{BIC}_{01} \quad (58)$$

we obtain the difference between the BIC_1 for our model and BIC_0 for the ΛCDM model is equal 5.11. We use the scale for interpretation of the twice natural logarithm of the Bayes factor proposed by Kass and Raftery [57]. Because the ΔBIC_{01} is between 2 and 6 it is a positive evidence in favor of the ΛCDM model.

From the statistical analysis we get that the model with the negative value of α^2 at $2\text{-}\sigma$ level, which means that dark matter particles decay.

5. Conclusions

The main goal of the paper was to investigate in details the dynamic of the model with matter and the running cosmological constant term with respect to the cosmological time. It was assumed that baryonic matter satisfies the equation of state for dust (i.e. is non-relativistic). We were interested how the running $\Lambda(t)$ influences on the scaling relation for energy density ρ_{dm} . We have found the deviation from standard scaling a^{-3} for this relation. We explained the source of this deviation showing that ρ_{dm} decreases more rapidly or slowly like $a^{-3+\delta}$ due to the energy transfer from dark matter to dark energy sector or in the opposite direction. The direction of the energy transfer crucially depends on the sign of α^2 constant in the model under consideration.

The value of α^2 can be theoretically calculated in the quantum formalism developed by Urbanowski and collaborators [15–18]. In their paper it was proposed a quantum mechanical effect which can be responsible for emission X or γ rays by charged unstable particles at sufficiently late times. The sign of α^2 constant is obtained from the analysis of the survival amplitude. In these calculations, the crucial role plays the Breit–Wigner distribution function which gives rise to a negative sign of the α^2 constant. For typical particles, decaying processes are describing through this distribution function.

From the cosmological point of view it is interesting that fluctuations of instantaneous energy of these unstable particles, which together with other stable particles form dark matter, can be manifested as fluctuations of the velocity of these particles [17]. As the result this effect may cause the emission of the electromagnetic radiation from radio up to ultra-high frequencies by unstable particles including the unstable components of dark matter. In the context of astrophysics important stays information can be obtained from the observation of X-rays or γ -rays. From X-ray CCD instruments, dark matter is searched in keV energy for looking for the non-baryonic X-ray signature [28].

On the other hand the α^2 constant is a dimensionless model parameter which value can be estimated from some astronomical data. Our estimations favor the negative α^2 constant, i.e. it is favored the decaying vacuum of dark matter particles and the radiative nature of the energy transfer to dark energy sector.

The survival amplitude of unstable particles is well described by the Breit–Wigner energy distribution function [17]. So it is very probable that the survival amplitude of the unstable components of the sterile neutrino sector is also described sufficiently well by this distribution function. Such a distribution function leads to negative α^2 [58]. The negative sign of the α^2 constant offers a new insight into the cosmological constant problem because the running Λ is the growing function of the cosmological time with asymptotic Λ_{bare} at $t \rightarrow \infty$. Therefore, the problem of different values of Λ at the early time universe and at the present epoch is solved by our model.

In our paper we have also found the physical background of the relation $\rho_{\text{dm}} \propto a^{-3+\lambda}$, where $\lambda = \text{const}$, plays an important role. Our observational analysis of the evolution this parameter during the cosmic evolution indicates that such an ansatz has a strongly physical justification.

In interacting cosmology the interacting term which is postulated in different physical forms is interpreted as a kind of non-gravitational interactions in the dark sector. We suggest that this interaction has the radiation nature and can be rather interpreted following the Urbanowski and Raczynska idea as a possible emission of cosmic X and γ rays by unstable particles [17] including unstable particles forming dark matter.

It is still an open discussion about the nature of dark matter: cold or warm dark matter [59]. Our results showed that in the model of dark matter decay dark matter particle being lighter than CDM particles. Therefore particles of warm dark matter remain relativistic longer during the cosmic evolution at the early universe. Our model is consistent with a conception of mixed dark matter (MDM) which is also called hot+cold dark matter [60,61].

In the investigating dynamics of the interacting cosmology the corresponding dynamical systems, which are determining the evolutionary paths, are not closed until one specify the form of the interacting term Q . Usually this form is postulated as a specific function of the Hubble parameter, energy density of matter or scalar field or their time derivatives [62–65]. Our model with decaying dark matter favors the choice of the interacting term in the form $Q \propto H\rho_m$.

The statistical analysis favored the model with the negative value of α^2 (the model with decaying dark matter particles). However there is a positive evidence in favor of the Λ CDM model with respect to the our model based on twice natural logarithm of Bayes factor calculated as the difference of BIC for both models.

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Cosmological implications of the transition from the false vacuum to the true vacuum state

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Abstract We study cosmology with running dark energy. The energy density of dark energy is obtained from the quantum process of transition from the false vacuum state to the true vacuum state. We use the Breit–Wigner energy distribution function to model the quantum unstable systems and obtain the energy density of the dark energy parametrization $\rho_{de}(t)$. We also use Krauss and Dent’s idea linking properties of the quantum mechanical decay of unstable states with the properties of the observed Universe. In the cosmological model with this parametrization there is an energy transfer between dark matter and dark energy. The intensity of this process, measured by a parameter α , distinguishes two scenarios. As the Universe starts from the false vacuum state, for the small value of α ($0 < \alpha < 0.4$) it goes through an intermediate oscillatory (quantum) regime of the density of dark energy, while for $\alpha > 0.4$ the density of the dark energy jumps down. In both cases the present value of the density of dark energy is reached. From a statistical analysis we find this model to be in good agreement with the astronomical data and practically indistinguishable from the Λ CDM model.

1 Introduction

The standard cosmological model (Λ CDM model), which describes the Universe, is the one most favored by astronomical observations such as supernovae of type Ia or measurements of CMB. In the Λ CDM model, the dark matter is treated as dust and dark energy has the form of the cosmological constant Λ_{bare} . We are looking for an alternative for the Λ CDM model by a modification of the dark energy term.

The standard cosmological model possesses the six parameters: the density of baryons $\Omega_b h^2$, the density of cold dark matter $\Omega_{dm} h^2$, the angular diameter of sound horizon at last scattering θ , the optical depth due to the reionization τ_R , the slope of the primordial power spectrum of fluctuations n_s , and the amplitude of the primordial power spectrum A_s , where $h = H_0$ (100 km s⁻¹ Mpc⁻¹).

From the methodological point of view, the standard cosmological model plays the role of an effective theory, which very well describes properties of the current Universe without explaining the nature of two components of the model: the dark energy and the dark matter. The nature of both components of the Universe has been unknown up to now but we describe these in terms of some useful fiction, the cosmological constant and the cold dark matter, which is a kind of a dust perfect fluid.

In this paper we concentrate on the interpretation of dark energy in terms of running cosmological constant rather than in terms of the pure cosmological constant parameter (Λ_{bare} in our approach). It is a consequence of some problems with the interpretation of the pure cosmological constant, namely:

1. One cannot explain why the cosmological constant is not large.
2. One does not know why it is not just equal zero.
3. One cannot explain why energy densities of both dark energy and dark matter, expressed in terms of dimensionless density parameters, are comparable in the current epoch (cosmic coincidence problem).

In our proposition of the explanation of these problems with the cosmological constant parameter, we base our ideas on the theories of the cosmological constant in which the vacuum energy is fixed by the fundamental theory [1]. Extending the Λ CDM model beyond the classical regime, we apply quantum mechanics as a fundamental theory, which deter-

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mines cosmological parameters and we explain how cosmological parameters vary during the cosmic evolution.

The cosmological constant is the source of two problems in modern cosmology. The first problem is the cosmological constant problem, which is consequence of the interpretation of dark energy as a vacuum energy. The observed present value of the cosmological constant is 120 orders of magnitude smaller than we expect from quantum physics. The second problem is the coincidence problem. If we assume that the dark energy is always constant, then the Λ CDM model cannot explain why the cosmological constant has the same order of magnitude as the density of matter today. If the model belongs to the class of running dark energy cosmologies then the first problem of cosmological constant can be solved.

This question seems to be crucial in contemporary physics because its solution would certainly mean a very crucial step forward in our attempts to understand physics from the boundary of particle physics and cosmology. A discussion as regards the cosmological constant problem can be found in Refs. [1–16].

In our model, the influence of running dark energy densities of both visible and invisible matter is very small. Thus we share Weinberg's opinion, according to which looking for a solution of the coincidence problem, we should consider the anthropic principle. According to Weinberg's argument, no observers at all should be in the Universe if the cosmological constant was even three orders of magnitude larger than it is now.

Coleman et al. [17–19] discussed the instability of a physical system, which is not at an absolute energy minimum, and which is separated from the minimum by an effective potential barrier. They showed that if the early Universe is too cold to activate the energy transition to the minimum energy state, then a quantum decay, from the false vacuum to the true vacuum, is still possible through a barrier penetration via macroscopic quantum tunneling.

The discovery of the Higgs-like resonance at 125–126 GeV [20–23] caused a discussion as regards the instability of the false vacuum. If we assume that the Standard Model well describes the evolution of the Universe up to the Planck epoch, then a Higgs mass $m_h < 126\text{GeV}$ causes the electroweak vacuum to be in a metastable state [21]. In consequence the instability of the Higgs vacuum should be considered in the cosmological models of the early time Universe.

The idea that properties of the quantum mechanical decay process of metastable states can help to understand the properties of the observed Universe was formulated in [24–26]. It is because the decay of the false vacuum is a quantum decay process [17–19]. This means that the state vector corresponding to the false vacuum is a quantum unstable (or metastable) state. Therefore all general properties of quantum unstable systems must also occur in the case of such a quantum unsta-

ble state as the false vacuum and, as a consequence, models of quantum unstable systems can be used to analyze properties of the systems of which the time evolution starts from the false vacuum state. Note that Landim and Abdalla built a model of metastable dark energy, in which the observed vacuum energy is the value of the scalar potential at the false vacuum [27].

In this paper, we assume the Breit–Wigner energy distribution function, which is very often used to model unstable quantum systems, as a model of the process of the energy transition from the false vacuum to the true vacuum. In consequence the parametrization of the dark energy is given by formula

$$\rho_{\text{de}} = E_0 + E_R \frac{\alpha}{1 - \alpha} \Re \left(\frac{J(t)}{I(t)} \right), \quad (1)$$

where α and E_R are model parameters describing the variation from the standard cosmological model. The values of the parameter α belong to interval $(0, 1)$. Note that if the parameter α or E_R is equal to zero, then the model is equivalent to the Λ CDM model.

Let $\Lambda_{\text{bare}} = E_0 - E_R$; then Eq. (1) can be rewritten in the equivalent form

$$\rho_{\text{de}} = \Lambda_{\text{bare}} + E_R \left[1 + \frac{\alpha}{1 - \alpha} \Re \left(\frac{J(t)}{I(t)} \right) \right]. \quad (2)$$

Here the units $8\pi G = c = 1$ are used.

The functions $J(t)$ and $I(t)$ are defined by the following expressions:

$$J(t) = \int_{-\frac{1-\alpha}{\alpha}}^{\infty} \frac{\eta}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta, \quad (3)$$

$$I(t) = \int_{-\frac{1-\alpha}{\alpha}}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta. \quad (4)$$

The integrals $J(t)$ and $I(t)$ can be expressed by the exact solutions of these integrals. The formula for $J(t)$ is the following expression:

$$J(\tau) = \frac{1}{2} e^{-\tau/2} \left(-2i\pi + e^{\tau} \text{Ei} \left(\left[\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) + \text{Ei} \left(\left[-\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right) \quad (5)$$

and $I(t)$ is expressed by

$$I(\tau) = 2\pi e^{-\tau/2} \left(1 + \frac{i}{2\pi} \left(-e^{\tau} \text{Ei} \left(\left[\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) + \text{Ei} \left(\left[-\frac{1}{2} - \frac{i(1-\alpha)}{\alpha} \right] \tau \right) \right) \right), \quad (6)$$

where $\tau = \frac{\alpha(E_0 - \Lambda_{\text{bare}})}{\hbar(1-\alpha)} V_0 t$ and V_0 is the volume of the Universe in the Planck epoch. In this paper we assume that

$V_0 = 1$. The function $E_1(z)$ is called the exponential integral and is defined by the formula: $E_1(z) = \int_z^\infty \frac{e^{-x}}{x} dx$ (see [28, 29]).

2 Preliminaries: unstable states

As mentioned in Sect. 1 we will use the parametrization of the dark energy transition from the false vacuum state to the true vacuum state following from the quantum properties of such a process. This process is a quantum decay process, so we need quantities characterizing decay processes of quantum unstable systems. The main information as regards properties of quantum unstable systems is contained in their decay law, that is, in their survival probability. So if one knows that the system is in the initial unstable state $|\phi\rangle \in \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space of states of the considered system), which was prepared at the initial instant $t_0 = 0$, then one can calculate its survival probability (the decay law), $\mathcal{P}(t)$, of the unstable state $|\phi\rangle$ decaying in vacuum, which equals

$$\mathcal{P}(t) = |A(t)|^2, \quad (7)$$

where $A(t)$ is the probability amplitude of finding the system at time t in the rest frame $\mathcal{O}_0$ in the initial unstable state $|\phi\rangle$,

$$A(t) = \langle \phi | \phi(t) \rangle, \quad (8)$$

and $|\phi(t)\rangle$ is the solution of the Schrödinger equation for the initial condition $|\phi(0)\rangle = |\phi\rangle$, which has the following form:

$$i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \mathfrak{H} |\phi(t)\rangle. \quad (9)$$

Here $|\phi\rangle, |\phi(t)\rangle \in \mathcal{H}$, and $\mathfrak{H}$ is the total self-adjoint Hamiltonian for the system considered. The spectrum of $\mathfrak{H}$ is assumed to be bounded from below, $E_{\min} > -\infty$ is the lower bound of the spectrum $\sigma_c(\mathfrak{H}) = [E_{\min}, +\infty)$ of $\mathfrak{H}$. Using the basis in $\mathcal{H}$ built from normalized eigenvectors $|E\rangle$, $E \in \sigma_c(\mathfrak{H})$ of $\mathfrak{H}$ and expanding $|\phi\rangle$ in terms of these eigenvectors one can express the amplitude $A(t)$ as the following Fourier integral:

$$A(t) \equiv \int_{E_{\min}}^{\infty} \omega(E) e^{-\frac{i}{\hbar} E t} dE, \quad (10)$$

where $\omega(E) > 0$ (see [30–32]).

So the amplitude $A(t)$ and, thus, the decay law $\mathcal{P}(t)$ of the unstable state $|\phi\rangle$ are completely determined by the density of the energy distribution $\omega(E)$ for the system in this state [30, 31]; see also [32–39] (this approach is also applicable in Quantum Field Theory models [40, 41]).

Note that in fact the amplitude $A(t)$ contains information as regards the decay law $\mathcal{P}_\phi(t)$ of the state $|\phi\rangle$, that is, as regards the decay rate Γ_ϕ^0 of this state, as well as the

energy E_ϕ^0 of the system in this state. This information can be extracted from $A(t)$. It can be done using the rigorous equation governing the time evolution in the subspace of unstable states, $\mathcal{H}_\parallel \ni |\phi\rangle_\parallel \equiv |\phi\rangle$. Such an equation follows from the Schrödinger equation (9) for the total state space $\mathcal{H}$.

The use of the Schrödinger equation (9) allows one to find that within the problem considered

$$i\hbar \frac{\partial}{\partial t} \langle \phi | \phi(t) \rangle = \langle \phi | \mathfrak{H} | \phi(t) \rangle. \quad (11)$$

This relation leads to the conclusion that the amplitude $A(t)$ satisfies the following equation:

$$i\hbar \frac{\partial A(t)}{\partial t} = h(t) A(t), \quad (12)$$

where

$$h(t) = \frac{\langle \phi | \mathfrak{H} | \phi(t) \rangle}{A(t)}, \quad (13)$$

and $h(t)$ is the effective Hamiltonian governing the time evolution in the subspace of unstable states $\mathcal{H}_\parallel = \mathbb{P}\mathcal{H}$, where $\mathbb{P} = |\phi\rangle\langle\phi|$ (see [42] and also [43, 44] and the references therein). The subspace $\mathcal{H} \ominus \mathcal{H}_\parallel = \mathcal{H}_\perp \equiv \mathbb{Q}\mathcal{H}$ is the subspace of decay products. Here $\mathbb{Q} = \mathbb{I} - \mathbb{P}$. We have the following equivalent formula for $h(t)$ [42–44]:

$$h(t) \equiv \frac{i\hbar}{A(t)} \frac{\partial A(t)}{\partial t}. \quad (14)$$

One meets the effective Hamiltonian $h(t)$ when one starts with the Schrödinger equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_\parallel \subset \mathcal{H}$ [39, 42]. In general $h(t)$ is a complex function of time and in the case of $\mathcal{H}_\parallel$ is dimension two or more the effective Hamiltonian governing the time evolution in such a subspace it is a non-hermitian matrix $H_\parallel$ or non-hermitian operator. We have

$$h(t) = E_\phi(t) - \frac{i}{2} \Gamma_\phi(t), \quad (15)$$

and

$$E_\phi(t) = \Re[h(t)], \quad \Gamma_\phi(t) = -2\Im[h(t)], \quad (16)$$

are the instantaneous energy (mass) $E_\phi(t)$ and the instantaneous decay rate, $\Gamma_\phi(t)$ [42–44]. Here $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z , respectively. Equations (12), (14) and (16) are convenient when the density $\omega(E)$ is given and one wants to find the instantaneous energy $E_\phi(t)$ and decay rate $\Gamma_\phi(t)$: Inserting $\omega(E)$ into (10) one obtains the amplitude $A(t)$ and then using (14) one finds $h(t)$ and thus

$E_\phi(t)$ and $\Gamma_\phi(t)$. The simplest choice is to take $\omega(E)$ in the Breit–Wigner form,

$$\omega(E) \equiv \omega_{\text{BW}}(E) \stackrel{\text{def}}{=} \frac{N}{2\pi} \frac{\Gamma_0 \Theta(E - E_{\min})}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}, \quad (17)$$

where N is a normalization constant and $\Theta(E) = 1$ for $E \geq 0$ and $\Theta(E) = 0$ for $E < 0$. The parameters E_0 and Γ_0 correspond to the energy of the system in the unstable state and its decay rate at the exponential (or canonical) regime of the decay process. $E_{\min}$ is the minimal (the lowest) energy of the system. Inserting $\omega_{\text{BW}}(E)$ into Eq. (10) for the amplitude $A(t)$ after some algebra one finds that

$$A(t) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right), \quad (18)$$

where

$$I_\beta(\tau) \stackrel{\text{def}}{=} \int_{-\beta}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta. \quad (19)$$

Here $\tau = \frac{\Gamma_0 t}{\hbar} \equiv \frac{t}{\tau_0}$, τ_0 is the lifetime and $\beta = \frac{E_0 - E_{\min}}{\Gamma_0}$. The integral $I_\beta(t)$ can be expressed in terms of special functions as follows:

$$I_\beta(\tau) = 2\pi e^{-\frac{\tau}{2}} + i \left\{ e^{-\frac{\tau}{2}} E_1 \left(-i \left(\beta - \frac{i}{2} \right) \tau \right) - e^{+\frac{\tau}{2}} E_1 \left(-i \left(\beta + \frac{i}{2} \right) \tau \right) \right\}, \quad (20)$$

where $E_1(z)$ denotes the integral–exponential function defined according to [28, 29] (z is a complex number).

Next using this $A(t)$ given by Eqs. (18), (19) and (14), defining the effective Hamiltonian $h_\phi(t)$, one finds that within the Breit–Wigner model considered

$$h(t) = i\hbar \frac{1}{A(t)} \frac{\partial A(t)}{\partial t} = E_0 + \Gamma_0 \frac{J_\beta(\frac{\Gamma_0 t}{\hbar})}{I_\beta(\frac{\Gamma_0 t}{\hbar})}, \quad (21)$$

where

$$J_\beta(\tau) = \int_{-\beta}^{\infty} \frac{x}{x^2 + \frac{1}{4}} e^{-ix\tau} dx. \quad (22)$$

It is important to be aware of the following problem: Namely from the definition of $J_\beta(\tau)$ one can conclude that $J_\beta(0)$ is undefined ($\lim_{\tau \rightarrow 0} J_\beta(\tau) = \infty$). This is because within the model defined by the Breit–Wigner distribution of the energy density, $\omega_{\text{BW}}(E)$, the expectation value of $\mathfrak{H}$, that is, $\langle \phi | \mathfrak{H} | \phi \rangle$, is not finite. So all the considerations based on the use of $J_\beta(\tau)$ are valid only for $\tau > 0$.

Note that simply

$$J_\beta(\tau) \equiv i \frac{\partial I_\beta(\tau)}{\partial \tau}, \quad (23)$$

which allows one to find analytical form of $J_\beta(\tau)$ having such a form for $I_\beta(\tau)$.

We need to know the energy of the system in the unstable state $|\phi\rangle$ considered. The instantaneous energy $E_\phi(t)$ of the system in the unstable state $|\phi\rangle$ is given by Eq. (16). So within the Breit–Wigner model one finds that

$$E_\phi(t) = E_0 + \Gamma_0 \Re \left[\frac{J_\beta(\frac{\Gamma_0 t}{\hbar})}{I_\beta(\frac{\Gamma_0 t}{\hbar})} \right], \quad (24)$$

or, equivalently,

$$\kappa(t) \stackrel{\text{def}}{=} \frac{E_\phi(t) - E_{\min}}{E_0 - E_{\min}} = 1 + \frac{1}{\beta} \Re \left[\frac{J_\beta(\frac{\Gamma_0 t}{\hbar})}{I_\beta(\frac{\Gamma_0 t}{\hbar})} \right]. \quad (25)$$

(This relation, i.e. $\kappa(t)$, was studied, for example in [45, 46].)

It is relatively simple to find asymptotic expressions $I_\beta\tau$ and $J_\beta(\tau)$ for $\tau \rightarrow \infty$ directly from (19) and (22), using, e.g., the method of integration by parts. We have, for $\tau \rightarrow \infty$,

$$I_\beta(\tau) \simeq \frac{i}{\tau} \frac{e^{i\beta\tau}}{\beta^2 + \frac{1}{4}} \left\{ -1 + \frac{2\beta}{\beta^2 + \frac{1}{4}} \frac{i}{\tau} + \left[\frac{2}{\beta^2 + \frac{1}{4}} - \frac{8\beta^2}{(\beta^2 + \frac{1}{4})^2} \right] \left(\frac{i}{\tau} \right)^2 + \dots \right\} \quad (26)$$

and

$$J_\beta(\tau) \simeq \frac{i}{\tau} \frac{e^{i\beta\tau}}{\beta^2 + \frac{1}{4}} \left\{ \beta + \left[1 - \frac{2\beta^2}{\beta^2 + \frac{1}{4}} \right] \frac{i}{\tau} + \frac{\beta}{\beta^2 + \frac{1}{4}} \left[\frac{8\beta^2}{\beta^2 + \frac{1}{4}} - 6 \right] \left(\frac{i}{\tau} \right)^2 + \dots \right\}. \quad (27)$$

These two last asymptotic expressions allow one to find for $\tau \rightarrow \infty$ the asymptotic form of the ratio $\frac{J_\beta(\tau)}{I_\beta(\tau)}$ used in Eqs. (21), (24) and (25); it has a much simpler form than the asymptotic expansions for $I_\beta(\tau)$ and $J_\beta(\tau)$. One finds that, for $\tau \rightarrow \infty$,

$$\frac{J_\beta(\tau)}{I_\beta(\tau)} \simeq -\beta - \frac{i}{\tau} - \frac{2\beta}{\beta^2 + \frac{1}{4}} \frac{1}{\tau^2} + \dots. \quad (28)$$

Starting from this asymptotic expression and Eq. (24) or making use of the asymptotic expansion of $E_1(z)$ [29] and (20),

$$E_1(z)_{|z| \rightarrow \infty} \sim \frac{e^{-z}}{z} \left(1 - \frac{1}{z} + \frac{2}{z^2} - \dots \right), \quad (29)$$

where $|\arg z| < \frac{3}{2}\pi$, one finds, e.g., that, for $t \rightarrow \infty$,

$$E_\phi(t)_{t \rightarrow \infty} \simeq E_{\min} - 2 \frac{E_0 - E_{\min}}{|h_\phi^0 - E_{\min}|^2} \left(\frac{\hbar}{t} \right)^2, \quad (30)$$

where $h_\phi^0 = E_0 - \frac{i}{2}\Gamma_0$. This last relation is valid for $t > T$, where T denotes the cross-over time, i.e. the time when exponential and late time inverse power law contributions to the survival amplitude begin to be comparable.

Some cosmological scenarios predict the possibility of decay of the Standard Model vacuum at an inflationary stage of the evolution of the Universe (see, e.g., [47] and also [48] and the references therein) or earlier. Of course this decaying Standard Model vacuum is described by the quantum state corresponding to a local minimum of the energy density, which is not the absolute minimum of the energy density of the system considered (see, e.g., Fig. 1). The scenario in which false vacuum may decay at the inflationary stage of the Universe corresponds with the hypothesis analyzed by Krauss and Dent [24, 25]. Namely in the mentioned papers the hypothesis that some false vacuum regions do survive well up to the cross-over time T or later was considered where T is the same cross-over time as is considered within the theory of evolving in time quantum unstable systems. The fact that the decay of the false vacuum is a quantum decay process means that the state vector corresponding to the false vacuum is a quantum unstable (or metastable) state. Therefore all the general properties of quantum unstable systems must also occur in the case of such a quantum unstable state as the false vacuum. This applies in particular to such properties as late time deviations from the exponential decay law and properties of the energy $E_0^{\text{false}}(t)$ of the system in the quantum false vacuum state at late times $t > T$. In [49] it was pointed out that the energy of those false vacuum regions which survived up to T and much later differs from E_0^{false} [49].

So within the cosmological scenario in which the decay of a false vacuum is assumed the unstable state $|\phi\rangle$ corresponds to the false vacuum state: $|\phi\rangle = |0\rangle^{\text{false}}$. Then $|0\rangle^{\text{true}}$ is the true vacuum state, that is, the state corresponding to the true minimal energy. In such a case $E_0 \rightarrow E_0^{\text{false}}$ is the energy of a state corresponding to the false vacuum measured at the canonical decay time (the exponential decay regime) and E_0^{true} is the energy of true vacuum (i.e., the true ground state of the system), so $E_0^{\text{true}} \equiv E_{\min}$. The corresponding quantum mechanical process looks as shown in Fig. 1.

If one wants to generalize the above results, obtained on the basis of quantum mechanics, to quantum field theory one should take into account among others a volume factor so that survival probabilities per unit volume should be considered and similarly the energies and the decay rate, $E \mapsto \rho(E) = \frac{E}{V_0}$, $\Gamma_0 \mapsto \gamma = \frac{\Gamma_0}{V_0}$, where $V_0 = V(t_0)$ is the volume of the considered system at the initial instant t_0 , when the time evolution starts. The volume V_0 is used in these considerations because the initial unstable state $|\phi\rangle \equiv |0\rangle^{\text{false}}$ at $t = t_0 = 0$ is expanded into eigenvectors $|E\rangle$ of $\hat{H}$ at this initial instant t_0 (where $E \in \sigma_c(\hat{H})$) and then this expansion is used to find the density of the energy distribution $\omega(E)$.

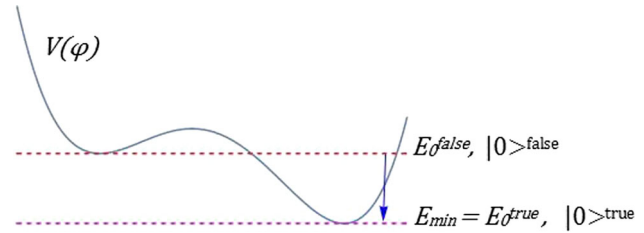


Fig. 1 Transition of the system from the false vacuum state $|0\rangle^{\text{false}}$ to the true ground state of the system, i.e. the true vacuum state $|0\rangle^{\text{true}}$. The states $|0\rangle^{\text{false}}$ and $|0\rangle^{\text{true}}$ correspond to the local minimum and to the true lowest minimum of the potential $V(\phi)$ of the scalar field ϕ , respectively

It is easy to see that the mentioned changes $E \mapsto \frac{E}{V_0}$ and $\Gamma_0 \mapsto \frac{\Gamma_0}{V_0}$ do not change the integrals $I_\beta(t)$ and $J_\beta(t)$ and Eq. (25). Similarly in such a situation the parameter $\beta = \frac{E_0 - E_{\min}}{\Gamma_0}$ does not change. This means that Eqs. (24), (25), (30) can be replaced by the corresponding relations for the densities ρ_{de} or Λ (see, e.g., [45, 51, 52]). Within such an approach $E(t)$ corresponds to the running cosmological constant $\Lambda(t)$ and $E_{\min}$ to the Λ_{bare} . The parametrization used in next sections is based on Eqs. (24) and (25). The integrals (3), (4) introduced in Sect. 1 are obtained from (22) and (19) replacing β by $\frac{1-\alpha}{\alpha}$. Similarly solutions (5) and (6) correspond to (20) and to the function $J_\beta(\tau)$ obtained from (20) using (23).

3 Cosmological equations with

$$\rho_{\text{de}} = \Lambda_{\text{bare}} + E_{\text{R}} \left[1 + \frac{\alpha}{1-\alpha} \Re \left(\frac{J(t)}{I(t)} \right) \right]$$

The cosmological model with the parametrization of the dark energy (1), belonging to the class of parametrizations proposed in [45] after putting $E_{\text{R}} = E_0 - \Lambda_{\text{bare}}$, assumes the following form of ρ_{de} (we use units $8\pi G = c = 1$):

$$\rho_{\text{de}} = \Lambda_{\text{bare}} + E_{\text{R}} \left[1 + \frac{\alpha}{1-\alpha} \Re \left(\frac{J(t)}{I(t)} \right) \right]. \quad (31)$$

It can be introduced as the covariant theory from the following action:

$$S = \int \sqrt{-g} (R + \mathcal{L}_{\text{m}}) d^4x, \quad (32)$$

where R is the Ricci scalar, $\mathcal{L}_{\text{m}}$ is the Lagrangian for the barotropic fluid and $g_{\mu\nu}$ is the metric tensor. We assume the signature of the metric tensor to be $(+, -, -, -)$ and, for simplicity, we assume that the constant curvature is zero (the flat model). The Ricci scalar for the Friedmann–Lemaître–Robertson–Walker (FLRW) metric is represented by the following formula:

$$R = -6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] \quad (33)$$

where a dot means differentiation with respect to the cosmological time t .

The Lagrangian for the barotropic fluid is expressed by the formula

$$\mathcal{L}_m = -\rho_{\text{tot}} \left(1 + \int \frac{p_{\text{tot}}(\rho_{\text{tot}})}{\rho_{\text{tot}}^2} d\rho_{\text{tot}} \right), \quad (34)$$

where ρ_{tot} is the total density of fluid and $p_{\text{tot}}(\rho_{\text{tot}})$ is the total pressure of fluid [53]. We assume that this fluid consists of three components: the baryonic matter ρ_b , the dark matter ρ_{dm} and the dark energy ρ_{de} . We treat the baryonic matter and the dark matter like dust. In consequence the equations of state for them are the following: $p_b(\rho_b) = 0$ and $p_{\text{dm}}(\rho_{\text{dm}}) = 0$. The equation of state for the dark energy is assumed in the form $p_{\text{de}}(\rho_{\text{de}}) = -\rho_{\text{de}}$.

Of course, the total density is expressed by $\rho_{\text{tot}} = \rho_b + \rho_{\text{dm}} + \rho_{\text{de}}$ and the total pressure is expressed by $p_{\text{tot}}(\rho_{\text{tot}}) = p_{\text{de}}(\rho_{\text{de}}) = -\rho_{\text{de}}$.

We can find the Einstein equations using the method of calculus of variations by variation of the action (32) by the metric $g_{\mu\nu}$. Then we get two equations: the Friedmann equation

$$3H^2 = 3\frac{\dot{a}^2}{a^2} = \rho_{\text{tot}} = \rho_b + \rho_{\text{dm}} + \rho_{\text{de}}, \quad (35)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function, and the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_{\text{tot}} + 3p_{\text{tot}}(\rho_{\text{tot}})) = \rho_b + \rho_{\text{dm}} - 2\rho_{\text{de}}. \quad (36)$$

From Eqs. (35) and (36) we can get the conservation equation

$$\dot{\rho}_{\text{tot}} = -3H(\rho_{\text{tot}} + p_{\text{tot}}(\rho_{\text{tot}})). \quad (37)$$

The above equation can be rewritten as

$$\dot{\rho}_m = -3H\rho_m - \dot{\rho}_{\text{de}}, \quad (38)$$

where $\rho_m = \rho_b + \rho_{\text{dm}}$.

Let Q be the interaction between the dark matter and the dark energy. Then Eq. (38) is equivalent to the following equations:

$$\dot{\rho}_b = -3H\rho_b, \quad (39)$$

$$\dot{\rho}_{\text{dm}} = -3H\rho_{\text{dm}} + Q, \quad (40)$$

and

$$\dot{\rho}_{\text{de}} = -Q, \quad (41)$$

where the interaction Q is defined by Eq. (41). The interaction between the dark matter and the dark energy can be interpreted as the energy transfer in the dark sector. If $Q > 0$ then the energy flow is from the dark energy to the dark matter. If $Q < 0$ then the energy flow is from the dark matter to the dark energy.

For the description of the evolution of the Universe it is necessary to use the Friedmann equation (35) and the conservation equation (38). These formulas can be rewritten in terms of dimensionless parameters. Let $\Omega_m = \frac{\rho_m}{3H_0^2}$ and $\Omega_{\text{de}} = \frac{\rho_{\text{de}}}{3H_0^2}$, where H_0 is the present value of the Hubble function. Then from Eqs. (35) and (38), we get

$$\frac{H^2}{H_0^2} = \Omega_m + \Omega_{\text{de}} \quad (42)$$

and

$$\dot{\Omega}_m = -3H\Omega_m - \dot{\Omega}_{\text{de}}. \quad (43)$$

The above equations are sufficient to find the behavior of the matter, the dark energy, the Hubble function and the scale factor as a function of cosmological time. We cannot find the exact solutions because these equations are too complicated. In this case we should search for numerical solutions. The behavior of the dark energy is presented in Figs. 2 and 3. Figure 2 shows the diagram of the dependence $\Omega_{\text{de}}(\tau)$ with respect of the rescaled time τ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. On the diagram we can see that the start value of the dark energy density, which is equal to $\Omega_{\text{de}} \approx 10^{120}$, is reduced to the present value of the dark energy density, which is $\Omega_{\text{de}} \approx 0.7$. This final value of Ω_{de} does not depend on the values of the parameters α and $\frac{E_0}{3H_0^2}$. Therefore, this mechanism makes an attempt of solving the cosmological

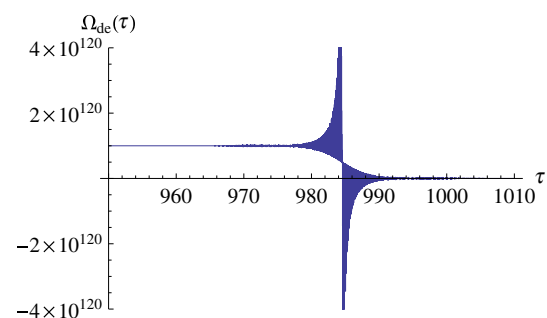


Fig. 2 The dependence $\Omega_{\text{de}}(\tau)$ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. The rescaled time τ is given in units of $[1.3 \times 10^{-40} \text{ s}]$

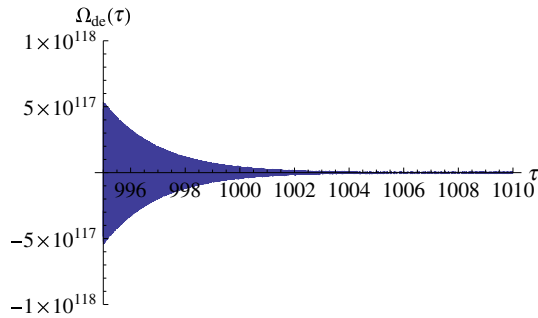


Fig. 3 The dependence $\Omega_{\text{de}}(\tau)$ during the intermediate phase of damped oscillations for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. The rescaled time τ is given in units of $[1.3 \times 10^{-40} \text{ s}]$

constant problem. For late time, dark energy can be treated as the cosmological constant. The characteristic of the intermediate oscillatory regime is depending on the parameter α . With the increasing value of α the number of oscillations, their amplitude, their period as well as the length of this regime decreases. If $\alpha > 0.4$ then oscillations begin to disappear and the value of Ω_{de} jumps to the constant value of 0.7.

Figure 3 shows the diagram of the dependence $\Omega_{\text{de}}(\tau)$ during the intermediate phase of damped oscillations with respect of the time τ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. Note that the dark energy oscillates and the amplitude of the oscillations decreases with time. In consequence the dark energy can be treated as the cosmological constant after the intermediate phase of oscillations. Figure 4 shows the diagrams of the dependence $\Omega_{\text{de}}(\tau)$ with respect of the time τ for different values of α ($\alpha = 0.2, 0.4, 0.8$) and $\frac{E_0}{3H_0^2} = 10^{120}$. This figure presents how the evolution of $\Omega_{\text{de}}(\tau)$ is dependent on the parameter α . Note that the oscillations disappear for $\alpha > 0.4$.

In general, if α decreases then the times when oscillatory regime takes place increase. This means that passage from the very high energies to the extremely small energies, which takes place at the oscillatory regime, moves in the direction of increasing time with decreasing α and for a suitable small value of α this oscillatory regime can occur at relatively late times.

Figure 5 presents the evolution of $\frac{d\Omega_{\text{de}}}{d\tau}$. The evolution of matter is demonstrated in Fig. 6 and the Hubble function is presented in Fig. 7. The diagram of the scale factor with respect to the cosmological time is presented in Fig. 8.

We have $\tau = \frac{\alpha(E_0 - \Lambda_{\text{bare}})}{h(1-\alpha)} V_0 t$; therefore if the value of the parameter α increases then the damping of oscillations should also be increased. In the limiting case, if α is equal zero then we get the Λ CDM model. This last conclusion can easily be drawn analyzing the late time properties of ρ_{de} .

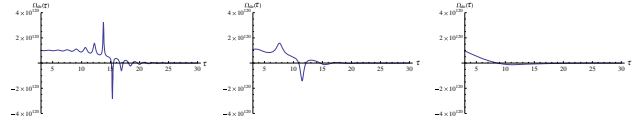


Fig. 4 The dependence $\Omega_{\text{de}}(\tau)$ for $\alpha = 0.2$ (left figure) and $\alpha = 0.4$ (medium figure) and $\alpha = 0.8$ (right figure) and $\frac{E_0}{3H_0^2} = 10^{120}$. The rescaled time τ for the left figure is given in units of $[5.3 \times 10^{-145} \text{ s}]$, for the center figure is given in units of $[2.0 \times 10^{-145} \text{ s}]$ and for the right figure is given in units of $[3.3 \times 10^{-146} \text{ s}]$

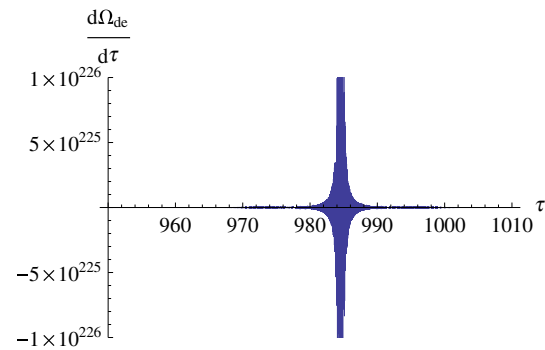


Fig. 5 The dependence $\frac{d\Omega_{\text{de}}}{d\tau}(\tau)$ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. Note that, for negative value of $\frac{d\Omega_{\text{de}}}{d\tau}$, the energy is transferred from the dark energy to the dark matter and for the positive value of $\frac{d\Omega_{\text{de}}}{d\tau}$, the energy is transferred from the dark matter to the dark energy. The rescaled time τ is given in unit $[1.3 \times 10^{-40} \text{ s}]$

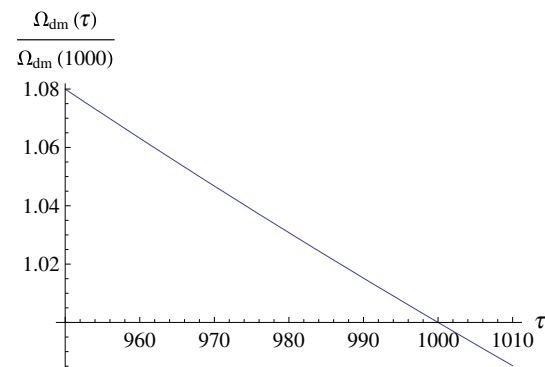


Fig. 6 The dependence Ω_{dm} for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. We include the influence of the radiation for the evolution of the matter. Note that the dark energy has a negligible influence on the evolution of the matter. The rescaled time τ is given in units of $[1.3 \times 10^{-40} \text{ s}]$

For the late time, $\tau \rightarrow \infty$, according to Eq. (28), the parametrization of dark energy (31) can be approximated by the following expression:

$$\rho_{\text{de}} = \Lambda_{\text{bare}} - 2E_{\text{R}} \frac{\alpha^2}{(1-\alpha)^2 + \frac{\alpha^2}{4}} \frac{1}{\tau^2} + \dots \quad (44)$$

From this relation the important observation follows: For any $\alpha > 0$ the Λ CDM model is the limiting case, when

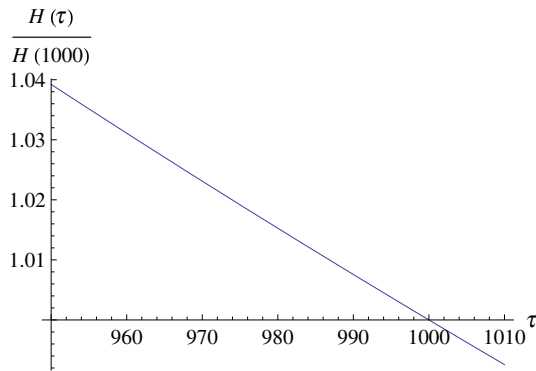


Fig. 7 The dependence $H(\tau)$ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. We include influence of the radiation for the evolution of the Hubble function. Note that dark energy has a negligible influence on the evolution of the Hubble function. The rescaled time τ is given in units of $[1.3 \times 10^{-40}s]$

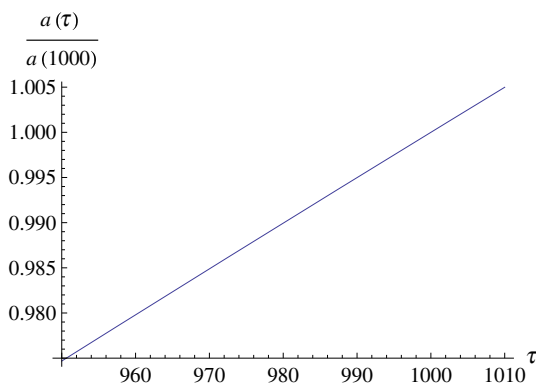


Fig. 8 The dependence $a(\tau)$ for $\alpha = 10^{-105}$ and $\frac{E_0}{3H_0^2} = 10^{120}$. We include the influence of the radiation on the evolution of the scale factor. Note that dark energy has negligible influence on the evolution of the scale factor. The rescaled time τ is given in units of $[1.3 \times 10^{-40}s]$

$\tau \rightarrow \infty$, of our model. So for very, very late times the results obtained within our model and within the Λ CDM model have to coincide. This parametrization of the dark energy was considered in [52,54,55].

The dark energy is significantly lower than the energy density of matter in the early Universe, which has the consequence that the transfer to the dark sector is negligible (see Fig. 5). Our model makes an attempt of solving the cosmological constant problem. In general, the amplitude of oscillations of the dark energy decreases with time.

Thus for the late time Universe, oscillations are negligible and the dark energy has the form of the cosmological constant.

The conservation equation for the dark energy (41) can be rewritten as

$$\dot{\rho}_{\text{de}} = -3H(\rho_{\text{de}} + p_{\text{de}}), \quad (45)$$

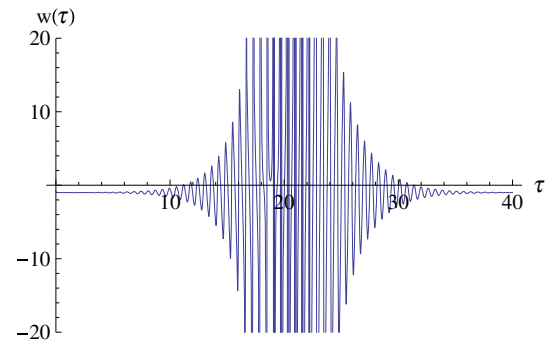


Fig. 9 The typical dependence $w(\tau)$. This example is for $\alpha = 0.09$ and $\frac{E_0}{3H_0^2} = 10^{120}$. Note that after the intermediate phase of oscillations, the function $w(\tau)$ can be treated as a constant, which is equal to -1 . The rescaled time τ is given in units of $[1.3 \times 10^{-144}s]$

where p_{de} is an effective pressure of the dark energy. In this case the equation of state for the dark energy is expressed by the following formula:

$$p_{\text{de}} = w(t)\rho_{\text{de}}, \quad (46)$$

where the function $w(t)$ is given by the expression

$$w(t) = -1 - \frac{\dot{\rho}_{\text{de}}}{\sqrt{3}\sqrt{\rho_{\text{m}} + \rho_{\text{de}}\rho_{\text{de}}}} = -1 - \frac{1}{3H} \frac{d \ln \rho_{\text{de}}}{dt}. \quad (47)$$

The diagram of coefficient equation of state $w(t)$ is presented in Fig. 9. The function $w(t)$, for the late time, is a constant and equals -1 , which means that it describes the cosmological constant parameter. Note that the function $w(t)$ is also equal -1 , which means that ρ_{de} is constant as a consequence of the conservation condition (transfer between the sectors is negligible). Therefore, the energy transfer is an effective process only during the intermediate oscillation period (quantum regime).

Let $\rho_{\text{de}} \gg \rho_{\text{m}}$. Then our model predicts inflation. The formula for the e-foldings $N = H_{\text{init}}(t_{\text{fin}} - t_{\text{init}})$ (see [56]) becomes the following expression for our model:

$$N = \sqrt{\frac{E_0}{3}}(t_{\text{fin}} - t_{\text{init}}), \quad (48)$$

where $t_{\text{init}} \approx 0$ and t_{fin} is the time of appearing of the intermediate phase of oscillations. Figure 10 presents the evolution of the scale factor a with respect to the cosmological time during inflation.

4 Statistical analysis

To estimate the model parameters we use the astronomical observations such as the supernovae of type Ia (SNIa), BAO,

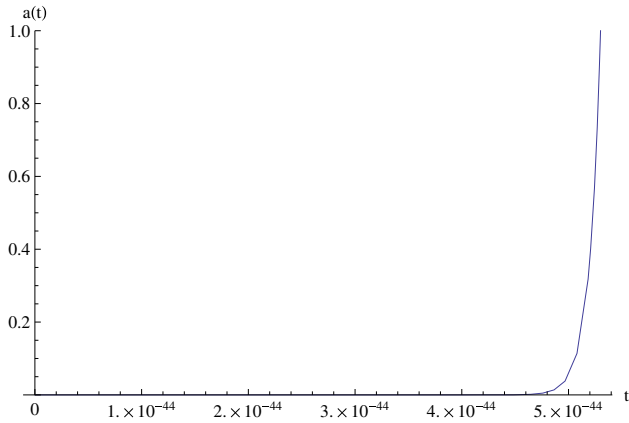


Fig. 10 The dependence $a(t)$ for $\frac{E_0}{3H_0^2} = 2 * 10^{125}$. We assume that $\rho_{de} \gg \rho_m$ and the intermediate phase of oscillations is after the Planck epoch. Note that, for the above assumptions, inflation appears after the Planck epoch. The characteristic number of e-foldings of this inflation is equal to 53 here. The cosmological time t is given in seconds

measurements of $H(z)$ for galaxies, the Alcock–Paczynski test and the measurements CMB.

The data of supernovae of type Ia, which were used in this paper, are taken from the Union 2.1 dataset [57]. In this context we use the following likelihood function:

$$\ln L_{\text{SNIa}} = -\frac{1}{2} [A - B^2/C + \log(C/(2\pi))], \quad (49)$$

where $A = (\mu^{\text{obs}} - \mu^{\text{th}})\mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $B = \mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $C = \text{Tr}\mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for SNIa. The observer distance modulus μ^{obs} is defined by the formula $\mu^{\text{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of SNIa). The theoretical distance modulus is given by $\mu^{\text{th}} = 5 \log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$).

We use the following BAO data: Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [58], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [59], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [60]. The likelihood function is defined by the expression

$$\ln L_{\text{BAO}} = -\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right), \quad (50)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch [61, 62].

Measurements of the Hubble parameter $H(z)$ of galaxies were taken from [63–65]. The likelihood function is given by the following formula:

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2. \quad (51)$$

The likelihood function for the Alcock–Paczynski test [66, 67] has the following form:

$$\ln L_{AP} = -\frac{1}{2} \sum_i \frac{(AP^{\text{th}}(z_i) - AP^{\text{obs}}(z_i))^2}{\sigma^2}, \quad (52)$$

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data [68–76].

In this paper, the likelihood function for the measurements of CMB [77] and lensing by Planck, and low- ℓ polarization from the WMAP (WP), has the following form:

$$\ln L_{\text{CMB+lensing}} = -\frac{1}{2} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \mathbb{C}^{-1} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}), \quad (53)$$

where $\mathbb{C}$ is the covariance matrix with the errors, $\mathbf{x}$ is a vector of the acoustic scale l_A , the shift parameter R and $\Omega_b h^2$ where

$$l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')} \quad (54)$$

$$R = \sqrt{\Omega_{m,0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}, \quad (55)$$

where z^* is the redshift of the epoch of the recombination [61].

In this paper, the final formula for the likelihood function is given in the following form:

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB+lensing}}. \quad (56)$$

The statistical analysis was done by our own code CosmoDarkBox. This code uses the Metropolis–Hastings algorithm [78, 79].

We estimated four cosmological parameters: H_0 , $\Omega_{m,0}$, α and the parameter E_0 . Our statistical results are completed in Table 1. We present intersections of the likelihood function with 68 and 95% confidence level projections in Figs. 11, 12, 13 and 14. PDF diagrams for α and $\frac{E_0}{3H_0^2}$ are presented in Figs. 15 and 16.

The values of the likelihood function are not always sensitive to changing of the parameters α and E_0 . The possible changing of the values of the likelihood function are beyond

Table 1 The best fit and errors for the estimated model for SNIa + BAO + $H(z)$ + AP + CMB test with H_0 from the interval (66.0, 72.0), $\Omega_{m,0}$ from the interval (0.27, 0.34). $\Omega_{b,0}$ is assumed as 0.048468

Parameter	Best fit	68% CL	95% CL
H_0	68.82 km/(s Mpc)	+0.61 −0.55	+0.98 −0.92
$\Omega_{m,0}$	0.3009	+0.0079 −0.0084	+0.0133 −0.0134

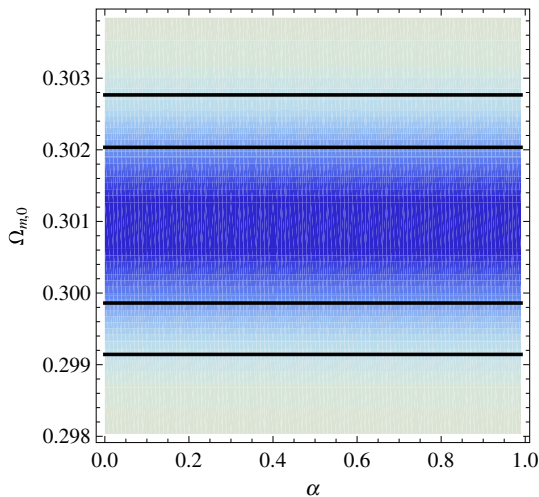


Fig. 11 The intersection of the likelihood function of two model parameters $(\Omega_{m,0}, \alpha)$, with the marked 68 and 95% confidence levels. The plane of the intersection is the best fit of H_0 ($H_0 = 68.82 \left[\frac{\text{km}}{\text{s} \times \text{Mpc}} \right]$). We assumed that $E_0/(3H_0^2)$ is equal to 10^{120} , but changing of the value of $E_0/(3H_0^2)$ does not influence the results. Note that the values of the likelihood function are not sensitive to changing of the parameter α

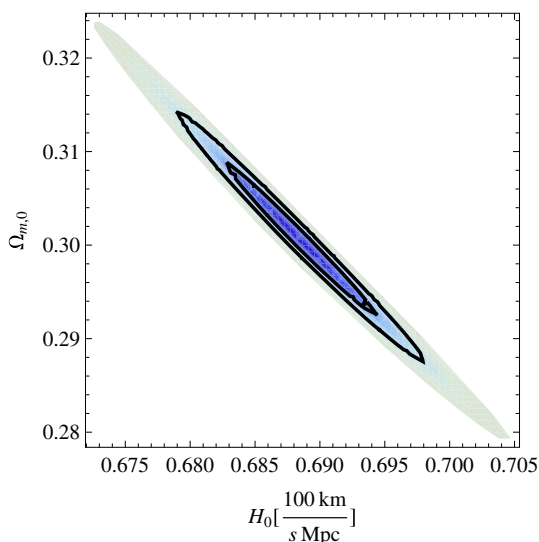


Fig. 12 The intersection of the likelihood function of two model parameters $(\Omega_{m,0}, H_0)$, with the marked 68 and 95% confidence levels. The plane of the intersection is $\alpha = 0.5$ and $E_0 = 10^{120}$

the capabilities of numerical methods. This fact can be interpreted as the lack of sensitivity of the present evolution of the Universe for changing of the parameters α and E_0 . The best fit values of H_0 and Ω_m for our model are equivalent of the best fit values for the Λ CDM model.

5 Conclusion

The main goal of our paper was to analyze the cosmological model with the running dark energy as well as the dark matter

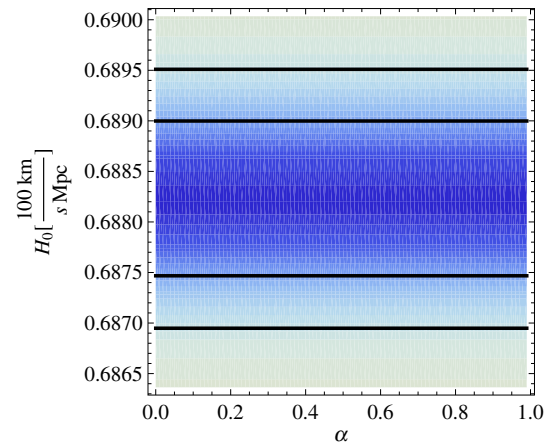


Fig. 13 The intersection of the likelihood function of two model parameters (H_0, α) , with the marked 68 and 95% confidence levels. The plane of the intersection is the best fit of $\Omega_{m,0}$ ($\Omega_{m,0} = 0.3009$). We assumed that $E_0/(3H_0^2)$ is equal to 10^{120} , but changing of the value of $E_0/(3H_0^2)$ does not influence the results. Note that the values of the likelihood function are not sensitive to changing of the parameter α

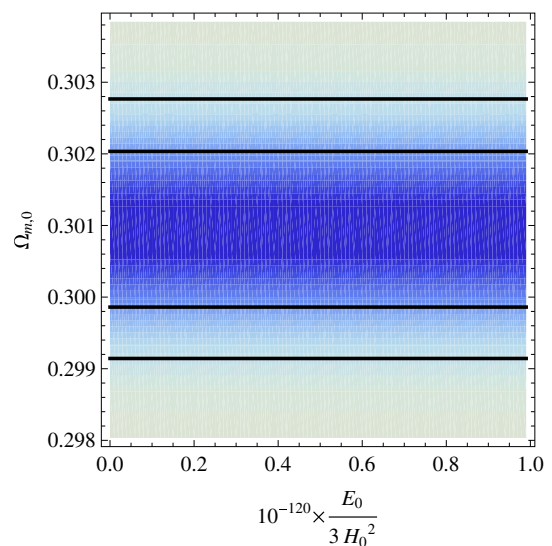


Fig. 14 The intersection of the likelihood function of two model parameters $(\Omega_{m,0}, \frac{E_0}{3H_0^2})$, with the marked 68 and 95% confidence levels. The plane of the intersection is the best fit of H_0 ($H_0 = 68.82 \left[\frac{\text{km}}{\text{s} \times \text{Mpc}} \right]$). We assumed that α is equal to 0.1, but changing of the value of α does not influence the results. Note that the values of the likelihood function are not sensitive to changing of $\frac{E_0}{3H_0^2}$

and the baryonic matter in the form of dust. We considered the evolution of the dark energy using the fact that the decay of a false vacuum to the true vacuum is a quantum decay process. From the cosmological point of view this model was formulated in terms of the cosmological model with the interaction between dark matter and dark energy.

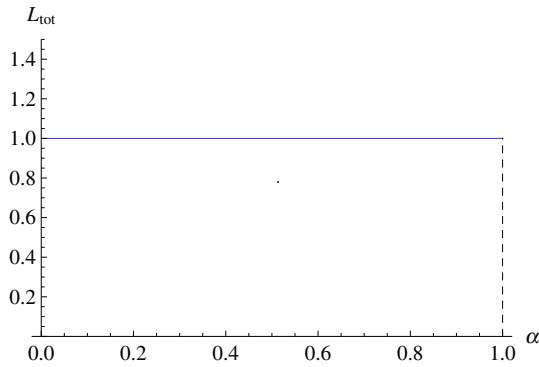


Fig. 15 Diagram of PDF for parameter α obtained as an intersection of the likelihood function. Two planes of the intersection likelihood function are $H_0 = 68.82 \left[\frac{\text{km}}{\text{s} \times \text{Mpc}} \right]$ and $\Omega_{m,0} = 0.3009$. The planes of intersection are constructed from the best fitting value of the model parameters. We assume the value of α from the interval (0, 1). Note that the values of the likelihood function are not sensitive to changing of α

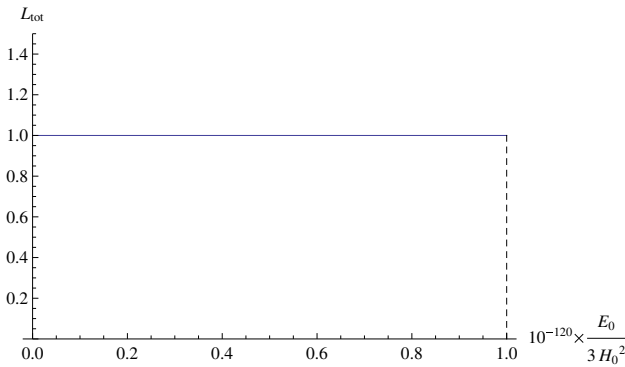


Fig. 16 Diagram of PDF for $\frac{E_0}{3H_0^2}$ obtained as an intersection of the likelihood function. Two planes of intersection likelihood function are $H_0 = 68.82 \left[\frac{\text{km}}{\text{s} \times \text{Mpc}} \right]$ and $\Omega_{m,0} = 0.3009$. The planes of intersection are constructed from the best fitting value of the model parameters. We assume the value of $\frac{E_0}{3H_0^2}$ from the interval (0, 10^{120}). Note that the values of the likelihood function are not sensitive to changing of $\frac{E_0}{3H_0^2}$

We detected the intermediate phase of oscillations between phases of constant dark energy. The preceding phase has $\rho_{\text{de}} = E_0$ and the following phase has $\rho_{\text{de}} = \Lambda_{\text{bare}}$. Defining this class of models parametrized with α (the deviation from the Λ CDM model) we have found two different types of dynamical behavior. Independently of $0 < \alpha < 1$ there is a universal mechanism of jumping of the value of energy density of dark energy from the initial value of $E_0 = 10^{120}$ to the present value of the cosmological constant of 0.7.

During this epoch there is the oscillatory behavior of energy density of dark energy as well as its coefficient equation of state. In this intermediate regime the amplitude of the

oscillations increases, then there is a jump down followed by the decreasing oscillations. This kind of oscillation appears for $0 < \alpha < 0.4$. The number, period and amplitude of oscillations as well as the length of this intermediate regime decreases as the parameter α grows. For $\alpha > 0.4$ the oscillations disappear and only the jump down of energy density of dark energy remains. The jump down mechanism is independent from the value of the parameter α , which leads to solving the cosmological constant problem.

In the early Universe the energy density of dark energy is significantly lower than the energy density of dark matter, therefore the change of energy density of the dark matter, which is caused by energy transfer in the dark sector, is negligible.

While our model makes an attempt of an explanation of the cosmological constant problem, the coincidence problem is still open as we forced the model to have an exit on the present value of the cosmological constant. In the early Universe, the dark energy oscillates. But the amplitude of the oscillations decreases with time. In consequence, for the late time Universe, oscillations are negligible and the dark energy can be described as the cosmological constant. Unfortunately our model cannot explain why the present value of dark energy and matter are comparable.

From the statistical analysis of the model we found that the model is generic in the sense that independently from the values of the parameters α and E_0 we can obtain the present value of the energy density of the dark energy. Therefore, the Λ CDM model is an attractor which all models with different values of parameters α and E_0 can reach. The final interval of evolution for which we have data at our disposal is identical for a whole class of models, therefore it is impossible to find best-fitted values of the model parameters and indicate one particular model (degeneration problem).

As should be expected it is difficult to discriminate the parameters of the early state of the Universe as there is no data for very high redshift. In Figs. 15 and 16 the likelihood functions for parameters of interest are flat, so there is no best fit value. That is why we take calibrated values of these parameters for further analysis in this paper. We assume that the false vacuum energy is 10^{120} as is indicated from the quantum field theory. On the other hand the parameter α should be chosen to get the decaying process of false vacuum to take place after the Planck era.

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Quantum mechanical look at the radioactive-like decay of metastable dark energy

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Abstract We derive the Shafieloo, Hazra, Sahni and Starobinsky (SHSS) phenomenological formula for the radioactive-like decay of metastable dark energy directly from the principles of quantum mechanics. To this aim we use the Fock–Krylov theory of quantum unstable states. We obtain deeper insight on the decay process as having three basic phases: the phase of radioactive decay, the next phase of damping oscillations, and finally the phase of power-law decay. We consider the cosmological model with matter and dark energy in the form of decaying metastable dark energy and study its dynamics in the framework of non-conservative cosmology with an interacting term determined by the running cosmological parameter. We study the cosmological implications of metastable dark energy and estimate the characteristic time of ending of the radioactive-like decay epoch to be 2.2×10^4 of the present age of the Universe. We also confront the model with astronomical data which show that the model is in good agreement with the observations. Our general conclusion is that we are living in the epoch of the radioactive-like decay of metastable dark energy which is a relict of the quantum age of the Universe.

1 Introduction

We follow Krauss and Dent's paper and apply the Fock–Krylov theory of unstable quantum states to analyze a cosmological scenario with decaying dark energy [1–5]. For this purpose we extend the Shafieloo, Hazra, Sahni and Starobinsky (SHSS) model of metastable dark energy with radioactive-like decay [6] and we give physical motivation arising directly from quantum mechanics for phenomenolog-

ical formulas for SHSS model of the dark energy. We replace the radioactive, classical physics constant decay rate by the decay rate derived using the Fock–Krylov theory of unstable quantum states.

As a result we obtain a logistic-type radiative decay of dark energy, which is followed by the much slower decay process than the radioactive one, known as the quantum Zeno effect. Within such an approach we find the energy of the system in the unstable state and the decay rate. The rigorous results show that these quantities both are time dependent. We find the exact analytical expression for them assuming that the density of the energy distribution, $\omega(E)$, in the unstable state has the Breit–Wigner form. Using these results we also find the late times asymptotic expressions of these quantities. Then we assume that the dark energy density decays and that this is a quantum process. Starting from these assumptions we use the derived decay rate to analyze the decay process of the dark energy density.

We study the cosmological implications of the derived formula for decaying dark energy in the framework of flat FRW cosmology. We find an extension of the standard cosmological model in the form of an interacting cosmology in which the energy-momentum tensor is not conserved due to the interaction between the dark energy and dark matter by energy transfer.

We consider the problem if the decay of the running lambda term can solve the cosmological constant problem and how it can modify the canonical scaling law of energy density for dark matter. We also test the model by astronomical observations.

Our statistical analysis gives the best fit values of the density parameters for each component of the decaying vacuum of the dark energy. Testing the model with observational data we have found that dark energy can decay in three distinguished ways: exponentially, by damping oscillation and in power-law decay. We show that the main contribution to the

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decay of the metastable vacuum is the dark energy decay of an exponential type and this type of decay dominates up to $2.2 \times 10^4 T_0$, where T_0 is the present age of the Universe. Our calculations show that the exponential decay has only an intermediate character and will be replaced in the future evolution of the Universe by an oscillation decay and decay of $1/t^2$ type. From the estimation of the model parameters we see that the decay half life should be much larger than the age of the Universe.

Today modern cosmology has the methodological status of some effective theory, which is described very well by current astronomical observations in terms of dark matter and dark energy. However, there are many open problems related to the unknown nature of dark energy. The cosmological parameter is a good effective description of the accelerating phase of the current Universe, but we do not understand why the today value of this parameter is so small in comparison with its value in the early Universe.

We look for an alternative cosmological model to supersede the Λ CDM model, the present standard cosmological model. Our main motivation is to check if the model considered in the next sections is able to solve the cosmological constant problem. In this paper, we consider the case when the cosmological constant parameter results from the assumption that the vacuum energy is given by the fundamental theory [7].

We assume quantum mechanics as a fundamental theory, which determines the cosmological parameters and explain how the cosmological parameters change during the cosmic evolution. The discussion of the cosmological constant problem is included in Refs. [7–20].

Krauss and Dent [1] analyzed the properties of the false vacuum state from the point of view of the quantum theory of decay processes. They assumed that the decay process of metastable vacuum is a quantum decay process realized as the transition from the state corresponding to the metastable (false) vacuum state to the state corresponding to the lowest energy of the Universe (that is, to the true vacuum state) and thus that this process can be described using the standard quantum formalism usually used to describe the decay of excited atomic levels or unstable particles. They used the Fock–Krylov theory of unstable quantum states [2–5]. One of the famous results of this theory is the proof that unstable quantum systems cannot decay exponentially at very late times and that in such a late time regime any decay process must run slower than any exponentially decreasing function of time [4]. Model calculations show that survival probability exhibits inverse power-law behavior at these times. Krauss and Dent [1] analyzing a false vacuum decay pointed out that in eternal inflation, many false vacuum regions can survive up to much later than the times when the exponential decay law holds. They formulated the hypothesis that some false vacuum regions survive well up to the cross-over time T or

later, where the cross-over time, T , is the time when contributions of the exponential and late time non-exponential parts of the survival probability are of the same order. They gave a simple explanation of such an effect. It may occur even though the regions of false vacua by assumption should decay exponentially, and gravitational effects force space in a region that has not decayed yet to grow exponentially fast. Such a cosmological scenario may be realized if the lifetime of the metastable vacuum state or the dark energy density is much, much shorter than the age of the Universe. It should be of order of times of the age of the inflationary stage of the Universe.

The possibility that our Universe (or some regions in our Universe) were able to survive up to times longer than the cross-over time T should be considered seriously was concluded by Krauss and Dent's analysis [1]. This is impossible within the standard approach of calculations of the decay rate Γ for the decaying vacuum state [21–25]. Calculations performed within this standard approach cannot lead to a correct description of the evolution of the Universe with false vacuum in all cases when the lifetime of the false vacuum state is so short that its survival probability exhibits an inverse power-law behavior at times comparable with the age of the Universe. This conclusion is valid not only when the dark energy density and its late time properties are related to the transition of the Universe from the false vacuum state to the true vacuum, but also when the dark energy is formed by unstable "dark particles". In both cases the decay of the dark energy density is the quantum decay process and only the formalism based on the Fock–Krylov theory of unstable quantum states and used by Krauss and Dent [1] is able to describe correctly such a situation. Note that Landim and Abdalla built a model of metastable dark energy, in which the observed vacuum energy is the value of the scalar potential at the false vacuum [26].

Models with metastable dark energy have recently been discussed in the context of the explanation of the H_0 tension problem [27]. Our model is a quantum generalization of Shafieloo et al.'s model [6] and contains a phase of radioactive-like decay valid in the context of solving this problem. Shafieloo et al. considered three different ways of dark energy decay. In our paper, we investigate the second way of the decay into dark matter. The models of the decay of the dark energy analyzed in [6] can be a useful tool for numerically testing decay processes discussed in [1] and for analyzing the properties of the decaying dark energy at times $t > T$. Namely, Shafieloo et al. [6] analyzed the properties of the model of the time evolution of the dark energy. Their model assumes a "radioactive decay" scheme for decaying dark energy in which the present value of the dark energy density, $\rho_{\text{DE}}(t_0)$, is related to its value at an earlier instant of time, $\rho_{\text{DE}}(t)$, by

$$\rho_{\text{DE}}(t) = \rho_{\text{DE}}(t_0) \times \exp[-\Gamma(t - t_0)] \equiv \rho_{\text{DE}}(t - t_0), \quad (1)$$

where the only free parameter is the decay rate Γ . Shafieloo et al. [6] derived this equation from the fundamental equation of the theory of radioactive decays,

$$\dot{\rho}_{\text{DE}}(t) = -\Gamma \rho_{\text{DE}}(t) \quad (2)$$

(see Eqs. (2.1) and (2.2) in [6]). These equations are known from the Rutherford theory of the decay of radioactive elements. Rutherford deriving these equations assumed that the number decaying radioactive elements at a given instant of time is proportional to a number of these elements at this moment of time [28–31] as in Eq. (2). So the Rutherford equations and thus also Eqs. (1)–(2) are the classical physics equations.

In the context of Eqs. (1)–(2) one may ask what $\rho_{\text{DE}}(t)$ is built from that decays according to radioactive decay law? For physicists the only reasonable explanation for this problem is the assumption that $\rho_{\text{DE}}(t)$ describes the energy of an extremely huge number of particles occupying a volume V_0 at the initial instant of time t_0 and decaying at later times. Of course when such particles can be considered as classical particles, then this process can be described using the classical radioactive decay law. Unfortunately the process of the creation of the Universe is not a classical physics process, but it is a quantum process and particles or states of the system created during such a process exhibit quantum properties and are subject to the laws of quantum physics. The same concerns $\rho_{\text{DE}}(t)$ generated by quantum fluctuations or excitations of a quantum scalar field, which can be described as excited metastable states of this field and the process of their decay is a quantum process. Therefore, as a quantum decay process it exhibits at late times completely different properties than the classical radioactive decay process, as pointed out by Krauss and Dent. Simply, if $\rho_{\text{DE}}(t)$ is related to the extremely huge number of metastable states (excitations of the scalar field or its fluctuations) generated at t_0 in a volume V_0 , it is very likely that many of them can be found undecayed at times longer than the cross-over time T . All this suggests that Eqs. (1) and (2) may not be used when one wants to describe such a processes.

It seems that a reasonable way to make these equations suitable for description of quantum decay processes is to replace the quantity (the decay rate) Γ appearing in Eqs. (1) and (2) by a corresponding decay rate derived using the quantum theory of unstable systems. The decay rate Γ used in Eqs. (1) and (2) is constant in time but the decay rate derived within the quantum theory is constant to a very good approximation only at the so-called “canonical decay regime” of times t (that is, when the quantum decay law has the exponential form, i.e. when $t < T$) and at times t much later than T it tends to zero as $1/t$ when time t tends to infinity

(see, e.g., [32]). This means that the decay process of an unstable quantum system is slower and slower for sufficiently late time, which was also pointed out in [1]. This and other properties of the quantum decay process seem to be important when considering the cosmological inflationary and late time (much later than the inflationary regime of times) processes including transition processes of the dark energy density from its early time extremely large values to its present small value. Therefore we need quantities characterizing the decay processes of unstable quantum systems.

The paper is organized as follows. Section 2 contains a brief introduction to the problems of unstable states and a description of quantities characterizing such states, which are used in the next sections. In Sect. 3 we analyze a possibility to describe metastable dark energy considering it as an unstable quantum system. Section 4 contains a discussion of the cosmological equations with decaying dark energy according to the quantum mechanical decay law, and the results of the numerical calculations are presented in graphical form. In Sect. 5 we present a statistical analysis. Section 6 contains the conclusions.

2 Preliminaries: unstable quantum states

The properties of unstable quantum systems are characterized by their survival probability (decay law). The survival probability can be found knowing the initial unstable state $|\phi\rangle \in \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space of states of the considered system) of the quantum system, which was prepared at the initial instant t_0 . The survival probability, $\mathcal{P}(t)$, of this state $|\phi\rangle$ decaying in vacuum equals $\mathcal{P}(t) = |A(t)|^2$, where $A(t)$ is the probability amplitude of finding the system at the time t in the rest frame $\mathcal{O}_0$ in the initial unstable state $|\phi\rangle$, $A(t) = \langle\phi|\phi(t)\rangle$. Here $|\phi(t)\rangle$ is the solution of the Schrödinger equation for the initial condition $|\phi(t_0)\rangle = |\phi\rangle$, which has the following form:

$$i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = \mathcal{H} |\phi(t)\rangle. \quad (3)$$

Here $|\phi\rangle, |\phi(t)\rangle \in \mathcal{H}$, and $\mathcal{H}$ denotes the total self-adjoint Hamiltonian for the system considered. The spectrum of $\mathcal{H}$ is assumed to be bounded from below: $E_{\min} > -\infty$ is the lower bound of the spectrum $\sigma_c(\mathcal{H}) = [E_{\min}, +\infty)$ of $\mathcal{H}$). Using the basis in $\mathcal{H}$ built from normalized eigenvectors $|E\rangle$, $E \in \sigma_c(\mathcal{H})$ of $\mathcal{H}$ and using the expansion of $|\phi\rangle$ in this basis one can express the amplitude $A(t)$ as the following Fourier integral:

$$A(t) \equiv A(t - t_0) = \int_{E_{\min}}^{\infty} \omega(E) e^{-\frac{i}{\hbar} E(t - t_0)} dE, \quad (4)$$

where $\omega(E) = \omega(E)^*$ and $\omega(E) > 0$ (see [2, 3, 5]). Note that from the normalization condition $\mathcal{P}(0) \equiv |A(0)|^2 = 1$

it follows that $\int_{E_{\min}}^{\infty} \omega(E) dE = 1$, which means that in the case of unstable states $\omega(E)$ is an absolutely integrable function. The consequence of this property is the conclusion following from the Riemann–Lebesgue lemma: we need to have $|A(t)| \rightarrow 0$ as $t \rightarrow \infty$. All these properties are the essence of the so-called Fock–Krylov theory of unstable states [2, 3, 5]. So within this approach the amplitude $A(t)$, and thus the decay law $\mathcal{P}(t)$ of the unstable state $|\phi\rangle$, are completely determined by the density of the energy distribution $\omega(E)$ for the system in this state [2, 3] (see also [4, 5, 33–37]). (This approach is also applicable in quantum field theory models [38, 39].)

Note that in fact the amplitude $A(t)$ contains information as regards the decay law $\mathcal{P}(t)$ of the state $|\phi\rangle$, that is, as regards the decay rate Γ_ϕ of this state, as well as the energy E_ϕ of the system in this state. This information can be extracted from $A(t)$. It can be done using the rigorous equation governing the time evolution in the subspace of unstable states, $\mathcal{H}_\parallel \ni |\phi\rangle_\parallel \equiv |\phi\rangle$. Such an equation follows from the Schrödinger equation (3) for the total state space $\mathcal{H}$.

Using the Schrödinger equation (3) one finds that for the problem considered

$$i\hbar \frac{\partial}{\partial t} \langle \phi | \phi(t) \rangle = \langle \phi | \mathfrak{H} | \phi(t) \rangle. \quad (5)$$

From this relation one can conclude that the amplitude $A(t)$ satisfies the following equation:

$$i\hbar \frac{\partial A(t)}{\partial t} = h(t) A(t), \quad (6)$$

where

$$h(t) = \frac{\langle \phi | \mathfrak{H} | \phi(t) \rangle}{A(t)} \equiv \frac{i\hbar}{A(t)} \frac{\partial A(t)}{\partial t} \quad (7)$$

and $h(t)$ is the effective Hamiltonian governing the time evolution in the subspace of unstable states $\mathcal{H}_\parallel = \mathbb{P}\mathcal{H}$, where $\mathbb{P} = |\phi\rangle\langle\phi|$ (see [32] and also [41, 42] and the references therein). The subspace $\mathcal{H} \ominus \mathcal{H}_\parallel = \mathcal{H}_\perp \equiv \mathbb{Q}\mathcal{H}$ is the subspace of decay products. Here $\mathbb{Q} = \mathbb{I} - \mathbb{P}$. One meets the effective Hamiltonian $h(t)$ when one starts with the Schrödinger equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_\parallel \subset \mathcal{H}$ [32, 37]. In general, $h(t)$ is a complex function of time and in the case of $\mathcal{H}_\parallel$ of dimension two or more the effective Hamiltonian governing the time evolution in such a subspace it is a non-hermitian matrix $H_\parallel$ or a non-hermitian operator. We have

$$h(t) = E_\phi(t) - \frac{i}{2} \Gamma_\phi(t), \quad (8)$$

and $E_\phi(t) = \Re[h(t)]$ and $\Gamma_\phi(t) = -2\Im[h(t)]$, are the instantaneous energy (mass) $E_\phi(t)$ and the instantaneous decay rate, $\Gamma_\phi(t)$ [32, 41, 42]. (Here $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z , respectively.) The quantity $\Gamma_\phi(t) = -2\Im[h(t)]$ is interpreted as the decay rate because it satisfies the definition of the decay rate used in quantum theory: $\frac{\Gamma_\phi(t)}{\hbar} \stackrel{\text{def}}{=} -\frac{1}{\mathcal{P}(t)} \frac{\partial \mathcal{P}(t)}{\partial t}$. Using (7) it is easy to check that

$$\begin{aligned} \frac{\Gamma_\phi(t)}{\hbar} &\equiv -\frac{1}{\mathcal{P}(t)} \frac{\partial \mathcal{P}(t)}{\partial t} = -\frac{1}{|A(t)|^2} \frac{\partial |A(t)|^2}{\partial t} \\ &\equiv -\frac{2}{\hbar} \Im[h(t)]. \end{aligned} \quad (9)$$

The use of the effective Hamiltonian $h(t)$ leads to the following form of the solutions of Eq. (6):

$$A(t) = e^{-i \frac{t}{\hbar} \overline{h(t)}} \equiv e^{-i \frac{t}{\hbar} \left(\overline{E_\phi(t)} - \frac{i}{2} \overline{\Gamma_\phi(t)} \right)}, \quad (10)$$

where $\overline{h(t)}$ is the average effective Hamiltonian $h(t)$ for the time interval $[0, t]$: $\overline{h(t)} \stackrel{\text{def}}{=} \frac{1}{t} \int_0^t h(x) dx$ (averages $\overline{E_\phi(t)}$, $\overline{\Gamma_\phi(t)}$ are defined analogously). Within a rigorous treatment of the problem it is straightforward to show that the basic assumptions of the quantum theory guarantee that (see, e.g. [32])

$$\lim_{t \rightarrow \infty} \Gamma_\phi(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \overline{\Gamma_\phi(t)} = 0. \quad (11)$$

These results are rigorous. For $\overline{E_\phi(t)}$ one can show that $\lim_{t \rightarrow \infty} \overline{E_\phi(t)} = E_{\min}$ (see [43]).

Equations (6) and (7) are convenient when the density $\omega(E)$ is given and one wants to find the instantaneous energy $E_\phi(t)$ and decay rate $\Gamma_\phi(t)$: Inserting $\omega(E)$ into (4) one obtains the amplitude $A(t)$ and then using (7) one finds the $h(t)$ and thus $E_\phi(t)$ and $\Gamma_\phi(t)$. In the general case the density $\omega(E)$ possesses properties analogous to the scattering amplitude, i.e., it can be decomposed into a threshold factor, a pole-function $P(E)$ with a simple pole and a smooth form factor $F(E)$. We have $\omega(E) = \Theta(E - E_{\min}) (E - E_{\min})^{\alpha_l} P(E) F(E)$, where α_l depends on the angular momentum l through $\alpha_l = \alpha + l$ (see Eq. (6.1) in [5]), $0 \leq \alpha < 1$ and $\Theta(E)$ is a step function: $\Theta(E) = 0$ for $E \leq 0$ and $\Theta(E) = 1$ for $E > 0$. The simplest choice is to take $\alpha = 0, l = 0, F(E) = 1$ and to assume that $P(E)$ has the Breit–Wigner (BW) form of the energy distribution density. (The mentioned Breit–Wigner distribution was found when the cross-section of slow neutrons was analyzed [44].) It turns out that the decay curves obtained in this simplest case are very similar in form to the curves calculated for the above described more general $\omega(E)$ (see [33] and the analysis in [5]). So to find the most typical properties of the decay process it is suf-

ficient to make the relevant calculations for $\omega(E)$ modeled by the Breit–Wigner distribution of the energy density: $\omega(E) \equiv \omega_{\text{BW}}(E) \stackrel{\text{def}}{=} \frac{N}{2\pi} \Theta(E - E_{\min}) \frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}$, where N is a normalization constant. The parameters E_0 and Γ_0 correspond to the energy of the system in the unstable state and its decay rate at the exponential (or canonical) regime of the decay process. $E_{\min}$ is the minimal (the lowest) energy of the system. Inserting $\omega_{\text{BW}}(E)$ into Eq. (4) for the amplitude $A(t)$ after some algebra one finds that

$$A(t) = A(t - t_0) = \frac{N}{2\pi} e^{-\frac{i}{\hbar} E_0 t} I_\beta \left(\frac{\Gamma_0(t - t_0)}{\hbar} \right), \quad (12)$$

where

$$I_\beta(\tau) \stackrel{\text{def}}{=} \int_{-\beta}^{\infty} \frac{1}{\eta^2 + \frac{1}{4}} e^{-i\eta\tau} d\eta. \quad (13)$$

Here $\tau = \frac{\Gamma_0(t - t_0)}{\hbar} \equiv \frac{t - t_0}{\tau_0}$, τ_0 is the lifetime, $\tau_0 = \frac{\hbar}{\Gamma_0}$, and $\beta = \frac{E_0 - E_{\min}}{\Gamma_0} > 0$. (The integral $I_\beta(\tau)$ can be expressed in terms of the integral–exponential function [40–42]; for a definition, see [45, 46].)

Note that the more convenient is to use $t' = (t - t_0)$ in (12), (13) or (4) and in a formula of this type, or to assume that $t_0 = 0$ in all formulas of this type, because this does not change the results of calculations but makes them easier. So from this point on we will assume that $t_0 = 0$.

Next using this $A(t)$ given by Eqs. (12), (13) and Eq. (7) defining the effective Hamiltonian $h_\phi(t)$ one finds that within the Breit–Wigner (BW) model considered

$$h(t) = E_0 + \Gamma_0 \frac{J_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)}{I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)}, \quad (14)$$

where

$$J_\beta(\tau) = \int_{-\beta}^{\infty} \frac{x}{x^2 + \frac{1}{4}} e^{-ix\tau} dx. \quad (15)$$

Working within the BW model and using $J_\beta(\tau)$ one should remember that $J_\beta(0)$ is undefined ($\lim_{\tau \rightarrow 0} J_\beta(\tau) = \infty$). Simply within the model defined by the Breit–Wigner distribution of the energy density, $\omega_{\text{BW}}(E)$, the expectation value of $\mathfrak{H}$, that is, $\langle \phi | \mathfrak{H} | \phi \rangle$, is not finite. So the whole consideration based on the use of $J_\beta(\tau)$ is valid only for $\tau > 0$.

It is relatively simple to find the analytical form of $J_\beta(\tau)$ using the following identity:

$$J_\beta(\tau) \equiv i \frac{\partial I_\beta(\tau)}{\partial \tau}. \quad (16)$$

We need to know the energy of the system in the unstable state $|\phi\rangle$ considered and its decay rate. The instantaneous

energy $E_\phi(t)$ of the system in the unstable state $|\phi\rangle$ has the following form within the BW model considered:

$$E_\phi(t) = \Re[h(t)] = E_0 + \Gamma_0 \Re \left[\frac{J_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)}{I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)} \right], \quad (17)$$

whereas the instantaneous decay rate looks as follows:

$$\begin{aligned} \Gamma_\phi(\tau) &= -2\Im[h(t)] = -2\Gamma_0 \Im \left[\frac{J_\beta(\tau)}{I_\beta(\tau)} \right] \\ &\equiv -2\Gamma_0 \Im \left[\frac{J_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)}{I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right)} \right]. \end{aligned} \quad (18)$$

It is relatively simple to find the asymptotic expressions $I_\beta\tau$ and $J_\beta(\tau)$ for $\tau \rightarrow \infty$ directly from (13) and (15) using, e.g., the method of integration by parts. We have for $\tau \rightarrow \infty$

$$\begin{aligned} I_\beta(\tau) &\simeq \frac{i}{\tau} \frac{e^{i\beta\tau}}{\beta^2 + \frac{1}{4}} \left\{ -1 + \frac{2\beta}{\beta^2 + \frac{1}{4}} \frac{i}{\tau} \right. \\ &\quad \left. + \left[\frac{2}{\beta^2 + \frac{1}{4}} - \frac{8\beta^2}{(\beta^2 + \frac{1}{4})^2} \right] \left(\frac{i}{\tau} \right)^2 + \dots \right\} \end{aligned} \quad (19)$$

and

$$\begin{aligned} J_\beta(\tau) &\simeq \frac{i}{\tau} \frac{e^{i\beta\tau}}{\beta^2 + \frac{1}{4}} \left\{ \beta + \left[1 - \frac{2\beta^2}{\beta^2 + \frac{1}{4}} \right] \frac{i}{\tau} \right. \\ &\quad \left. + \frac{\beta}{\beta^2 + \frac{1}{4}} \left[\frac{8\beta^2}{\beta^2 + \frac{1}{4}} - 6 \right] \left(\frac{i}{\tau} \right)^2 + \dots \right\}. \end{aligned} \quad (20)$$

These two last asymptotic expressions allow one to find for $\tau \rightarrow \infty$ the asymptotic form of the ratio $\frac{J_\beta(\tau)}{I_\beta(\tau)}$ used in Eqs. (14), (17) and (18), having a much simpler form than asymptotic expansions for $I_\beta(\tau)$ and $J_\beta(\tau)$. One finds that, for $\tau \rightarrow \infty$,

$$\frac{J_\beta(\tau)}{I_\beta(\tau)} \simeq -\beta - \frac{i}{\tau} - \frac{2\beta}{\beta^2 + \frac{1}{4}} \frac{1}{\tau^2} + \dots \quad (21)$$

Starting from this asymptotic expression and Eq. (17) one finds, e.g. that, for $t \rightarrow \infty$,

$$E_\phi(t)|_{t \rightarrow \infty} \simeq E_{\min} - 2 \frac{E_0 - E_{\min}}{|h_\phi^0 - E_{\min}|^2} \left(\frac{\hbar}{t} \right)^2, \quad (22)$$

where $h_\phi^0 = E_0 - \frac{i}{2}\Gamma_0$, and

$$\Gamma_\phi(t)|_{t \rightarrow \infty} \simeq 2\Gamma_0 \frac{1}{t} + \dots = 2 \frac{\hbar}{t} + \dots \quad (23)$$

The last two relations are valid for $t > T$, where T denotes the cross-over time, i.e. the time when exponential and late

time inverse power-law contributions to the survival amplitude become comparable.

3 Metastable dark energy with a decay law from quantum mechanics

Note that the model described by Eqs. (1)–(2) is the classical physics model and therefore it cannot be applied directly when one would like to follow Krauss and Dent and to consider the decay of the dark energy density $\rho_{\text{DE}}(t)$ as the quantum decay process. For example, the late time effects discussed in [1] can never occur in the SHSS model. The simplest way to extend models considered in [6] so that they might be used to describe the decay of $\rho_{\text{DE}}(t)$ as a quantum process seems to be a replacement of the classical decay rate Γ in Eqs. (1), (2) by the decay rate $\Gamma_\phi(t)/\hbar$ appearing in the quantum theoretical considerations. It is because the classical decay rate $\Gamma_{\text{class}} = \Gamma$ corresponds to the quantum physics decay rate $\Gamma_{\text{quant}} = \Gamma_\phi(t)$ divided by $\hbar$ (that is, to $\Gamma_\phi(t)/\hbar$) and using $\Gamma_\phi(t)$ one can insert it into Eq. (2) to obtain

$$\dot{\rho}_{\text{DE}}(t) = -\frac{1}{\hbar} \Gamma_\phi(t) \rho_{\text{DE}}(t), \quad (24)$$

instead of the classical fundamental equation of the radioactive decays theory. In fact this equation is a simple improvement of models discussed in [6], and it can be considered as the use of quantum corrections in the models mentioned. In such a case Eq. (1) takes the following form:

$$\rho_{\text{DE}}(t) = \rho_{\text{DE}}(t_0) \times \exp\left[-\frac{t}{\hbar} \overline{\Gamma_\phi(t)}\right] \quad (25)$$

$$\equiv \rho_{\text{DE}}(t_0) \times \exp\left[-\frac{1}{\hbar} \int_{t_0}^t \Gamma_\phi(x) dx\right], \quad (26)$$

where $\Gamma_\phi(t)$ is given by Eq. (18) and $\overline{\Gamma_\phi(t)} \stackrel{\text{def}}{=} \frac{1}{t} \int_{t_0}^t \Gamma_\phi(x) dx$ is the average decay rate for the time interval $[0, t]$. These relations, replacing Eq. (1), contain quantum corrections connected with the use of the quantum theory decay rate.

Note that using the identity (9) and Eq. (12) one can rewrite Eq. (26) as follows:

$$\rho_{\text{DE}}(t) \equiv \frac{N^2}{4\pi^2} \rho_{\text{DE}}(t_0) \left| I_\beta \left(\frac{\Gamma_0(t-t_0)}{\hbar} \right) \right|^2, \quad (27)$$

which can make simpler numerical calculations.

Now in order to obtain analytical or numerical results having Eqs. (24)–(26) one needs a quantum mechanical model of the decay process, that is, one needs $\omega(E)$ (see (4)). We begin our considerations using the Breit–Wigner model analyzed in the previous section. Inserting $\Gamma_\phi(t)$ given by (18) into Eq. (24), or Eqs. (25) and (26) we can analyze the decay

process of $\rho_{\text{DE}}(t)$. One can notice that performing the calculations, e.g. using the Breit–Wigner model, it is more convenient to use Eq. (27) with $I_\beta(t)$ given by Eq. (13) than using Eqs. (25) and (26) with $\Gamma_\phi(t)$ given by Eq. (18).

Note that one of the parameters appearing in the quantum mechanical formula (18) for $\Gamma_\phi(t)$ is Γ_0 . This parameter can be eliminated if we notice that $\beta = \frac{E_0 - E_{\text{min}}}{\Gamma_0} > 0$. Hence $\Gamma_0 \equiv \frac{E_0 - E_{\text{min}}}{\beta}$, and therefore one can rewrite (18) as

$$\Gamma_\phi(\tau) = -2 \frac{E_0 - E_{\text{min}}}{\beta} \Im \left[\frac{J_\beta(\tau)}{I_\beta(\tau)} \right], \quad (28)$$

or

$$\Gamma_\phi(\tau) = -2 \frac{\frac{E_0}{V_0} - \frac{E_{\text{min}}}{V_0}}{\beta} V_0 \Im \left[\frac{J_\beta(\tau)}{I_\beta(\tau)} \right], \quad (29)$$

where V_0 is the volume of the system at $t = t_0$. We have $\frac{E_0}{V_0} = \rho_{\text{DE}}^{qft} \stackrel{\text{def}}{=} \rho_{\text{DE}}^0$ and $\frac{E_{\text{min}}}{V_0} = \rho_{\text{bare}}$, (where ρ_{DE}^{qft} is the energy density calculated using quantum field theory methods), so Eq. (29) can be rewritten as follows:

$$\Gamma_\phi(\tau) = -2 \frac{\rho_{\text{DE}}^0 - \rho_{\text{bare}}}{\beta} V_0 \Im \left[\frac{J_\beta(\tau)}{I_\beta(\tau)} \right]. \quad (30)$$

The parameter τ used in (28)–(30) denotes time t measured in lifetimes as mentioned after Eq. (13): $\tau = \frac{t}{\tau_0}$. Using the parameter β the lifetime τ_0 can be expressed as follows: $\tau_0 = \frac{\beta}{\rho_{\text{DE}}^0 - \rho_{\text{bare}}} \frac{\hbar}{V_0}$.

The asymptotic form (23) indicates one of the main differences between the SHSS model and our improvement of this model. Namely, from Eq. (1) it follows that

$$\lim_{t \rightarrow \infty} \rho_{\text{DE}}(t) = 0. \quad (31)$$

From (1) one sees that $\rho_{\text{DE}}(t)$ is an exponentially decreasing function of time.

It is interesting to consider a more general form of the energy density,

$$\tilde{\rho}_{\text{DE}}(t) = \rho_{\text{DE}}(t) - \rho_{\text{bare}}, \quad (32)$$

where $\rho_{\text{bare}} = \text{const}$ is the minimal value of the dark energy density. Inserting the density $\tilde{\rho}_{\text{DE}}(t)$ into Eq. (1) one concludes that $\rho_{\text{DE}}(t)$ tends to ρ_{bare} exponentially fast as $t \rightarrow \infty$.

Let us see now what happens when we insert $\tilde{\rho}_{\text{DE}}(t)$ into our Eq. (24) and consider only the asymptotic behavior of $\rho_{\text{DE}}(t)$ for times $t \geq T_0 \gg T$. In such a case inserting the late time asymptotic expression of Eq. (23) into Eq. (24) one finds for very late times $t > T_0$ that

$$\ln \frac{\tilde{\rho}_{\text{DE}}(t)}{\tilde{\rho}_{\text{DE}}(T_0)} = \ln \left(\frac{t}{T_0} \right)^{-2}, \quad (33)$$

that is, for $t > T_0 \gg T$,

$$\rho_{\text{DE}}(t) \simeq \rho_{\text{bare}} + D \frac{1}{t^2}, \quad (34)$$

where $D = \text{const}$. Note that the same result follows directly from (27) when one inserts there $A(t)$ given by Eq. (12) and uses the asymptotic expression of Eq. (19) for $I_\beta(\tau)$, which shows that our approach is self-consistent. The result (34) means that quantum corrections do not allow $\rho_{\text{DE}}(t)$ to tend to ρ_{bare} exponentially fast when $t \rightarrow \infty$, but $\rho_{\text{DE}}(t)$ must tend to ρ_{bare} as $1/t^2$, for $t \rightarrow \infty$, which is in the full agreement with our earlier results, presented, e.g., in [20, 47–50]. So in fact, as one can see, the SSHS model is the classical physics approximation of the model discussed in our papers mentioned, where the cosmological parametrization resulting from the quantum mechanical treatment of unstable systems was used.

4 Cosmological equations

We introduce our model as the covariant theory with the interaction term [51]. We consider the flat cosmological model (the constant curvature is equal zero).

The total density of energy consists of the baryonic matter ρ_B , the dark matter ρ_{DM} and the dark energy ρ_{DE} . We assume, for the baryonic matter and the dark matter, the equation of state for dust ($p_B(\rho_B) = 0$ and $p_{\text{DM}}(\rho_{\text{DM}}) = 0$). Also we consider the equation of state for the dark energy to be $p_{\text{DE}}(\rho_{\text{DE}}) = -\rho_{\text{DE}}$.

The cosmological equations such as the Friedmann and acceleration equations are found by the variation action by the metric $g_{\mu\nu}$ [51]. In consequence we get the equations

$$3H^2 = 3\frac{\dot{a}^2}{a} = \rho_{\text{tot}} = \rho_B + \rho_{\text{DM}} + \rho_{\text{DE}} \quad (35)$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_{\text{tot}} + 3p_{\text{tot}}(\rho_{\text{tot}})) = \rho_B + \rho_{\text{DM}} - 2\rho_{\text{DE}}, \quad (36)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function. Here, we assume $8\pi G = c = 1$.

Equations (35) and (36) give the conservation equation in the following form:

$$\dot{\rho}_{\text{tot}} = -3H(\rho_{\text{tot}} + p_{\text{tot}}(\rho_{\text{tot}})) \quad (37)$$

or in the equivalent form

$$\dot{\rho}_{\text{M}} = -3H\rho_{\text{M}} - \dot{\rho}_{\text{DE}}, \quad (38)$$

where $\rho_{\text{M}} = \rho_B + \rho_{\text{DM}}$.

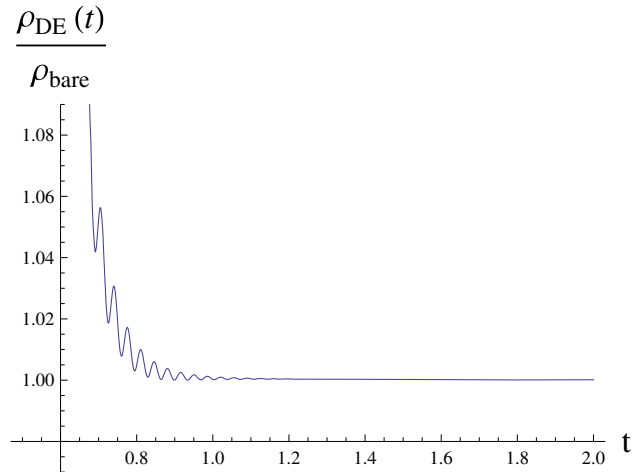


Fig. 1 The dependence $\rho_{\text{DE}}(t)$ [from Eq. (40)]. For illustration we put $\beta = 800$, $\Gamma_0 = 20\hbar$ and $\epsilon = 1000\rho_{\text{bare}}$. The qualitative behavior of ρ_{DE} does not depend on ϵ . The units of time t are determined by the choice of units of Γ_0 because $\frac{\Gamma_0 t}{\hbar}$ is dimensionless

Let Q denote the interaction term. Equation (38) can be rewritten as

$$\dot{\rho}_b = -3H\rho_b, \quad \dot{\rho}_{\text{DM}} = -3H\rho_{\text{DM}} + Q \text{ and } \dot{\rho}_{\text{DE}} = -Q. \quad (39)$$

If $Q > 0$ then the energy flows from the dark energy sector to the dark matter sector. If $Q < 0$ then the energy flows from the dark matter sector to the dark energy sector.

Figure 1 shows the diagrams of the evolution of $\rho_{\text{DE}}(t)$. Note that the oscillatory phase appears in the evolution of $\rho_{\text{DE}}(t)$. Figure 2 presents the evolution of the $\bar{\Gamma}_\phi(t)$. At the initial period we obtain a logistic-type decay of dark energy. The period when $\bar{\Gamma}_\phi(t)$ grows to a plateau is characteristic for the so-called Zeno time [52]. It increases slowly about 0.0004 (the slope of this curve is 0.0001) with the cosmic time t in the interval $(0, 4)$. Then in the interval $(4, 30000)$ it becomes strictly constant. This behavior justifies a radioactive approximation given in Ref. [6]. For the late time, $\bar{\Gamma}_\phi(t)$ approaches zero.

Using (27) we get the final formula for $\rho_{\text{DE}}(t)$,

$$\rho_{\text{DE}}(t) = \rho_{\text{bare}} + \epsilon \left| I_\beta \left(\frac{\Gamma_0 t}{\hbar} \right) \right|^2, \quad (40)$$

where $\epsilon \equiv \epsilon(\beta) = \frac{\rho_{\text{DE}}(0) - \rho_{\text{bare}}}{|I_\beta(0)|^2}$ measures the deviation from the Λ CDM model ($I_\beta(0) \equiv \frac{2\pi}{N} = \pi + 2 \arctan(2\beta)$ and $\beta > 0$).

The canonical scaling law for cold dark matter should be modified. In this case

$$\rho_{\text{DM}} = \rho_{\text{DM},0} a^{-3+\delta}, \quad (41)$$

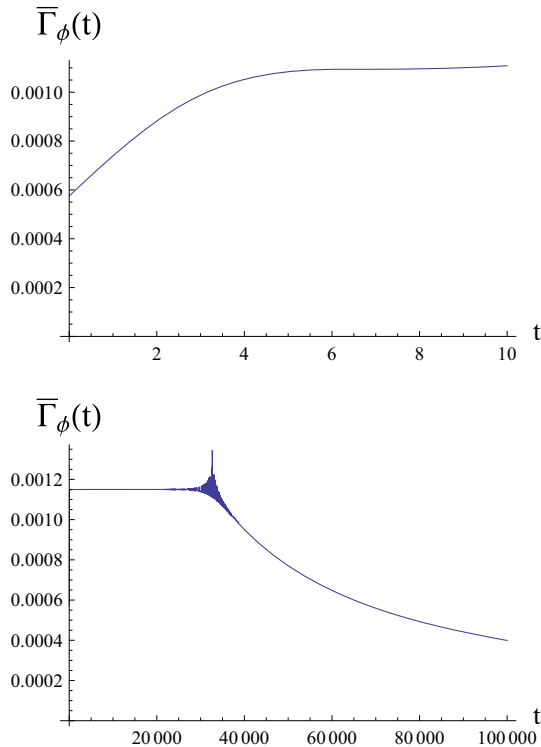


Fig. 2 The dependence $\bar{\Gamma}_\phi(t)$ for the best fit values (see Table 1). The upper panel presents the evolution of $\bar{\Gamma}_\phi(t)$ for the early Universe and the present epoch. The lower panel presents evolution of $\bar{\Gamma}_\phi(t)$ for the late time Universe. The cosmological time t is expressed in $\frac{s \times \text{Mpc}}{100 \text{ km}}$. In these units, the age of the Universe is equal $1.41 \frac{s \times \text{Mpc}}{100 \text{ km}}$

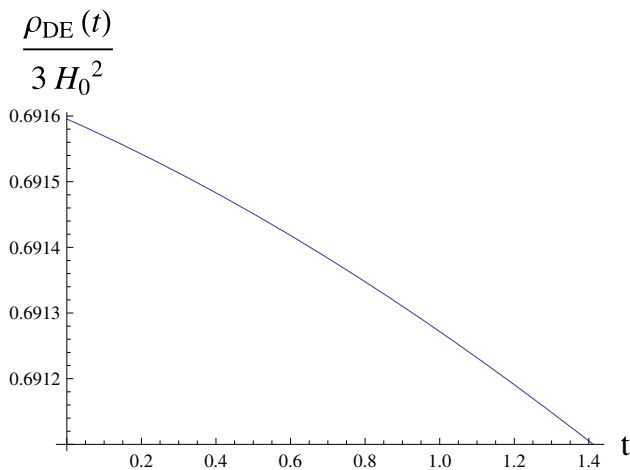


Fig. 3 The dependence $\rho_{DE}(t)$ (from Eq. (40)) for the best fit value of model parameter (see Table 1). The cosmological time t is expressed in $\frac{s \times \text{Mpc}}{100 \text{ km}}$. The present epoch is for $t = 1.41 \frac{s \times \text{Mpc}}{100 \text{ km}}$. Note that, in the Planck epoch, the value of $\frac{\rho_{DE}(t_{Pl})}{3H_0^2}$ is equal to 0.6916

where $\delta = \frac{1}{\ln a} \int \frac{Q}{H \rho_{DM}} d \ln a$. The dependence $\rho_{DE}(t)$ [from Eq. (40)] for the best fit value of model parameter (see Table 1) is presented in Fig. 3 and the evolution of $\delta(t)$ is shown in Fig. 4.

Assuming that $\beta > 0$ one obtains for $t > t_L = \frac{\hbar}{T_0} \frac{2\beta}{\beta^2 + \frac{1}{4}}$ (see [40]) the approximation of (40) in the following form:

$$\rho_{DE}(t) \approx \rho_{bare} + \epsilon \left(4\pi^2 e^{-\frac{T_0}{\hbar} t} + \frac{4\pi e^{-\frac{T_0}{2\hbar} t} \sin\left(\beta \frac{T_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar} t} + \frac{1}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar} t\right)^2} \right). \quad (42)$$

For the best fit value (see Table 1) $t_L \approx 2T_0$.

From Eq. (42), it results that, for the late time, the behavior of dark energy can be described by the following formula:

$$\rho_{DE}(t) \approx \rho_{bare} + \frac{\epsilon}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar}\right)^2} \frac{1}{t^2}. \quad (43)$$

This case was considered in [53, 54].

If we use Eq. (42) in the Friedmann equation (35), we get

$$3H^2 = \rho_{tot} = \rho_B + \rho_{DM} + \rho_{bare} + \rho_{rad.dec.} + \rho_{dam.osc.} + \rho_{pow.law}, \quad (44)$$

where $\rho_{rad.dec.} = 4\pi^2 \epsilon e^{-\frac{T_0}{\hbar} t}$ is the radioactive-like decay

part of the dark energy, $\rho_{dam.osc.} = \frac{4\pi \epsilon e^{-\frac{T_0}{2\hbar} t} \sin\left(\beta \frac{T_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar} t}$ rep-

resents the damping oscillations part of the dark energy and $\rho_{pow.law} = \frac{\epsilon}{\left(\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar}\right)^2}$ represents the power-law part of the

dark energy. Using dimensionless parameters, $\Omega_i = \frac{\rho_i}{3H_0^2}$, where H_0 is the present value of the Hubble constant, Eq. (44) can be rewritten as

$$\frac{H^2}{H_0^2} = \Omega_B + \Omega_{DM} + \Omega_{bare} + \Omega_{rad.dec.} + \Omega_{dam.osc.} + \Omega_{pow.law}. \quad (45)$$

If the radioactive-like decay dominates then one can define the e-folding time λ and half life time $T_{1/2} = \lambda \ln 2 = \frac{\hbar \ln 2}{T_0}$.

The evolution of $\Omega_{rad.dec.}$, $\Omega_{dam.osc.}$, $\Omega_{pow.law}$ with respect to time, for the best fit value (see Table 1), is presented in Fig. 5.

In the moment when the period of the radioactive-like decay $T_{end \text{ rad.dec.}}$ finishes, the value of $\rho_{rad.dec.}$ is equal to the value of $\rho_{dam.osc.}$. It leads us to the condition

$$4\pi^2 \epsilon e^{-\frac{T_0}{\hbar} t} = \frac{4\pi \epsilon e^{-\frac{T_0}{2\hbar} t} \sin\left(\beta \frac{T_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar} t}, \quad (46)$$

or, after simplifying,

$$\pi e^{-\frac{T_0}{2\hbar} t} = \frac{\sin\left(\beta \frac{T_0}{\hbar} t\right)}{\left(\frac{1}{4} + \beta^2\right) \frac{T_0}{\hbar} t}. \quad (47)$$

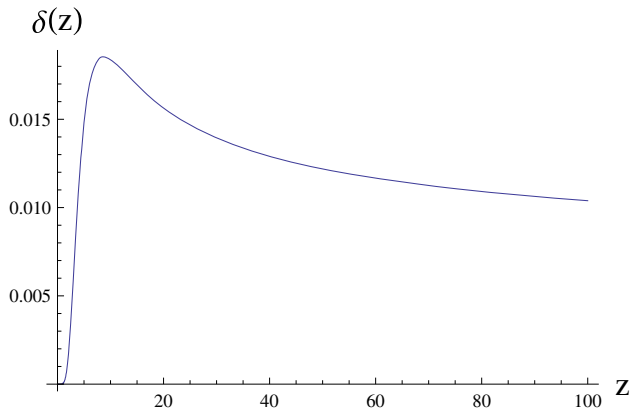


Fig. 4 A diagram of the evolution of $\delta(z)$, where z is redshift. For illustration we put $\beta = 800$, $\Gamma_0 = 20\hbar$ and $\epsilon = 1000\rho_{\text{bare}}$. The function $\delta(z)$ reaches the maximum for $z = z_0$, which is a solution of equation $\delta(z_0) = \frac{Q(z_0)}{H(z_0)\rho_{\text{DM}}(z_0)}$

Equation (47) has infinitely many solutions but $T_{\text{end rad.dec.}}$ is equal to the least positive real solution of (47) because the period of the radioactive-like decay is before the period of the damping oscillation decay.

Searching for the value of $T_{\text{end rad.dec.}}$ can be simplified by using of the upper envelope of oscillations of $\rho_{\text{dam.osc.}}$, which is given by

$$e_{\text{upper}}(t) = \frac{4\pi\epsilon e^{-\frac{\Gamma_0}{2\hbar}t}}{\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar}t}. \quad (48)$$

Then we get an approximation of Eq. (47) in the form $\rho_{\text{rad.dec.}} = e_{\text{upper}}$ or after simplifying

$$\pi e^{-\frac{\Gamma_0}{2\hbar}t} = \frac{1}{\left(\frac{1}{4} + \beta^2\right) \frac{\Gamma_0}{\hbar}t}. \quad (49)$$

The solution of Eq. (49) gives us the approximated value of $T_{\text{end rad.dec.}}$.

Note that a solution of Eq. (49) cannot be less than the value of $T_{\text{end rad.dec.}}$ having subtracted the value of one period of oscillation of $\rho_{\text{dam.osc.}}$ (i.e., $T_{\text{dam.osc.}} = \frac{2\pi\hbar}{\beta\Gamma_0}$) and cannot be greater than the value of $T_{\text{end rad.dec.}}$. In consequence for $\beta > 29$, the error of the approximation is less than 1%. The dependence $T_{\text{end rad.dec.}}(\beta)$ is presented in Fig. 6.

From the statistical analysis (see Sect. 5), we have the best fit values of $\Gamma_0/\hbar = 0.00115$ and $\beta = \frac{1}{\alpha} - 1 = 799$ (see Table 1) and Eq. (49) gives $T_{\text{end rad.dec.}} = 2.2 \times 10^4 T_0$, where T_0 is the present age of the Universe.

5 Statistical analysis

In our statistical analysis, we used the following astronomical data: supernovae of type Ia (SNIa) (Union 2.1

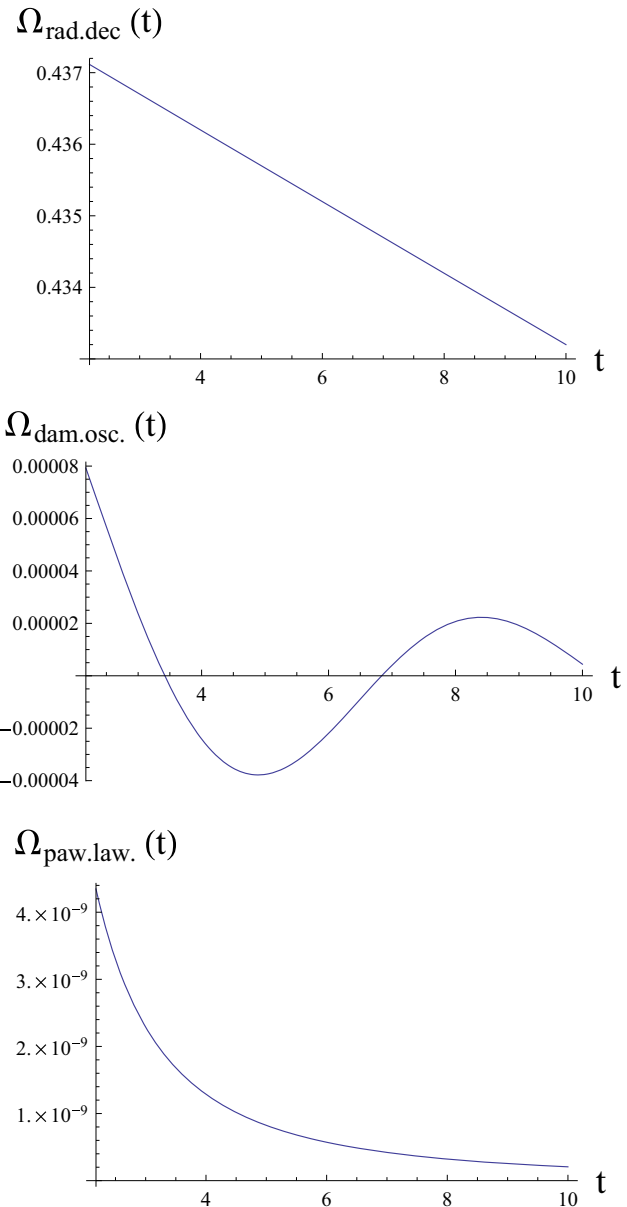


Fig. 5 The dependence $\Omega_{\text{rad.dec.}}$, $\Omega_{\text{dam.osc.}}$, $\Omega_{\text{pow.law}}$ with respect to the cosmological time t for the best fit value of model parameter (see Table 1). The cosmological time t is expressed in $\frac{\text{s} \times \text{Mpc}}{100 \text{ km}}$. In these units, the present epoch is for $t = 1.41 \frac{\text{s} \times \text{Mpc}}{100 \text{ km}}$. Let us note that while the density parameters do not change practically during the cosmic evolution for the cases shown in the upper and middle panels, the density parameters are lowered by many orders of magnitude for the case presented in the lower panel [20]

dataset [55]), BAO data (Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [56], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [57], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [58]), measurements of the Hubble parameter $H(z)$ of galaxies [59–61], the Alcock–Paczynski test (AP)[62,63] (data from [64–72].) and measurements of CMB by Planck

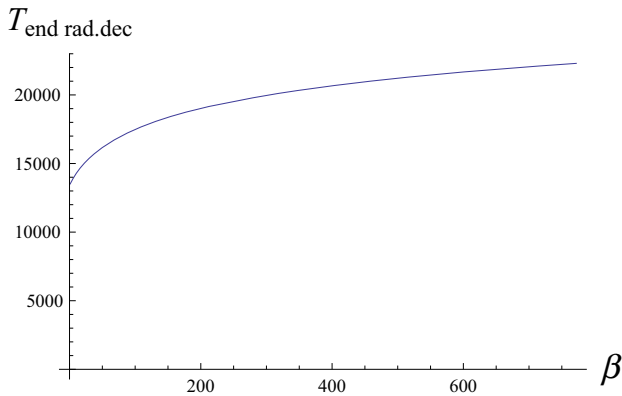


Fig. 6 A diagram presents a dependence $T_{\text{end rad.dec.}}(\beta)$ given by Eq. (49) for $\beta > 29$. For illustration we put the best fit value of Γ_0 (see Table 1). The values of $T_{\text{end rad.dec.}}$ are expressed in terms of the present age of the Universe T_0

[73]. The equation for the likelihood function is given by

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB}}. \quad (50)$$

The likelihood function for SNIa has the form

$$L_{\text{SNIa}} = \exp \left[-\frac{1}{2} \left[A - B^2/C + \log(C/(2\pi)) \right] \right], \quad (51)$$

where $A = (\mu^{\text{obs}} - \mu^{\text{th}}) \mathbb{C}^{-1} (\mu^{\text{obs}} - \mu^{\text{th}})$, $B = \mathbb{C}^{-1} (\mu^{\text{obs}} - \mu^{\text{th}})$, $C = \text{Tr } \mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for SNIa, μ^{obs} is the observer distance modulus and μ^{th} is the theoretical distance modulus.

The likelihood function for BAO is described by the equation

$$L_{\text{BAO}} = \exp \left[-\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \right], \quad (52)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch [74, 75].

The likelihood function

$$L_{H(z)} = \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2 \right] \quad (53)$$

is for measurements of the Hubble parameter $H(z)$ of galaxies.

The likelihood function for AP is given by

$$L_{AP(z)} = \exp \left[-\frac{1}{2} \sum_{i=1}^N \left(\frac{AP(z_i)^{\text{obs}} - AP(z_i)^{\text{th}}}{\sigma_i} \right)^2 \right], \quad (54)$$

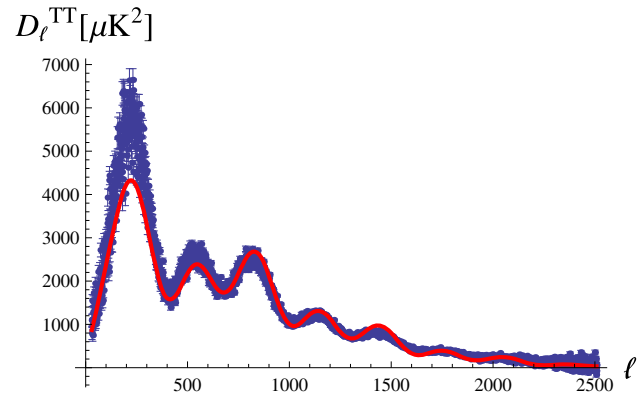


Fig. 7 Diagram of the temperature power spectrum of CMB for the best fit values (red line). The error bars from the Planck data are presented by the color blue

Table 1 The best fit and errors for the estimated model with α from the interval (0.0, 0.033), $\Gamma_0/\hbar$ from the interval (0.00 $\frac{100 \text{ km}}{\text{s} \times \text{Mpc}}$, 0.036 $\frac{100 \text{ km}}{\text{s} \times \text{Mpc}}$) and $\epsilon/3H_0^2$ from the interval (0.00, 0.0175). We assumed that $\Omega_{b,0} = 0.048468$, $H_0 = 67.74 \frac{\text{km}}{\text{s} \times \text{Mpc}}$ and $\Omega_{m,0} = 0.3089$. In the table, the values of $\Gamma_0/\hbar$ are expressed in $\frac{100 \text{ km}}{\text{s} \times \text{Mpc}}$. Because $\alpha = \frac{1}{1+\beta}$, the best fit value of β parameter is equal to 799

Parameter	Best fit	68% CL	95% CL
α	0.00125	+0.00104 −0.00125	+0.01777 −0.00125
$\Gamma_0/\hbar$	0.00115	+0.00209 −0.00115	+0.2123 −0.00115
$\epsilon/3H_0^2$	0.0111	+0.0064 −0.0083	+0.0064 −0.0093

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data. The likelihood function for CMB is given by

$$L_{\text{CMB}} = \exp \left[-\frac{1}{2} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \mathbb{C}^{-1} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \right], \quad (55)$$

where $\mathbb{C}$ is the covariance matrix with the errors, $\mathbf{x}$ is a vector of the acoustic scale l_A , the shift parameter R and $\Omega_b h^2$ where $l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')}$ and $R = \sqrt{\Omega_{m,0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}$, where z^* is the redshift of the epoch of the recombination [74].

In this paper, we used our own code CosmoDarkBox in the estimation of the model parameters. Our code uses the Metropolis–Hastings algorithm [76, 77].

In the statistical analysis, we estimated three model parameters: $\alpha = \frac{1}{1+\beta}$, Γ_0 , $\epsilon/3H_0^2$. Our statistical results are completely presented in Table 1. The diagram of the temperature power spectrum for the best fit values is presented in Fig. 7. Therefore the radioactive type of decay gives the most effective mechanism of the decaying metastable dark energy. We estimated also that the decay half life time $T_{1/2}$ of dark energy is equal to $8503 \text{ Gyr} \approx 616 \times T_0$.

6 Discussion and conclusions

The main aim of this paper was to study the implication of the derived form of the running dark energy. In our approach the formula for the parametrization of this dark energy is derived directly from quantum mechanics rather than being postulated in a phenomenological way. The evolution scenario of dark energy contains three different phases: a phase of radioactive-like decay in the early Universe, a phase of damping oscillations and finally a phase of the power-law type of decay.

We investigated the cosmological evolution caused by such a variability of dark energy and matter. The dynamics of the model is governed by a cosmological dynamical system with an interacting term because the energy-momentum tensor is not conserved in this case.

Using results of the investigation of variability of dark energy with the cosmological time, we analyzed the issue of whether the problem of the cosmological constant could be solved within the considered model based on the assumption that the decay process of the dark energy is the quantum decay process having the same form as the decay process of the unstable quantum systems or not. For simplicity it was assumed that this decay process is determined by Eq. (4) with the distribution of the energy density $\omega(E)$ in the unstable quantum state having the Breit–Wigner form $\omega(E) = \omega_{\text{BW}}(E)$. We show that within such a model dark energy decays and then the canonical scaling law for cold dark matter a^{-3} should be modified. Unfortunately, from our analysis it follows that within the considered model, where $\omega(E) = \omega_{\text{BW}}(E)$ and the assumptions leading to estimations presented in Table 1 are used, there is a very small difference between $\rho_{\text{DE}}(0)$ and $\rho_{\text{DE}}(T_0)$, which cannot be considered as a solution of the cosmological constant problem. On the other hand one cannot exclude that $\omega(E)$ has such a form as will lead by (4), for such a decay law, to $\rho_{\text{DE}}(0) \gg \rho_{\text{DE}}(T_0)$ for suitably chosen parameters of the model.

Using astronomical data we tested the model and see that it is in good agreement with the data. Our estimation also shows that the fraction of all components of the dynamical dark energy in the whole dark energy is larger than the contribution of the cosmological constant term.

In our model it is calculated that the Λ term has a dynamical nature as a consequence of a decaying of the dark energy. In consequence the conservation of the energy-momentum tensor (EMT) is violated. Recently, Josset and Perez [51] have demonstrated the model in which the violation of EMT can be achieved in the context of the unimodular gravity and how it leads to the emergence of the effective cosmological constant in Einstein's equations. In our approach the violation of the conservation of EMT is rather a consequence of the quantum mechanical nature of the metastable vacuum, rather than a modification of the gravity theory.

In our approach the concrete form of the decaying dark energy is derived directly from a quantum mechanical consideration of unstable states. We obtain a more complex form of decaying dark energy in which we have found a radioactive type of its decay. We also estimated the model parameters as well as fractions of three different forms of decaying: radioactive type, damping oscillating type and power-law type. From the astronomical data we see that the radioactive type of decay is favored and 44% of the energy budget of the Universe corresponds with a radioactive-like decay.

In our paper we investigate the second way of the decay of dark energy into dark matter from the three different ways of dark energy decay considered by Shafieloo et al. [6]. They proposed a class of metastable dark energy models in which dark energy decays according to the radioactive law. They assumed a phenomenological form of the decay, studying observational constraints for the cosmological model. In our paper, it is derived directly from quantum mechanics. Our results are complementary to their results because they justify the phenomenological choice of the exponential decay as a major mechanism of dark energy decay. Moreover, our calculation of the decay half life is in agreement with Shafieloo et al.'s calculation. We see that the radioactive-like decay dominates up to $2.2 \times 10^4 T_0$. Our calculations show that the radioactive-like decay has only an intermediate character and will be replaced in the future evolution of the Universe by an oscillation decay and then decay of $1/t^2$ type.

One of the differences between our approach and the theory developed by Shafieloo et al. is that they consider only decay of the dark energy into dark constituents assuming that the decay rate Γ of the dark energy is constant and depends only on its internal composition. The latter assumption is approximately true only if one considers decay processes as classical physics processes. The detailed analysis of decay processes of unstable quantum systems shows that the basic principles of the quantum theory do not allow them to be described by an exponential decay law at very late times as well as at initial stage of the decay process (see, e.g., [5] and the references therein, or [78]) and that the decay law can be described by the exponentially decreasing function of time only at “canonical decay regime” of the decay process, that is, at intermediate times (at times longer than the initial stage of the evolution of the unstable quantum system and shorter than the cross-over time T). These properties of quantum decay processes mean that in general the decay rate cannot be constant in time, $\Gamma = \Gamma(t) \neq \text{const}$ (see, e.g., [32, 37, 42, 78]), and at the “canonical decay” stage $\Gamma(t) \simeq \Gamma_0$, to a very good approximation.

These properties of the decay rate were used in our paper. The advantage of the use of the decay rate following from the quantum properties of the decaying systems is that such an approach allows one to describe correctly the initial stage of the dark energy decay process, and at very late times.

It is impossible to realize this within the approach used by Shafieloo et al. Moreover, the use of $\Gamma = \text{const}$ may lead to the results which need not be correct. The example of such a situation is the analysis performed in Appendix A, Section A1, of Ref. [6], where the authors considered the case $\Gamma t \ll 1$ and then applied the results obtained within such an assumption for the analysis of properties of their Model I. Namely, there are many reasons for drawing the conclusion that the decay of the dark energy must be a quantum decay process (see the discussion in Sect. 1) and that it cannot be a classical physics process. So when one wants to describe the early stage of the decay process of the dark energy, which mathematically can be expressed by the assumption that $\Gamma t \ll 1$ one should not use a relation of the type (1) but the relation

$$\rho_{\text{DE}}(t) = \rho_{\text{DE}}(0) |A(t)|^2, \quad (56)$$

resulting from the quantum mechanical treatment of the decay process. Instead of considering the relation of this type, the authors of [6] used Eq. (1), which leads to Eq. (A1) in [6] for $\Gamma t \ll 1$, that is, to

$$\rho_{\text{DE}} = \epsilon_0 e^{-\Gamma t} \simeq \epsilon_0 (1 - \Gamma t) \quad (57)$$

(ϵ_0 is defined in [6]), which is mathematically correct but it is not correct when one considers the decay of the dark energy as a quantum process. In the case of a quantum decay process one should use a relation of the type (56) and the approximate form of $|A(t)|^2$ for very short times. In such a case (see, e.g. [5, 32])

$$|A(t)|^2 \simeq 1 - d^2 t^2, \quad \text{for } t \rightarrow 0, \quad (58)$$

where $d = \text{const}$ and it does not depend on Γ . Therefore we should have

$$\rho_{\text{DE}}(t) \simeq \rho_{\text{DE}}(0) (1 - d^2 t^2), \quad \text{for } t \rightarrow 0, \quad (59)$$

for short times t , when the decay of the dark energy is a quantum decay process. The difference between Eqs. (57) (i.e., (A1) in [6]) and (59) is dramatic (the use ϵ_0 in (57) and $\rho_{\text{DE}}(0)$ in (59) is not the point). The problem is that the authors of [6] use their result (A1) (that is, Eq. (57)) in Eq. (A2) and then all considerations related to their Model I in Section A I of Appendix A are founded on Eqs. (A1) and (A2). This means that the conclusions drawn in [6] (based on the analysis performed in Sect. A I of Appendix A) may not reflect real properties of the decaying dark energy. It should be noted that our analysis performed in this paper is free of this defect.

Note also that Shafieloo et al. [6] considered only the decays of the dark energy into dark components: dark mat-

ter and dark radiation, whereas we consider the general case (that is, in our approach the decay of the dark energy into a visible baryonic matter is also admissible, which cannot be excluded in the light of the recently reported discovery of baryonic spindles linking galaxies [79–81]).

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Dynamics of the diffusive DM-DE interaction – Dynamical system approach

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Abstract. We discuss dynamics of a model of an energy transfer between dark energy (DE) and dark matter (DM). The energy transfer is determined by a non-conservation law resulting from a diffusion of dark matter in an environment of dark energy. The relativistic invariance defines the diffusion in a unique way. The system can contain baryonic matter and radiation which do not interact with the dark sector. We treat the Friedman equation and the conservation laws as a closed dynamical system. The dynamics of the model is examined using the dynamical systems methods for demonstration how solutions depend on initial conditions. We also fit the model parameters using astronomical observation: SNIa, $H(z)$, BAO and Alcock-Paczynski test. We show that the model with diffuse DM-DE is consistent with the data.

Keywords: cosmology of theories beyond the SM, dark energy theory, dark matter theory, modified gravity

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1 Introduction

In spite of an excellent agreement of the Λ CDM model with observational data some basic assumptions of this model need justification. There are some ingredients in the model which could hardly be derived from a certain fundamental theory. The presence of dark energy (DE) with its currently small value is difficult to explain in the standard model of elementary particles [1]. Then, the relation of dark energy to the dark matter (DM) seems accidental (coincidence problem). That these components are of the same order suggests that there may be certain dynamical relation between them. We suggest a model describing an irreversible flow of DE to DM. We assume that the total mass of the dark matter does not change. These assumptions lead to the unique model of the DM-DE interaction.

The Einstein equations are

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}, \quad (1.1)$$

where $R^{\mu\nu}$ is the Ricci tensor, $g^{\mu\nu}$ the metric and $8\pi G = c = \hbar = 1$, we can decompose the right-hand sides of (1.1) as

$$T^{\mu\nu} = T_b^{\mu\nu} + T_R^{\mu\nu} + T_{de}^{\mu\nu} + T_{dm}^{\mu\nu}, \quad (1.2)$$

where the absence of an interaction between baryonic matter T_b , radiation T_R and the dark component means

$$\nabla_\mu (T_R^{\mu\nu} + T_b^{\mu\nu}) = 0. \quad (1.3)$$

The conservation of the total energy gives

$$\nabla_\mu T_{de}^{\mu\nu} = -\nabla_\mu T_{dm}^{\mu\nu} \equiv -3\kappa^2 J^\nu \quad (1.4)$$

with a current J^ν and a certain constant κ which can be calculated when the model of $T_{dm}^{\mu\nu}$ is defined.

The relation between the non-conservation law (1.4) can explain the coincidence between DM and DE densities as well as the relevance of the dark energy exactly at the present epoch. We need a model for $T_{de}^{\mu\nu}$ and $T_{dm}^{\mu\nu}$. We assume that the gain of energy of the dark matter consisting of particles of mass m results from a diffusion in an environment described by an ideal fluid. There is only one diffusion which is relativistic invariant and preserves the particle mass m [2]. The corresponding energy-momentum satisfies the conservation law (1.4). The current J^ν in eq. (1.4) is conserved [3–5]

$$\nabla_\mu J^\mu = 0. \quad (1.5)$$

This is a realization of the conservation law of the total mass of the dark matter. In a homogeneous universe the current conservation implies

$$J^0 = \frac{\gamma}{3\kappa^2} a^{-3}, \quad (1.6)$$

where a is the scale factor of an expanding metric and another constant γ .

In a homogeneous space-time we can represent the DM as well as DE energy-momentum as the energy-momentum of an ideal fluid. The conservation law (1.4) leads to a particular interaction among the fluids. An interaction which is a linear combination of the DM and DE fluids has been discussed in [6]. Non-linear interactions are discussed in [7–10]. Our formula for the DM dissipation (1.4) follows from the assumption that the dissipation results from a relativistic motion in a DE fluid. It cannot be expressed as a polynomial formula in DM and DE fluids as it is in the above mentioned references. Nevertheless, we are able to express the dynamics of the model as a quadratic dynamical system what makes our approach similar to that of refs. [7, 8, 10].

Methods of dynamical systems [11] have been recently used in a cosmological model with diffusion described by a cosmological scalar field [12]. A similar analysis of the dynamics has been also explored in the context of Bianchi cosmological models [13] as well as in a description of non-homogeneous and anisotropic cosmological models [14]. In this paper we intend to explore cosmological models as closed dynamical systems with matter (dark and baryonic) and dark energy in the form of ideal fluids whose interaction is determined by the current J^ν (1.4). In contradistinction to above mentioned models our model does not contain non-physical trajectories passing through $\rho_m = 0$ line [15].

The plan of the paper is the following. In section 2 we review the model of a relativistic diffusion and explain eq. (1.4). In section 3 we derive exactly soluble limits relevant for early and late universe. We discuss energy-momentum conservation and Einstein equations in section 4. In section 5 we formulate the cosmological equations of section 4 as a closed dynamical system. We determine its critical points and the phase portrait. In section 6 we fit the parameters of the model to the observational data.

2 Relativistic diffusion

In this section we consider a Markovian approximation of an interaction of the system with an environment which leads to the description of this interaction by a diffusion. We consider

a relativistic generalization of the Krammers diffusion defined on the phase space. It is determined in the unique way by the requirement that the diffusing particle moves on the mass-shell (see [2, 16–18]).

Let us choose the contravariant spatial coordinates p^j on the mass shell and define the Riemannian metric

$$ds^2 = g_{\mu\nu} dp^\mu dp^\nu = -G_{jk} dp^j dp^k,$$

where Greek indices $\mu, \nu = 0, 1, 2, 3$, Latin indices $j, k = 1, 2, 3$ and p_0 is expressed by p^j from $p^2 = m^2$. We have (we assumed that $g_{0k} = 0$)

$$G_{jk} = -g_{jk} + p_j p_k \omega^{-2},$$

where

$$\omega^2 = m^2 - g_{jk} p^j p^k.$$

Then, the inverse matrix is

$$G^{jk} = -g^{jk} + m^{-2} p^j p^k.$$

Next,

$$G \equiv -\det(G_{jk}) = -m^2 \det(g_{jk}) \omega^{-2}.$$

We define diffusion as a stochastic process generated by the Laplace-Beltrami operator Δ_H^m on the mass shell

$$\Delta_H^m = \frac{1}{\sqrt{G}} \partial_j G^{jk} \sqrt{G} \partial_k, \quad (2.1)$$

where $\partial_j = \frac{\partial}{\partial p^j}$ and $G = \det(G_{jk})$ is the determinant of G_{jk} .

The transport equation for the diffusion generated by Δ_H reads

$$(p^\mu \partial_\mu^x - \Gamma_{\mu\nu}^k p^\mu p^\nu \partial_k) \Omega = \kappa^2 \Delta_H^m \Omega, \quad (2.2)$$

where $\Gamma_{\mu\nu}^k$ are the Christoffel symbols, ∂_μ^x are space-time derivatives and κ^2 is the diffusion constant.

Then, we can define the current

$$J^\mu = \sqrt{g} \int \frac{d\mathbf{p}}{(2\pi)^3} p_0^{-1} p^\mu \Omega, \quad (2.3)$$

where $g = |\det[g_{\mu\nu}]|$ and the energy momentum

$$T_{de}^{\mu\nu} = \sqrt{g} \int \frac{d\mathbf{p}}{(2\pi)^3} p_0^{-1} p^\mu p^\nu \Omega. \quad (2.4)$$

It can be shown using eq. (2.2) that [3–5]

$$\nabla_\mu T_{\text{dm}}^{\mu\nu} = 3\kappa^2 J^\nu \quad (2.5)$$

and

$$\nabla_\mu J^\mu = g^{-\frac{1}{2}} \partial_\mu (g^{\frac{1}{2}} J^\mu) = 0. \quad (2.6)$$

Hence,

$$g^{-\frac{1}{2}} \partial_t (g^{\frac{1}{2}} J^0) = -\partial_j J^j. \quad (2.7)$$

This implies (1.6) if the metric is homogeneous and Ω does not depend on x . The constant γ can be expressed from eq. (2.3) as

$$\frac{\gamma}{3\kappa^2} = g \int \frac{d\mathbf{p}}{(2\pi)^3} \Omega \equiv Z. \quad (2.8)$$

3 The limits $ma \rightarrow 0$ and $ma \rightarrow \infty$

Most of our subsequent results hold true for a general FWR metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2 h_{jk} dx^j dx^k, \quad (3.1)$$

but for simplicity of our analysis we restrict ourselves to the flat space $h_{jk} = \delta_{jk}$. We rewrite the diffusion equation in terms of the covariant momenta

$$q_j = g_{ij} p^j \quad (3.2)$$

Then, Δ_H^m in eq. (2.1) depends on $\sqrt{m^2 a^2 + \mathbf{q}^2}$ and on a . The assumption $\mathbf{q}^2 \gg m^2 a^2$ (high energy approximation) is equivalent to the limit

$$m^2 a^2 \rightarrow 0. \quad (3.3)$$

Let

$$\nu = \int dt a \quad (3.4)$$

Then, in the limit $m^2 a^2 \rightarrow 0$ and in a homogeneous universe (Ω independent of spatial coordinates) we obtain

$$\kappa^{-2} |\mathbf{q}| \partial_\nu \Omega = q_i q_j \frac{\partial^2}{\partial q_i \partial q_j} \Omega + 3 q_j \frac{\partial}{\partial q_j} \Omega \quad (3.5)$$

or in the original contravariant coordinates

$$a \kappa^{-2} |\mathbf{p}| \left(\partial_t - 2 H p^j \frac{\partial}{\partial p^j} \right) \Omega = p^i p^j \frac{\partial^2}{\partial p^i \partial p^j} \Omega + 3 p^j \frac{\partial}{\partial p^j} \Omega. \quad (3.6)$$

If in $\sqrt{m^2 a^2 + \mathbf{q}^2}$ we assume $\mathbf{q}^2 \ll m^2 a^2$ (low energy approximation, i.e., we neglect $\mathbf{q}$) then in the limit

$$m^2 a^2 \rightarrow \infty$$

eq. (2.2) simplifies to

$$m^{-1} \kappa^{-2} \partial_\sigma \Omega = \frac{1}{2} \Delta_{\mathbf{q}} \Omega, \quad (3.7)$$

where

$$\sigma = 2 \int_{t_0}^t ds a^2 \quad (3.8)$$

and $\Delta_{\mathbf{q}}$ is the Laplacian. This is the non-relativistic diffusion equation. In terms of the original contravariant momenta eq. (3.7) takes the form

$$m^{-1} a^2 \kappa^{-2} \left(\partial_t - 2 H p^j \frac{\partial}{\partial p^j} \right) \Omega = \frac{1}{2} \Delta_{\mathbf{p}} \Omega. \quad (3.9)$$

4 Current conservation and Einstein equations

The energy-momentum (2.4) in a homogeneous space-time can be expressed as an energy-momentum of a fluid

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - g^{\mu\nu}p, \quad (4.1)$$

where

$$g_{\mu\nu}u^\mu u^\nu = 1. \quad (4.2)$$

The divergence equations (1.4) in the frame $u = (1, \mathbf{0})$ takes the form

$$\partial_t \rho_{dm} + 3H(1 + \tilde{w})\rho_{dm} = \gamma a^{-3}, \quad (4.3)$$

where

$$\tilde{w} = \frac{p_{dm}}{\rho_{dm}}.$$

We assume that the energy-momentum tensor of the dark energy has also the form of an ideal fluid (4.1). Then, from eqs. (1.4) and (1.6)

$$\partial_t \rho_{de} + 3H(1 + w)\rho_{de} = -\gamma a^{-3}. \quad (4.4)$$

On the basis of observational data we choose $w = -1$ in eq. (4.4). In the diffusion model $\tilde{w}$ depends on time as follows from the formula

$$\begin{aligned} \tilde{w} &= \frac{1}{3} \int d\mathbf{p} \frac{1}{p_0} a^2 \mathbf{p}^2 \Omega_t \left(\int d\mathbf{p} p_0 \Omega_t \right)^{-1} \\ &= \frac{1}{3} - \frac{m^2}{3} \int d\mathbf{p} \frac{1}{p_0} \Omega_t \left(\int d\mathbf{p} p_0 \Omega_t \right)^{-1} \equiv \frac{1 - \omega}{3}. \end{aligned} \quad (4.5)$$

In an expansion in m we can apply the explicit solution [19] of eq. (3.6) ($m = 0$) and calculate

$$\omega = \frac{m^2 a^2}{6(T_0 + \kappa^2 \nu)},$$

where T_0 is a parameter which has the meaning of the temperature of the DM fluid at $t = t_0$. In the ultrarelativistic (massless) case (3.6) we have $\omega = 0$, hence $\tilde{w} = \frac{1}{3}$.

We can express the solution of eq. (4.3) as

$$\begin{aligned} \rho_{dm}(t) &= \rho_{dm}(0) a^{-4} \exp \left(\int_{t_0}^t d\tau H \omega \right) \\ &\quad + \gamma a^{-4} \exp \left(\int_{t_0}^t d\tau H \omega \right) \int_{t_0}^t ds a(s) \exp \left(- \int_{t_0}^s d\tau H \omega \right). \end{aligned} \quad (4.6)$$

For $w = -1$

$$\rho_{de}(t) = \rho_{de}(0) - \gamma \int_{t_0}^t a^{-3}(s) ds. \quad (4.7)$$

We still consider the non-relativistic limit of the energy-momentum (2.4)

$$\begin{aligned} \rho_{dm} = \tilde{T}^{00} &= \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} p^0 \Omega = g^{-\frac{1}{2}} Z m + \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \frac{a^2 \mathbf{p}^2}{2m} \Omega \\ &\equiv Z m a^{-3} + a^{-2} \rho_{nr}, \end{aligned} \quad (4.8)$$

where

$$\rho_{nr} = \sqrt{g}(2\pi)^{-3} \int d\mathbf{p} \Omega a^4 \frac{\mathbf{p}^2}{2m}. \quad (4.9)$$

Using the non-relativistic diffusion equation (3.9) we can show that ρ_{nr} satisfies the non-conservation equation

$$\partial_t a^{-2} \rho_{nr} + 3H(1 + \tilde{w}_{nr}) a^{-2} \rho_{nr} = \gamma a^{-3}, \quad (4.10)$$

where $\tilde{w}_{nr} = \frac{2}{3}$. The non-relativistic diffusive energy in eq. (4.8) is a sum of two terms Zma^{-3} which describes a conservative non-relativistic total rest mass and $a^{-2}\rho_{nr}$ describing the diffusive energy gained from the motion in an environment of the dark energy. The sum of these energies satisfies the equation (which is not of the form) (1.4))

$$\partial_t \rho_{dm} + 5H\rho_{dm} = 3Z\kappa^2 a^{-3} + 2ZmHa^{-3}. \quad (4.11)$$

The solution of eq. (4.10) is

$$\tilde{\rho}_{nr}(t) = a^{-3} \left(\rho_{nr}(0) + \frac{1}{2} \gamma \sigma \right), \quad (4.12)$$

where σ is defined in eq. (3.8). As a consequence of eqs. (4.8), (4.10) and (1.4) the non-relativistic dark energy satisfies the same eqs. (4.4) and (4.7) as the relativistic dark energy.

The Friedman equation in the FRW metric (3.1) with the dark matter, dark energy and baryonic matter ρ_b reads

$$H^2 = \frac{1}{3}(\rho_{dm} + \rho_{de} + \rho_b). \quad (4.13)$$

By differentiation

$$\dot{H} = -\frac{1}{2} \left((1 + \tilde{w})\rho_{dm} + \rho_b \right). \quad (4.14)$$

In eqs. (4.13)–(4.14) we should insert the general expressions for DM and DE. We need an approximation for $\tilde{w}(t)$. There we shall discuss approximations to eq. (4.6). In a subsequent section we study the relativistic homogeneous dynamical system (4.3), (4.4) and (4.14) under the assumption that $\tilde{w}$ is time independent. The non-relativistic (low z) approximation (4.8) when inserted in eq. (4.13) gives the Friedmann equation

$$H^2 = \frac{1}{3} \left(a^{-5} \left(\rho_{dm}(0) + \frac{3}{2} Z\kappa^2 \sigma \right) + Zma^{-3} + \rho_b(0)a^{-3} + \rho_{de}(0) - 3Z\kappa^2 \int_{t_0}^t ds a(s)^{-3} \right). \quad (4.15)$$

Eqs. (4.7), (4.11) and (4.15) form a system of ordinary differential equations which is expressed by means of new (energetic) variables into a quadratic dynamical system in the next section.

5 Dynamical system approach to the DM-DE interaction

In this section we reduce the dynamics of the diffusive DM-DE interaction to the form of autonomous dynamical system $\frac{dx}{dt} = \dot{x} = f(x)$, where x is a state variable and t is time. In this approach one describes the evolution of the diffusive DM-DE interaction in terms of trajectories situated in a space of all states of the system, i.e., a phase space. This space

possesses the geometric structure which is a visualization of a global dynamics, i.e. it is the space of all evolutionary paths of the physical system, which are admissible for all initial conditions. The equivalence of phase portraits is established by means of a homeomorphism (topological equivalence) which is mapping trajectories of the system while preserving their orientation. The phase space is organized by critical points, which from the physical point of view represent stationary states of the system. From the mathematical point of view they are singular solutions of the system $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ is a vector state, corresponding to vanishing right-hand sides of the system, i.e. $\mathbf{f} = [f^1(x), \dots, f^n(x)]$ and $\forall_i f^i(x) = 0$. The final outcome of any dynamical system analysis is the phase portrait of the system from which one can easily obtain the information about the stability and genericity of particular solutions.

The methods of dynamical systems [11], which enable us to investigate the dynamics of the system without the knowledge of its exact solutions, have been recently applied in a similar context of cosmological models with diffusion [12]. An analysis of cosmological dynamics has also been explored in Bianchi cosmological models [13]. Some of these methods are applicable to non-homogeneous and anisotropic cosmological models [14] as well. In this paper we intend to explore an energy exchange in models describing matter (dark and baryonic) and dark energy in the form of the cosmological ideal fluids. In contrast to Alho et al. [12] our model does not contain non-physical trajectories passing through $\rho_m = 0$ line [15].

5.1 Cosmological models with constant equation of state for DM and cosmological constant — Dynamical system analysis

Let us consider the continuity equations for the model with $\tilde{w} = \text{const}$ and $w = -1$ (dark energy in the form of the cosmological constant). The corresponding continuity equations take the form (4.3)–(4.4)

$$a^{-3(\tilde{w}+1)} \frac{d}{dt}(\rho_{\text{dm}} a^{3(\tilde{w}+1)}) = \gamma a^{-3} > 0, \quad (5.1)$$

$$\frac{d\rho_{\text{de}}}{dt} = -\gamma a^{-3} < 0, \quad (5.2)$$

where $\gamma > 0$. To formulate the dynamics in the form of a dynamical system, we rewrite in a suitable way equation (5.1)

$$J = \frac{d\rho_{\text{dm}}/\rho_{\text{dm}}}{da/a} \equiv \frac{d \ln \rho_{\text{dm}}}{d(\ln a)} = -3(1 + \tilde{w}) + \frac{\gamma a^{-3}}{H \rho_{\text{dm}}}. \quad (5.3)$$

Next we define a dimensionless quantity, which measures the strength of the interaction

$$\delta \equiv \frac{\gamma a^{-3}}{H \rho_{\text{dm}}}. \quad (5.4)$$

Clearly, in general δ is time dependent. Let us consider that $\delta = \delta(a(t))$. If this quantity is constant during the cosmic evolution, then the solution of eq. (5.3) has a simple form

$$\rho_{\text{dm}} = \rho_{\text{dm},0} a^{-3(1+\tilde{w})+\delta}. \quad (5.5)$$

Our aim is to study the dynamics of the energy transfer from the DE to DM sector. The corresponding system assumes the form of a three-dimensional dynamical system.

Recalling that $8\pi G = c = 1$ we define

$$x \equiv \frac{\rho_{\text{dm}}}{3H^2}, \quad y \equiv \frac{\rho_{\Lambda}}{3H^2}, \quad (5.6)$$

where $H = \frac{d \ln a}{dt}$ is the Hubble parameter and t is the cosmological time. The differentiation with respect to the cosmological time t will be denoted by a dot ($\dot{} \equiv \frac{d}{dt}$). The variables x and y have the meaning of dimensionless density parameters.

For simplicity of presentation it is assumed that FRW space is flat (zero curvature in the Friedmann equations (4.13)). In this case the acceleration equation assumes the following form

$$\dot{H} = -\frac{1}{2}(\rho_{\text{eff}} + p_{\text{eff}}), \quad (5.7)$$

where $\rho_{\text{eff}} = \rho_{\text{dm}} + \rho_{\text{de}}$ and $p_{\text{eff}} = \tilde{w}\rho_{\text{dm}} - \rho_{\text{de}}$ are the effective energy density and pressure of the matter filling the universe, $\tilde{w} = \frac{p_{\text{dm}}}{\rho_{\text{dm}}}$.

Taking a natural logarithm of the state variables (5.6) and the interaction effect variable (5.4) and performing the differentiation with respect to the cosmological time t we obtain

$$\frac{\dot{x}}{x} = \frac{\dot{\rho}_{\text{dm}}}{\rho_{\text{dm}}} - 2\frac{\dot{H}}{H} = -3H(1 + \tilde{w}) + \delta H - 2\frac{\dot{H}}{H}, \quad (5.8)$$

$$\frac{\dot{y}}{y} = \frac{\dot{\rho}_{\Lambda}}{\rho_{\Lambda}} - 2\frac{\dot{H}}{H} = -\delta H \alpha - 2\frac{\dot{H}}{H}, \quad (5.9)$$

$$\frac{\dot{\delta}}{\delta} = -3H - \frac{\dot{H}}{H} - \frac{\dot{\rho}_{\text{dm}}}{\rho_{\text{dm}}} = 3\tilde{w}H - \delta H - \frac{\dot{H}}{H}, \quad (5.10)$$

where $\alpha = \frac{\rho_{\text{dm}}}{\rho_{\Lambda}}$.

It would be convenient to divide both sides of the system (5.8)–(5.10) by H and then reparameterize the original time variable t following the rule

$$t \rightarrow \tau = \ln a. \quad (5.11)$$

The differentiation with respect to the parameter τ will be denoted by a prime ($' \equiv \frac{d}{d\tau}$). Note that $\frac{d\tau}{da} = a^{-1}$ is a strictly monotonic function of the scale factor a .

After the time reparameterization (5.11) the system (5.8)–(5.10) can be expressed as the three-dimensional system of equations

$$x' = x \left(-3(1 + \tilde{w}) + \delta - 2\frac{\dot{H}}{H^2} \right), \quad (5.12)$$

$$y' = y \left(-\delta\alpha - 2\frac{\dot{H}}{H^2} \right), \quad (5.13)$$

$$\delta' = \delta \left(3\tilde{w} - \delta - \frac{\dot{H}}{H^2} \right), \quad (5.14)$$

where $\frac{\dot{H}}{H^2}$ can be determined from the formula (5.7)

$$\dot{H} = -\frac{1}{2}(1 + \tilde{w})\rho_{\text{dm}} = -\frac{3}{2}(1 + \tilde{w})H^2x, \quad (5.15)$$

i.e.,

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + \tilde{w})x. \quad (5.16)$$

In this way the dynamics of the process of decaying cold dark matter satisfying the equation of state $p_{\text{dm}} = \tilde{w}\rho_{\text{dm}}$ in the background of the flat FRW metric can be described by

means of the dynamical system theory. The resulting three-dimensional dynamical system has the form

$$x' = x(-3(1 + \tilde{w}) + \delta + 3(1 + \tilde{w})x), \quad (5.17)$$

$$y' = x(-\delta + 3(1 + \tilde{w})y), \quad (5.18)$$

$$\delta' = \delta \left(3\tilde{w} - \delta + \frac{3}{2}(1 + \tilde{w})x \right), \quad (5.19)$$

where $\alpha = x/y$.

Note that the right-hand sides of the dynamical system (5.17)–(5.19) are of a polynomial form. Therefore all methods of dynamical system analysis, especially analysis of the behavior on the Poincaré sphere, can be adopted; both in a finite domain as well as at infinity. One can see that the system (5.17)–(5.19) has as an invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$, the set $\{x: x = 0\}$, corresponding to the case of the vanishing dark matter energy density.

Clearly, the system (5.17)–(5.19) has also an invariant submanifold $\{\delta: \delta = 0\}$ corresponding to the case of the vanishing interacting term $Q = \delta$ as it appears in the Λ CDM model.

Another interesting submanifold is the plane $\{y: y = \frac{\delta}{3(1+\tilde{w})}\}$.

5.2 Dynamics of the model for dust matter

Let $\tilde{w}$ be equal to zero. Then, the equation of state for matter is of the form of a dust. Because $x + y = 1$ ($\Omega_{\text{dm}} + \Omega_{\text{de}} = 1$) then the dynamical system (5.17)–(5.19) reduces to the two-dimensional dynamical system in the following form

$$x' = x(-3 + \delta + 3x), \quad (5.20)$$

$$\delta' = \delta \left(-\delta + \frac{3}{2}x \right). \quad (5.21)$$

The phase portrait for the dynamical system (5.20)–(5.21) is presented in figure 1. On this phase portrait the deS₊ universe is a global attractor for expanding universes. On another hand the critical point (3) is a global repeller representing the Einstein-de Sitter universe. The saddle point is representing the static Einstein universe.

For the analysis of the behavior of trajectories at infinity we use the following sets of two projective coordinates: $\tilde{x} = \frac{1}{x}$, $\tilde{\delta} = \frac{\delta}{x}$ and $\tilde{X} = \frac{x}{\delta}$, $\tilde{\Delta} = \frac{1}{\delta}$.

The dynamical system in variables $\tilde{x}$ and $\tilde{\delta}$ covers the behavior of trajectories at infinity

$$\tilde{x}' = \tilde{x}(3\tilde{x} - \tilde{\delta} - 3), \quad (5.22)$$

$$\tilde{\delta}' = \tilde{\delta} \left(3\tilde{x} - 2\tilde{\delta} - \frac{3}{2} \right), \quad (5.23)$$

where $' \equiv \tilde{x} \frac{d}{d\tau}$. The phase portrait for the dynamical system (5.22)–(5.23) is presented in figure 2.

The dynamical system for variables $\tilde{X}$ and $\tilde{\Delta}$ is described by the following equations

$$\tilde{X}' = \tilde{X} \left(-3\tilde{\Delta} + \frac{3}{2}\tilde{X} + 2 \right), \quad (5.24)$$

$$\tilde{\Delta}' = \tilde{\Delta} \left(1 - \frac{3}{2}\tilde{X} \right), \quad (5.25)$$

No.	critical point	type of critical point	type of universe
1	$x = 0, \delta = 0$	saddle-node	de Sitter universe without diffusion effect
2	$x = 2/3, \delta = 1$ $(\tilde{x} = 3/2, \tilde{u} = 3/2)$ $(\tilde{X} = 2/3, \tilde{U} = 1)$	saddle	scaling universe
3	$x = 1, \delta = 0$ $(\tilde{x} = 1, \tilde{u} = 0)$	unstable node	Einstein-de Sitter universe without diffusion effect
4	$\tilde{x} = 0, \tilde{\delta} = 0$	stable node	static universe
5	$\tilde{X} = 0, \tilde{\Delta} = -3/4$	saddle	static universe
6	$\tilde{X} = 0, \tilde{\Delta} = 0$	unstable node	de Sitter universe with diffusion effect

Table 1. Critical points for autonomous dynamical systems (5.20)–(5.21), (5.22)–(5.23), (5.24)–(5.25), their type and cosmological interpretation.

where $' \equiv \tilde{\Delta} \frac{d}{d\tau}$. The phase portrait for the dynamical system (5.24)–(5.25) is presented in figure 3.

We use also the Poincaré sphere to analyze critical points in the infinity. We define variables

$$X = \frac{x}{\sqrt{1+x^2+\delta^2}}, \quad \Delta = \frac{\delta}{\sqrt{1+\delta^2+x^2}} \quad (5.26)$$

and in these variables the dynamical system has the following form

$$X' = X \left[-\Delta^2 \left(\frac{3}{2}X - \Delta \right) + (1 - X^2)(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (5.27)$$

$$\Delta' = \Delta \left[(1 - \Delta^2) \left(\frac{3}{2}X - \Delta \right) - X^2(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (5.28)$$

where $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{d\tau}$. The phase portrait for the dynamical system (5.27)–(5.28) is presented in figure 4. Critical points for autonomous dynamical systems (5.20)–(5.21), (5.22)–(5.23), (5.24)–(5.25) are completed in table 1.

5.3 Dynamics of the model at the late time ($ma \rightarrow \infty$)

As can be seen from eq. (4.8) the relativistic model of the dark matter consists of two fluids first with $\tilde{w} = 0$ and the second with $\tilde{w} = \frac{2}{3}$. So, in the approximation $ma \rightarrow \infty$ and $w = -1$ we have according to eqs. (4.7), (4.11) and (4.15) the following DM and DE continuity equations

$$\dot{\rho}_{\text{dm}} + 5\rho_{\text{dm}}H = \gamma a^{-3} + 2ZmHa^{-3}, \quad (5.29)$$

$$\dot{\rho}_{\text{de}} = -\gamma a^{-3}. \quad (5.30)$$

We define the variables

$$x = \frac{\rho_{\text{dm}}}{3H^2}, \quad y = \frac{\rho_{\text{de}}}{3H^2}, \quad u = \frac{(2Zm)a^{-3}}{\rho_{\text{dm}}} \quad \text{and} \quad \delta = \frac{\gamma a^{-3}}{H\rho_{\text{dm}}}. \quad (5.31)$$

If we use the variables (5.31) and time $\tau = \ln a$ then we obtain the following dynamical system

$$x' = x \left(-5 + \delta + u - 2 \frac{\dot{H}}{H^2} \right), \quad (5.32)$$

$$y' = -x(\delta + u) - 2y \frac{\dot{H}}{H^2}, \quad (5.33)$$

$$u' = u(2 - \delta - u), \quad (5.34)$$

$$\delta' = \delta \left(2 - \delta - u - \frac{\dot{H}}{H^2} \right), \quad (5.35)$$

where $' \equiv \frac{d}{d\tau}$ and $\frac{\dot{H}}{H^2} = -\frac{1}{2}x(5 - u)$. Because $x + y = 1$ ($\Omega_{\text{dm}} + \Omega_{\text{de}} = 1$) then the dynamical system (5.32)–(5.35) reduces to the three-dimensional dynamical system.

The dynamical system (5.32)–(5.35) has the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$, which is the set $\{x: x = 0\}$ or $\{u: u = 5\}$. There is also an interesting submanifold $\delta = 0$. On the invariant submanifold $\delta = 0$ the dynamical system (5.32)–(5.35) reduces to

$$x' = x(u + 5(x - 1) - xu), \quad (5.36)$$

$$u' = u(2 - u). \quad (5.37)$$

The phase portrait for the dynamical system (5.36)–(5.37) is presented in figure 5. Note that critical point (1) is representing the deS₊ universe without the diffusion effect. On the other hand the de Sitter universe without diffusion is represented by saddle critical point. Therefore the model with diffusion is generic in the class of all trajectories.

For the analysis the behavior of trajectories at infinity we use the following two sets of projective coordinates: $\tilde{x} = \frac{1}{x}$, $\tilde{u} = \frac{u}{x}$ and $\tilde{X} = \frac{x}{u}$, $\tilde{U} = \frac{1}{u}$.

The dynamical system for variables $\tilde{x}$ and $\tilde{u}$ is expressed by

$$\tilde{x}' = \tilde{x}(5\tilde{x}(\tilde{x} - 1) + \tilde{u}(1 - \tilde{x})), \quad (5.38)$$

$$\tilde{u}' = \tilde{u}(\tilde{x}(7\tilde{x} - 5) + \tilde{u}(1 - 2\tilde{x}) + \tilde{u}), \quad (5.39)$$

where $' \equiv \tilde{U}^2 \frac{d}{d\tau}$. The phase portrait for above dynamical system is presented in figure 6. In comparison to the phase portrait in figure 5 a new critical point (5) is emerging. It is representing the Einstein-de Sitter universe fully dominated by dark matter.

The dynamical system for variables $\tilde{X}$ and $\tilde{\Delta}$ is described by the following equations

$$\tilde{X}' = \tilde{X} \left(\tilde{U}(2 - 7\tilde{U}) + \tilde{X}(5\tilde{U} - 1) \right), \quad (5.40)$$

$$\tilde{U}' = \tilde{U}^2 \left(1 - 2\tilde{U} \right), \quad (5.41)$$

where $' \equiv \tilde{U}^2 \frac{d}{d\tau}$. The phase portrait for the dynamical system (5.40)–(5.41) is presented in figure 7. Note the de Sitter universe represented by critical point (1) which is a stationary universe without effect of diffusion is a global attractor.

Critical points for autonomous dynamical systems (5.36)–(5.37), (5.38)–(5.39), (5.40)–(5.41) are completed in table 2.

We apply also the Poincaré sphere to this system in order to analyze critical points at infinity. We define the variables

$$X = \frac{x}{\sqrt{1 + x^2 + u^2}}, \quad U = \frac{u}{\sqrt{1 + x^2 + u^2}} \quad (5.42)$$

No	critical point	type of critical point	type of universe
1	$x = 0, u = 0$	saddle	de Sitter universe without diffusion effect
2	$x = 1, u = 2$ $(\tilde{x} = 1, \tilde{u} = 2)$ $(\tilde{X} = 1/2, \tilde{U} = 1/2)$	saddle	scaling universe
3	$x = 1, u = 0$ $(\tilde{x} = 1, \tilde{u} = 0)$	unstable node	Einstein-de Sitter universe without diffusion effect
4	$x = 0, u = 2$ $(\tilde{X} = 0, \tilde{U} = 1/2)$	stable node	de Sitter universe without diffusion effect
5	$\tilde{x} = 0, \tilde{u} = 0$	stable node	static universe
6	$\tilde{X} = 0, \tilde{U} = 0$	unstable node	de Sitter universe without diffusion effect

Table 2. Critical points for autonomous dynamical systems (5.36)–(5.37), (5.38)–(5.39), (5.40)–(5.41), their types and cosmological interpretations.

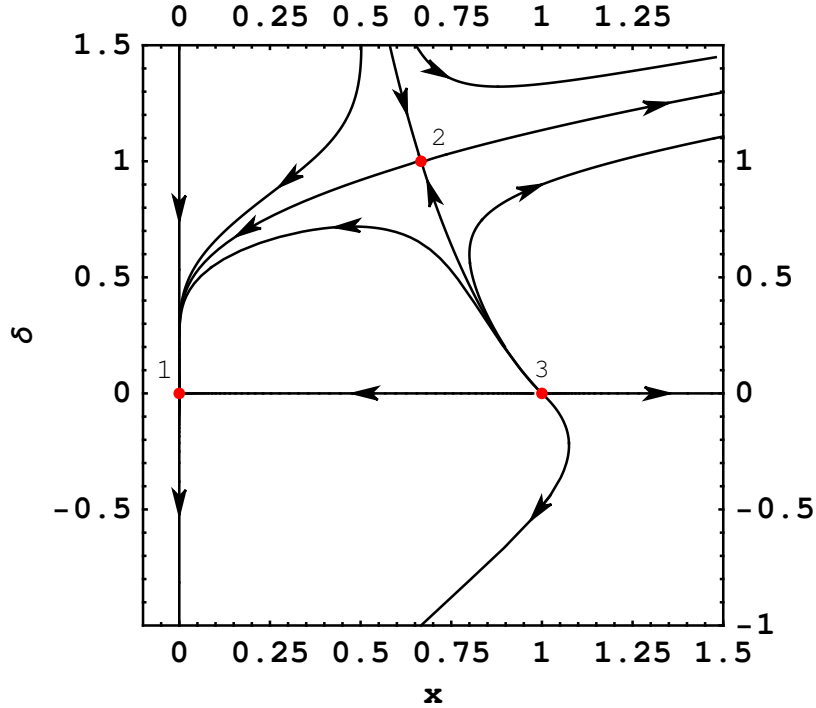


Figure 1. A phase portrait for dynamical system (5.20)–(5.21). Critical point (1) ($x = 0, \delta = 0$) represents the de Sitter universe. Critical point (2) ($x = 2/3, \delta = 1$) is a saddle and represents the scaling universe. Critical point (3) ($x = 1, \delta = 0$) is an unstable node and represents the Einstein-de Sitter universe. The critical point (1) is a complex type of saddle-node.

In these variables the dynamical system has the following form

$$X' = X \left[U^2 \sqrt{1 - X^2 - U^2} (U - 2\sqrt{1 - X^2 - U^2}) + (1 - X^2)(\sqrt{1 - X^2 - U^2}(5X + U) - 5(1 - X^2 - U^2) - XU) \right], \quad (5.43)$$

$$U' = U \left[(1 - U^2)\sqrt{1 - X^2 - U^2}(2\sqrt{1 - X^2 - U^2} - U) - X^2(\sqrt{1 - X^2 - U^2}(5X + U) - 5(1 - X^2 - U^2) - XU) \right], \quad (5.44)$$

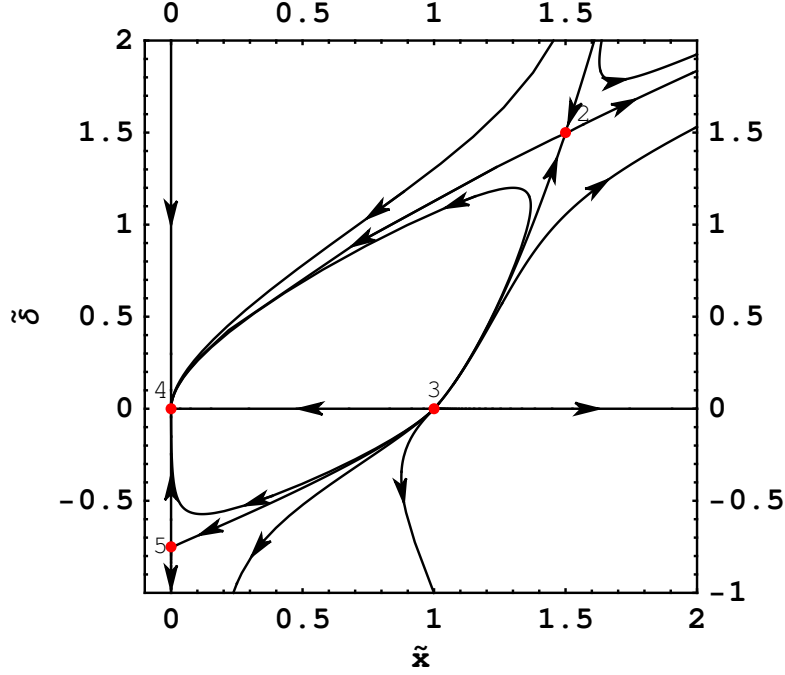


Figure 2. A phase portrait for dynamical system (5.22)–(5.23). Critical point (4) ($\tilde{x} = 0, \tilde{\delta} = 0$) and (5) ($\tilde{x} = 0, \tilde{\delta} = -3/4$) and represents the static universe. Critical point (2) ($\tilde{x} = 3/2, \tilde{\delta} = 3/2$) is a saddle and represents the scaling universe. Critical point (3) ($\tilde{x} = 1, \tilde{\delta} = 0$) is an unstable node and represents the Einstein-de Sitter universe.

where $' \equiv (1 - X^2 - U^2) \frac{d}{d\tau}$. The phase portrait for the dynamical system (5.43)–(5.44) is presented in figure 8.

6 Statistical analysis

6.1 Introduction

In this section we use astronomical observations for low redshifts such as the SNIa, BAO, measurements of $H(z)$ for galaxies and the Alcock-Paczyński test. We do not use the observation for high redshifts such as CMB.

$$\ln L_{\text{SNIa}} = -\frac{1}{2}[A - B^2/C + \ln(C/(2\pi))], \quad (6.1)$$

where $A = (\mu^{\text{obs}} - \mu^{\text{th}})\mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $B = \mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $C = \text{Tr}\mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for SNIa. The distance modulus is expressed by $\mu^{\text{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of SNIa) and $\mu^{\text{th}} = 5 \log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$).

We also use BAO observations such as Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [20], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [21], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [22]. The likelihood function is expressed by the formula

$$\ln L_{\text{BAO}} = -\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right), \quad (6.2)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch [23].

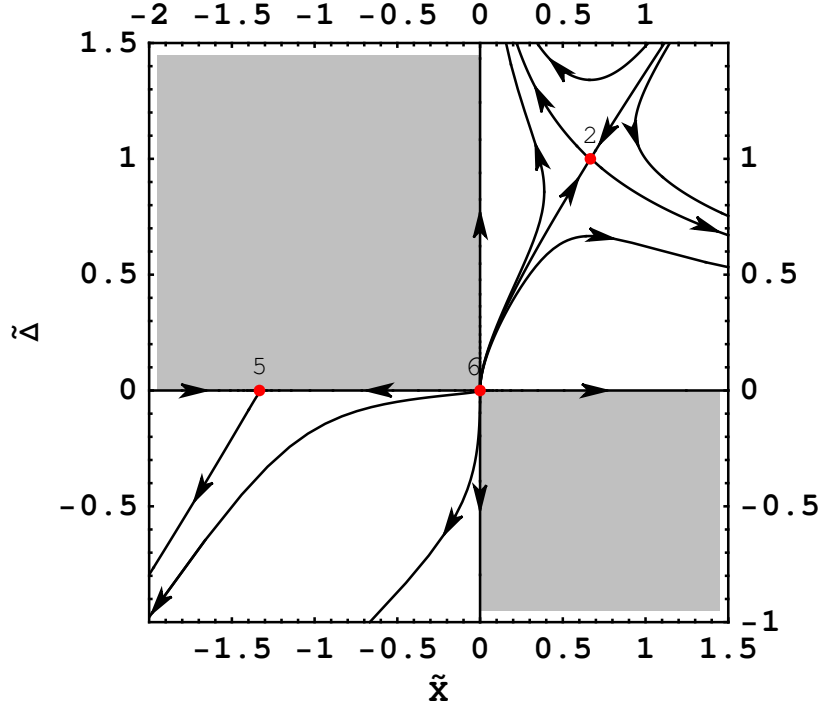


Figure 3. A phase portrait for dynamical system (5.24)–(5.25). Critical point (5) ($\tilde{X} = -4/3$, $\tilde{\Delta} = 0$) represents the static universe. Critical point (2) ($\tilde{X} = 2/3$, $\tilde{\Delta} = 1$) is a saddle and represents the scaling universe. Critical point (6) ($\tilde{X} = 0$, $\tilde{\Delta} = 0$) is an unstable node and represents the de Sitter universe. Note that if $\tilde{\Delta} < 0$ the arrow of time indicates how the scale factor is decreasing during the evolution.

For the Alcock-Paczynski test [24, 25] we use the likelihood function

$$\ln L_{AP} = -\frac{1}{2} \sum_i \frac{(AP^{th}(z_i) - AP^{obs}(z_i))^2}{\sigma^2}. \quad (6.3)$$

where $AP(z)^{th} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{obs}$ are observational data [26–34].

In addition, we are applying measurements of the Hubble parameter $H(z)$ of galaxies from [35–37]. In this case the likelihood function is expressed by

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{obs} - H(z_i)^{th}}{\sigma_i} \right)^2. \quad (6.4)$$

The final likelihood function is in the following form

$$L_{tot} = L_{SNIa} L_{BAO} L_{AP} L_{H(z)}. \quad (6.5)$$

We use our own code CosmoDarkBox to estimate the model parameters. This code applies the Metropolis-Hastings algorithm [38, 39] and the dynamical system formulation of model dynamics to obtain the likelihood function [23, 40]. The dynamical system formulation of the cosmological dynamics developed in section 5 plays a crucial role in our method of estimation. We solve the system numerically using the Monte Carlo method and then put this solution to the corresponding expression for observables in our model.

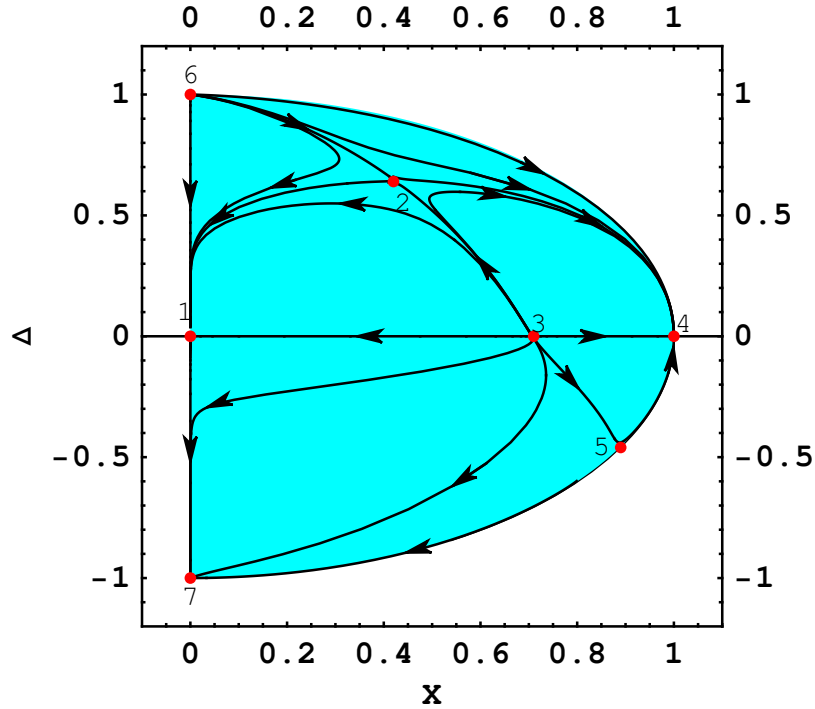


Figure 4. A phase portrait for dynamical system (5.27)–(5.28). Critical point (1) represents the de Sitter universe. Critical point (2) is a saddle and represents the scaling universe. Critical point (3) is an unstable node and represents the Einstein-de Sitter universe. Critical point (4) represents the static universe. Critical point (5) represents the static universe. Critical point (6) is an unstable node and represents the de Sitter universe. Note that if $\Delta < 0$ the arrow of time indicates how the scale factor is decreasing during the evolution.

For comparison models with diffusion with the Λ CDM model, we use Bayesian information criterion (BIC) [41, 42]. The BIC is defined as

$$\text{BIC} = -2 \ln L + j \ln n, \quad (6.6)$$

where L is the maximum of the likelihood function, j is the number of model parameters (in this paper for our models $j = 3$ and for Λ CDM $j = 2$) and n is number of data points (in this paper $n = 622$).

6.2 Model of DM-DE interaction and $\tilde{w} = 0$

Let us consider the model of DM-DE interaction and with dark matter in the form of dust. We present a statistical analysis of the model parameters such as H_0 , $\Omega_{\text{dm},0} = \frac{\rho_{\text{dm},0}}{3H_0^2}$, where $\rho_{\text{dm},0}$ is the present value of dark matter and $\Omega_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T dt$, where T is the present age of the Universe. We must have $\Omega_{\gamma} \geq 0$ because $\gamma \geq 0$ for a diffusion.

The Friedmann equation for $\tilde{w} = 0$ in terms of the present values of the density parameters takes the form

$$\frac{H^2}{H_0^2} = \Omega_{\text{cm},0} a^{-3} + \frac{\Omega_{\gamma,0}}{\int^T dt} a^{-3} \int^t dt + \Omega_{\text{b},0} a^{-3} + \Omega_{\text{de}}(0) - \frac{\bar{\Omega}_{\gamma,0}}{\int^T a^{-3} dt} \int^t a^{-3} dt, \quad (6.7)$$

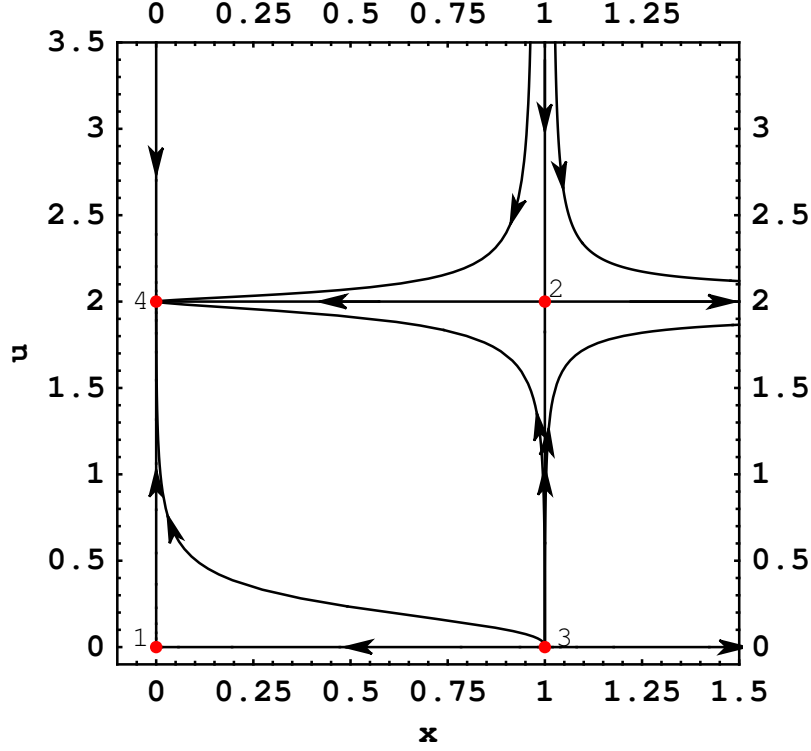


Figure 5. A phase portrait for dynamical system (5.36)–(5.37). Critical point (1) ($x = 0, u = 0$) represents the de Sitter universe without the diffusion effect. Critical point (2) ($x = 1, u = 2$) is a saddle type and represents the scaling universe. Critical point (3) ($x = 1, u = 0$) is an unstable node and represents the Einstein-de Sitter universe without the diffusion effect. The critical point (4) is representing the Einstein-de Sitter without the diffusion effect.

where $\Omega_{\text{cm},0} = \frac{\rho_{\text{cm},0}}{3H_0^2}$, where $\rho_{\text{cm},0}$ is the present value of the conservative part of dark matter, which scales as a^{-3} , $\bar{\Omega}_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T a^{-3} dt$.

In these estimation we use formulation of dynamics in the form of a two-dimensional non-autonomous system with the redshift variable z . This model possesses three parameters γ, H_0 and $\Omega_{\text{m}}(z=0) = \Omega_{\text{m},0}$

$$\begin{aligned}\Omega_{\text{m}}' &= \frac{3}{1+z}\Omega_{\text{m}} - \frac{\gamma}{3H_0^3}P(1+z)^2, \\ P' &= -\frac{3}{2}\frac{1}{1+z}\Omega_{\text{m},0}P^3,\end{aligned}\tag{6.8}$$

where $P = \frac{H_0}{H}$ and $' \equiv z$.

Statistical results are presented in table 3. Figure 9 shows the likelihood function with 68% and 95% confidence level projections on the plane $(\Omega_{\text{dm},0}, \Omega_{\gamma})$. For this case the value of reduced χ^2 is equal 0.187767.

The value of BIC, for this model is equal $\text{BIC}_1=135.527$. Because BIC for the ΛCDM model is equal $\text{BIC}_{\Lambda\text{CDM}}=129.105$, $\Delta\text{BIC} = \text{BIC}_1 - \text{BIC}_{\Lambda\text{CDM}}$ is equal 6.421. If that a value of ΔBIC is more than 6, the evidence for the model is strong [42]. Consequently, the evidence in favor of the ΛCDM model is strong in comparison to our model.

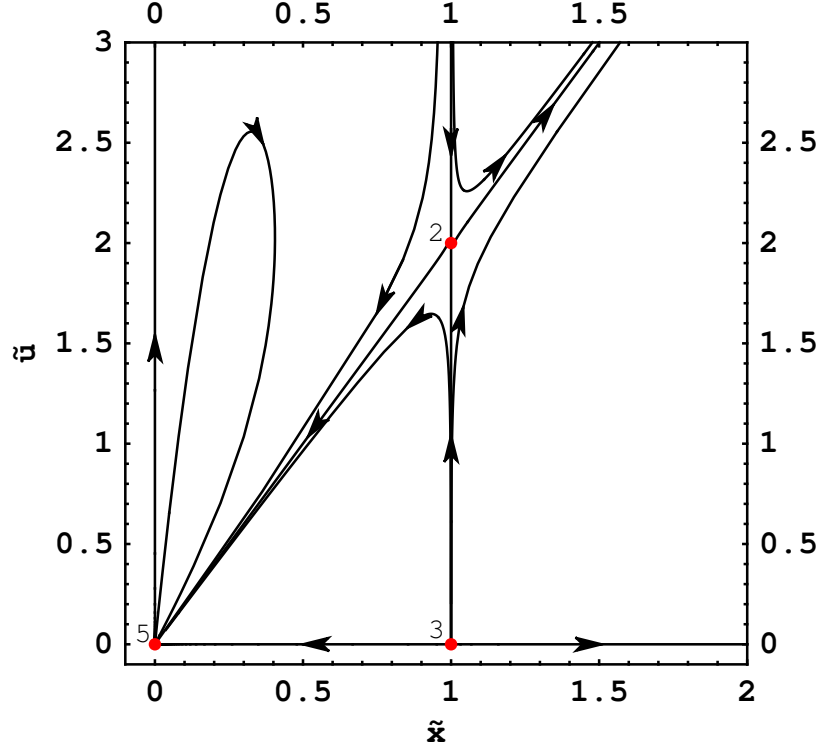


Figure 6. A phase portrait for dynamical system (5.38)–(5.39). Critical point (5) ($\tilde{x} = 0$, $\tilde{u} = 0$) represents the static universe. Critical point (2) ($\tilde{x} = 1$, $\tilde{u} = 2$) is a saddle and represents the scaling universe. Critical point (3) ($\tilde{x} = 1$, $\tilde{u} = 0$) is an unstable node and represents the Einstein-de Sitter universe. Note that the Einstein-de Sitter universe is fully dominated by dark matter. It is an attractor solution as well as the de Sitter which one can see in figure 5.

parameter	best fit	68% CL	95% CL
H_0	67.97 km/(s Mpc)	+0.75 −0.72	+1.57 −1.45
$\Omega_{\text{dm},0}$	0.2658	+0.0223 −0.0208	+0.0485 −0.0415
$\Omega_{\gamma,0}$	0.0135	+0.0735 −0.0135	+0.1570 −0.0135

Table 3. The best fit and errors for the estimated model for SNIa+BAO+ $H(z)$ +AP test with H_0 from the interval (65.0 (km/(s Mpc)), 71.0 (km/(s Mpc))), $\Omega_{\text{dm},0}$ from the interval (0.25, 0.40), $\Omega_{\gamma,0}$ from the interval (0.00, 0.20) $\Omega_{\text{b},0}$ is assumed as 0.048468. The value of reduced χ^2 is equal 0.187767.

6.3 Model with DM-DE interaction for $ma \rightarrow \infty$

Let us consider a late time behavior of the universe. For the case $ma \rightarrow \infty$ we estimated values of cosmological parameters such as $\Omega_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T a^2 dt$, $\Omega_{Zm,0} = \frac{Z_m}{3H_0^2}$, H_0 and γ . The formula for the Friedmann equation in terms of the present values of the density parameters

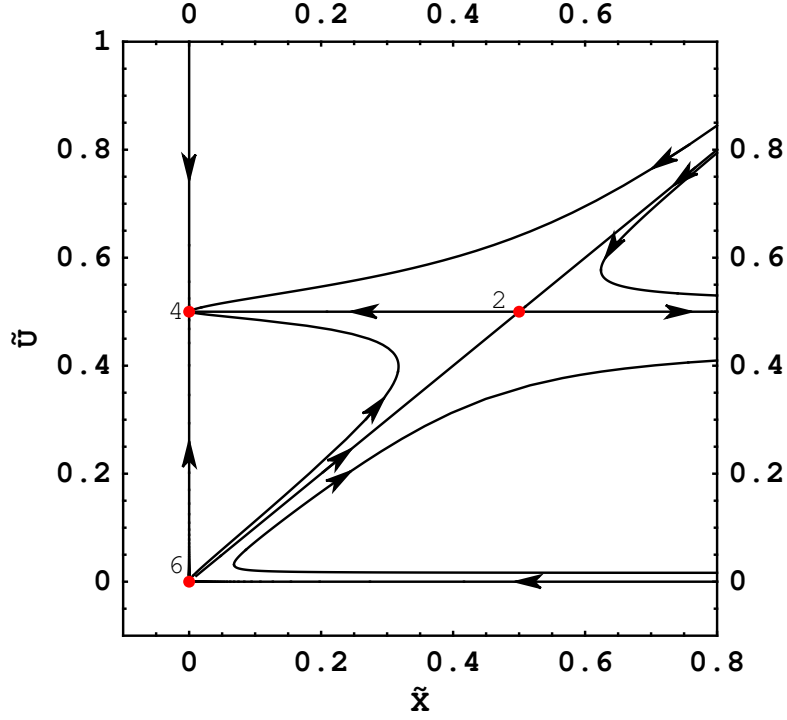


Figure 7. A phase portrait for dynamical system (5.40)–(5.41). Critical point (2) ($\tilde{X} = 1/2, \tilde{U} = 1/2$) is a saddle and represents the scaling universe. Critical point (6) ($\tilde{X} = 0, \tilde{U} = 0$) is an unstable node and represents the de Sitter universe. At this critical point the effect of diffusion are important. On the other hand the critical point (6) is an unstable stationary solution in which the effect of the non-zero term (Zm) vanishes.

is in the form

$$\frac{H^2}{H_0^2} = \Omega_{\text{dm},0} a^{-5} + \frac{\Omega_{\gamma,0}}{\int^T a^2 dt} a^{-5} \int^t a^2 dt + \Omega_{Zm} a^{-3} + \Omega_{\text{de}}(0) - \frac{\bar{\Omega}_{\gamma,0}}{\int^T a^{-3} dt} \int^t a^{-3} dt, \quad (6.9)$$

where $\bar{\Omega}_{\gamma,0} = \frac{\gamma}{3H_0^2} \int^T a^{-3} dt$ and $\Omega_{\text{dm},0}$ is the present value of the part of dark matter, which scales as a^{-5} .

The results of our analysis of the model are completed in table 4. Figure 10 shows the likelihood function with the 68% and 95% confidence level projections on the plane $(\Omega_{\text{dm},0}, \Omega_{\gamma})$. For this case the value of reduced χ^2 is equal 0.188201.

The value of BIC, for this model is equal $\text{BIC}_2=135.795$ Because BIC for the ΛCDM model is equal $\text{BIC}_{\Lambda\text{CDM}} = 129.105$, $\Delta\text{BIC} = \text{BIC}_2 - \text{BIC}_{\Lambda\text{CDM}}$ is equal 6.690. If that a value of ΔBIC is more than 6, the evidence for the model is strong [42]. Consequently, the evidence in favor of the ΛCDM model is strong in comparison to our model.

We can compare the behavior of Ω_{de} for our models with others models of the early dark energy. In Doran and Robbers model [43] the fractional dark energy density is assumed as a constant, which is different from zero, for the early time universe. This means that $\Omega_{\text{de}}(z)$ cannot be negligible for the early universe for this model. In our models, Ω_{de} approaches zero for the high redshifts (see figure 11) and Ω_{de} is negligible for the early universe. In consequence, we do not use the high redshift astronomical observations, such as CMB, to fit values of model parameters for our models.

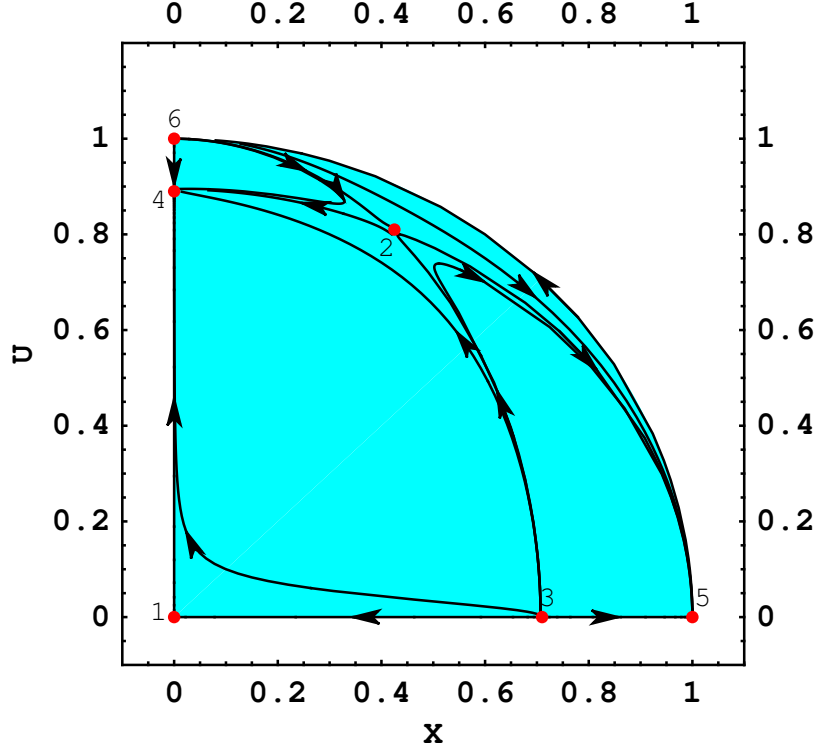


Figure 8. A phase portrait for dynamical system (5.43)–(5.44). Critical point (1) represents the de Sitter universe without the diffusion effect. Critical point (2) is a saddle type and represents the scaling universe. Critical point (3) is an unstable node and represents the Einstein-de Sitter universe without the diffusion effect. The critical point (4) is representing the Einstein-de Sitter with the diffusion effect. Critical point (5) represents the static universe. Critical point (6) is an unstable node and represents the de Sitter universe.

parameter	best fit	68% CL	95% CL
H_0	68.04	+0.73 −0.70	+1.27 −1.25
$\Omega_{\gamma,0}$	0.0106	+0.0082 −0.0106	+0.0137 −0.0106
$\Omega_{Zm,0}$	0.2943	+0.0356 −0.0077	+0.0536 −0.0231
γ	0.0299	+0.2198 −0.0299	+0.4555 −0.0299

Table 4. The best fit and errors for the estimated model with $w = 2/3$ for SNIa+BAO+ $H(z)$ +AP test with $\Omega_{Zm,0}$ from the interval (0.22, 0.38), $\Omega_{\gamma,0}$ from the interval (0.0, 0.03), γ from the interval $(0.00(100 \text{ km}/(\text{s Mpc}))^3, 0.500(100 \text{ km}/(\text{s Mpc}))^3)$ and H_0 from the interval $(65.0 (\text{km}/(\text{s Mpc})), 71.0 (\text{km}/(\text{s Mpc})))$. $\Omega_{b,0}$ is assumed as 0.048468. The value of reduced χ^2 is equal 0.188201.

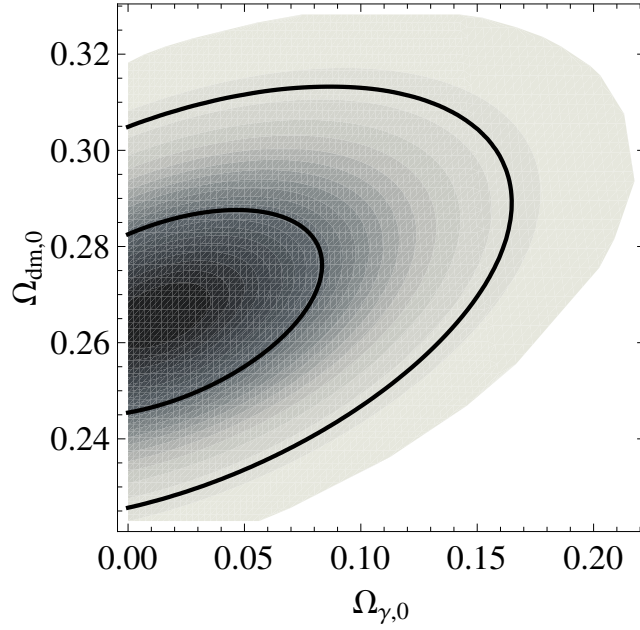


Figure 9. The intersection of the likelihood function of two model parameters $(\Omega_{\text{dm},0}, \Omega_{\gamma,0})$, for the case of the model of DM-DE interaction and $\tilde{w} = 0$, with the marked 68% and 95% confidence levels for SNIa+BAO+ $H(z)$ +AP test. $\Omega_{\text{dm},0}$ is the present value of dark matter.

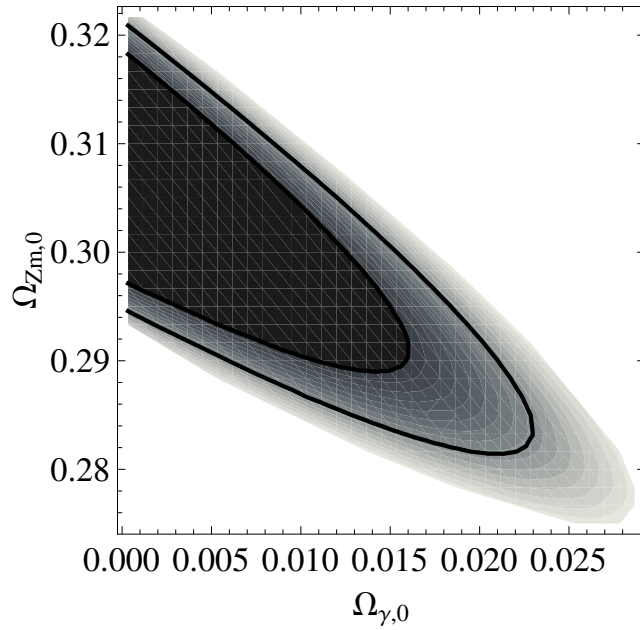


Figure 10. The intersection of the likelihood function of two model parameters $(\Omega_{\text{Zm},0}, \Omega_{\gamma,0})$, for the case of the model with DM-DE interaction for $ma \rightarrow \infty$, with the marked 68% and 95% confidence levels for SNIa+BAO+ $H(z)$ +AP test.

7 Conclusion

In this paper we studied the dynamics of DM-DE interaction with the relativistic diffusion process. For this aim we used the dynamical system methods, which enable us to study

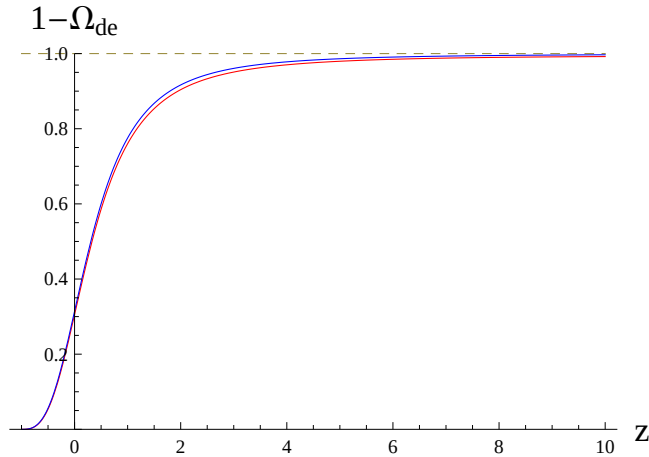


Figure 11. The diagram presents the evolution of $1 - \Omega_{de}(z) = \frac{\Omega_{m,0}f(z)}{H^2(z)/H_0^2}$, where z is redshift, $\Omega_{m,0}f(z) = \frac{\rho_m(z)}{3H_0^2}$ and $f(0) = 1$, for the first model (blue line) and for the second model (red line). We assume the best fit values of model parameters (see table 3 and 4). Note that, for the early universe, for the both models, $1 - \Omega_{de}(z)$ is going to a constant (the horizontal asymptotics equals one). This means, that $\Omega_{de}(z)$ for the high redshifts is negligible.

all evolutionary scenarios admissible for all initial conditions. We show that dynamics of our model reduces to the three-dimensional dynamical system, which in order is investigated on an invariant two-dimensional submanifold. From our dynamical analysis the dynamics is free from the difficulties, which are present in Alho et al.'s models with diffusion [12], namely there is no non-physical trajectories crossing the boundary set $\rho_m = 0$ [15].

The model is tested by astronomical data in two cases of dark matter in the domain of low redshifts (SNIa, BAO, $H(z)$ for galaxies and AP test).

In the model under consideration the energy density of dark matter is a growing function with the cosmological time on the cost of dark energy sector. In the basic formulas on $H^2(z)$ some additional terms appear related with the diffusion process itself. These contributions can be interpreted as the running Lambda term ($\bar{\Omega}_{\gamma,0} \neq 0$) and a correction to the standard scaling law $\propto a^{-3}$ for dark matter. At the present epoch the value of the density parameter related with the dark matter correction is about 1% of total energy budget.

In the first model it is assumed dark matter in the form of dust. The estimated values of the model parameters are comparable with the parameters for the Λ CDM model and the value of reduced chi-square of this model is 0.187767. We also studied the second model with diffusion in a late time approximation: $ma \rightarrow \infty$. The value of density parameter of $\Omega_{\gamma,0}$ related with diffusion is equal 0.0106. In this case the value of reduced chi-square is 0.188201. For comparison, the value of reduced chi-square of the Λ CDM model is 0.187483.

The value of $\Delta\text{BIC} = \text{BIC}_i - \text{BIC}_{\Lambda\text{CDM}}$ for the first model is 6.421 and for the second model is equal 6.690. While the evidence is strong in favor of the Λ CDM model in comparison to our model, our model cannot be rejected based on our statistical analysis.

Acknowledgments

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Does the diffusion dark matter-dark energy interaction model solve cosmological puzzles?

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We study dynamics of cosmological models with diffusion effects modeling dark matter and dark energy interactions. We show the simple model with diffusion between the cosmological constant sector and dark matter, where the canonical scaling law of dark matter ($\rho_{dm,0}a^{-3}(t)$) is modified by an additive $\epsilon(t) = \gamma ta^{-3}(t)$ to the form $\rho_{dm} = \rho_{dm,0}a^{-3}(t) + \epsilon(t)$. We reduced this model to the autonomous dynamical system and investigate it using dynamical system methods. This system possesses a two-dimensional invariant submanifold on which the dark matter-dark energy (DM-DE) interaction can be analyzed on the phase plane. The state variables are density parameter for matter (dark and visible) and parameter δ characterizing the rate of growth of energy transfer between the dark sectors. A corresponding dynamical system belongs to a general class of jungle type of cosmologies represented by coupled cosmological models in a Lotka-Volterra framework. We demonstrate that the de Sitter solution is a global attractor for all trajectories in the phase space and there are two repellers: the Einstein-de Sitter universe and the de Sitter universe state dominating by the diffusion effects. We distinguish in the phase space trajectories, which become in good agreement with the data. They should intersect a rectangle with sides of $\Omega_{m,0} \in [0.2724, 0.3624]$, $\delta \in [0.0000, 0.0364]$ at the 95% CL. Our model could solve some of the puzzles of the Λ CDM model, such as the coincidence and fine-tuning problems. In the context of the coincidence problem, our model can explain the present ratio of ρ_m to ρ_{de} , which is equal $0.4576^{+0.1109}_{-0.0831}$ at a 2σ confidence level.

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I. INTRODUCTION

Thanks to astronomical observations the modern cosmology has elaborated on the concept of the standard cosmological model called the cold dark matter model with the cosmological constant (Λ CDM model). From a methodological point of view cosmology has achieved a similar status as the particle physics with its standard model of particles. The Λ CDM model acts as an effective theory describing the Universe from redshift $z = 0$ (today) to $z = 10^9$ (the epoch of primordial nucleosynthesis). In the very structure of the Λ CDM model there are essentially two components: matter (dark matter and baryonic matter) and dark energy. It is assumed that the universe is spatially homogeneous and isotropic, and that its evolution is governed by Einstein field equations with the energy momentum tensor for the ideal fluid satisfying the barotropic equation of state $p = p(\rho)$, where ρ is the energy density of fluid. From the observational point of view it is convenient to use dimensionless density parameters Ω_i ,

defined as the fractions of critical density $3H_0^2$ which corresponds to a flat model. These parameters are observables which can be determined from astronomical observations [1]. In the Λ CDM model it is assumed that all fluids are noninteracting.

The natural interpretation of the cosmological constant is to treat it as the energy of the quantum vacuum [2]. The cosmological model with the cosmological constant term and pressureless matter fits well to the observation data of measurements of SNIa luminosity distance as a function of redshift [3] and observations microwave relic radiation (WMAP, Planck). Measurements of large-scale structures also remain consistent with the Λ CDM model. Although the Λ CDM model describes well the present Universe the nature of its basic constituents (dark energy and dark matter) remains unknown. So dark energy and dark matter are like some useful fictions in the terminology of Nancy Cartwright [4]. Comparing the value of the cosmological constant required us to explain the effect of the accelerated expansion of the Universe observations of distant supernovae type SNIa with the value of the cosmological constant interpreted as the energy of the quantum vacuum, we get the most incredible gap in history of physics

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$\rho_{vac}/\rho_{\Lambda} = 10^{143}$. In this context, the question is born why the value of the cosmological constant is the small, why not simply a zero? This is the known problem of the cosmological constant. In the model under consideration, the comparison of $\rho_{vac}/\rho_{\Lambda}$ still gives 83 orders of magnitude from the measurement value because $\rho_{vac}/\rho_{\Lambda} \approx 10^{60}$.

Another problem related closely with the problem of the cosmological constant is the coincidence problem [5]. This problem is caused by the lack of explanation of why in today's era the density of dark matter and dark energy are comparable although it is assumed that they have different time of origin. In this paper we construct a cosmological model in which it is assumed that the process of interaction between sectors of dark matter and dark energy is continuous. Relativistic diffusion describes the transfer of energy to the sector of dark matter. As a result, we go beyond the standard model assuming from the outset that dark matter and dark energy interact. This effect is described by the running cosmological constant and the modification of the standard scaling law of the dark matter density.

If we assume that general relativity is an effective theory which can be extrapolated to the Planck epoch, then the interpretation of the cosmological constant parameter appeared in the Λ CDM model as a vacuum energy seems to be natural. By equating this density to the energy density of the zero point energy that is left in a volume after removing all particles, then we obtain that its value is about 120 orders of magnitude higher than the corresponding value required for explanation of acceleration of the Universe in the current epoch. In the Universe with such a high value of cosmological constant (dark energy) we have a rapid inflation and galaxies would have no time to form. The lack of explanation of this difference is called the cosmological constant problem.

Its solution can be possible if we can find some physical mechanism lowering dramatically this value during the cosmic evolution. Of course this process should be defined in a covariant way following general relativity principles.

Our hypothesis is that diffusion cosmology can offer the possibility of obtaining a low value of the cosmological constant today because the effects of diffusion effectively produce the running cosmological constant.

The first model with diffusion, dark matter-dark energy (DM-DE) interaction was constructed by Calogero and Velten [6–8]. In Calogero and Velten's paper, dark matter is modeled by dust matter and dark energy by the scalar field. In Haba *et al.*'s approach [9], dark matter is modeled by a coefficient of the equation of state w as a function of redshift and dark energy assumes the form of the decaying Λ parameter. In Calogero and Velten's approach, the model is based in a modification of the geometric side of Einstein's equations. Haba *et al.*'s approach is consistent with general relativity at very beginning. Taking into account the cosmological equations there is no difference

between these approaches if we replace $\rho_{de}(a)$ by $\phi(t)$ and $w = \text{const}$.

We study how a value of effective running cosmological constant parameter changes during the cosmic evolution and for late time is going to be a small constant value.

From the astronomical observations of distant supernovae SNIa and measurements of CMB by Planck, measurement of BAO and other astronomical observations we obtain that the present value of the energy densities of both dark energy and dark matter are of the same order of magnitude [10]. If we assume that the standard cosmological model (Λ CDM model) is an adequate way to describe the cosmic evolution, then the value ρ_{de}/ρ_{dm} will depend on the cosmological time or redshift and the question arises: Why are two quantities with different time of origin comparable at the present epoch? It is called the cosmic coincidence problem.

We are looking for some physical relativistic mechanism which gives rise to this coincidence observed for the current Universe. In the opposite case very special initial conditions are required for its realization (fine tuning problem). In the framework of diffusion cosmology our investigation of this problem shows that while the values of dark matter and dark energy densities are comparable today they were significantly different in the past history of the Universe. Because the diffusion effects effectively act for fluids which interact with each other during the cosmic evolution. As a consequence dark energy is running and a canonical rule of scaling dark matter ρ_{dm} proportional to a^{-3} is adjusted.

The main aim of our paper is to demonstrate how the coincidence problem can be naturally solved in the framework of diffusion cosmology. The interacting dark energy models have been considered by many authors in the context of this problem. One of the reasons to study these models is to solve the cosmic coincidence problem [11–15]. To this aim different *ad hoc* proposed models of an interacting term were postulated *a priori*. In these models the covariance of general relativity is usually violated and therefore they have limited application to cosmology. In the present work we consider a unique relativistic diffusion model where an interaction mechanism is motivated physically.

In the study of evolutionary scenarios of the model under consideration we apply the dynamical systems methods [16]. Our model belongs to a general class of jungle type of cosmologies represented by coupled cosmological models in a Lotka-Volterra framework [17].

The crucial role in the organization of the phase space plays the critical point located inside the physical region. The possible bifurcation of this point is studied in detail for extracting variability of DE and DM density as the function of the cosmological time. It is interesting that at this critical point $\rho_{dm} \propto \rho_{de}$ (scaling type solution).

II. FRIEDMANN EQUATION FOR DIFFUSION INTERACTING OF DARK MATTER WITH DARK ENERGY

Haba *et al.* postulated a particular model of an energy-momentum exchange between DM and DE sectors, while a baryonic sector was preserved [9]. In this approach, it is assumed that the whole number of particles is conserved in the dark sector. In this paper we reconsider this model in the light of aforementioned cosmological problems.

We assume Einstein equations in the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = T^{\mu\nu}, \quad (1)$$

where $g^{\mu\nu}$ is the metric, $R^{\mu\nu}$ is the Ricci tensor. In this paper we use the natural units $8\pi G = c = 1$.

Because of a cosmological application we assume that the universe has topology $R \times M^3$, where M^3 is homogeneous and isotropic space. Then the spacetime metric depends only on one function of the cosmic time t —the scale factor $a(t)$. Additionally for simplicity we also assume flatness ($k = 0$) of sections $t = \text{const}$. We decompose the energy-momentum tensor on two parts

$$T^{\mu\nu} = T_{de}^{\mu\nu} + T_m^{\mu\nu}. \quad (2)$$

We assume the conservation of the total energy momentum, which gives

$$-\nabla_\mu T_{de}^{\mu\nu} = \nabla_\mu T_m^{\mu\nu} \equiv 3\kappa^2 J^\nu, \quad (3)$$

where κ^2 is the diffusion constant and J^ν is the current, which represents a flow of stream of particles.

We also assume that energy density of the dark matter consisting of particles of mass m is transferred by a diffusion mechanism in an environment described by a perfect fluid. There is only unique diffusion which is relativistic invariant and preserves the particle-mass m [18]. The corresponding energy-momentum satisfies the conservation law (3).

The Friedmann equation in the Friedmann-Robertson-Walker (FRW) metric with baryonic matter, dark matter and dark energy reads

$$3H^2 = \rho_b + \rho_{dm} + \rho_{de}, \quad (4)$$

where ρ_m and ρ_{de} are determined by relations

$$\rho_m = \rho_{b,0}a^{-3} + \rho_{dm,0}a^{-3} + \gamma(t - t_0)a^{-3}, \quad (5)$$

$$\rho_{de} = \rho_{de}(0) - \gamma \int_0^t a^{-3} dt. \quad (6)$$

The current J^μ in Eq. (3) is conserved [19–21]

$$\nabla_\mu J^\mu = 0. \quad (7)$$

The above conservation condition for the FRW metric reduces to

$$J^0 = \gamma/3\kappa^2 a^{-3} \quad (8)$$

with a positive constant γ which can be computed from the phase space distribution $\Omega(p, x)$ of diffusing particles [9].

The condition (3) after calculation of divergence reduces to the continuity conditions for energy density of both matter and dark energy

$$\dot{\rho}_m = -3H\rho_m + \gamma a^{-3}, \quad (9)$$

$$\dot{\rho}_{de} = -\gamma a^{-3}, \quad (10)$$

where we assume the equation of state for dark energy as $p_{de} = -\rho_{de}$ and for matter as $p_m = 0$; a dot denotes differentiation with respect to the cosmological time t . Eq. (9) can be rewritten as

$$a^{-3} \frac{d}{dt}(\rho_m a^3) = \gamma a^{-3} \Leftrightarrow \frac{d}{dt}(E) = \gamma, \quad (11)$$

where E is the total energy of matter in the comoving volume $V \sim a^3$. From relation (11), we can obtain that $E = \gamma(t - t_0)$.

In our paper [9] we considered one unique model of an energy transfer from dark energy (DE) to dark matter (DM) with the diffusive interaction in the dark sector where DE and DM can be treated as ideal fluids. Particles are scattering in an environment of other particles. If we assume that the subsequent scattering events are independent, the particle motion is described by a Markov process. In order, the assumption that the energy of the particle remains finite leads to the conclusion that the Markov process must be a diffusion.

Therefore diffusion is in some sense unique because there is only one diffusion which is relativistic invariant and preserves the mass [18,22,23]. In consequence the interaction between DM and DE fluids is defined in a unique way.

III. DIFFUSION COSMOLOGY

In the investigation, the dark energy and dark matter interaction plays a role in the continuity equation. This equation is a special case of jungle cosmological models [17]. We assume that $\rho_m = \rho_b + \rho_{dm}$, where ρ_b is density of baryonic matter and ρ_{dm} is density of dark matter. The equation of state for dark energy is expressed by $p_{de} = -\rho_{de}$ in our model, where p_{de} is pressure of dark energy and the equation of state for matter is given by $p_m = 0$, where p_m is pressure of matter.

The Friedmann equation is expressed in the following form

$$3H^2 = \rho_{b,0}a^{-3} + \rho_{dm,0}a^{-3} + \gamma(t - t_0)a^{-3} + \rho_{de}(0) - \gamma \int^t a^{-3} dt, \quad (12)$$

where $\rho_{b,0}a^{-3} \equiv \rho_b$, $\rho_{dm,0}a^{-3} + \gamma(t - t_0)a^{-3} \equiv \rho_{dm}$, $\rho_{de}(0) - \gamma \int^t a^{-3} dt \equiv \rho_{de}$. From the Friedmann formula we get a condition

$$1 = \Omega_m + \Omega_{de}, \quad (13)$$

where $\Omega_m = \frac{\rho_m}{3H^2}$ and $\Omega_{de} = \frac{\rho_{de}}{3H^2}$ are dimensionless density parameters.

We can obtain Eqs. (9)–(12) in the form of the dynamical system $x' = f_x(x, y, \delta)$, $y' = f_y(x, y, \delta)$ and $\delta' = f_\delta(x, y, \delta)$, where $x = \Omega_m$, $y = \Omega_{de}$, $\delta = \frac{\gamma a^{-3}}{H\rho_m}$ and $' \equiv \frac{d}{d \ln a}$ is a differentiation with respect to the reparametrized time $\ln a(t)$. For these variables, the dynamical system is in the following form

$$x' = x(-3 + \delta + 3x), \quad (14)$$

$$y' = x(-\delta + 3y), \quad (15)$$

$$\delta' = \delta \left(-\delta + \frac{3}{2}x \right). \quad (16)$$

From Eq. (13), we have the following relation

$$x + y = 1. \quad (17)$$

Then dynamical system (14)–(16) is reduced to the two-dimension dynamical system.

For analysis of the critical points in the infinity, we use the Poincaré sphere. Let us introduce new variables in which one can study dynamical behavior at infinity

$$X = \frac{x}{\sqrt{x^2 + \delta^2}}, \quad \Delta = \frac{\delta}{\sqrt{x^2 + \delta^2}}. \quad (18)$$

For these variables we get the dynamical system

$$X' = X \left[-\Delta^2 \left(\frac{3}{2}X - \Delta \right) + (1 - X^2) \times (3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (19)$$

$$\Delta' = \Delta \left[(1 - \Delta^2) \left(\frac{3}{2}X - \Delta \right) - X^2(3X + \Delta - 3\sqrt{1 - X^2 - \Delta^2}) \right], \quad (20)$$

where $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{d \ln a}$. Critical points, for Eqs. (19)–(20), are presented in Table I. The phase portrait for the dynamical system (19)–(20) is presented in Fig. 1.

In the phase portrait there is an interesting class of trajectories labeled as “I”, starting from the critical point 6 and approaching the de Sitter state. Because the diffusion has the physical sense only for interval $t > t_0$ the corresponding cosmological solution should be cut off for any $t < t_0$ from the de-Sitter solution. Hence, we obtain that $a(t_0)$ is a positive number, i.e., a solution which represents the critical point (6) is nonsingular. All trajectories starting from the de Sitter state can be treated as a models of an extended idea of emergent cosmology [24] [25].

TABLE I. Critical points for dynamical system (19)–(20), their type and cosmological interpretation.

No.	Critical point	Type of critical point	Type of universe	Dominating part in the Friedmann equation	$H(t)$	$a(t)$
1	$X_0 = 0,$ $\Delta_0 = 0$	Saddle	de Sitter universe without diffusion effect	Cosmological constant	$H(t) = \sqrt{\frac{\Lambda_{\text{bare}}}{3}}$	$a(t) \propto e^{\sqrt{\frac{\Lambda_{\text{bare}}}{3}}t}$
2	$X_0 = \sqrt{2/11},$ $\Delta_0 = 3/\sqrt{22}$	Saddle	Scaling universe ($\rho_m \propto \rho_{de}$)	Matter and dark energy	$H(t) = (t - t_0)^{-1}$	$a(t) \propto (t - t_0)$
3	$X_0 = 1/\sqrt{2},$ $\Delta_0 = 0$	Unstable node	Einstein-de Sitter universe	Matter	$H(t) = \frac{2}{3}(t - t_0)^{-1}$	$a(t) \propto (t - t_0)^{2/3}$
4	$X_0 = 1,$ $\Delta_0 = 0$	Stable node	Static universe	Matter and running dark energy	$H(t) = 0$	$a(t) = \text{const}$
5	$X_0 = 4/5,$ $\Delta_0 = -3/5$	Saddle	Static universe	Matter and running dark energy	$H(t) = 0$	$a(t) = \text{const}$
6	$X_0 = 0,$ $\Delta_0 = 1$	Unstable node	de Sitter universe with diffusion effect	Running dark energy	$H(t) = \sqrt{\frac{\rho_{de}(0)}{3}}$	$a(t) \propto e^{\sqrt{\frac{\rho_{de}(0)}{3}}t}$
7	$X_0 = 0,$ $\Delta_0 = -1$	Stable node	de Sitter universe with diffusion effect	Running dark energy	$H(t) = -\sqrt{\frac{\Lambda_{\text{bare}}}{3}}$	$a(t) \propto e^{-\sqrt{\frac{\Lambda_{\text{bare}}}{3}}t}$

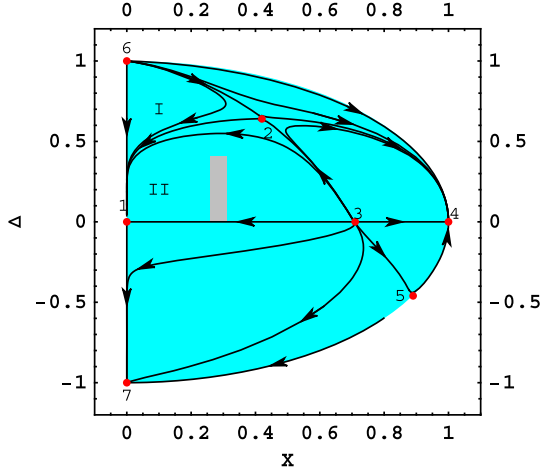


FIG. 1. Phase portrait of dynamical system $x' = x(\delta + 3x - 3)$, $\delta' = \delta(-\delta + \frac{3}{2}x)$, where $x = \Omega_m = \frac{\rho_m}{3H^2}$, $\delta = \frac{\gamma a^{-3}}{H\rho_m}$ and $l \equiv \frac{d}{d \ln a}$ on the Poincaré sphere coordinates are $X = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$, $\Delta = \frac{\delta}{\sqrt{x^2 + y^2 + z^2}}$. Critical point (1) represents the de Sitter universe—a global attractor for all physical trajectories. Critical point (2) represents the scaling universe. Critical point (3) represents the Einstein-de Sitter universe. Critical points (4) and (5) represent the static universe. Critical point (6) represents the de Sitter universe. The gray region represents the domain of the present value of X and Δ , which is distinguished by astronomical data. Let us note trajectories lie in the domain with $\Delta < 0$ represent the contracting model but there is no symmetry with respect to the Δ -axis. At critical point (6), energy density of baryonic matter is negligible as well as density of dark matter and only effects of the relativistic diffusion are important.

While in the standard emergent cosmology universe is starting from the static Einstein model, trajectories of type I are simple realization of extended idea of emergent Universe in which Universe is starting rather from the stationary state.

The results of our previous paper [9] show that the density parameter for total (dark and visible) x and dimensionless parameter δ are constrained to $x \in (0.2724, 0.3624)$, $\delta \in (0.0000, 0.0364)$ at the 95% confidence level. This domain is represented in the phase space by a shaded rectangle. Only these trajectories which intersect this domain are in good agreement with the observation at a 2σ confidence level. Therefore observation favored the cosmological models starting from the Einstein-de Sitter solution and going toward the de Sitter attractor (trajectory II on the phase portrait).

Note that on the phase portrait there are trajectories labeled as “I” starting from the de Sitter state and approaching the de Sitter state at late times. They are going toward a saddle point—representing a nonsingular solution. However all these trajectories do not intersect the rectangle and therefore they are not favored by observation.

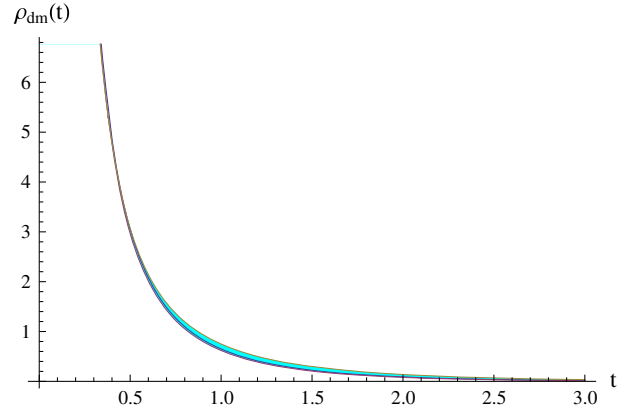


FIG. 2. The evolution of dark matter energy density for trajectories of type II (for the best fitted values of model parameter together with confidence level at 95%). Dark matter ρ_{dm} is expressed in $[100 \times \text{km}/(\text{s Mpc})]^2$. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

The saddle point in the phase space is representing the Milne universe (see Table I). Therefore, the interacting term is proportional to t^{-3} and consequently is of the form $\rho_{dm} = \Lambda_{\text{bare}} + \alpha^2 t^{-2}$. The cosmological model with such a parametrization of dark energy was studied by Szydlowski and Stachowski [26,27].

Figures 2 and 3 present the evolution of dark matter ρ_{dm} as a function of the cosmological time t for trajectories of type II. The evolution of the cosmological time for matter is determined by the following formula

$$\rho_m(t) = \rho_{m,0} a(t - t_0)^{-3} + \gamma(t - t_0) a(t - t_0)^{-3}. \quad (21)$$

The additive form of the scaling relation for dark matter (21) suggests that dark matter consists of two components: the first term scaling like $\rho_{m,0} a(t)^{-3}$ and

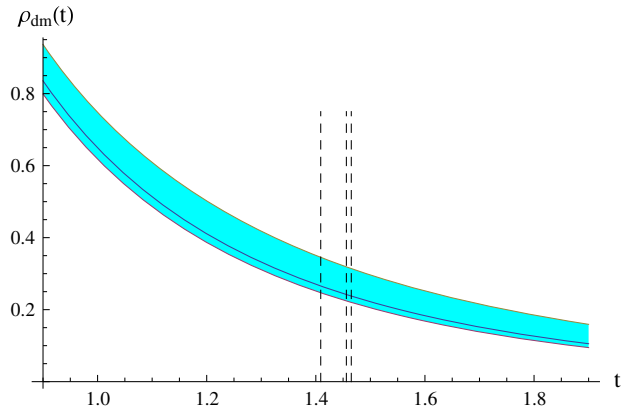


FIG. 3. The evolution of dark matter energy density for trajectories of type II (for the best fitted values of model parameter together with confidence level at 95% for the present epoch). Dark matter ρ_{dm} is expressed in $[100 \times \text{km}/(\text{s Mpc})]^2$. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t . The value of the age of the Universe for the best fit with errors are presented by the dashed lines.

the second term scaling like $\gamma ta(t)^{-3}$. The latter describes an amount of dark energy density which is transferred to dark energy sector by the diffusion process. In the unit of Ω_{total} the canonically scaling dark matter is 25.23% while transferred dark energy is about 1.35%. The amount of transferred dark energy today is of the order γT where T is the age of the Universe.

In consequence of Eq. (21), $\delta(t)$ can be rewritten as

$$\delta(t) = \frac{1}{H(\frac{\rho_{m,0}}{\gamma} + t - t_0)}. \quad (22)$$

Therefore at the present epoch we have

$$\delta(T) = \frac{1}{H_0(\frac{\rho_{m,0}}{\gamma} + T)}, \quad (23)$$

where $t_0 = 0$ and $t = T$ is the present age of the Universe.

Note that while at late time $\delta(t) = \sqrt{\frac{3}{\Lambda}} \frac{1}{t}$ for small time $\delta(t) = \frac{3\gamma}{2\rho_{m,0}}(t - t_0)$.

If we give $\gamma = 0$ in Eq. (21) then ρ_m is scaling in the canonical way. From Eq. (21) one can simply obtain that the density of dark matter is

$$\rho_{dm} = (\rho_{m,0} - \rho_{b,0})a(t - t_0)^{-3} + \gamma(t - t_0)a(t - t_0)^{-3}. \quad (24)$$

Note that the interval of the values of ρ_{dm} is $(0, +\infty)$ or $(0, \rho_{dm}^{\text{max}})$, which depends on a type of trajectory. The evolution of the scale factor $a(t)$ with respect to the cosmological time, for trajectories of type II, is demonstrated in Fig. 4. The function $\delta(t)$, for trajectories of type II, is presented in Fig. 5 and the consideration for the maximum of this function is in the form

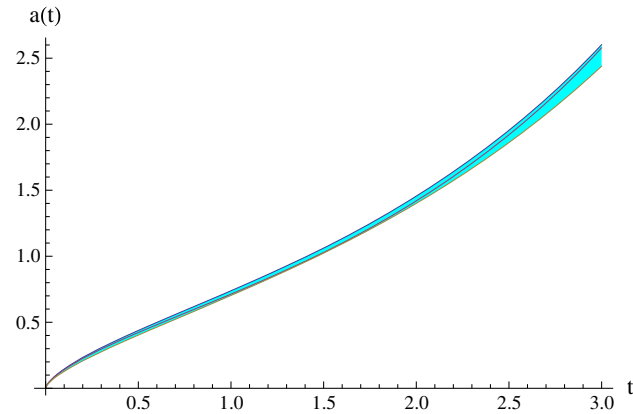


FIG. 4. Diagram of scale factor as a function of cosmological time t for trajectories of type II (for the best fitted values of model parameter together with confidence level at 95%). For the present epoch $T a(T) = 1$. A universe is starting from the initial singularity toward a de Sitter universe. This type of behavior is favored by the observational data. We choose (s Mpc)/(100 × km) as a unit of the cosmological time t .

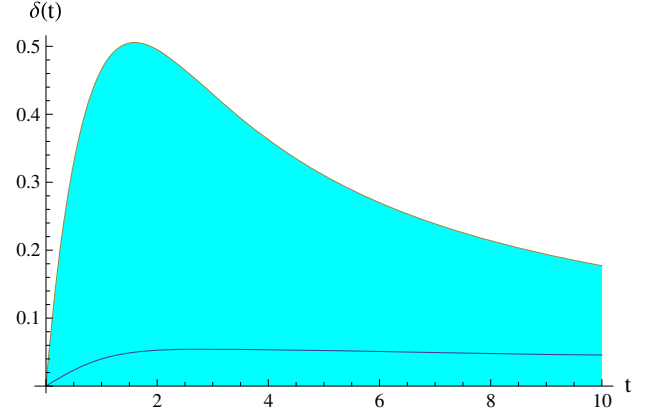


FIG. 5. Evolution of dimensionless parameter δ of cosmological time t for trajectories of type II (for the best fitted values of model parameter together with confidence level at 95%). Note that as trajectory in the phase space achieved the state of the pericentrum located in the saddle point, this state is corresponding on the diagram the maximum. Note that the existence of a maximum value of δ parameter $(\frac{\rho_{m,0}}{\gamma} + t_{\text{max}} - t_0)\rho_m(t_{\text{max}}) = 2H(t_{\text{max}})$. For the late time $\delta(t)$ function is decreasing function of t and $\rho(\infty) = 0$. We choose (s Mpc)/(100 × km) as a unit of the cosmological time t .

$$\left(\frac{\rho_{m,0}}{\gamma} + t_{\text{max}}\right)\rho_m(t_{\text{max}}) = 2H(t_{\text{max}}), \quad (25)$$

where t_{max} is corresponding to the value of the cosmological time at the maximum.

The Hubble function $H(t)$, for trajectories of type II, is presented in Fig. 6. Note that the Hubble function in the late time is constant. The evolution of ρ_{de} , for trajectories of type II, is shown in Fig. 7 and for late time ρ_{de} is going toward constant value. The evolution of Ω_m/Ω_{de} , for

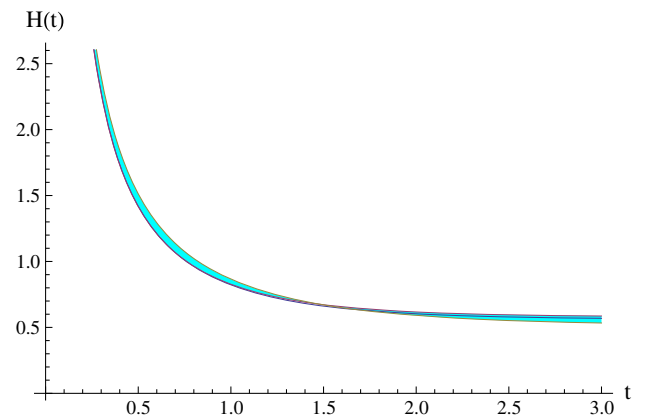


FIG. 6. Dependence of Hubble function of trajectories of type II (for the best fitted values of model parameter together with confidence level at 95%). For late times $H(t)$ goes to constant values (deS₊). The Hubble function $H(t)$ is expressed in [100 × km/(s Mpc)]. We choose (s Mpc)/(100 × km) as a unit of the cosmological time t .

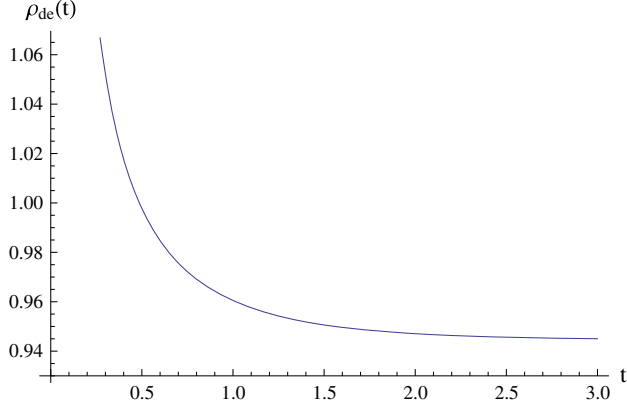


FIG. 7. The evolution of dark energy density for the best fitted values of model parameter for trajectories of type II. Dark energy ρ_{de} is expressed in $[100 \times \text{km}/(\text{s Mpc})]^2$. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

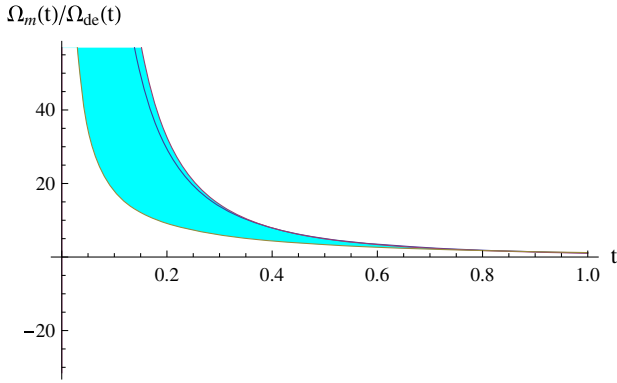


FIG. 8. Diagram of relation Ω_m/Ω_{de} for trajectories of type II (for the best fitted values of model parameter together with confidence level at 95%). We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

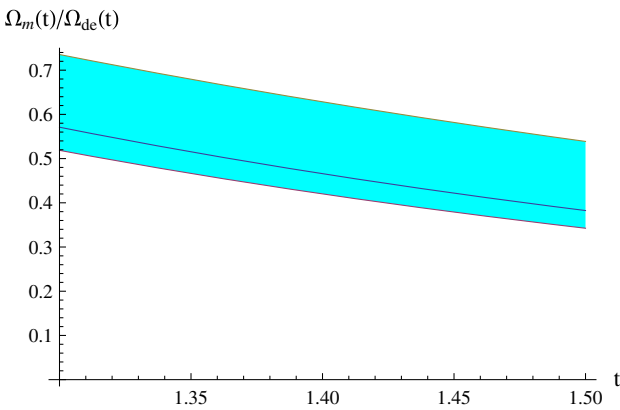


FIG. 9. Diagram of relation Ω_m/Ω_{de} for trajectories of type II for the present epoch. Note that at the present epoch $\rho_{m,0} \propto \rho_{de,0}$ (therefore coincidence problem is solved). We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t . The value of the age of the Universe for the best fit with errors are presented by the dashed lines.

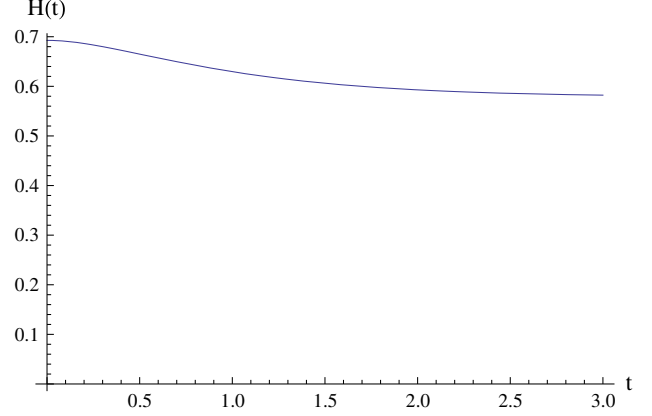


FIG. 10. The relation of $H(t)$ for typical trajectory of type I. The $H(t)$ function is expressed in $[100 \times \text{km}/(\text{s Mpc})]$. Note that $H(0)$ is finite therefore it is not a singularity. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

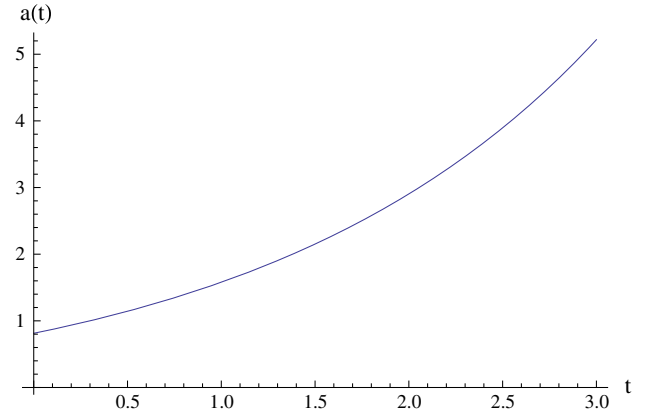


FIG. 11. Diagram of $a(t)$ for typical trajectory of type I. Note that $a(0)$ is finite therefore it is not a singularity. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

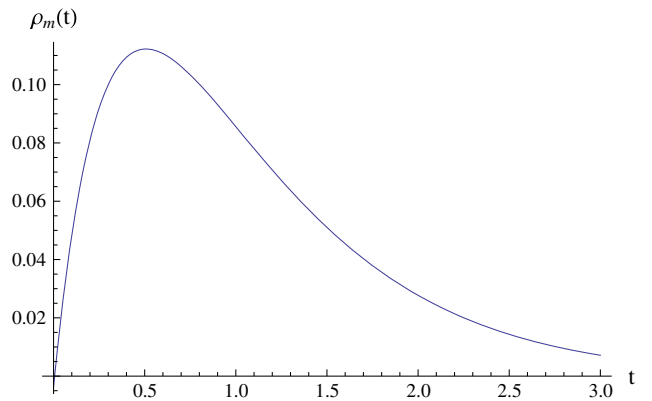


FIG. 12. Diagram of $\rho_m(t)$ for typical trajectory of type I. Note that $\rho_m(0)$ is equal zero therefore it is not a singularity. Matter ρ_m is expressed in $[100 \times \text{km}/(\text{s Mpc})]^2$. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

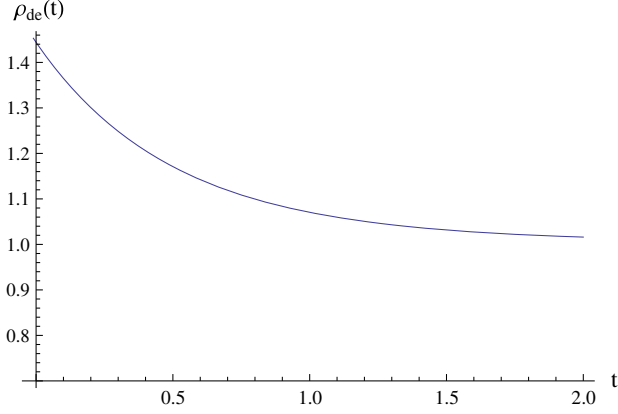


FIG. 13. Diagram of $\rho_{de}(t)$ for typical trajectory of type I. Note that $\rho_{de}(0)$ is finite therefore it is not a singularity. Dark energy ρ_{de} is expressed in $[100 \times \text{km}/(\text{s Mpc})]^2$. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

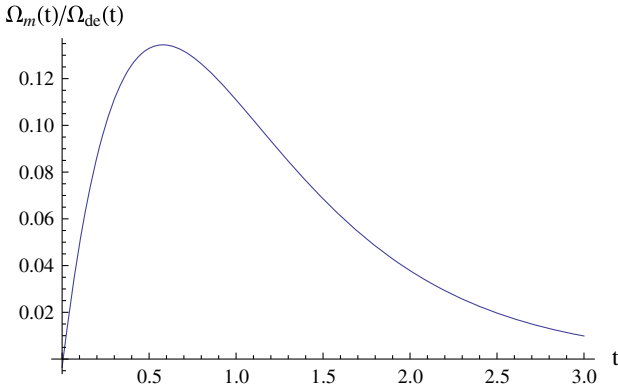


FIG. 14. Diagram of $\Omega_m(t)/\Omega_{de}(t)$ for typical trajectory of type I. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

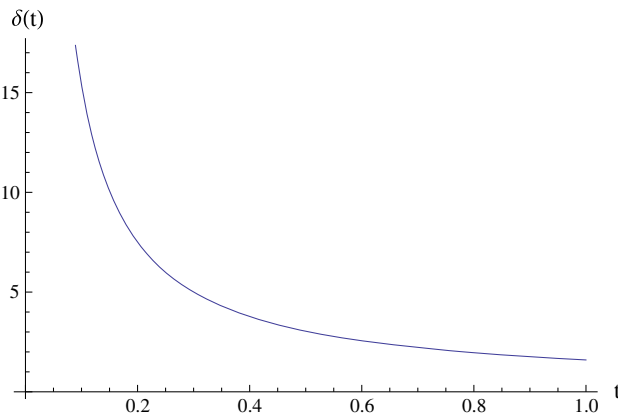


FIG. 15. Diagram of $\delta(t)$ for typical trajectory of type I. We choose $(\text{s Mpc})/(100 \times \text{km})$ as a unit of the cosmological time t .

trajectories of type II, is demonstrated in Figs. 8 and 9. For the trajectories of type I, the functions $H(t)$, $a(t)$, ρ_{dm} , ρ_{de} , Ω_m/Ω_{de} and $\delta(t)$ are presented in Figs. 10, 11, 12, 13, 14, 15. These figures show that there are two distinct behaviors of trajectories of type I and II. While the trajectory of type II represents a matter dominating model with a singularity the trajectories of type I represents the model without an initial singularity.

IV. GENERALIZED DIFFUSION COSMOLOGY

Dynamical system methods are especially suitable in investigation of dynamics of both fluids, dark energy and dark matter. Presented here the dynamical system approach to the study of DM-DE interaction in diffusion cosmology can be simply generalized to the case when both dark energy and dark matter satisfy a general form of the equation of state

$$p_{de} = w\rho_{de}, \quad (26)$$

$$p_{dm} = \tilde{w}\rho_{dm}, \quad (27)$$

$$p_b = 0, \quad (28)$$

where w and $\tilde{w}$ are constant coefficients equation of state for dark energy and matter respectively. Then the continuity equations for baryonic and dark matter and dark energy are presented by

$$\dot{\rho}_{dm} = -3(1 + \tilde{w})H\rho_{dm} + \gamma a^{-3}, \quad (29)$$

$$\dot{\rho}_{de} = -3(1 + w)H\rho_{de} - \gamma a^{-3}, \quad (30)$$

$$\dot{\rho}_b = -3H\rho_b. \quad (31)$$

The corresponding dynamical system assumes the form of a 3-dimensional autonomous dynamical system

$$\frac{dx}{d \ln a} = 3x \left[(1 + \tilde{w})(x - 1) + (1 + w)y + \frac{z}{3} \right], \quad (33)$$

$$\frac{dy}{d \ln a} = 3y \left[(1 + w)(y - 1) + (1 + \tilde{w})x \right] - xz, \quad (34)$$

$$\frac{dz}{d \ln a} = z \left[3\tilde{w} - z + \frac{3}{2}[(1 + \tilde{w})x + (1 + w)y] \right], \quad (35)$$

where we choose state variables $x = \Omega_m$, and $y = \Omega_{de}$ and $z = \delta$ like in a previously considered case.

Because $1 = x + y$ the above dynamical system reduces to

$$\frac{dx}{d \ln a} = 3x \left[(\tilde{w} - w)(x - 1) + \frac{z}{3} \right], \quad (36)$$

TABLE II. Critical points for dynamical system (36)–(37), their positions, types and cosmological interpretation.

No.	Critical point	Type of the universe
1	$x_0 = 0, z_0 = 0$	de Sitter universe without diffusion
2	$x_0 = 1, z_0 = 0$	Einstein-de Sitter
3	$x_0 = -\frac{1+3w}{3(\tilde{w}-w)}, z_0 = 1 + 3\tilde{w}$	Scaling universe ($\rho_m \propto \rho_{de}$)
4	$x_0 = 0, z_0 = 3/2(1 + 2\tilde{w} + w)$	de Sitter universe with diffusion

$$\frac{dz}{d \ln a} = z \left[3\tilde{w} - z + \frac{3}{2}[(1 + \tilde{w})x + (1 + w)(1 - x)] \right]. \quad (37)$$

Critical points of the dynamical system (36)–(37) are completed in Table II. Especially there is an interesting critical point inside the admissible region $D = \{(x, z) : x \geq 0, z \geq 0\}$ representing scaling solution: $\rho_{dm} \propto \rho_{de}$. It is a saddle fixed point in the phase space D . This critical point is important in the context of the solution of the cosmic coincidence problem as well as the scaling solution in the context of the quintessence idea.

The above system possesses critical points on the planes of the coordinate system or inside the phase space $D = \{(x, z) : x, z \geq 0\}$. Of course the system under consideration is restricted to the submanifold $x + y = 1$, because the constraint condition $\Omega_m + \Omega_{de} = 1$.

The behavior of trajectories of the dynamical system (36)–(37) depends on the values of parameters $w, \tilde{w}$. By choosing different values of these parameters one can study how phase space structure changes under change of values of parameters. The equivalence of the phase portraits is established following homeomorphism preserving direction of time along the trajectories. If there exists a value of parameter for which phase is not topologically equivalent, then such value is the bifurcation value.

The stability of critical points depends on the linearization matrix. At the critical point (3), the linearization matrix has the following form

$$A = \begin{pmatrix} \left. \frac{\partial f_x(x, z)}{\partial x} \right|_{x_0, z_0} & \left. \frac{\partial f_x(x, z)}{\partial z} \right|_{x_0, z_0} \\ \left. \frac{\partial f_z(x, z)}{\partial x} \right|_{x_0, z_0} & \left. \frac{\partial f_z(x, z)}{\partial z} \right|_{x_0, z_0} \end{pmatrix} = \begin{pmatrix} -1 - 3w & \frac{1+3w}{-3\tilde{w}+3w} \\ \frac{3}{2}(1+3\tilde{w})(\tilde{w}-w) & -1 - 3\tilde{w} \end{pmatrix}, \quad (38)$$

where $f_x(x, z)$ and $f_z(x, z)$ are the right sides of Eqs. (36) and (37) and x_0 and z_0 are coordinates of critical point (3) (see Table II).

The determinant of matrix (38) can be expressed by the formula

$$\det A = \frac{3}{2}(1 + 3\tilde{w})(1 + 3w) \quad (39)$$

and the trace of matrix (38) is described by

$$\text{tr} A = -2 - 3(\tilde{w} + w). \quad (40)$$

Therefore the critical point (3) is stable when $w + \tilde{w} > -2/3$.

The characteristic equation for matrix A at critical point (3) is in the following form

$$\begin{aligned} \lambda^2 - \text{tr} A \lambda + \det A &= \lambda^2 + (2 + 3\tilde{w} + 3w)\lambda \\ &\quad + \frac{3}{2}(1 + 3\tilde{w})(1 + 3w) \\ &= 0. \end{aligned} \quad (41)$$

From the characteristic Eq. (41), we can obtain the eigenvalues for critical point (3) (see Table II). In Fig. 16 we demonstrate the stability of critical point (3), depending on w and $\tilde{w}$.

The linearized Eqs. (36)–(37) at critical point (3) is given by the following formulas

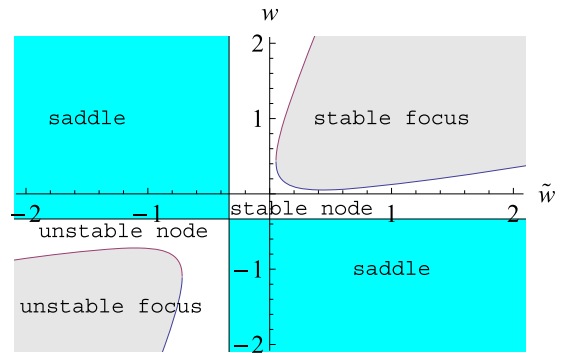


FIG. 16. Diagram of stability of critical point (3), depending on w and $\tilde{w}$. In the gray domains there is the focus type of critical point and the boundaries of this domains is given by the lines $w = \frac{1}{3}(1 + 6\tilde{w} + \sqrt{-1 + 9\tilde{w}(2 + 3\tilde{w})})$ and $w = \frac{1}{3}(1 + 6\tilde{w} - \sqrt{-1 + 9\tilde{w}(2 + 3\tilde{w})})$. In the blue regions there are the saddle type of critical point and is limited by lines $w = -1/3$ and $\tilde{w} = -1/3$. In the white top and bottom regions there are the stable and unstable nodes, respectively.

$$\begin{aligned}
(x - x_0)' &= A_{11}(x - x_0) + A_{12}(z - z_0) \\
&= (-1 - 3w) \left(x + \frac{1 + 3w}{3(\tilde{w} - w)} \right) \\
&\quad + \left(\frac{1 + 3w}{-3\tilde{w} + 3w} \right) (z - 1 - 3\tilde{w}), \quad (42)
\end{aligned}$$

$$\begin{aligned}
(z - z_0)' &= A_{21}(x - x_0) + A_{22}(z - z_0) \\
&= \frac{3}{2}(1 + 3\tilde{w})(\tilde{w} - w) \left(x + \frac{1 + 3w}{3(\tilde{w} - w)} \right) \\
&\quad + (-1 - 3\tilde{w})(z - 1 - 3\tilde{w}), \quad (43)
\end{aligned}$$

where $x_0 = -\frac{1+3w}{3(\tilde{w}-w)}$ and $z_0 = 1 + 3\tilde{w}$. The solutions of the above equations are presented by formulas

$$x = C_1 a^{(-2-3\tilde{w}-3w-\alpha)/2} (a^\alpha + C_2) - \frac{1 + 3w}{3(\tilde{w} - w)}, \quad (44)$$

$$\begin{aligned}
z &= C_1 \frac{3(\tilde{w} - w)}{2 + 6w} a^{(-2-3\tilde{w}-3w-\alpha)/2} ((3\tilde{w} - 3w - \alpha)a^\alpha \\
&\quad + C_2(3\tilde{w} - 3w + \alpha)) + 1 + 3\tilde{w}, \quad (45)
\end{aligned}$$

where $\alpha = \sqrt{-1 + 9\tilde{w}^2 + 9w^2 - (1 + 6\tilde{w})(1 + 6w)}$.

It is interesting to check how the structure of phase space changes under changing coefficient equation of state for dark matter from 0 (cold dark matter) to $\tilde{w} = 1/3$ (hot dark matter).

Results of dynamical investigation show that structure of the phase space is preserved under changes of the model parameter. Let us considered some details.

V. DIFFUSION COSMOLOGY WITH THE HOT RELATIVISTIC DARK MATTER

In this section we consider the case with relativistic dark matter ($\tilde{w} = 1/3$) and $w = -1$. Then the equation of state for dark matter is in the form $p_{dm} = \frac{1}{3}\rho_{dm}$, where p_{dm} is the pressure of dark matter. We get the following equations

$$x' = x(-4 + z + 4x), \quad (46)$$

$$z' = z(1 - z + 2x). \quad (47)$$

We can analyze the critical points in the infinity. In this case we use the Poincaré sphere. Let $X = \frac{x}{\sqrt{x^2 + \delta^2}}$, $\Delta = \frac{\delta}{\sqrt{x^2 + \delta^2}}$. For variables X and Δ , we get the dynamical system

$$\begin{aligned}
X' &= X \left[-\Delta^2 \left(\sqrt{1 - X^2 - \Delta^2} + \frac{3}{2}X - \Delta \right) \right. \\
&\quad \left. + (1 - X^2)(3X + \Delta - 4\sqrt{1 - X^2 - \Delta^2}) \right], \quad (48)
\end{aligned}$$

$$\begin{aligned}
\Delta' &= \Delta \left[(1 - \Delta^2) \left(\sqrt{1 - X^2 - \Delta^2} + \frac{3}{2}X - \Delta \right) \right. \\
&\quad \left. - X^2(3X + \Delta - 4\sqrt{1 - X^2 - \Delta^2}) \right], \quad (49)
\end{aligned}$$

where $' \equiv \sqrt{1 - X^2 - \Delta^2} \frac{d}{dt}$. Critical points, for the above equation, are presented in Table III. The phase portrait for the dynamical system (48)–(49) is demonstrated in Fig. 17.

It is interesting to see how different solutions with and without an initial singularity are distributed in the phase space. To answer this question it would be useful to consider a phase space structure of the models under consideration. For this aim we reduce the dynamics to the form of an autonomous 2D dynamical system. In such a system the state variables are dimensionless parameters: the density parameter for radiation dark matter and the parameter δ characterizing the rate of energy transfer to the dark matter sector.

The main advantage of visualization global dynamics on the phase portrait is the possibility to see all solution of the system admitted for all initial conditions. On the phase portrait there is the geometric representation of evolutionary paths of both solution types. Critical points are representing asymptotic states of the system, i.e. stationary states. In order, trajectories joining different critical points are representing the evolution of the system.

Similarly to dynamical investigations presented in our previous paper [9] we added to the plane circle at infinity

TABLE III. Critical points for dynamical system (48)–(49), their type and cosmological interpretation.

No.	Critical point	Type of critical point	Type of universe
1	$X_0 = 0, \Delta_0 = 0$	Saddle	de Sitter universe without diffusion effect
2	$X_0 = 2/7, \Delta_0 = 6/7$	Saddle	Scaling universe
3	$X_0 = 4/5, \Delta_0 = 0$	Unstable node	Einstein-de Sitter universe
4	$X_0 = 1, \Delta_0 = 0$	Stable node	Static universe
5	$X_0 = 4/5, \Delta_0 = -3/5$	Saddle	Static universe
6	$X_0 = 0, \Delta_0 = 1$	Unstable node	de Sitter universe with diffusion effect
7	$X_0 = 0, \Delta_0 = -1$	Stable node	de Sitter universe with diffusion effect
8	$X_0 = 0, \Delta_0 = 1/\sqrt{2}$	Stable node	de Sitter universe with diffusion effect

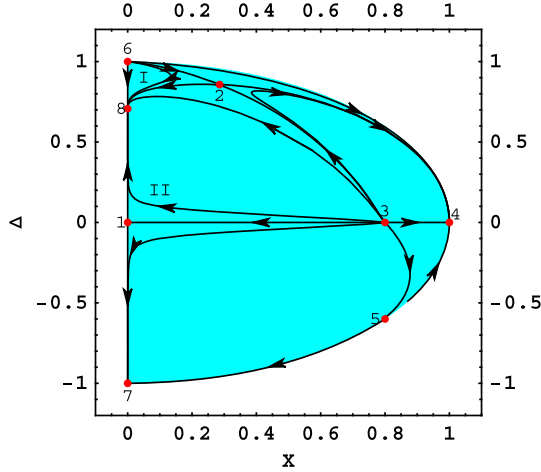


FIG. 17. Phase portrait of the dynamical system (36)–(37). Note that trajectories for the $\Delta < 0$ represent solutions with the negative value of H . From the cosmological point of view trajectories representing expanding models with $\Delta > 0$ are physical. Critical point (1) represents the de Sitter universe without diffusion effect. Critical point (2) represents the scaling universe. Critical point (3) represents the Einstein-de Sitter universe. Critical points (4) and (5) represent the static universe. Critical point (6) and (8) represent the de Sitter universe with diffusion effect.

via the construction of the Poincaré sphere. Hence we obtain a compact phase space and consequently a global phase portrait. In Fig. 17 we have identified linear solutions without the initial singularity as the representing by saddle critical point (2). In the phase portrait there are critical points at a finite domain as well as located on the boundary at infinity.

Note that the phase portrait has no symmetry with respect to the x -axis. Critical point (6) is representing an expanding stationary de Sitter type solution determined by diffusion effects. We denote as typical trajectories starting from this critical point and going toward the de Sitter empty universe. These trajectories we called trajectories of type I. In the phase portrait there are also present trajectories of type II. These trajectories are starting from the Einstein-de Sitter universe with the initial singularity and coming toward the de Sitter universe labeled as critical point (8).

Looking at the phase portrait one can observe that only critical points of type unstable and stable node (global attractors and global repellers) and saddle appear in the phase space. Therefore the model obtained is structurally stable, i.e. any small change of its r.h.s does not disturb the global phase portrait. Physically this means that corresponding model is realistic. Mathematically this fact has a nice interpretation in the context of the Peixoto

theorem [16] that they are generic in this sense because they form open and dense subsets in the space of dynamical systems on the plane.

VI. CONCLUSION

The standard cosmological model (Λ CDM model) is widely accepted but it has still some problems, namely the cosmological constant problem and the coincidence problem. In the standard cosmological model (Λ CDM model) it is assumed that all fluids are noninteracting. In this paper we construct a cosmological model in which it is assumed that the process of interaction between sectors of dark matter and dark energy is continuous. Relativistic diffusion describes the transfer of energy to the sector of dark matter. This effect is described by the running cosmological constant and the modification of the standard scaling law of the dark matter density to the form $\rho_{dm,0}a(t)^{-3} + \gamma ta(t)^{-3}$. The dynamics of this model is studied for possible explanations of cosmological puzzles: the cosmological constant problem and the coincidence problem.

In the context of the coincidence problem, our model can explain the present ratio of ρ_m to ρ_{de} , which is equal $0.4576^{+0.1109}_{-0.0831}$ at a 2σ confidence level. In our model, the canonical scaling law of dark matter ($\rho_{dm,0}a^{-3}(t)$) is modified by an additive $\epsilon(t) = \gamma ta^{-3}(t)$ to the form $\rho_{dm} = \rho_{dm,0}a^{-3}(t) + \epsilon(t)$.

The analysis of the time dependence of density of dark energy and dark matter, we conclude that the value of effective energy of vacuum runs from an infinite value to a constant value, and the delta amendment to the scaling law goes from zero to zero and being different from zero in a long intermediate period. This characteristic type of behavior is controlled by the diffusion effect.

The paper presents a detailed study of the behavior of a state of the system represented by the state variables (x, δ) . In this context, it was natural to consider the diffusion mechanism which controls the change of the ratio of both energy densities and the very dynamics of this process remains in analogy to the description of population changes of competing species [17]. A crucial role plays the saddle critical point $Ha = \text{const}$, which is a scaling type of the solution ($\rho_m \propto \rho_{de}$). The position of this point cannot be disturbed by a small perturbation (structurally stable point as well as whole system).

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Dynamical system approach to running Λ cosmological models

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Abstract We study the dynamics of cosmological models with a time dependent cosmological term. We consider five classes of models; two with the non-covariant parametrization of the cosmological term Λ : $\Lambda(H)$ CDM cosmologies, $\Lambda(a)$ CDM cosmologies, and three with the covariant parametrization of Λ : $\Lambda(R)$ CDM cosmologies, where $R(t)$ is the Ricci scalar, $\Lambda(\phi)$ -cosmologies with diffusion, $\Lambda(X)$ -cosmologies, where $X = \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$ is a kinetic part of the density of the scalar field. We also consider the case of an emergent $\Lambda(a)$ relation obtained from the behaviour of trajectories in a neighbourhood of an invariant submanifold. In the study of the dynamics we used dynamical system methods for investigating how an evolutionary scenario can depend on the choice of special initial conditions. We show that the methods of dynamical systems allow one to investigate all admissible solutions of a running Λ cosmology for all initial conditions. We interpret Alcaniz and Lima's approach as a scaling cosmology. We formulate the idea of an emergent cosmological term derived directly from an approximation of the exact dynamics. We show that some non-covariant parametrization of the cosmological term like $\Lambda(a)$, $\Lambda(H)$ gives rise to the non-physical behaviour of trajectories in the phase space. This behaviour disappears if the term $\Lambda(a)$ is emergent from the covariant parametrization.

1 Introduction

Our understanding of the properties of the current universe is based on the assumption that gravitational interactions, which are extrapolated at the cosmological scales, are described successfully by the Einstein general relativity theory with the cosmological term Λ . If we assume that the geometry of the universe is described by the Robertson–Walker metric, i.e., the universe is spatially homogeneous and

isotropic, then we obtain the model of the current universe in the form of standard cosmological model (the Λ CDM model). From the methodological point of view this model plays the role of an effective theory which describes well the current universe in the present accelerating epoch.

If we compare the Λ CDM model with the observational data, then we find that more than 70% of the energy budget is in the form of dark energy and well modelled in terms of an effective parameter of the cosmological constant term.

If we assume that the SCM (standard cosmological model) is an effective field theory which is valid up to a certain cutoff of mass M , and if we extrapolate of the SCM up to the Planck scale then we should have $\Lambda \sim 1$. On the other hand from the observations we find that both density parameters $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$ and $\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}$ are order one, which implies $\Lambda \propto H_0^2 \sim 10^{-120}$. We assume the natural units $G = c = \hbar = 1$ here.

In consequence we obtain the huge discrepancy between the expected and observed values of the term Λ . It is just what is called the cosmological constant problem requiring the explanation why the cosmological constant assumes such a small value today.

In this context an idea of a running cosmological constant term appears. It was developed in a series of papers by Shapiro et al. [1–4]. Shapiro and Solà [5] showed neither there is the rigorous proof indicating that the cosmological constant is running, nor there are strong arguments for a non-running one. Therefore one can study different theoretical possibilities of the running Λ term given in a phenomenological form and investigate cosmological implications of such an assumption. Such models are a simple generalization of the standard cosmological model in which the term Λ is constant.

The corresponding form of the $\Lambda(t)$ dependence can be motivated by quantum field theory [5–7] or by some theoretical motivations [8,9]. Padmanabhan [10] and Vishwakarma [11] also suggested that $\Lambda \propto H^2$ from the dimensional considerations.

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The relation $\Lambda(t)$ is not given directly but through a function which describes the evolution of the universe. One can consider two classes of models with the non-covariant parametrizations of the Λ term:

- the cosmological models in which dependence on time is hidden and $\Lambda(t) = \Lambda(H(t))$ or $\Lambda(t) = \Lambda(a(t))$ depends on the time through the Hubble parameter $H(t)$ or scale factor $a(t)$,

and three classes of models with covariant parametrizations of the Λ term:

- the Ricci scalar of the dark energy model, i.e., $\Lambda = \Lambda(R)$,
- the parametrization of the Λ term through the scalar field $\phi(t)$ with a self-interacting potential $V(\phi)$,
- as the special case of the previous one, the Λ term can be parametrized by a kinetic part of the energy density of the scalar field $X = \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$.

Note that some parametrizations of the Λ term can also arise from another theory beyond general relativity. For example Shapiro and Solà [5] suggested that a solution, which is derived from the form of $\rho_\Lambda(H) = \rho_\Lambda^0 + \alpha(H^2 - H_0^2) + \mathcal{O}(H^4)$, can be a solution of the fundamental general relativity equations.

Another problem, which is related to the standard cosmological problem, is the problem of coincidence [12]. From the cosmological data such as measurements of distant SNIa, CMB, BAO and other astronomical observations, we find that we live in the very special age of the universe when $\rho_\Lambda \sim \rho_{\text{dm}} \sim \rho_{\text{b}}$. The appearance of any epoch with this coincidence is puzzling and we should explain why we live in such a special epoch.

The appearance of scaling solutions in the phase space suggests that in this model the problem of cosmic coincidence can be solved because during the whole evolution $\rho_\Lambda \sim \rho_{\text{m}}$.

The motivation for studying cosmology with the decaying vacuum comes from the solution of the cosmological constant problem as well as the cosmic coincidence problem – the main problems which standard cosmological model struggles. In this context, different propositions of parametrization of the Λ term are postulated. As mentioned above both the covariant contributions to the general relativity action and others violate this covariance. We study cosmological implications of such choices. And the methods of dynamical systems will be used to help us to understand better the dynamical aspects of this problem.

We are looking for such parametrizations of the Λ term for which in the phase space the de Sitter stationary state is a global attractor and a generic class of initial conditions gives rise in this attractor. It is a consequence of the fact that we are

going toward a solution of the standard cosmological model without an idea of the fine tuning.

The main aim of this paper is to study dynamics of the cosmological models with the running cosmological term and dust matter. We apply dynamical systems methods to investigate theoretically possible dynamics of these models. The main advantage of these methods is the possibility of studying all solutions (cosmological evolutionary scenarios) for admissible initial conditions. The phase space is a geometrization of the dynamics whose structure informs us how generic are solutions with desired properties. In this approach we are looking for attractor solutions in the phase space representing generic solutions for the problem which gives such a parametrization of $\Lambda(t)$ which explain how the value of cosmological term achieves a small value for the current universe. We search for such an evolutionary scenario for which the Λ_{bare} is an attractor in the phase space.

The dynamics of both the above mentioned subclasses of the $\Lambda(t)$ CDM cosmologies is investigated by dynamical system methods. Bonanno and Carloni have recently used these methods to study the qualitative behaviour of FRW cosmologies with time-dependent vacuum energy on cosmological scales [6]. Of course, the methods of dynamical systems are not a way to solve problems of the cosmological constant. It is only a useful tool for the visualization of the dynamics in a geometrical way which can help us to better understand the term Λ during the cosmic evolution.

We also develop the idea of an emergent relation $\Lambda(a)$ obtained from the behaviour of the trajectories of the dynamical system near the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. By the emerging of a running parametrization $\Lambda(a)$ we understand its derivation directly from the true dynamics. Therefore, the corresponding parametrization is obtained from the entry of trajectories in a de Sitter state.

Measurements of the cosmic microwave background anisotropy are considered in the background of the Λ CDM model and indicates that the cosmological spatial hypersurface of the FRW geometry is very close to flat [13, 14]. On the other hand, under of the assumption of flatness, the data favour rather the time-independent dark energy [15].

It is well known that if a spatially curved time variable dark energy model is used to analyse the CMB anisotropy measurements then there is a degeneracy between the spatial curvature and the parameters which govern the dark energy time variability. For this reason it seems that an in-depth analysis should be performed of the influence of curvature effects on the dynamical scenarios of different cosmological models.

For this aim we consider the following issues.

- We explore idea of the reducing dynamics to the form of the 2D dynamical system of the Newtonian type as soon as possible. In this system, the energy integral is related

with the curvature index (or density parameter for the curvature fluid) and therefore energy levels will determine evolutionary paths in the configuration space. All information, which is concerning these types of evolution, can be directly taken from the geometry of potential function $V(a)$ because the curvature effect of new types of evolution emerges. For example we can obtain oscillating models, models with bounce, oscillating models without the initial and final singularity etc.

- From the cosmological point of view, it is interesting to find in the phase space attractors, which position is caused by curvature effects. In the generic case these attractors lie on the invariant submanifold, which represents the surface of the flat model. However, our dynamical analysis gives us an opportunity to detect curvature attractors beyond the invariant submanifold, which represents the evolution of the flat models, which are studied in detail by the phase portraits of the lower dimension.

2 $\Lambda(H)$ CDM cosmologies as dynamical systems

From the theoretical point of view if we do not know the exact form of the $\Lambda(t)$ relation we study the dynamical properties of cosmological models in which the Λ -dependence on the cosmological time t is through the Hubble parameter or scale factor, i.e. $\Lambda(t) = \Lambda(H(t))$ or $\Lambda(t) = \Lambda(a(t))$. The connection of such models with the mentioned ones in the previous section will be demonstrated, in which the choice of a $\Lambda(t)$ form was motivated by physics. Cosmological models with a quadratic Λ -dependence on the cosmological time are revealed as a special solution in the phase space.

In the investigation of the dynamics of $\Lambda(H)$ cosmologies we apply the dynamical system methods [16]. We investigate all solutions which are admissible for all physically admitted initial conditions. The global characteristics of the dynamics are given in the form of phase portraits, which reflect the phase space structure of all solutions of the problem. The phase space structure contains all information as regards dependence of solutions on initial conditions, its stability, genericity, etc. Then we can distinguish some generic (typical) cases as well as non-generic (fine-tuned) ones, which physical realizations require a tuning of the initial conditions. The methods of dynamical systems allow us to study the stability of the solutions in a simple way by investigation of the linearization of the system around the non-degenerate critical points of the system.

If the dynamical system is in the form $\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and f is of class C^∞ , then the solution of this system is a vector field $\mathbf{x}(t; \mathbf{x}_0)$ where $\mathbf{x}(t_0)$ is a vector of initial conditions. Beyond this regular solution there are singular ones. They are special and obtained from the condition of vanishing of its right-hand sides.

The $\Lambda(H)$ CDM cosmological models have recently been investigated intensively in the contemporary cosmology [8, 17–19]. Among these class of models there is one with a particular form of $\Lambda(t) = \Lambda + \alpha H^2$. It was studied in detail in [17]. Its generalization to the relation of $\Lambda(H)$ given in the form of a Taylor series of the Hubble parameter can be found in [20].

It is also interesting that motivations for studying such a class of models can be taken from Urbanowski's expansion formula for decaying false vacuum energy, which can be identified with the cosmological constant term [7]. It is sufficient to interpret the time t in terms of the Hubble time scale $t = t_H \equiv \frac{1}{H}$. Therefore, $\Lambda(H)$ CDM cosmologies can be understood as some kind of effective theories of the influence of vacuum decay in the universe [21]. This approach is interesting especially in the context of both the dark energy and the dark matter problem because the problem of cosmological constant cannot be investigated in isolation from the problem of dark matter.

In $\Lambda(H)$ cosmologies, in general, a scaling relation on matter is modified and differs from the canonical relation $\rho_m = \rho_{m,0} a^{-3}$ in the Λ CDM model. The deviation from the canonical relation here is characterized by a positive constant ϵ such that $\rho_m = \rho_{m,0} a^{-3+\epsilon}$ [22].

FRW cosmologies with a running cosmological term $\Lambda(t)$ such that $\rho_{\text{vac}} = \Lambda(t)$ and $p_{\text{vac}} = -\Lambda(t)$ can be formulated in the form of a non-autonomous dynamical system,

$$\frac{dH}{dt} \equiv \dot{H} = -H^2 - \frac{1}{6}(\rho_m + 3p_m) + \frac{1}{3}\Lambda(t) \quad (1)$$

$$\frac{d\rho_m}{dt} \equiv \dot{\rho}_m = -3H(\rho_m + p_m) - \dot{\Lambda}, \quad (2)$$

where ρ_m and p_m are the energy density and the pressure of matter, respectively, and a dot denotes differentiation with respect to the cosmological time t . In this paper, we assume that $8\pi G = c = 1$. In this model the energy-momentum tensor is not conserved because of the presence of an interaction in both matter and dark energy sector. System (1)–(2) has a first integral called the conservation condition in the form

$$\rho_m - 3H^2 = -\Lambda(t). \quad (3)$$

Note that the solution $\rho_m = 0$ is a solution of (2) only if $\Lambda = \text{const}$. Of course system (1)–(2) does not form a closed dynamical system, while a concrete form of the $\Lambda(t)$ relation is not postulated. Therefore, this cosmology belongs to a more general class of models in which the energy-momentum tensor of matter is not conserved.

Let us consider that both visible matter and dark matter are given in the form of dust, i.e. $p_m = 0$ and

$$\Lambda(t) = \Lambda(H(t)). \quad (4)$$

Due to the above simplifying assumption (4), system (1)–(2) with the first integral in the form (3) assumes the form of a two-dimensional closed dynamical system,

$$\dot{H} = -H^2 - \frac{1}{6}\rho_m + \frac{1}{3}\Lambda(t), \quad (5)$$

$$\dot{\rho}_m = -3H\rho_m - \Lambda'(H) \left(-H^2 - \frac{1}{6}\rho_m + \frac{\Lambda(H)}{3} \right), \quad (6)$$

where $\Lambda'(H) = \frac{d\Lambda}{dH}$ and $\rho_m - 3H^2 = -\Lambda(H)$ are the first integrals of system (5)–(6).

Let us consider $\Lambda(H)$ given in the form of a Taylor series with respect to the Hubble parameter H , i.e.

$$\Lambda(H) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dH^n} \Lambda(H)|_0 H^n. \quad (7)$$

We assume additionally that the model dynamics has a reflection symmetry, $H \rightarrow -H$, i.e., $a(t)$ is a solution of the system and $a(-t)$ is also its solution. Therefore, only even terms of type H^{2n} are present in the expansion series (7). Finally, we assume the following form of the energy density parametrization through the Hubble parameter H [1]:

$$\rho_\Lambda(H) = \Lambda_{\text{bare}} + \alpha_2 H^2 + \alpha_4 H^4 + \dots \quad (8)$$

There are also some physical motivations for such a choice of the $\Lambda(H)$ parametrization (see [19]).

It would be useful for the further dynamical analysis of the system under consideration to re-parametrize the time variable

$$\tau \mapsto \tau = \ln a \quad (9)$$

and to rewrite the dynamical system (5)–(6) in the new variables

$$x = H^2, \quad y = \rho_m. \quad (10)$$

Then we obtain the following dynamical system:

$$x' \equiv \frac{dx}{d \ln a} = 2 \left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) \right], \quad (11)$$

$$y' \equiv \frac{dy}{d \ln a} = -3y - \frac{1}{3}(\alpha_2 + 2\alpha_4 x + \dots) \times \left[-x - \frac{1}{6}y + \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) \right] \quad (12)$$

and

$$y - 3x = -(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) \quad (13)$$

where instead of Λ_{bare} we write simply Λ , which represents a constant contribution to the $\Lambda(H)$ given by the expansion in the Taylor series (7).

Now, with the help of the first integral (13) we rewrite system (11)–(12) to the new form

$$x' = 2 \left(-x - \frac{1}{6}y + \frac{3x - y}{3} \right) = -y, \quad (14)$$

$$y' = -3y - (\alpha_2 + 2\alpha_4 x + \dots) \frac{3x - y}{9}. \quad (15)$$

Therefore, all trajectories of the system on the plane (x, y) are determined by the first integral (13).

The dynamical system (11)–(12) in a finite domain has a critical point of the one type: a stationary solution $x = x_0$, $y = y_0 = 0$ representing a de Sitter universe. In the original variables (H, ρ_m) we have two solutions: the stable expanding de Sitter universe and the unstable contracting de Sitter universe, both lying on the H axis. Note that if stationary solutions exist then they always lie on the intersection of the x axis ($y = 0$) with the trajectory of the flat model represented by the first integral (13), i.e., they are solutions of the following polynomial equation:

$$x - \frac{1}{3}(\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots) = 0 \quad (16)$$

and $y = 0$ (empty universe).

Note that the static critical point which represents the static Einstein universe does not satisfy the first integral (13) because both y and Λ are positive. Let us notice that if we substitute y into (11) then the dynamics is reduced to a form of a one-dimensional dynamical system,

$$\frac{dx}{d\tau} = -(3x - \Lambda - \alpha_2 x - \alpha_4 x^2 - \dots). \quad (17)$$

$$y = 3x - (\Lambda + \alpha_2 x + \alpha_4 x^2 + \dots). \quad (18)$$

Following the Hartmann–Grobman theorem [16] a system in the neighbourhood of critical points is well approximated by its linear part obtained by its linearization around this critical point.

On the other hand, a linear part dominates for small x in a right-hand side. Let us consider the dynamical system (17) truncated on this linear contribution, then the Hartman–Grobman theorem [16] guarantees us that the dynamical system in a neighbourhood of the critical point is a good approximation of the behaviour near the critical points. This system has the simple form

$$\frac{dx}{d\tau} = x(\alpha_2 - 3) + \Lambda, \quad (19)$$

$$y = (3 - \alpha_2)x - \Lambda. \quad (20)$$

System (19)–(20) has the single critical point of the form

$$x_0 = \frac{\Lambda}{3 - \alpha_2}, \quad y = 0. \quad (21)$$

It represents an empty de Sitter universe.

Let us now shift the position of this critical point to the origin by introducing the new variable $x \rightarrow X = x - x_0$. Then we obtain

$$\frac{dX}{d\tau} = (\alpha_2 - 3)X, \quad (22)$$

which possesses the exact solution of the form

$$X = X_0 e^{\tau(\alpha_2 - 3)} = X_0 a^{-3 + \alpha_2}, \quad (23)$$

where α_2 is constant. Of course this critical point is asymptotically stable if $\alpha_2 < 3$. The trajectories approaching this critical point at $\tau = \ln a \rightarrow \infty$ has the attractor solution $X = X_0 a^{\alpha_2 - 3}$ or $x = X + x_0$, where $x_0 = \frac{\Lambda}{3 - \alpha_2}$ or $X = 0$ (see Fig. 1). This attractor solution is crucial for the construction of a new model of a decaying Lambda effect strictly connected with the dark matter problem [9, 21].

The solution (23) has a natural interpretation: in a neighbourhood of a global attractor of system (17), trajectories behave as the universal solution, which motivates the Alcaniz–Lima approach in which

$$x = H^2 = \frac{\tilde{\rho}_{m,0}}{3} a^{-3 + \alpha_2} + \frac{\rho_{\Lambda,0}}{3}, \quad (24)$$

where $\tilde{\rho}_{m,0} = \frac{3}{3 - \alpha_2} \rho_{m,0}$.

We can rewrite Eq. (1) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$

$$\ddot{a} = -\frac{\partial V(a)}{\partial a}. \quad (25)$$

In our case the potential $V(a)$ is given in the following form:

$$V(a) = \frac{1}{2} \left(\frac{\Lambda}{3 - \alpha_2} - H_0^2 \right) a^{-1 + \alpha_2} - \frac{1}{2} \frac{\Lambda}{3 - \alpha_2} a^2. \quad (26)$$

The first integral of (25) can be expressed by

$$\frac{\dot{a}^2}{2} + V(a) = E = \text{const}, \quad (27)$$

where E is the value of the energy level (for the positive curvature $E = -1/2$, for the negative curvature $E = 1/2$ and for the flat universe $E = 0$). Figure 2 presents the evolution of $V(a)$ for $\alpha_2 = 0.1$ and for $\alpha_2 = 1$.

In our case if we consider the curvature in the dynamical analysis then we get new solutions for the positive curvature

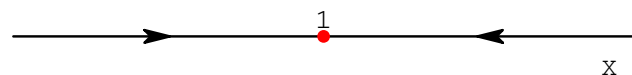


Fig. 1 A one-dimensional phase portrait of the FRW model with $\Lambda = \Lambda(H)$. Note the existence of universal behaviour of the $H^2(a)$ relation near the stable critical point (1) of the type of stable node. In a neighbourhood of this attractor we have the solution $X = H^2 - \frac{\Lambda}{3 - \alpha_2} = X_0 a^{\alpha_2 - 3}$ and $\rho_m = (3 - \alpha_2)H^2 - \Lambda = X_0 a^{\alpha_2 - 3}$. Therefore both ρ_m and $\rho_\Lambda - \Lambda$ are proportional (scaling solution)

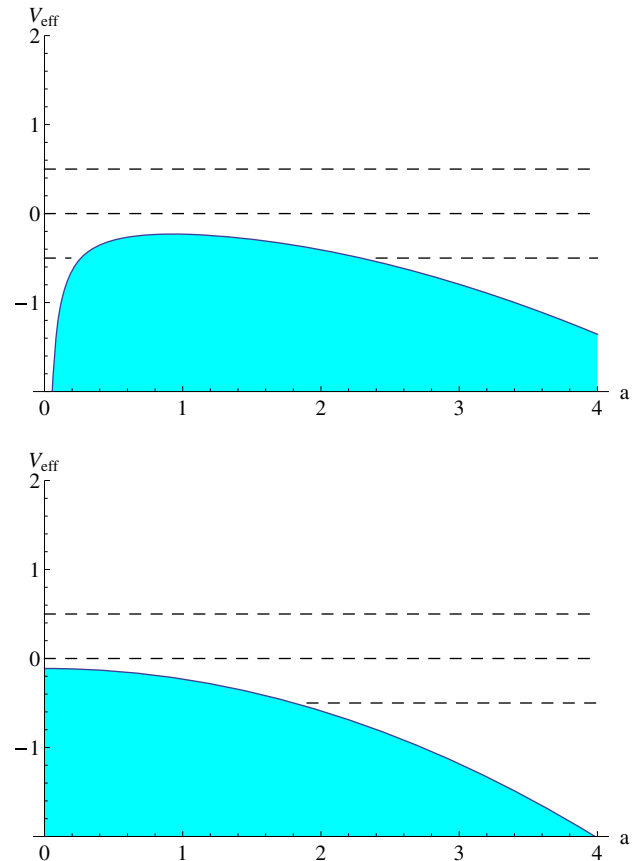


Fig. 2 The potential $V_{\text{eff}}(a)$ for $\alpha_2 = 0.1$ (top diagram) and $\alpha_2 = 1$ (bottom diagram). The top dashed lines ($V_{\text{eff}} = 1/2$) represent the energy level, which corresponds with the negative curvature. The bottom dashed lines ($V_{\text{eff}} = -1/2$) represent the energy level, which corresponds with the positive curvature. The middle dashed lines ($V_{\text{eff}} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is colored. The maximum of the potential is corresponding to a static Einstein universe in the phase space. Note that, for the case of positive curvature, the universe can oscillate with the initial singularity (the left bottom part of the top diagram) or be a universe with a bounce (the right bottom part of both diagrams)

such as the oscillating universe with the initial singularity and the universe with the bounce. But if we perturb solutions for the flat universe by a small spatial curvature then these solutions do not change qualitatively (see Fig. 2).

3 $\Lambda(a(t))$ CDM cosmologies as a dynamical system

In their construction many cosmological models of a decaying Λ make the ansatz $\Lambda(t) = \Lambda(a(t))$. For a review of different approaches in which ansatzes of this type appear, see Table 1.

In this section, we would like to discuss some general properties of the corresponding dynamical systems which model a decaying Λ term. It would be convenient to introduce the dynamical system in the state variables (H, ρ) ,

Table 1 Different choices of the $\Lambda(a)$ parametrization for different cosmological models appearing in the literature

$\Lambda(a)$ Parametrization	References
$\Lambda \sim a^{-m}$	[23,24]
$\Lambda = M_{\text{pl}}^4 \left(\frac{r_{\text{pl}}}{R}\right)^n$	[25]
$\Lambda = \frac{c^5}{hG^2} \left(\frac{l_{\text{pl}}}{a}\right)^n$	[26]
$\rho_\Lambda = \tilde{\rho}_{v,0} + \frac{\epsilon \rho_{m,0}}{3-\epsilon} a^{-3+\epsilon}$	[9]
$\rho_{de} = a^{-(4+\frac{2}{\alpha})}$, $\alpha = 2(1 - \Omega_{m,0} - \Omega_{\Lambda,0})$	[27]
$\rho_\Lambda = 3\alpha^2 M_p^2 a^{-2(1+\frac{1}{\alpha})}$	[28]
$\Lambda = \frac{\Lambda_{\text{pl}}}{(a/l_{\text{pl}})^2} \propto a^{-2}$	[29]
$\Lambda = \Lambda_1 + \Lambda_2 a^{-m}$	[30]

$$\dot{H} = -H^2 - \frac{1}{6}\rho_m + \frac{\Lambda_{\text{bare}}}{3} + \frac{\Lambda(a)}{3}, \quad (28)$$

$$\dot{\rho}_m = -3H\rho_m - \frac{d\Lambda}{da}(Ha) \quad (29)$$

or

$$\frac{dH^2}{d \ln a} = 2 \left(-H^2 - \frac{1}{6}\rho_m + \frac{1}{3}\Lambda_{\text{bare}} + \frac{1}{3}\Lambda(a) \right), \quad (30)$$

$$\frac{d\rho_m}{d\tau} = \frac{d\rho_m}{d \ln a} = -3\rho_m - a \frac{d\Lambda}{da} \quad (31)$$

with the first integral of the form

$$\rho_m = 3H^2 - \Lambda_{\text{bare}} - \Lambda(a). \quad (32)$$

As we have prescribed the form of the $\Lambda(a)$ relation, we can start the dynamical analysis with Eq. (28). It would be convenient to rewrite it to the form of the acceleration equation, i.e.,

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\rho_m(a) + \frac{\Lambda_{\text{bare}}}{3} + \frac{\Lambda(a)}{3}, \quad (33)$$

where $\rho_m(a)$ is determined by Eq. (31) which is a linear non-homogeneous differential equation which can be solved analytically

$$\frac{d\rho_m}{d\tau} = -3\rho_m - \frac{d\Lambda}{d\tau}(a) \quad (34)$$

and

$$\rho_m = - \left(\int^a a^3 d\Lambda(a) + C \right) a^{-3} \quad (35)$$

Equation (33) can be rewritten in an analogous form to the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)), where

$$V(a) = \frac{1}{6}a^{-1} \left(\int^a a^3 \frac{d\Lambda}{da} da + C \right) - \frac{\Lambda_{\text{bare}}}{6}a^2 - \frac{1}{6}a^2 \Lambda(a).$$

The integration of the above function gives the form of the potential.

Of course, Eq. (25) can be rewritten as the Newtonian two-dimensional dynamical system

$$\dot{a} = p, \quad \dot{p} = -\frac{\partial V}{\partial a}, \quad (37)$$

where the first integral has the form of Eq. (27). The integral of energy (27) should be consistent with the first integral (32), i.e.,

$$\rho_m + \rho_\Lambda = 3H^2, \quad (38)$$

$$a^{-3} \int a^3 \Lambda'(a) da + 3 \frac{\dot{a}^2}{a^2} = \Lambda_{\text{bare}} + \Lambda(a). \quad (39)$$

Because the system under consideration is a conservative system, centres or saddles can appear in the phase space. If the potential function $V(a)$ possesses a maximum, then in the phase space we obtain a saddle type critical point. If $V(a)$ has a minimum, this point is a centre.

As an example of adopting the method of the effective potential, which is presented here, let us consider the parametrization of $\Lambda(a)$ like in the Alcaniz–Lima model of decaying vacuum [9]. They assumed that energy density of vacuum is of the form (see Table 1)

$$\rho_\Lambda = \rho_{v,0} + \frac{\epsilon \rho_{m,0}}{3-\epsilon} a^{-3+\epsilon}, \quad (40)$$

where $\rho_{v,0}$ is vacuum energy $\rho_{m,0}$ is the energy density of matter at the present moment for which we choose $a = 1 = a_0$. Because $\dot{\rho}_{\text{vac}} < 0$, i.e., the energy of vacuum is decaying, from the conservation condition

$$\dot{\rho}_m = -3H\rho_m - \dot{\rho}_{\text{vac}} \quad (41)$$

we obtain

$$\dot{\rho}_{\text{vac}} = -\dot{\rho}_m - 3H\rho_m = -\rho_m \left(\frac{\dot{\rho}_m}{\rho_m} + 3H \right) \quad (42)$$

and the vacuum is decaying if

$$\frac{d \ln \rho_m}{d \ln a} > -3. \quad (43)$$

Let us notice that $\rho_m = 0$ is a solution of the system (41) only if Λ is constant. It is a source of some difficulties in the phase space because the trajectories can pass through the line $\rho_m = 0$. As a consequence of decaying vacuum energy density of matter will dilute more slowly compared to the corresponding canonical relation in the Λ CDM model, i.e., the energy density of matter is scaling following the rule

$$\rho_m = \rho_{m,0} a^{-3+\epsilon}, \quad (44)$$

where $\epsilon > 0$.

The dynamical system obtained from Eqs. (30)–(31) with the parametrization (40) has the following form:

$$(36) \quad x' = -3x + (y - \Lambda)(3 - \epsilon), \quad (45)$$

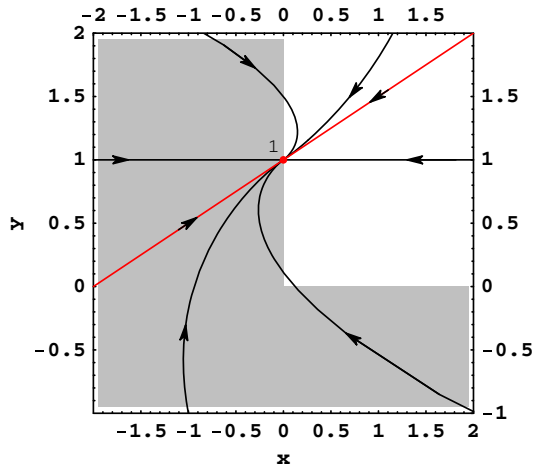


Fig. 3 A phase portrait for the dynamical system (45)–(46). The critical point (1) at $x = 0$, $y = \Lambda$ is a stable node. It represents a de Sitter universe. The red line represents the solutions of scaling type $y = \frac{\epsilon}{3-\epsilon}x + \Lambda$. The grey region represents a non-physical domain excluded by the condition $\rho_m = x > 0$, $\rho_\Lambda = y > 0$. Note that trajectories approach the attractor along a straight line. Let us note the existence of trajectories coming to the physical region from the non-physical one. We treated this type of behaviour as a difficulty related to an appearance of ghost trajectories, which emerges from the non-physical region

$$y' = -(y - \Lambda)(3 - \epsilon), \quad (46)$$

$$z' = -z - \frac{x}{6} + \frac{y}{3}, \quad (47)$$

with the condition $y = \Lambda + \frac{\epsilon}{3-\epsilon}x$, where $x = \rho_m$, $y = \rho_\Lambda$, $z = H^2$ and $' \equiv \frac{d}{d\tau}$. The above dynamical system contains the autonomous two-dimensional dynamical system (45)–(46). Therefore this system has an invariant two-dimensional submanifold. A phase portrait with this invariant submanifold is demonstrated in Fig. 3.

For a deeper analysis of the system, the investigation of trajectories at the circle $x^2 + y^2 = \infty$ at infinity is required. For this aim the dynamical system (45)–(46) is rewritten in projective coordinates. Two maps (X, Y) and $(\tilde{X}, \tilde{Y})$ cover the circle at infinity. In the first map we use the following projective coordinates: $X = \frac{1}{x}$, $Y = \frac{y}{x}$ and in the second one $\tilde{X} = \frac{x}{y}$, $\tilde{Y} = \frac{1}{y}$. System (45)–(46) rewritten in coordinates X and Y has the following form:

$$X' = X((Y - \Lambda X)(-3 + \epsilon) + 3), \quad (48)$$

$$Y' = (Y + 1)(Y - \Lambda X)(-3 + \epsilon) + 3Y \quad (49)$$

and for variables $\tilde{X}$, $\tilde{Y}$, we obtain

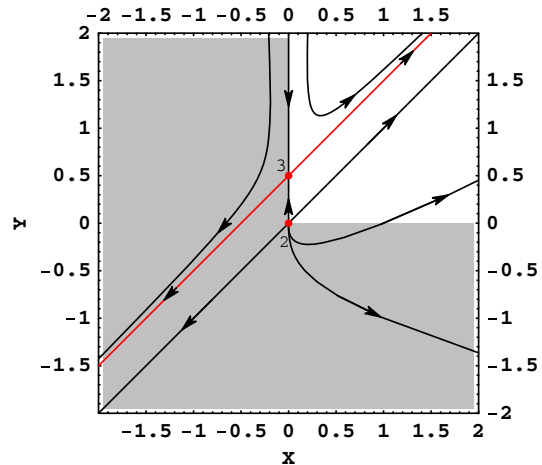


Fig. 4 A phase portrait for the dynamical system (48)–(49). Both the critical point (2) at $X = 0$, $Y = 0$ and the critical point (3) at $X = 0$, $Y = \frac{\epsilon}{3-\epsilon}$ present nodes. The red line represents the solutions of a scaling type $Y = \frac{\epsilon}{3-\epsilon} + \Lambda X$. The grey region represents a non-physical domain excluded by the condition $X > 0$, $Y > 0$

Table 2 Critical points for autonomous dynamical systems (45)–(46), (48)–(49), (50)–(51), their eigenvalues and cosmological interpretation

No.	Critical point	Eigenvalues	Type of critical point	Type of universe
1	$x = 0, y = \Lambda$	$-3, -3 + \epsilon$	Stable node	de Sitter
2	$X = 0, Y = 0$	$3, \epsilon$	Unstable node	Einstein–de Sitter
3	$X = 0, Y = \frac{\epsilon}{3-\epsilon}$	$3 - \epsilon, -\epsilon$	Saddle	Scaling universe ρ_m Is proportional to ρ_Λ

$$\tilde{X}' = (1 + \tilde{X})(1 - \Lambda \tilde{Y})(3 - \epsilon) - 3\tilde{X}, \quad (50)$$

$$\tilde{Y}' = \tilde{Y}(1 - \Lambda \tilde{Y})(3 - \epsilon). \quad (51)$$

The phase portraits for dynamical systems (48)–(49) and (50)–(51) are demonstrated in Figs. 4 and 5. The critical points for the above dynamical system are presented in Table 2 (Fig. 3, 4).

The reduction of the dynamics to the particle-like description with the effective potential enables us to treat the evolution of the universe in manners of classical mechanics. One treats the scale factor as a positional variable and

$$V_{\text{eff}}(a) = -\frac{\rho_{\text{eff}}(a)a^2}{6} = -\frac{1}{6}a^2 \left(\rho_m a^{-3+\epsilon} + \rho_{v,0} + \frac{\epsilon \rho_{m,0} a^{-3+\epsilon}}{3 - \epsilon} \right), \quad (52)$$

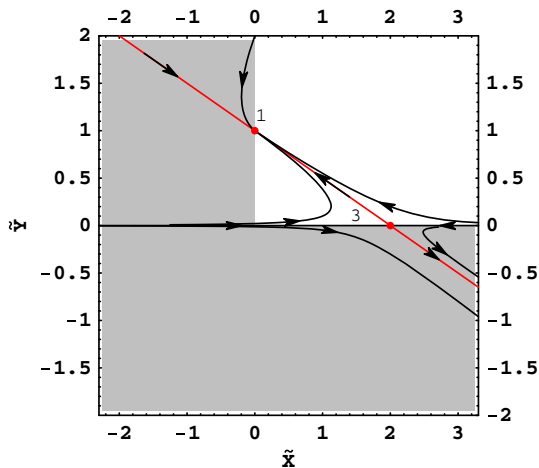


Fig. 5 A phase portrait for the dynamical system (50)–(51). The critical point (1) at $\tilde{X} = 0, \tilde{Y} = 1/\Lambda$ presents a stable node and the critical point (3) is at $\tilde{X} = \frac{3-\epsilon}{3}, \tilde{Y} = 0$ presents a saddle type point. The red line represents the solutions of a scaling type $\tilde{Y} = \left(1 - \frac{\epsilon}{3-\epsilon}\tilde{X}\right)/\Lambda$. The grey region represents non-physical domain excluded by the condition $\tilde{X} > 0, \tilde{Y} > 0$

where $\rho_{\text{eff}} = \rho_m + \rho_{\text{vac}}(a)$ and

$$\frac{\dot{a}^2}{2} + V_{\text{eff}} = -\frac{k}{2}. \quad (53)$$

The motion of a particle (the universe, that is; it mimics a unit-mass particle in that description) is restricted to the zero energy level $E = 0$ (because we considered a flat model). The evolutionary paths of the model can be directly determined from the diagram of the effective potential $V_{\text{eff}}(a)$.

Figure 6 demonstrates the diagram $V_{\text{eff}}(a)$ for values $\epsilon = 0.1$ and 1. In general, for the phase portrait in the plane $(a, \dot{a})$ the maximum of $V(a)$ corresponds to the static Einstein universe. This critical point is situated on the a -axis and it is always of the saddle type. Of course, it is only admissible for closed universes. In that case a minimum corresponds to a critical point of a centre type. If we include the curvature in the dynamical analysis then we get new solutions for the positive curvature such as the oscillating universe with an initial singularity and the universe with a bounce. But if we perturb solutions for the flat universe by a small spatial curvature then these solutions do not change qualitatively (see Fig. 6).

The Alcaniz–Lima model behaves in the phase space $(a, \dot{a})$ like the Λ CDM one [9]. Trajectories start from $(a, \dot{a}) = (0, \infty)$ (corresponding to the big bang singularity), approach the static universe and then evolve to infinity. Note that if $0 < \epsilon < 1$ then dynamics is qualitatively equivalent to the Λ CDM model.

The Eq. (29) can be written as

$$\dot{\rho}_m = -3H\rho_m - 3H\rho_m\delta(t), \quad (54)$$

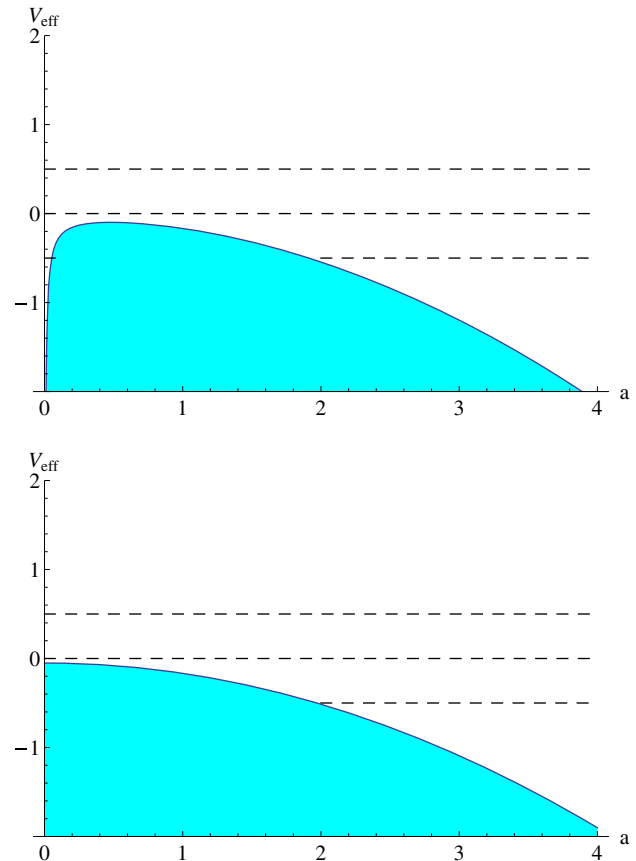


Fig. 6 The potential $V_{\text{eff}}(a)$ for $\epsilon = 0.1$ (top diagram) and for $\epsilon = 1$ (bottom diagram). The top dashed lines ($V_{\text{eff}} = 1/2$) represent the energy level, which corresponds with the negative curvature. The bottom dashed lines ($V_{\text{eff}} = -1/2$) represent the energy level, which corresponds with the positive curvature. The middle dashed lines ($V_{\text{eff}} = 0$) represent the energy level, which corresponds with the flat universe. The colored region represents a forbidden domain for the motion. The shape of diagram of the potential determines the phase space structure. The maximum of the potential is corresponding to a static Einstein universe in the phase space. Note that the universe with the positive curvature is an oscillating universe with the initial singularity (the left bottom part of the top diagram) or is a universe with a bounce (the right bottom part of both diagrams)

where $\delta(t) = -\frac{1}{3\rho_m} \frac{d\Lambda}{da} a$. Therefore,

$$\dot{\rho}_m = -3H\rho_m(1 + \delta(t)), \quad (55)$$

where $-3H\rho_m\delta(t) = \frac{d\Lambda}{da}Ha$, i.e., $\delta(t) = -\frac{\frac{d\Lambda}{da}a}{3\rho_m} \propto -\frac{\rho_\Lambda}{\rho_m}$. If $\delta(t)$ is a slowly changing function of time, i.e., $\delta(t) \simeq \delta$ then (55) has the solution $\rho_m = \rho_{m,0}a^{-3+\delta}$.

4 $\Lambda(R)$ CDM cosmologies as a dynamical system

The Ricci scalar dark energy idea has been recently considered in the context of the holographic principle [31]. In this case dark energy can depend on time t through the Ricci

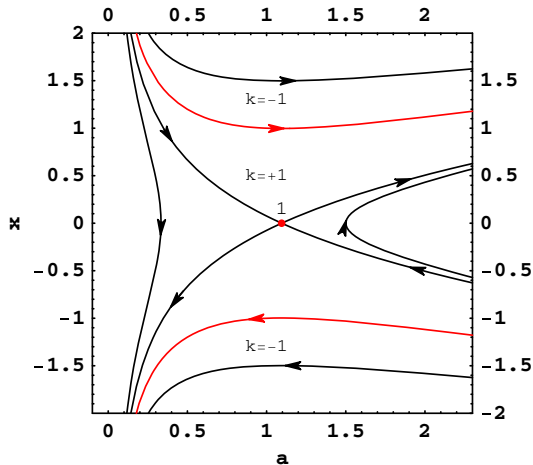


Fig. 7 A phase portrait for dynamical system (59)–(60) with $\alpha = 2/3$. The critical point (1), which is located on a -axis, ($a = \sqrt{\frac{3\rho_{m,0}}{2\rho_{\Lambda,0}-\rho_{m,0}}}$, $x = 0$), is a saddle point and represents a static Einstein universe. The red lines represent the trajectories of the flat universe. They separate the regions in which closed and open models lie. In the region, at the right from the critical point (1), bounded by the incoming separatrix from above and the outgoing separatrix from below, trajectories are going out from the contracting Milne solution, reaching the $a_{\min}$ and coming into the expanding Milne solution

scalar $R(t)$, i.e., $\Lambda(t) = \Lambda(R(t))$. Such a choice does not violate covariance of general relativity. A special case is the parametrization $\rho_{\Lambda} = -\frac{\alpha}{2}R = 3\alpha(\dot{H} + 2H^2 + \frac{k}{a^2})$ [27]. Then the cosmological equations are also formulated in the form of a two-dimensional dynamical system,

$$\dot{H} = -H^2 - \frac{1}{6}(\rho_m + \rho_{\Lambda}), \quad (56)$$

$$\dot{\rho} = -3H\rho_m \quad (57)$$

with the first integral of the form

$$H^2 = \frac{1}{3} \left(-\frac{3k}{a^2} + \frac{2}{2-\alpha} \rho_{m,0} a^{-3} + f_0 a^{2\frac{1-2\alpha}{\alpha}} \right), \quad (58)$$

where f_0 is an integration constant.

From the above equations, we can obtain a dynamical system in the state variables a , $x = \dot{a}$,

$$\dot{a} = x, \quad (59)$$

$$\dot{x} = -\Omega_{m,0} \frac{1}{2-\alpha} a^{-2} + \left(\frac{1}{\alpha} - 1 \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) a^{\frac{2}{\alpha}-3}. \quad (60)$$

The phase portrait on the plane (a, x) is shown in Fig. 7.

In order to analyze the trajectories behaviour at infinity we use the following sets of projective coordinates: $A = \frac{1}{a}$, $X = \frac{x}{a}$.

The dynamical system for variables A and X is expressed by

$$\dot{A} = -XA, \quad (61)$$

$$\dot{X} = A^3 \left[-\Omega_{m,0} \frac{1}{2-\alpha} + \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) A^{\frac{\alpha-2}{\alpha}} \right] - X^2. \quad (62)$$

We can use also the Poincaré sphere to identify the critical points at infinity. We introduce the following new variables: $B = \frac{a}{\sqrt{1+a^2+x^2}}$, $Y = \frac{x}{\sqrt{1+a^2+x^2}}$. In the variables (B, Y) , we obtain a dynamical system of the form

$$\begin{aligned} B' &= YB^2(1-B^2) \\ &- BY \left[-\Omega_{m,0} \frac{1}{2-\alpha} (1-B^2-Y^2)^{3/2} \right. \\ &+ \left. \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) B^{-1+2/\alpha} \right. \\ &\times \left. (1-B^2-Y^2)^{2-1/\alpha} \right], \end{aligned} \quad (63)$$

$$\begin{aligned} Y' &= \left[-\Omega_{m,0} \frac{1}{2-\alpha} (1-B^2-Y^2)^{3/2} \right. \\ &+ \left. \left(\frac{1-\alpha}{\alpha} \right) \left(\Omega_{\Lambda,0} - \Omega_{m,0} \frac{\alpha}{2-\alpha} \right) B^{-1+2/\alpha} \right. \\ &\times \left. (1-B^2-Y^2)^{2-1/\alpha} \right] (1-Y^2) - Y^2 B^3, \end{aligned} \quad (64)$$

where $' \equiv B^2 \frac{d}{dt}$.

The phase portraits for the dynamical systems (61)–(62) and (63)–(64) are demonstrated in Figs. 8 and 9, respectively.

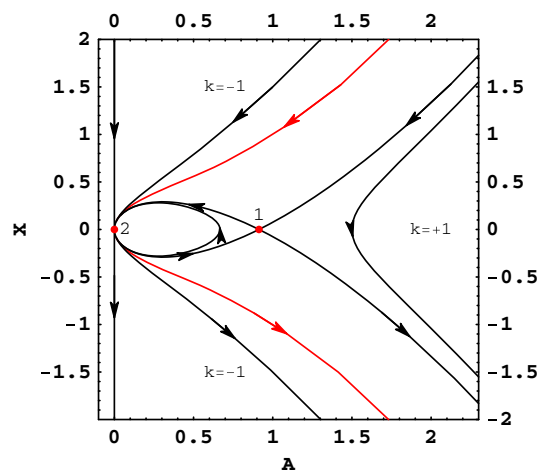


Fig. 8 A phase portrait for dynamical system (61)–(62) with $\alpha = 2/3$. The critical point (1) on the A -axis, ($A = \sqrt{\frac{2\rho_{\Lambda,0}-\rho_{m,0}}{3\rho_{m,0}}}$, $X = 0$), is a saddle and represents a static Einstein universe. The red lines represent the trajectories of a flat universe and they separate the regions in which closed and open models lie. The critical point (2) is a degenerate point at which the expanding and contracting Milne solutions are glued

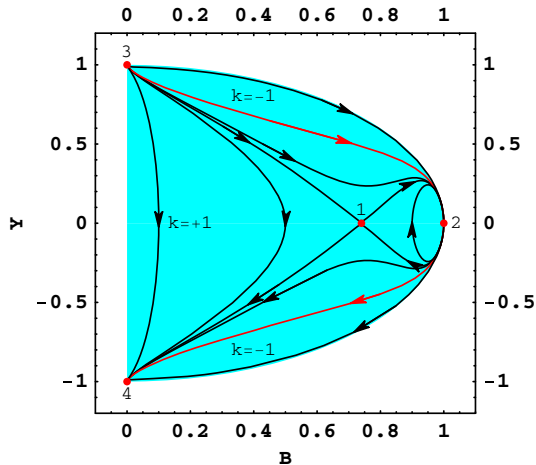


Fig. 9 A phase portrait for dynamical system (63)–(64) with $\alpha = 2/3$.

The critical point (1) is at $(B = 1/\sqrt{(\frac{2\rho_\Lambda - \rho_{m,0}}{3\rho_{m,0}})^2 + 1}, Y = 0)$ is a saddle and represents a static Einstein universe. The critical point (2) at the B -axis, $Y = 0$ is a stable node and represents a Milne universe. The critical points (3) and (4) at $(B = 0, Y = 1)$ and $(B = 0, Y = -1)$ are nodes and represent Einstein–de Sitter universes. The blue region represents a physical domain restricted to $B^2 + Y^2 \leq 0, B \geq 0$. The red lines represent the flat universe and they separate the regions in which closed and open models lie

If we include the curvature in the dynamical analysis then we get new types of universes. In the phase space in the positive curvature domain, new trajectories appear which represent the oscillating universe with an initial singularity and the universe with a bounce. For this model, the universe with the bounce start from the Milne universe and is the universe without the initial singularity. A similar situation holds for many $f(R)$ models, where the de Sitter universe is at the infinite past. Non-singular solutions of this type were found by Starobinsky [32].

5 Cosmology with emergent $\Lambda(a)$ relation from exact dynamics

In order to illustrate the idea of an emergent $\Lambda(a)$ relation let us consider cosmology with a scalar field which is non-minimal coupled to gravity. For simplicity, without loss of generality of our consideration, we assume that the non-minimal coupling ξ is constant like the conformal coupling. It is also assumed that dust matter, present in the model, does not interact with the scalar field. Since we would like to nest the Λ CDM model in our model we postulate that the potential of the scalar field is constant. We also assume a flat geometry with the R-W metric. The action for our model assumes the following form:

$$S = S_g + S_\phi + S_m, \quad (65)$$

where

$$S_g + S_\phi = \frac{1}{2} \int \sqrt{g} (R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2 - 2V(\phi)) d^4x, \quad (66)$$

$$S_m = \int \sqrt{g} \mathcal{L}_m d^4x, \quad (67)$$

where the metric signature is $(-, +, +, +)$, $R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})$ is the Ricci scalar and the dot denotes the differentiation with respect to the cosmological time t , i.e., $' \equiv \frac{d}{dt}$ and $\mathcal{L}_m = -\rho_m (1 + \int \frac{p_m(\rho_m)}{\rho_m^2} d\rho_m)$.

After skipping the full derivatives with respect to the time, the equation of motion for the scalar field is obtained after the variation over the scalar field and metric,

$$\frac{\delta S}{\delta \phi} = 0 \Leftrightarrow \ddot{\phi} + 3H\dot{\phi} + \xi R\phi + V'(\phi) = 0, \quad (68)$$

where $' \equiv \frac{d}{d\phi}$ and

$$\frac{\delta S}{\delta g} = 0 \Leftrightarrow \mathcal{E} = \frac{1}{2}\dot{\phi}^2 + 3\xi H^2 \phi^2 + 6\xi H\phi\dot{\phi} + V(\phi) - 3H^2 \equiv 0. \quad (69)$$

Additionally, from the conservation condition of the equation of state $p_m = p_m(\rho_m)$ for the barotropic matter we have

$$\dot{\rho}_m = -3H(\rho_m + p_m(\rho_m)). \quad (70)$$

Because we assume dust matter ($p_m=0$), Eq. (70) has a simple scaling solution of the form

$$\rho_m = \rho_{m,0} a^{-3}, \quad (71)$$

where $a = a(t)$ is the scale factor from the R-W metric $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$.

Analogously, the effects of the homogeneous scalar field satisfy the conservation condition

$$\dot{\rho}_\phi = -3H(\rho_\phi + p_\phi), \quad (72)$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 6\xi H\phi\dot{\phi} + 3\xi H^2 \phi^2, \quad (73)$$

$$p_\phi = \frac{1}{2}(1 - 4\xi)\dot{\phi}^2 - V(\phi) + 2\xi H\phi\dot{\phi} - 2\xi(1 - 6\xi)\dot{H}\phi^2 - 3\xi(1 - 8\xi)H^2 \phi^2 + 2\xi\phi V'(\phi). \quad (74)$$

In the investigation of the dynamics it would be convenient to introduce the so-called energetic state variables [33]

$$x \equiv \frac{\dot{\phi}}{\sqrt{6H}}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3H}}, \quad z \equiv \frac{\phi}{\sqrt{6}}. \quad (75)$$

The choice of such state variables (75) is suggested by the energy constraint $\mathcal{E} = 0$ (69).

The energy constraint condition can be rewritten in terms of dimensionless density parameters

$$\Omega_m + \Omega_\phi = 1 \quad (76)$$

then

$$\begin{aligned} \Omega_\phi &= 1 - \Omega_m = (1 - 6\xi)x^2 + y^2 + 6\xi(x+z)^2 \\ &= 1 - \Omega_{m,0}a^{-3} \end{aligned} \quad (77)$$

and the formula $H(x, y, z, a)$ rewritten in the terms of state variables x, y, z assumes the following form:

$$\begin{aligned} \left(\frac{H}{H_0}\right)^2 &= \Omega_\phi + \Omega_m \\ &= (1 - 6\xi)x^2 + y^2 + 6\xi(x+z)^2 + \Omega_{m,0}a^{-3}. \end{aligned} \quad (78)$$

Equation (78) is crucial for the model testing and estimation of the model parameters using astronomical data.

Because we try to generalize the Λ CDM model it is natural to interpret the additional contribution beyond Λ_{bare} as a running Λ term in (78). In our further analysis we will call this term ‘emergent Λ term’. Therefore,

$$\Omega_{\Lambda, \text{emergent}} = (1 - 6\xi)x^2 + y^2 + 6\xi(x+z)^2. \quad (79)$$

Of course, state variables satisfy a set of the differential equations in the consequence of Einstein equations. We try to organize them in the form of autonomous differential equations, i.e., some dynamical system.

For this aim let us start from the acceleration equation,

$$\dot{H} = -\frac{1}{2}(\rho_{\text{eff}} + p_{\text{eff}}) = -\frac{3}{2}H^2(1 + w_{\text{eff}}), \quad (80)$$

where ρ_{eff} and p_{eff} are the effective energy density and the pressure, while $w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}}$ is an effective coefficient of equation of state. Moreover, $\rho_{\text{eff}} = \rho_m + \rho_\phi$ and $p_{\text{eff}} = 0 + p_\phi$.

The coefficient equation of state w_{eff} is given by the formula

$$\begin{aligned} w_{\text{eff}} &= \frac{1}{1 - 6\xi(1 - 6\xi)z^2} \\ &\times \left[(1 - 4\xi)x^2 - y^2(1 + 2\xi\lambda z) + 4\xi xz + 12\xi^2 z^2 \right], \end{aligned} \quad (81)$$

where $\lambda \equiv -\sqrt{6} \frac{V'(\phi)}{V(\phi)}$ is related to geometry of the potential, where $' \equiv \frac{d}{d\phi}$.

The dynamical system which describes the evolution in the phase space is in the form

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x - 12\xi z + \frac{1}{2}\lambda y^2 - (x + 6\xi z) \frac{\dot{H}}{H^2}, \quad (82)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = -\frac{1}{2}\lambda xy - y \frac{\dot{H}}{H^2}, \quad (83)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (84)$$

$$\frac{d\lambda}{d(\ln a)} = \frac{d\lambda}{d\tau} = -\lambda^2(\Gamma(\lambda) - 1)x, \quad (85)$$

where $\Gamma = \frac{V''(\phi)V(\phi)}{V'^2(\phi)}$ and

$$\begin{aligned} \frac{\dot{H}}{H^2} &= \frac{1}{H^2} \left[-\frac{1}{2}(\rho_\phi + p_\phi) - \frac{1}{2}\rho_{m,0}a^{-3} \right] \\ &= \frac{1}{6\xi z^2(1 - 6\xi) - 1} \left[-12\xi(1 - 6\xi)z^2 - 3\xi\lambda y^2 z \right. \\ &\quad \left. + \frac{3}{2}(1 - 6\xi)x^2 + 3\xi(x+z)^2 + \frac{3}{2} - \frac{3}{2}y^2 \right]. \end{aligned} \quad (86)$$

Let us notice that the dynamical system (82)–(85) is closed if we only assume that $\Gamma = \Gamma(\lambda)$.

From the form of system (82)–(85) one can observe that it admits the invariant submanifold $\left\{ \frac{\dot{H}}{H^2} = 0 \right\}$ for which the equation in the phase space is of the form

$$\begin{aligned} &-12\xi(1 - 6\xi)z^2 - 3\xi\lambda y^2 z + \frac{3}{2}(1 - 6\xi)x^2 \\ &+ 3\xi(x+z)^2 + \frac{3}{2} - \frac{3}{2}y^2 = 0. \end{aligned} \quad (87)$$

Therefore, there are no trajectories which intersect this invariant surface in the phase space. From the physical point of view the trajectories are stationary solutions and on this invariant submanifold they satisfy the condition

$$\frac{\dot{H}}{H^2} = 0 \Leftrightarrow -\frac{1}{2}(\rho_\phi + p_\phi) - \frac{1}{2}\rho_{m,0}a^{-3} = 0. \quad (88)$$

If we look at the trajectories in the whole phase in the neighbourhood of this invariant submanifold, then we can observe that they will be asymptotically reached at an infinite value of time $\tau = \ln a$. They are tangent asymptotically to this surface. Note that in many cases the system on this invariant submanifolds can be solved and the exact solutions can be obtained.

As an illustration of the idea of the emergent $\Lambda(a)$ relation we consider two cases of cosmologies for which we derive $\Lambda = \Lambda(a)$ formulae. Such parametrizations of $\Lambda(a)$ arise if we consider the behaviour of trajectories near the invariant submanifold of dynamical systems

1. $V = \text{const}$ or $\lambda = 0$, the case of minimal coupling, $\xi = 0$;
2. $V = \text{const}$, the case of conformal coupling, $\xi = \frac{1}{6}$.

In these cases the dynamical system (82)–(85) reduces to

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x - x \frac{\dot{H}}{H^2}, \quad (89)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = -y \frac{\dot{H}}{H^2}, \quad (90)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (91)$$

where

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}x^2 - \frac{3}{2} + \frac{3}{2}y^2. \quad (92)$$

and

$$\frac{dx}{d\tau} = -3x - 2z - \frac{\dot{H}}{H^2}(x + z), \quad (93)$$

$$\frac{dy}{d\tau} = -y \frac{\dot{H}}{H^2}, \quad (94)$$

$$\frac{dz}{d\tau} = x, \quad (95)$$

where

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}(x + z)^2 - \frac{3}{2} + \frac{3}{2}y^2. \quad (96)$$

The dynamical system (93)–(95) can be rewritten using the variables $X = x + z$, $Y = y$ and $Z = z$. Then we get

$$\frac{dX}{d\tau} = -2X - \frac{\dot{H}}{H^2}X, \quad (97)$$

$$\frac{dY}{d\tau} = -Y \frac{\dot{H}}{H^2}, \quad (98)$$

$$\frac{dZ}{d\tau} = X - Z, \quad (99)$$

where

$$\frac{\dot{H}}{H^2} = -\frac{1}{2}X^2 - \frac{3}{2} + \frac{3}{2}Y^2. \quad (100)$$

The next step in a realization of our idea of the emergent Λ is to solve the dynamical system on invariant submanifold and then to substitute this solution into Eq. (79).

For the first case ($\xi = 0$, $V = \text{const}$), the dynamical system (89)–(91) has the following form:

$$\frac{dx}{d(\ln a)} = \frac{dx}{d\tau} = -3x, \quad (101)$$

$$\frac{dy}{d(\ln a)} = \frac{dy}{d\tau} = 0, \quad (102)$$

$$\frac{dz}{d(\ln a)} = \frac{dz}{d\tau} = x, \quad (103)$$

with the condition

$$0 = x^2 - y^2 + 1. \quad (104)$$

The solution of the dynamical system (101)–(103) is $x = C_1 a^{-3}$, $y = \text{const}$ and $z = -\frac{1}{3}C_1 a^{-3} + C_2$.

The phase portraits and a list of critical points for the dynamical system (89)–(91) is presented in Figs. 10, 11 and Table 3, respectively. The critical point (1) represents the matter dominating universe – an Einstein–de Sitter universe.

Finally, for first case $\Omega_{\Lambda, \text{emergent}}$ is given as

$$\Omega_{\Lambda, \text{emergent}} = \Omega_{\Lambda, \text{emergent}, 0} a^{-6} + \Omega_{\Lambda, 0}. \quad (105)$$

Now, let us concentrate on the second case ($\xi = 1/6$, $V = \text{const}$). The system (93)–(95) assumes the following form:

$$\frac{dx}{d\tau} = -3x - 2z, \quad (106)$$

$$\frac{dy}{d\tau} = 0 \Rightarrow y = \text{const}, \quad (107)$$

$$\frac{dz}{d\tau} = x \quad (108)$$

with the condition

$$0 = (x + z)^2 - 3y^2 + 3. \quad (109)$$

The dynamical system (106)–(108) is linear and can be simply integrated. The solution of the above equations are $x = -2C_1 a^{-2} - C_2 a^{-1}$, $y = \text{const}$ and $z = C_1 a^{-2} + C_2 a^{-1}$.

The phase portrait and critical points for the dynamical system (93)–(95) are presented in Figs. 12, 13 and Table 4.

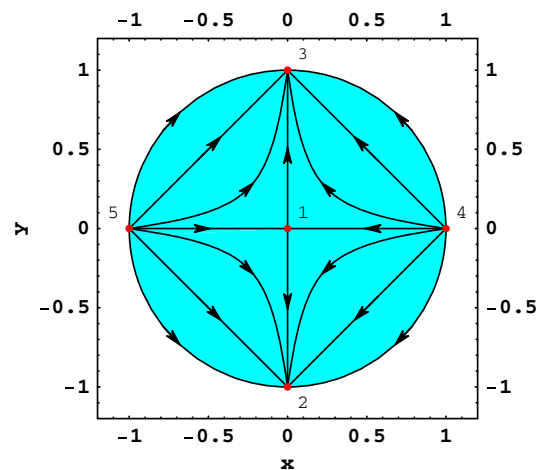


Fig. 10 The phase portrait for autonomous dynamical system (89)–(90). The critical point (1) represents a Einstein–de Sitter universe. The critical points (4) and (5) represent a Zeldovich stiff matter universe. The critical point (2) represents a contracting de Sitter universe. The critical point (3) represents stable de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. The blue region presents the physical region restricted by the condition $x^2 + y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

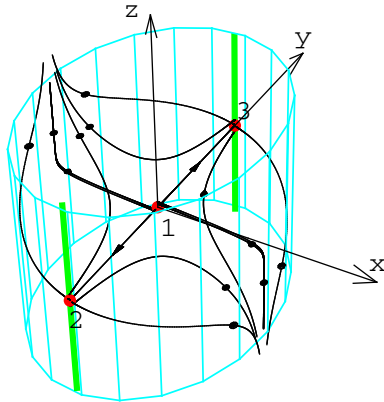


Fig. 11 The phase portrait for dynamical system (89)–(91). The critical point (1) represents the Einstein–de Sitter universe. Note that time $dt = Hd\tau$ is measured along trajectories, therefore in the region $H < 0$ (contracting model) time τ is reversed to the original time t . Hence, the critical point (2) represents an unstable de Sitter universe. Point (3) is opposite to the critical point (2) which represents a contracting de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$, which is an element of a cylinder and is presented by *green lines*. The surface of the cylinder presents a boundary of the physical region restricted by the condition $x^2 + y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

To illustrate the trajectories' behaviour close to the invariant submanifold (represented by the green lines) in the phase portrait (13) we construct two-dimensional phase portraits; see Fig. 14. In the latter trajectories reach the stationary states along tangential vertical lines (green lines).

On invariant submanifold (109) the dynamical system (106)–(108) reduces to

$$\frac{dx}{d\tau} = -x, \quad (110)$$

$$\frac{dz}{d\tau} = -z. \quad (111)$$

The solutions of (110)–(111) are $x = C_1 a^{-1}$ and $z = C_2 a^{-1}$.

Finally, we have

$$\Omega_{\Lambda, \text{emergent}} = \Omega_{\Lambda, 0} + \Omega_{\Lambda, \text{emergent}, 0} a^{-4}, \quad (112)$$

Table 3 The complete list of critical points of the autonomous dynamical system (89)–(90) which are shown in Figs. 10 and 11

Critical point	Coordinates	Eigenvalues	Type of critical point	Type of universe
1	$x = 0, y = 0$	$3, -3$	Saddle	Einstein–de Sitter
2	$x = 0, y = -1$	$-3, -3$	Stable node	Contracting de Sitter
3	$x = 0, y = 1$	$-3, -3$	Stable node	de Sitter
4	$x = 1, y = 0$	$3, 3$	Unstable node	Zeldovich stiff
				Matter dominating
5	$x = -1, y = 0$	$3, 3$	Unstable node	Zeldovich stiff
				Matter dominating

Coordinates, eigenvalues of the critical point as well as its type and cosmological interpretation are given

i.e., the relation $\Lambda(a) \propto a^{-4}$ appears if we consider the behaviour of trajectories in the neighbourhood of an unstable de Sitter state $\frac{\dot{H}}{H^2} = 0$. Therefore, the emergent term is of the type 'radiation'. In the scalar field cosmology there is a phase of evolution during each effective coefficient e.o.s. is 1/3 like for radiation. If we find a trajectory in a neighbourhood of a saddle point then such a type of behaviour appears [33] (Fig. 14).

We can rewrite Eq. (86) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)). On the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$ the above equation gives the following form of the potential:

$$V(a) = -\frac{1}{2} H_0^2 a^2. \quad (113)$$

Figure 15 presents the evolution of $V(a)$. For the positive curvature we get new solution which is the universe with the bounce. If we perturb solutions for the flat universe by a small negative spatial curvature then these solutions do not change qualitatively (see Fig. 15). But for the positive curvature, we always get the solutions, which represents the universe with bounce.

6 How to constrain emergent running $\Lambda(a)$ cosmologies?

Dark energy can be divided into two classes: with or without early dark energy [34]. Models without early dark energy behave like the Λ CDM model in the early time universe. For models with early dark energy, dark energy plays an important role in evolution of early universe. The second type models should have a scaling or attractor solution where the fraction of dark energy follows the fraction of the dominant matter or radiation component. In this case, we use the fractional early dark energy parameter Ω_d^e to measure a ratio of dark energy to matter or radiation.

The model with $\xi = 1/6$ (conformal coupling) and $V = \text{const}$ belongs to a class of models with early constant ratio dark energy in which $\Omega_{\text{de}} = \text{const}$ during the radiation dominated stage. In this case we can use the fractional

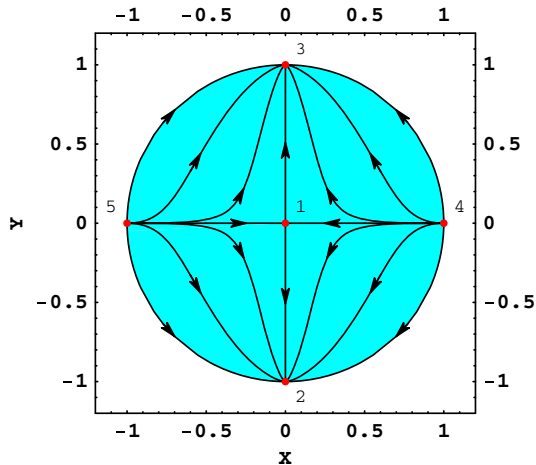


Fig. 12 The phase portrait for dynamical system (97)–(98). The critical point (1) represents a Einstein–de Sitter universe. The critical point (4) and (5) represent Zeldovich stiff matter universes. The critical points (2) represents a contracting de Sitter universe. The critical point (3) represents a stable de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$. The blue region presents the physical region restricted by the condition $X^2 + Y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

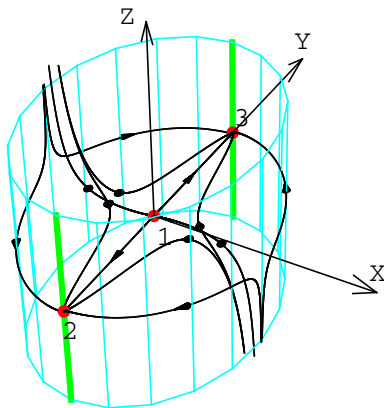


Fig. 13 The phase portrait for dynamical system (97)–(99). The critical point (1) represents an Einstein–de Sitter universe. Note that time $d\tau = H dt$ is measured along trajectories, therefore in the region $H < 0$ (contracting model) time τ is reversed to the original time t . Hence, the critical point (2) represents an unstable de Sitter universe. Point (3) is opposite to critical points (2) which represents a contracting de Sitter universe. The de Sitter universe is located on the invariant submanifold $\frac{\dot{H}}{H^2} = 0$, which is the element of a cylinder and is presented by green lines. The surface of the cylinder presents a boundary of the physical region restricted by the condition $X^2 + Y^2 \leq 1$, which is a consequence of $\Omega_m \geq 0$

early dark energy parameter Ω_d^e [34,35] which is constant for models with constant dark energy in the early universe. The fractional density of early dark energy is defined by the expression $\Omega_d^e = 1 - \frac{\Omega_m}{\Omega_{\text{tot}}}$, where Ω_{tot} is the sum of dimensionless density of matter and dark energy. In this case, there exist strong observational upper limits on this quantity [14].

For this aim let us notice that during the ‘radiation’ epoch we can apply this limit $\Omega_d^e < 0.0036$ [14] and

$$1 - \Omega_d^e = \frac{\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4}}{\Omega_{m,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \Omega_{\text{emergent},0}a^{-4}}. \quad (114)$$

Let us consider a radiation dominating phase $a(t) \propto t^{\frac{1}{2}}$ ($p_{\text{eff}} = \frac{1}{3}\rho_{\text{eff}}$) [33],

$$1 - \Omega_d^e = \frac{\Omega_{m,0}t^{-\frac{3}{2}} + \Omega_{r,0}t^{-2}}{\Omega_{m,0}t^{-\frac{3}{2}} + \Omega_{r,0}t^{-2} + \Omega_{\Lambda,0} + \Omega_{\text{emergent},0}t^{-2}} \underset{\text{at early universe}}{\simeq} \frac{\Omega_{r,0}}{\Omega_{r,0} + \Omega_{\text{emergent},0}}. \quad (115)$$

Ω_d^e at the early universe is constant and

$$\Omega_d^e = 1 - \frac{\Omega_{r,0}}{\Omega_{r,0} + \Omega_{\text{emergent},0}} < 0.0036. \quad (116)$$

From the above formula we get $\frac{\Omega_{\text{emergent},0}}{\Omega_{r,0}} < 0.003613$. In consequence we have a strict limit on a strength of the running Λ parameter in the present epoch, $\Omega_{\text{emergent},0} < 3.19 \times 10^{-7}$.

7 Cosmology with non-canonical scalar field

The dark energy can also be parameterized in a covariant way by a non-canonical scalar field ϕ [36]. The main difference between canonical and non-canonical description of the scalar field is in the generalized form of the pressure p_ϕ of the scalar field. For the canonical scalar field, the pressure p_ϕ is expressed by the formula $p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$, where $\dot{\phi} \equiv \frac{d\phi}{dt}$ and $V(\phi)$ is the potential of the scalar field. In the non-canonical case, the pressure is described by the expression $p_\phi = \left(\frac{\dot{\phi}^2}{2}\right)^\alpha - V(\phi)$, where α is an additional parameter. If α is equal 1 then the pressure of the non-canonical scalar field represents the canonical case.

The theory of the non-canonical scalar field is of course a covariant formulation because this theory can be obtained from the action, which is described by the following formula:

$$S = \int \sqrt{-g} \left(R + \left(\frac{\dot{\phi}^2}{2}\right)^\alpha - V(\phi) + \mathcal{L}_m \right) d^4x, \quad (117)$$

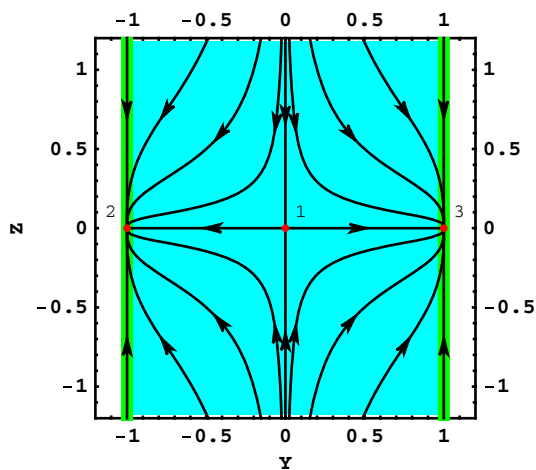
where $\mathcal{L}_m$ is the Lagrangian for the matter. Note that if $V(\phi)$ is constant then the model is equivalent to the model which is filled with an ideal fluid with the equation of state $p = w\rho$ (where w is determined by α) and the cosmological constant. After variation of the Lagrangian $\mathcal{L}$ with respect to the metric we get the Friedmann equations in the following form:

$$3H^2 = \rho_m + (2\alpha - 1) \left(\frac{\dot{\phi}^2}{2}\right)^\alpha + V(\phi) - \frac{3k}{a^2}, \quad (118)$$

Table 4 The list of critical points for the autonomous dynamical system (97)–(98) which are shown in Fig. 12 and 13

Critical point	Coordinates	Eigenvalues	Type of critical point	Type of universe
1	$X = 0, Y = 0$	$3/2, -1/2$	Saddle	Einstein–de Sitter
2	$X = 0, Y = -1$	$-3, -2$	Stable node	Contracting de Sitter
3	$X = 0, Y = 1$	$-3, -2$	Stable node	de Sitter
4	$X = 1, Y = 0$	$1, 2$	Unstable node	Zeldovich stiff
5	$X = -1, Y = 0$	$1, 2$	Unstable node	Matter dominating Zeldovich stiff Matter dominating

Coordinates, eigenvalues of the critical point as well as its type and cosmological interpretation are given

**Fig. 14** The phase portrait of the invariant submanifold $X = 0$ of the dynamical system (97)–(99). The critical point (1) represents a Einstein–de Sitter universe. The critical points (3) represents a stable de Sitter universe. The critical point (2) represents a contracting de Sitter universe. Note that because of time parametrization $dt = Hd\tau$ in the region $X < 0$, the cosmological time t is reversed. In consequence, the critical point (2) is unstable. The de Sitter universe is located on the invariant submanifold $\{\frac{\dot{H}}{H^2} = 0\}$, which is represented by green vertical lines. By identification of green lines of the phase portrait one can represent the dynamics on the cylinder. The boundary of the physical region is restricted by the condition $Y^2 \leq 1$, which is a consequence $\Omega_m \geq 0$. Note that trajectories reach the de Sitter states along tangential vertical lines

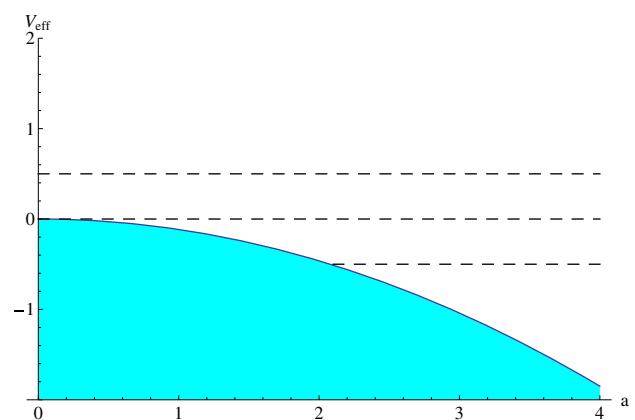
$$-3\frac{\ddot{a}}{a} = \frac{\rho_m}{2} + (\alpha + 1) \left(\frac{\dot{\phi}^2}{2} \right)^\alpha - V(\phi). \quad (119)$$

We obtain an additional equation of motion for a scalar field after the variation of the Lagrangian $\mathcal{L}$ with respect to the scalar field ϕ ,

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} + \left(\frac{V'(\phi)}{\alpha(2\alpha - 1)} \right) \left(\frac{2}{\dot{\phi}^2} \right)^{\alpha-1} = 0, \quad (120)$$

where $' \equiv \frac{d}{d\phi}$.

For $\alpha = 1$, Eqs. (118), (119) and (120) reduce to the case of the canonical scalar field. For $\alpha = 0$ we have the case

**Fig. 15** The figure presents a potential $V_{\text{eff}}(a)$. The top dashed lines ($V_{\text{eff}} = 1/2$) represent the energy level, which corresponds with the negative curvature. The bottom dashed lines ($V_{\text{eff}} = -1/2$) represent the energy level, which corresponds with the positive curvature. The middle dashed lines ($V_{\text{eff}} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is colored. Note that, for the case of the positive curvature, the universe is with the bounce (the right bottom part of the diagram)

with the constant scalar field. The case $\alpha = 2$ with the constant potential V is interesting since the scalar field imitates radiation because $\phi^{2\alpha} \propto a^{-4}$ in the Friedmann equation.

For the constant potential $V = \Lambda$, Eq. (120) reduces to

$$\ddot{\phi} + \frac{3H\dot{\phi}}{2\alpha - 1} = 0. \quad (121)$$

Equation (121) has the following solution:

$$\dot{\phi} = \phi_0 a^{\frac{-3}{2\alpha-1}}. \quad (122)$$

We can obtain from (118), (119) and (120) the dynamical system for the non-canonical scalar field with the constant potential in the variables a and $x = \dot{a}$,

$$a' = xa^2, \quad (123)$$

$$x' = -\frac{\rho_{m,0}}{6} - \frac{\alpha+1}{3} a^{\frac{3}{1-2\alpha}} + \frac{\Lambda}{3} a^3, \quad (124)$$

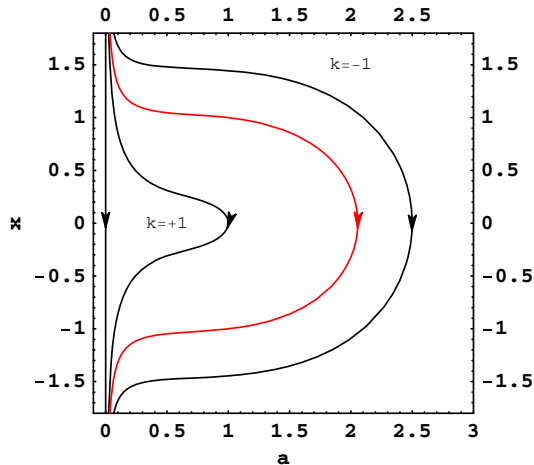


Fig. 16 A phase portrait for the dynamical system (123)–(124) for example with $\alpha = 1/8$. The red lines represent the flat universe and these trajectories separates the regions in which closed and open models lie. Note that all models independence on curvature are oscillating

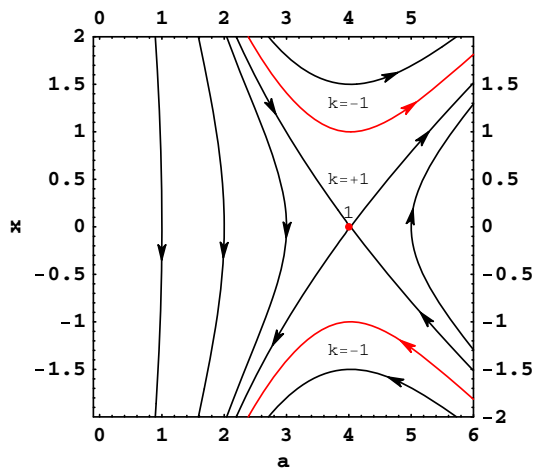


Fig. 17 A phase portrait for the dynamical system (125)–(126) with $\alpha = 1/8$ as an example. The red lines represent the flat universe and these trajectories separates the regions in which closed and open models lie

where $' \equiv a^2 \frac{d}{dt}$. The phase portrait for the dynamical system (123)–(124) is presented in Figs. 16 and 17.

The system (123)–(124) possesses critical points which belong to two types:

1. static critical points $x_0 = 0$,
2. non-static critical points $a_0 = 0$ (Big Bang singularity).

If we assume the matter in the form of dust ($p = 0$) then non-static critical points cannot exist at a finite domain of the phase space. The Big Bang singularity corresponds to a critical point at infinity.

Note that, if $\alpha > \frac{1}{2}$, then the eigenvalues for the critical point $(a_0, 0)$ are real and correspond to a saddle type of criti-

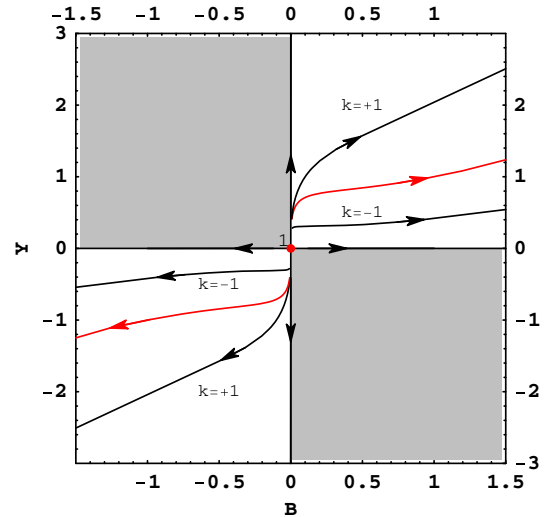


Fig. 18 A phase portrait for the dynamical system (127)–(128) with $\alpha = 1/8$ as an example. The critical point (I) at the origin $B = 0$, $Y = 0$ presents a stable node and Einstein–de Sitter universe. The grey region represents a non-physical domain excluded by the condition $\tilde{X}\tilde{Y} > 0$. The red lines represent the flat universe and these trajectories separates the regions in which closed and open models lie

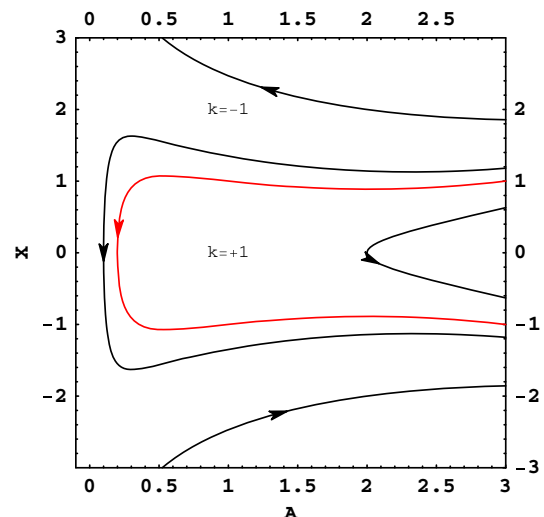


Fig. 19 A phase portrait for the dynamical system (123)–(124) with $\alpha = 50$ as an example. The critical point (I) is a saddle and represents a static Einstein universe. The red lines represent the flat universe and these trajectories separates the regions in which closed and open models lie. Note that all models have independence on curvature and are oscillating

cal point. Therefore, for $\alpha > \frac{1}{2}$ the qualitative structure of the phase space is topologically equivalent (by homeomorphism) to the Λ CDM model. Hence, the phase space portrait is structurally stable, i.e., it is not disturbed under small changes of the right-hand side of the system.

For the analysis of the behaviour of trajectories at infinity we use the following sets of projective coordinates:

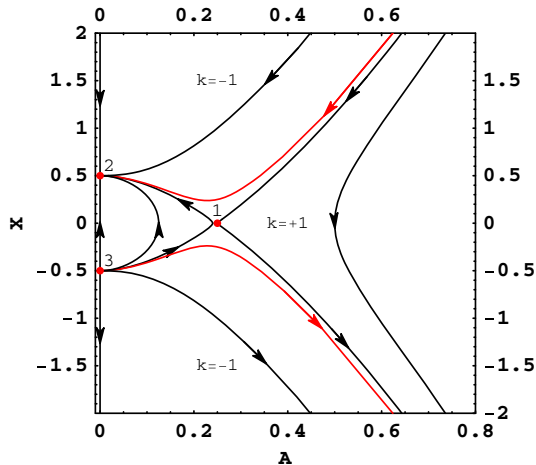


Fig. 20 A phase portrait for the dynamical system (125)–(126) with $\alpha = 50$ as an example. The critical point (1) is a saddle and represents a static Einstein universe. The critical point (2) represents a stable de Sitter universe. The critical point (3) represents a contracting de Sitter universe. The red lines represent the flat universe and these trajectories separate the regions in which closed and open models lie

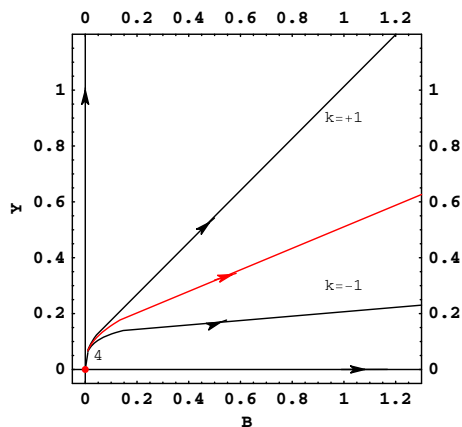


Fig. 21 A phase portrait for dynamical system (127)–(128) for example with $\alpha = 50$. The critical point (4) at the origin ($B = 0, Y = 0$) is an unstable node and represents an Einstein–de Sitter universe. The red lines represent the flat universe and these trajectories separate the regions in which closed and open models lie

1. $A = \frac{1}{a}, X = \frac{x}{a},$
2. $B = \frac{a}{x}, Y = \frac{1}{x}.$

Two maps cover the behaviour of trajectories at the circle at infinity.

The dynamical system for variables A and X is expressed by

$$A' = -XA^2, \quad (125)$$

$$X' = A^4 \left(-\frac{\rho_{m,0}}{6} - \frac{\alpha+1}{3} A^{\frac{3}{2\alpha-1}} \right) + A \left(\frac{\Lambda}{3} - X^2 \right), \quad (126)$$

where $' \equiv A \frac{d}{dt}$. The dynamical system for variables B and Y is given by

$$\dot{B} = BY \left[B + \left(\frac{\rho}{6} Y^3 + \frac{\alpha+1}{3} B^{\frac{3}{1-2\alpha}} Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3} B^3 \right) \right], \quad (127)$$

$$\dot{Y} = Y^2 \left(\frac{\rho}{6} Y^3 + \frac{\alpha+1}{3} B^{\frac{3}{1-2\alpha}} Y^{\frac{6\alpha}{2\alpha-1}} - \frac{\Lambda}{3} B^3 \right), \quad (128)$$

where $\dot{} \equiv B^2 Y \frac{d}{dt}$.

From the analysis of the dynamical system (127)–(128) we find one critical point ($B = 0, Y = 0$) which represents the Einstein–de Sitter universe. The phase portraits for dynamical system (125)–(126) and (127)–(128) are depicted in Figs. 18, 19, 20, and 21.

Let us consider the curvature in the dynamical analysis. Then in the phase space in the positive curvature domain, we find new trajectories which represent an oscillating universe with the initial singularity and a universe with a bounce (Figs. 17, 20, 21).

8 Cosmology with diffusion

The parametrization of dark energy can also be described in terms of the scalar field ϕ [37, 38]. As an example of such a covariant parametrization of Λ let us consider cosmological models with diffusion. In this case the Einstein equations and equations of the current density J^μ are the following:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \phi g_{\mu\nu} = T_{\mu\nu}, \quad (129)$$

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu, \quad (130)$$

$$\nabla_\mu J^\mu = 0, \quad (131)$$

where σ is a positive parameter.

From the Bianchi identity, $\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$, and Eqs. (129) and (130) we get the following expression for $\Lambda(a(t))$:

$$\nabla_\mu \phi = \sigma J_\mu. \quad (132)$$

We assume also that matter is a perfect fluid. Then the energy-momentum tensor is expressed in the following form:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p (g_{\mu\nu} + u_\mu u_\nu), \quad (133)$$

where u_μ is the 4-velocity and the current density is expressed by

$$J^\mu = n u^\mu. \quad (134)$$

Under these considerations Eqs. (130), (132) and (131) are described by the following expressions:

$$\nabla_\mu (\rho u^\mu) + p \nabla_\mu u^\mu = \sigma n, \quad (135)$$

$$\nabla_\mu (n u^\mu) = 0, \quad (136)$$

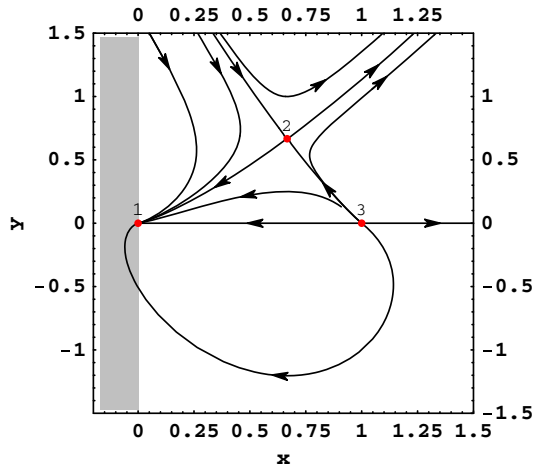


Fig. 22 A phase portrait for the dynamical system (142)–(142). The critical point (1) ($x = 0, y = 0$) is a stable node and represents the de Sitter universe. The critical point (2) ($x = 2/3, y = 2/3$) is a saddle and represents the Milne universe. The critical point (3) ($x = 1, y = 0$) is an unstable node type and represents the Einstein–de Sitter universe. Note the existence of trajectories crossing the boundary $x = \rho_m = 0$ in a non-physical region

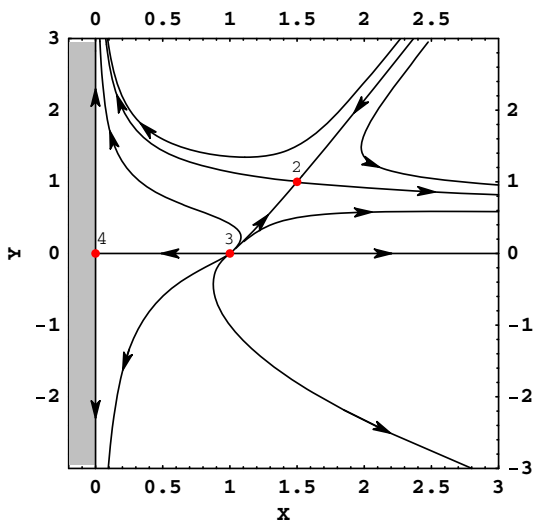


Fig. 23 A phase portrait for the dynamical system (144)–(145). The critical point (4) ($X = 0, Y = 0$) is a saddle and represents the static universe. The critical point (2) ($X = 3/2, Y = 1$) is a saddle and represents the Milne universe

and

$$\nabla_\mu \phi = \sigma n u_\mu. \quad (137)$$

We consider for simplicity the case of cosmological equations with the zero curvature. Equation (136) is now

$$n = n_0 a^{-3}. \quad (138)$$

In this case we have the following cosmological equations:

$$3H^2 = \rho_m + \Lambda(a(t)), \quad (139)$$

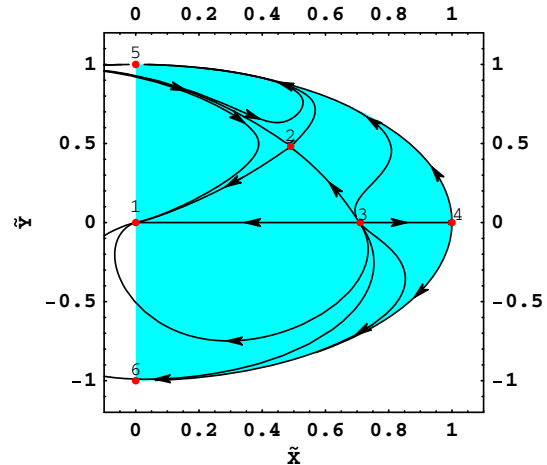


Fig. 24 A phase portrait for the dynamical system (146)–(147). The blue region represents the physical domain. The critical points (5) and (6) ($\tilde{X} = 0, \tilde{Y} = 1$) and ($\tilde{X} = 0, \tilde{Y} = -1$) represent the de Sitter universe with diffusion. The blue region represents a physical domain restricted by $B^2 + Y^2 \leq 0, B \geq 0$

$$\dot{\rho}_m = -3H\rho_m + \sigma n_0 a^{-3}, \quad (140)$$

$$\frac{d\phi}{dt} = -\sigma n_0 a^{-3}. \quad (141)$$

If we choose the dimensionless state variables $x = \frac{\rho_m}{3H^2}$ and $y = \frac{\sigma n_0 a^{-3}}{3H^3}$ and the parametrization of time as $' \equiv \frac{d}{d \ln a}$ then we get the following dynamical system:

$$x' = 3x(x - 1) + y, \quad (142)$$

$$y' = 3y\left(\frac{3}{2}x - 1\right). \quad (143)$$

The phase portrait for (142)–(143) is demonstrated in Fig. 22.

The dynamical system (142)–(143) can be rewritten in the projective variables for the analysis of critical points in infinity. In this case we use the following projective coordinates: $X = \frac{1}{x}, Y = \frac{y}{x}$. For the new variables X and Y , we obtain

$$X' = X(X(3 - Y) - 3), \quad (144)$$

$$Y' = Y\left(\frac{3}{2} - XY\right) \quad (145)$$

where $' \equiv X \frac{d}{d \ln a}$.

We can use also the Poincaré sphere to search critical points in infinity. We introduce the following new variables: $\tilde{X} = \frac{x}{\sqrt{1+x^2+y^2}}, \tilde{Y} = \frac{y}{\sqrt{1+x^2+y^2}}$. In the variables $\tilde{X}, \tilde{Y}$, we obtain the dynamical system of the form

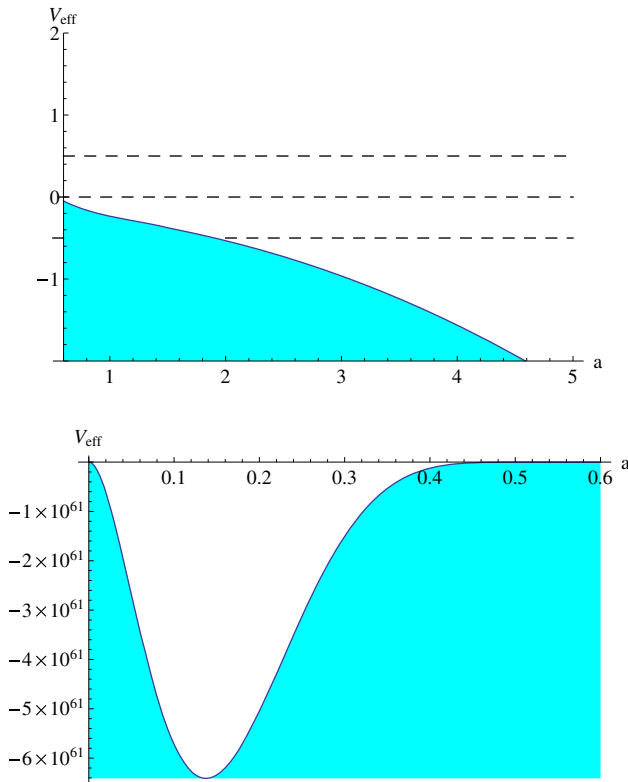


Fig. 25 The potential $V_{\text{eff}}(a)$ for $a > 0.6$ (top diagram) and for $a < 0.6$ (bottom diagram). The top dashed lines ($V_{\text{eff}} = 1/2$) represent the energy level, which corresponds with the negative curvature. The bottom dashed lines ($V_{\text{eff}} = -1/2$) represent the energy level, which corresponds with the positive curvature. The middle dashed lines ($V_{\text{eff}} = 0$) represent the energy level, which corresponds with the flat universe. The forbidden domain for the motion is colored. Note that, for the case of the positive curvature, the universe is oscillating (the left bottom part of the top diagram) or is the universe with the bounce (the right bottom part of both diagrams)

$$\begin{aligned} \tilde{X}' &= (1 - \tilde{X}^2)(3\tilde{X}^2 + (\tilde{Y} - 3\tilde{X})\sqrt{1 - \tilde{X}^2 - \tilde{Y}^2}) \\ &\quad - 3\tilde{X}\tilde{Y}^2 \left(\frac{3}{2}\tilde{X} - \sqrt{1 - \tilde{X}^2 - \tilde{Y}^2} \right), \end{aligned} \quad (146)$$

$$\begin{aligned} \tilde{Y}' &= -\tilde{X}\tilde{Y}(3\tilde{X}^2 + (\tilde{Y} - 3\tilde{X})\sqrt{1 - \tilde{X}^2 - \tilde{Y}^2}) \\ &\quad + 3(1 - \tilde{Y}^2)\tilde{Y} \left(\frac{3}{2}\tilde{X} - \sqrt{1 - \tilde{X}^2 - \tilde{Y}^2} \right), \end{aligned} \quad (147)$$

where $' \equiv \sqrt{1 - \tilde{X}^2 - \tilde{Y}^2} \frac{d}{d \ln a}$. The phase portraits for (144)–(145) and (146)–(147) are demonstrated in Figs. 23 and 24.

We can rewrite Eqs. (139–141) as the Newtonian equation of motion for a particle of unit mass moving in the potential $V(a)$ (Eq. (25)) for finding the potential $V(a)$. The potential $V(a)$ has the following form:

$$V(a) = -\frac{1}{2}H(a)^2a^2. \quad (148)$$

Figure 25 presents the evolution of $V(a)$. For the curvature we get new solutions which are for a universe with a bounce and an oscillating universe without the initial singularity.

9 Conclusion

In this paper we have studied the dynamics of cosmological models with the running cosmological constant term using the dynamical system methods. We considered different parametrizations of the Λ term which are used in the cosmological applications. The most popular approaches are to parametrize the Λ term through the scale factor a or the Hubble parameter H . We considered cosmological models for which the energy-momentum tensor of matter (we assume dust matter) is not conserved. In this case there is an interaction between both dark matter and dark energy sectors.

There is a class of parameterizations of the Λ term through the Ricci scalar (or the trace of the energy-momentum tensor), the energy density of the scalar field or their kinetic part, and a scalar field ϕ minimally or non-minimally coupled to gravity. These choices are consistent with the covariance of general relativity.

We have discovered a new class of the emergent Λ parameterizations (in the case of $\Lambda(a)$) obtained directly from the exact dynamics, which does not violate the covariance of general relativity.

In consequence, the energy density deviates from the standard dilution. Due to decaying vacuum energy the standard relation $\rho_m \propto a^{-3}$ is modified. From the cosmological point of view this class of models is a special case of cosmology with the interacting term $Q = -\frac{d\Lambda}{dr}$.

The main motivation for studying such models comes from the solution of the cosmological constant problem, i.e., explanations why the cosmological upper bound ($\rho_\Lambda \leq 10^{-47}$ GeV) dramatically differs from theoretical expectations ($\rho_\Lambda \sim 10^{71}$ GeV) by more than 100 orders of magnitude [39]. In this context the running Λ cosmology is some phenomenological approach toward finding the relation $\Lambda(t)$ lowering the value of cosmological constant during the cosmic evolution.

In the study of the $\Lambda(t)$ CDM cosmology different parametrizations of the Λ term are postulated. Some of them like $\Lambda(\phi)$, $\Lambda(R)$ or $\Lambda(\text{tr } T_\nu^\mu)$, $\Lambda(T)$, where $T = \frac{1}{2}\dot{\phi}^2$ are consistent with the principle of covariance of general relativity. Others, like $\Lambda = \Lambda(H)$, are motivated by the quantum field theory.

We demonstrated that the parameterization $\Lambda = \Lambda(a)$ can be obtained from the exact dynamics of the cosmological models with scalar field and the potential by taking approximation of trajectories in a neighbourhood of the invariant submanifold $\frac{\dot{H}}{H^2}$ of the original system. The trajectories approaching

a stable de Sitter state are mimicking the effects of the running $\Lambda(a)$ term. The arbitrary parametrizations of $\Lambda(a)$, in general, violate the covariance of general relativity. However, some of them which emerge from the covariant theory are an effective description of the behaviour of trajectories in the neighbourhood of a stable de Sitter state.

In the paper we have studied the dynamics of these cosmological models in detail. We have examined the structure of the phase space which is organized by critical points representing stationary states, invariant manifolds, etc. We have explored the dynamics at finite domains of the phase space as well as at infinity using the projective coordinates.

The detailed results obtained from the dynamical system analysis are as follows:

- We have found that Alcaniz and Lima's solution in the exploration of the conception of $\Lambda(H)$ cosmology represents the scaling solution $\rho_\Lambda(a) \sim \rho_m(a)$. For this trajectory deS_+ is a global attractor.
- The non-covariant $\Lambda(a)$ parametrization can be obtained from the covariant action for the scalar field as an emergent parameterization.
- We have found strong evidence for the tuned-in Λ term in the $\Lambda(a)$ cosmology: $\Omega_{\Lambda,0} < 3.19 \times 10^{-7}$. This limit was obtained on the base of Ade et al.'s estimation of the constant early dark energy fraction [14].
- We have shown that trajectories in the phase space for which $\rho_\Lambda \sim \rho_m$ represent scaling solutions.

Due to the dynamical system analysis we can reveal the physical status of the Alcaniz–Lima ansatz in the $\Lambda(H)$ approach. From the point of view of dynamical system theory this solution is a universal asymptote for trajectories which go toward a global attractor, i.e. a de Sitter state. In this regime both $\rho_\Lambda - \Lambda_{\text{bare}}$ and ρ_m are proportional, i.e., it is a scaling solution.

The detailed studies of the dynamics on the phase portraits showed how 'large' is the class of running Λ cosmological models for which the concordance Λ CDM model is a global attractor.

We also demonstrated on the example of cosmological models with non-minimal coupling constant and constant potential that a running part of the Λ term can be constrained by the Planck data. Applying the idea of constant early dark energy fraction and Ade et al.'s bound we have found a convincing constraint on the value of the running Λ term.

In the paper we considered some parametrization of the Λ term, which violates the covariance of the Lagrangian like $\Lambda(H)$, $\Lambda(a)$ parameterization but it is used as a some kind of an effective description. In the phase space of cosmological models with such a parametrization we observe some difficulties which are manifested by trajectories crossing the boundary line of zero energy density invariant submanifold.

It is a consequence of the fact that $\rho_m = 0$ is not a trajectory of the dynamical system. On the other hand the $\Lambda(a)$ parametrization can emerge from the basic covariant theory as some approximation of the true dynamics.

We illustrated such a possibility for the scalar field cosmology with a minimal and non-minimal coupling to gravity. In the phase space of evolutionary scenarios the difficulties disappear. Trajectories depart from the invariant submanifold $\frac{\dot{H}}{H^2} = 0$ of the corresponding dynamical system and this behaviour can be approximated by a running cosmological term such as a slow roll parameter $\epsilon_1 = \frac{\dot{H}}{H^2} \ll 1$.

We included the curvature in the dynamical analysis. In the phase space in the positive curvature domain, we found new trajectories which represent an oscillating universe with the initial singularity and without the initial singularity and a universe with a bounce. For models in this paper, perturbations of the flat model, by the negative curvature, do not change qualitatively this model in contrast to a closed model.

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Starobinsky cosmological model in Palatini formalism

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Abstract We classify singularities in FRW cosmologies, which dynamics can be reduced to the dynamical system of the Newtonian type. This classification is performed in terms of the geometry of a potential function if it has poles. At the sewn singularity, which is of a finite scale factor type, the singularity in the past meets the singularity in the future. We show that such singularities appear in the Starobinsky model in $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$ in the Palatini formalism, when dynamics is determined by the corresponding piecewise-smooth dynamical system. As an effect we obtain a degenerate singularity. Analytical calculations are given for the cosmological model with matter and the cosmological constant. The dynamics of model is also studied using dynamical system methods. From the phase portraits we find generic evolutionary scenarios of the evolution of the universe. For this model, the best fit value of $\Omega_\gamma = 3\gamma H_0^2$ is equal 9.70×10^{-11} . We consider a model in both Jordan and Einstein frames. We show that after transition to the Einstein frame we obtain both the form of the potential of the scalar field and the decaying Lambda term.

1 Introduction

The main aim of the paper is the construction of the Starobinsky model with a squared term $\hat{R}^2$ in the Palatini formalism and the investigation of cosmological implications of this model. In this model the inflation phase of evolution of the universe can be obtained by the modification of general relativity in the framework of $f(\hat{R})$ modified gravity theories [1]. In this context, historically the first theory of inflation was proposed by Starobinsky [2]. In the original Starobinsky model the term $R^2/6M^2$ was motivated by the conformal anomaly in the quantum gravity. The problem of

inflation in an $f(R)$ cosmological model is strictly related with the choice of frames. The authors of [1] show that CMB spectra in both Einstein and Jordan frames are different functions of the number of e-foldings until the end of inflation.

Inflation is a hypothesis about the existence of a short but very fast (of exponential type) accelerated growth of the scale factor $a(t)$ during the early evolution of the universe, after the Big-Bang but before the radiation-dominated epoch [3, 4]. It implies $\ddot{a}(t) > 0$. Irregularities in the early epoch may lead to the formation of structures in the universe due to the appearance of inflation.

Starobinsky [2] was the first who proposed a very simple theoretical model with one parameter M (energy scale M) of such inflation and which is in good agreement with astronomical data and CMB observation. The Starobinsky model is representing the simplest version of $f(R)$ gravity theories which have been developed considerably in the last decade [1, 5, 6], whose extra term in the Lagrangian is quadratic in the scalar curvature. This model predicts the value of spectral index $n_s = 0.9603 \pm 0.0073$, at the 68% CL, with deviation from scale-invariance of the primordial power spectrum [7, 8].

The Starobinsky model is also compatible with Planck 2015 data [9] and nicely predicts the number $N = 50$ –60 e-folds between the start and the end of inflation [10].

It has been recently investigated some generalization of the Starobinsky inflationary model with a polynomial form of $f(R) = R + \frac{R^2}{6M^2} + \frac{\lambda_n}{2n} \frac{R^n}{(3M^2)^{n-1}}$. It was demonstrated that the slow-roll inflation can be achieved as long as the dimensionless coupling λ_n is sufficiently small [11].

The Starobinsky model becomes generic because the smallness of the dimensionless coupling constant λ_n does not imply that fine-tuning is necessary [11]. The Starobinsky model was developed in many papers [8, 12–17].

In this paper we develop the idea of endogenous inflation as an effect of modification of the FRW equation after

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the formulation of $f(R)$ cosmological model in the Palatini formalism.

We are looking for an inflation mechanism as a pure dynamical mechanism driven by the presence of the additional term (square of the Ricci scalar) in the Lagrangian, without necessity of the choice of a frame (Einstein vs. Jordan frame) [16–18].

In modern cosmology, a most popular trend is to explain the dark energy and the dark matter in terms of some substances, of which the nature is unknown up to now. Einstein was representing the opposite relational point of view on the description of gravity, in which all substantial forms should be eliminated. Such a point of view is called anti-substantialism. Extended $f(R)$ gravity models [6, 19] offer intrinsic or geometric models of both dark matter and dark energy—the key elements of Standard Cosmological Model. Therefore, the Einstein idea of relational gravity, in which dark matter and dark energy can be interpreted as geometric objects, is naturally realized in $f(\hat{R})$ extended gravity. The methods of dynamical system in the context of investigation dynamics of $f(R)$ models are used since Carroll [19, 20].

Unfortunately, the metric formulation of extended gravity gives rise to fourth order field equations. To avoid this difficulty, the Palatini formalism can be applied where both the metric g and the symmetric connection Γ are assumed to be independent dynamical variables. In consequence, one gets a system of second order partial differential equations. The Palatini approach reveals that the early universe inherits properties of the global Λ CDM evolution.

The Palatini approach has become of some interest lately. An excellent review of the Palatini $f(R)$ theories can be found in Olmo's paper [21]. He has published many other papers on this topic, namely, about the scalar–tensor representation of the Palatini theories [22, 23]. The other important papers were on the existence of non-singular solutions in the Palatini gravity [24, 25]. Some more recent papers concentrate on studying black holes and their singularities in the Palatini approach [26–30]. Other work which is important to mention is Flanagan's papers on the choice of a conformal frame [31, 32]. Pannia et al. considered the impact of the Starobinsky model in compact stars [33].

In the Palatini gravity action for $f(\hat{R})$ gravity is introduced to be

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \quad (1)$$

where $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\Gamma)$ is the generalized Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ . In this paper, we assume that $8\pi G = c = 1$. The equation of motion obtained from the first order Palatini formalism reduces to

$$f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

$$\hat{\nabla}_\alpha (\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \quad (3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g_{\mu\nu}}$ is matter energy-momentum tensor, i.e. one assumes that the matter minimally couples to the metric. As a consequence the energy-momentum tensor is conserved, i.e.: $\nabla^\mu T_{\mu\nu} = 0$ [34]. In Eq. (3) $\hat{\nabla}_\alpha$ means the covariant derivative calculated with respect to Γ . In order to solve Eq. (3) it is convenient to introduce a new metric,

$$\sqrt{h} h_{\mu\nu} = \sqrt{-g} f'(\hat{R}) g_{\mu\nu} \quad (4)$$

for which the connection $\Gamma = \Gamma_{LC}(h)$ is a Levi-Civita connection. As a consequence in $\dim M = 4$ one gets

$$h_{\mu\nu} = f'(\hat{R}) g_{\mu\nu}, \quad (5)$$

i.e. both metrics are related by the conformal factor. For this reason one should assume that the conformal factor $f'(\hat{R}) \neq 0$, so it has strictly positive or negative values.

Taking the trace of (2), we obtain additional so called structural equation

$$f'(\hat{R}) \hat{R} - 2f(\hat{R}) = T. \quad (6)$$

where $T = g^{\mu\nu} T_{\mu\nu}$. Because of cosmological applications we assume that the metric g is FRW metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{1}{1 - kr^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (7)$$

where $a(t)$ is the scale factor, k is a constant of spatial curvature ($k = 0, \pm 1$), t is the cosmological time. For simplicity of presentation we consider the flat model ($k = 0$).

As a source of gravity we assume a perfect fluid, with the energy-momentum tensor

$$T^\mu_\nu = \text{diag}(-\rho, p, p, p), \quad (8)$$

where $p = w\rho$, $w = \text{const}$ is a form of the equation of state ($w = 0$ for dust and $w = 1/3$ for radiation). Formally, effects of the spatial curvature can also be included into the model by introducing a curvature fluid $\rho_k = -\frac{k}{2}a^{-2}$, with the barotropic factor $w = -\frac{1}{3}$ ($p_k = -\frac{1}{3}\rho_k$). From the conservation condition $T^\mu_{\nu;\mu} = 0$ we obtain $\rho = \rho_0 a^{-3(1+w)}$. Therefore the trace T reads

$$T = \sum_i \rho_{i,0} (3w_i - 1) a(t)^{-3(1+w_i)}. \quad (9)$$

In what follows we consider visible and dark matter ρ_m in the form of dust $w = 0$, dark energy ρ_Λ with $w = -1$ and radiation ρ_r with $w = 1/3$.

Because a form of the function $f(\hat{R})$ is unknown, one needs to probe it via ensuing cosmological models. Here we choose the simplest modification of the general relativity Lagrangian,

$$f(\hat{R}) = \hat{R} + \gamma \hat{R}^2, \quad (10)$$

induced by the first three terms in the power series decomposition of an arbitrary function $f(R)$. In fact, since the terms $\hat{R}^n$ have different physical dimensions, i.e. $[\hat{R}^n] \neq [\hat{R}^m]$ for $n \neq m$, one should take instead the function $\hat{R}_0 f(\hat{R}/\hat{R}_0)$ for constructing our Lagrangian, where $\hat{R}_0$ is a constant and $[\hat{R}_0] = [\hat{R}]$. In this case the power series expansion reads $\hat{R}_0 f(\hat{R}/\hat{R}_0) = \hat{R}_0 \sum_{n=0} \alpha_n (\hat{R}/\hat{R}_0)^n = \sum_{n=0} \tilde{\alpha}_n \hat{R}^n$, where the coefficients α_n are dimensionless, while $[\tilde{\alpha}_n] = [\hat{R}]^{1-n}$ are dimension full.

From the other hand the Lagrangian (10) can be viewed as a simplest deviation, by the quadratic Starobinsky term, from the Lagrangian $\hat{R}$ which provides the standard cosmological model a.k.a. Λ CDM model. A corresponding solution of the structural equation (6)

$$\hat{R} = -T \equiv 4\rho_{\Lambda,0} + \rho_{m,0}a^{-3}. \quad (11)$$

is, in fact, exactly the same as for the Λ CDM model, i.e. with $\gamma = 0$. However, the Friedmann equation of the Λ CDM model (with dust matter, dark energy and radiation)

$$H^2 = \frac{1}{3} \left(\rho_{r,0}a^{-4} + \rho_{m,0}a^{-3} + \rho_{\Lambda,0} \right) \quad (12)$$

is now hardly affected by the presence of quadratic term. More exactly a counterpart of the above formula in the model under consideration looks as follows:

$$\begin{aligned} \frac{H^2}{H_0^2} &= \frac{b^2}{(b + \frac{d}{2})^2} \left[\Omega_\gamma (\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0})^2 \right. \\ &\quad \times \frac{(K-3)(K+1)}{2b} + (\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}) \\ &\quad \left. + \frac{\Omega_{r,0}a^{-4}}{b} + \Omega_k \right], \end{aligned} \quad (13)$$

where

$$\Omega_k = -\frac{k}{H_0^2 a^2}, \quad (14)$$

$$\Omega_{r,0} = \frac{\rho_{r,0}}{3H_0^2}, \quad (15)$$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}, \quad (16)$$

$$\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{3H_0^2}, \quad (17)$$

$$K = \frac{3\Omega_{\Lambda,0}}{(\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0})}, \quad (18)$$

$$\Omega_\gamma = 3\gamma H_0^2, \quad (19)$$

$$b = f'(\hat{R}) = 1 + 2\Omega_\gamma (\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0}), \quad (20)$$

$$d = \frac{1}{H} \frac{db}{dt} = -2\Omega_\gamma (\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0})(3 - K) \quad (21)$$

From the above one can check that the standard model (12) can be reconstructed in the limit $\gamma \mapsto 0$. The study of this generalized Friedmann equation is a main subject of our research.

The paper is organized as follows. In Sect. 2, we consider the Palatini approach in the Jordan and Einstein frame. In Sect. 3, we present some generalities concerning dynamical systems of Newtonian type, and their relations with the Palatini–Starobinsky model. Section 4, is devoted to the classification of cosmological singularities with special attention on Newtonian type systems represented by potential function $V(a)$. We adopt the Fernandes–Jambrina and Lazkoz classification of singularities [35] to these systems using the notion of elasticity of the potential function with respect the scale factor. In Sect. 5, we will analyze the singularities in the Starobinsky model in the Palatini formalism. This system requires the form of piecewise-smooth dynamical system. Statistical analysis of the model is presented in Sect. 6. In Sect. 7, we shall summarize obtained results and draw some conclusions.

2 The Palatini approach in different frames (Jordan vs. Einstein frame)

Because the effect of acceleration can depend on a choice of a frame [36] this section is devoted to showing the existence of the inflation effect if the model is considered in the Einstein frame.

The action (1) is dynamically equivalent to the first order Palatini gravitational action, provided that $f''(\hat{R}) \neq 0$ [1, 6, 17]

$$\begin{aligned} S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) &= \frac{1}{2} \int d^4x \sqrt{-g} \left(f'(\chi)(\hat{R} - \chi) + f(\chi) \right) \\ &\quad + S_m(g_{\mu\nu}, \psi), \end{aligned} \quad (22)$$

Introducing a scalar field $\Phi = f'(\chi)$ and taking into account the constraint $\chi = \hat{R}$, one gets the action (22) in the following form:

$$\begin{aligned} S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) &= \frac{1}{2} \int d^4x \sqrt{-g} \left(\Phi \hat{R} - U(\Phi) \right) \\ &\quad + S_m(g_{\mu\nu}, \psi), \end{aligned} \quad (23)$$

where the potential $U(\Phi)$ is defined by

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \quad (24)$$

with $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

The Palatini variation of the action (23) gives rise to the following equations of motion:

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - T_{\mu\nu} = 0, \quad (25a)$$

$$\hat{\nabla}_\lambda (\sqrt{-g} \Phi g^{\mu\nu}) = 0, \quad (25b)$$

$$\hat{R} - U'(\Phi) = 0. \quad (25c)$$

Equation (25b) implies that the connection $\hat{\Gamma}$ is a metric connection for a new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$; thus $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $\bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. The g -trace of (25a) produces a new structural equation

$$2U(\Phi) - U'(\Phi)\Phi = T. \quad (26)$$

Now Eqs. (25a) and (25c) take the following form:

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{U}(\Phi), \quad (27)$$

$$\Phi \bar{R} - (\Phi^2 \bar{U}(\Phi))' = 0, \quad (28)$$

where we introduce $\bar{U}(\Phi) = U(\Phi)/\Phi^2$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$ and the structural equation can be replaced by

$$\Phi \bar{U}'(\Phi) + \bar{T} = 0. \quad (29)$$

In this case, the action for the metric $\bar{g}_{\mu\nu}$ and scalar field Φ is given by

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2} \int d^4x \sqrt{-\bar{g}} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1} \bar{g}_{\mu\nu}, \psi), \quad (30)$$

where we have to take into account a non-minimal coupling between Φ and $\bar{g}_{\mu\nu}$

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} S_m = (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} \bar{g}^{\mu\nu} = \Phi^{-3} T^{\mu\nu}, \quad (31)$$

$\bar{u}^\mu = \Phi^{-\frac{1}{2}} u^\mu$, $\bar{\rho} = \Phi^{-2} \rho$, $\bar{p} = \Phi^{-2} p$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$ (see e.g. [17, 37]).

In FRW case, one gets the metric $\bar{g}_{\mu\nu}$ in the following form:

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(\bar{t}) \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (32)$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}} dt$ and new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$. Ensuing cosmological equations (in the case of the barotropic matter) are given by

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \quad 6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_m(1 + 3w) \quad (33)$$

where

$$\bar{\rho}_\Phi = \frac{1}{2} \bar{U}(\Phi), \quad \bar{\rho}_m = \rho_0 \bar{a}^{-3(1+w)} \Phi^{\frac{1}{2}(3w-1)} \quad (34)$$

and $w = \bar{p}_m/\bar{\rho}_m = p_m/\rho_m$. In this case, the conservation equations has the following form:

$$\dot{\bar{\rho}}_m + 3\bar{H} \bar{\rho}_m(1 + w) = -\dot{\bar{\rho}}_\Phi. \quad (35)$$

Let us consider the Starobinsky–Palatini model in the above framework. The potential $\bar{U}$ is described by the following formula:

$$\bar{U}(\Phi) = 2\bar{\rho}_\Phi(\Phi) = \left(\frac{1}{4\gamma} + 2\lambda \right) \frac{1}{\Phi^2} - \frac{1}{2\gamma} \frac{1}{\Phi} + \frac{1}{4\gamma}. \quad (36)$$

Figure 1 presents the relation $\bar{\rho}_\Phi(\Phi)$. Note that the function $\bar{\rho}_\Phi$ has the same shape like the Starobinsky potential. The function $\bar{\rho}_\Phi(\Phi)$ has the minimum for

$$\Phi_{\min} = 1 + 8\gamma\lambda. \quad (37)$$

In general, the scalar field $\Phi(\bar{a})$ is given by (cf. (11))

$$\Phi = 1 + 2\gamma \hat{R} = 1 + 8\gamma\lambda + 2\gamma\rho_m - 6\gamma p_m. \quad (38)$$

Because $\bar{\rho}_m = \Phi^{-2} \rho_m$, $\bar{p}_m = \Phi^{-2} p_m$, and taking into account (34) one gets

$$2\gamma(1 - 3w)\rho_0 \bar{a}^{-3(1+w)} \Phi^{\frac{3}{2}(w+1)} - \Phi + 1 + 8\gamma\lambda = 0. \quad (39)$$

the algebraic equation determining the function $\Phi(\bar{a})$ for a given barotropic factor w . This provides an implicit dependence $\Phi(\bar{a})$. In order to get it more explicit one needs to solve (39) for some interesting values w . For example in the case of dust we obtain the third order polynomial equation

$$\left(\frac{1}{2\gamma} + 4\lambda \right) y^3 - \frac{1}{2\gamma} y + \rho_{0w} \bar{a}^{-3} = 0$$

where $y = \Phi^{-\frac{1}{2}}$.

The evolution of $\Phi(\bar{a})$, at the beginning of the inflation epoch, is presented in Fig. 2.

For $\gamma \approx 0$, the potential $\bar{U}$ can be approximated as $\bar{U} = -\bar{\rho}_m + \frac{1}{4\gamma}$. In this case the Friedmann equation can be written as

$$3\bar{H}^2 = \frac{\bar{\rho}_m}{2} + \frac{1}{8\gamma}. \quad (40)$$

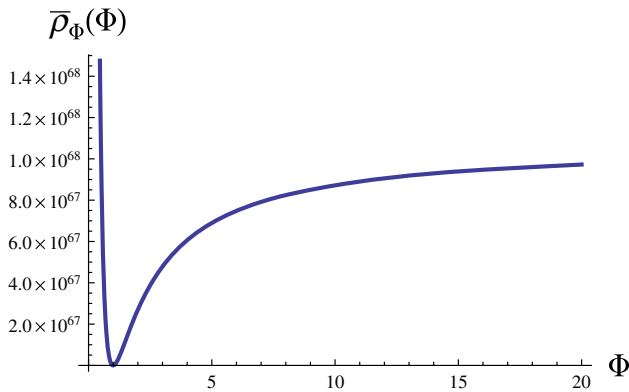


Fig. 1 Illustration of the dependence $\bar{\rho}_\Phi$ of Φ . We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$. $\bar{\rho}_\Phi$ is expressed in units of $\frac{\text{km}^2}{\text{s}^2\text{Mpc}^2}$. Note that this potential has the same shape like the Starobinsky potential

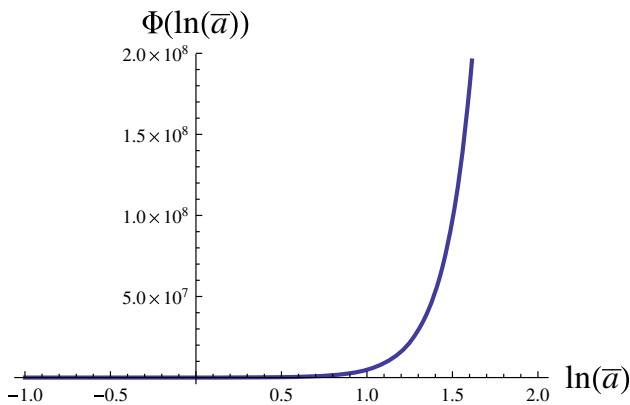


Fig. 2 Illustration of the typical evolution of Φ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch

In the case of $\bar{\rho}_m = 0$, $\bar{\rho}_\Phi$ is constant and the Friedmann equation has the following form:

$$3\bar{H}^2 = \frac{1}{8\gamma}. \quad (41)$$

In this model the inflation phenomenon appears when the value of the parameter γ is close to zero and the matter $\bar{\rho}_m$ is negligible with comparison to $\bar{\rho}_\Phi$. In this case the approximate number of e-foldings is given by the following formula:

$$N = H_{\text{init}}(\bar{t}_{\text{fin}} - \bar{t}_{\text{init}}) = \frac{\bar{t}_{\text{fin}} - \bar{t}_{\text{init}}}{\sqrt{24\gamma}}. \quad (42)$$

The number of e-folds N should be equal $50 \sim 60$ in the inflation epoch [10]. In this model we obtain $N = 60$, when $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and the timescale of the inflation is equal 10^{-32} s [38]. The relation between γ and the approximate number of e-foldings N is presented in Fig. 3.

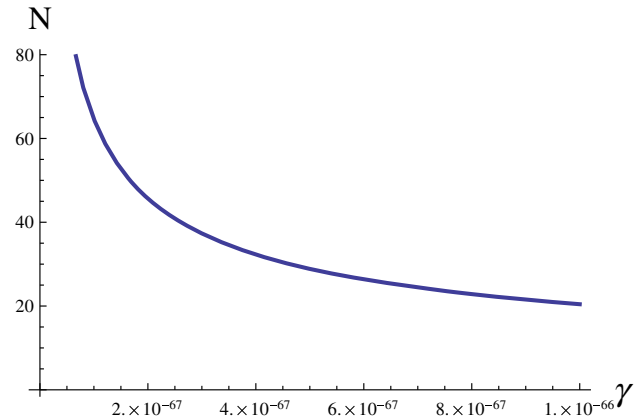


Fig. 3 The diagram of the relation between γ and the approximate number of e-foldings $N = \bar{H}_{\text{init}}(\bar{t}_{\text{fin}} - \bar{t}_{\text{init}})$ from $\bar{t}_{\text{init}}$ to $\bar{t}_{\text{fin}}$. We assume that $\bar{t}_{\text{fin}} - \bar{t}_{\text{init}} \approx 10^{-32} \text{ s}$. The parameter γ is expressed in units of s^2 . Note that the number of e-foldings grows when the parameter γ decreases and $N = 60$ when $\gamma = 1.16 \times 10^{-69} \text{ s}^2$

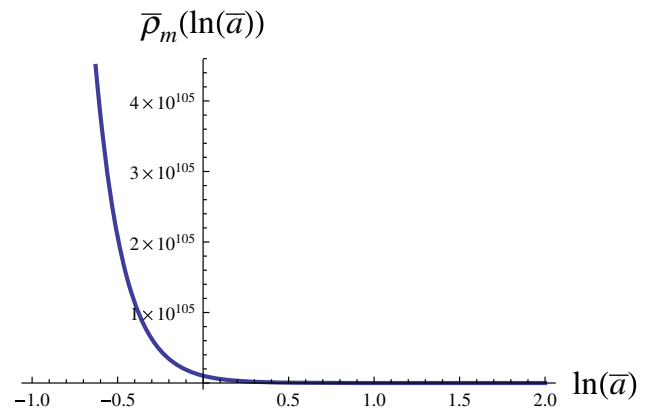


Fig. 4 Illustration of the typical evolution of $\bar{\rho}_m$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. $\bar{\rho}_m$ is expressed in units of $\frac{\text{km}^2}{\text{s}^2\text{Mpc}^2}$

The condition for appearing of the inflation is for the value of the parameter γ to be close to zero, hence the influence of the parameter λ on the evolution of the universe is negligible.

In Fig. 4 the typical evolution is demonstrated of $\bar{\rho}_m(\bar{a})$ at the beginning of the inflation epoch. The typical evolution of $\bar{\rho}_\Phi$, at the beginning of the inflation epoch, is presented in Fig. 5. Note that, for the late time universe, $\bar{\rho}_\Phi$ can be approximated as a constant. Figure 6 presents the evolution of the scale factor $\bar{a}(\bar{t})$ during the inflation. Figure 7 shows the Hubble function $\bar{H}$ during the inflation epoch.

The conservation equation for $\bar{\rho}_\Phi$ can be written

$$\dot{\bar{\rho}}_\Phi = -3\bar{H}(\bar{\rho}_\Phi + \bar{p}_\Phi), \quad (43)$$

where $\bar{p}_\Phi$ is an effective pressure. In this case the equation of state for the dark energy is expressed by the following formula:

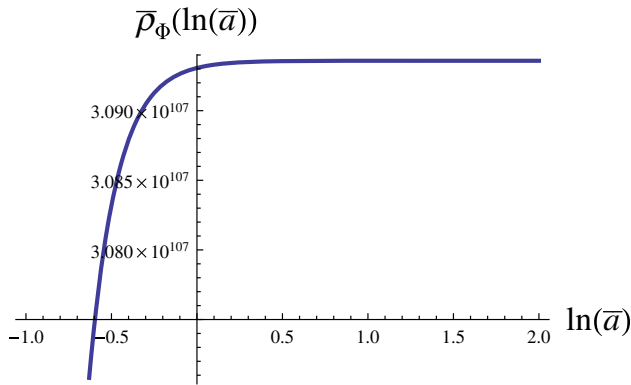


Fig. 5 Illustration of the typical evolution of $\bar{\rho}_\phi$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. $\bar{\rho}_\phi$ is expressed in units of $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. Note that during the inflation $\bar{\rho}_\phi \approx \text{const}$

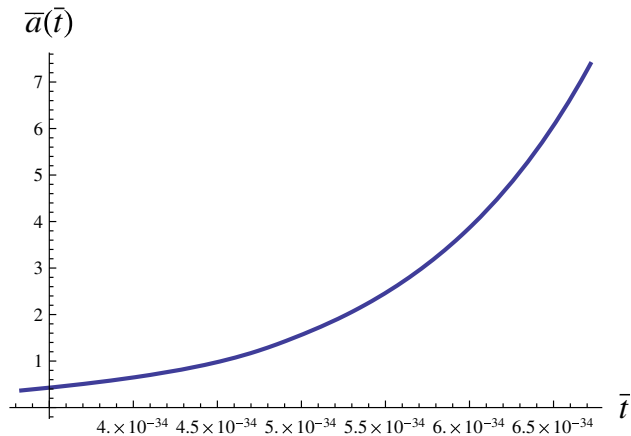


Fig. 6 Illustration of the typical evolution of $\bar{a}$ with respect to $\bar{t}$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. The time $\bar{t}$ is expressed in seconds

$$\bar{\rho}_\Phi = w(a)\bar{\rho}_\Phi, \quad (44)$$

where the function $w(a)$ is given by the expression

$$w(a) = -1 - \frac{\dot{\bar{\rho}}_\Phi}{\sqrt{3}\sqrt{\bar{\rho}_m + \bar{\rho}_\Phi\rho_\Phi}} = -1 - \frac{1}{3\bar{H}} \frac{d \ln \rho_\Phi}{d\bar{t}}. \quad (45)$$

The diagram of the coefficient of equation of state $w(a)$, at the beginning the inflation epoch, is presented in Fig. 8. Note that the function $w(a)$, for the late time, is a constant and equal -1 .

The action (23) can be rewritten in the Jordan frame $(g_{\mu\nu}, \Phi)$ as

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} \left(\Phi R + \frac{3}{2\Phi} \partial_\mu \Phi \partial^\mu \Phi - U(\Phi) \right), \quad (46)$$

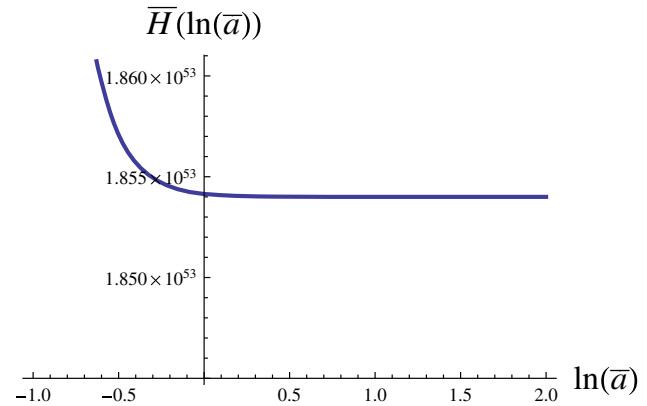


Fig. 7 Illustration of the typical evolution of $\bar{H}$ with respect to $\ln(\bar{a})$ at the beginning of the inflation epoch. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. $\bar{H}$ is expressed in units of $\frac{\text{km}}{\text{s Mpc}}$. Note that, for the late time, $\bar{H}$ can be treated as a constant

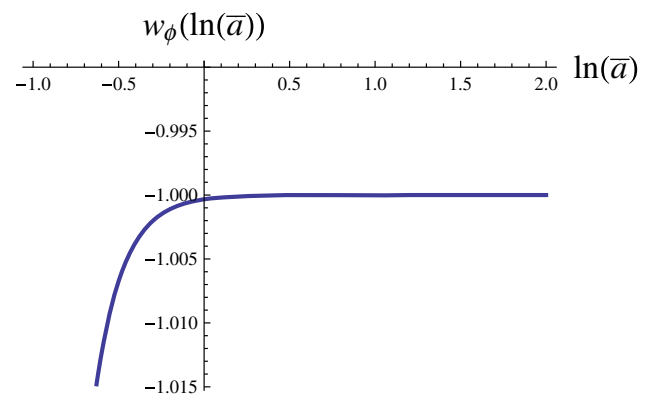


Fig. 8 Illustration of the typical evolution of w_ϕ with respect to $\ln(\bar{a})$. We assume that $\gamma = 1.16 \times 10^{-69} \text{ s}^2$ and $\bar{a}_0 = 1$ at the beginning of the inflation epoch. Note that during the inflation $w_\phi \approx -1$

where R is the metric Ricci scalar, $\Phi = f'(\hat{R})$, $\hat{R} = \chi(\Phi)$.

We obtain the Brans–Dicke action with the coupling parameter $\omega = -\frac{3}{2}$ in the Jordan frame. The equations of motion take the form

$$\Phi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{3}{4\Phi} g_{\mu\nu} \nabla_\sigma \Phi \nabla^\sigma \Phi + \frac{3}{2\Phi} \nabla_\mu \Phi \nabla_\nu \Phi + g_{\mu\nu} \square \Phi - \nabla_\mu \nabla_\nu \Phi + \frac{1}{2} g_{\mu\nu} U(\phi) - \kappa T_{\mu\nu} = 0, \quad (47a)$$

$$R - \frac{3}{\Phi} \square \Phi + \frac{3}{2\Phi^2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{2} U'(\Phi) = 0. \quad (47b)$$

In this case the dynamics of the metric g is exactly the same as described by the original Palatini equations (2)–(6). On cosmological grounds it means that the scale factor $a(t)$ evolves according to the Friedmann equation (13). It has recently been shown that cosmological data favor the value $\omega \approx -1$ on the 3σ level [39].

3 Singularities in cosmological dynamical systems of Newtonian type

There is a class of cosmological models, of which the dynamics can be reduced to a dynamical system of the Newtonian type. Let consider a homogeneous and isotropic universe with a spatially flat space-time metric of the form

$$ds^2 = dt^2 - a^2(t) \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (48)$$

where $a(t)$ is the scale factor and t is the cosmological time.

Let us consider the energy-momentum tensor T_{ν}^{μ} for the perfect fluid with energy density $\rho(t)$ and pressure $p(t)$ as a source of gravity. In this case the Einstein equations assume the form of the Friedmann equations,

$$\rho = 3H^2 = \frac{3\dot{a}^2}{a^2}, \quad (49)$$

$$p = -\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}, \quad (50)$$

where dot denotes differentiation with respect to the cosmic time t , $H \equiv \frac{\dot{a}}{a}$ is the Hubble function. In our notation we use the natural system of units in which $8\pi G = c = 1$.

We assume $\rho(t) = \rho(a(t))$ as well as $p(t) = p(a(t))$, i.e. both energy density and pressure depend on the cosmic time through the scale factor $a(t)$. The conservation condition $T_{;\mu}^{\mu\nu} = 0$ reduces to

$$\dot{\rho} = -3H(\rho + p). \quad (51)$$

It would be convenient to rewrite (49) in the equivalent form

$$\dot{a}^2 = -2V(a), \quad (52)$$

where

$$V(a) = -\frac{\rho(a)a^2}{6}. \quad (53)$$

In (53) $\rho(a)$ plays the model role of an effective energy density. For example for the standard cosmological model (12)

$$V = -\frac{\rho_{\text{eff}} a^2}{6} = -\frac{a^2}{6} \left(\rho_{m,0} a^{-3} + \rho_{\Lambda,0} \right), \quad (54)$$

where $\rho_{\text{eff}} = \rho_m + \Lambda$ and $\rho_m = \rho_{m,0} a^{-3}$. Equation (50) is equivalent to

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p), \quad (55)$$

which is called the acceleration equation. It is easily to check that

$$\ddot{a} = -\frac{\partial V}{\partial a}, \quad (56)$$

where $V(a)$ is given by (53) provided that the conservation equation (51) is fulfilled.

Due to Eq. (56) the evolution of the universe can be interpreted as the motion of a fictitious particle of unit mass in the potential $V(a)$. Here $a(t)$ plays the role of a position variable. The equation of motion (56) assumes a form analogous to the Newtonian equation of motion.

If we know the form of the effective energy density then we can construct the form of the potential $V(a)$, which determines the whole dynamics in the phase space $(a, \dot{a})$. In this space the Friedmann equation (52) plays the role of a first integral and determines the phase space curves representing the evolutionary paths of the cosmological models. The diagram of the potential $V(a)$ contains all information needed to construction a phase space portrait. In this case the phase space is two-dimensional,

$$\left\{ (a, \dot{a}) : \frac{\dot{a}^2}{2} + V(a) = -\frac{k}{2} \right\}. \quad (57)$$

In the general case of an arbitrary potential, the dynamical system which describes the evolution of a universe takes the form

$$\dot{a} = x, \quad (58)$$

$$\dot{x} = -\frac{\partial V(a)}{\partial a}. \quad (59)$$

We shall study the system above using the theory of piecewise-smooth dynamical systems. Therefore it is assumed that the potential function, except some isolated (singular) points, belongs to the class $C^2(\mathbb{R}_+)$.

The lines $\frac{\dot{a}^2}{2} + V(a) = -\frac{k}{2}$ represent possible evolutions of the universe for different initial conditions. Equations (58) and (59) can be rewritten in terms of dimensionless variables if we replace the effective energy density ρ_{eff} by the density parameter:

$$\Omega_{\text{eff}} = \frac{\rho_{\text{eff}}}{3H_0^2}. \quad (60)$$

Then

$$\frac{1}{H_0^2} \frac{\dot{a}^2}{2} = -\frac{\Omega_{\text{eff}} a^2}{2}, \quad (61)$$

$$\frac{d^2 a}{d\tau^2} = -\frac{\partial \tilde{V}}{\partial a}, \quad (62)$$

where $t \rightarrow \tau = |H_0|t$ and

$$\tilde{V}(a) = -\frac{\Omega_{\text{eff}} a^2}{2}. \quad (63)$$

Any cosmological model can be identified by its form of the potential function $V(a)$ depending on the scale factor a . From the Newtonian form of the dynamical system (58)–(59) one can see that all critical points correspond to vanishing of r.h.s. of the dynamical system $(x_0 = 0, \frac{\partial V(a)}{\partial a}|_{a=a_0})$. Therefore all critical points are localized on the x -axis, i.e. they represent a static universe.

Because of the Newtonian form of the dynamical system the character of critical points is determined from the characteristic equation of the form

$$a^2 + \det A|_{x_0=0, \frac{\partial V(a)}{\partial a}|_{a_0}=0} = 0, \quad (64)$$

where $\det A$ is the determinant of the linearization matrix calculated at the critical points, i.e.

$$\det A = \frac{\partial^2 V(a)}{\partial a^2}|_{a_0, \frac{\partial V(a)}{\partial a}|_{a_0}=0}. \quad (65)$$

From Eqs. (64) and (65) one can conclude that only admissible critical points are of saddle type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} < 0$ or of center type if $\frac{\partial^2 V(a)}{\partial a^2}|_{a=a_0} > 0$.

If the shape of the potential function is known (from knowledge of the effective energy density), then it is possible to calculate the cosmological functions in exact form,

$$t = \int^a \frac{da}{\sqrt{-2V(a)}}, \quad (66)$$

$$H(a) = \pm \sqrt{-\frac{2V(a)}{a^2}}, \quad (67)$$

the deceleration parameter, the effective barotropic factor

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{1}{2} \frac{d \ln(-V)}{d \ln a}, \quad (68)$$

$$w_{\text{eff}}(a(t)) = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{1}{3} \left(\frac{d \ln(-V)}{d \ln a} + 1 \right), \quad (69)$$

the parameter of deviation from de Sitter universe [35]

$$h(t) \equiv -(q(t) + 1) = \frac{1}{2} \frac{d \ln(-V)}{d \ln a} - 1 \quad (70)$$

(note that if $V(a) = -\frac{\Lambda a^2}{6}$, $h(t) = 0$), the effective matter density and pressure

$$\rho_{\text{eff}} = -\frac{6V(a)}{a^2}, \quad (71)$$

$$p_{\text{eff}} = \frac{2V(a)}{a^2} \left(\frac{d \ln(-V)}{d \ln a} + 1 \right), \quad (72)$$

and, finally, the Ricci scalar curvature for the FRW metric (48),

$$R = \frac{6V(a)}{a^2} \left(\frac{d \ln(-V)}{d \ln a} + 2 \right). \quad (73)$$

From the formulas above one can observe that the most of them depend on the quantity

$$I_v(a) = \frac{d \ln(-V)}{d \ln a}. \quad (74)$$

This quantity measures the elasticity of the potential function, i.e. indicates how the potential $V(a)$ changes if the scale factor a changes. For example, for the de Sitter universe $-V(a) \propto a^2$, the rate of growth of the potential is 2% as the rate of growth of the scale factor is 1%.

In the classification of the cosmological singularities by Fernandez-Jambrina and Lazkoz [35] a crucial role is played by the parameter $h(t)$. Note that in a cosmological sense this parameter is

$$h(t) = \frac{1}{2} I_v(a) - 1. \quad (75)$$

In this approach the classification of singularities is based on generalized power and asymptotic expansion of the barotropic index w in the equation of state (or equivalently of the deceleration parameter q) in terms of the time coordinate.

4 Degenerated singularities—new type (VI) of singularity—sewn singularities

Recently, due to the discovery of an accelerated phase in the expansion of our universe, many theoretical possibilities for future singularities are seriously considered. If we assume that the universe expands following the Friedmann equation, then this phase of expansion is driven by dark energy—a hypothetical fluid, which violates the strong energy condition. Many of the new types of singularities were classified by Nojiri et al. [40]. Following their classification the type of singularity depends on the singular behavior of the cosmological quantities like the scale factor a , the Hubble parameter H , the pressure p and the energy density ρ :

- Type 0: ‘Big crunch’. In this type, the scale factor a is vanishing and there is blow-up of the Hubble parameter H , energy density ρ and pressure p .

- Type I: ‘Big rip’. In this type, the scale factor a , energy density ρ and pressure p are blown up.
- Type II: ‘Sudden’. The scale factor a , energy density ρ and Hubble parameter H are finite and $\dot{H}$ and the pressure p are divergent.
- Type III: ‘Big freeze’. The scale factor a is finite and the Hubble parameter H , energy density ρ and pressure p are blown up [41] or divergent [42].
- Type IV. The scale factor a , Hubble parameter H , energy density ρ , pressure p and $\dot{H}$ are finite but higher derivatives of the scale factor a diverge.
- Type V. The scale factor a is finite but the energy density ρ and pressure p vanish.

Following Królak [43], big rip and big crunch singularities are strong whereas sudden, big freeze and type IV are weak singularities.

In the model under consideration the potential function and/or its derivative can diverge at isolated points (value of the scale factor). Therefore the classification mentioned before has application only for a single component of piecewise-smooth potential. In our model the dynamical system describing the evolution of a universe belongs to the class of a piecewise-smooth dynamical systems. As a consequence new types of singularities at finite scale factor a_s can appear for which $\frac{\partial V}{\partial a}(a_s)$ does not exist (is not determined). This implies that the classification of singularities should be extended to the case of non-isolated singularities.

Let us illustrate this idea on the example of a freeze singularity in the Starobinsky model with the Palatini formalism (previous section). Such a singularity has a complex character and in analogy to the critical point we called it degenerate. Formally it is composed of two types III singularities: one in the future and another one in the past. If we consider the evolution of the universe before this singularity we detect an isolated singularity of type III in the future. Conversely if we consider the evolution after the singularity, then going back in time we meet a type III singularity in the past. Finally, at the finite scale factor the two singularities will meet. For a description of behavior near the singularity one considers the $t = t(a)$ relation. This relation has a horizontal inflection point and it is natural to expand this relation in a Taylor series near this point at which $\frac{dt}{da} = \frac{1}{Ha}$ is zero. For the freeze singularity, the scale factor remains constant a_s , ρ and H blow up and $\ddot{a}$ is undefined. In this case, the degenerate singularity of type III is called sewn (non-isolated) singularity. We, therefore, obtain [44]

$$t - t_s \simeq \pm \frac{1}{2} \frac{d^2 t}{da^2} \big|_{a=a_{\text{sing}}} (a - a_{\text{sing}})^2. \quad (76)$$

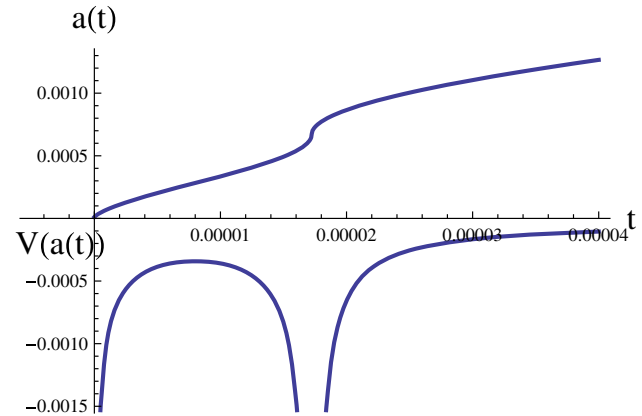


Fig. 9 Illustration of sewn freeze singularity, when the potential $V(a)$ has a pole

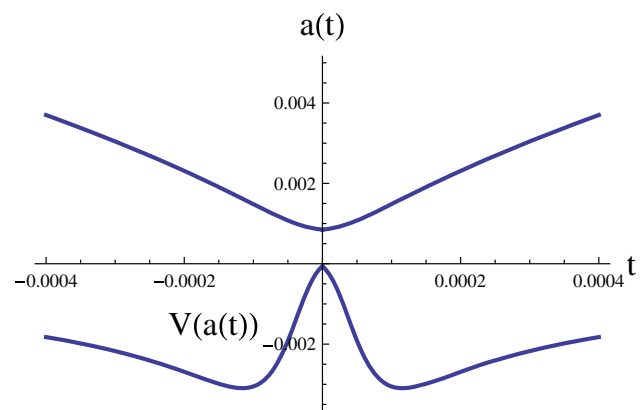


Fig. 10 Illustration of a sewn sudden singularity. The model with negative Ω_γ has a mirror symmetry with respect to the cosmological time. Note that the spike on the diagram shows a discontinuity of the function $\frac{\partial V}{\partial a}$. Note the existence of a bounce at $t = 0$

The above formula combines two types of behavior near the freeze singularities in the future,

$$a - a_{\text{sing}} \propto -(t_{\text{sing}} - t)^{1/2} \text{ for } t \rightarrow t_{\text{sing}}^- \quad (77)$$

and in the past

$$a - a_{\text{sing}} \propto +(t - t_{\text{sing}})^{1/2} \text{ for } t \rightarrow t_{\text{sing}}^+. \quad (78)$$

Figure 9 illustrates the behavior of the scale factor in cosmological time in neighborhood of a pole of the potential function. Diagram of $a(t)$ is constructed from the dynamics in two disjoint region $\{a: a < a_s\}$ and $\{a: a > a_s\}$. Figure 10 presents the behavior of the scale factor in the cosmological time in a neighborhood of the sudden singularity.

In the model under consideration another type of sewn singularity also appears. It is a composite singularity with two sudden singularities glued at the bounce when $a = a_{\text{min}}$. In this singularity the potential itself is a continuous func-

tion while its first derivative has a discontinuity. Therefore, the corresponding dynamical system represents a piecewise-smooth dynamical system.

The problem of C^0 metric extension beyond the future Cauchy horizon, when the second derivative of the metric is inextendible, was discussed in work of Sbierski [45]. In the context of FLRW cosmological models, Sbierski's methodology was considered in [46].

5 Singularities in the Starobinsky model in the Palatini formalism

In our model, one finds two types of singularities, which are a consequence of the Palatini formalism: the freeze and sudden singularity. The freeze singularity appears when the multiplicative expression $\frac{b}{b+d/2}$, in the Friedmann equation (13), is equal to infinity. So we get a condition for the freeze singularity: $2b+d=0$, which produces a pole in the potential function. It appears that the sudden singularity appears in our model when the multiplicative expression $\frac{b}{b+d/2}$ vanishes. This condition is equivalent to the case $b=0$.

The freeze singularity in our model is a solution of the algebraic equation

$$2b+d=0 \implies f(K, \Omega_{\Lambda,0}, \Omega_\gamma) = 0 \quad (79)$$

or

$$-3K - \frac{K}{3\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})\Omega_{\Lambda,0}} + 1 = 0, \quad (80)$$

where $K \in [0, 3)$.

The solution of the above equation is

$$K_{\text{freeze}} = \frac{1}{3 + \frac{1}{3\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})\Omega_{\Lambda,0}}}. \quad (81)$$

From Eq. (81), we can find an expression for a value of the scale factor for the freeze singularity

$$a_{\text{freeze}} = \left(\frac{1 - \Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})}} \right)^{\frac{1}{3}}. \quad (82)$$

The relation between a_{freeze} and positive Ω_γ is presented in Fig. 11.

The sudden singularity appears when $b=0$. This leaves us with the following algebraic equation:

$$1 + 2\Omega_\gamma(\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0})(K+1) = 0. \quad (83)$$

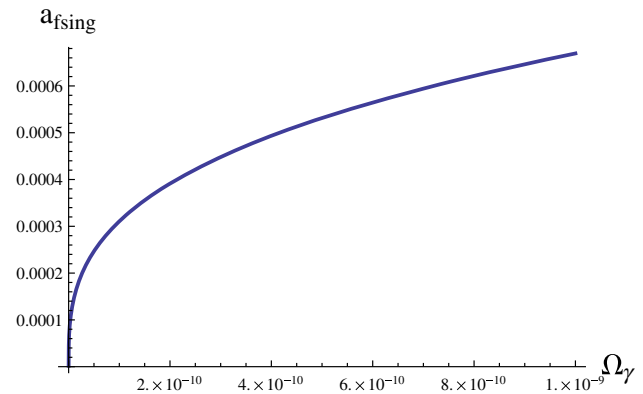


Fig. 11 Diagram of the relation between a_{sing} and positive Ω_γ . Note that in the limit $\Omega_\gamma \mapsto 0$ the singularity overlaps with a big-bang singularity

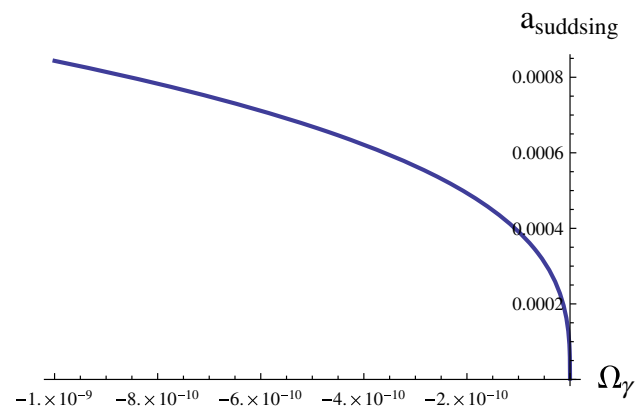


Fig. 12 Diagram of the relation between a_{sing} and negative Ω_γ . Note that in the limit $\Omega_\gamma \mapsto 0$ the singularity overlaps with a big-bang singularity

The above equation can be rewritten as

$$1 + 2\Omega_\gamma(\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0}) = 0. \quad (84)$$

From Eq. (84), we have the formula for the scale factor for a sudden singularity,

$$a_{\text{sudden}} = \left(-\frac{2\Omega_{m,0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}} \right)^{1/3}, \quad (85)$$

which, in fact, becomes a (degenerate) critical point and a bounce at the same time. The relation between a_{sing} and negative Ω_γ is presented in Fig. 12.

Let $V = -\frac{a^2}{2} \left(\Omega_\gamma \Omega_{\text{ch}}^2 \frac{(K-3)(K+1)}{2b} + \Omega_{\text{ch}} + \Omega_k \right)$. We can rewrite dynamical system (58)–(59) as

$$a' = x, \quad (86)$$

$$x' = -\frac{\partial V(a)}{\partial a}, \quad (87)$$

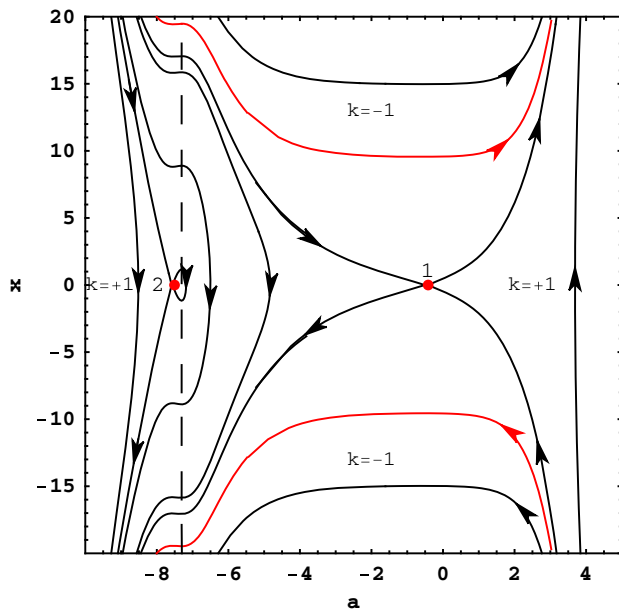


Fig. 13 The figure represents the phase portrait of the system (86)–(87) for positive Ω_γ . The scale factor a is in the logarithmic scale. The red trajectories represent the spatially flat universe. Trajectories under the top red trajectory and below the bottom red trajectory represent models with the negative spatial curvature. Trajectories between the top and bottom red trajectory represent models with the positive spatial curvature. The dashed line $2b + d = 0$ corresponds to the freeze singularity. The critical points (1) and (2) present two static Einstein universes. The phase portrait belongs to the class of sewn dynamical systems [49]

where $' \equiv \frac{d}{d\sigma} = \frac{b+\frac{d}{2}}{b} \frac{d}{d\tau}$ is a new parametrization of time.

We can treat the dynamical system (86)–(87) as a sewn dynamical system [47,48]. In this case, we divide the phase portrait into two parts: the first part is for $a < a_{\text{sing}}$ and the second part is for $a > a_{\text{sing}}$. Both parts are glued along the singularity.

For $a < a_{\text{sing}}$, dynamical system (86)–(87) can be rewritten in the corresponding form,

$$a' = x, \quad (88)$$

$$x' = -\frac{\partial V_1(a)}{\partial a}, \quad (89)$$

where $V_1 = V(-\eta(a - a_s) + 1)$ and $\eta(a)$ notes the Heaviside function.

For $a > a_{\text{sing}}$, in an analogous way, we get the following equations:

$$a' = x, \quad (90)$$

$$x' = -\frac{\partial V_2(a)}{\partial a}, \quad (91)$$

where $V_2 = V\eta(a - a_s)$. The phase portraits, for dynamical system (86)–(87), are presented in Figs. 13 and 14. Figure

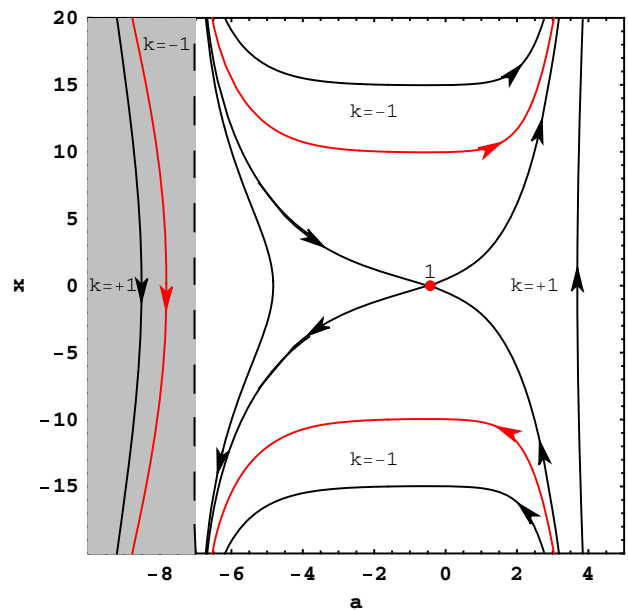


Fig. 14 The phase portrait of the system (86)–(87) for negative Ω_γ . The scale factor a is in logarithmic scale. The red trajectories represent a spatially flat universe. Trajectories under the top red trajectory and below the bottom red trajectory represent models with a negative spatial curvature. Trajectories between the top and bottom red trajectory represent models with the positive spatial curvature. The dashed line $b = 0$ corresponds to the sudden singularity. The shaded region represents trajectories with $b < 0$. If we assume that $f'(R) > 0$ then this region can be removed. The phase portrait possesses the symmetry $\dot{a} \rightarrow -\dot{a}$ and in consequence this singularity presents a bounce. This symmetry can be used to identify the corresponding points on the b -line. The critical point (1) represents the static Einstein universe. The phase portrait belongs to the class of sewn dynamical systems [49]

13 shows the phase portrait for positive Ω_γ , while Fig. 14 shows the phase portrait for negative Ω_γ .

In Fig. 13 there are two critical points labeled '1' and '2' at the finite domain. They are both saddle points. These critical points correspond to a maximum of the potential function. The saddle point '2' possesses the homoclinic closed orbit starting from it and returning to it. This orbit represents an emerging universe from the static Einstein universe and approaching it again. During the evolution this universe (orbit) goes two times through the freeze singularity. The region bounded by the homoclinic orbit contains closed orbits representing the oscillating universes. A diagram of the evolution of scale factor for closed orbit is presented by Fig. 15. It is also interesting that trajectories in the neighborhood of straight vertical line of freeze singularities undergo short time inflation $x = \text{const}$. The characteristic number of e-foldings from t_{init} to t_{fin} of this inflation period $N = H_{\text{init}}(t_{\text{fin}} - t_{\text{init}})$ (see Eq. (3.13) in [1]) with respect to Ω_γ is shown in Fig. 16. This figure illustrates the number of e-foldings is too small to obtain the inflation effect.

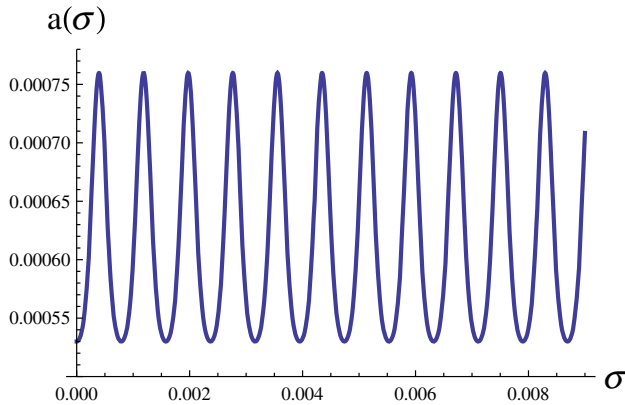


Fig. 15 Illustration of the evolution of $a(\sigma)$ for closed orbit which is contained by the homoclinic orbit, where $\sigma = \frac{b}{b+\frac{d}{2}}t$ is a reparametrization of time. We choose $\text{sMpc}/(100 \times \text{km})$ as a unit of σ

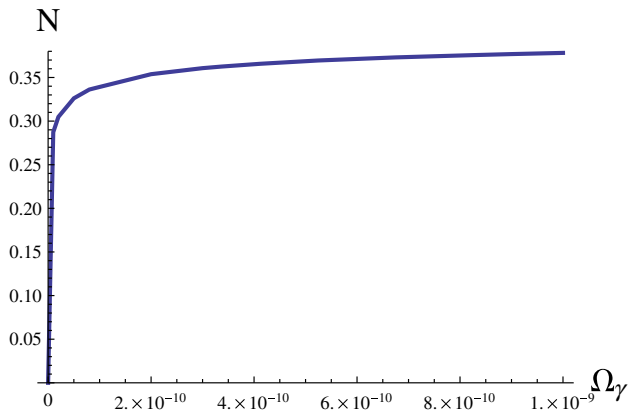


Fig. 16 Diagram of the relation between positive Ω_γ and the approximate number of e-foldings $N = H_{\text{init}}(t_{\text{fin}} - t_{\text{init}})$ from t_{init} to t_{fin}

6 Observations

In this paper we perform statistical analysis using the following astronomical observations: observations of 580 supernovae of type Ia, BAO, measurements of $H(z)$ for galaxies, Alcock–Paczyński test, measurements of CMB and lensing by Planck and low ℓ by WMAP.

The likelihood function for observations of supernovae of type Ia [50] is given by the following expression:

$$\ln L_{\text{SNIa}} = -\frac{1}{2}[A - B^2/C + \ln(C/(2\pi))], \quad (92)$$

where $A = (\mu^{\text{obs}} - \mu^{\text{th}})\mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $B = \mathbb{C}^{-1}(\mu^{\text{obs}} - \mu^{\text{th}})$, $C = \text{Tr}\mathbb{C}^{-1}$ and $\mathbb{C}$ is a covariance matrix for observations of supernovae of type Ia. The distance modulus is defined by the formula $\mu^{\text{obs}} = m - M$ (where m is the apparent magnitude and M is the absolute magnitude of observations of supernovae of type Ia) and $\mu^{\text{th}} = 5 \log_{10} D_L + 25$ (where the luminosity distance is $D_L = c(1+z) \int_0^z \frac{dz'}{H(z')}$).

BAO observations such as Sloan Digital Sky Survey Release 7 (SDSS DR7) dataset at $z = 0.275$ [51], 6dF Galaxy Redshift Survey measurements at redshift $z = 0.1$ [52], and WiggleZ measurements at redshift $z = 0.44, 0.60, 0.73$ [53] have the following likelihood function:

$$\ln L_{\text{BAO}} = -\frac{1}{2} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right) \mathbb{C}^{-1} \left(\mathbf{d}^{\text{obs}} - \frac{r_s(z_d)}{D_V(\mathbf{z})} \right), \quad (93)$$

where $r_s(z_d)$ is the sound horizon at the drag epoch [54,55].

For the Alcock–Paczynski test [56,57] we used the following expression for the likelihood function:

$$\ln L_{\text{AP}} = -\frac{1}{2} \sum_i \frac{(AP^{\text{th}}(z_i) - AP^{\text{obs}}(z_i))^2}{\sigma^2}. \quad (94)$$

where $AP(z)^{\text{th}} \equiv \frac{H(z)}{z} \int_0^z \frac{dz'}{H(z')}$ and $AP(z_i)^{\text{obs}}$ are observational data [58–66].

The likelihood function for measurements of the Hubble parameter $H(z)$ of galaxies from [67–69] is given by the expression

$$\ln L_{H(z)} = -\frac{1}{2} \sum_{i=1}^N \left(\frac{H(z_i)^{\text{obs}} - H(z_i)^{\text{th}}}{\sigma_i} \right)^2. \quad (95)$$

In this paper, we use the likelihood function for observations of CMB [9] and lensing by Planck, and low- ℓ polarization from the WMAP (WP) in the following form:

$$\ln L_{\text{CMB+lensing}} = -\frac{1}{2} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}) \mathbb{C}^{-1} (\mathbf{x}^{\text{th}} - \mathbf{x}^{\text{obs}}), \quad (96)$$

where $\mathbb{C}$ is the covariance matrix with the errors, $\mathbf{x}$ is a vector of the acoustic scale l_A , the shift parameter R and $\Omega_b h^2$ where

$$l_A = \frac{\pi}{r_s(z^*)} c \int_0^{z^*} \frac{dz'}{H(z')} \quad (97)$$

$$R = \sqrt{\Omega_{m,0} H_0^2} \int_0^{z^*} \frac{dz'}{H(z')}, \quad (98)$$

where z^* is the redshift of the epoch of the recombination [54].

The total likelihood function is expressed in the following form:

$$L_{\text{tot}} = L_{\text{SNIa}} L_{\text{BAO}} L_{\text{AP}} L_{H(z)} L_{\text{CMB+lensing}}. \quad (99)$$

To estimate model parameters, we use our own code CosmoDarkBox. The Metropolis–Hastings algorithm [70,71] is used in this code.

Table 1 The best fit and errors for the estimated model for the positive Ω_γ with $\Omega_{m,0}$ from the interval (0.27, 0.33), Ω_γ from the interval (0.0, 2.6×10^{-9}) and H_0 from the interval [66.0 (km/(s Mpc)), 70.0 (km/(s Mpc))]. $\Omega_{b,0}$ is assumed as 0.048468. The redshift of matter–

radiation equality is assumed as 3395. H_0 , in the table, is expressed in km/(s Mpc). The value of reduced χ^2 of the best fit of our model is equal 0.187066 (for the Λ CDM model 0.186814)

Parameter	Best fit	68% CL	95% CL
H_0	68.10	+1.07 −1.24	+1.55 1.82
$\Omega_{m,0}$	0.3011	+0.0145 −0.0138	+0.0217 −0.0201
Ω_γ	9.70×10^{-11}	$+1.3480 \times 10^{-9}$ -9.70×10^{-11}	$+2.2143 \times 10^{-9}$ -9.70×10^{-11}

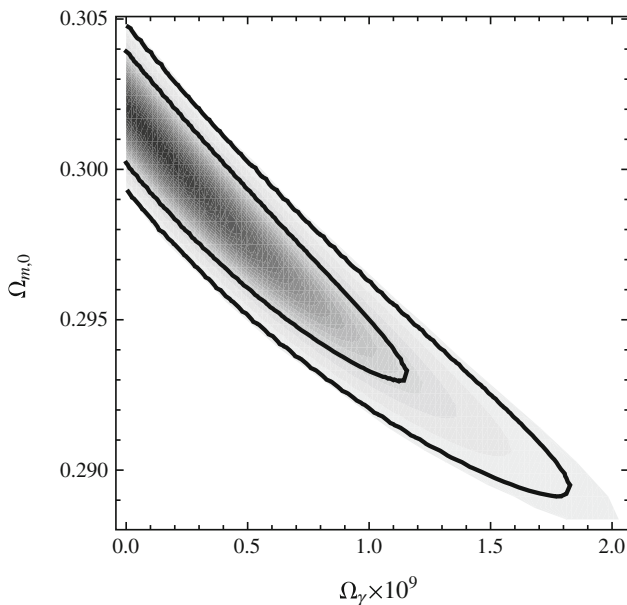


Fig. 17 The intersection of the likelihood functions of two model parameters (Ω_γ , $\Omega_{m,0}$) with the marked 68 and 95% confidence levels

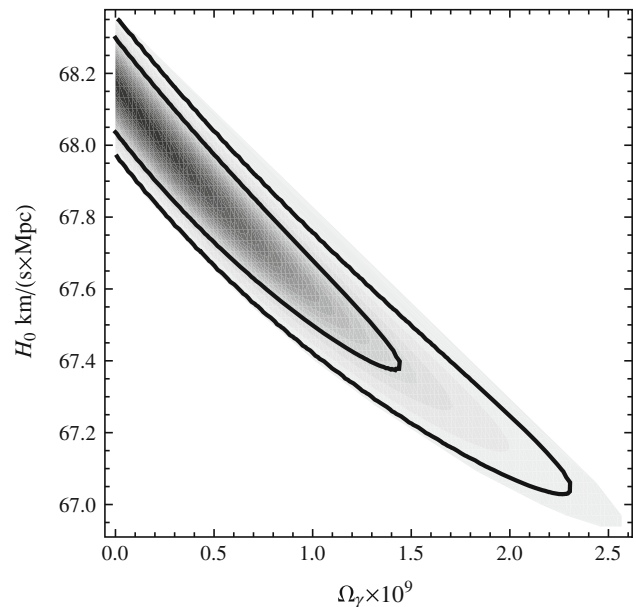


Fig. 18 The intersection of the likelihood functions of two model parameters (Ω_γ , H_0) with the marked 68 and 95% confidence levels

Table 1 shows the values of parameters for the best fit with errors. Figures 17 and 18 show the intersection of a likelihood function with the 68 and 95% confidence level projections on the (Ω_γ , $\Omega_{m,0}$) and (Ω_γ , H_0) planes.

In this paper, we use the Bayesian information criterion (BIC) [72, 73], for comparison of our model with the Λ CDM model. The expression for BIC is defined as

$$\text{BIC} = \chi^2 + j \ln n, \quad (100)$$

where χ^2 is the value of χ^2 in the best fit, j is the number of model parameters (our model has three parameters, Λ CDM model has two parameters) and n is the number of data points ($n = 625$) which are used in the estimation.

For our model, the value of BIC is equal 135.668 and for the Λ CDM model $\text{BIC}_{\Lambda\text{CDM}} = 129.261$. So $\Delta\text{BIC} = \text{BIC} - \text{BIC}_{\Lambda\text{CDM}}$ is equal 6.407. The evidence for the model is

strong [73] if ΔBIC is higher than 6. So, in comparison to our model, the evidence in favor of the Λ CDM model is strong, but we cannot absolutely reject our model.

7 Conclusions

In this paper, we demonstrated that evolution of the Starobinsky model with a quadratic term R^2 gives rise to the description of dynamics in terms of piecewise-smooth dynamical systems, i.e., systems whose the phase space is partitioned into different regions, each of them associated to a different smooth functional form of the system of a Newtonian type. Different regions of the phase space correspond to different forms of the potential separated by singularities of the type of poles.

Our idea was to obtain inflation as an endogenous effect of the dynamics in the Palatini formalism. While the effect of

inflation appears in the model under consideration a sufficient number of e-folds are not achieved and the additional effect of amplification is required. Note that this type of inflation is a realization of the idea of singular inflation [74–77]. In our model inflation is driven by the freeze degenerate singularity (the extension of a type III isolated singularity).

We show that the dynamics of the model can be analyzed in terms of two-dimensional dynamical systems of the Newtonian type. In this approach, in the diagram of the potential of a fictitious particle, the evolution of the universe contains all information which is needed for an investigation of singularities in the model. Note that they are not isolated singularities which were classified into five types but rather double singularities glued in one point of the evolution at $a = a_{\text{sing}}$. Appearance of such types of singularities is typical for piecewise-smooth dynamics describing the model evolution. We call this type sewn singularities in analogy to sewn dynamical systems [78, 79].

We investigated the model with $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$, where γ assumes the positive or negative values. While the dynamics of this class of models depend crucially on the sign of the parameter γ in the early universe for the late time we obtain the behavior consistent with the Λ CDM model.

Note that in the model with positive γ , the phase space is a sum of two disjoint domains which boundary represents the double freeze singularity (cf. Fig. 13). In the first domain the evolution starts from a big bang followed by the deceleration phase; then it changes to acceleration (early acceleration $\equiv$ inflation) after reaching a maximum of the potential function. In the second domain, on the right from the vertical line of the freeze singularity, the universe decelerates and after reaching another maximum starts to accelerate again. This last eternal acceleration corresponds to the present day epoch called the dark energy domination epoch. Two phases of deceleration and two phases of acceleration are key ingredients of our model. While the first phase models a transition from the matter domination epoch to inflation the second phase models a transition from the second matter dominated epoch toward the present day acceleration.

As De Felice and Tsujikawa have noted [1, p. 24] the applications of $f(R)$ theories should be focused on construct of viable cosmological models, for which a sequence of radiation, matter and accelerating epochs is realized. All these epochs are also presented in the model under consideration but, for negative γ (negative squared M^2 for the scalar field), some difficulties appear in the interpretation of the phase space domain $\{a: a < a_{\text{sing}}\}$. The size of this domain will depend on the value of the parameter Ω_γ and this domain vanishes as we are going toward Ω_γ equal zero.

On the other hand it is well known that violation of condition $f''_{\hat{R}\hat{R}} > 0$ gives rise to the negative values of M^2 . We do not assume this condition but we require that $f'_{\hat{R}} > 0$ to avoid the appearance of ghosts (see Sect. 7.4 in [1]). In our case,

statistical analysis favors a model with $f'_{\hat{R}} > 0$ ($\Omega_\gamma > 0$) rather than a model with $f'_{\hat{R}} < 0$ ($\Omega_\gamma < 0$). In other words, statistical analysis favors the case without ghosts.

In order to obtain deeper insight into the model we have also performed complementary investigations in the Einstein frame. In this case we find that the model is reduced to the FRW cosmological model with the selfinteracting scalar field and the vanishing part of the kinetic energy. Therefore from the Palatini formulation we obtain directly the form of the potential and the (implicit) functional dependence between the scalar field and the scale factor. Moreover, we obtain the parametrization of the decaying cosmological constant.

Due to a time-dependent cosmological constant the model evolution can be described in terms of an interaction between the matter and the decaying lambda terms. We study how the energy is transferred between the sectors and how the standard scaling relation for matter is modified.

We pointed out that the consideration of the Starobinsky model in the Einstein frame gives rise to new interesting properties from the cosmological point of view; similar to the original (metric) the Starobinsky model is very important for the explanation of inflation. The model under the consideration gives rise analogously to the running cosmological term. This fact seems to be interesting in the context of an explanation of the cosmological constant problem.

Detailed conclusions coming from our analysis are the following:

- We show that the interaction between two sectors: the matter and the decaying vacuum, appears naturally in the Einstein frame. For the model formulated in the Jordan frame this interaction is absent.
- Inflation appears in our model formulated in the Einstein frame, when the parameter γ is close to zero and the density of matter is negligible in comparison to $\bar{\rho}_\Phi$.
- In our model in the Einstein frame, the potential $\bar{U}(\Phi)$ has the same shape as the Starobinsky potential and has the minimum for $\Phi = 1 + 8\gamma\lambda$.
- While the freeze double singularities appear in our model in the Jordan frame there are no such singularities in the dynamics of the model in the Einstein frame.
- If Ω_γ is small, then $a_{\text{sing}} = \left(-\frac{2\Omega_{m,0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}}\right)^{1/3}$ for negative Ω_γ and $a_{\text{sing}} = \left(\frac{1 - \Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})}}\right)^{1/3}$ for positive Ω_γ . These values define the natural scale at which singularities appear in the model under consideration with the negative or positive value of γ parameter. It seems to be natural to identify this scale with a cut off at which the model can be treated as some kind of effective theory.
- In both the cases of a negative and positive γ one deals with a finite scale factor singularity. For negative γ it

is a double sudden singularity which meets the future singularity of a contracting model before the bounce with the initial singularity in the expanding model. The sewn evolutionary scenarios reveal the presence of a bounce during the cosmic evolution.

- In the context of the Starobinsky model in the Palatini formalism we found a new type of double singularity beyond the well-known classification of isolated singularities.
- The phase portrait for the model with a positive value of γ is equivalent to the phase portrait of the Λ CDM model (following dynamical system theory [80] equivalence assumes the form of topological equivalence established by a homeomorphism). There is only a quantitative difference related with the presence of the non-isolated freeze singularity. The scale of the appearance of this type singularity can also be estimated and be cast in terms of the redshift $z_{\text{freeze}} = \Omega_\gamma^{-1/3}$.
- We estimated the model parameters using astronomical data and conclude that positive Ω_γ is favored by the best fit value; still the model without $\hat{R}^2$ term is statistically admitted.

In our model, the best fit value of Ω_γ is equal 9.70×10^{-11} and positive Ω_γ parameter belongs to the interval $(0, 2.2143 \times 10^{-9})$ at $2\text{-}\sigma$ level. This mean that the positive value of Ω_γ is more favored by astronomical data than the negative value of Ω_γ . The difference between values of BIC for our model and the Λ CDM model is equal 6.407. So, in comparison to our model, the evidence in favor of the Λ CDM model is strong. But one cannot absolutely reject the model.

Note added in proof After completing the paper we found a paper by Faraoni and Cardini where freeze singularities have been analyzed in a different context, both from point particle and cosmological perspectives [81].

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Emergence of running dark energy from polynomial $f(R)$ theory in Palatini formalism

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Abstract We consider FRW cosmology in $f(R) = R + \gamma R^2 + \delta R^3$ modified framework. The Palatini approach reduces its dynamics to the simple generalization of Friedmann equation. Thus we study the dynamics in two-dimensional phase space with some details. After reformulation of the model in the Einstein frame, it reduces to the FRW cosmological model with a homogeneous scalar field and vanishing kinetic energy term. This potential determines the running cosmological constant term as a function of the Ricci scalar. As a result we obtain the emergent dark energy parametrization from the covariant theory. We study also singularities of the model and demonstrate that in the Einstein frame some undesirable singularities disappear.

1 Introduction

A variety of explanations have been proposed for the accelerating expansion of the universe at the current epoch. Among them, the idea of positive cosmological constant Λ , as one of the simplest candidates, seems to be viable. However, it is only an economical description (with the help of one free parameter) of observational facts rather than an effective explanation. The simplest alternative candidate for the constant cosmological parameter being a key element in the standard cosmological model (called Λ CDM model) is a time-dependent (or running) cosmological term. It is crucial for avoiding fine-tuning and coincidence problems [1, 2].

It would be crucial to derive the dynamics of the running cosmological term as an emergent phenomenon from a more fundamental theory, e.g., from string theory or from the first principles of quantum mechanics [3]. In this context,

it is important to formulate a dynamical cosmological term without violating the covariance of the action. For example, models with a slowly rolling homogeneous cosmological scalar field provide a popular alternative to the standard time-independent cosmological constant. We can study the simultaneous evolution of the background expansion and an evolution of the scalar field with the self-interacting potential [4].

In this paper we are going to push forward the idea of the emergent running cosmological term from a covariant theory [5]. Parametrization of the cosmological term is derived directly from a formulation of the model in the Einstein frame by means of the Palatini variational approach. In analogy with Starobinsky's purely metric formulation [6], we obtain the parametrization of the cosmological term directly from the potential of the scalar field which appears after formulation of the specific FRW model in the Einstein frame. As a next step we investigate the dynamics of the model with such a form of the dark energy.

In this letter, we demonstrate how $f(R)$ model is modified in the Palatini formulation. Our construction provides a simple model of an evolving dark energy (running cosmological term) to explain a dynamical relaxation of the vacuum energy (gravitational repulsive pressure) to a very small value today (cosmological constant problem [7]). This model, when studied in the Einstein frame, leads also to a small deviation from the $w = -1$ prediction of the non-running dark energy.

2 Cosmological equations for the polynomial $f(R)$ theory in the Palatini formalism

The Palatini gravity action for $f(\hat{R})$ gravity is given by

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \quad (1)$$

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where $\hat{R}$ is the generalized Ricci scalar [8,9]. From the action (1) we get

$$f'(\hat{R})\hat{R}_{(\mu\nu)} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

$$\hat{\nabla}_\alpha \left(\sqrt{-g} f'(\hat{R}) g^{\mu\nu} \right) = 0, \quad (3)$$

where $T_{\mu\nu}$ is energy-momentum tensor and $\hat{\nabla}_\alpha$ is the covariant derivative calculated with respect to Γ .

If we take the trace of Eq. (2), we get a structural equation, which is given by

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T, \quad (4)$$

where $T = g^{\mu\nu}T_{\mu\nu}$. We assume the FRW metric in the following form:

$$ds^2 = dt^2 - a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (5)$$

where $a(t)$ is the scale factor, k is a constant of spatial curvature ($k = 0, \pm 1$) and t is the cosmological time. Thereafter, we assume the flat model ($k = 0$).

We assume the energy-momentum tensor for a perfect fluid,

$$T^\mu_\nu = \text{diag}(-\rho, p, p, p), \quad (6)$$

where $p = w\rho$ with $w = \text{const}$. The conservation condition $T^\mu_{\nu;\mu} = 0$ [10] gives

$$\dot{\rho}_m = -3(1+w)H\rho_m, \quad (7)$$

where H is the Hubble function and ρ_m is the density of baryonic and dark matter which is assumed to be in the form of dust ($w = 0$).

In our paper the function $f(\hat{R})$ is assumed in the polynomial form as

$$f(\hat{R}) = \sum_{i=1}^n \gamma_i \hat{R}^i, \quad (8)$$

where γ_i are some dimensionful parameters.

Therefore, we introduce more convenient dimensionless functions and parameters,

$$\Omega_R = \frac{\hat{R}}{3H_0^2}, \quad \Omega_{\gamma_i} = 3^{i-1} \gamma_i H_0^{2(i-1)},$$

$$\Omega_{\text{tot}} = \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}, \quad b = f'(\hat{R}) = \sum_{i=1}^n i \Omega_{\gamma_i} \Omega_R^{i-1},$$

$$d = -3 \left(\sum_{i=1}^n (i-2) \Omega_{\gamma_i} \Omega_R^{i-1} + \frac{4\Omega_\Lambda}{\Omega_R} \right) \times \frac{\sum_{i=1}^n i(i-1) \Omega_{\gamma_i} \Omega_R^{i-1}}{\sum_{i=1}^n i(i-2) \Omega_{\gamma_i} \Omega_R^{i-1}}, \quad (9)$$

where H_0 is the present value of the Hubble function, $\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}$, $\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{3H_0^2}$.

For the function (8) the structural equation (4) is in the following form:

$$\sum_{i=1}^n (i-2) \Omega_{\gamma_i} \Omega_R^i = -\Omega_m - 4\Omega_\Lambda. \quad (10)$$

The Friedmann equation for the function (8) has the following form:

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left[\frac{1}{2b} \left[\sum_{i=1}^n \Omega_{\gamma_i} \Omega_R^{i-1} (\Omega_R - 2i\Omega_{\text{tot}}) + \Omega_{\text{tot}} - 3\Omega_\Lambda \right] + \Omega_{\text{tot}} \right]. \quad (11)$$

3 Singularities in the polynomial $f(R)$ theory in the Palatini formalism

The Friedmann equation (11) can be rewritten in the equivalent form

$$a'^2 = -2V(a), \quad (12)$$

where $' = \frac{d}{d\tau} = \frac{|b+d/2|}{|b|} \frac{d}{dt}$ is a new parametrization of time (this parametrization is not a diffeomorphism) and

$$V(a) = -\frac{H_0^2 a^2}{2} \left[\frac{1}{2b} \left[\sum_{i=1}^n \Omega_{\gamma_i} \Omega_R^{i-1} (\Omega_R - 2i\Omega_{\text{tot}}) + \Omega_{\text{tot}} - 3\Omega_\Lambda \right] + \Omega_{\text{tot}} \right]. \quad (13)$$

The potential $V(a)$ can be used to construction of a phase space portrait. In this case the phase space is two-dimensional,

$$\left\{ (a, a') : \frac{a'^2}{2} + V(a) = -\frac{k}{2} \right\}. \quad (14)$$

¹ For the sake of generality (following the standard cosmological model) the presence of the cosmological constant is also assumed.

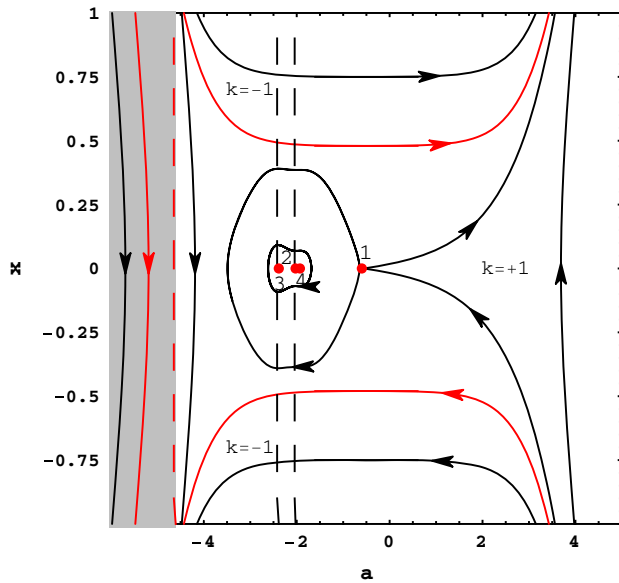


Fig. 1 The phase portrait for system (15)–(16) with $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = 10^{-6} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = -10^{-14} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Critical points (1), (2), (3) and (4) represent the static Einstein universes. Critical points (1) and (2) are of saddle type and critical points (3) and (4) are of center type. The red dashed line presents the sudden singularity. The black dashed lines present the freeze singularities. The gray color marks the non-physical domain ($f'(R) < 0$). The red trajectories represent the path of evolution for the flat universe. These trajectories separate the domain with the negative curvature ($k = -1$) from the domain with the positive curvature ($k = +1$). The scale factor is expressed in a logarithmic scale

The dynamical system has the following form:

$$a' = x, \quad (15)$$

$$x' = -\frac{\partial V(a)}{\partial a}. \quad (16)$$

We assume that the potential function, except some isolated (singular) points, belongs to the class $C^2(\mathbb{R}_+)$.

The example phase portraits for the dynamical system (15)–(16) are presented in Figs. 1, 2 and 3.

The evolution of a universe can be treated as a motion of a fictitious particle of unit mass in the potential $V(a)$. Here $a(t)$ plays the role of a positional variable. Equation of motion (16) assumes the form analogous to the Newtonian equation of motion. In this case the lines $\frac{x^2}{2} + V(a) = -\frac{k}{2}$ represent possible evolutions of the universe for different initial conditions.

In our model, there are two types of singularities: the freeze and sudden singularities. They are a consequence of the Palatini formalism. We get the freeze singularity when $b + d/2 = 0$. The sudden singularity appears when $b = 0$ or $b + d/2$ is equal to infinity.

For the case when the positive part of $f(\hat{R})$ dominates after the domination of the negative part of $f(\hat{R})$, it is possible that

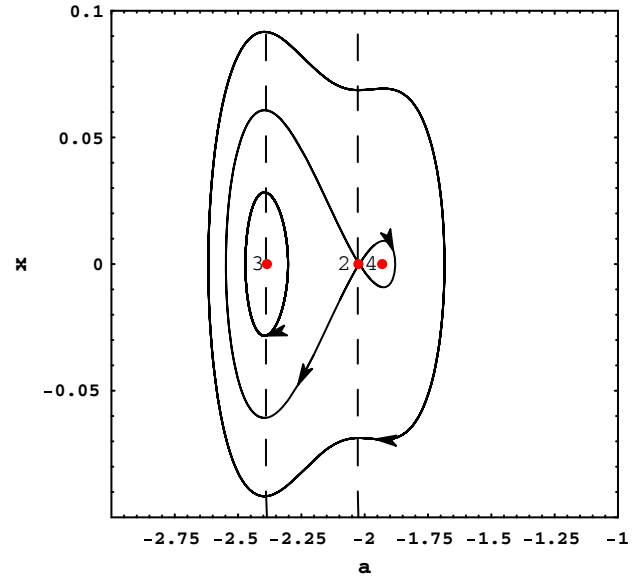


Fig. 2 The zoomed region of Fig. 1. The behavior of trajectories in the neighborhood of critical points (2), (3) and (4) which represent the static Einstein universes. Critical point (2) is of saddle type and critical points (3) and (4) are of center type. The black dashed lines present the freeze singularities. The scale factor is expressed in a logarithmic scale. The homoclinic orbits represent the bouncing models, which evolution starts and ends at the Einstein universe (critical point 2). In the domain bounded by the homoclinic orbits the oscillating models present cases without the initial singularity

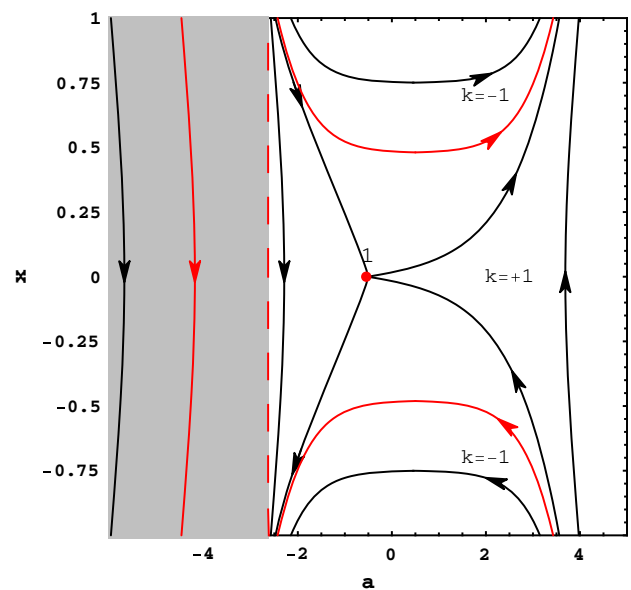


Fig. 3 The phase portrait for system (15)–(16) with $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = -10^{-6} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = -10^{-14} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Critical point (1), which is of saddle type, represents the static Einstein universe. The red dashed line presents the sudden singularity. The gray color presents the non-physical domain ($f'(R) < 0$). The red trajectories represent the path of evolution for the flat universe. These trajectories separate the domain with the negative curvature ($k = -1$) from the domain with the positive curvature ($k = +1$). The scale factor is expressed in a logarithmic scale

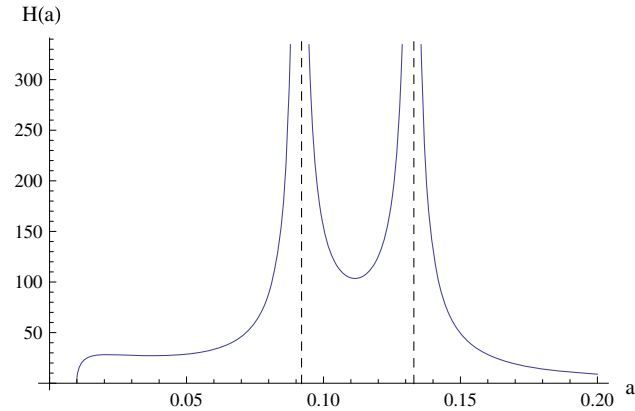


Fig. 4 The evolution of the Hubble function for $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = 10^{-6} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = -10^{-14} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. The black dashed lines present the freeze singularities. Note that the singularity of the big bang type does not appear here. $H(a)$ is expressed in units of $\frac{100 \text{ km}}{\text{s Mpc}}$

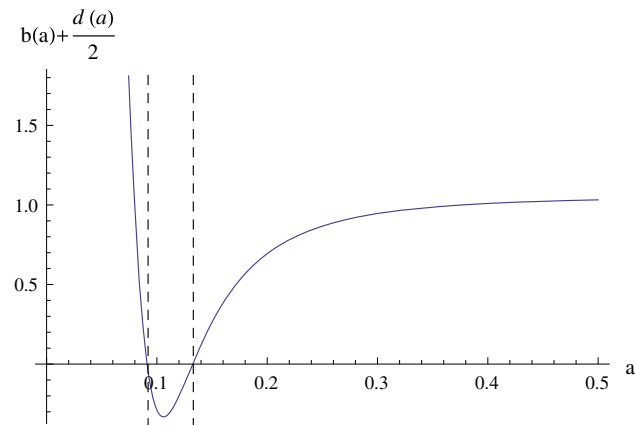


Fig. 5 The evolution of $b(a) + \frac{d(a)}{2}$. For values of the scale factor for which the equation $b(a) + \frac{d(a)}{2} = 0$ has roots, the freeze singularities appear (black dashed lines). This figure corresponds with Fig. 4

two freeze singularities appear. This situation is presented in Fig. 4 for $f(\hat{R}) = \hat{R} + 10^{-2} \hat{R}^2 - 10^{-6} \hat{R}^3$. In this case they appear two freeze singularities and one sudden singularity. The evolution of $b(a) + \frac{d(a)}{2}$, which corresponds with Fig. 4, is presented in Fig. 5. Note that, for values of the scale factor for which the function $b(a) + \frac{d(a)}{2}$ has roots, the freeze singularities appear. $V(a)$ potential, which corresponds with Fig. 4, is presented in Figs. 6 and 7.

4 Singularities in the Palatini $f(R) = R + \gamma R^2 + \delta R^3$ model

For the special case of polynomial $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, one gets the following structural equation:

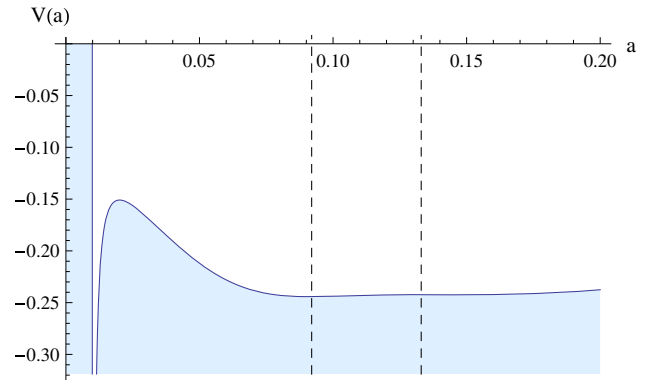


Fig. 6 The evolution of $V(a)$. This figure corresponds with Fig. 4. The black dashed lines present the freeze singularities. The potential is regular at these singularities while its higher derivative blows up. The pole of $V(a)$ represents the sudden singularity. The potential $V(a)$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

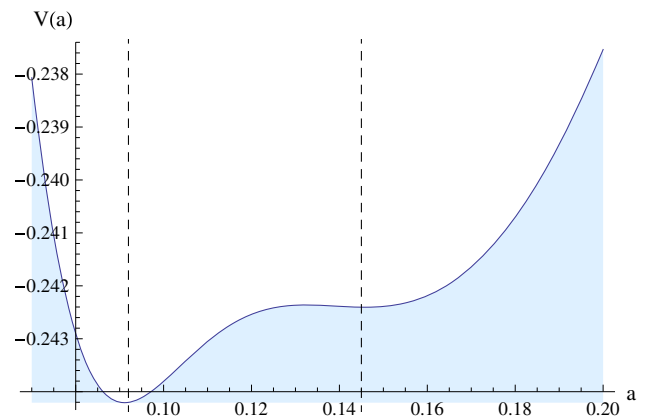


Fig. 7 The zoomed region of Fig. 6. The extrema of $V(a)$ are presented. The black dashed lines represent the freeze singularities. The potential $V(a)$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

$$\Omega_R - \Omega_\delta \Omega_R^3 = \Omega_m + 4\Omega_\Lambda, \quad (17)$$

where $\Omega_\gamma = 3\gamma H_0^2$ and $\Omega_\delta = 9\delta H_0^4$.

The Friedmann equation takes the form

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \times \left[\frac{\Omega_R}{2b} \left[\Omega_\gamma (\Omega_R - 4\Omega_{\text{tot}}) + 2\Omega_\delta \Omega_R (\Omega_R - 3\Omega_{\text{tot}}) \right] + \Omega_{\text{tot}} + \Omega_k \right], \quad (18)$$

where

$$\begin{aligned} \Omega_{\text{tot}} &= \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}, \\ b &= f'(\hat{R}) = 1 + \Omega_R [2\Omega_\gamma + 3\Omega_\delta \Omega_R], \\ d &= \frac{1}{H} \frac{db}{dt} = 6 \frac{\Omega_\gamma + 3\Omega_\delta \Omega_R}{3\Omega_\delta \Omega_R^2 - 1} [\Omega_R (1 - \Omega_\delta \Omega_R^2) - 4\Omega_{\Lambda,0}]. \end{aligned} \quad (19)$$

The condition for the appearance of the freeze singularity is $b + \frac{d}{2} = 0$ and in this case it has the form

$$3\Omega_\gamma\Omega_\delta\Omega_R^3 + 9\Omega_\delta\Omega_R^2 + (\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda)\Omega_R - 12\Omega_\gamma\Omega_\Lambda - 1 = 0. \quad (20)$$

Equation (20) has the following solution:

$$\Omega_{R_{\text{sing}}} = \Omega_\gamma^{-1} \left[-1 + \frac{r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda)}{92^{1/3}\Omega_\delta} - \frac{2^{1/3}(-81\Omega_\delta^2 + 9\Omega_\gamma\Omega_\delta(\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda))}{9r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda)\Omega_\delta} \right], \quad (21)$$

where

$$r(\Omega_\gamma, \Omega_\delta, \Omega_\Lambda) = 2[243\Omega_\gamma^2\Omega_\delta^2(1 + 6\Omega_\gamma\Omega_\Lambda) - 729\Omega_\delta^3(1 + 6\Omega_\gamma\Omega_\Lambda) + (59049(\Omega_\gamma^2 - 3\Omega_\delta)^2\Omega_\delta^4(1 + 6\Omega_\gamma\Omega_\Lambda)^2 - (81\Omega_\delta^2 - 9\Omega_\gamma\Omega_\delta(\Omega_\gamma - 36\Omega_\delta\Omega_\Lambda))^3)^{1/2}]^{1/3}. \quad (22)$$

For the sudden singularity the condition $b = 0$ provides the equation

$$1 + \Omega_R[2\Omega_\gamma + 3\Omega_\delta\Omega_R] = 0. \quad (23)$$

which has the following solutions:

$$\Omega_{R_{\text{sing}}} = \frac{-\Omega_\gamma \pm \sqrt{\Omega_\gamma^2 - 3\Omega_\delta}}{3\Omega_\delta}. \quad (24)$$

5 The Palatini approach in the Einstein frame

If $f''(\hat{R}) \neq 0$ then the action (1) can be rewritten in dynamically equivalent form of the first order Palatini gravitational action [11–13]

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) = \frac{1}{2} \int d^4x \sqrt{-g} (f'(\chi)(\hat{R} - \chi) + f(\chi)) + S_m(g_{\mu\nu}, \psi). \quad (25)$$

The conditions that allow for the change of variables and lead to Eq. (25) were discussed in the well-known paper of Olmo [14], who clarified the issues raised by Faraoni [15].

Let $\Phi = f'(\chi)$ be a scalar field, where $\chi = \hat{R}$. Then the action (25) takes the form

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) = \frac{1}{2} \int d^4x \sqrt{-g} (\Phi \hat{R} - U(\Phi)) + S_m(g_{\mu\nu}, \psi), \quad (26)$$

where the potential $U(\Phi)$ is given as

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \quad (27)$$

with $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

After the Palatini variation of the action (26) we get the following equations of motion:

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - T_{\mu\nu} = 0, \quad (28a)$$

$$\hat{\nabla}_\lambda (\sqrt{-g} \Phi g^{\mu\nu}) = 0, \quad (28b)$$

$$\hat{R} - U'(\Phi) = 0. \quad (28c)$$

As a consequence of (28b) the connection $\hat{\Gamma}$ is a metric connection for a new (conformally related) metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$; thus $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $\bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. The g -trace of (28a) gives a new structural equation

$$2U(\Phi) - U'(\Phi)\Phi = T. \quad (29)$$

Equations (28a) and (28c) can be rewritten in the following form:

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{U}(\Phi), \quad (30)$$

$$\Phi \bar{R} - (\Phi^2 \bar{U}(\Phi))' = 0, \quad (31)$$

where $\bar{U}(\Phi) = U(\Phi)/\Phi^2$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$. In this case, the structural equation is given by the following formula:

$$\Phi \bar{U}'(\Phi) + \bar{T} = 0. \quad (32)$$

The action for the metric $\bar{g}_{\mu\nu}$ and the scalar field Φ can be recast into the Einstein frame form

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2} \int d^4x \sqrt{-\bar{g}} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1} \bar{g}_{\mu\nu}, \psi) \quad (33)$$

with non-minimal coupling between Φ and $\bar{g}_{\mu\nu}$

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} S_m = (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} \bar{g}^{\mu\nu} = \Phi^{-3} T^{\mu\nu}, \quad (34)$$

$\bar{u}^\mu = \Phi^{-\frac{1}{2}} u^\mu$, $\bar{\rho} = \Phi^{-2} \rho$, $\bar{p} = \Phi^{-2} p$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$ (see e.g. [13, 16]).

The metric $\bar{g}_{\mu\nu}$ takes the standard FRW form

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(\bar{t})[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad (35)$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}} dt$ and a new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$. In the case of barotropic matter, the cosmological equations are

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \quad 6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_m(1 + 3w) \quad (36)$$

where

$$\bar{\rho}_\Phi = \frac{1}{2} \bar{U}(\Phi), \quad \bar{\rho}_m = \rho_0 \bar{a}^{-3(1+w)} \Phi^{\frac{1}{2}(3w-1)} \quad (37)$$

and $w = \bar{p}_m/\bar{\rho}_m = p_m/\rho_m$. In this case, the conservation equation has the following form:

$$\dot{\bar{\rho}}_m + 3\bar{H}\bar{\rho}_m(1+w) = -\dot{\bar{\rho}}_\Phi. \quad (38)$$

Let us consider our Palatini model $f(\hat{R}) = \sum_{i=1}^n \gamma_i \hat{R}^i$ in the Einstein frame, where $\gamma_1 = 1$. The potential $\bar{U}$ is given by the following formula:

$$\bar{U}(\hat{R}) = 2\bar{\rho}_\Phi(\hat{R}) = \frac{\sum_{i=1}^n (i-1)\gamma_i \hat{R}^i}{\left(\sum_{i=1}^n i\gamma_i \hat{R}^{i-1}\right)^2}. \quad (39)$$

The scalar field Φ can be parametrized by $\hat{R}$ in the following way:

$$\Phi(\hat{R}) = \frac{df(\hat{R})}{d\hat{R}} = \sum_{i=1}^n i\gamma_i \hat{R}^{i-1}. \quad (40)$$

The relation between $\bar{U}$ and $\hat{R}$ for the case $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$ is presented in Fig. 8.

In this frame, two scenarios of cosmic evolution may appear. In the first one the evolution of the universe starts from the generalized sudden singularity. The second case is when it starts from the freeze singularity. The diagrams of the corresponding Newtonian potentials $V(\bar{a})$ are presented in Figs. 9 and 10. We can use the potential $V(\bar{a})$ to construct phase space portraits analogous to the ones in Sect. 3 (see Figs. 11, 12).

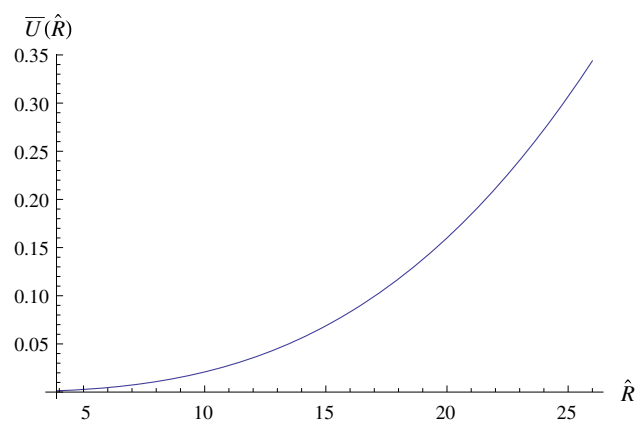


Fig. 8 The evolution of $\bar{U}(\hat{R})$ in the Einstein frame in the case when the evolution of the universe starts from the generalized sudden singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = 10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}\right)$ and $\delta = 10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4}\right)$. $\bar{U}(\hat{R})$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

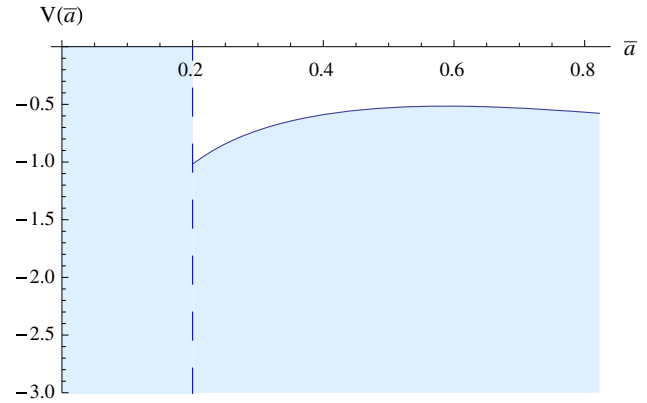


Fig. 9 The evolution of $V(\bar{a})$ in the Einstein frame in the case when the evolution of the universe starts from the generalized sudden singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$. This figure corresponds to Fig. 11. The blue dashed line presents the generalized sudden singularity. Note that the undesirable freeze singularity disappears. The potential $V(a)$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

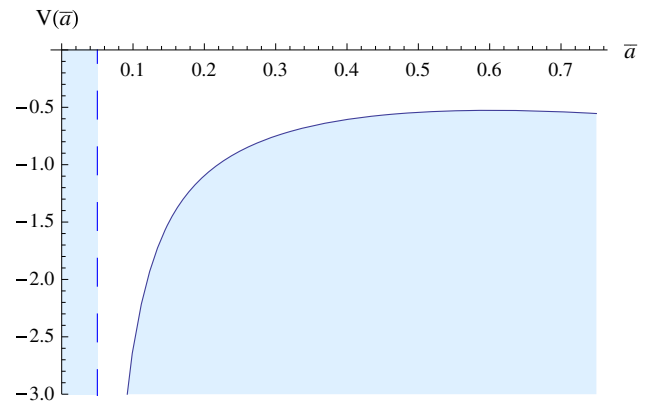


Fig. 10 The evolution of $V(\bar{a})$ in the Einstein frame in the case when the evolution of the universe starts from the freeze singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$. This figure corresponds with Fig. 12. The blue dashed line presents the freeze singularity. The potential $V(a)$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

The evolution of the scalar field potential $\bar{U}(\bar{r})$, which plays a role of dynamical cosmological constant, is presented in Fig. 13 for the case with the generalized sudden singularity. Note that for the late time the potential $\bar{U}(\bar{r})$ is constant. The evolution of $\bar{U}(\bar{r})$, for the case when the freeze singularity appears, is presented in Fig. 14. For the late time the potential $\bar{U}(\bar{a})$ can be approximated by

$$\bar{U}(\bar{a}) = \frac{\gamma \hat{R}(\bar{a})^2}{1 + 4\gamma \hat{R}(\bar{a})} = \frac{\gamma(4\Lambda + \bar{\rho}_{m,0}\bar{a}^{-3})^2}{1 + 4\gamma(4\Lambda + \bar{\rho}_{m,0}\bar{a}^{-3})}. \quad (41)$$

From the structural equation (32) for $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$ case, we get the parameterization of the dust matter density with respect to $\hat{R}$,

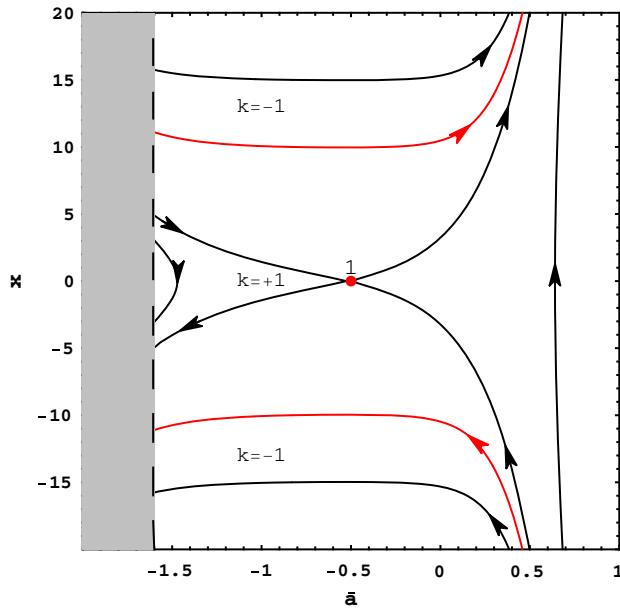


Fig. 11 The phase portrait for system (15)–(16) in the Einstein frame in the case when the evolution of the universe starts from the generalized sudden singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = 10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = 10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Critical point (1) represents the static Einstein universe and is a saddle. The black dashed line presents the generalized sudden singularity. The gray color presents the non-physical domain ($\bar{a} < \bar{a}_s$). The red trajectories represent the path of evolution for the flat universe. These trajectories separate the domain with negative curvature ($k = -1$) from the domain with positive curvature ($k = +1$). The scale factor is expressed in a logarithmic scale

$$\bar{\rho}_m = \frac{\hat{R} - \delta \hat{R}^3}{(1 + 2\gamma \hat{R} + 3\delta \hat{R}^2)^2} - 4\Lambda. \quad (42)$$

It is interesting that in the Einstein frame the interaction between dark matter and dark energy naturally appears as a physical phenomenon. This interaction modifies the original scaling law for dust matter by a function $\epsilon(\bar{t})$. We have

$$\bar{\rho}_m = \bar{\rho}_{m,0} \bar{a}(\bar{t})^{-3+\epsilon(\bar{t})}, \quad (43)$$

where $\epsilon = \frac{1}{\ln \bar{a}} \int \frac{Q}{H \bar{\rho}_m} d \ln \bar{a}$ and $Q = -\dot{\bar{\rho}}_\phi = \bar{H}(\hat{R}) \times \bar{\rho}_m(\hat{R}) \frac{3\hat{R}(\gamma + 3\delta \hat{R})(\delta \hat{R}^2 - 1)}{\hat{R}(\gamma + 3\delta \hat{R}(3 + \gamma \hat{R})) - 1}$ for the case $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$. The evolution of $\epsilon(\bar{t})$ is presented in Fig. 15.

6 Conclusions

The main goal of the paper was to point out some advantages of the formulation of Palatini FRW cosmology in the Einstein frame. The most crucial one is that in the Einstein frame the

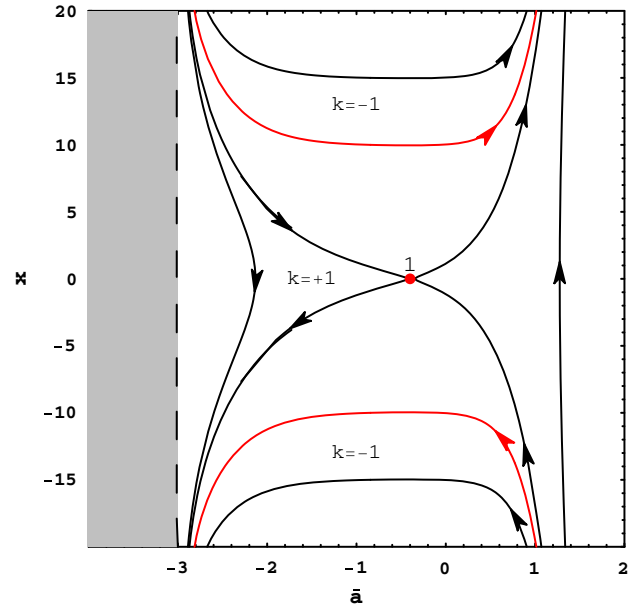


Fig. 12 The phase portrait for system (15)–(16) in the Einstein frame in the case when the evolution of the universe starts from the freeze singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = -10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = -10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Critical point (1) represents the static Einstein universe and is a saddle. The black dashed line presents the freeze singularity. The gray color presents the non-physical domain ($\bar{a} < \bar{a}_s$). The red trajectories represent the path of evolution for the flat universe. These trajectories separate the domain with negative curvature ($k = -1$) from the domain with positive curvature ($k = +1$). The scale factor is expressed in a logarithmic scale

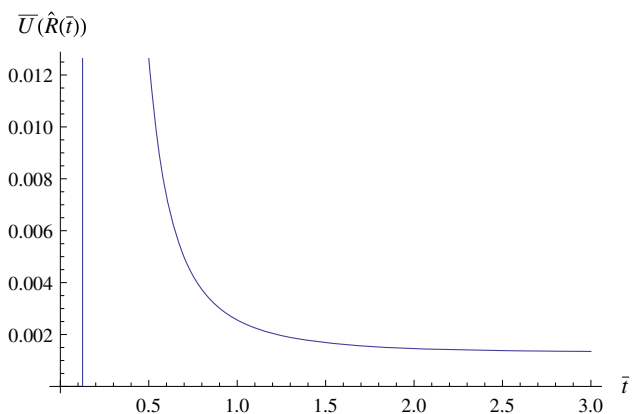


Fig. 13 The evolution of $\bar{U}(\hat{R}(\bar{t}))$ in the Einstein frame in the case when the evolution of the universe starts from the generalized sudden singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$, where $\gamma = 10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = 10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Note that for the late time the potential $\bar{U}(\bar{t})$ goes to a constant value at late time. Time is expressed in units of $\frac{\text{s Mpc}}{100 \text{ km}}$ and $\bar{U}(\hat{R}(\bar{t}))$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

parametrization of dark energy is uniquely determined. In general it is obtained in the covariant form as a function of the Ricci scalar.

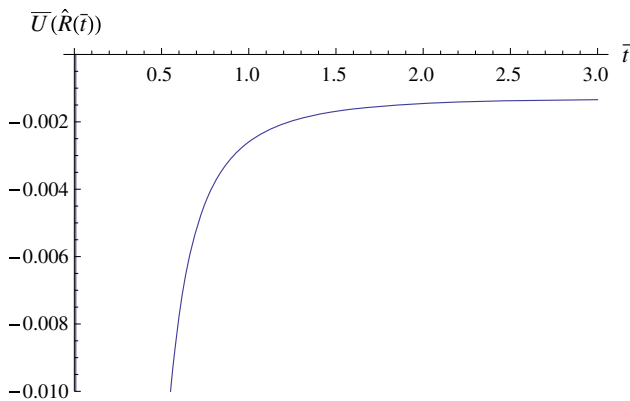


Fig. 14 The evolution of $\bar{U}(\hat{R}(\tilde{t}))$ in the Einstein frame in the case when the evolution of the universe starts from the freeze singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2 + \delta\hat{R}^3$, where $\gamma = -10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = -10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Time is expressed in units of $\frac{\text{s Mpc}}{100 \text{ km}}$ and $\bar{U}(\hat{R}(\tilde{t}))$ is expressed in units of $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$

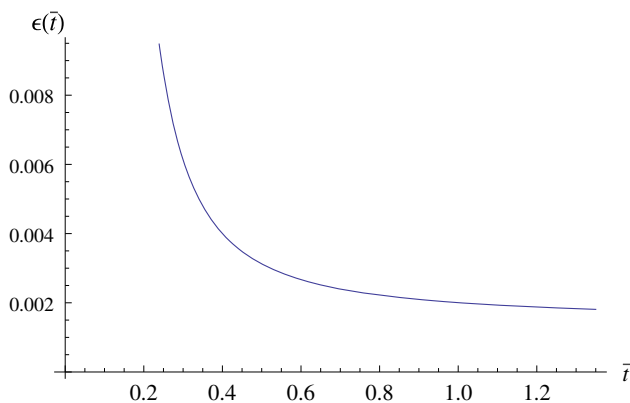


Fig. 15 The evolution of $\epsilon(\tilde{t})$ in the Einstein frame in the case when the evolution of the universe starts from the generalized sudden singularity. For illustration it is assumed that $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2 + \delta\hat{R}^3$, where $\gamma = 10^{-9} \left(\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} \right)$ and $\delta = 10^{-13} \left(\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4} \right)$. Note that for the late time $\epsilon(\tilde{t})$ is constant. Time is expressed in units of $\frac{\text{s Mpc}}{100 \text{ km}}$

It is well known that scalar–tensor theories of gravity can be formulated both in the Jordan and in the Einstein frame. These frames are conformally related [17]. We also know that the formulations of a scalar–tensor theory in two different conformal frames, although mathematically equivalent, are physically inequivalent.

In recent years significant progress has been achieved in the understanding of the geometric features of the Palatini theories and the role of the choice of the frame [18, 19]. In particular, through the analysis of the tensorial perturbations, it is shown that it is the auxiliary (conformal in this case) metric, which determines the propagation of the gravitational waves, while the geodesic motion of the particles is dictated by the Jordan frame metric. A discussion, in this direction,

seems to be important as it would help to eliminate the need to choose between the frames.

Faraoni and Gunzing gave a simple argument which favors the Einstein frame over the Jordan frame: in the latter one should potentially detect the time-dependent amplification induced by gravitational waves [20].

An analogous problem has been detected in $f(R)$ gravity: the Jordan frames could be physically non-equivalent, although they are connected by a conformal transformation [21, 22]. In principle, there are two types of admissible arguments for favoring one frame over another: coming from observations (for example astronomical observations) or being of a theoretical nature (e.g. showing that some obstacles or pathologies will vanish in the privileged frame).

From our investigation of the model in an Einstein frame we found that some pathologies, like degenerate multiple freeze singularities, [23] disappear in a generic case. The big bang singularity is replaced by the singularity of a finite scale factor. The subtle issue of what a singularity is in the context of Palatini theories has been discussed in recent work by Olmo et al. [24–26]. We are using singularities in a cosmological framework rather as a theoretical discriminator for the optimal choice of the frame. We pointed out that the Einstein frame is favored in this context.

Because the potential $\bar{U}(\hat{R}(\tilde{t}))$ is constant for the late time, in the case when matter is negligible, the inflation appears like in the case $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2$ [23].

There are also some other advantages when transforming to an Einstein frame, namely that in this frame one naturally obtains the formula for dynamical dark energy which is going at late time toward the cosmological constant. It is important that the corresponding parametrization of dark energy is not postulated ad hoc but emerges from first principles – which is the formulation of the problem in the Einstein frame. It is important that the parametrization of dark energy (energy density as well as pressure) in terms of the Ricci scalar is given in a covariant form from the structure equation.

After a transition to the Einstein frame the model evolution is governed by the Friedmann equation with two interacting fluids: dark energy and dark matter. This interaction modifies the standard scaling of the redshift relation for dark matter.

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Simple cosmological model with inflation and late times acceleration

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Abstract In the framework of polynomial Palatini cosmology, we investigate a simple cosmological homogeneous and isotropic model with matter in the Einstein frame. We show that in this model during cosmic evolution, early inflation appears and the accelerating phase of the expansion for the late times. In this frame we obtain the Friedmann equation with matter and dark energy in the form of a scalar field with a potential whose form is determined in a covariant way by the Ricci scalar of the FRW metric. The energy density of matter and dark energy are also parameterized through the Ricci scalar. Early inflation is obtained only for an infinitesimally small fraction of energy density of matter. Between the matter and dark energy, there exists an interaction because the dark energy is decaying. For the characterization of inflation we calculate the slow roll parameters and the constant roll parameter in terms of the Ricci scalar. We have found a characteristic behavior of the time dependence of density of dark energy on the cosmic time following the logistic-like curve which interpolates two almost constant value phases. From the required numbers of N -folds we have found a bound on the model parameter.

1 Introduction

While current astronomical observations favour the standard cosmological model [1], the Λ CDM model plays only the role an effective theory of the Universe which offers rather the description of the current properties of the Universe than its explanations. The origin of properties of the current Universe we should find in the very early Universe. In this context a very simple inflation model was proposed by Starobinsky in 1980 [2]. This model attracted attention of cosmologists because it can explain some troubles of the Λ CDM model

in a very simple way. Moreover, this evolutionary scenario is generic and emerged in cosmology in different contexts [1]. In this model, the inflationary scenario of the Universe is driven by the higher quadratic term in the action which takes the form $S = \int \sqrt{-g} \left(R + \frac{R^2}{6M^2} \right) d^4x$.

This model [3,4] predicts that the slow roll parameters $n_s = 1 - \frac{2}{N}$ and $r = \frac{12}{N^2}$ where $N = 50 \sim 60$ is the number of e-folds before the end of inflation, are in good agreement with Planck 2015 data [1].

On the other hand, from the viewpoint of the complete quantum theory of gravity, higher order corrections $\alpha' = 1/M_s^2$ to the Einstein–Hilbert action are always expected i.e.

$$S = \int \sqrt{-g} (R + c_2 \alpha' R^2 + \sum_{i=3} c_i \alpha'^{i-1} R^i + \text{other higher derivative terms}) d^4x, \quad (1)$$

where c_i are the dimensionless couplings.

The higher derivative terms in the action may also originate from supergravity [5,6].

The problem of inflation in polynomial $f(R)$ cosmology was investigated in the metric formalism in [7], where the spectral index and tensor-to-scalar ratio were calculated in the $f(R)$ inflation model.

In this paper we will phenomenologically investigate the inflation model with a polynomial form of the potential in the Palatini formalism in the Einstein frame [8,9]. For simplicity we truncate a Taylor series on the term R^3 .

In the present paper we consider cosmological models of modified gravity which are the polynomial extensions of the Starobinsky model because our aim is to study how tuned is this model and in consequence its prediction—inflation. However, we must remember that the exact form of the function $f(R)$ can be different from such a choice. In particular the adding of negative powers in a $f(R)$ series is also

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very interesting [10]. The treating of the relation $f(R)$ in the form of a series with respect to R guarantees that it is simple enough to handle it easily in the study of physical effects of modified gravity [11]. On the other hand, the introduction of negative powers of R may lead to instabilities [12].

Therefore, it is interesting to investigate some stable isotropic cosmological models describing both inflation and present acceleration in $f(R)$ gravity. In this context the idea of quintessential cosmology seems to be interesting [10, 13]. In the metric approach a more complicated, non-polynomial form of the function $f(R)$ is required at low curvature [14].

The main aim of the paper is to investigate how rigid the Starobinsky model of inflation is and if it can be disturbed by switching higher order terms. Therefore, our study is motivated by a stability investigation. If the Starobinsky model is stable it is in some sense generic. The standard Starobinsky model of inflation is formulated in the background of a metric formulation of $f(R)$ modified gravity. In this paper we formulate $f(R)$ theory in the Palatini formalism which gives us an equation of motion in the form of a second order equation. The inflation similarly to the Starobinsky approach is obtained after transition to the Einstein frame. We obtain the form of the potential for the scalar field in the covariant form directly parameterized by the Ricci scalar in the Palatini formulation.

We investigate how the shape of the potential changes under changing of the parameter which measures the fraction of the higher order term in the assumed $f(R)$ formula.

In modern cosmology, the Starobinsky model of inflation plays a crucial role [2]. This model of the cosmic inflation is considered as a source of the inflaton field—higher curvature corrections with respect to the Ricci scalar R in the Einstein–Hilbert action of gravity of the type R^2 .

The Starobinsky model seems to be distinguished among different alternative models of inflation in predicting a low value of the scalar-to-tensor ratio r ; namely, it predicts that $r \sim 12/N^2$, where N is the number of e-foldings during inflation [15].

The Starobinsky model is also favoured by experimental results [1, 16–19] which give an upper bound on r around the value of 0.1. What it is important from the observational point of view the Starobinsky model is the model with the highest Bayesian evidence [17]. It is characteristic that the other types of models which also fit the data are actually equivalent to the Starobinsky model during inflation [15].

From the methodological point of view it is important that the Starobinsky model can be embedded in different domains of fundamental physics. The situation is in some sense similar to what happens in mathematics, where an important theorem has many references to it in different areas of mathematics. Here, one can distinguish embedding into the supergravity [20, 21] and embedding into the superstring theory [22–26].

In our paper we consider a new embedding of the Starobinsky model into cosmology of Palatini gravity. The emergence of inflation will be demonstrated as an endogenous dynamical effect in the Palatini formulation of gravity applied to FRW cosmology.

2 Cosmological equations for the polynomial $f(\hat{R})$ theory in the Palatini formalism in the Einstein frame

In the Palatini formalism, the gravity action for $f(\hat{R})$ gravity has the following form:

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \quad (2)$$

where $\hat{R}$ is the generalized Ricci scalar $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\hat{\Gamma})$ in the Palatini formalism [27, 28]. In this approach the torsionless connection $\hat{\Gamma}$ is treated as a variable independent of the spacetime metric $g_{\mu\nu}$ and it is used to construct the Riemann and Ricci tensor.

Let $f''(\hat{R}) \neq 0$. In this case, the action (2) has the equivalent form [11, 29, 30]

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) = \frac{1}{2} \int d^4x \sqrt{-g} \left(f'(\chi)(\hat{R} - \chi) + f(\chi) \right) + S_m(g_{\mu\nu}, \psi). \quad (3)$$

We introduce a scalar field $\Phi = f'(\chi)$, where $\chi = \hat{R}$. Then the action (3) is given by the following form:

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) = \frac{1}{2} \int d^4x \sqrt{-g} \left(\Phi \hat{R} - U(\Phi) \right) + S_m(g_{\mu\nu}, \psi), \quad (4)$$

where the potential $U(\Phi)$ is defined as

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)) \quad (5)$$

with $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

The equations of motion are obtained after the Palatini variation of the action (4),

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - T_{\mu\nu} = 0, \quad (6a)$$

$$\hat{\nabla}_\lambda (\sqrt{-g} \Phi g^{\mu\nu}) = 0, \quad (6b)$$

$$\hat{R} - U'(\Phi) = 0. \quad (6c)$$

From Eq. (6b) we see that a metric connection $\hat{\Gamma}$ is a new (conformally related) metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$; thus $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $\bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. We can obtain from the g -trace of Eq. (6a) a new structural equation,

$$2U(\Phi) - U'(\Phi)\Phi = T. \quad (7)$$

Let $\bar{U}(\phi) = U(\phi)/\Phi^2$, $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$. Then Eq. (6a) and (6c) can be rewritten in the following form:

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{U}(\Phi), \quad (8)$$

$$\Phi\bar{R} - (\Phi^2\bar{U}(\Phi))' = 0, \quad (9)$$

and we get the following structural equation:

$$\Phi\bar{U}'(\Phi) + \bar{T} = 0. \quad (10)$$

In this case the action for the metric $\bar{g}_{\mu\nu}$ and the scalar field Φ has the following form in the Einstein frame:

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2} \int d^4x \sqrt{-\bar{g}} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1}\bar{g}_{\mu\nu}, \psi) \quad (11)$$

with a non-minimal coupling between Φ and $\bar{g}_{\mu\nu}$,

$$\begin{aligned} \bar{T}^{\mu\nu} &= -\frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} S_m = (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu} \\ &= \Phi^{-3}T^{\mu\nu}, \end{aligned} \quad (12)$$

$\bar{u}^\mu = \Phi^{-\frac{1}{2}}u^\mu$, $\bar{\rho} = \Phi^{-2}\rho$, $\bar{p} = \Phi^{-2}p$, $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$, $\bar{T} = \Phi^{-2}T$ (see e.g. [30,31]).

We take the metric $\bar{g}_{\mu\nu}$ in the standard form of the FRW metric,

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(\bar{t}) \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (13)$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}}dt$ and a new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}}a(\bar{t})$. We assume the cosmological equations for the barotropic matter in the following form:

$$\begin{aligned} 3\bar{H}^2 &= 3\left(\frac{\dot{\bar{a}}}{\bar{a}}\right)^2 = \bar{\rho}_\phi + \bar{\rho}_m + \Lambda, \quad 6\frac{\ddot{\bar{a}}}{\bar{a}} \\ &= 2\bar{\rho}_\phi - \bar{\rho}_m(1+3w) \end{aligned} \quad (14)$$

where

$$\bar{\rho}_\phi = \frac{1}{2}\bar{U}(\Phi), \quad \bar{\rho}_m = \rho_0\bar{a}^{-3(1+w)}\Phi^{\frac{1}{2}(3w-1)} \quad (15)$$

and $w = \bar{p}_m/\bar{\rho}_m = p_m/\rho_m$. The conservation equation has the following form:

$$\dot{\bar{\rho}}_m + 3\bar{H}\bar{\rho}_m(1+w) = -\dot{\bar{\rho}}_\phi. \quad (16)$$

In this paper, we consider the Palatini model, $f(\hat{R}) = \sum_{i=1}^n \gamma_i \hat{R}^i$, in the Einstein frame, where $\gamma_1 = 1$. In this case, the potential $\bar{U}$ is given by the following formula:

$$\bar{U}(\hat{R}) = 2\bar{\rho}_\phi(\hat{R}) = \frac{\sum_{i=1}^n (i-1)\gamma_i \hat{R}^i}{\left(\sum_{i=1}^n i\gamma_i \hat{R}^{i-1}\right)^2} \quad (17)$$

and the scalar field Φ has the following form:

$$\Phi(\hat{R}) = \frac{df(\hat{R})}{d\hat{R}} = \sum_{i=1}^n i\gamma_i \hat{R}^{i-1}. \quad (18)$$

3 Inflation in $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2 + \delta\hat{R}^3$ theory in the Palatini formalism in the Einstein frame

Let $f(\hat{R}) = \hat{R} + \gamma\hat{R}^2 + \delta\hat{R}^3$. For this case

$$\bar{U}(\hat{R}) = \frac{\hat{R}^2(\gamma + 2\delta\hat{R})}{(1 + 2\gamma\hat{R} + 3\delta\hat{R}^2)^2} \quad (19)$$

and

$$\Phi = 1 + 2\gamma\hat{R} + 3\delta\hat{R}^2. \quad (20)$$

For this parameterization, we can obtain, from the structural equation (10), a parameterization of $\bar{\rho}_m$ with respect to $\hat{R}$,

$$\bar{\rho}_m(\hat{R}) = \frac{\hat{R} - \delta\hat{R}^3}{(1 + 2\gamma\hat{R} + 3\delta\hat{R}^2)^2} - 4\Lambda. \quad (21)$$

In consequence, the Friedmann equation is given by the following equation:

$$\begin{aligned} 3\bar{H}^2 &= \bar{\rho}_m(\hat{R}) + \frac{\bar{U}(\hat{R})}{2} + \Lambda \\ &= \frac{\hat{R}(2 + \gamma\hat{R})}{2(1 + 2\gamma\hat{R} + 3\delta\hat{R}^2)^2} - 3\Lambda. \end{aligned} \quad (22)$$

As a reminder, the Hubble function in the Einstein frame $\bar{H}$ is defined by Eq. (14) and the generalized Ricci scalar in the Palatini formalism is $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma})$.

In this model inflation appears when matter $\bar{\rho}_m$ is negligible with comparison to $\bar{\rho}_\phi$.

In the statistical analysis the slow roll parameters are helpful in the estimation of the model parameter in the inflation period [1]. These parameters are defined as

$$\epsilon = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta = 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}. \quad (23)$$

In our model the slow roll parameters have the following form in the case when $\delta = 0$:

$$\epsilon = \frac{3}{2} \frac{\hat{R} - 4\Lambda(1 + 2\gamma\hat{R})^2}{\hat{R} + \frac{\gamma}{2}\hat{R}^2 - 3\Lambda(1 + 2\gamma\hat{R})^2}, \quad (24)$$

$$\eta = 5 + \frac{3}{2(\gamma\hat{R} - 1)} + \frac{\hat{R}(1 + 2\gamma\hat{R})}{6\Lambda(1 + 2\gamma\hat{R})^2 - \hat{R}(2 + \gamma\hat{R})}. \quad (25)$$

From the Planck observations, we know the limits at a 2- σ level of the values of the scalar spectral index n_s and the tensor-to-scalar ratio r ($n_s = 0.9667 \pm 0.0040$ and $r < 0.113$ [1]). The relations between the scalar spectral index and the tensor-to-scalar ratio and the slow roll parameters are the following:

$$n_s - 1 = -6\epsilon + 2\eta \quad \text{and} \quad r = 16\epsilon. \quad (26)$$

Because the slow roll parameters ϵ and η cannot be treated as constant parameters in our model (see Figs. 1 and 2), we cannot use these parameters to find the restriction on the parameter γ from astronomical observations [1].

For example, if we assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$ [1], then we get $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2} < \gamma < 3.285 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$, $0 < \Omega_m = \frac{\bar{\rho}_m}{3\dot{H}^2} < 0.0047$ and $\Omega_\phi = \frac{\bar{\rho}_\phi}{3\dot{H}^2} \approx 0.50$. But this value of the parameter γ is too large for explaining the present evolution of the Universe. In consequence, the slow roll parameters are useless in the estimation of the parameter γ .

The slow roll parameter approximation is more restrictive than the constant roll condition [32,33]. The constant roll condition has the following form:

$$\beta = \frac{\ddot{\phi}}{H\dot{\phi}} = \text{const}. \quad (27)$$

When $\beta \ll 1$ then we get the slow roll approximation.

In our case $\frac{\ddot{\phi}}{H\dot{\phi}}$ is given by

$$\begin{aligned} \frac{\ddot{\phi}}{H\dot{\phi}} = & 4 - 240\gamma\Lambda + \frac{2}{1 - 24\gamma\Lambda} - 192\gamma^2\Lambda\hat{R} \\ & + \frac{9(36\gamma\Lambda - 1)}{(\gamma\hat{R} - 1)^2} \\ & + \frac{12\Lambda + 3(8\gamma\Lambda - 1)\hat{R}}{(24\gamma\Lambda - 1)(6\Lambda + \hat{R}(24\gamma\Lambda - 2 + \gamma(24\gamma\Lambda - 1)\hat{R}))}, \end{aligned} \quad (28)$$

when $\delta = 0$. $\frac{\ddot{\phi}}{H\dot{\phi}}$ is not constant (see Fig. 3) at all times, but beyond the logistic-like type transition it can be well approximated by a constant value. At this intermediate interval the effects of matter do not become negligible. The constant roll inflation approximation is approximately valid beyond a short time during which the effects of matter stay very important (in consequence of the interaction between matter and dark energy).

Figure 1 presents the evolution of ϵ with respect to the cosmological time $\bar{t}$. We can see that ϵ is not a constant function when matter is not negligible (see Fig. 4).

Figure 2 demonstrates the evolution of η with respect to the cosmological time $\bar{t}$. Note that η is not a constant function when matter is not negligible (see Fig. 4). The characteristic

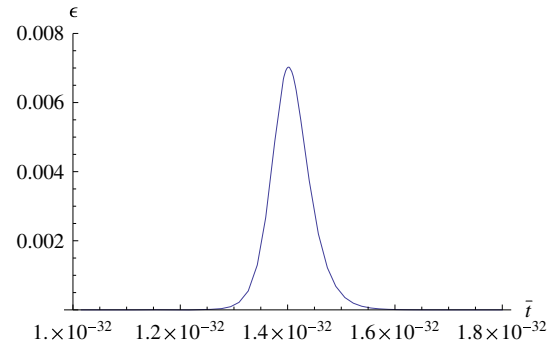


Fig. 1 The diagram presents the evolution of ϵ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$. Note that ϵ is not a constant function when matter is not negligible (see Fig. 4)

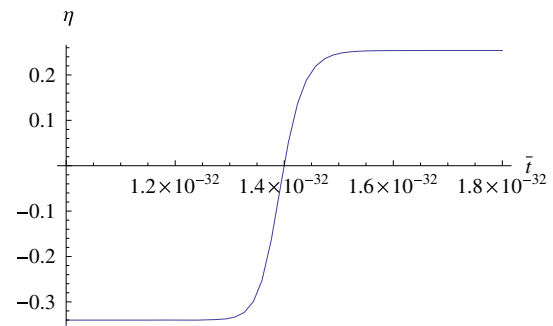


Fig. 2 The evolution of η with respect to the cosmological time $\bar{t}$. The time is expressed in seconds. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$. Note that η is not a constant function when matter is not negligible (see Fig. 4). It is interesting that the function η is of logistic-like function type

attribute of the function η is the shape of the logistic-like function.

Figure 3 presents the evolution of $\frac{\ddot{\phi}}{H\dot{\phi}}$ with respect to the cosmological time $\bar{t}$. It is important that $\frac{\ddot{\phi}}{H\dot{\phi}}$ is not a constant function when matter is not negligible (see Fig. 4). It is interesting that the $\frac{\ddot{\phi}}{H\dot{\phi}}$ function is of the logistic-like function type.

Note that $\beta = \frac{d \ln \dot{\phi}}{d \ln a} = \frac{\ddot{\phi}}{H\dot{\phi}}$ measures the elasticity of $\dot{\phi}$ with respect to the scale factor. When β is constant then

$$\dot{\phi} \propto a^\beta. \quad (29)$$

Therefore, if β is positive then $\dot{\phi}$ is a growing function of the scale factor. In the opposite case ($\beta < 0$) $\dot{\phi}$ is an increasing function of the scale factor and goes to zero for large values of the scale factor.

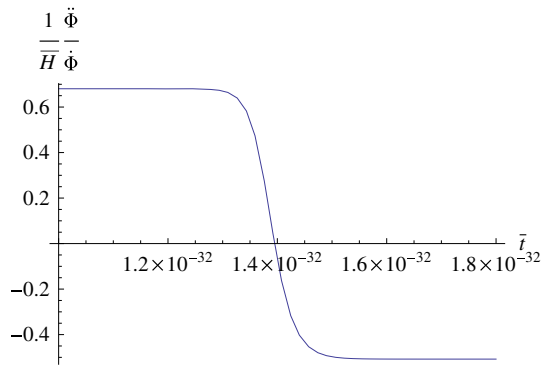


Fig. 3 The diagram presents the evolution of $\frac{\ddot{\Phi}}{H\dot{\Phi}}$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$. Note that $\frac{\ddot{\Phi}}{H\dot{\Phi}}$ is not a constant function when matter is not negligible (see Fig. 4). It is interesting that $\frac{\ddot{\Phi}}{H\dot{\Phi}}$ function is of the logistic-like function type

The slow roll approximation is achieved in our model when matter is negligible. Of course, the constant roll condition is respected automatically.

The evolution of matter in the inflation period can be divided into four phases. The first phase is when matter is negligible and the density of $\bar{\rho}_m$ increases by the interaction with the potential $\bar{\rho}_\Phi$. The second phase is when the matter cannot be negligible and its density still increases. In this phase the injection of matter is the most effective. After achieving of the maximum of the density of $\bar{\rho}_m$ the third phase appears. In this phase matter still cannot be negligible but its density decreases. The last phase is when matter density decreases and is negligible.

The evolution of matter in the inflation period is presented in Fig. 4. We see all four phases of the evolution of matter. The maximum is achieved when

$$\hat{R} = \frac{1}{2\gamma}. \quad (30)$$

In the maximum, the value of $\bar{\rho}_m$ is equal to $\frac{1}{8\gamma} - 4\Lambda$.

In detail, the behaviour of the potential function $\bar{U}(\Phi)$ depends on the form of $f(\hat{R})$. For the polynomial form of $f(\hat{R})$, there are two cases. In the first case $f(\hat{R})$ is in the form $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$. The typical behaviour of the potential $\bar{U}(\Phi)$ for $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$ is present in Fig. 5. The characteristic attribute is a plateau for a large value of Φ like for the Starobinsky potential [2]. In this case the formula for the potential $\bar{U}(\Phi)$ has the following form:

$$\bar{U}(\Phi) = \gamma \left(\frac{\Phi - 1}{2\gamma\Phi} \right)^2. \quad (31)$$

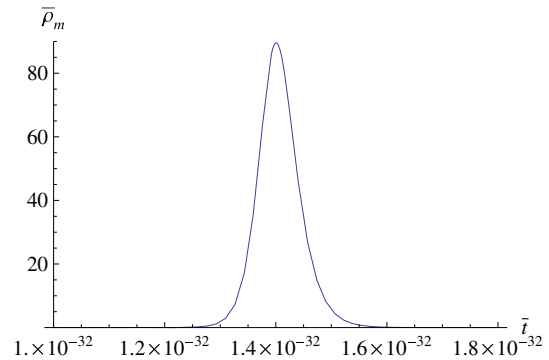


Fig. 4 The diagram presents the evolution of $\bar{\rho}_m$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds and $\bar{\rho}_m$ is expressed in $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$. Note that the maximum of this function is achieved when $\hat{R} = \frac{1}{2\gamma}$

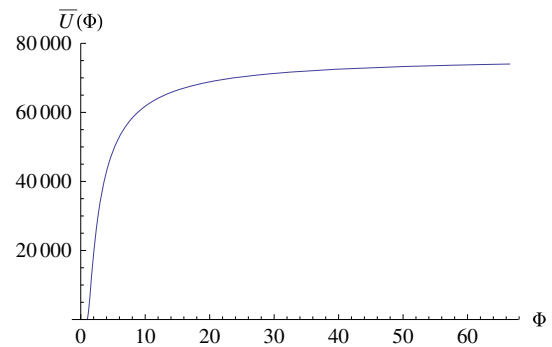


Fig. 5 The diagram presents the typical behaviour of the function $\bar{U}(\Phi)$ for the case $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$. The potential $\bar{U}(\Phi)$ is expressed in $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. Note that, for the large value of Φ , the function $\bar{U}(\Phi)$ has the plateau

The second case is when $f(\hat{R})$ is of the form $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \sum_{i=2}^n \delta_i \hat{R}^{i+1}$. Then the potential $\bar{U}(\Phi)$ has no plateau and decreases asymptotically to zero when Φ goes to infinity. This situation is presented in Fig. 6. The formula for the potential $\bar{U}(\Phi)$ for $f(\hat{R}) = \hat{R} + \gamma \hat{R} + \delta \hat{R}^2$ has the following form:

$$\begin{aligned} \bar{U}(\Phi) &= \frac{\left(\gamma - \sqrt{\gamma^2 + 3\delta(\Phi - 1)} \right)^2 \left(\gamma + 2\sqrt{\gamma^2 + 3\delta(\Phi - 1)} \right)}{27\delta^2\Phi^2}. \end{aligned} \quad (32)$$

In the context of inflation Ijjas et al. [34] pointed out the problem with the desired plateau in the behaviour of the potential of the scalar field. Such a choice seems to be unjustified because it requires that the power series expansion of potential U with respect to Φ is cancelled at a precise order in Φ to make the plateau appear.

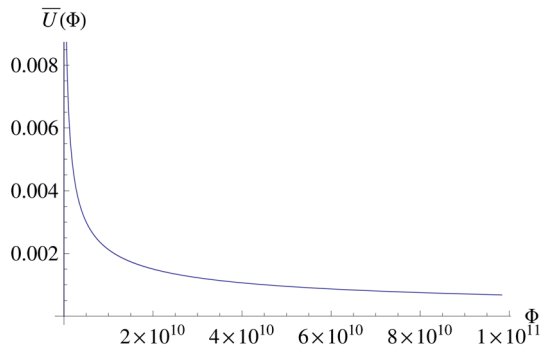


Fig. 6 The diagram presents the typical behaviour of the function $\bar{U}(\Phi)$ for the case $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \delta \hat{R}^3$. The potential $\bar{U}(\Phi)$ is expressed in $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. Note that, for a large value of Φ , the function $\bar{U}(\Phi)$ decreases asymptotically to zero

In agreement with Ijjas et al. we obtain the plateau of the potential $\bar{U}(\Phi)$ only when $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$. For the higher order terms in the expansion of the $f(\hat{R})$, i.e., $\hat{R}^3$ and higher, the potential monotonically decreases to zero.

Now, we consider in detail inflation in the two above-mentioned cases with the potential expanded to second order and third order with respect to Φ . In consequence, we study whether the plateau is necessary for the appearance of inflation in our model and whether inflation is possible for the model with a cut-off in a higher order ($\hat{R}^3$ and higher) expansion.

In the inflation period when the matter is negligible, the Ricci scalar $\hat{R}$ is constant. The evolution of the Ricci scalar $\hat{R}$ is presented in Fig. 7. We can see three phases of the evolution of the Ricci scalar $\hat{R}$. The first phase is when matter is negligible and the density of $\bar{\rho}_m$ is increased by an interaction with the potential $\bar{\rho}_\phi$. Then the Ricci scalar $\hat{R}$ is constant and is described by the following formula when $\delta = 0$:

$$\hat{R} = \frac{1 - 16\gamma\Lambda + \sqrt{1 - 32\gamma\Lambda}}{32\gamma^2\Lambda}. \quad (33)$$

The second phase is when the matter is not negligible. In this case, the Ricci scalar $\hat{R}$ decreases. The last phase is when matter density decreases and is negligible. Then the Ricci scalar $\hat{R}$ is constant and is equal to

$$\hat{R} = \frac{1 - 16\gamma\Lambda - \sqrt{1 - 32\gamma\Lambda}}{32\gamma^2\Lambda}, \quad (34)$$

when $\delta = 0$. The function which describes the evolution of the Ricci scalar $\hat{R}$ has the shape of a logistic-like function.

The evolution of $\bar{\rho}_\phi$, in the inflation period, similar qualitatively to the evolution of the Ricci scalar $\hat{R}$. We can find three phases. In the first phase, $\bar{\rho}_\phi$ is constant and is equal to

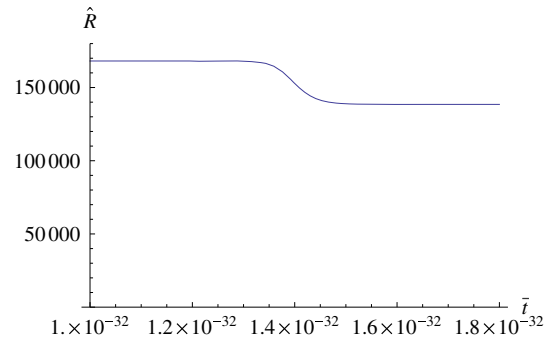


Fig. 7 The diagram presents the evolution of the Ricci scalar $\hat{R}$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds and the Ricci scalar $\hat{R}$ is expressed in $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{s Mpc}}$. The transition phase is of logistic-like behaviour and is strictly correlated with a peak of the matter density, as shown in Fig. 4

$$\bar{\rho}_\phi = \frac{1 - 16\gamma\Lambda + \sqrt{1 - 32\gamma\Lambda}}{16\gamma} \quad (35)$$

and in the last phase when $\bar{\rho}_\phi$ is also constant,

$$\bar{\rho}_\phi = \frac{1 - 16\gamma\Lambda - \sqrt{1 - 32\gamma\Lambda}}{16\gamma} \quad (36)$$

for $\delta = 0$. The difference between $\bar{\rho}_\phi$ in the first and in the last phase is equal to

$$\Delta\bar{\rho}_\phi = \frac{\sqrt{1 - 32\gamma\Lambda}}{8\gamma} \approx \frac{1}{8\gamma}. \quad (37)$$

The evolution of $\bar{\rho}_\phi$ is presented in Fig. 8. Our model predicts a phase of the early constant dark energy which is correlated with inflation [35,36].

When $\delta = 0$ the number of e-folds in the first phase is equal to

$$N = \frac{1}{4\sqrt{3}} \sqrt{\frac{1 + \sqrt{1 - 32\gamma\Lambda}}{\gamma}} (\bar{t}_{\text{fin}} - \bar{t}_{\text{ini}}) \approx \frac{\bar{t}_{\text{fin}} - \bar{t}_{\text{ini}}}{4\sqrt{3\gamma}}, \quad (38)$$

where $\bar{t}_{\text{fin}}$ is the time of the end of inflation and $\bar{t}_{\text{ini}}$ is the time of the beginning of inflation. In the last phase

$$N = \frac{1}{4\sqrt{3}} \sqrt{\frac{1 - \sqrt{1 - 32\gamma\Lambda}}{\gamma}} (\bar{t}_{\text{fin}} - \bar{t}_{\text{ini}}). \quad (39)$$

Figures 9 and 10 present the number of e-folds in the first phase with respect to the parameters γ and δ . In our model, inflation appears only when $\delta \geq 0$.

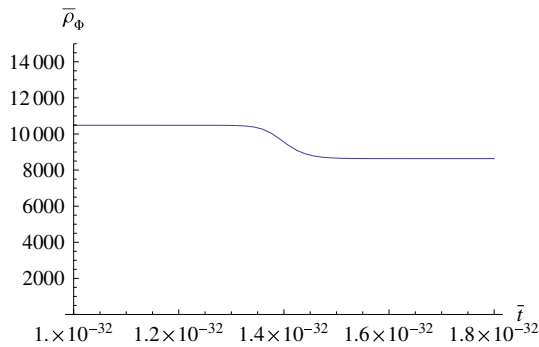


Fig. 8 The diagram presents the evolution of $\bar{\rho}_\Phi$ with respect to the cosmological time $\bar{t}$. The time is expressed in seconds and $\bar{\rho}_\Phi$ is expressed in $\frac{\text{km}^2}{\text{s}^2 \text{Mpc}^2}$. The value of the parameter γ is assumed as $3.277 \times 10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We also assume that $\frac{\Lambda}{3H_0^2} = 0.6911$, where $H_0 = 67.74 \frac{\text{km}}{\text{sMpc}}$. Note that $\bar{\rho}_\Phi$ is not a constant function when matter is not negligible (see Fig. 4). It is interesting that the function $\bar{\rho}_\Phi$ is of the logistic-like function type

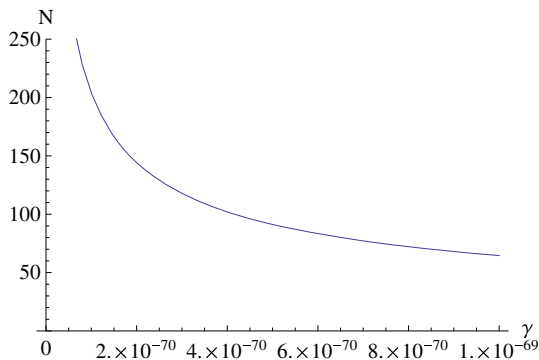


Fig. 9 The diagram presents the relation between the number of e-folds N and the parameter γ . The parameter γ is given in $\frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. We assume that $\delta = 0$ and the inflation time is of order 10^{-32} s [38]

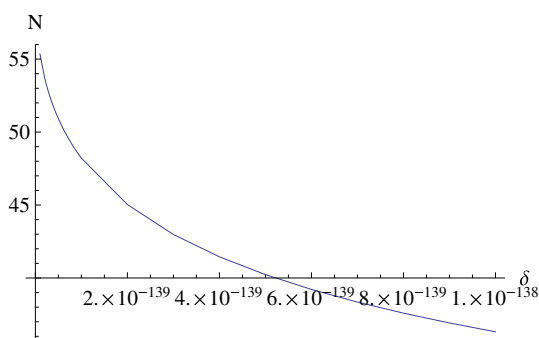


Fig. 10 The diagram presents the relation between the number of e-folds N and the parameter δ . The parameter δ is given in $\frac{\text{s}^4 \text{Mpc}^4}{\text{km}^4}$. We assume that $\gamma = 1.16 \times 10^{-69} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$ and the inflation time is of order 10^{-32} s [38]

If we assume that the parameter δ is equal to zero and $N = 50$ – 60 [37] and the inflation time is of order 10^{-32} s

[38] then the parameter γ belongs to the interval $(1.16 \times 10^{-69}, 1.67 \times 10^{-69})$. In consequence, the present value of $\frac{\bar{\rho}_\Phi}{3H_0^2}$ belongs to the interval $(3.41 \times 10^{-61}, 4.90 \times 10^{-61})$. This means that the running dark energy is negligible in the present epoch and does not influence the acceleration of the present Universe.

If the parameter $\delta \neq 0$ the number of e-folds is modified. For the parameter γ belonging to the interval $(1.16 \times 10^{-69}, 1.67 \times 10^{-69})$, we get the number of e-folds $N = 50$ – 60 , when the value of δ parameter belongs to the interval $(0, 6.4 \times 10^{-140})$.

4 Conclusions

We are looking for a cosmological model in which one can see both the early inflation and the late times acceleration phase of the expansion in a unique evolutionary scenario. To this aim we study the cosmological model of polynomial $f(R)$ gravity cut on the R^3 term in the Palatini formalism in the Einstein frame. This model can be treated as an extension of the Starobinsky model which is formulated in the metric formalism. Our model is formulated in the Palatini formalism, but it possesses analogous features and its main advantage is simplicity. The model is reduced to the FRW model with matter and dark energy in the form of the homogeneous scalar field. Both energy densities of the matter and dark energy are determined by the Ricci scalar of the FRW metric. Therefore they are given in the covariant way. In the Einstein frame the energy density of the dark energy is fully determined by the potential of the scalar field. Because the density of dark energy is running, the interaction appears naturally between the matter and dark energy which can also be parametrized in a covariant way through the Ricci scalar. It is interesting that in our model it is possible to achieve some analytic formulae on the energy densities of dark energy and dark matter.

While the Hilbert–Einstein action and the $f(R)$ -action can be related by a conformal transformation [39–41], the corresponding equations are connected by the same transformation. This fact shows that the Einstein frame and the Jordan frame are mathematically equivalent [42] but they could not be physically equivalent as pointed out in several papers (see e.g. [41, 43, 44]).

Our investigation confirms that theories equivalent mathematically on the classical level can be non-equivalent physically [45]. However, we observe in the context of our model that the Einstein frame is privileged in this sense that some strong singularities can be cured in the cosmological evolution [14]. A detailed discussion of the meaning of conformal transformations is in [46].

In our model, we have found that the plateau of the potential $\bar{U}(\Phi)$ is not necessary for the appearance of inflation

[34]. In the expansion of the function $f(\hat{R})$, the coefficient δ of the term $\hat{R}^3$ affects the number of e-folds. The number of e-folds decreases for $\delta > 0$ with respect to the number of e-folds obtained for the model with the $f(\hat{R})$ expansion cut off at a quadratic term. In our model, inflation appears only when $\delta \geq 0$.

In the model if the matter is vanishing we obtain eternal inflation following the stationary solution $H = \text{const}$. This result is valid for the function $f(\hat{R})$ given by the polynomial form $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2 + \sum_{i=2}^n \delta_i \hat{R}^{i+1}$. Only for an infinitesimally small fraction of matter inflation take places. The early inflation is studied in detail in terms of slow roll parameters as well as using the conception of constant roll inflation. We calculate the constant roll parameter $\beta = \frac{d \ln \phi}{d \ln a}$, which measures the elasticity of ϕ with respect to the scale factor. We have found the characteristic type of the behaviour of the parameter β following the logistic-like curve. One can distinguish four different phases in the time behaviour of the parameter β . In the first phase, the effects of matter are negligible but due to the interaction with the dark energy sector, the energy density of matter grows. As inflation progresses, matter is created, it disturbs the inflation phenomenon at the point when matter cannot be neglected. In consequence the first phase of inflation becomes unstable and the second phase appears. During the second and third phase, the effects of matter are not negligible. Finally, the fourth phase is characterized by diminishing effects of matter and the constant value of the Ricci scalar (and in consequence the constant value of energy density). During this phase dark energy dominates and the Universe behaves following the standard cosmological Λ CDM model.

Because the slow roll parameters are inadequate to constrain the model parameter we have found a bound on the model parameter γ from the numbers of required N -folds. If we assume that $N = 50\text{--}60$ [37] then the parameter γ belongs to the interval $(1.16 \times 10^{-69}, 1.67 \times 10^{-69})$. For this interval of the parameter γ , we get the number of e-folds $N = 50\text{--}60$, when the value of the δ parameter belongs to the interval $(0, 6.4 \times 10^{-140})$.

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Polynomial $f(R)$ Palatini cosmology: Dynamical system approach

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We investigate cosmological dynamics based on $f(R)$ gravity in the Palatini formulation. In this study, we use the dynamical system methods. We show that the evolution of the Friedmann equation reduces to the form of the piecewise smooth dynamical system. This system is reduced to a 2D dynamical system of the Newtonian type. We demonstrate how the trajectories can be sewn to guarantee C^0 extendibility of the metric similarly as “Milne-like” Friedmann-Lemaître-Robertson-Walker spacetimes are C^0 -extendible. We point out that importance of the dynamical system of the Newtonian type with nonsmooth right-hand sides in the context of Palatini cosmology. In this framework, we can investigate singularities which appear in the past and future of the cosmic evolution. We consider cosmological systems in both Einstein and Jordan frames. We show that at each frame the topological structures of phase space are different.

DOI: [10.1103/PhysRevD.97.103524](https://doi.org/10.1103/PhysRevD.97.103524)**I. INTRODUCTION**

Extended $f(R)$ gravity models [1–14] are intrinsic or geometric models of both dark matter and dark energy. Therefore, the idea of relational gravity, in which dark matter and dark energy can be interpreted as geometric objects, is naturally realized in $f(R)$ extended gravity.

The metric formulation of the extended gravity model gives the fourth order field equations, except for some cases, namely the Lovelock family of Lagrangians, where the field equations are second order [15]. This difficulty is solved by the Palatini formalism where the metric g and symmetric connection Γ are assumed to be independent variables. In this case, we get a system of second order partial differential equations [16]. This formalism also yields vacuum general relativity equations [17].

There are many papers about the Palatini formalism. In Olmo’s paper [16], a review of the Palatini $f(R)$ theories appears. Papers [18,19] are about the scalar-tensor representation of the Palatini theories. Studies about the existence of nonsingular solutions in Palatini gravity are in [20,21]. The papers [22–26] are about black holes and their singularities in the Palatini approach. Studies about the choice of a conformal frame in the Palatini gravity are in [27,28]. Compact stars in the Starobinsky model are discussed in [29].

Conformal transformations became interesting after the formulation of Weyl’s theory [30] aimed at unifying gravitation and electromagnetism. A conformally invariant version of special relativity was formulated in [31–33]. Another example of the development of Weyl’s theory is the self-consistent, scale-invariant theory of Canuto *et al.* [34]. In this theory, the astronomical unit of length is related to the atomic unit by a scalar function which depends on the spacetime point. This theory contains a running cosmological “constant” $\Lambda(t) = \Lambda_0 \frac{t_0}{t^2}$.

Recently, the most significant and important achievements appear in the context of the understanding of the Palatini theory and their application to the cosmological problem description of the evolution of the Universe [1,12,16,35–38]. If we consider Friedmann-Robertson-Walker (FRW) cosmological models in the Palatini framework in the Einstein frame, one can obtain the exact formula for the running cosmological constant parameter [39].

Cosmology is the physics of the Universe but in opposition to the physical system; we do not know the initial conditions for the Universe. Therefore, to explain the current state of the Universe we consider all admissible physically initial conditions and study all evolutionary paths for the evolution of the Universe in the universal cosmological time.

For this investigation of dynamics, the tools of the dynamical system theory are especially interesting. Dynamical system methods in the context of investigation dynamics of $f(R)$ gravity models have been used since Carroll [14,40]. The dynamical system is a system of

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differential equations which describes the motion of the points in the phase space [41]. In this approach, the evolution of the Universe is represented by trajectories in the phase space (spaces of all states of the system at any time). The phase space is organized by the singular solution represented by critical points (points in which the derivative of solutions of the dynamical system is zero), invariant submanifolds (submanifolds which are invariant under the action of the dynamical system) and trajectories (geometrical representations of solutions of the dynamical system). Whole dynamics can be visualized in a geometrical way on the phase portrait—a phase space of all evolutionary paths for all initial conditions [41]. We are looking for attractors (repellers) in the phase space to distinguish some generic evolution scenarios for the Universe [42].

We describe effectively the cosmic evolution in terms of the dynamical system of the Newtonian type. In this language, the motion of a fictitious particle mimics the evolution of the Universe and the potential contains all information needed for studying its dynamics. The right-hand side of the system cannot be a smooth function like for the cosmological evolution governed by general relativity. However, in any case, they are piecewise smooth functions. The context of the application of the Palatini formalism in the investigation of cosmological dynamics discovers the significance of new types of dynamical systems with nonsmooth right-hand sides [43]. It is interesting that cosmological singularities can be simply characterized in terms of the geometry of the potential $V(a)$, where a is the scale factor [43].

In this geometrical framework, singularities are manifested by a lack of analyticity of a potential itself or its derivatives with respect to the scale factor a and a diagram of the potential function (or its derivatives) possesses poles at some values of scale factor $a = a_{\text{sing}}$. Because the potential function is an additive function of energy density components, the discontinuities appearing on a diagram of the potential $V(a)$ can be interpreted as a discontinuous jumping of a potential part. This idea that a potential form possesses some part which contains jump discontinuities can be applied in different cosmological contexts. For example, it was considered to characterize singularities in phantom cosmologies [44].

Finite late-time singularities can be classified into six categories according to divergences of physical characteristics [45,46]:

- (a) Type 0: “Big crunch.” The scale factor a is vanishing and the Hubble parameter H , effective energy density ρ and pressure p are blown up.
- (b) Type I: “Big rip.” The scale factor a , ρ and p are blown up. They are classified as strong [47,48].
- (c) Type II: “Typical sudden.” The scale factor a , ρ and H are finite, and $\dot{H}$ and p are divergent. Geodesics are not incomplete in this case [49–51].
- (d) Type III: “Big freeze.” The scale factor a is finite and H , ρ and p are blown up [49] or divergent [52]. In this

case, there is no geodesic incompleteness and they can be classified as weak or strong [52].

- (e) Type IV: “Generalized sudden.” The scale factor a , H , ρ , p and $\dot{H}$ are finite but higher derivatives of the scale factor a diverge. These singularities are weak [53].
- (f) Type V: “ w singularities.” The cosmological time t is finite, the scale factor a and ρ blow up, p vanishes and a coefficient of the equation of state $w = \frac{p}{\rho}$ diverges.

These singularities are weak [54–56].

Following Królak [57], types 0 and I are strong, whereas types II, III and IV are weak singularities.

The main aim of the paper is a study of the cosmological equations based on $f(R)$ gravity in the Palatini formalism in both Einstein and Jordan frames. We want to show that the topological structures of phase space are different in these frames.

The order of this paper is as follows. In Sec. II, we introduce the Palatini formalism in the context of cosmology. We consider the Palatini formalism in cosmology in the Jordan frame in Sec. III and in the Einstein frame in Sec. IV. Section V is about differences between these frames. The last section is our conclusions.

II. PALATINI FORMALISM: INTRODUCTION

The Palatini gravity action of $f(\hat{R})$ gravity in the Jordan frame is given by

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} f(\hat{R}) d^4x + S_m, \quad (1)$$

where $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\Gamma)$ is the Ricci scalar and $\hat{R}_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ [16,58]. To simplify, we assume that $8\pi G = c = 1$.

After variation with respect to both dynamical variables g and Γ we obtain the field equations ($\delta S = 0$), which are the counterparts of the Einstein equations in the Palatini formalism, and an additional equation which establishes some relation between the metric and the connection,

$$f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = T_{\mu\nu}, \quad (2)$$

$$\hat{\nabla}_\alpha (\sqrt{-g} f'(\hat{R}) g^{\mu\nu}) = 0, \quad (3)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}$ is the matter energy momentum tensor and $\nabla^\mu T_{\mu\nu} = 0$ and $\hat{\nabla}_\alpha$ means that the covariant derivative is calculated with respect to connection Γ . The conservation equation $\nabla^\mu T_{\mu\nu} = 0$ is obtained from the Bianchi's identities $\nabla^\mu (f'(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu}) = 0$.

From the trace of the metric field equation (2), we get an additional equation, which is called the structural equation,

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = T, \quad (4)$$

where $T = g^{\mu\nu}T_{\mu\nu}$.

The metric g is the FRW metric for which the line element is given in the following form:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{1}{1-kr^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (5)$$

where $a(t)$ is the scale factor, k is a constant of spatial curvature ($k = 0, \pm 1$) and t is the cosmological time.

In this paper, we assume perfect fluid with the energy-momentum tensor

$$T^\mu_\nu = \text{diag}(-\rho, p, p, p), \quad (6)$$

where $p = w\rho$, $w = \text{const}$ is a form of the equation of state. From the conservation equation $T^\mu_{\nu;\mu} = 0$ we get that $\rho = \rho_0 a^{-3(1+w)}$. As a result, trace T is in the form

$$T = \sum_i \rho_{i,0} (3w_i - 1) a(t)^{-3(1+w_i)}. \quad (7)$$

In the above equation, parameters w_i correspond to different fluids described by the equation of state $p_i = w_i \rho_i$. We assume baryonic and dark matter ρ_m in the form of dust $w = 0$ and dark energy $\rho_\Lambda = \Lambda$ with $w = -1$.

A form of the function $f(\hat{R})$ is unknown. In this paper we assume that the polynomial form of the $f(\hat{R})$ function is in the form

$$f(\hat{R}) = \hat{R} + \gamma \hat{R}^2. \quad (8)$$

The Lagrangian (8) can be treated as a deviation from the lambda cold dark matter (Λ CDM) model by the quadratic Starobinsky term. The Starobinsky model in the Palatini formalism in the cosmological context is considered in [21,43].

A solution of the structural equation (4) has the following form:

$$\hat{R} = -T \equiv 4\Lambda + \rho_{m,0} a^{-3}. \quad (9)$$

Note that solution (9) has the same form in our model as in the Λ CDM model.

The Friedmann equation in our model is given by

$$\frac{H^2}{H_0^2} = \frac{b^2}{(b + \frac{d}{2})^2} \left[\Omega_\gamma (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0})^2 \frac{(K-3)(K+1)}{2b} + (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}) + \frac{\Omega_{r,0} a^{-4}}{b} + \Omega_k \right], \quad (10)$$

where $\Omega_k = -\frac{k}{H_0^2 a^2}$, $\Omega_{r,0} = \frac{\rho_{r,0}}{3H_0^2}$, $\Omega_{m,0} = \frac{\rho_{m,0}}{3H_0^2}$, $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$, $K = \frac{3\Omega_{\Lambda,0}}{(\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0})}$, $\Omega_\gamma = 3\gamma H_0^2$, $b = f'(\hat{R}) = 1 + 2\Omega_\gamma (\Omega_{m,0} a^{-3} + 4\Omega_{\Lambda,0})$, $d = \frac{1}{H} \frac{db}{dt} = -2\Omega_\gamma (\Omega_{m,0} a^{-3} + \Omega_{\Lambda,0})$

$(3-K)$, H_0 is the present value of Hubble function, $\rho_{r,0}$ is the present value of the energy density of radiation and $\rho_{m,0}$ is the present value of the density of matter. For simplicity, henceforth, we consider the model without radiation ($\rho_{r,0} = 0$). Note that for $\gamma = 0$, we get the Λ CDM model.

III. TYPES OF SINGULARITIES IN COSMOLOGY IN THE PALATINI FORMALISM IN THE JORDAN FRAME

In our model, new types of singularities appear which are not contained in the classification of Nojiri *et al.* They are nonisolated singularities. Our model with such singularities is an example of a piecewise smooth dynamical system of the cosmological origin.

Recently, a physically relevant solution of general relativity of the typical black hole spacetimes which admit C^0 -metric extensions beyond the future Cauchy horizon has focused mathematicians' attention [59] because this discovery is related to the fundamental issues concerning the strong cosmic censorship conjecture. In his paper, Sbierski [59] noted that the Schwarzschild solution in the global Kruskal-Szekeres coordinates is C^0 -extendible.

Galloway and Ling [60] reviewed some aspects of Sbierski's methodology in the general relativity context of cosmological solutions, and use similar techniques to Sbierski in the investigation of the C^0 extendibility of open Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models. They found that a certain special class of open FLRW spacetimes, which we have dubbed "Milne-like," actually admits the C^0 extension through the big bang. [60–62].

Sbierski has presented recently a new version of his original proof of the C^0 inextendibility of the maximal analytic Schwarzschild spacetime [63]. He deviates from his original proof by using the result, established in collaboration with Galloway and Ling [63], that given the C^0 extension of a globally hyperbolic spacetime, one can find a timelike geodesic that leaves this spacetime. Consequently, this result simplifies greatly the Sbierski proof of the inextendibility through the exterior region of the Schwarzschild spacetime.

The above-mentioned fact and phase portraits suggest that models with the sewn type of singularity can belong to a new class of metrics which admits C^0 extension like in the Milne-like model.

In our model, we find two new types of singularities, which are a consequence of the Palatini formalism: the sewn freeze and sewn sudden singularity. Generally, the freeze singularity takes place when the scale factor a is finite and H , ρ and p are blown up [49] or divergent [52], and the sudden singularity is when the scale factor a , ρ and H are finite and $\dot{H}$ and p are divergent [45]. The freeze

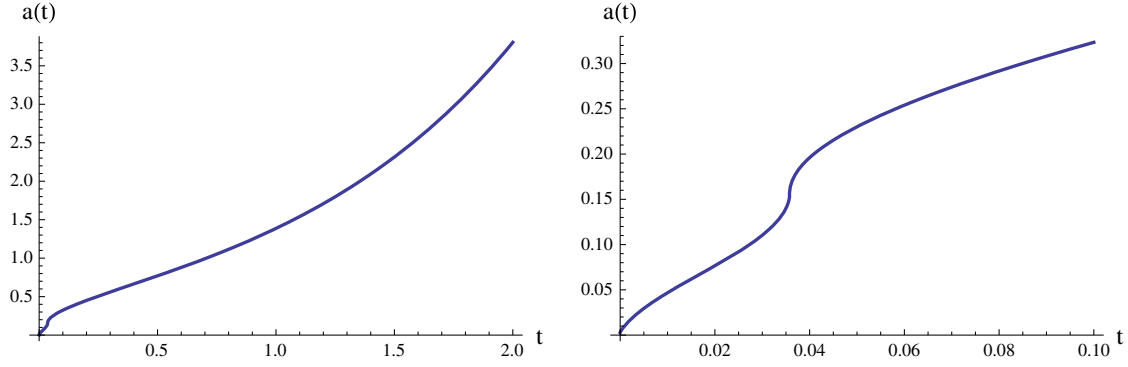


FIG. 1. The left panel presents the illustration of the evolution of the scale factor of the model (10) for the positive parameter γ for the flat universe. The right panel presents a close-up of the left panel in the neighborhood of the sewn freeze singularity (at the vertical inflection point). The value of the parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The cosmological time is expressed in $\frac{\text{sMpc}}{100 \text{ km}}$.

singularity appears when the expression $\frac{b}{b+d/2}$, in the Friedmann equation (10), is equal to infinity. The evolution of the scale factor of the model (10) through the sewn freeze singularity is presented in Fig. 1. The sewn sudden singularity appears when $\frac{b}{b+d/2}$ is equal to zero. This condition is equivalent to $b = 0$. The evolution of the scale factor of the model (10) through the sewn sudden singularity is presented in Fig. 2.

When the parameter γ is positive, the sewn freeze singularity appears. In this case, the evolution of the universe in our model and Λ CDM model are equivalent, except the freeze singularity. The evolution starts from the big bang and follows by the deceleration phase. Then the acceleration phase appears in the neighborhood of the sewn freeze singularity. In this singularity, the Hubble function

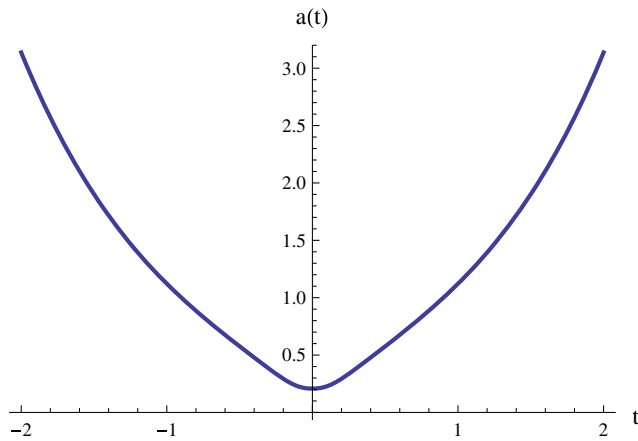


FIG. 2. The illustration of the evolution of the scale factor of the model (10) through the sewn sudden singularity (at the inflection point) for the flat universe. The model with the negative parameter Ω_γ has a mirror symmetry with respect to the cosmological time t . The bounce is at $t = 0$. The value of parameter γ is chosen as $-10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The cosmological time is expressed in $\frac{\text{sMpc}}{100 \text{ km}}$.

reaches the infinity value, which corresponds to the pole of the potential function. In this time, the inflation appears. After the inflation phase, the universe decelerates and the evolution is similar to the evolution in the Λ CDM model. The main physical effect of the sewn freeze singularity is the inflation, but its influence on the evolution of the universe is minor because the number of e -folds is too small [64].

In the case of the negative parameter γ , the big bang does not appear because it is replaced by the bounce, which corresponds with the sewn sudden singularity. In this singularity, the value of the Hubble function is zero. When the bounce is reached, the acceleration and next the deceleration phase appears. Afterwards, the behavior of the universe is like that in the Λ CDM model.

After an explicit application of geodesic equation to the Friedmann cosmology, one can find out whether geodesics can be prolonged through a singularity, i.e., about the geodesic incompleteness of the spacetime. Let us note that geodesics do not feel a singularity at all—they are not singular there since, for example $a_s = a(t_s) = \text{const}$ at $t = t_s$ being the time of a singularity, and there is no geodesic incompleteness [65].

A deeper insight in the structure of singularities can be obtained from the geodesic deviation equation (which measures the behavior of a bunch of geodesics). It is important that this equation does feel singularities since at $t = t_s$ the Riemann tensor $R_{\alpha\beta\mu\nu} \rightarrow \infty$. As an example we see that with the sudden singularity it is possible to “go through” the singularity since we have

$$R^\alpha_{0\beta 0} = -\frac{\ddot{a}}{a} \delta^\alpha_\beta, (\dots) = \frac{\partial}{\partial t}, \quad (11)$$

$$\ddot{u}^\alpha = -R^\alpha_{0\beta 0} n^\beta \propto \ddot{a} \propto -\frac{\partial V}{\partial a}, \quad (12)$$

where δ^α_β is the Kronecker delta, u^α is the four-velocity vector and n^α is the deviation vector separating neighboring

geodesics (particle worldlines) which describes the propagation of the distance between geodesics.

The curvature tensor feels, for example, the sudden singularity because the Riemann tensor diverges to minus infinity at $t = t_s$.

Physically, it means that the tidal forces which manifest here as the (infinite) impulse which reverses (or stops) the increase of separation of geodesics and the geodesics themselves can evolve further—the universe can continue its evolution through a singularity.

In our model, the sewn freeze singularity is a solution of the following algebraic equation:

$$2b + d = 0 \quad (13)$$

or

$$-3K - \frac{K}{3\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})\Omega_{\Lambda,0}} + 1 = 0, \quad (14)$$

where $K \in [0, 3)$.

The solution of Eq. (14) is

$$K_{\text{freeze}} = \frac{1}{3 + \frac{1}{3\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})\Omega_{\Lambda,0}}}. \quad (15)$$

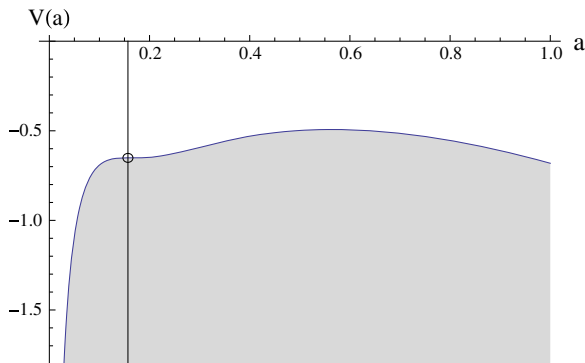
We obtain an expression for a value of the scale factor at the freeze singularity from Eq. (15),

$$a_{\text{freeze}} = \left(\frac{1 - \Omega_{\Lambda,0}}{8\Omega_{\Lambda,0} + \frac{1}{\Omega_\gamma(\Omega_m + \Omega_{\Lambda,0})}} \right)^{\frac{1}{3}}. \quad (16)$$

We get the sewn sudden singularity when $b = 0$. This gets us the following algebraic equation:

$$1 + 2\Omega_\gamma(\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0}) = 0. \quad (17)$$

From Eq. (17), we get the formula for the scale factor at a sewn sudden singularity,



$$a_{\text{sudden}} = \left(-\frac{2\Omega_{m,0}}{\frac{1}{\Omega_\gamma} + 8\Omega_{\Lambda,0}} \right)^{1/3}. \quad (18)$$

We can rewrite Eq. (10) as a dynamical system. We choose a and $y = a'$, where $' \equiv \frac{d}{d\sigma} = \frac{b+d}{b} \frac{d}{dH_0 t}$ is a new parametrization of time, as the variables of the dynamical system. We derive these variables with respect to the σ time using Eq. (10) and we get the following equations of the dynamical system:

$$a' = y, \quad (19)$$

$$y' = -\frac{\partial V(a)}{\partial a}, \quad (20)$$

where

$$V = -\frac{a^2}{2} \left[\Omega_\gamma(\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0})^2 \frac{(K-3)(K+1)}{2b} + (\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0}) \right]. \quad (21)$$

We can treat the dynamical system [(19)–(20)] as a sewn dynamical system [66,67]. In this case, the phase portrait is divided into two regions: the first part is for $a < a_{\text{sing}}$ and the second part is for $a > a_{\text{sing}}$. Both parts are sewn along the singularity.

For $a < a_{\text{sing}}$, we can rewrite the dynamical system [(19)–(20)] in the corresponding form

$$a' = y, \quad (22)$$

$$y' = -\frac{\partial V_1(a)}{\partial a}, \quad (23)$$

where $V_1 = V(-\eta(a - a_s) + 1)$ and $\eta(a)$ denotes the Heaviside function.

For $a > a_{\text{sing}}$, we get in an analogous way the following equations:

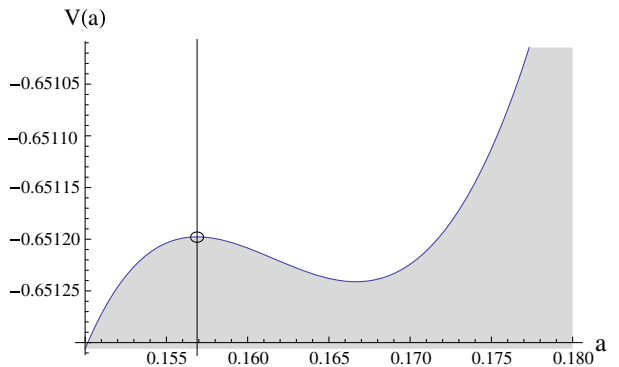


FIG. 3. The left panel presents the diagram of the potential $V(a)$ (21) for the positive parameter γ . The right panel presents a close-up of the left diagram in the neighborhood of the sewn singularity. The vertical line represents the sewn freeze singularity. The parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. Note that for $a = a_{\text{sing}}$, $V(a)$ is undefined.

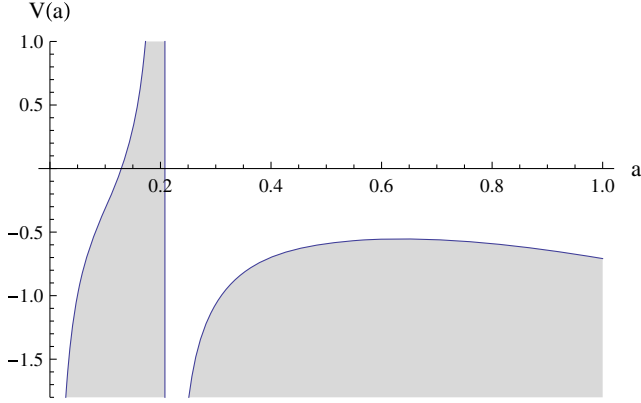


FIG. 4. The diagram of the potential $V(a)$ (21) for the negative parameter γ . The vertical line represents the sewn sudden singularity. The parameter γ is chosen as $-10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$.

$$a' = y, \quad (24)$$

$$y' = -\frac{\partial V_2(a)}{\partial a}, \quad (25)$$

where $V_2 = V\eta(a - a_s)$.

The diagrams of the potential function $V(a)$ (21) are presented in Fig. 3 for the positive parameter γ and in Fig. 4 for the negative parameter γ . The phase portraits of the

system can be constructed similarly as in classical mechanics due to the particlelike description of dynamics. Phase trajectories representing evolutionary paths can be obtained directly from the geometry of potential function $V(a)$ by consideration of constant energy levels $(a')^2/2 + V(a) = E = \text{const} = -k/2$. The reparametrized time parameter σ is measured along the trajectories of the corresponding dynamical system. It has a sense of a diffeomorphic transformation beyond the singularity vertical line.

The potential function (21) is undefined at the singularity point $a = a_{\text{sing}}$. Therefore, in phase portraits of the system in the Jordan frame, there are two domains separated by a line of singularity points. These phase portraits are constructed by the application of the diffeomorphic reparametrization of cosmological time beyond this singularity line and then C^1 sewing of trajectories. As a result, we obtain that only one unique trajectory moves at any point in the phase space.

The phase portraits for the system [(19)–(20)] for positive Ω_γ are presented in Fig. 5 and for negative Ω_γ in Fig. 6. The line of singularity points is represented by a dashed line.

We find that the system [(19)–(20)] for positive Ω_γ has a sequence of three critical points located on the a axis (saddle-center-saddle sequence). To clarify the behavior of trajectories in the neighborhood of the saddle located at the singularity line we present a close-up of this area in Fig. 5 (see the right panel).

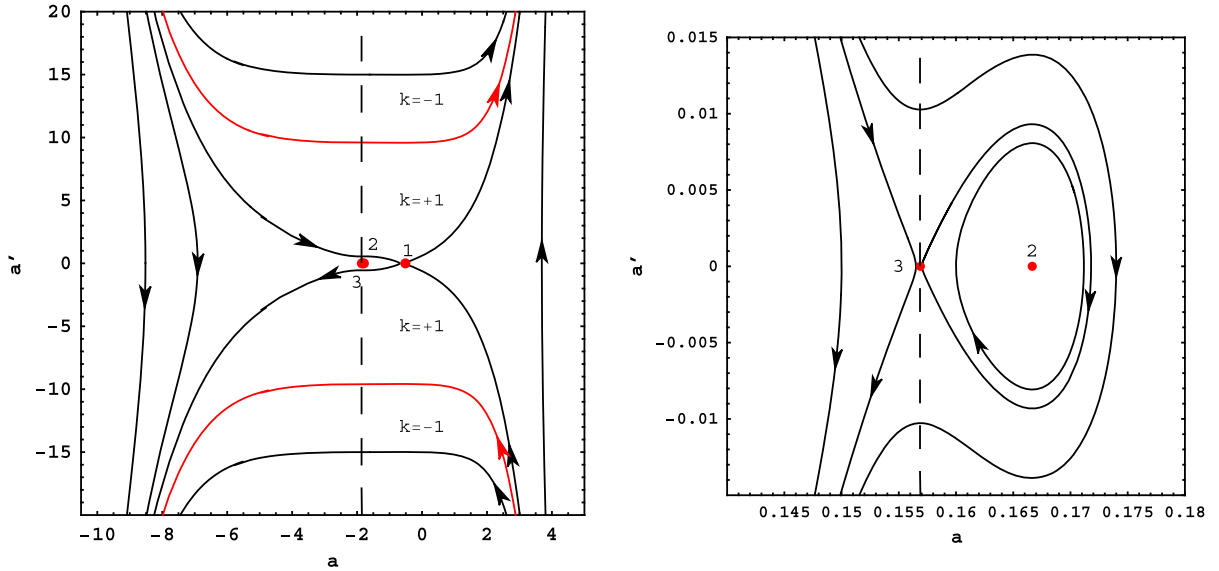


FIG. 5. The left panel is the phase portrait of the system [(19)–(20)] with the positive parameter Ω_γ . The right panel is a close-up of the left panel in the neighborhood of critical points 2 and 3. The value of parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The value of $\Omega_{\Lambda,0}$ is chosen as 0.7 and the present value of the Hubble function is chosen as $68 \frac{\text{km}}{\text{sMpc}}$. The scale factor a is presented in the natural logarithmic scale. The spatially flat universe is represented by the red trajectories. The dashed line $2b + d = 0$ represents the freeze singularity. The critical points 1, 2 and 3 represent the static Einstein universes. The phase portrait belongs to the class of sewn dynamical systems. Note that the existence of the homoclinic orbit which starts at $t = -\infty$ and approach at $t = +\infty$. In the interior of this orbit, there are located trajectories representing oscillating cosmological models. They are free from initial and final singularities.

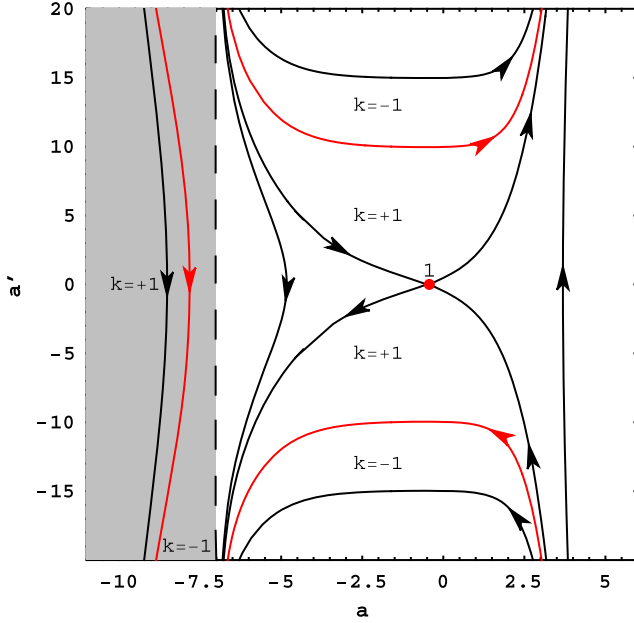


FIG. 6. The phase portrait of the system [(19)–(20)] with the negative parameter Ω_γ . The value of the parameter γ is chosen as $-10^{-13} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The value of $\Omega_{\Lambda,0}$ is chosen as 0.7 and the present value of the Hubble function is chosen as $68 \frac{\text{km}}{\text{s Mpc}}$. The scale factor a is presented in the natural logarithmic scale. The spatially flat universe is represented by the red trajectories. The dashed line separates the domain where $a < a_{\text{sing}}$ from the domain where $a > a_{\text{sing}}$. The shaded region represents trajectories with $b < 0$. If we assume that $f'(R) > 0$, then this region can be removed. Critical point 1 represents the static Einstein universe. The critical points at infinity, $a = a_{\text{sing}}$, $a' = \pm\infty$ represent typical sudden singularities. The phase portrait belongs to the class of sewn dynamical systems.

In Fig. 6 the critical points at infinity, $a = a_{\text{sing}}$, $a' = \pm\infty$ represent typical sudden singularities. There are two types of sewn trajectories: one homoclinic orbit and infinity of periodic orbits. The homoclinic orbit starts from the neighborhood of critical point 1, goes to the singularity at $a' = -\infty$ and, after sewing with the trajectory which comes from the singularity at $a' = +\infty$, finishes at the saddle point 1. The periodic orbits are situated inside the domain bounded by the homoclinic orbit. Similarly to the homoclinic orbit, the periodic orbits are sewn when going to the

minus infinity singularity and going out from the plus infinity singularity. Note that these periodic orbits are possible only in the $k = +1$ universe. There are also nonperiodic trajectories which lie inside the two regions bounded by the separatrices of the saddle 1. The trajectories start at $a' = -\infty$, approach saddle 1, go to the minus infinity singularity after sewing go out from the plus infinity singularity, approach saddle 1 and then continue to $a' = +\infty$. This kind of evolution is possible for the flat universe as well as $k = -1$ and $k = +1$ universes. Finally, in the region on the right of the separatrices of saddle 1, the trajectories start at $a' = -\infty$ and go to $a' = +\infty$, representing the evolution without a sewn sudden singularity of the $k = +1$ universes.

The critical points of the dynamical system [(19)–(20)] are completed in Table I.

The action (1) can be rewritten as

$$S = S_g + S_m = \frac{1}{2} \int \sqrt{-g} \phi \hat{R} d^4x + S_m, \quad (26)$$

where $\phi = \frac{f(\hat{R})}{\hat{R}}$. Let G_{eff} mean the effective gravitational constant. Then $\phi = \frac{1}{8\pi G_{\text{eff}}}$ and in the consequence $G_{\text{eff}}(\hat{R}) = \frac{\hat{R}}{8\pi f(\hat{R})}$ and especially for $f(\hat{R}) = \hat{R} + \gamma \hat{R}^2$ has the following form:

$$\frac{G_{\text{eff}}(\hat{R})}{G} = \frac{1}{1 + \gamma \hat{R}}. \quad (27)$$

The evolution of G_{eff} is presented in Fig. 7. Note that the value of G_{eff} for $t = 0$ is equal to zero and approaches asymptotically to the value of gravitational constant.

IV. THE PALATINI FORMALISM IN THE EINSTEIN FRAME

Scalar-tensor theories of gravity can be formulated in the Jordan and in the Einstein frames. These frames are conformally related [68]. We know that the formulations of a scalar-tensor theory in two different conformal frames are physically inequivalent. There was a remarkable progress in the understanding of the geometric features of the Palatini theories and the role of the choice of a frame in the last years [69,70]. Considering the model in the

TABLE I. Critical points of the dynamical system [(19)–(20)]. They are also presented in Fig. 5. All three critical points represent a static Einstein universe.

No. of critical point	Coordinates of critical point	Type of critical point
1	$\left(a = \left(\frac{8\gamma\Lambda^2 - \Lambda + 3H_0^2(1-8\gamma\Lambda) + (3H_0^2 - \Lambda)\sqrt{(1-24\gamma\Lambda)}}{4\Lambda(1+8\gamma\Lambda)} \right)^{1/3}, a' = 0 \right)$	saddle
2	$\left(a = \left(\frac{8\gamma\Lambda^2 - \Lambda + 3H_0^2(1-8\gamma\Lambda) - (3H_0^2 - \Lambda)\sqrt{(1-24\gamma\Lambda)}}{4\Lambda(1+8\gamma\Lambda)} \right)^{1/3}, a' = 0 \right)$	center
3	$\left(a = \frac{(\gamma(3H_0^2 - \Lambda))^{1/3}}{(1+8\gamma\Lambda)^{1/3}}, a' = 0 \right)$	saddle

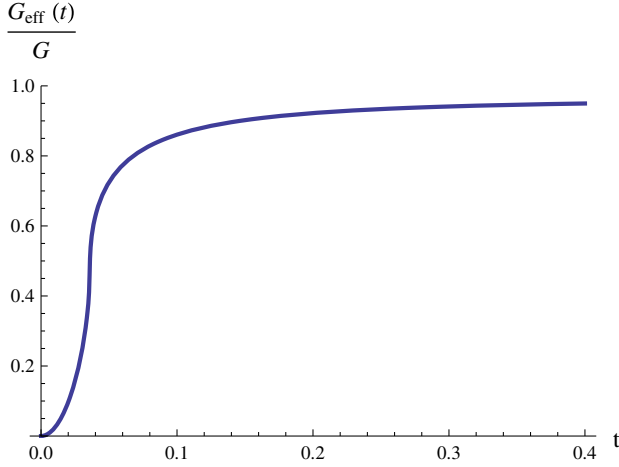


FIG. 7. The evolution of G_{eff} for the positive parameter γ and the flat universe. The cosmological time t is expressed in $\frac{\text{sMpc}}{100 \text{ km}}$. The parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. Note that when $t \rightarrow \infty$ then $\frac{G_{\text{eff}}(t)}{G} \rightarrow \frac{1}{1+4\gamma\Lambda}$.

Einstein frame in the Palatini formalism, we find that the big bang singularity is replaced by the singularity of the finite scale factor and that some pathologies, like degenerated multiple freeze singularities [64], disappear in a generic case.

If $f''(\hat{R}) \neq 0$, then action (1) is dynamically equivalent to the first order Palatini gravitational action [1,12,36]

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \chi) = \frac{1}{2} \int d^4x \sqrt{-g} (f'(\chi)(\hat{R} - \chi) + f(\chi)) + S_m(g_{\mu\nu}, \psi). \quad (28)$$

Let $\Phi = f'(\chi)$ be a scalar field and $\chi = \hat{R}$. Then action (28) can be rewritten in the following form:

$$S(g_{\mu\nu}, \Gamma_{\rho\sigma}^\lambda, \Phi) = \frac{1}{2} \int d^4x \sqrt{-g} (\Phi \hat{R} - U(\Phi)) + S_m(g_{\mu\nu}, \psi), \quad (29)$$

where the potential $U(\Phi)$ is defined by

$$U_f(\Phi) \equiv U(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)), \quad (30)$$

where $\Phi = \frac{df(\chi)}{d\chi}$ and $\hat{R} \equiv \chi = \frac{dU(\Phi)}{d\Phi}$.

We can get from the Palatini variation of the action (29) the following equations of motion:

$$\Phi \left(\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} \right) + \frac{1}{2} g_{\mu\nu} U(\Phi) - T_{\mu\nu} = 0, \quad (31a)$$

$$\hat{\nabla}_\lambda (\sqrt{-g} \Phi g^{\mu\nu}) = 0, \quad (31b)$$

$$\hat{R} - U'(\Phi) = 0. \quad (31c)$$

From Eq. (31b), we get that the connection $\hat{\Gamma}$ is a metric connection for a new metric $\bar{g}_{\mu\nu} = \Phi g_{\mu\nu}$; thus, $\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}$, $\bar{R} = \bar{g}^{\mu\nu} \bar{R}_{\mu\nu} = \Phi^{-1} \hat{R}$ and $\bar{g}_{\mu\nu} \bar{R} = g_{\mu\nu} \hat{R}$. The g trace of (31a) gives a new structural equation,

$$2U(\Phi) - U'(\Phi)\Phi = T. \quad (32)$$

The question of whether the metric $g_{\mu\nu}$ or $\bar{g}_{\mu\nu}$ has the physical meaning is a problem of the interpretation of these functions. It is strictly related to the problem of a choice of the frame (Einstein frame or Jordan frame). Some people claim that a conformally rescaled metric by a scalar field is only an mathematical trick without a physical meaning. However, the objectivity of investigation requires the consideration of both cases. In our opinion, only astronomical observations can resolve this question [71]. In this section, we also consider that $\bar{g}_{\mu\nu}$ has the physical meaning in the Einstein frame. We are looking for such a choice of the frame in which inflation can be reproduced in analogy to the Starobinsky model. Unfortunately, it is not the case of the Jordan case. Azri [72] tried to answer the question about the reality of conformal frames in the context of the nonminimal coupling dynamics of a single scalar field in purely affine gravity. In this approach, the coupling is performed via an affine connection and its associated curvature without referring to any metric tensor. It is interesting that in affine gravity the transition from nonminimal to minimal couplings is realized by only field redefinition of the scalar field. As a result, the inflationary models gain a unique description in this context where observed parameters are invariant under a field reparametrization. The inflation in the Starobinsky model is realized in the Einstein frame but it would be nice to find the realization of the inflation as a phenomenon which is invariant under the redefinition of the scalar field.

Now Eqs. (31a) and (31c) take the following forms:

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{U}(\Phi), \quad (33)$$

$$\Phi \bar{R} - (\Phi^2 \bar{U}(\Phi))' = 0, \quad (34)$$

where $\bar{U}(\phi) = U(\phi)/\Phi^2$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$ and the structural equation can be replaced by

$$\Phi \bar{U}'(\Phi) + \bar{T} = 0. \quad (35)$$

As a result, the action for the metric $\bar{g}_{\mu\nu}$ and scalar field Φ is given in the following form:

$$S(\bar{g}_{\mu\nu}, \Phi) = \frac{1}{2} \int d^4x \sqrt{-\bar{g}} (\bar{R} - \bar{U}(\Phi)) + S_m(\Phi^{-1} \bar{g}_{\mu\nu}, \psi), \quad (36)$$

where a nonminimal coupling is between Φ and $\bar{g}_{\mu\nu}$,

$$\bar{T}^{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}} \delta \bar{g}_{\mu\nu}} S_m = (\bar{\rho} + \bar{p}) \bar{u}^\mu \bar{u}^\nu + \bar{p} \bar{g}^{\mu\nu} = \Phi^{-3} T^{\mu\nu}, \quad (37)$$

$\bar{u}^\mu = \Phi^{-\frac{1}{2}} u^\mu$, $\bar{\rho} = \Phi^{-2} \rho$, $\bar{p} = \Phi^{-2} p$, $\bar{T}_{\mu\nu} = \Phi^{-1} T_{\mu\nu}$, $\bar{T} = \Phi^{-2} T$ [12,73].

In the FRW metric case, metric $\bar{g}_{\mu\nu}$ has the following form:

$$d\bar{s}^2 = -d\bar{t}^2 + \bar{a}^2(\bar{t})[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (38)$$

where $d\bar{t} = \Phi(t)^{\frac{1}{2}} dt$ and new scale factor $\bar{a}(\bar{t}) = \Phi(\bar{t})^{\frac{1}{2}} a(\bar{t})$. Because we assume the barotropic matter, the cosmological equations are given by

$$3\bar{H}^2 = \bar{\rho}_\Phi + \bar{\rho}_m, \quad 6\frac{\ddot{\bar{a}}}{\bar{a}} = 2\bar{\rho}_\Phi - \bar{\rho}_m(1+3w), \quad (39)$$

where

$$\bar{\rho}_\Phi = \frac{1}{2}\bar{U}(\Phi), \quad \bar{\rho}_m = \rho_0 \bar{a}^{-3(1+w)} \Phi^{\frac{1}{2}(3w-1)} \quad (40)$$

and $w = \bar{p}_m/\bar{\rho}_m = p_m/\rho_m$. The conservation equation gets the following form:

$$\dot{\bar{\rho}}_m + 3\bar{H}\bar{\rho}_m(1+w) = -\dot{\bar{\rho}}_\Phi. \quad (41)$$

In the case of the Starobinsky-Palatini model, the potential $\bar{U}$ is described by the following formula:

$$\bar{U}(\Phi) = 2\bar{\rho}_\Phi(\Phi) = \left(\frac{1}{4\gamma} + 2\lambda\right) \frac{1}{\Phi^2} - \frac{1}{2\gamma} \frac{1}{\Phi} + \frac{1}{4\gamma}. \quad (42)$$

V. A COMPARISON OF THE JORDAN FRAME AND THE EINSTEIN FRAME IN THE PALATINI FORMALISM

If we consider dynamics in the Jordan frame, then one can use a formula for H^2 to reduce the dynamics to the dynamical system of the Newtonian type which possesses the first integral $\frac{1}{2}(\frac{da}{dt})^2 + V(a) = 0$, where $V(a) = -\frac{1}{2}H^2 a^2$. In this representation of dynamics, singularities for the finite value of the scale factor $a = a_s$ are poles of $V(a)$ potential or their derivatives. Stachowski *et al.* [64] investigated these type of singularities in detail. The generic feature of the formulation of dynamics is the appearance of the freeze or typical sudden type of singularity in the past. At the freeze singularity point while the scale factor is finite, its second derivative with respect to the time blows up, i.e., $\frac{d^2 a}{dt^2} = \pm\infty$. In general, all singularities can be detected from the diagram of the potential function.

If we consider dynamics in the Einstein frame, there are no such singularities. The big bang singularity present in the Λ CDM model is replaced by the generalized sudden singularity of the finite scale factor. Beyond this singularity, the phase portrait is equivalent to the Λ CDM model.

Two dynamical systems in the phase space are equivalent if there is a homeomorphism transforming all trajectories with the preserving of the direction of time measured along the trajectories. The comparison of dynamics in both the Jordan and Einstein frames explicitly shows that

corresponding dynamical systems are not topologically equivalent. Consequently, the physics in both frames is different.

The cosmological equation for the Starobinsky-Palatini model in the Einstein frame can be rewritten to the form of the dynamical system with the Hubble parameter $\hat{H}(\bar{t})$ and the Ricci scalar $\hat{R}(\bar{t})$ as variables,

$$\dot{\hat{H}}(\bar{t}) = \frac{1}{6(1+2\gamma\hat{R}(\bar{t}))^2} (6\Lambda - 6\bar{H}(\bar{t})^2(1+2\gamma\hat{R}(\bar{t}))^2 + \hat{R}(\bar{t})(-1+24\gamma\Lambda + \gamma(1+24\gamma\Lambda)\hat{R}(\bar{t}))), \quad (43)$$

$$\dot{\hat{R}}(\bar{t}) = -\frac{3}{(-1+\gamma\hat{R}(\bar{t}))} \bar{H}(\bar{t})(1+2\gamma\hat{R}(\bar{t}))(4\Lambda + \hat{R}(\bar{t})) \times (-1+16\gamma\Lambda + 16\gamma^2\Lambda\hat{R}(\bar{t})), \quad (44)$$

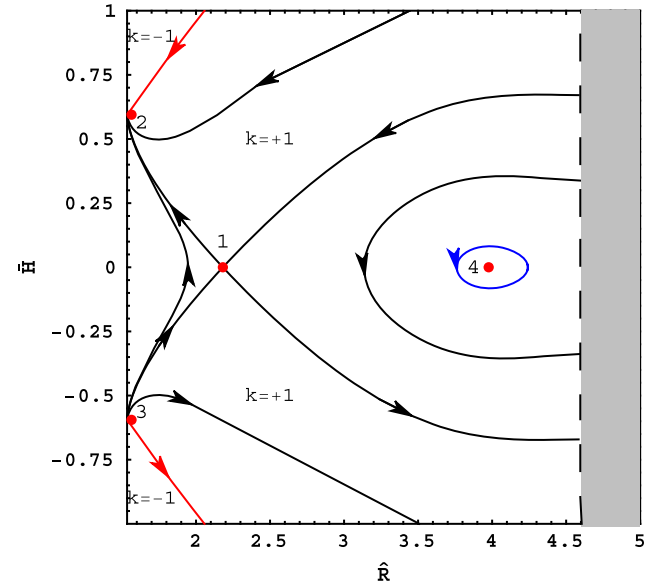


FIG. 8. The phase portrait of system (43)–(44). There are four critical points: point 1 represents the Einstein universe, point 2 represents the stable de Sitter universe, point 3 represents the unstable de Sitter universe and point 4 represents the Einstein universe. The value of the parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The value of $\Omega_{\Lambda,0}$ is chosen as 0.7 and the present value of the Hubble function is chosen as $68 \frac{\text{km}}{\text{s Mpc}}$. The values of the Hubble function are given in $\frac{100 \text{ km}}{\text{s Mpc}}$ and the values of the Ricci scalar are given in $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$ in the natural logarithmic scale. The gray color represents the nonphysical domain. The dashed line represents the generalized sudden singularity. Note that for the Starobinsky-Palatini model in the Einstein frame for the positive parameter γ , the sewn freeze singularity is replaced by the generalized sudden singularity. A typical trajectory in the neighborhood of the trajectory of the flat model (represented by the red trajectory) starts from the generalized sudden singularity then goes to the de Sitter attractor. The position of this attractor is determined by the cosmological constant parameter. Oscillating models (blue trajectory) are situated around critical point 4.

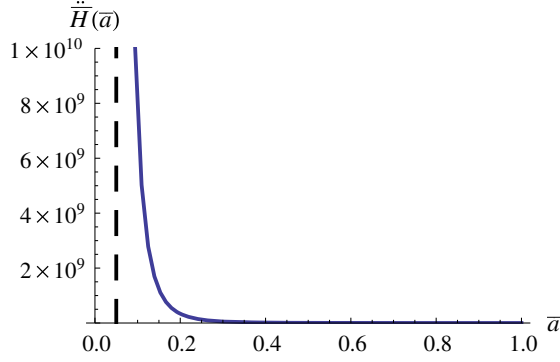


FIG. 9. The relation $\ddot{H}(\bar{a})$ for the Palatini formalism in the Einstein frame. The value of the parameter γ is chosen as $10^{-9} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The values of the $\ddot{H}(\bar{a})$ are given in $\frac{\text{km}^3}{\text{s}^3 \text{Mpc}^3}$. The dashed line represents the generalized sudden singularity. Note that, in the generalized sudden singularity, H and $\dot{H}$ are finite but $\ddot{H}$ and its derivatives are divergent.

where a dot denotes the differentiation with respect to the time $\bar{t}$. The phase portrait for the dynamical system [(43)–(44)] is presented in Fig. 8. Here, the periodic orbits appear around critical point 4. In the Starobinsky-Palatini model in the Einstein frame appears the generalized sudden singularity, for which H and $\dot{H}$ are finite but $\ddot{H}$ and its derivatives are diverge (see Fig. 9). The evolution of the scale factor begins from a finite value different from zero (see Fig. 10). In terms of the scale factor, at the singularity for the finite value of the scale factor $\bar{a}$, a third time derivative (and higher orders) of the scale factor in Einstein frame blows up, while first and second order time derivatives behave regularly. The evolution of the scale factor for one of these periodic orbits is presented in Fig. 11. When matter is negligible, then the inflation appears. In this case, $a \approx a_0 \exp\left(\frac{t}{4} \sqrt{\frac{1+\sqrt{1-32\gamma\Lambda}}{3\gamma}}\right)$, where $a_0 = a(0)$ and $R(t) \approx \frac{1-16\gamma\Lambda+\sqrt{1-32\gamma\Lambda}}{32\gamma^2\Lambda}$ [74]. If $\gamma > \frac{1}{36\Lambda}$, then the non-physical domain appears for $\hat{R} < \frac{1-16\gamma\Lambda+\sqrt{1-32\gamma\Lambda}}{32\gamma^2\Lambda}$ for which $\rho_m < 0$.

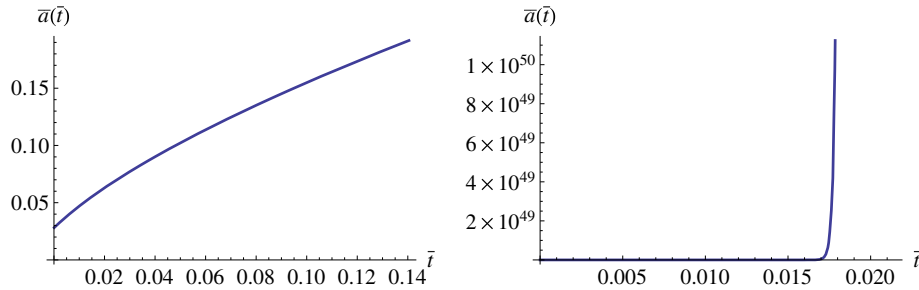


FIG. 10. The illustration of the evolution of the scale factor for the Palatini formalism in the Einstein frame for the flat universe. The left panel presents the case when matter is not negligible. The right panel presents the case when matter is negligible. The value of parameter γ is chosen as $10^{-9} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The cosmological time is expressed in $\frac{\text{sMpc}}{\text{km}}$. Note that the evolution of the scale factor begins from a finite value different from zero. Note that when matter is negligible, then the inflation appears (see the right panel). In this case, the number of e -folds is equal to 50.

For comparison of the dynamical systems in both frames, we obtain the dynamical system for the Starobinsky-Palatini model in the Jordan frame in the variables $H(t)$ and $\hat{R}(t)$

$$\dot{H}(t) = -\frac{1}{6} \left[6(2\Lambda + H(t)^2) + \hat{R}(t) + \frac{18(1+8\gamma\Lambda)(\Lambda - H(t)^2)}{-1-12\gamma\Lambda + \gamma\hat{R}(t)} - \frac{18(1+8\gamma\Lambda)H(t)^2}{1+2\gamma\hat{R}(t)} \right], \quad (45)$$

$$\dot{\hat{R}}(t) = -3H(t)(\hat{R}(t) - 4\Lambda), \quad (46)$$

where a dot means the differentiation with respect to time t . The phase portrait for the dynamical system (45)–(46) is shown in Fig. 12 (see left panel). This phase portrait represents all evolutionary paths of the system in the Jordan frame without adopting the time reparametrization. Along the trajectories is measured the original cosmological time t . The system [(45)–(46)] constitutes a two-dimensional autonomous dynamical system. Let us note that while the Ricci scalar $\hat{R}$ is related with a second time derivative of the scale factor a , the Hubble function H is related with a first time derivative of the scale factor a . The oscillating orbits appear around critical point 4 (see Fig. 12). The evolution of the scale factor for one of these periodic orbits is presented in Fig. 13.

For a deeper analysis of the behavior of the trajectories of system (45)–(46) in the infinity, we introduce variables $\hat{R}$ and $W = \frac{H}{\sqrt{1+H^2}}$ and rewrite Eqs. (45)–(46) in these variables. Then we get the following dynamical system:

$$\dot{W}(t) = \frac{\dot{H}(t)}{(1+H(t)^2)^{3/2}} = -\frac{(1-W(t)^2)^{3/2}}{6} \left[6\left(2\Lambda + \frac{W(t)^2}{1-W(t)^2}\right) + \hat{R}(t) + \frac{18(1+8\gamma\Lambda)\left(\Lambda - \frac{W(t)^2}{1-W(t)^2}\right)}{-1-12\gamma\Lambda + \gamma\hat{R}(t)} - \frac{18(1+8\gamma\Lambda)\frac{W(t)^2}{1-W(t)^2}}{1+2\gamma\hat{R}(t)} \right], \quad (47)$$

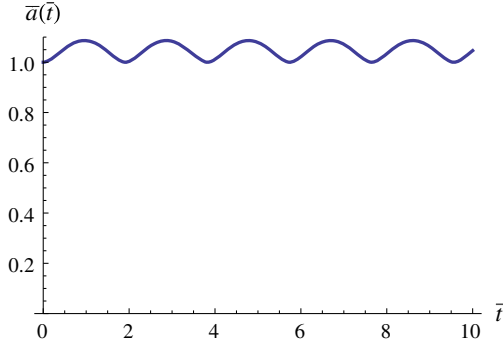


FIG. 11. The diagram presents the evolution of the scale factor for trajectory of the oscillating orbit in the neighborhood of critical point 4 (see Fig. 8). The cosmological time is expressed in $\frac{\text{s Mpc}}{100 \text{ km}}$. Here, $a_{\min} = 1$.

$$\dot{\hat{R}}(t) = -3 \frac{W(t)}{\sqrt{1 - W(t)^2}} (\hat{R}(t) - 4\Lambda). \quad (48)$$

The phase portrait for dynamical system (47)–(48) is presented in Fig. 12 (the right panel). This portrait is a good illustration of how trajectories are sewn at the points at infinity (points 5 and 6). For expanding models situated on

the upper part of the domain, where W is positive, all the trajectories pass through point 6. This continuation of trajectories is the class of C^0 . The singularity line represents the freeze type of singularity. There are some differences in the behavior of trajectories of the same model represented in Figs. 5 and 12. While the continuation on the singularity line in Fig. 5 is smooth of C^1 class and the Cauchy problem is correctly solved in Fig. 12, all trajectories from separated regions focused at the degenerated point 6 (and point 5 for contracting models) represent the freeze type of singularity. It has a consequence for the solution of the Cauchy problem. Therefore, the representation of dynamics in the reparametrized time seems to be more suitable than in the original cosmological time.

For the Eqs. (43)–(44) and (45)–(46), we can find the first integrals. In the case of Eqs. (43)–(44), the first integral has the following form:

$$\bar{H}(\bar{t})^2 + \Lambda - \frac{\hat{R}(\bar{t})(2 + \gamma \hat{R}(\bar{t}))}{6(1 + 2\gamma \hat{R}(\bar{t}))^2} + \frac{k}{\bar{a}^2} = 0. \quad (49)$$

Because

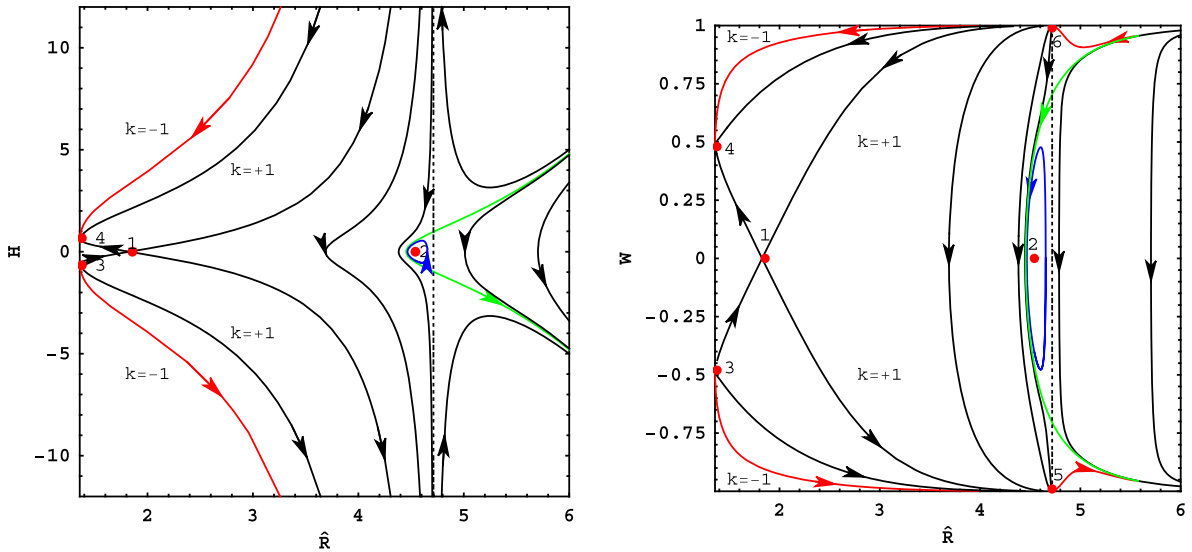


FIG. 12. The left panel is the phase portrait of system (45)–(46) and the right one is the phase portrait of system (47)–(48). There are four critical points in both systems: point 1 and 2 represent the Einstein universe, point 3 represents the unstable de Sitter universe and point 4 represents the stable de Sitter universe. For illustration, the value of the parameter γ is chosen as $10^{-6} \frac{\text{s}^2 \text{ Mpc}^2}{\text{km}^2}$. The value of $\Omega_{\Lambda,0}$ is chosen as 0.7 and the present value of the Hubble function is chosen as $68 \frac{\text{km}}{\text{s Mpc}}$. The values of the Hubble function are given in $\frac{100 \text{ km}}{\text{s Mpc}}$ and the values of the Ricci scalar are given in $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{ Mpc}^2}$ in the natural logarithmic scale. The dotted line, representing a line of discontinuity, separates the domain where $\hat{R} < \hat{R}_{\text{sing}} = \hat{R}(a_{\text{sing}})$ from the domain where $\hat{R} > \hat{R}_{\text{sing}} = \hat{R}(a_{\text{sing}})$. In the right panel, points 5 and 6 represent points where the right and left side of the phase space is sewn (some trajectories pass through the sewn singularity—points 5 and 6). Note that oscillating models exist (blue trajectory) and are situated around critical point 2. They represent oscillating models without the initial and final singularities. The green line represents the separatrix trajectory, which represents the only case for which the trajectory can pass from the left side of the phase portrait to the right one without the appearance of the sewn freeze singularity during the evolution. It joins saddle points in a circle at infinity. This line separates trajectories going to the freeze singularity from the bouncing solutions. For this case $\Omega_k = -\Omega_\gamma(\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0})^2 \frac{(K-3)(K+1)}{2b} - (\Omega_{m,0}a^{-3} + 4\Omega_{\Lambda,0})$ when $a = a_{\text{sing}}$.

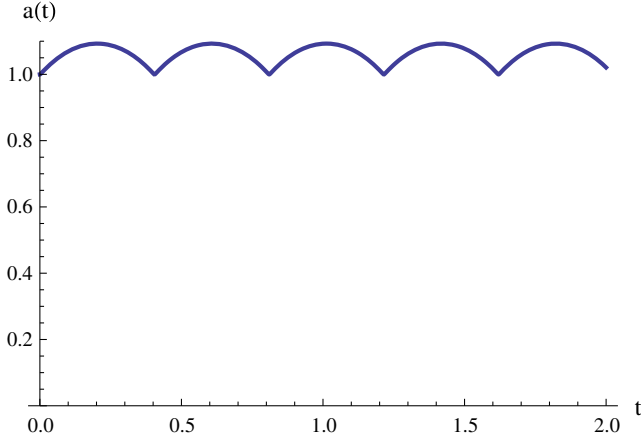


FIG. 13. The diagram presents the evolution of the scale factor for the trajectory of the oscillating orbit in the neighborhood of critical point 2 (see Fig. 12). The cosmological time is expressed in $\frac{\text{sMpc}}{100 \text{ km}}$. Here, $a_{\min} = 1$.

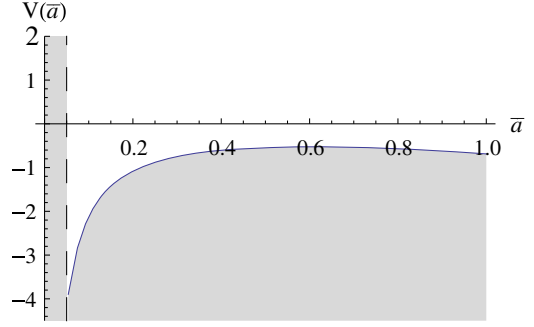


FIG. 14. The potential $V(\bar{a})$ for the Palatini formalism in the Einstein frame. The value of the parameter γ is chosen as $10^{-9} \frac{\text{s}^2 \text{Mpc}^2}{\text{km}^2}$. The values of the $V(\bar{a})$ are given in $\frac{10^4 \text{ km}^2}{\text{s}^2 \text{Mpc}^2}$. The dashed line represents the generalized sudden singularity. The value of the potential at the singularity is finite.

$$\bar{a} = \sqrt{\frac{C_0(1 + 2\gamma\hat{R}(\bar{t}))}{e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t})}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}} \sqrt{4\Lambda + \hat{R}(\bar{t})(-1 + 16\gamma\Lambda + 16\gamma^2\Lambda\hat{R}(\bar{t}))}}}, \quad (50)$$

where $C_0 = \frac{\bar{a}_0^2 e^{\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t}_0)}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}} \sqrt{4\Lambda + \hat{R}(\bar{t}_0)(-1 + 16\gamma\Lambda + 16\gamma^2\Lambda\hat{R}(\bar{t}_0))}}{(1+2\gamma\hat{R}(\bar{t}_0))}$ with $\bar{a}_0$ as the present value of the scale factor, we get the first integral in the following form:

$$\bar{H}(\bar{t})^2 + \Lambda - \frac{\hat{R}(\bar{t})(2 + \gamma\hat{R}(\bar{t}))}{6(1 + 2\gamma\hat{R}(\bar{t}))^2} + k \frac{e^{-\frac{\arctan\left(\frac{-1+16\gamma\Lambda+32\gamma^2\Lambda\hat{R}(\bar{t})}{\sqrt{-1+32\gamma\Lambda}}\right)}{3\sqrt{-1+32\gamma\Lambda}}} \sqrt{4\Lambda + \hat{R}(\bar{t})(-1 + 16\gamma\Lambda + 16\gamma^2\Lambda\hat{R}(\bar{t}))}}{C_0(1 + 2\gamma\hat{R}(\bar{t}))} = 0. \quad (51)$$

As a result, the potential $V(\hat{R})$ is given by

$$V(\hat{R}) = \frac{a^2}{2} \left(\Lambda - \frac{\hat{R}(\bar{t})(2 + \gamma\hat{R}(\bar{t}))}{6(1 + 2\gamma\hat{R}(\bar{t}))^2} \right). \quad (52)$$

Because we know the form of $V(\hat{R})$ and $\bar{a}(\hat{R})$, we can get the potential $V(\bar{a})$ in a numerical way. $V(\bar{a})$ potential is demonstrated in Fig. 14.

Equations (45)–(46) have the following first integral given by

$$H(t)^2 - \frac{(1 + 2\gamma\hat{R}(t))^2 \left(-3\Lambda + \hat{R}(t) - \frac{k(-4\Lambda + \hat{R}(t))^{2/3}}{C_0} + \frac{\gamma(12\Lambda - 3\hat{R}(t))\hat{R}(t)}{2(1+2\gamma\hat{R}(t))} \right)}{(1 + 2\gamma\hat{R}(t) - 3\gamma(-4\Lambda + \hat{R}(t)))^2} = 0, \quad (53)$$

where $C_0 = a_0^2(-4\Lambda + \hat{R}(t_0))^{2/3}$. Here, a_0 is the present value of the scale factor.

VI. CONCLUSIONS

In this paper, the main conclusion is that the Starobinsky models in the Palatini formalism in the Jordan and Einstein

frames are not physically equivalent. There are a few qualitative differences between the models in these frames. The most important difference is that the sewn freeze singularity in the Jordan frame is replaced by the generalized sudden singularity in the Einstein frame. Other differences between these frames are the lack of the big bang in our model in the Einstein frame and the fact that

phase portraits in these frames are not qualitatively equivalent. It is consistent with results obtained that models in the Jordan frame are not physically equivalent to those in the Einstein frame [75–79].

From the detailed analysis of cosmological dynamics in the Palatini formulation we derive the following conclusions:

- (1) If we consider the cosmic evolution in the Einstein frame we obtain inflation as an endogenous effect from the dynamical formulation in the Palatini formalism [74].
- (2) If we consider the cosmic evolution in the Jordan frame we obtain an exact and covariant formula for the variability of the gravitational constant G_{eff} parametrized by the Ricci scalar.
- (3) Given two representations of our model in the Einstein and Jordan frames, we found that its dynamics are simpler in the Einstein frame as being free from some obstacles related to an appearance of bad singularities. It is an argument for the choice of the Einstein frame as physical.
- (4) In our model considered in the Einstein frame, we have both the inflation as well as the acceleration [74]. While the inflation in the model is obtained as an inherited dynamical effect, the acceleration is driven by the cosmological constant term.
- (5) In the model under consideration, we include effects of matter. This enables us to study the fragility of the inflation with respect to small changes of the energy density of matter [74].

- (6) In the obtained evolutionary scenario of the evolution of the Universe in the Einstein frame in the Palatini formalism we found the singularity of the finite scale factor (generalized sudden singularity) and the phase of the acceleration of the current Universe. Note that in [74] it was found the inflation in this model with the sufficient number of e -folds in the case when the matter is negligible.
- (7) In the context of the Starobinsky model in the Palatini formalism we found a new type of double singularities beyond the well-known classification of isolated singularities.
- (8) The phase portrait for the Starobinsky model in the Palatini formalism with a positive value of γ is equivalent to the phase portrait of the Λ CDM model. There is only a quantitative difference related to the presence of the nonisolated freeze singularity.
- (9) For the Starobinsky-Palatini model in the Einstein frame for the positive parameter γ , a sewn freeze singularity is replaced by a generalized sudden singularity. As a result, this model is not equivalent to the phase portrait of the Λ CDM model.

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