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Numerical Analysis and Simulations of Mechanical Contact Problems

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I would like to thank
my supervisor for everything she has taught me,
my aunt for inspiring me to study mathematics,
my parents and grandparents for always rooting for me,
and my wonderful wife for her continuous support.

Abstract

This dissertation focuses on numerical analysis and simulations of soft-body mechanical contact problems using the finite element method. It is an interdisciplinary study that combines computer science, applied mathematics, and solid mechanics. The main goal of this dissertation is to reduce the gap between the theoretical analysis of physical models and their practical implementations.

We present theoretical results in the field of contact mechanics based on a series of five academic articles published in international journals. In these articles, we focus on theoretical analysis of convergence and error of a discretized numerical scheme. We cover static, quasistatic, and dynamic models with various additional effects, such as temperature, wear of the material, and damage to the body.

We also introduce open-source software "Connech3D", an important component of the study, which allowed us to perform simulations presented in the included papers. Our software provides GPU acceleration to increase simulation speed, simple numerical precision configuration, and automatic differentiation support. Using this software, we simulate concrete instances of abstract problems and empirically validate theoretical results.

Contents

Abstract	1
1 Overview of the study	4
1.1 Outline	4
1.2 Motivation	4
1.3 Contents	5
1.4 Scope	5
1.5 Classification	6
1.6 Contributions	7
1.7 Other publications	8
1.8 Future work	9
2 Background of the study	10
2.1 Physical simulations	10
2.2 Contact mechanics	11
2.3 Perspectives and applications	11
2.3.1 Theoretical analysis	11
2.3.2 Engineering applications	12
2.3.3 Computer graphics	12
2.3.4 Machine learning	13
3 Description of the software	15
3.1 Overview	15
3.2 Implementation details	15
3.3 Implemented options	16
3.4 Employed algorithms	18
4 Theoretical introduction	20
4.1 Main framework	20
4.1.1 Modeling	20
4.1.2 Theoretical analysis	21
4.1.3 Numerical analysis	21
4.1.4 Computational simulations	22
4.2 Sample contact problem	22
4.2.1 Derivation of the model	22
4.2.2 Classical statement	24
4.2.3 Weak formulation	25

5	Overview of the contact problems	27
5.1	Abstract static problem	27
5.2	Problem with wear of the material	29
5.3	Problem with damage to the body	31
5.4	Problem with thermal effects	32
6	Contents of the publications	33
I	Numerical analysis of a contact problem with wear	34
II	Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage	45
III	A nonsmooth optimization approach for hemivariational inequalities with applications to contact mechanics	61
IV	A survey of numerical methods for hemivariational inequalities with applications to contact mechanics	83
V	Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem	98
A	Statements of the authors	119
B	Theorems and lemmas	127
	Bibliography	134

Chapter 1

Overview of the study

1.1 Outline

We begin with the outline of this dissertation. In Chapter 1 we provide an overview, presenting motivation, scope, classification, main contributions, and future directions. Chapter 2 is a short introduction to the fields of physical simulations, contact mechanics, and their applications. Its objective is to provide background information on the problems considered in this study. Chapter 3 contains a description of the software that was used to obtain published results of physical simulations. In Chapter 4 we introduce the mathematical notation and provide a derivation of the classical and weak formulation of a sample mechanical contact problem. Chapter 5 summarizes the problems considered in each of the publications constituted in this study and Chapter 6 contains their published texts. The statements of the authors are included in Appendix A, and theorems and lemmas commonly used throughout this study are collected in Appendix B.

1.2 Motivation

Simulation of physics is widely used in various fields of science, engineering, and entertainment. It enables us to recreate real-world experiences in virtual reality and explore physical phenomena that are otherwise difficult to analyze. Additionally, the analysis and implementation of physical simulations can be viewed from multiple perspectives and requires knowledge of computer science, classical physics, and applied mathematics. As a result, it is an interesting and challenging field of study where established methods and new ideas meet. This creates a potential for mixing various approaches and combining knowledge from different fields.

The main goal of this dissertation is to **reduce the gap between the theoretical analysis of physical models and their practical implementations**. We aim to do that by concentrating on mechanical problems encountered in engineering and computer graphics. In addition to advancing research, this study can be used for educational or commercial purposes and can contribute to the popularization of physical modeling and simulations.

1.3 Contents

This dissertation consists of two main components.

1. **Theoretical results** in the field of contact mechanics, that is, five academic articles published in international journals. The papers are listed in chronological order.

- [I] Danfu Han, Weimin Han, Michał Jureczka, Anna Ochal, **Numerical analysis of a contact problem with wear**, *Computers and Mathematics with Applications*, 79(10), 2020, 2942–2951, doi:10.1016/j.camwa.2019.12.027
- [II] Weimin Han, Michał Jureczka, Anna Ochal, **Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage**, *Journal of Computational and Applied Mathematics*, 377, 2020, 112886, doi:10.1016/j.cam.2020.11288
- [III] Michał Jureczka, Anna Ochal, **A nonsmooth optimization approach for hemivariational inequalities with applications to contact mechanics**, *Applied Mathematics and Optimization*, 83, 2021, 1465–1485, doi:10.1007/s00245-019-09593-y
- [IV] Anna Ochal, Michał Jureczka, Piotr Bartman, **A survey of numerical methods for hemivariational inequalities with applications to contact mechanics**, *Communications in Nonlinear Science and Numerical Simulation*, 114, 2022, 106563, doi.org/10.1016/j.cnsns.2022.106563
- [V] Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafraniec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870, doi.org/10.1016/j.nonrwa.2023.103870

2. **Open-source software "Conmech3D"**, which is an important component of the study. Using this software, we simulate concrete instances of abstract problems and empirically validate the theoretical results. In addition, it has capabilities that extend beyond the theory presented in the included papers (see Chapter 3 for more details). The current version of the software is available at

<https://github.com/KOS-UJ/conmech3d>

1.4 Scope

In this study, we performed a numerical analysis and simulations of soft-body mechanical contact problems using the finite element method (FEM). Such problems are, in general, more challenging than rigid-body problems. We consider a wide range of models that involve different types of materials, boundary conditions, and loading scenarios. The influence of various physical phenomena such as temperature, material wear, and body damage is studied. In the process, we investigate static, quasistatic,

and dynamic problems that lead to the so-called variational and hemivariational inequalities.

In the included papers, we provide a detailed numerical analysis along with theoretical and empirical error estimates for the FEM discretization. We also conduct a theoretical and empirical comparison of algorithms used to solve mechanical contact problems. We supplement this by including or referencing proofs concerning the existence and uniqueness of the solutions to the problems considered.

It should be noted that in our theoretical analysis we focus on physical models that employ a small strain tensor. As this tensor is a linear approximation of the complete Green strain tensor, we are required to assume a small deformation and rotation of the body. This assumption limits application of theoretical results, especially in cases where the body is not clamped at any point and can freely move. At the same time, in "Conmech3D", we also allow the use of nonlinear constitutive relations and implement the Saint Venant-Kirchhoff model, which uses the Green strain tensor.

The main focus of this dissertation is on problems with general contact conditions. Such relations can be multivalued, and we express them using the notion of the Clarke subdifferential (more details are provided in Chapter 4). In each paper, we present assumptions on functions that describe contact conditions, such as e.g. assumption of sublinear growth. These assumptions are satisfied for a wide range of relations used in practice, with examples provided in the simulations included in each paper.

The generality of contact conditions is reflected in the implementation, which allows us to simply express it as a function of displacement or velocity on the boundary of the body. As a result, we do not simplify the contact laws for practical purposes (for example, by using penalization forces), as many theoretical papers and implementations in the field of physical simulations do. This is one of the distinctive features of this study and in some sense provides a counterbalance for limitations of employed small strain theory. It is also part of the reason why we decided to introduce our own implementation of FEM, although there are other open-source solutions (other factors are listed in Section 1.6).

1.5 Classification

As mentioned previously, the subject of this study is interdisciplinary and cannot be easily classified. We believe that because of the amount of time spent on implementation of the software and focus on numerical analysis and empirical error estimations, the contributions of the author of this dissertation are best described as applied (or technical) computer science. This is in contrast to physics, which is used to formulate the problems considered, applied mathematics, which provides the main theoretical framework and theorems on existence and uniqueness of a solution (the main focus of some of the other authors of the included papers), or theoretical computer science that deals with abstract theorems concerning complexity of algorithms employed. The author, by curiosity and necessity, has some experience in all of these domains, but is mainly interested in numerical and computational aspects and their applications. We also support this view by noting that all journals that published papers included in this dissertation have a classification of technical computer science by the Polish Ministry of Science and Higher Education.

1.6 Contributions

We claim that the main contributions of this thesis are the open-source software created for mechanical contact simulations and the numerical analysis of selected contact problems. In this section, we summarize the main results obtained in those directions. We also include a brief description motivating each point.

- i) **We provide an open-source implementation of FEM for contact problems with general boundary conditions.** As mentioned previously, contrary to other practical implementations, in our software we do not simplify contact conditions and allow for general functions that depend on body displacement and velocity on the boundary. The code is written in Python, which is more user-friendly than lower-level languages. Our software has many useful features, such as GPU acceleration to increase simulation speed, simple numerical precision configuration, and automatic differentiation support.
- ii) **We implement simulations of two-dimensional and three-dimensional bodies, and in the paper [V] we provide examples for both cases.** Most articles that focus on the theoretical analysis of contact problems include the results of simulations only in the two-dimensional case, as they usually serve as only a simple example. We believe that the inclusion of three-dimensional experiments reduces the gap between theoretical analysis and practical applications and serves to popularize both the obtained theoretical results and the implemented software.
- iii) **Papers [I]-[III] and [V] include a numerical analysis of the problems considered.** In many cases, the existence and uniqueness of a solution to a particular contact problem are proven in the literature. At the same time, the numerical analysis that describes the relationship between the abstract model and its discretized version is not performed. This analysis should lead to a theorem that guarantees convergence or even provides the order of this convergence. Such a theorem is important from a theoretical point of view because, without error estimation, the FEM is not guaranteed to converge to the right solution for a problem under consideration.
- iv) **Papers [II]-[V] provide empirical estimation of the numerical error.** The numerical analysis described above requires theoretical assumptions that may not be easy to verify in practice. It also does not take into account inaccuracies incurred in a practical implementation, e.g. by floating-point arithmetic. The precise theoretical analysis of such factors is beyond the scope of this study. Instead, in all included papers, we present an empirical error estimation that validates the theoretical order of convergence in practice. Because the exact solution is not available, we use a common methodology and instead take the reference solution obtained by simulation with the smallest discretization parameters (highest precision attainable).
- v) **Paper [III] provides complete self-contained analysis of a static mechanical contact problem.** Usually, when a numerical analysis is performed, abstract theorems of existence and uniqueness of a solution are cited from other publications. In this paper, we provide optimization-based reformulation that allows us to obtain such results. As a result, this article contains proofs of

existence and uniqueness of a solution, numerical analysis, empirical error estimation, and simulation results. We believe that it can serve as an introduction to the entire spectrum of theoretical and practical analysis of mechanical contact problems.

- vi) **In paper [IV] we review and compare three algorithms that are commonly used to solve static mechanical problems under general contact conditions.** Algorithms that solve hemivariational inequalities are used throughout studies of mechanical contact problems to illustrate theoretical results, but often without providing a detailed description. In this paper, we provide some guidance on how such algorithms can be implemented in practice and what their relative performance is.
- vii) **Paper [V] provides numerical analysis of the unclamped body without requiring the smallness assumption.** In most theoretical papers, the body is assumed to be clamped, meaning that a part of its boundary of a positive measure has to satisfy the Dirichlet condition with displacement equal to zero. Also, in most cases, the so-called smallness assumption must be satisfied. This assumption is difficult to verify in practice for irregular geometries of the body. In this paper, we do not require any smallness assumption and consider a body that can move freely, which increases the applicability of the introduced theory.

1.7 Other publications

The author has published papers during his Ph.D. studies that are not included in this dissertation. These papers are related to the same topic, and we present them here for the sake of completeness. The paper [VII] deals with the same contact problem with foundation wear as [I], but in [I] we extend the results obtained previously by providing a theoretical and empirical analysis of the order of convergence. The paper [VIII] applies a similar approach as presented in [III] to a quasistatic problem instead of a static one. The paper [VI] deals with a simpler formulation of the thermal contact problem than [V] and provides only basic simulations on a small mesh. Results presented there can be seen as a starting point in the development of simulating software that was improved with each publication.

- [VI] Anna Ochal, Michał Jureczka, **Numerical treatment of contact problems with thermal effect**, *Discrete and Continuous Dynamical Systems - Series B*, 23(1), 2018, 387–400, doi:10.3934/dcdsb.2018027
- [VII] Michał Jureczka, Anna Ochal, **Numerical analysis and simulations of contact problem with wear**, *Computers and Mathematics with Applications*, 77(11), 2018, 2980–2988, doi:10.1016/j.camwa.2018.08.044
- [VIII] Michał Jureczka, Anna Ochal, Piotr Bartman, **A nonsmooth optimization approach for time-dependent hemivariational inequalities**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103871, doi:10.1016/j.nonrwa.2023.103871

1.8 Future work

We believe that the results presented in this study constitute a coherent whole. At the same time, we know that more work can be done to bring theoretical analysis closer to its applications and increase the versatility and performance of the presented software. As an example, after the conclusion of this study, the author began preliminary work on the application of machine learning to the domain of model reduction, taking advantage of existing features of the program "Conmech3D". The initial implementation of this technique is included in the code repository, and more information is provided in Chapter 3. For future work, we plan to pursue the following directions

- i) continue the research on application of machine learning techniques for model reduction and end-to-end simulations, presented in Section 2.3.4,
- ii) perform analysis of simulation techniques alternative to FEM, presented in Section 2.3.3,
- iii) research feasibility of theoretical treatment of contact problems with nonlinear constitutive laws based on the Green stress tensor,
- iv) increase quality and completeness of the software presented in this dissertation and its documentation.

Chapter 2

Background of the study

2.1 Physical simulations

In this section, we briefly introduce the main ideas behind physical simulations and contact mechanics. The aim is to look at the results presented in this dissertation from various perspectives and to identify areas of application.

Physical simulations are computer-based methods that aim to reproduce the behavior of real-world physical systems. They can be used for various purposes, such as studying natural phenomena, designing and testing engineering products or creating realistic animations and games. Depending on the application domain, the behavior of rigid and soft bodies, cloths, gases, and fluids in one, two, or three dimensions can be investigated.

Physical simulations employ mathematical models that describe how the state variables of the system change. Such models are usually based on partial differential equations (PDEs) that describe how physical quantities vary over space and time. However, solving these PDEs analytically is often impossible or impractical for realistic problems. Therefore, numerical methods such as FEM are used to discretize PDEs into a system of algebraic equations that can be solved by computers. A practical introduction to FEM with the focus on implementation can be found in [1].

Models of physical systems incorporate constitutive laws that describe the relationship between stress, strain, and other factors. For example, linear elasticity assumes that the stress is proportional to the strain; plasticity accounts for permanent deformation beyond a certain threshold; viscoelasticity incorporates both elastic and viscous effects. Such mathematical descriptions approximate the properties of real-world materials with varying degrees of accuracy. For example, linear elasticity, known also as Hooke's law, is a first-order approximation to the real response of springs and other elastic bodies to applied forces.

Additional factors can also be considered that affect the behavior of objects and materials. These factors include, for example, temperature, damage, or piezoelectricity. Temperature affects the thermal expansion or contraction of materials and their phase transitions. Damage refers to the degradation or fracture of materials due to excessive stress. Piezoelectricity is a phenomenon in which certain materials generate electric voltage when subjected to mechanical stress or vice versa.

Another important aspect of physical simulations is handling the interactions between different objects in the system. These interactions can include collision, pene-

tration, friction, cohesion, adhesion, and wear. Such phenomena can affect the motion, deformation, and energy transfer of the objects involved.

One of the main challenges of physical simulations is to achieve a balance between accuracy and efficiency. Some physical simulations require high-performance computing and sophisticated numerical methods to handle complex and nonlinear problems. Other applications can be implemented with simpler algorithms and run on personal computers or mobile devices.

2.2 Contact mechanics

As mentioned previously, physical simulations have many branches and domains of application. The main focus of this dissertation is on a subfield called contact mechanics that deals with the behavior of rigid and soft bodies that are in contact with each other at parts of their boundaries.

Depending on the particular setting in this framework, we can consider many different physical problems that can be experienced in everyday life. To name just some examples, in the automotive industry, we observe contact between the tire and the road, the brake pad and the rotor, or the piston and the cylinder.

Contact mechanics deals mainly with two effects: normal and tangential. Normal effects, such as compressive effect of the material or adhesion, are caused by forces that act perpendicularly to the surfaces of bodies in contact. Such effects can be modeled, for example, by the Signiorini contact law or a normal compliance condition. The tangential effects caused by forces that act tangentially between surfaces must also be taken into account. Friction can be modeled by the classical Coulomb friction law and its extensions.

2.3 Perspectives and applications

Physical simulations, and by extension contact mechanics, can be viewed from different perspectives based on the application and methodology used. The perspectives listed below are neither exhaustive nor exclusive, but are presented to reflect the broadness of this topic.

2.3.1 Theoretical analysis

In this approach, the models and algorithms used in the simulations are studied using rigorous mathematical theory. The aim is to assume as little as possible about the functions and spaces used to obtain general theorems that can be applied to a wide range of situations. However, sometimes these models are not (yet) motivated by practice or do not capture all the relevant features of reality.

As part of the theoretical analysis of a considered physical model, we can reformulate it to the abstract weak form and prove the existence and uniqueness of a solution. Other studies concentrate on numerical analysis of the discretized scheme or the application of the FEM and its properties, such as conforming or nonconforming variants, different weak formulations and test spaces, or extensions, such as the recently introduced Virtual Element Method (VEM). In scientific publications that take

this point of view, computational experiments are usually only used to illustrate the model considered.

The application of theoretical analysis is the main approach in the research presented in this thesis. Therefore, a more detailed description of such an approach to mechanical contact problems is provided in Chapter 4. For an introduction to VEM, we refer to [2].

2.3.2 Engineering applications

Engineering applications to physical simulations are driven by practical requirements and constraints, rather than by the elegance of theoretical theorems. The foundations laid by the theoretical results are still used, especially the numerical analysis that provides error bounds and convergence rates for applied methods. Some techniques that work well in practice, but are not necessarily mathematically proven, are applied to enhance the efficiency and accuracy of the simulations. At the same time, there is less emphasis on the mathematical treatment of abstract generalizations or proofs of the existence or uniqueness of solutions to the problems considered.

Applications of physical simulations to engineering span real-world problems in various domains, such as the construction, automotive, and aerospace industries, and require accurate results. Therefore, well-established methods, such as FEM, are applied to solve the discretized problem. Moreover, meshes that represent physical bodies in the scene are usually larger and have more complex geometries than those used in academic research or teaching.

As high resolution or accuracy in certain regions or aspects of interest is usually required, special techniques, such as local adaptivity with respect to element sizes or degrees of polynomial shape functions, are employed. More details on the implementation of a technique called hp-FEM that uses both of these approaches together can be found in [3].

Applied models may be highly nonlinear and may involve discontinuities or other singularities that make numerical methods complicated. Specialized methods such as the discontinuous Petrov-Galerkin (DPG) method were introduced to address such problems. A review of DPG methods is presented in [4].

Robust implementations proven in practice are needed to ensure the efficiency, reliability, and validity of simulations. Commercial software that has been used for years by engineers and scientists is usually applied for this purpose. Examples of such software are [5, 6, 7].

2.3.3 Computer graphics

Physical simulations are widely used in computer graphics to create realistic visualizations of objects' motion in games and movies. To ensure smooth animation in games, such simulations must be executed in real time. Furthermore, we cannot repeat the simulation, so the applied methods must be unbreakable, meaning that the solutions cannot explode or contain visual artifacts. Fortunately, in computer graphics, even inexact results of simulations that look convincing to the viewer are usually sufficient. Because of this, in contrast to engineering applications, speed and stability are more important than accuracy.

There are various approaches to physical simulations in computer graphics. FEM is used in this domain to achieve realistic results, but because of its computational cost, usually only small meshes and simple constitutive laws are considered. As an example, linear elasticity can be computed efficiently but produces artifacts when large rotational deformations are present. Rotational invariance can be obtained by applying corotated linear elasticity, which requires polar decomposition of the deformation gradient.

Another approach is to use position-based dynamics (PBD), which iteratively displaces mesh nodes based on their inertia and corrects their positions to satisfy other physical constraints. PBD can be faster and more stable than FEM and thus be better suited for real-time applications. On the other hand, it is a new technique that is not as well established as FEM and depends on the particular mesh geometry and the number of substeps used. A variant of PBD called extended position-based dynamics (XPBD) aims to address some of its shortcomings. PBD was introduced in [8] and extended to XPBD in [9].

At the same time, to increase the speed, various techniques can be applied to decrease the number of degrees of freedom in the simulation. As an example, model reduction with principal component analysis (PCA) can be used to decompose the model and take into account only the most important deformation modes of the body. Another approach, called tetrahedral skinning, requires the simulation of a simplified model and the retrieval of original displacements using barycentric coordinates. An example of the application of PCA to perform model reduction can be found in [10], and tetrahedral skinning is described in [11].

In the domain of computer graphics, the treatment of theoretically involved and computationally expensive contact phenomena, such as friction and compliance, is usually simplified. This can be achieved by applying penalty forces that replace contact conditions at the cost of accuracy of the model.

For a practical overview of the methods used in computer graphics, we refer to [11]. An example of physics engine that is used in games is Nvidia PhysX [12].

2.3.4 Machine learning

In recent years, many approaches have been proposed to integrate machine learning techniques into physical simulations. Such methods aim to enhance or completely replace classical approaches. In this section, we provide some examples of such techniques.

Physics-Informed Neural Networks (PINNs) can leverage knowledge of any physical laws along with measurement data in the learning process. PINNs are trained to satisfy both the given training data and the imposed governing equations, thus overcoming the low availability of data for some biological and engineering systems. Furthermore, PINNs can naturally form PDEs using automatic differentiation. PINNs were introduced in [13].

Deep learning models, called autoencoders, can also be used to perform sophisticated model reduction to increase the simulation speed. The autoencoder consists of two parts: an encoder that maps the high-dimensional state to a low-dimensional latent variable and a decoder that reconstructs the state from this learned latent representation. The advantage of using autoencoders for model reduction is that they can

learn complex nonlinear mappings between the state space and the latent space based only on examples of ground-truth simulation results. This approach was presented in [14] and was later extended in [15].

Graph Neural Networks (GNNs) are another family of models that can operate on graph-structured physical data that represent, for example, molecules and their interactions, or solid bodies as meshes. In the context of physical simulations, GNNs work by encoding node information (e.g. force, velocity) and edge information (e.g. relative position and relative deformation) into latent vectors and propagating encoded information to neighboring nodes using a process called message passing. The latent vectors are then aggregated and decoded to obtain dependent variables such as the acceleration of the nodes. This approach to particle simulation was presented in [16], later applied to solid bodies [17] and extended to multiple layers of graphs in [18].

These examples show that machine learning can offer new ways to solve existing problems in the space of physical simulations. However, such techniques also pose new challenges and limitations. Currently, machine learning methods do not offer the same reliability and interpretability as well-studied and practically proven approaches like FEM, especially for critical applications such as modeling of airplane components. In addition, such models have mostly been applied to solve sample problems that are less complex than many current engineering problems. At the same time, there is ongoing active research in this domain that shows that machine learning can complement traditional approaches in many applications in the future. An example of such application in computer graphics is presented in [19].

Chapter 3

Description of the software

3.1 Overview

As part of the effort to bring theoretical analysis closer to practical applications, we developed an open-source software "Conmech3D" to simulate the deformation of soft bodies in contact with rigid obstacles using FEM. It is written in Python to make it more accessible to other researchers and practitioners and to lower the barrier of entry for didactic purposes. The program has undergone several iterations and improvements since its initial version. Increasing the versatility and performance of this software allowed us to improve the results presented in each published paper.

The latest version of our software was used to obtain results in [V] and is available on Github as an open-source project. However, not all the options and functionalities of the original code have yet been ported, and the software is still under active development. In Section 3.2 we provide more details of the implementation. Information about the usage of "Conmech3D" can be found on the Github page.

The current version of our program can handle a single soft body that interacts with multiple stationary rigid obstacles, which are treated as part of the environment and not simulated dynamically. However, this can be extended in the future to include multiple soft bodies and moving obstacles. The program has several options to model different aspects of the deformation problem, such as the inclusion of various physical phenomena, constitutive laws, and boundary conditions. These options are described in Section 3.3.

In the implementation, we use triangular and tetrahedral meshes to discretize the simulated body spatially. For the time-dependent case, we employ a fully discrete scheme that computes the solution at each time step. In order to solve this problem, we reformulate it as a more basic computational task, for example, optimization or root finding. Various algorithms used in our software are described in Section 3.4.

3.2 Implementation details

The latest version of our program takes advantage of JAX [20], a library for high-performance numerical computing. Its application allows us to perform simulations in three dimensions and to run our program on various devices, such as a CPU, GPU, or TPU. We remark that many implementations of FEM do not support computations

on devices other than a CPU, and in our experiments the usage of GPU increased performance significantly. JAX supports single (32-bit) and double (64-bit) modes of numerical precision. Double precision provides more accurate results due to reduction of roundoff errors; however, it is much slower to compute, especially on consumer-grade GPUs that are not focused on double precision performance. JAX also implements automatic differentiation that is used to calculate derivatives in the optimization routine and can be leveraged in the future, for example, as part of the machine learning pipeline (e.g. by setting the energy function as loss and employing supervised learning of underlying physics).

Our program relies on various other external libraries for different purposes. For the generation of meshes specified by the geometry, we use pygmsh [21], a Python interface of Gmsh [22]. For the generation of a tetrahedral mesh based on a surface mesh, we use TetWild [23]. To increase the speed of initial setup and object creation, we use Numba [24] for CPU operations. The main FEM engine that performs optimization is implemented from scratch using JAX, Numpy [25] and Scipy [26]. We provide different options to visualize the results of our simulations, using Matplotlib [27], Blender [28], and Three.js [29]. These libraries offer different levels of convenience, quality, and interactivity for displaying the motion of the simulated body.

In addition to the work presented in our papers, we also conducted some preliminary research on applying machine learning techniques to contact problems. Specifically, we have explored the use of GNNs introduced in Section 2.3.4 to perform model reduction for soft body deformation. The implementation of GNN, also available on Github, uses JAX, Flax [30] and data loading from Pytorch Geometric [31]. As a baseline, we also provide an implementation of tetrahedral skinning using Cython [32] and an initial implementation of PCA for model reduction (see Section 2.3.3 for description and references). We are also working on implementing other features, such as corotated linear elasticity and Neo-Hookean elasticity.

3.3 Implemented options

In this section, we list the options implemented in "Conmech3D". Most of them can be configured independently to solve a specific soft body mechanical contact problem. We also refer to the papers in which we presented the results of simulations using the described option.

i) Dimension

- (a) two dimensional problems - papers [I]-[V],
- (b) three dimensional problems - paper [V].

ii) Physical phenomena

- (a) displacement of the body; we can equivalently solve for velocity or acceleration of the body - papers [I]-[V],
- (b) temperature of the body - paper [V].

In the previous versions of the software we also consider

- (c) wear of the layer of the material that covers the foundation - paper [I],
- (d) damage to the material of the body - paper [II].

iii) Dependence on time

- (a) static, where we assume that the body is in a steady state making the problem time independent - papers [III] and [IV],
- (b) quasistatic, where we assume that the acceleration is negligible (e.g ice melting) - papers [I] and [II],
- (c) dynamic, where we take into account acceleration of the body - paper [V].

iv) Constitutive laws

- (a) linearized elasticity using the Hooke's law based on the small strain tensor,
- (b) linearized viscoelasticity using the Kelvin-Voigt model based on the small strain tensor,
- (c) hyperelasticity using the Saint Venant-Kirchhoff model based on the Green strain tensor.

Other options, such as the corotated and neo-Hookean constitutive laws, can be added and are in development at the time of writing.

v) Boundary conditions - the boundary can be divided into the following regions

- (a) contact boundary (denoted Γ_C) that represents contact with another body, called obstacle or foundation, and allows us to specify friction and normal compliance conditions (described below),
- (b) Neumann boundary (denoted Γ_N) representing the situation when external force is applied on the boundary,
- (c) Dirichlet boundary (denoted Γ_D) with displacement equal to zero - representing the situation when the body is clamped and cannot move; at the time of writing, this option is only implemented in the two-dimensional case.

Regions of various types are required to have an empty intersection and to sum to the entire boundary, but this limitation is introduced only for simplicity and can be removed in the future.

vi) Contact laws - we represent conditions on contact boundary as a function of the displacement and velocity of the body at current timestep. This general formulation allows us to easily implement the desired configuration. In our experiments, we focus on functions that satisfy the assumptions introduced in each publication.

We differentiate between

- (a) functions that model friction between the body and the foundation. In practice, we use extensions of the standard Coulomb law (which itself is a modification of the Tresca friction law). For example, in the paper [V] we used its nonconvex modification in order to present generality of the assumptions considered in theoretical part,
- (b) functions that model dependence between penetration of the foundation and reaction force in normal direction. Usually, we apply no force when penetration is negative (the body and foundation are separated), and increasing linear response when penetration is positive, but other options are also possible.

In previous versions of the software, we also included the Signiorini contact condition. This condition prevents the body from penetrating the foundation and is implemented using the constrained optimization algorithm.

- vii) Obstacles - the implementation allows for configuration of multiple obstacles, like other bodies or planes. Bodies are represented internally as meshes, and planes are represented by coordinates of any point on the plane and its normal vector. We also provide an option to factor in self-collisions that can be parameterized in the same way as in the case of contact with an external obstacle.

3.4 Employed algorithms

In order to solve the mechanical contact problem specified by the options listed in the previous section, we discretize it using the FEM and the implicit Euler method. The implicit method is commonly used in physical simulations, as such systems are usually stiff, and instabilities may occur when using explicit methods. This discretization can then be viewed as either a minimization or a root-finding problem. In the case of time-independent models, such a problem is then solved once, and in the case of time-dependent models multiple times for each timestep. Our software offers several strategies and algorithms that can be used to solve discretized problems. We provide a brief overview below. We do not include descriptions of time and space complexity, as we mostly use well-studied and established algorithms, and such an analysis is available in some of the included references.

1. Linear solvers

If the formulation of a mechanical problem is simple, i.e. it does not involve contact and uses a linear constitutive law based on the small strain tensor, the entire problem becomes linear. This allows us to use the conjugate gradient method, implemented in JAX or SciPy with the use of preconditioners (such as the Jacobi preconditioner) to speed up convergence. That approach is several orders of magnitude faster than any optimization method, but linear approximations of constitutive laws introduce visible artifacts when the body rotates (see [11] for an example).

2. Optimization methods

If either the constitutive law is nonlinear (for example, it is based on the Green strain tensor) or we consider contact with the foundation, the problem requires the use of more sophisticated tools, such as optimization algorithms. Furthermore, a general contact condition can potentially cause the optimization problem under consideration to become nonsmooth or nonconvex. In this case, we use optimization methods such as, for example, L-BFGS implemented in JAX or the Powell method from Scipy. From a theoretical point of view, the Powell method introduced in [33] is well suited in this case, as it does not require the assumption of smoothness. Depending on the problem, in practice, methods such as L-BFGS that exploit the assumption of smoothness can also be successfully used, which is confirmed empirically, e.g. by [34] and by our own experiments in the case of contact problems.

3. Alternative methods

In the paper [IV] we compare optimization with the augmented Lagrange method and the primal-dual active set strategy. These methods provide alternative approaches to solving contact problems. Their main drawback is that the formulation of both methods depends on a particular contact conditions and, especially in the case of augmented Lagrange methods, is more mathematically involved. The comparison was made using a previous version of the code, and as of now, both algorithms are not available in the new implementation. The augmented Lagrange method was introduced in the context of contact mechanics in [35]. For details of the primal-dual active set strategy for hemivariational inequalities, we refer to [36].

4. Algorithms used for other purposes

Other algorithms implemented include the Uzawa method and the Schur complement method. The Uzawa approach is applied to solve coupled equations for unknown variables, such as displacement and temperature. The main idea of this method is to solve equations iteratively in turns and converge to the final solution. Schur complement method can be applied when we consider the linear constitutive law and contact with the foundation. In this case, we can split the problem into two parts, one involving the use of optimization on the contact boundary and the other requiring solving a system of linear equations for other vertices of the mesh. This substantially increases the performance of the simulation. For a more detailed discussion of the Schur complement method in this context, we refer to [37]. As mentioned previously, we also provide initial implementations of several methods of model reduction - based on tetrahedral skinning, PCA, and a new approach using GNNs.

Chapter 4

Theoretical introduction

4.1 Main framework

Most publications in the field of theoretical contact mechanics, including those that comprise this dissertation, share a similar framework for analyzing physical problems. Usually, the authors of a publication focus only on a specific task, leaving the prior and subsequent analysis to other studies. In this section, we present a brief introduction to this framework. Analysis of a particular contact problem can be divided into the following steps:

- i) **modeling** - description of physical phenomena that lead to a classical statement of the problem,
- ii) **theoretical analysis** - introduction of a weak formulation, proof of existence, and possibly uniqueness of a weak solution,
- iii) **numerical analysis** - introduction of a discrete scheme and estimation of numerical error,
- iv) **computational simulations** - conduction of computational calculations to illustrate the considered model and confirm its correctness.

All of these steps are represented in the papers that constitute this thesis and are described in more detail in the following subsections.

4.1.1 Modeling

First, the physical properties of the body, the conditions enforced on the boundary in contact with the obstacle, the external forces, and other key factors are modeled. The problem considered is formulated using differential equations or inclusions that describe body behavior, for example, its displacement and temperature at any given time. Sometimes these conditions are expressed using Clarke's subdifferential of non-convex functions, which will be referenced below. For a given mechanical contact problem, from both a practical and a theoretical point of view, we are interested in calculating its solution. Unfortunately, the introduced problem usually does not admit a solution in the classical sense, and our only possibility is to obtain a solution in the so-called weak sense.

4.1.2 Theoretical analysis

In order to obtain a weak formulation of a considered problem, we introduce suitable function spaces and apply the standard approach, in particular, the Green theorem. When the potentials present in the problem are convex functions, this procedure leads to a variational inequality (VI). When potentials are nonconvex functions, we obtain the hemivariational inequality (HVI) or, for both convex and nonconvex functions, the variational-hemivariational inequality (V-HVI). The study of VIs in the context of contact mechanics can be found, for example, in the monograph [38]. The notion of HVI was presented in the early 1980s by P.D. Panagiotopoulos in [39]. This theory is based on the generalized gradient introduced by F.C. Clarke in the 1970s and can be found in [40]. Early studies on HVI applications are summarized in [37, 41], and recent references include [42, 43] and, for V-HVIs, [44].

When other physical factors are considered, such as temperature or body damage, a weak formulation is supplemented by an additional equation or inequality. These relations are often coupled and cannot be solved separately. Our aim is to prove the theorem concerning the existence and regularity of the solution to the problem under suitable assumptions on operators and initial data. Usually, when static or quasistatic problems are concerned, to obtain the uniqueness of the solution, we require a so-called smallness assumption. This assumption binds constants from the hypothesis to the functions present in the problem. The proof of existence requires various mathematical tools, for example, convex and non-smooth analysis or functional analysis, in particular, the properties of different function spaces, such as Lebesgue and Sobolev spaces. Because a solution to a specific contact problem is not given by an analytical formula, obtaining the existence and uniqueness of this solution does not end the analysis of a physical problem.

4.1.3 Numerical analysis

In applications, our aim is to calculate the solution to the physical problem for specific input data, and as a result the third step is to numerically approximate this solution using a discrete scheme. Usually, the existence of a unique solution to this scheme follows from the same arguments as in the continuous case. To use this approach, a theorem on numerical error estimation is required for the applied scheme. In some cases, to obtain such a theorem, we have to strengthen the assumptions from previous steps. After we prove the theorem on numerical error estimation, we can employ the Finite Element Method. This method allows us to discretize the function spaces present in the weak formulation of the problem. An early discussion of applying FEM to solve VIs can be found in [45], and for HVIs in [46]. Essential examples of monographs in which numerical error estimations have been provided for various models are [47, 48, 49]. For a more recent review of the numerical analysis of HVIs and V-HVIs, we refer to [50].

In some cases, the existence and uniqueness of the solution to a particular problem is proved in the literature, but a numerical estimation of the Finite Element Method error is not conducted. Most of the estimates and simulations obtained refer to variational problems and not hemivariational problems. From a mathematical point of view, FEM in isolation from error estimation cannot be used directly, as it does not

guarantee correct results. This shows that numerical error estimations of solutions to contact problems connect an abstract mathematical model with a real-world application.

4.1.4 Computational simulations

Finally, discretization of the problem allows us to implement it and perform simulations for specific parameters. At this stage, the problem is reduced to either a problem of solving a finite system of equations or an optimization problem, depending on the approach. Simulations allow us to experiment with all parameters that describe the behavior of the body, as long as the adopted assumptions are satisfied. As an example, the exact statement of constitutive law and the magnitude of forces acting in a physical system can be changed. We can then confront our expectations with the results obtained. It is also a chance to evaluate the introduced model and present its strengths compared to other models. The main ideas and algorithms for the computational approach to contact mechanics are presented in [51], and for the survey of nonsmooth optimization methods that can be used in the process, we refer to [52]. From a theoretical point of view, proceeding with numerical analysis allows us to conduct practical computational simulations.

As stated above, the main focus of research presented in this thesis is numerical analysis and computational simulations of contact problems. Motivation for this is the fact that theoretical contact mechanics is motivated by the need to solve real physical problems. Numerical estimation of solutions to introduced problems and presentation of computational simulations are therefore valuable not only from the theoretical point of view but also have practical value.

4.2 Sample contact problem

We now present a sample mechanical contact model in its classical formulation and then derive its weak version. We also introduce some common notation used in the field of theoretical contact mechanics. This notation may slightly differ between the papers that constitute this dissertation. The language used here and in the included papers is rather mathematical and detailed compared to that commonly used in publications that focus more on applications. However, the core problems considered are the same in all domains of physical simulations, and the results obtained can be applied in each domain.

4.2.1 Derivation of the model

Let us denote by “ $\cdot$ ” and $\|\cdot\|$ the scalar product and the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{S}^d$, respectively, where $\mathbb{S}^d = \mathbb{R}_{sym}^{d \times d}$. The indexes i and j run from 1 to d and the index after a comma represents the partial derivative with respect to the corresponding component of the independent variable. Summation over repeated indices is implied.

A deformable physical body considered is represented by an open, bounded, and connected set $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$ in the application, with the boundary indicated by Γ . In the case of a time-dependent process, in which the data and vari-

ables depend on time, we are interested in modeling the behavior of the body on the time interval $[0, T]$, with $T > 0$. In this sample model, the unknowns are functions $\mathbf{u}: \Omega \times (0, T) \rightarrow \mathbb{R}^d$ that describe body displacement and $\boldsymbol{\sigma}: \Omega \times (0, T) \rightarrow \mathbb{S}^d$ that describe stresses within the body. This model can be extended with various unknowns, such as temperature or damage, to name some examples. According to Newton's law, the equation of motion has the following form

$$\rho \mathbf{u}''(t) = \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega \times (0, T),$$

where $\rho > 0$ is the mass density, $\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})$ is the divergence operator, and $\mathbf{f}_0: \Omega \times (0, T) \rightarrow \mathbb{R}^d$ is a function that represents forces acting inside the body, for example gravitational force. Processes modeled by this equation for ρ are called dynamic. Under the assumption that changes in the introduced model occur slowly and that the acceleration of the body is negligibly small, we can modify this equation by taking $\mathbf{u}'' \equiv \mathbf{0}$. Then we obtain a quasistatic process with equilibrium equation

$$\mathbf{0} = \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } \Omega \times (0, T).$$

When all data and variables are time independent, we have the case of a static process, and the equilibrium equation takes the form

$$\mathbf{0} = \text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega.$$

The relationship between stress that occurs in the body and its displacement can be described by the so-called constitutive law. The constitutive law for a viscoelastic body can take the form of

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}'(t))) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{in } \Omega \times (0, T).$$

This law reflects the properties of the particular body and employs the concept of linearized small strain tensor $\boldsymbol{\varepsilon}$ with dependence on displacement $\mathbf{u}$ defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

The operator $\mathcal{A}$ describes in this case the viscous properties of the body, whereas the operator $\mathcal{B}$ describes the elastic properties. Note that, generally, these operators may not be linear and may depend on time and spatial variables. In a static setting, we can consider a general elastic constitutive law of the form

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega,$$

where $\mathcal{A}$ is an elasticity operator. Again, this operator may additionally depend on the spatial variable reflecting different physical properties of the material in different parts of the body.

The next step in modeling body behavior is the introduction of boundary conditions, which are briefly presented here. We usually assume that the boundary Γ is divided into three disjoint measurable parts: Γ_D, Γ_C and Γ_N . Additionally, Γ is assumed to be Lipschitz continuous and therefore the outward normal vector $\boldsymbol{\nu}$ to Γ exists a.e. on the boundary. The body is clamped on Γ_D which is described by

a homogeneous Dirichlet condition. In many cases, we assume $meas(\Gamma_D) > 0$, which allows us to use Korn's inequality. On Γ_N body behavior is influenced by the external force, which can cause displacement on this part of the boundary. This behavior is modeled by a Neumann-type condition. Finally, let $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\sigma_\nu = \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\nu}$ be the normal components of $\mathbf{u}$ and $\boldsymbol{\sigma}$, and let $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ be their tangential components.

From the perspective of contact mechanics, we are especially interested in the behavior of the body on the part of the boundary denoted by Γ_C , where the body is in contact with the foundation. This contact typically generates two types of forces that act on the body - frictional forces (in the tangential direction) and forces caused by the pressure of the body on the foundation (in the normal direction). A wide variety of contact conditions have been introduced in the literature. An example of a friction law is the classical Coulomb law, and various modifications of this law can also be used.

In many cases, we cannot use a single value function to describe the relationship between body displacement $\mathbf{u}$ and stress field $\boldsymbol{\sigma}$ on Γ_C . This is because some laws, for example Coulomb's law, admit multivalence. To include such general cases, Clarke's subdifferential and subgradient may be used in order to introduce conditions on Γ_C (see Definitions 3 and 4 in Appendix B).

Let us now present the notion of generalized directional derivative and subgradient in the sense of Clarke. Assume $\psi: X \rightarrow \mathbb{R}$ is locally Lipschitz continuous on a reflexive Banach space X . The generalized directional derivative of ψ at $x \in X$ in the direction $v \in X$ is defined by

$$\psi^0(x; v) := \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{\psi(y + \lambda v) - \psi(y)}{\lambda}.$$

The generalized subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial\psi(x) := \{\zeta \in X^* \mid \psi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \ \forall v \in X\}.$$

4.2.2 Classical statement

Combining the correlations given by the equation of motion, constitutive law, and boundary conditions, we obtain a classical formulation of the contact mechanics problem. For simplicity in what follows, we do not explicitly indicate the dependence of various functions on the spatial variable $\mathbf{x}$. We consider a static mechanical contact problem with the classical formulation of the following form

Problem P_{clas} . Find a displacement field $\mathbf{u}: \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}: \Omega \rightarrow \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega \quad (4.1)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \quad (4.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad (4.3)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_N \quad \text{on } \Gamma_N \quad (4.4)$$

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_C \quad (4.5)$$

$$-\boldsymbol{\sigma}_\tau \in h_\tau(u_\nu) \partial j_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_C. \quad (4.6)$$

Here, equation (4.3) represents the clamped boundary condition on Γ_D and (4.4) represents the traction condition on Γ_N . Inclusion (4.5) describes the response of the foundation in a normal direction, while friction is modeled by inclusion (4.6), where j_ν and j_τ are given superpotentials, and h_τ is a given friction bound.

4.2.3 Weak formulation

In order to obtain weak formulation for Problem P , we introduce an additional notation. We use the standard notions of the Lebesgue and Sobolev spaces

$$L^2(\Omega)^d = L^2(\Omega; \mathbb{R}^d), \quad H^1(\Omega)^d = H^1(\Omega; \mathbb{R}^d).$$

We consider the following Hilbert spaces

$$\begin{aligned} \mathcal{H} &= L^2(\Omega; \mathbb{S}^d), \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \\ V &= \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \quad (\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}. \end{aligned}$$

The expression “ $\mathbf{v} = \mathbf{0}$ on Γ_D ” is understood as “ $\mathbf{v} = \mathbf{0}$ a.e. on Γ_D ” (the same applies for other similar expressions). Since we assume that $\text{meas}(\Gamma_D) > 0$, the Korn inequality

$$\|\mathbf{v}\|_{H^1(\Omega)^d} \leq c \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}$$

holds for all $\mathbf{v} \in V$, where $c > 0$ depends only on Ω and Γ_D . This implies that the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)^d}$ are equivalent on V . We take the trace operator $\gamma: V \rightarrow X := L^2(\Gamma_C)^d$. We denote by $\gamma^*: X^* \rightarrow V^*$ the adjoint operator to γ . By the Sobolev trace theorem, we know that $\gamma \in \mathcal{L}(V, X)$ with a norm equal to c_γ . Moreover, we denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between a dual space V^* and V .

We operate under assumptions presented in [III] (Section 4) and take sufficiently smooth functions $\boldsymbol{\sigma}$ and $\mathbf{u}$ which solve (4.1) - (4.6). Let us take $\mathbf{v} \in V$. We use the Green formula (Lemma 12) and equation (4.2) to obtain

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \gamma \mathbf{v} \, da \quad (4.7)$$

We now use conditions (4.3) and (4.4) to calculate

$$\int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \gamma \mathbf{v} \, da = \int_{\Gamma_N} \mathbf{f}_N \cdot \gamma \mathbf{v} \, da + \int_{\Gamma_C} (\sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau) \, da \quad (4.8)$$

From inclusions (4.5), (4.6) and definition of the Clarke subdifferential we have

$$-\sigma_\nu v_\nu \leq j_\nu^0(u_\nu; v_\nu), \quad -\boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau \leq h_\tau(u_\nu) j_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau) \quad \text{on } \Gamma_C,$$

and this implies that

$$\int_{\Gamma_C} (\sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau) \, da \geq - \int_{\Gamma_C} \left(j_\nu^0(u_\nu; v_\nu) + h_\tau(u_\nu) j_\tau^0(\mathbf{u}_\tau; \mathbf{v}_\tau) \right) \, da. \quad (4.9)$$

Let now $F: V \rightarrow V^*$ and $\mathbf{f} \in V^*$ be defined for all $\mathbf{w}, \mathbf{v} \in V$ as follows

$$\begin{aligned}\langle F\mathbf{w}, \mathbf{v} \rangle_{V^* \times V} &= (\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{w})), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \\ \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V} &= \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \boldsymbol{\gamma} \mathbf{v} \, da.\end{aligned}$$

From (4.1), (4.7), (4.8) and (4.9) we obtain the weak variational formulation of Problem $\mathbf{P}_{clas}$ in the following form of the hemivariational inequality

Problem $\mathbf{P}_{weak}$. *Find a displacement $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$*

$$\langle F\mathbf{u}, \mathbf{v} \rangle_{V^* \times V} + \int_{\Gamma_C} \left(j_{\nu}^0(u_{\nu}; v_{\nu}) + h_{\tau}(u_{\nu}) j_{\tau}^0(\mathbf{u}_{\tau}; \mathbf{v}_{\tau}) \right) da \geq \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V}. \quad (4.10)$$

The classical statement of the static mechanical contact problem presented here - Problem $\mathbf{P}_{clas}$ and its weak formulation - Problem $\mathbf{P}_{weak}$ are used as an application to abstract theoretical results in the paper [III].

Chapter 5

Overview of the contact problems

In what follows, we briefly describe the content and main results of each paper forming this dissertation. We group the papers by the type of contact problem considered and present problems in order of increasing complexity.

5.1 Abstract static problem

[III] Michał Jureczka, Anna Ochal, **A nonsmooth optimization approach for hemivariational inequalities with applications to contact mechanics**, *Applied Mathematics and Optimization*, 83, 2021, 1465–1485, doi:10.1007/s00245-019-09593-y

[IV] Anna Ochal, Michał Jureczka, Piotr Bartman, **A survey of numerical methods for hemivariational inequalities with applications to contact mechanics**, *Communications in Nonlinear Science and Numerical Simulation*, 114, 2022, 106563, doi.org/10.1016/j.cnsns.2022.106563

In the paper [III] we consider the following abstract operator inclusion problem. Let V be a reflexive Banach space, and let X be a Banach space. Let $A: V \rightarrow V^*$, $J: X \times X \rightarrow \mathbb{R}$, $\gamma \in \mathcal{L}(V, X)$ and $f \in V^*$. The functional J is assumed to be locally Lipschitz continuous with respect to its second argument.

Problem P_{incl} . Find $u \in V$ such that

$$Au + \gamma^* \partial_2 J(\gamma u, \gamma u) \ni f.$$

Here, $\partial_2 J$ denotes the Clarke subdifferential with respect to second variable of J . The main idea of this paper is to present a self-contained analysis of Problem P_{incl} and apply it to a sample mechanical contact problem. In order to do so, we introduce the following nonsmooth optimization problem. Let a functional $\mathcal{L}: V \times V \rightarrow \mathbb{R}$ be defined for all $w, v \in V$ as follows

$$\mathcal{L}(w, v) = \frac{1}{2} \langle Av, v \rangle_{V^* \times V} - \langle f, v \rangle_{V^* \times V} + J(\gamma w, \gamma v). \quad (5.1)$$

and consider the problem

Problem P_{opti} . Find $u \in V$ such that

$$0 \in \partial_2 \mathcal{L}(u, u).$$

We show that Problem P_{incl} and Problem P_{opti} are equivalent. Next, we prove the existence of a unique solution to P_{opti} under suitable assumptions. As the proof of this theorem is not constructive, we consider a finite-dimensional subspace $V^h \subset V$ with a discretization parameter $h > 0$ for which we present a discrete version of P_{opti} called Problem P_{opti}^h . We provide the following numerical error estimation

Theorem. *Under assumptions presented in the paper [III], for the unique solutions u and u^h to Problems P_{opti} and P_{opti}^h , respectively, there exists a constant $c > 0$ such that*

$$\|u - u^h\|_V^2 \leq c \inf_{v^h \in V^h} \left\{ \|u - v^h\|_V^2 + \|\gamma u - \gamma v^h\|_X + R(u, v^h) \right\}, \quad (5.2)$$

where a residual quantity is given by

$$R(u, v^h) = \langle Au, v^h - u \rangle_{V^* \times V} + \langle f, u - v^h \rangle_{V^* \times V}. \quad (5.3)$$

We now apply these abstract results to Problem P_{clas} and its weak formulation Problem P_{weak} that were already described in Section 4.2. We show that, under suitable assumptions, P_{weak} is equivalent to P_{incl} . This entire line of reasoning allows us to treat P_{opti}^h as a weak discrete version of P_{clas} .

Finally, we simulate P_{opti}^h for sets of sample data. As this scheme is in a form of nonsmooth minimization problem, we use a suitable algorithm to obtain an approximation of the solution. We additionally present a theoretical and empirical analysis of the convergence order for the introduced scheme.

In the paper [IV], we analyze three computational methods to solve contact mechanical problems: the direct optimization method, the augmented Lagrangian method, and the primal-dual active set strategy. We describe the details of the implementation and compare the results of the error estimate obtained for each method. To do this, we use a simplified version of Problem P_{clas} , where the function h_τ does not depend on u_ν .

5.2 Problem with wear of the material

[I] Danfu Han, Weimin Han, Michał Jureczka, Anna Ochal, **Numerical analysis of a contact problem with wear**, *Computers and Mathematics with Applications*, 79(10), 2020, 2942–2951, doi:10.1016/j.camwa.2019.12.027

In the paper [I] we present an elastic mechanical contact problem with an additional component of foundation wear. The body considered is in contact with a moving foundation, and we assume that this foundation is made of a hard, perfectly rigid material. Part of the body in contact with the foundation is covered by a layer of soft material that can wear out due to friction. Moreover, we assume that the soft layer is deformable and that the foundation can penetrate it.

Let us introduce the unknown wear function $w: \Gamma_C \times [0, T] \rightarrow \mathbb{R}_+$. The contact process on Γ_C is modeled using the normal compliance condition with unilateral constraint and Coulomb's law of dry friction. The model is supplemented by the ordinary differential equation and the initial condition for the wear function. These conditions are a consequence of Archard's law and describe the increase in foundation wear over time. The classical formulation reads as follows

Problem P_{clas}^w . Find a displacement field $\mathbf{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times [0, T] \rightarrow \mathbb{S}^d$ and a wear function $w: \Gamma_C \times [0, T] \rightarrow \mathbb{R}_+$ such that for all $t \in [0, T]$

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}(t))) \quad \text{in } \Omega \quad (5.4)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega \quad (5.5)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_D \quad (5.6)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_N(t) \quad \text{on } \Gamma_N \quad (5.7)$$

$$\left. \begin{aligned} u_\nu(t) &\leq g, \quad \sigma_\nu(t) + p(u_\nu(t) - w(t)) \leq 0, \\ (u_\nu(t) - g)(\sigma_\nu(t) + p(u_\nu(t) - w(t))) &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_C \quad (5.8)$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu p(u_\nu(t) - w(t))\mathbf{n}^*(t) \quad \text{on } \Gamma_C \quad (5.9)$$

$$w'(t) = \alpha(t)p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_C \quad (5.10)$$

$$w(0) = 0 \quad \text{on } \Gamma_C \quad (5.11)$$

where $\mathbf{n}^*(t) = -\frac{\mathbf{v}^*(t)}{\|\mathbf{v}^*(t)\|}$, $\alpha(t) = \kappa\|\mathbf{v}^*(t)\|$ and $\mathbf{v}^*$ denotes the velocity of the foundation.

Here, equation (5.4) represents an elastic constitutive law and $\mathcal{A}$ is an elasticity operator. Equilibrium equation (5.5) corresponds to the assumption that the acceleration of the body is negligible. Equation (5.6) describes the fact that the body is clamped on Γ_D and (5.7) represents outside forces acting on Γ_N . Equation (5.8) describes the damping response of the foundation with $g > 0$ being the thickness of a soft layer that covers Γ_C . The friction is modeled by equation (5.9). Here, it is assumed that $\mathbf{v}^*$ is significantly higher than the tangential body velocity $\mathbf{u}'_\tau$, $\kappa \geq 0$ represents the wear coefficient, and $\mu \geq 0$ is the friction coefficient. Equations (5.10) and (5.11) govern the evolution of the wear function.

The weak formulation of Problem $\mathbf{P}_{clas}^w$ and the proof of the existence and uniqueness of its solution are presented in [53]. In the paper [VII], we conducted an error estimation of a discrete numerical scheme and presented the results of computational simulations. Finally, the paper [I] that forms this dissertation contains a theoretical analysis of the order of convergence of this numerical scheme under additional assumptions. This analysis is supplemented by an empirical estimation based on computational simulations.

5.3 Problem with damage to the body

[II] Weimin Han, Michał Jureczka, Anna Ochal, **Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage**, *Journal of Computational and Applied Mathematics*, 377, 2020, 112886, doi:10.1016/j.cam.2020.11288

In the paper [II] we study the body made of material that is prone to damage. This damage (e.g. cracks in the concrete) is created by strain inside the body and can influence elastic properties of the body. The body is in contact with the rigid foundation.

Let us introduce a damage field $\zeta : \Omega \times [0, T] \rightarrow \mathbb{R}$ that is the ratio between the elastic moduli of damaged and damage-free materials. In applications, the function ζ obtains values between 0 and 1. When $\zeta = 0$ at a point of Ω the material is completely damaged and when $\zeta = 1$ there is no damage to the material. In the constitutive law, we introduce the dependence of the elasticity operator $\mathcal{B}$ on ζ . Furthermore, we consider partial differential inclusion with Clarke subdifferential and initial conditions on ζ . Here, $I_{[0,1]}$ denotes the indicator function of the interval $[0, 1]$. This model is based on one introduced in the monograph [49].

Problem P_{clas}^d . Find a displacement field $\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$, and a damage field $\zeta : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that for all $t \in (0, T)$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)) \quad \text{in } \Omega, \quad (5.12)$$

$$\zeta'(t) - \kappa \Delta \zeta(t) + \partial I_{[0,1]}(\zeta(t)) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)) \quad \text{in } \Omega, \quad (5.13)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (5.14)$$

$$\nabla \zeta(t) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma, \quad (5.15)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_D, \quad (5.16)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_N(t) \quad \text{on } \Gamma_N, \quad (5.17)$$

$$u_\nu(t) = 0, \quad -\boldsymbol{\sigma}_\tau(t) \in \partial j(\mathbf{u}'_\tau(t)) \quad \text{on } \Gamma_C, \quad (5.18)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \quad (5.19)$$

In the paper [II] we provide a complete mathematical treatment of Problem P_{clas}^d . We present a proof of the existence and uniqueness of the solution, introduce a fully discrete scheme for the problem considered, and derive an optimal-order error estimate. Finally, to illustrate the results obtained, we include computational simulations for sets of sample data along with a comparison of empirical convergence orders with those predicted theoretically.

5.4 Problem with thermal effects

[V] Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafraniec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870, doi.org/10.1016/j.nonrwa.2023.103870

In the paper [V] we consider a dynamic viscoelastic model that includes the influence of temperature. In this paper, in contrast to previous ones, we do not assume that the body is clamped, and, as a result, it can move freely through space. Conditions describing the evolution of displacement and temperature are coupled and cannot be solved separately. On the one hand, when the body heats up, it expands and, on the other hand, friction and deformation generate heat.

We introduce a temperature field $\theta: \Omega \times (0, T) \rightarrow \mathbb{R}$ and include the influence of temperature in the constitutive law using the expansion tensor $C = (c_{ij})$. The problem is complemented by Fourier's law of heat conduction, where $\mathcal{K}$ is the thermal conductivity tensor and $c_p \rho$ is the thermal mass of the material. We also include the influence of body deformation on its temperature. On the contact boundary Γ_C , we take into account the influence of heat transfer described by the function r , and temperature induced by friction described by the function h .

The statement of the problem reads as follows

Problem P_{clas}^t . Find a displacement field $\mathbf{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times [0, T] \rightarrow \mathbb{S}^d$ and a temperature field $\theta: \Omega \times [0, T] \rightarrow \mathbb{R}$ such that for all $t \in (0, T)$

$$\boldsymbol{\sigma}(t) = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u}'(t))) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + C\theta(t) \quad \text{in } \Omega \quad (5.20)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{u}''(t) \quad \text{in } \Omega \quad (5.21)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_N(t) \quad \text{on } \Gamma_N \quad (5.22)$$

$$-\sigma_\nu(t) = p(u_\nu(t)) \quad \text{on } \Gamma_C \quad (5.23)$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial j(\mathbf{u}'_\tau(t)) \quad \text{on } \Gamma_C \quad (5.24)$$

$$\rho c_p \theta'(t) - \text{Div}(K \nabla \theta(t)) = c_{ij} u'(t)_{i,j} + g(t) \quad \text{in } \Omega \quad (5.25)$$

$$-K \nabla \theta \cdot \boldsymbol{\nu}(t) = -r(\theta(t)) - h(\|\mathbf{u}'_\tau(t)\|) \quad \text{on } \Gamma_C \quad (5.26)$$

$$\theta(t) = 0 \quad \text{on } \Gamma_N \quad (5.27)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega \quad (5.28)$$

The paper [V] contains a weak formulation of Problem P_{clas}^t and a reference to the proof of the existence of a unique solution to this formulation. We discretize the problem and provide a detailed numerical error estimation. We also present results of computational simulations in two and three dimensions.

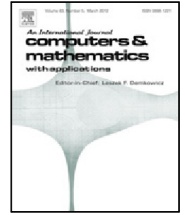
Chapter 6

Contents of the publications

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Danfu Han, Weimin Han, Michał Jureczka, Anna Ochal

Numerical analysis of a contact problem with wear



Numerical analysis of a contact problem with wear

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ABSTRACT

This paper represents a sequel to Jureczka and Ochal (2019) where numerical solution of a quasistatic contact problem is considered for an elastic body in frictional contact with a moving foundation. The model takes into account wear of the contact surface of the body caused by the friction. Some preliminary error analysis for a fully discrete approximation of the contact problem was provided in Jureczka and Ochal (2019). In this paper, we consider a more general fully discrete numerical scheme for the contact problem, derive optimal order error bounds and present computer simulation results showing that the numerical convergence orders match the theoretical predictions.

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1. Introduction

Contact phenomenon is common in engineering applications. Mathematical studies and numerical analysis of contact problems are most suitably carried out within the frameworks of variational inequalities or hemivariational inequalities, which have attracted the attention of many researchers. The related mathematical literature grows rapidly. Some representative comprehensive references in this area are [1–8] in the context of variational inequalities and [9–13] in the context of hemivariational inequalities.

For a contact problem, the mathematical model is constructed based on considerations of various aspects of the contact process. Factors to be taken into account include the type of the contact process (static, quasistatic or dynamic), constitutive relations of the deformable bodies, contact conditions of various application-specific forms. In certain applications, it is important to consider heating or thermal effects [14], or piezoelectricity effects [15]. Since the contact process inevitably causes material wear or even damage, it is not surprising that the wear effect has been built into mathematical models for a variety of contact processes, cf. [16–22]. In a recent paper [23], a mathematical model is proposed and studied for contact with wear described by Archard's law of surface wear. In this model, the friction between a deformable body and the foundation leads to wear of the contact surface of the body over time. Solution existence and uniqueness for the model are provided in [23]. Numerical approximation of the contact problem is the subject of [24] where some error bounds are derived for a fully discrete scheme. In this paper, we take a further step by considering a more general fully discrete numerical scheme for the contact problem that allows an arbitrary partition of the time interval, providing optimal order error estimates of the fully discrete scheme to solve the contact problem. Moreover, we present numerical results showing deformation of the contact body and numerical convergence orders of the fully discrete solutions that confirm the theoretical error bounds.

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The remainder of this paper is organized as follows. In Section 2 we introduce the contact problem and its variational formulation. In Section 3, we study a fully discrete numerical scheme and derive optimal order error bounds. In Section 4, we present computational simulation results for numerical convergence orders that match the theoretical predictions.

2. The contact problem and its variational formulation

First, we describe the physical setting of the contact problem. Consider a deformable body that occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in application. The body is subject to the action of volume forces with a total density $\mathbf{f}_0$. The boundary Γ of the domain Ω is assumed to be Lipschitz continuous and is divided into three disjoint measurable parts Γ_D , Γ_N and Γ_C , with $\text{meas}(\Gamma_D) > 0$. Denote by $\mathbf{v}$ the unit outward normal vector on Γ that is defined a.e. on Γ . The body is clamped on Γ_D , i.e., the displacement is equal to $\mathbf{0}$ on Γ_D . Surface tractions of a total density $\mathbf{f}_N$ act on the boundary Γ_N . The contact boundary is Γ_C where the contact is modeled by a normal compliance condition with a unilateral constraint and Coulomb's law of dry friction. Following [23], we assume that the body is elastic, in contact with a moving obstacle (foundation) made of a hard perfectly rigid material, and assume that the contact surface of the body Γ_C is covered by a layer of soft material. This layer is deformable and the foundation may penetrate it. Frictional contact with the foundation may cause this layer to wear over time.

We assume that the acceleration of the body is negligible and so the problem is quasistatic. In our model, the framework of the small strain theory is employed. We are interested in the body displacement and foundation wear in a time interval $[0, T]$, with $T > 0$. We denote by “ $\cdot$ ” and $\|\cdot\|$ the scalar product and the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{S}^d$, respectively, where $\mathbb{S}^d$ is the space of symmetric matrices of order d . The indices i and j run from 1 to d and the index after a comma represents the partial derivative with respect to the corresponding component of the independent variable. Summation convention over repeated indices is adopted. We denote the divergence operator by $\text{Div} \sigma = (\sigma_{ij,j})$ for an $\mathbb{S}^d$ -valued field σ . Standard Lebesgue and Sobolev spaces will be used, such as $L^2(\Omega)^d = L^2(\Omega; \mathbb{R}^d)$ and $H^1(\Omega)^d = H^1(\Omega; \mathbb{R}^d)$. Recall that the linearized strain tensor of a displacement field $\mathbf{u} \in H^1(\Omega)^d$ is

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

Let $u_\nu = \mathbf{u} \cdot \mathbf{v}$ and $\sigma_\nu = \boldsymbol{\sigma} \mathbf{v} \cdot \mathbf{v}$ be the normal components of $\mathbf{u}$ and $\boldsymbol{\sigma}$, respectively, and let $\mathbf{u}_\tau = \mathbf{u} - u_\nu \mathbf{v}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}$ be their tangential components, respectively. To simplify the notation, we will usually not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x}$.

Denote by $\mathbf{v}^*(t) \neq \mathbf{0}$ the velocity of the foundation. Let

$$\mathbf{n}^*(t) = -\mathbf{v}^*(t)/\|\mathbf{v}^*(t)\|, \quad \alpha(t) = \kappa \|\mathbf{v}^*(t)\|, \quad (2.1)$$

where κ represents the wear coefficient, and let μ be the friction coefficient. The classical formulation of the contact problem with wear is as follows.

Problem 1. Find a displacement field $\mathbf{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times [0, T] \rightarrow \mathbb{S}^d$, and a wear function $w: \Gamma_C \times [0, T] \rightarrow \mathbb{R}_+ = [0, \infty)$ such that for all $t \in [0, T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{F} \boldsymbol{\varepsilon}(\mathbf{u}(t)) \quad \text{in } \Omega, \quad (2.2)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (2.3)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_D, \quad (2.4)$$

$$\boldsymbol{\sigma}(t) \mathbf{v} = \mathbf{f}_N(t) \quad \text{on } \Gamma_N, \quad (2.5)$$

$$\left. \begin{aligned} u_\nu(t) &\leq g, \quad \sigma_\nu(t) + p(u_\nu(t) - w(t)) \leq 0, \\ (u_\nu(t) - g)(\sigma_\nu(t) + p(u_\nu(t) - w(t))) &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_C, \quad (2.6)$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu p(u_\nu(t) - w(t)) \mathbf{n}^*(t) \quad \text{on } \Gamma_C, \quad (2.7)$$

$$w'(t) = \alpha(t) p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_C, \quad (2.8)$$

$$w(0) = 0 \quad \text{on } \Gamma_C. \quad (2.9)$$

In Problem 1, Eq. (2.2) represents an elastic constitutive law with an elasticity operator $\mathcal{F}$. Eq. (2.3) is the equilibrium equation. The equality (2.4) describes the fact that body is clamped on Γ_D and (2.5) represents the traction boundary condition in which $\mathbf{f}_N$ denotes the density of external forces acting on Γ_N . The relations in (2.6) describe the damping response of the foundation, $g > 0$ being the thickness of a soft layer covering Γ_C and $p \geq 0$ being a normal compliance function which represents the reaction of the soft layer depending on the current value of the penetration. The friction is modeled by Eq. (2.7). Here, the size of $\mathbf{v}^*$ is assumed to be significantly larger than that of the tangential body velocity $\mathbf{u}'_\tau$. Eqs. (2.8) and (2.9) govern the evolution of the wear function. Detailed derivation of this model is presented in [23]. We remark that when $u_\nu = w = g$, the frictional force vanishes and the value of μ does not play a role. To change this, (2.6) and (2.7) can be replaced by more complicated relations, and analysis of such contact models may present substantial difficulty.

The contact problem will be studied in its variational formulation. For this purpose, we introduce function spaces and hypotheses on the problem data. We recall that for a normed space X , $C([0, T]; X)$ is the space of continuous functions from $[0, T]$ to X . We will use the following Hilbert spaces:

$$\mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad V = \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$$

endowed with the inner scalar products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} dx, \quad (\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}$$

with the corresponding norms. Denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between a dual space V^* and V . The set of admissible displacements is

$$U = \{\mathbf{v} \in V \mid v_\nu \leq g \text{ on } \Gamma_C\}.$$

For a function $\mathbf{v} \in V$, we use the same symbol $\mathbf{v}$ for its trace on the boundary Γ . By the Sobolev trace theorem, there exists a constant $c_0 > 0$ depending only on Ω , Γ_D and Γ_C such that

$$\|\mathbf{v}\|_{L^2(\Gamma_C)^d} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (2.10)$$

Now we introduce the hypotheses on the data needed in the study of [Problem 1](#).

$H(\mathcal{F})$: For the elasticity operator $\mathcal{F}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$,

- (a) $\mathcal{F}(\cdot, \boldsymbol{\varepsilon})$ is measurable on Ω for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\mathcal{F}(\cdot, \mathbf{0}) \in \mathcal{H}$;
- (b) $\exists L_{\mathcal{F}} > 0$ s.t. $\|\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$;
- (c) $\exists m_{\mathcal{F}} > 0$ s.t. $(\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.

$H(p)$: For the normal compliance function $p: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}_+$,

- (a) $p(\cdot, r)$ is measurable on $\Gamma_C \quad \forall r \in \mathbb{R}$;
- (b) $\exists L_p > 0$ s.t. $|p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$;
- (c) $(p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$;
- (d) $p(\mathbf{x}, r) = 0 \quad \forall r \leq 0$, a.e. $\mathbf{x} \in \Gamma_C$.

Note that $H(p)$ (b) and (d) imply

$$|p(\mathbf{x}, r)| \leq L_p |r| \quad \forall r \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C.$$

$H(\mathbf{f})$: For the densities of body and traction forces,

$$\mathbf{f}_0 \in C([0, T]; L^2(\Omega)^d), \quad \mathbf{f}_N \in C([0, T]; L^2(\Gamma_N)^d).$$

H_0 : For the friction and wear coefficients, and the foundation velocity,

- (a) $\mu \in L^\infty(\Gamma_C)$, $\mu(\mathbf{x}) \geq 0$ a.e. $\mathbf{x} \in \Gamma_C$;
- (b) $\kappa \in L^\infty(\Gamma_C)$, $\kappa(\mathbf{x}) \geq 0$ a.e. $\mathbf{x} \in \Gamma_C$;
- (c) $\mathbf{v}^* \in C([0, T]; \mathbb{R}^d)$, $\|\mathbf{v}^*(t)\| \geq v_0 > 0 \quad \forall t \in [0, T]$.

We notice that hypotheses H_0 imply the following regularities:

$$\mathbf{n}^* \in C([0, T]; \mathbb{R}^d), \quad \alpha \in C([0, T]; L^\infty(\Gamma_C)),$$

where $\mathbf{n}^*$ and α are defined in [\(2.1\)](#).

Finally, we will need a smallness assumption on the combined effect of the Lipschitz constant of the normal compliance function p and the friction coefficient μ . Recall that c_0 is the constant in the inequality [\(2.10\)](#).

H_s : $c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_C)} < m_{\mathcal{F}}$.

Now we define some operators and functions needed in the variational formulation of [Problem 1](#). Let $F: V \rightarrow V^*$, $\mathbf{f}: [0, T] \rightarrow V^*$ and $\varphi: [0, T] \times L^2(\Gamma_C) \times V \times V \rightarrow \mathbb{R}$ be defined for all $\mathbf{u}, \mathbf{v} \in V$, $w \in L^2(\Gamma_C)$, $t \in [0, T]$ as follows:

$$\begin{aligned} \langle F\mathbf{u}, \mathbf{v} \rangle_{V^* \times V} &= (\mathcal{F}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \\ \langle \mathbf{f}(t), \mathbf{v} \rangle_{V^* \times V} &= \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v} dx + \int_{\Gamma_N} \mathbf{f}_N(t) \cdot \mathbf{v} da, \\ \varphi(t, w, \mathbf{u}, \mathbf{v}) &= \int_{\Gamma_C} p(u_\nu - w) [v_\nu + \mu \mathbf{n}^*(t) \cdot \mathbf{v}_\tau] da. \end{aligned}$$

Let $W = L^2(\Gamma_C)$ be the space for the wear variable w . Using the standard procedures in the mathematical theory of contact mechanics, we obtain the weak formulation of [Problem 1](#).

Problem 2. Find $\mathbf{u}: [0, T] \rightarrow U$ and $w: [0, T] \rightarrow W$ such that for all $t \in [0, T]$,

$$\begin{aligned} \langle F\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t) \rangle_{V^* \times V} + \varphi(t, w(t), \mathbf{u}(t), \mathbf{v}) - \varphi(t, w(t), \mathbf{u}(t), \mathbf{u}(t)) &\geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{u}(t) \rangle_{V^* \times V} \quad \forall \mathbf{v} \in U, \\ w(t) &= \int_0^t \alpha(s) p(u_v(s) - w(s)) ds. \end{aligned} \quad (2.11)$$

We recall the following existence and uniqueness result for [Problem 2](#) from [\[23\]](#).

Theorem 3. Assume $\underline{H}(\mathcal{F})$, $\underline{H}(p)$, $\underline{H}(\mathbf{f})$, $\underline{H}_0$ and $\underline{H}_s$. Then [Problem 2](#) has a unique solution with the regularity

$$\mathbf{u} \in C([0, T]; V), \quad \sigma \in C([0, T]; \mathcal{H}), \quad w \in C^1([0, T]; W).$$

In addition, $w(t) \geq 0$ for all $t \in [0, T]$, a.e. on Γ_C .

3. Numerical analysis

We turn to the numerical solution of [Problem 2](#). Let $V^h \subset V$ and $W^h \subset W$ be two families of finite dimensional subspaces with a discretization parameter $h > 0$. Then define $U^h = U \cap V^h$. Let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the time interval $[0, T]$. Denote $k_n = t_{n+1} - t_n$, $0 \leq n \leq N-1$, and $k = \max_{0 \leq n \leq N-1} k_n$ for the time step size. For a function z continuous in t , we write $z_n = z(t_n)$.

We make the following additional assumptions on the solution $\mathbf{u}$ to [Problem 2](#) and the velocity of the foundation $\mathbf{v}^*$.

$$\underline{H}_1: \mathbf{u} \in H^1(0, T; V), \quad \mathbf{v}^* \in W^{1,\infty}(0, T; \mathbb{R}^d).$$

Note that assumptions $\underline{H}_1$ and $\underline{H}_0$ (b) imply that

$$\alpha \in W^{1,\infty}(0, T; L^\infty(\Gamma_C)).$$

Consider the following fully discrete scheme for solving [Problem 2](#).

Problem 4. Find $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=0}^N \subset U^h$ and $w^{hk} = \{w_n^{hk}\}_{n=0}^N \subset W^h$, $w_0^{hk} = 0$ such that for $0 \leq n \leq N$,

$$\langle F\mathbf{u}_n^{hk}, \mathbf{v}^h - \mathbf{u}_n^{hk} \rangle_{V^* \times V} + \varphi(t_n, w_n^{hk}, \mathbf{u}_n^{hk}, \mathbf{v}^h) - \varphi(t_n, w_n^{hk}, \mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) \geq \langle \mathbf{f}_n, \mathbf{v}^h - \mathbf{u}_n^{hk} \rangle_{V^* \times V} \quad \forall \mathbf{v}^h \in U^h, \quad (3.1)$$

and for $1 \leq n \leq N$,

$$w_n^{hk} = \sum_{j=0}^{n-1} k_j \alpha_j p(u_{j,v}^{hk} - w_j^{hk}).$$

We remark that existence of a unique solution to [Problem 4](#) follows from an application of discrete version of [Theorem 3](#). We also remark that the numerical scheme considered in [\[24\]](#) is a special case of [Problem 4](#) where a uniform partition of the time interval $[0, T]$ is used. For a uniform partition of $[0, T]$ into N equal size sub-intervals, we let $k = T/N$ be the time step and $t_n = nk$, $0 \leq n \leq N$, the node points.

We will make use of the following discrete Gronwall inequality [\[4, Lemma 7.25\]](#).

Lemma 5. Assume $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of non-negative numbers satisfying

$$e_n \leq c g_n + c \sum_{j=1}^{n-1} k_j e_j, \quad n = 1, \dots, N.$$

Then

$$e_n \leq c \left(g_n + \sum_{j=1}^{n-1} k_j g_j \right), \quad n = 1, \dots, N.$$

Therefore,

$$\max_{1 \leq n \leq N} e_n \leq c \max_{1 \leq n \leq N} g_n.$$

We have Ce a's inequality useful for error estimation.

Theorem 6. Under the assumptions stated in [Theorem 3](#) and the additional hypothesis H_1 , there exists a constant $c > 0$ such that for any $\mathbf{v}_n^h \in U^h$, $1 \leq n \leq N$,

$$\begin{aligned} \max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|w_n - w_n^{hk}\|_W^2) &\leq c k^2 + c k \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 \\ &+ c \max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{v}_n^h\|_V^2 + |R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h)|) \end{aligned} \quad (3.2)$$

where

$$R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h) = \langle F\mathbf{u}_n, \mathbf{v}_n^h - \mathbf{u}_n \rangle_{V^* \times V} + \varphi(t_n, w_n, \mathbf{u}_n, \mathbf{v}_n^h) - \varphi(t_n, w_n, \mathbf{u}_n, \mathbf{u}_n) - \langle \mathbf{f}_n, \mathbf{v}_n^h - \mathbf{u}_n \rangle_{V^* \times V}.$$

Proof. By modifying the proof of Theorem 4 in [\[24\]](#), we can establish the inequality

$$\begin{aligned} \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|w_n - w_n^{hk}\|_W^2 &\leq c \|\mathbf{u}_n - \mathbf{v}_n^h\|_V^2 + |R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h)| + c k^2 + c k \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 \\ &+ c \sum_{j=1}^{n-1} k_j (\|\mathbf{u}_j - \mathbf{u}_j^{hk}\|_V^2 + \|w_j - w_j^{hk}\|_W^2). \end{aligned} \quad (3.3)$$

Applying [Lemma 5](#) on (3.3), we get the inequality (3.2). ■

Note that from $H(p)$ and H_0 , we have (cf. [\[24, \(27\)\]](#)), for $t \in [0, T]$,

$$\begin{aligned} &\varphi(t, w_1, \mathbf{u}_1, \mathbf{v}_2) + \varphi(t, w_2, \mathbf{u}_2, \mathbf{v}_1) - \varphi(t, w_1, \mathbf{u}_1, \mathbf{v}_1) - \varphi(t, w_2, \mathbf{u}_2, \mathbf{v}_2) \\ &\leq L_p (c_0 \|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|w_1 - w_2\|_W) (c_0 \|\mu\|_{L^\infty(\Gamma_C)} \|\mathbf{v}_1 - \mathbf{v}_2\|_V + \|w_1 - w_2\|_W) \\ &\quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V, w_1, w_2 \in W. \end{aligned} \quad (3.4)$$

The inequality (3.2) is the starting point for further error estimation. For simplicity, we assume Ω is a polygonal/polyhedral domain. Then Γ_D , Γ_N and Γ_C can be expressed as unions of flat components (line segments for $d = 2$ and polygons for $d = 3$) that have pairwise disjoint interiors. In particular, we write $\overline{\Gamma_C} = \bigcup_{i=1}^{i_0} \Gamma_{C,i}$, where each component $\Gamma_{C,i}$ is a line segment if $d = 2$ or a polygon if $d = 3$. Consider a regular family of finite element partitions $\{\mathcal{T}^h\}$ of the domain Ω into triangular or tetrahedral elements such that if the intersection of one side/face of an element with one flat component of the boundary has a positive relative measure, then the side/face lies entirely in that flat component. Corresponding to $\mathcal{T}^h$, we define the linear element space

$$V^h = \{\mathbf{v}^h \in C(\overline{\Omega})^d \mid \mathbf{v}^h|_T \in \mathbb{P}_1(T)^d, T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D\}. \quad (3.5)$$

Then we define the discrete admissible finite element set

$$U^h = \{\mathbf{v}^h \in V^h \mid v_v^h \leq g \text{ at all nodes on } \Gamma_C\}. \quad (3.6)$$

We assume g is a concave function. Then, $U^h = V^h \cap U \subset U$. We proceed to derive an optimal order error estimate for the finite element solution defined by [Problem 4](#).

Theorem 7. Keep the assumptions stated in [Theorem 6](#). Assume further the solution regularities

$$\mathbf{u} \in C([0, T]; H^2(\Omega)^d), \quad \mathbf{u}|_{\Gamma_{C,i}} \in C([0, T]; H^2(\Gamma_{C,i})^d), \quad 1 \leq i \leq i_0, \quad (3.7)$$

$$\sigma \mathbf{v}|_{\Gamma} \in C([0, T]; L^2(\Gamma)^d). \quad (3.8)$$

Then we have the optimal order error estimate

$$\max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \|w_n - w_n^{hk}\|_W) \leq c(k + h). \quad (3.9)$$

Proof. By following the arguments presented in [\[4, Section 8.1\]](#), it can be shown that under the stated regularity assumptions, the solution of [Problem 2](#) satisfies, for $t \in [0, T]$,

$$\begin{aligned} \operatorname{Div} \sigma(t) + \mathbf{f}_0(t) &= \mathbf{0} \quad \text{a.e. in } \Omega, \\ \sigma(t) \mathbf{v} &= \mathbf{f}_N(t) \quad \text{a.e. on } \Gamma_N, \end{aligned}$$

where

$$\sigma(t) = \mathcal{F} \boldsymbol{\varepsilon}(\mathbf{u}(t)).$$

Using these relations we find that

$$R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h) = \int_{\Gamma_C} \{\sigma_n \mathbf{v} \cdot (\mathbf{v}_n^h - \mathbf{u}_n) + p(u_{n,v}^{hk} - w_n) [v_{n,v}^h - u_{n,v} + \mu \mathbf{n}_n^* \cdot (\mathbf{v}_{n,\tau}^h - \mathbf{u}_{n,\tau})]\} da.$$

Thus,

$$|R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h)| \leq c \|\mathbf{u}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_C)^d}. \quad (3.10)$$

This provides an upper bound for the term $|R_n(w_n, \mathbf{u}_n, \mathbf{v}_n^h)|$ on the right hand side of (3.2).

Now we bound the error $\|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V$. For simplicity, we denote

$$\varphi_0(\mathbf{u}, \mathbf{v}) := \varphi(0, 0, \mathbf{u}, \mathbf{v}).$$

Write

$$\begin{aligned} \langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{u}_0^{hk} \rangle_{V^* \times V} &= \langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{v}_0^h \rangle_{V^* \times V} + \langle F\mathbf{u}_0, \mathbf{v}_0^h - \mathbf{u}_0 \rangle_{V^* \times V} \\ &\quad + \langle F\mathbf{u}_0, \mathbf{u}_0 - \mathbf{u}_0^{hk} \rangle_{V^* \times V} - \langle F\mathbf{u}_0^{hk}, \mathbf{v}_0^h - \mathbf{u}_0^{hk} \rangle_{V^* \times V}. \end{aligned} \quad (3.11)$$

From (2.11) with $t = 0$,

$$\langle F\mathbf{u}_0, \mathbf{v} - \mathbf{u}_0 \rangle_{V^* \times V} + \varphi_0(\mathbf{u}_0, \mathbf{v}) - \varphi_0(\mathbf{u}_0, \mathbf{u}_0) \geq \langle \mathbf{f}_0, \mathbf{v} - \mathbf{u}_0 \rangle_{V^* \times V} \quad \forall \mathbf{v} \in U. \quad (3.12)$$

From (3.1) with $n = 0$,

$$\langle F\mathbf{u}_0^{hk}, \mathbf{v}_0^h - \mathbf{u}_0^{hk} \rangle_{V^* \times V} + \varphi_0(\mathbf{u}_0^{hk}, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0^{hk}, \mathbf{u}_0^{hk}) \geq \langle \mathbf{f}_0, \mathbf{v}_0^h - \mathbf{u}_0^{hk} \rangle_{V^* \times V} \quad \forall \mathbf{v}_0^h \in U^h. \quad (3.13)$$

Take $\mathbf{v} = \mathbf{u}_0^{hk}$ in (3.12), and use the resulting inequality and the inequality (3.13) in (3.11) to obtain

$$\begin{aligned} \langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{u}_0^{hk} \rangle_{V^* \times V} &\leq \langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{v}_0^h \rangle_{V^* \times V} + R_0(0, \mathbf{u}_0, \mathbf{v}_0^h) \\ &\quad + \varphi_0(\mathbf{u}_0, \mathbf{u}_0^{hk}) + \varphi_0(\mathbf{u}_0^{hk}, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0^{hk}, \mathbf{u}_0^{hk}). \end{aligned} \quad (3.14)$$

By $H(\mathcal{F})(c)$,

$$m_{\mathcal{F}} \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 \leq \langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{u}_0^{hk} \rangle_{V^* \times V}.$$

By $H(\mathcal{F})(b)$,

$$\langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{v}_0^h \rangle_{V^* \times V} \leq L_{\mathcal{F}} \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V \|\mathbf{u}_0 - \mathbf{v}_0^h\|_V.$$

Then, for an arbitrarily small $\epsilon > 0$, there is a constant c depending on ϵ such that

$$\langle F\mathbf{u}_0 - F\mathbf{u}_0^{hk}, \mathbf{u}_0 - \mathbf{v}_0^h \rangle_{V^* \times V} \leq \epsilon \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 + c \|\mathbf{u}_0 - \mathbf{v}_0^h\|_V^2.$$

By (3.10),

$$R_0(0, \mathbf{u}_0, \mathbf{v}_0^h) \leq c \|\mathbf{u}_0 - \mathbf{v}_0^h\|_{L^2(\Gamma_C)^d}.$$

By (3.4),

$$\varphi_0(\mathbf{u}_0, \mathbf{u}_0^{hk}) + \varphi_0(\mathbf{u}_0^{hk}, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0^{hk}, \mathbf{u}_0^{hk}) \leq c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_C)} \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V \|\mathbf{u}_0^{hk} - \mathbf{v}_0^h\|_V.$$

Since

$$\|\mathbf{u}_0^{hk} - \mathbf{v}_0^h\|_V \leq \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V + \|\mathbf{u}_0 - \mathbf{v}_0^h\|_V,$$

for the arbitrarily small $\epsilon > 0$, there is a constant c depending on ϵ such that

$$\begin{aligned} \varphi_0(\mathbf{u}_0, \mathbf{u}_0^{hk}) + \varphi_0(\mathbf{u}_0^{hk}, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0, \mathbf{v}_0^h) - \varphi_0(\mathbf{u}_0^{hk}, \mathbf{u}_0^{hk}) \\ \leq (c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_C)} + \epsilon) \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 + c \|\mathbf{u}_0 - \mathbf{v}_0^h\|_V^2. \end{aligned}$$

Using these relations in (3.14), we obtain

$$(m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_C)} - 2\epsilon) \|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 \leq c (\|\mathbf{u}_0 - \mathbf{v}_0^h\|_V^2 + \|\mathbf{u}_0 - \mathbf{v}_0^h\|_{L^2(\Gamma_C)^d}^2).$$

Recall the condition H_S ; choosing $\epsilon = (m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_C)}) / 4$ we obtain from the above inequality that

$$\|\mathbf{u}_0 - \mathbf{u}_0^{hk}\|_V^2 \leq c (\|\mathbf{u}_0 - \mathbf{v}_0^h\|_V^2 + \|\mathbf{u}_0 - \mathbf{v}_0^h\|_{L^2(\Gamma_C)^d}^2). \quad (3.15)$$

Using (3.15) and (3.10) in (3.2), we have

$$\begin{aligned} \max_{0 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|w_n - w_n^{hk}\|_W^2) &\leq c k^2 + c k (\|\mathbf{u}_0 - \mathbf{v}_0^h\|_V^2 + \|\mathbf{u}_0 - \mathbf{v}_0^h\|_{L^2(\Gamma_C)^d}^2) \\ &\quad + c \max_{1 \leq n \leq N} (\|\mathbf{u}_n - \mathbf{v}_n^h\|_V^2 + \|\mathbf{u}_n - \mathbf{v}_n^h\|_{L^2(\Gamma_C)^d}^2) \end{aligned} \quad (3.16)$$

for any $\mathbf{v}_n^h \in U^h$.

Thus, by applying the finite element interpolation theory (e.g., [25,26]), we have the optimal order error bound (3.9) from (3.16), under the solution regularities (3.7) and (3.8). ■

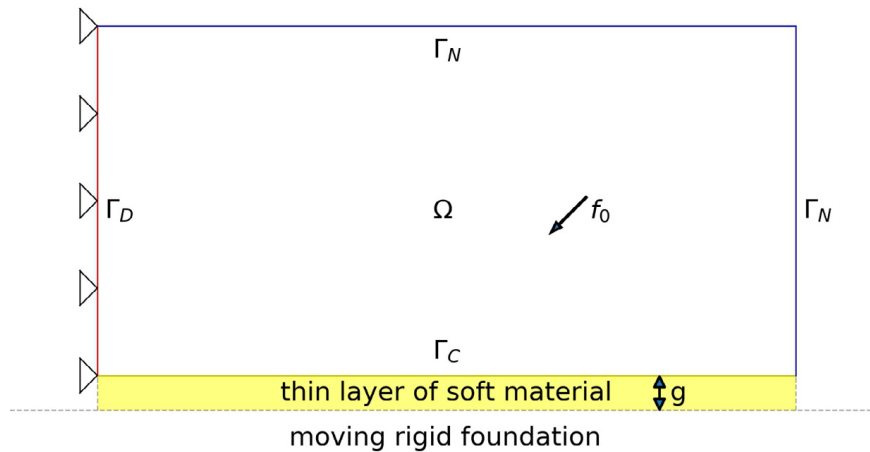


Fig. 1. Initial setting.

We comment that if $\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon})$ is a smooth function of $\mathbf{x}$, in particular if $\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon})$ does not depend on $\mathbf{x}$, then (3.8) follows from (3.7) and thus there is no need to assume (3.8). Moreover, if quadrilateral elements (in 2D) or hexahedral elements (in 3D) with lowest degree conforming finite element functions are used, we again have the optimal order error estimate stated in Theorem 7.

4. Numerical results

In this section, we report computer simulation results on a numerical example. Let $d = 2$ and consider a rectangular-shaped set $\Omega = (0, 2) \times (0, 1)$ presented in Fig. 1 with the following partition of the boundary

$$\Gamma_D = \{0\} \times [0, 1], \quad \Gamma_N = ([0, 2] \times \{1\}) \cup (\{2\} \times [0, 1]), \quad \Gamma_C = [0, 2] \times \{0\}.$$

The linear elasticity operator $\mathcal{F}$ is defined by

$$\mathcal{F}(\boldsymbol{\tau}) = 2\eta\boldsymbol{\tau} + \lambda\text{tr}(\boldsymbol{\tau})I, \quad \boldsymbol{\tau} \in \mathbb{S}^2.$$

Here I denotes the identity matrix, tr denotes the trace of the matrix, $\lambda > 0$ and $\eta > 0$ are the Lamé coefficients. In our simulations, we choose $\lambda = \eta = 4$, $T = 1$ and take the following data

$$\begin{aligned} \mathbf{u}_0(\mathbf{x}) &= (0, 0), \quad \mathbf{x} \in \Omega, \\ p(r) &= \begin{cases} 70r, & r \in [0, \infty), \\ 0, & r \in (-\infty, 0), \end{cases} \\ \mathbf{f}_N(\mathbf{x}, t) &= (0, 0), \quad \mathbf{x} \in \Gamma_N, \quad t \in [0, T], \\ \mathbf{f}_0(\mathbf{x}, t) &= (0, -2), \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \\ g &= 0.1. \end{aligned}$$

We use the linear finite element space V^h defined in (3.5) and its subset U^h defined in (3.6), based on uniform triangular partitions of $\overline{\Omega}$. We use the uniform partition of the time interval $[0, 1]$ with the time step size $k = 1/N$ for a positive integer N .

We first demonstrate the effect of some input data on the deformation of the body. In all cases, we show the shape of the body at final time $t = 1$ as well as the contact interface forces on Γ_C . We also present plots of wear w and penetration $u_v - w$ of the soft layer of material covering Γ_C at $t = 0.25$, $t = 0.5$, $t = 0.75$ and $t = 1$. The numerical solutions correspond to the time step size $1/64$ and where the boundary Γ_C of the body is divided into 64 equal parts.

In Fig. 2 we show the deformed configuration for $\mu(\mathbf{x}) = 0.2$, $\kappa(\mathbf{x}) = 0.02$ and $\mathbf{v}^*(\mathbf{x}, t) = (1, 0)$. We push the body down towards the moving foundation with a force $\mathbf{f}_0$, and as a result of friction, the soft layer of material covering Γ_C wears out allowing the body to move downward. We observe that in this case coefficient κ , governing the rate of wear, is not big enough to cause the body to touch the foundation. Because of the friction, the body moves in the same direction as the foundation, i.e. to the right.

We then increase the wear coefficient κ to $\kappa(\mathbf{x}) = 0.06$. The deformed configuration is shown in Fig. 3. We observe that the layer of soft material on part of the boundary Γ_C wears out, allowing the body to rest on the rigid foundation as it cannot penetrate it further. In this case, by condition (2.6), plotted normal components of contact interface forces represent a lower bound for the actual force.

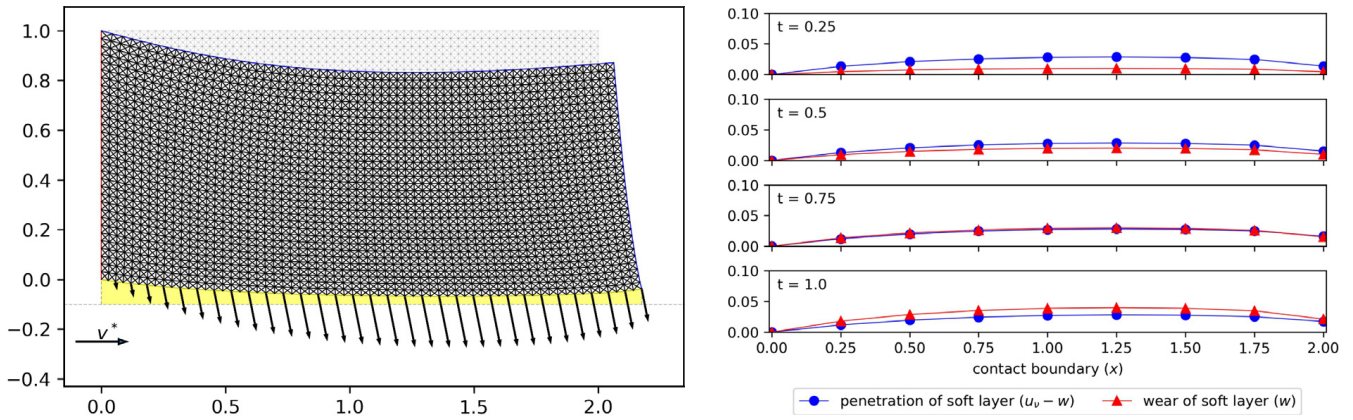


Fig. 2. Deformed configuration at $t = 1$, wear and penetration for $\mu = 0.2$, $\kappa = 0.02$, $\mathbf{v}^* = (1, 0)$.

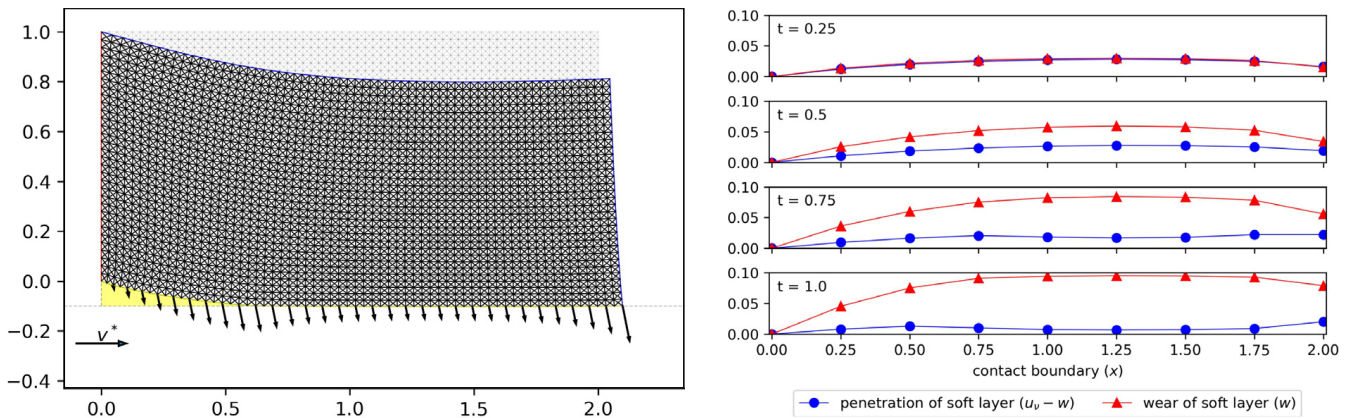


Fig. 3. Deformed configuration at $t = 1$, wear and penetration for $\mu = 0.2$, $\kappa = 0.06$, $\mathbf{v}^* = (1, 0)$.

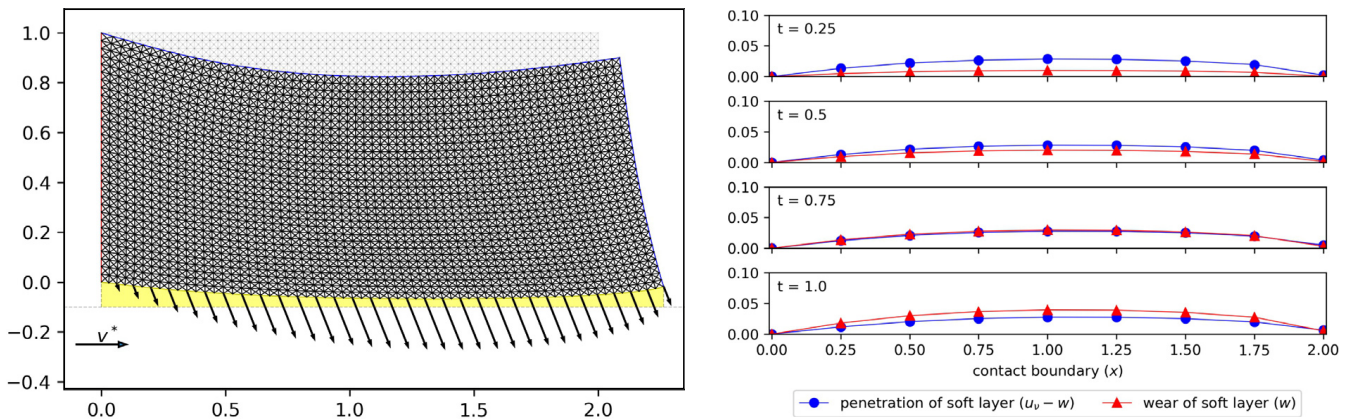


Fig. 4. Deformed configuration at $t = 1$, wear and penetration for $\mu = 1$, $\kappa = 0.02$, $\mathbf{v}^* = (1, 0)$.

In Fig. 4, we show the deformed configuration for $\mu(\mathbf{x}) = 1$, $\kappa(\mathbf{x}) = 0.02$ and $\mathbf{v}^*(\mathbf{x}, t) = (1, 0)$. We observe that the body moves further to the right, which is a result of increased friction between soft layer of material covering Γ_C and the rigid foundation.

The result in Fig. 5 corresponds to $\mu(\mathbf{x}) = 0.3$, $\kappa(\mathbf{x}) = 0.02$ and $\mathbf{v}^*(\mathbf{x}, t) = (-1, 0)$. Note that the direction of the motion of the foundation is reversed. As a result, the lower part of the body squeezes to the left and we observe that the boundary Γ_C is slightly curled. We conclude that all those modifications lead to results that can be expected.

Finally, we explore the numerical convergence orders of the numerical method on the model problem with $\mathbf{f}_N(\mathbf{x}, t) = (-0.5, -0.5)$, $\mathbf{f}_0(\mathbf{x}, t) = (-0.5, -2)$, $\mu(\mathbf{x}) = 0.6$, $\kappa(\mathbf{x}) = 0.01$, and $\mathbf{v}^*(\mathbf{x}, t) = (1, 0)$. We present a comparison of numerical errors $\|\mathbf{u} - \mathbf{u}^{hk}\|_V$ and $\|w - w^{hk}\|_W$ computed for a sequence of solutions to discretized problems. We use a uniform discretization of the problem domain and time interval according to the spatial discretization parameter h and time step

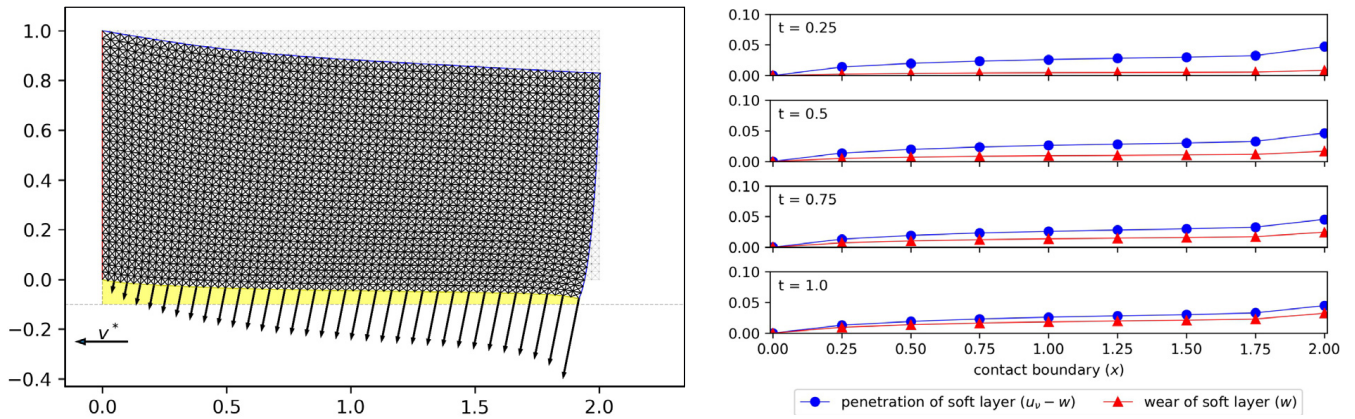


Fig. 5. Deformed configuration at $t = 1$, wear and penetration for $\mu = 0.3$, $\kappa = 0.02$, $v^* = (-1, 0)$.

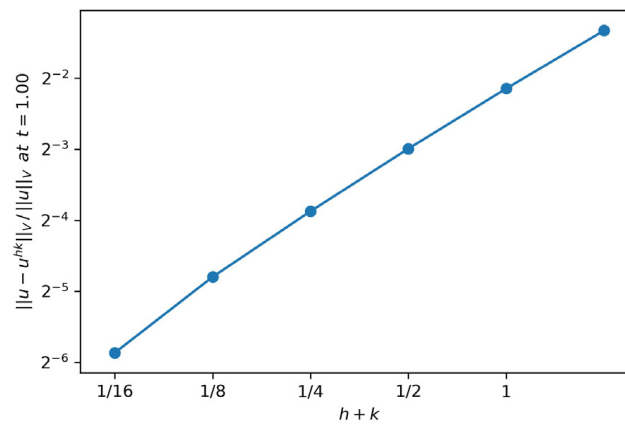


Fig. 6. Error estimate $\|u - u^{hk}\|_V / \|u\|_V$.

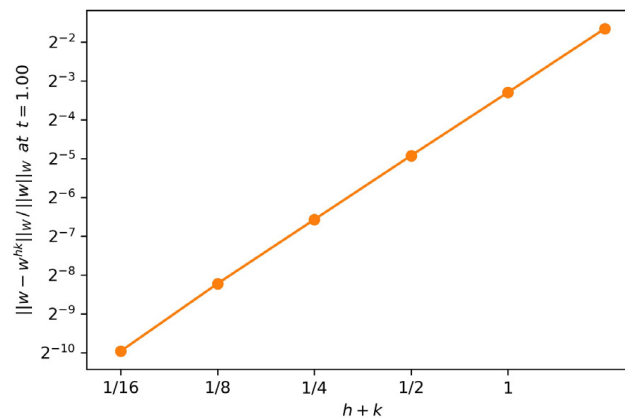


Fig. 7. Error estimate $\|w - w^{hk}\|_W / \|w\|_W$.

size k , respectively. The boundary Γ_C of Ω is divided into $1/h$ equal parts. We start with $h = 1/2$ and $k = 1/2$, which are successively halved. The numerical solution corresponding to $h = 1/128$ and $k = 1/128$ is taken as the “exact” solution $\mathbf{u}$ and w with $\|\mathbf{u}\|_V \doteq 0.30081$ and $\|w\|_W \doteq 0.03011$. The results are presented in Table 1 and Figs. 6 and 7, where the dependence of the relative error estimates $\|\mathbf{u} - \mathbf{u}^{hk}\|_V / \|\mathbf{u}\|_V$ and $\|w - w^{hk}\|_W / \|w\|_W$ with respect to $h + k$ are plotted on a log–log scale. A first order convergence is clearly observed for the numerical solutions of the displacement. The numerical convergence orders for the numerical solutions of the wear function are somewhat higher than 1.

Table 1

Numerical errors.

$h + k$	1	1/2	1/4	1/8	1/16	1/32
$\ u - u^{hk}\ _V / \ u\ _V$	$3.9800e^{-1}$	$2.2588e^{-1}$	$1.2565e^{-1}$	$6.8223e^{-2}$	$3.5958e^{-2}$	$1.7146e^{-2}$
Convergence order		0.8172	0.8462	0.8811	0.9239	1.0684
$\ w - w^{hk}\ _W / \ w\ _W$	$3.1733e^{-1}$	$1.0178e^{-1}$	$3.2950e^{-2}$	$1.0516e^{-2}$	$3.3533e^{-3}$	$1.0045e^{-3}$
Convergence order		1.6405	1.6271	1.6476	1.6490	1.7391

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II

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Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage



Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage

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ABSTRACT

This paper is devoted to the study of a hemivariational inequality modeling the quasistatic bilateral frictional contact between a viscoelastic body and a rigid foundation. The damage effect is built into the model through a parabolic differential inclusion for the damage function. A solution existence and uniqueness result is commented. A fully discrete scheme is introduced with the time derivative of the damage function approximated by the backward finite difference and the spatial derivatives approximated by finite elements. An optimal order error estimate is derived for the fully discrete scheme when linear elements are used for the velocity and displacement variables, and piecewise constants are used for the damage function. Simulation results on numerical examples are reported illustrating the performance of the fully discrete scheme and the theoretically predicted convergence orders.

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1. Introduction

In this paper, we study a mathematical model in the form of a hemivariational inequality for a quasistatic bilateral frictional contact problem between a viscoelastic body and a rigid foundation. The friction law is given in the form of a subdifferential condition. Damage of the material is incorporated. Modeling, variational analysis and numerical solution of contact problems have been studied extensively; in this regard, a few comprehensive references are [1–4] in the context of variational inequalities (VIs), and [5,6] in the context of hemivariational inequalities (HVIs).

The notion of hemivariational inequalities (HVIs) was introduced in early 1980s to model mechanical problems involving non-smooth, non-monotone or multi-valued relations [7]. Early results on modeling, mathematical analysis and engineering applications of HVIs are summarized in [8,9]; recent summarized accounts include [5,6,10]. Since there are no solution formulas for HVIs in applications, numerical simulation is the only feasible approach to solving HVIs. Detailed discussion of the finite element method for solving HVIs can be found in [11]. More recently, there has been substantial progress in numerical analysis of HVIs, especially on optimal order error estimates for numerical solutions of HVIs, starting with the paper [12], followed by a sequence of papers, e.g., [13–16]; the reader is referred to [17] for a recent survey.

Many contact processes are accompanied with material damage. In applications, it is very important to consider the damage effect. General mathematical models for damage were derived in [18,19]; see also [20]. In [21], a quasistatic

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contact problem for a viscoelastic material is studied variationally and numerically, where the damage effect of the viscoelastic material is taken into account. Systematic variational analysis and numerical analysis of contact problems with damage effect is summarized in [4]. The mathematical problems investigated in these references are in the form of VIs. For studies of contact problems with damage in the form of HVIs, the reader is referred to [22].

This is the first paper devoted to numerical analysis of an HVI arising in a contact problem with damage. The rest of the paper is organized as follows. In Section 2, we introduce the contact problem, present its weak formulation as an HVI and comment on the solution existence and uniqueness. In Section 3, we consider a fully discrete numerical scheme for the contact problem and derive an optimal order error estimate under appropriate solution regularity assumptions. In Section 4, we report computer simulation results on a numerical example and illustrate numerical convergence orders that match the theoretical error bound.

2. The contact problem

We first introduce the pointwise formulation of the quasistatic contact problem for the contact between a viscoelastic body and a rigid foundation. The initial configuration of the body is Ω , a Lipschitz bounded domain in $\mathbb{R}^d$ ($d \leq 3$ in applications). The body is subject to the action of volume forces of a total density $\mathbf{f}_0$. The boundary Γ of the domain Ω is split into three disjoint measurable parts, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ such that Γ_1 and Γ_3 are non-trivial. We will assume the body is fixed along Γ_1 , is subject to the action of surface traction with a total density $\mathbf{f}_2$ on Γ_2 . Along the contact boundary Γ_3 , the body and the foundation are in bilateral contact and the frictional process is described by a generalized subdifferential inclusion.

Following [3,23], we consider a viscoelastic constitutive law with the damage effect in the form

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta),$$

where $\mathbf{u}$ is the displacement field, ζ is the damage function, $\boldsymbol{\sigma}$ is the stress field, $\mathcal{A}$ and $\mathcal{B}$ are the viscosity operator and the elasticity operator. These operators are allowed to depend on the spatial location. For convenience, we use the shorthand notation $\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}})$ and $\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta)$ for $\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}(\dot{\mathbf{u}}))$ and $\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{u}), \zeta)$, respectively. The symbol $\dot{\mathbf{u}}$ denotes the time derivative of $\mathbf{u}$. The time interval of interest is $[0, T]$ for some $T > 0$.

The notion of the damage function was introduced in [18,19] to quantify the damage to the material. It is defined to be the ratio between the elastic modulus of the damaged material and that of the original material. The value of the damage function ζ lies in $[0, 1]$. The value $\zeta = 1$ indicates that there is no damage in the material, whereas the value $\zeta = 0$ corresponds to a completely damaged material. When $0 < \zeta < 1$, there is a partial damage and the system has a reduced load carrying capacity. A popular model for the evolution of the damage function is given by a parabolic differential inclusion:

$$\dot{\zeta} - \kappa \Delta \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta),$$

where $\kappa > 0$ is a constant microcrack diffusion coefficient, $I_{[0,1]}$ is the indicator function of the interval $[0, 1]$, $\partial I_{[0,1]}$ is the convex subdifferential of $I_{[0,1]}$, and ϕ is the mechanical source of damage, depending on the strain and the damage itself. On the boundary Γ , a homogeneous Neumann condition is described for ζ .

For a vector $\mathbf{v}$ defined on Γ , we let $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ be its normal component, and let $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ be its tangential component. For a stress tensor $\boldsymbol{\sigma}$ defined on Γ , we let $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ be its normal and tangential components, respectively.

Denote by $\mathbf{u}_0$ and ζ_0 the initial values of the displacement and the damage function. The pointwise formulation of the contact problem is as follows.

Problem 2.1. Find a displacement field $\mathbf{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times [0, T] \rightarrow \mathbb{S}^d$, and a damage field $\zeta: \Omega \times [0, T] \rightarrow \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega \times (0, T), \quad (2.1)$$

$$\dot{\zeta} - \kappa \Delta \zeta + \partial I_{[0,1]}(\zeta) \ni \phi(\boldsymbol{\varepsilon}(\mathbf{u}), \zeta) \quad \text{in } \Omega \times (0, T), \quad (2.2)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \times (0, T), \quad (2.3)$$

$$\frac{\partial \zeta}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma \times (0, T), \quad (2.4)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \quad (2.5)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \quad (2.6)$$

$$u_\nu = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j(\dot{\mathbf{u}}_\tau) \quad \text{on } \Gamma_3 \times (0, T), \quad (2.7)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega. \quad (2.8)$$

We already know that (2.1) is the viscoelastic constitutive law with damage, and (2.2) is the evolution relation for the damage function. We consider a quasistatic contact process and (2.3) is the corresponding equilibrium equation. The initial conditions for the displacement field and the damage function are given by (2.8). The relations (2.4)–(2.7) are the boundary conditions for the damage function, the displacement boundary condition on Γ_1 , the traction boundary condition on Γ_2 , and the bilateral friction contact condition on Γ_3 . Here, the friction dissipation pseudopotential j will be assumed to be Lipschitz continuous, and ∂j represents the generalized subdifferential in the sense of Clarke (cf. [24,25]). We will also need the notion of the generalized directional derivative in the sense of Clarke. Let V be a Banach space and let $\psi: V \rightarrow \mathbb{R}$ be a locally Lipschitz continuous functional. Recall that the generalized directional derivative of ψ at $u \in V$ in the direction $v \in V$ is

$$\psi^0(u; v) := \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{\psi(w + \lambda v) - \psi(w)}{\lambda},$$

whereas the generalized subdifferential of ψ at $u \in V$ is

$$\partial \psi(u) := \{ \xi \in V^* \mid \psi^0(u; v) \geq \langle \xi, v \rangle_{V^* \times V} \quad \forall v \in V \}.$$

We note the following properties:

$$\psi^0(u; t v) = t \psi^0(u; v) \quad \forall u, v \in V, t \geq 0, \quad (2.9)$$

$$\psi^0(u; v_1 + v_2) \leq \psi^0(u; v_1) + \psi^0(u; v_2) \quad \forall u, v_1, v_2 \in V, \quad (2.10)$$

$$\psi^0(u; v) = \max \{ \langle \zeta, v \rangle_{V^* \times V} \mid \zeta \in \partial \psi(u) \} \quad \forall u, v \in V, \quad (2.11)$$

$$u_n \rightarrow u \text{ and } v_n \rightarrow v \text{ in } V \implies \limsup_{n \rightarrow \infty} \psi^0(u_n; v_n) \leq \psi^0(u; v). \quad (2.12)$$

Problem 2.1 will be studied in its weak form. For this purpose, we first need to introduce some function spaces. Let

$$Q = L^2(\Omega)_{\text{sym}}^{d \times d},$$

which is a Hilbert space with the inner product

$$(\sigma, \tau)_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} dx, \quad \sigma, \tau \in Q.$$

This will be the space for stress and strain fields. The function space for the displacement field is the Hilbert space

$$V = \{ \mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1, \quad v_\nu = 0 \text{ on } \Gamma_3 \}$$

with the inner product $(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q$ and the associated norm $\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q$. The space for the damage field is

$$Z = H^1(\Omega).$$

For convenience, we let

$$Z_0 = L^2(\Omega).$$

The spaces Z and Z_0 are endowed with their canonical inner products and norms. Let $\mathbb{S}^d$ be the space of second-order symmetric tensors on $\mathbb{R}^d$.

In the study of the contact problem, we assume that the operator $\mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies the following conditions:

$$\left. \begin{array}{l} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \quad \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q. \end{array} \right\} \quad (2.13)$$

Similarly, we assume the operator $\mathcal{B}: \Omega \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{S}^d$ has the following properties:

$$\left. \begin{array}{l} \text{(a) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2, \zeta_2)\| \leq L_{\mathcal{B}} (\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|) \\ \quad \quad \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \zeta_1, \zeta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}, \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}, \zeta) \text{ is measurable on } \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } Q. \end{array} \right\} \quad (2.14)$$

As an example of the viscoelastic constitutive law with damage, we consider

$$\sigma = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \eta (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}_{K(\zeta)}(\boldsymbol{\varepsilon}(\mathbf{u}))), \quad (2.15)$$

where the viscosity tensor $\mathcal{A}$ satisfies (2.13), η is a positive coefficient, $K(\zeta)$ is a damage dependent elasticity set, which is assumed to be convex and $\mathcal{P}_{K(\zeta)}$ is the projection operator onto the set $K(\zeta)$. We require the properties $\mathbf{0} \in K(\zeta)$ and $\zeta_1 \geq \zeta_2$ implies $K(\zeta_1) \subset K(\zeta_2)$. The second property implies that as the damage of the material increases, i.e., the value of the damage function ζ decreases, the elasticity convex set expands, and the material resembles a purely viscous one. A concrete example is given by the von Mises convex set

$$K(\zeta) = \{ \boldsymbol{\tau} \in \mathbb{S}^d \mid \|\boldsymbol{\tau}^D\| \leq \zeta \sigma_Y \}, \quad (2.16)$$

where $\boldsymbol{\tau}^D = \boldsymbol{\tau} - (\text{tr } \boldsymbol{\tau}/d)\mathbf{I}$ is the deviatoric part of $\boldsymbol{\tau}$, and $\sigma_Y > 0$ is the yield limit of the damage-free material. Since the projection operator is nonexpansive, it can be verified that $\mathcal{B}(\boldsymbol{\varepsilon}, \zeta) = \eta (\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{P}_{K(\zeta)}(\boldsymbol{\varepsilon}(\mathbf{u})))$ satisfies (2.14).

On the damage source function $\phi: \Omega \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{R}$, the assumptions are

$$\left. \begin{array}{l} \text{(a) There exists } L_\phi > 0 \text{ such that} \\ \quad |\phi(\mathbf{x}, \boldsymbol{\varepsilon}_1, \zeta_1) - \phi(\mathbf{x}, \boldsymbol{\varepsilon}_2, \zeta_2)| \leq L_\phi (\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + |\zeta_1 - \zeta_2|) \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \zeta_1, \zeta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \zeta \in \mathbb{R}, \mathbf{x} \mapsto \phi(\mathbf{x}, \boldsymbol{\varepsilon}, \zeta) \text{ is measurable on } \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \phi(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } L^2(\Omega). \end{array} \right\} \quad (2.17)$$

On the friction dissipation pseudopotential $j: \Gamma_3 \times \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\left. \begin{array}{l} \text{(a) } \mathbf{x} \mapsto j(\mathbf{x}, \boldsymbol{\xi}) \text{ is measurable on } \Gamma_3 \forall \boldsymbol{\xi} \in \mathbb{R}^d \\ \quad \text{and } j(\mathbf{x}, \mathbf{0}) \in L^2(\Gamma_3). \\ \text{(b) } \boldsymbol{\xi} \mapsto j(\mathbf{x}, \boldsymbol{\xi}) \text{ is locally Lipschitz in } \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) There exist constants } c_{0\tau}, c_{1\tau} \geq 0 \text{ such that} \\ \quad \|\partial j(\mathbf{x}, \boldsymbol{\xi})\| \leq c_{0\tau} + c_{1\tau} \|\boldsymbol{\xi}\| \forall \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(d) There is a constant } c_{2\tau} \geq 0 \text{ such that} \\ \quad j^0(\mathbf{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0(\mathbf{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq c_{2\tau} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\ \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right\} \quad (2.18)$$

Moreover, we assume

$$\kappa > 0 \quad (2.19)$$

on the microcrack diffusion coefficient,

$$\mathbf{f}_0 \in C([0, T]; L^2(\Omega)^d), \quad \mathbf{f}_2 \in C([0, T]; L^2(\Gamma_2)^d) \quad (2.20)$$

on the densities of forces and tractions,

$$\mathbf{u}_0 \in V, \quad \zeta_0 \in K \quad (2.21)$$

on the initial data. Here K represents the set of admissible damage functions defined by

$$K = \{ \xi \in Z \mid \xi \in [0, 1] \text{ a.e. in } \Omega \}. \quad (2.22)$$

By the Riesz representation theorem, we can define $\mathbf{f}: [0, T] \rightarrow V$ by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V, \, t \in [0, T]. \quad (2.23)$$

Then conditions (2.20) imply

$$\mathbf{f} \in C([0, T]; V). \quad (2.24)$$

Let $a: Z \times Z \rightarrow \mathbb{R}$ be the bilinear form

$$a(\xi, \eta) = \kappa \int_{\Omega} \nabla \xi \cdot \nabla \eta \, dx, \quad \xi, \eta \in Z. \quad (2.25)$$

Let us introduce a weak formulation of the problem (2.1)–(2.8).

Problem 2.2. Find a displacement field $\mathbf{u}: [0, T] \rightarrow V$, a stress field $\boldsymbol{\sigma}: [0, T] \rightarrow Q$, and a damage field $\zeta: [0, T] \rightarrow Z$ such that for all $t \in [0, T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \quad (2.26)$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_Q + \int_{\Gamma_3} j^0(\dot{\mathbf{u}}_\tau(t); \mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t)) \, da \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall \mathbf{v} \in V, \quad (2.27)$$

$$\zeta(t) \in K, \quad (\dot{\zeta}(t), \xi - \zeta(t))_{Z_0} + a(\zeta(t), \xi - \zeta(t)) \geq (\phi(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \zeta(t)), \xi - \zeta(t))_{Z_0} \quad \forall \xi \in K, \quad (2.28)$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \zeta(0) = \zeta_0. \quad (2.29)$$

Denote by c_τ the smallest constant in the trace inequality

$$\|\mathbf{v}_\tau\|_{L^2(\Gamma_3)^d} \leq c_\tau \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (2.30)$$

Similar to [22, Theorem 5.1], we can prove the following result.

Theorem 2.3. Assume (2.13), (2.14), (2.17)–(2.21),

$$c_{2\tau} c_\tau^2 < m_A, \quad (2.31)$$

and either $\sqrt{2} c_{1\tau} c_\tau^2 < m_A$ or for a constant $d_\tau \geq 0$, $j^0(\mathbf{x}, \xi; -\xi) \leq d_\tau (1 + \|\xi\|)$ for all $\xi \in \mathbb{R}^d$ and a.e. $\mathbf{x} \in \Omega$. Then Problem 2.2 has a unique solution $\mathbf{u} \in C^1([0, T]; V)$, $\sigma \in C([0, T]; Q)$ and $\zeta \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Moreover, $\text{Div } \sigma \in L^2(\Omega)^d$.

In the numerical solution of Problem 2.2, it will be convenient to introduce the velocity variable

$$\mathbf{w}(t) = \dot{\mathbf{u}}(t). \quad (2.32)$$

Given the velocity $\mathbf{w}(t)$ and the initial displacement $\mathbf{u}(0) = \mathbf{u}_0$ from (2.29), we can recover the displacement by the formula

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{w}(s) ds. \quad (2.33)$$

Then (2.26)–(2.27) can be rewritten as

$$\sigma(t) = \mathcal{A}\varepsilon(\mathbf{w}(t)) + \mathcal{B}(\varepsilon(\mathbf{u}(t)), \zeta(t)), \quad (2.34)$$

$$(\sigma(t), \varepsilon(\mathbf{v} - \mathbf{w}(t)))_Q + \int_{\Gamma_3} j^0(\mathbf{w}_\tau(t); \mathbf{v}_\tau - \mathbf{w}_\tau(t)) da \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{w}(t))_V \quad \forall \mathbf{v} \in V, \quad (2.35)$$

for all $t \in [0, T]$.

3. Numerical analysis of the weak formulation

In this section, we introduce and study a fully discrete numerical scheme to solve Problem 2.2. We assume the conditions stated in Theorem 2.3 are valid so that Problem 2.2 has a unique solution.

For the approximation of the time derivative of the damage function, we use finite difference. We divide the time interval $[0, T]$ uniformly and comment that much of the discussion of the numerical method below can be extended straightforward to the case of general partition of the time interval. Thus, let N be a positive integer, and define $k = T/N$ the step-size. Then $0 = t_0 < t_1 < \dots < t_N = T$ is a uniform partition of $[0, T]$ with the nodes $t_n = nk$, $n = 0, 1, \dots, N$. For a function $z(t)$ continuous on $[0, T]$, we write $z_n = z(t_n)$. We use the backward difference approximation

$$\dot{\zeta}(t_n) \approx \delta \zeta_n := \frac{\zeta_n - \zeta_{n-1}}{k}, \quad 1 \leq n \leq N. \quad (3.1)$$

For the spatial discretization, we use the finite element method. For simplicity, we assume Ω is a polygonal/polyhedral domain, and express the three parts of the boundary, Γ_k , $1 \leq k \leq 3$, as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \bigcup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \leq k \leq 3.$$

Let $\{\mathcal{T}^h\}_h$ be a regular family of finite element partitions of $\overline{\Omega}$ into triangular/tetrahedral elements, compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \leq i \leq i_k$, $1 \leq k \leq 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Here $h \rightarrow 0$ denotes the finite element mesh-size. Corresponding to the partition $\mathcal{T}^h$, we introduce the linear finite element space

$$V^h = \{\mathbf{v}^h \in C(\overline{\Omega})^d \mid \mathbf{v}^h|_T \in \mathbb{P}_1(T)^d \quad \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_1, v_v^h = 0 \text{ on } \Gamma_3\} \quad (3.2)$$

for the displacement field, the piecewise constant finite element space

$$Q^h = \{\tau^h \in Q \mid \tau^h|_T \in \mathbb{R}^{d \times d} \quad \forall T \in \mathcal{T}^h\} \quad (3.3)$$

for the stress field, and the linear finite element space

$$Z^h = \{\xi^h \in C(\overline{\Omega}) \mid \xi^h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}^h\} \quad (3.4)$$

for the damage field. Define the constrained subset of Z^h :

$$K^h = \{\xi^h \in Z^h \mid \xi^h|_T \in [0, 1] \quad \forall T \in \mathcal{T}^h\}. \quad (3.5)$$

Let $\mathbf{u}_0^h \in V^h$ and $\zeta_0^h \in K^h$ be appropriate approximations of $\mathbf{u}_0$ and ζ_0 such that

$$\|\mathbf{u}_0 - \mathbf{u}_0^h\|_V \leq c h, \quad \|\zeta_0 - \zeta_0^h\|_{Z_0} \leq c h. \quad (3.6)$$

These conditions are valid if, e.g., $\mathbf{u}_0 \in H^2(\Omega)^d$, $\zeta_0 \in H^1(\Omega)$, and we define $\mathbf{u}_0^h \in V^h$ to be the interpolant or $H^1(\Omega)^d$ - or $L^2(\Omega)^d$ -projection of $\mathbf{u}_0$ onto V^h , define $\zeta_0^h \in K^h$ to be the $L^2(\Omega)$ -projection of ζ_0 onto K^h . The smoothness conditions $\mathbf{u}_0 \in H^2(\Omega)^d$ and $\zeta_0 \in H^1(\Omega)$ will follow from the solution regularities (3.14) and (3.16).

The discrete velocity and displacement approximations are denoted by $\{\mathbf{w}_n^{hk}\}_{n=1}^N \subset V^h$ and $\{\mathbf{u}_n^{hk}\}_{n=0}^N \subset V^h$, whereas the discrete stress and damage function approximations are denoted by $\{\boldsymbol{\sigma}_n^{hk}\}_{n=1}^N \subset Q^h$ and $\{\zeta_n^{hk}\}_{n=0}^N \subset K^h$. Let $\mathcal{P}_{Q^h}: Q \rightarrow Q^h$ be the orthogonal projection from Q to Q^h , defined by

$$\mathcal{P}_{Q^h} \boldsymbol{\sigma} \in Q^h, \quad (\mathcal{P}_{Q^h} \boldsymbol{\sigma}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\sigma}, \boldsymbol{\tau}^h)_Q \quad \forall \boldsymbol{\sigma} \in Q, \boldsymbol{\tau}^h \in Q^h. \quad (3.7)$$

Then a fully discrete scheme for Problem 2.2 is the following.

Problem 3.1. Find a discrete displacement field $\mathbf{u}^{hk} = \{\mathbf{u}_n^{hk}\}_{n=0}^N \subset V^h$, a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=1}^N \subset Q^h$, and a discrete damage field $\zeta^{hk} = \{\zeta_n^{hk}\}_{n=0}^N \subset K^h$ such that for $n = 1, 2, \dots, N$,

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h} \mathcal{A} \boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}) + \mathcal{P}_{Q^h} \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \quad (3.8)$$

$$(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_\tau^h - \mathbf{w}_{n\tau}^{hk}) da \geq (\mathbf{f}_n, \mathbf{v}^h - \mathbf{w}_n^{hk})_V \quad \forall \mathbf{v}^h \in V^h, \quad (3.9)$$

$$(\delta \zeta_n^{hk}, \xi^h - \zeta_n^{hk})_{Z_0} + a(\zeta_n^{hk}, \xi^h - \zeta_n^{hk}) \geq (\phi(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \xi^h - \zeta_n^{hk})_{Z_0} \quad \forall \xi^h \in K^h, \quad (3.10)$$

and

$$\mathbf{u}_0^{hk} = \mathbf{u}_0^h, \quad \zeta_0^{hk} = \zeta_0^h. \quad (3.11)$$

Here $\{\mathbf{u}_n^{hk}\}_{n=0}^N$ and $\{\mathbf{w}_n^{hk}\}_{n=1}^N$ are related by the equalities

$$\mathbf{w}_n^{hk} = \delta \mathbf{u}_n^{hk} \quad \text{and} \quad \mathbf{u}_n^{hk} = \mathbf{u}_0^h + k \sum_{i=1}^n \mathbf{w}_i^{hk}. \quad (3.12)$$

Note that for implementation, (3.8) and (3.9) are combined together to give

$$\begin{aligned} & (\mathcal{A} \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q + (\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_\tau^h - \mathbf{w}_{n\tau}^{hk}) da \\ & \geq (\mathbf{f}_n, \mathbf{v}^h - \mathbf{w}_n^{hk})_V \quad \forall \mathbf{v}^h \in V^h. \end{aligned} \quad (3.13)$$

The solution existence and uniqueness of Problem 3.1 can be proved by an induction argument. The focus of the rest of this section is to bound the numerical solution errors. For this purpose, we assume the following additional solution regularities:

$$\mathbf{u} \in W^{2,1}(0, T; V) \cap C([0, T]; H^2(\Omega)^d), \quad (3.14)$$

$$\boldsymbol{\sigma} \in C([0, T]; H^1(\Omega)^{d \times d}), \quad (3.15)$$

$$\zeta \in H^2(0, T; Z_0) \cap C^1([0, T]; Z) \cap C([0, T]; H^2(\Omega)). \quad (3.16)$$

Then following the argument in [3, Section 8.1], we can show that for all $t \in (0, T)$,

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{a.e. in } \Omega, \quad (3.17)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_2 \quad \text{a.e. on } \Gamma_2, \quad (3.18)$$

$$u_v = 0 \quad \text{a.e. on } \Gamma_3. \quad (3.19)$$

We first show the uniform boundedness of the numerical solution.

Lemma 3.2. *There exists a constant $M > 0$, independent of h and k , such that*

$$\max_{1 \leq n \leq N} \|\mathbf{w}_n^{hk}\|_V \leq M.$$

Proof. Let us fix $n \in \{1, \dots, N\}$. We take $\mathbf{v}^h = \mathbf{0} \in V^h$ in (3.13),

$$(\mathcal{A} \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q \leq -(\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; -\mathbf{w}_{n\tau}^{hk}) da + (\mathbf{f}_n, \mathbf{w}_n^{hk})_V.$$

By (2.13)(b),

$$m_{\mathcal{A}} \|\mathbf{w}_n^{hk}\|_V^2 \leq (\mathcal{A} \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}) - \mathcal{A}(\mathbf{0}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q.$$

Combining these relations, we obtain

$$m_{\mathcal{A}} \|\mathbf{w}_n^{hk}\|_V^2 \leq -(\mathcal{A}(\mathbf{0}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q - (\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}, -\mathbf{w}_{n\tau}^{hk}) da + (\mathbf{f}_n, \mathbf{w}_n^{hk})_V. \quad (3.20)$$

Let us treat each of the terms on the right side of (3.20). Let $\delta > 0$ be a small constant to be chosen later. We recall the modified Cauchy–Schwarz inequality: for any $a, b \in \mathbb{R}$,

$$a b \leq \delta a^2 + c b^2, \quad c = \frac{1}{4\delta}. \quad (3.21)$$

Then,

$$-(\mathcal{A}(\mathbf{0}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q \leq \delta \|\mathbf{w}_n^{hk}\|_V^2 + c \|\mathcal{A}(\mathbf{0})\|_Q^2. \quad (3.22)$$

Similarly,

$$-(\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q \leq \delta \|\mathbf{w}_n^{hk}\|_V^2 + c \|\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk})\|_Q^2.$$

By (2.14) (a),

$$\begin{aligned} \|\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk})\| &\leq \|\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}) - \mathcal{B}(\mathbf{0}, 0)\| + \|\mathcal{B}(\mathbf{0}, 0)\| \\ &\leq L_{\mathcal{B}} (\|\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk})\| + |\zeta_{n-1}^{hk}|) + \|\mathcal{B}(\mathbf{0}, 0)\|. \end{aligned}$$

Then, by noting that $|\zeta_{n-1}^{hk}| \leq 1$, we have

$$\|\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk})\|_Q \leq c (\|\mathbf{u}_{n-1}^{hk}\|_V + 1).$$

Now

$$\mathbf{u}_{n-1}^{hk} = \mathbf{u}_0^h + k \sum_{i=1}^{n-1} \mathbf{w}_i^{hk},$$

and so

$$\begin{aligned} \|\mathbf{u}_{n-1}^{hk}\|_V &\leq \|\mathbf{u}_0^h\|_V + k \sum_{i=1}^{n-1} \|\mathbf{w}_i^{hk}\|_V, \\ \|\mathbf{u}_{n-1}^{hk}\|_V^2 &\leq c + c k \sum_{i=1}^{n-1} \|\mathbf{w}_i^{hk}\|_V^2. \end{aligned}$$

Hence,

$$-(\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}))_Q \leq \delta \|\mathbf{w}_n^{hk}\|_V^2 + c k \sum_{i=1}^{n-1} \|\mathbf{w}_i^{hk}\|_V^2 + c. \quad (3.23)$$

Write

$$j^0(\mathbf{w}_{n\tau}^{hk}, -\mathbf{w}_{n\tau}^{hk}) = [j^0(\mathbf{w}_{n\tau}^{hk}, -\mathbf{w}_{n\tau}^{hk}) + j^0(\mathbf{0}; \mathbf{w}_{n\tau}^{hk})] - j^0(\mathbf{0}; \mathbf{w}_{n\tau}^{hk}).$$

By (2.18) (d),

$$j^0(\mathbf{w}_{n\tau}^{hk}, -\mathbf{w}_{n\tau}^{hk}) + j^0(\mathbf{0}; \mathbf{w}_{n\tau}^{hk}) \leq c_{2\tau} \|\mathbf{w}_{n\tau}^{hk}\|^2.$$

By (2.18) (c),

$$-j^0(\mathbf{0}; \mathbf{w}_{n\tau}^{hk}) \leq c_{0\tau} \|\mathbf{w}_{n\tau}^{hk}\| \leq \delta \|\mathbf{w}_n^{hk}\|^2 + c.$$

Thus,

$$\int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}, -\mathbf{w}_{n\tau}^{hk}) da \leq (c_{2\tau} c_{\tau}^2 + \delta) \|\mathbf{w}_n^{hk}\|_V^2 + c. \quad (3.24)$$

Finally,

$$(\mathbf{f}_n, \mathbf{w}_n^{hk})_V \leq \delta \|\mathbf{w}_n^{hk}\|_V^2 + c \|\mathbf{f}_n\|_V^2. \quad (3.25)$$

Use (3.22)–(3.25) in (3.20),

$$(m_{\mathcal{A}} - c_{2\tau} c_{\tau}^2 - 4\delta) \|\mathbf{w}_n^{hk}\|_V^2 \leq c k \sum_{i=1}^{n-1} \|\mathbf{w}_i^{hk}\|_V^2 + c,$$

where the constant c depends on $\|\mathcal{A}(\mathbf{0})\|_Q$, $\|\mathcal{B}(\mathbf{0}, \mathbf{0})\|_Q$, $\|\mathbf{f}\|_{C([0,T];V)}$, and an upper bound on $\|\mathbf{u}_0^{hk}\|_V$. By choosing $\delta = (m_{\mathcal{A}} - c_{2\tau}c_{\tau}^2)/8$, we have

$$\|\mathbf{w}_n^{hk}\|_V^2 \leq c k \sum_{i=1}^{n-1} \|\mathbf{w}_i^{hk}\|_V^2 + c.$$

Applying the Gronwall inequality, we get

$$\|\mathbf{w}_n^{hk}\|_V^2 \leq c, \quad 1 \leq n \leq N.$$

The proof is completed. ■

We will make use of several relations presented in [4]. By [4, (3.25)],

$$\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\|_V^2 \leq c(h^2 + k^2) + c k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2. \quad (3.26)$$

By [4, (3.27)],

$$\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 \leq c(h^2 + k^2) + c k \sum_{i=1}^n \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2. \quad (3.27)$$

By [4, (3.59)],

$$\|\zeta_n - \zeta_n^{hk}\|_{Z_0}^2 + k \sum_{i=1}^n |\zeta_i - \zeta_i^{hk}|_Z^2 \leq c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{u}_i - \mathbf{u}_i^{hk}\|_V^2 \right). \quad (3.28)$$

By [4, (3.71)],

$$\|\sigma_n - \sigma_n^{hk}\|_Q^2 \leq c(\|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}^2) + c k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 + c(h^2 + k^2). \quad (3.29)$$

From the regularity assumption (3.16), we see that

$$\|\zeta_n - \zeta_{n-1}\|_{Z_0} = O(k). \quad (3.30)$$

By (2.13)(b),

$$m_{\mathcal{A}} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \leq (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n - \mathbf{w}_n^{hk}))_Q. \quad (3.31)$$

For any $\mathbf{v}_n^h \in V^h$, write

$$(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n - \mathbf{w}_n^{hk}))_Q = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{w}_n - \mathbf{v}_n^h))_Q + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q.$$

From (2.34) at $t = t_n$,

$$\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) = \sigma_n - \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_n), \zeta_n).$$

From (3.8),

$$\mathcal{P}_{Q^h} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}) = \sigma_n^{hk} - \mathcal{P}_{Q^h} \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}).$$

Thus,

$$\mathcal{P}_{Q^h} (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk})) = (\sigma_n - \sigma_n^{hk}) - (I - \mathcal{P}_{Q^h})\sigma_n - \mathcal{P}_{Q^h} (\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_n), \zeta_n) - \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk})).$$

Then, from (3.7),

$$\begin{aligned} & (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \\ &= (\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q - ((I - \mathcal{P}_{Q^h})\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \\ & \quad - (\mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_n), \zeta_n) - \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u}_{n-1}^{hk}), \zeta_{n-1}^{hk}), \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \\ & \leq (\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q + c(\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}) \|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\|_V \\ & \quad + \|(I - \mathcal{P}_{Q^h})\sigma_n\|_Q \|\boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk})\|_Q. \end{aligned}$$

Apply the above relations in (3.31) to obtain

$$\begin{aligned} m_{\mathcal{A}} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 & \leq c \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V \|\mathbf{w}_n - \mathbf{v}_n^h\|_V + (\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \\ & \quad + c(\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\|_V + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}) \|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\|_V + c h \|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\|_V. \end{aligned} \quad (3.32)$$

By the triangle inequality of the norm,

$$\|\mathbf{v}_n^h - \mathbf{w}_n^{hk}\|_V \leq \|\mathbf{w}_n - \mathbf{v}_n^h\|_V + \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V.$$

For any $\delta > 0$, we apply (3.21) and derive from (3.32) that

$$(m_{\mathcal{A}} - 2\delta) \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \leq c \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q + c(h^2 + \|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}^2). \quad (3.33)$$

Using (3.26) to bound the term $\|\mathbf{u}_n - \mathbf{u}_{n-1}^{hk}\|_V^2$, we get from (3.33) that

$$(m_{\mathcal{A}} - 2\delta) \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \leq c \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q + c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}^2 \right). \quad (3.34)$$

We take $\mathbf{v} = \mathbf{w}_n^{hk} \in V^h$ in (2.35) at $t = t_n$ to get

$$-(\sigma_n, \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk} - \mathbf{w}_n))_Q \leq \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}; \mathbf{w}_{n\tau}^{hk} - \mathbf{w}_{n\tau}) da - (\mathbf{f}_n, \mathbf{w}_n^{hk} - \mathbf{w}_n)_V. \quad (3.35)$$

From (3.9),

$$-(\sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{w}_n^{hk}))_Q \leq \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}^{hk}) da - (\mathbf{f}_n, \mathbf{v}^h - \mathbf{w}_n^{hk})_V \quad (3.36)$$

Write

$$(\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q = (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{w}_n^{hk} - \mathbf{w}_n))_Q - (\sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q.$$

By using (3.35) and (3.36) with $\mathbf{v}^h = \mathbf{v}_n^h$, we then have

$$(\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \leq (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V + \int_{\Gamma_3} [j^0(\mathbf{w}_{n\tau}; \mathbf{w}_{n\tau}^{hk} - \mathbf{w}_{n\tau}) + j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}^{hk})] da.$$

By the sub-additivity of the generalized directional derivative,

$$j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}^{hk}) \leq j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) + j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{w}_{n\tau} - \mathbf{w}_{n\tau}^{hk}).$$

By (2.18)(d),

$$j^0(\mathbf{w}_{n\tau}; \mathbf{w}_{n\tau}^{hk} - \mathbf{w}_{n\tau}) + j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{w}_{n\tau} - \mathbf{w}_{n\tau}^{hk}) \leq c_{2\tau} \|\mathbf{w}_{n\tau} - \mathbf{w}_{n\tau}^{hk}\|^2.$$

Hence,

$$(\sigma_n - \sigma_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n^{hk}))_Q \leq (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da + c_{2\tau} c_\tau^2 \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2.$$

Use this inequality in (3.34),

$$\begin{aligned} & (m_A - c_{2\tau} c_\tau^2 - 2\delta) \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \\ & \leq c \|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V + \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da \\ & \quad + c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}^2 \right). \end{aligned} \quad (3.37)$$

Recall the smallness assumption (2.31). We can choose $\delta > 0$ sufficiently small and derive from (3.37) that

$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 & \leq c [\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V] + c \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da \\ & \quad + c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 + \|\zeta_n - \zeta_{n-1}^{hk}\|_{Z_0}^2 \right). \end{aligned}$$

With the use of (3.30), we rewrite the above inequality as

$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 & \leq c [\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V] + c \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da \\ & \quad + c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 + \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_{Z_0}^2 \right). \end{aligned} \quad (3.38)$$

From (3.28),

$$\|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_{Z_0}^2 \leq c \left(h^2 + k^2 + k \sum_{i=1}^{n-2} \|\mathbf{u}_i - \mathbf{u}_i^{hk}\|_V^2 \right).$$

Use (3.27),

$$k \sum_{i=1}^{n-2} \|\mathbf{u}_i - \mathbf{u}_i^{hk}\|_V^2 \leq c (h^2 + k^2) + c k \sum_{i=1}^{n-2} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2.$$

Hence, from (3.38),

$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 & \leq c [\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V] \\ & \quad + c \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da + c \left(h^2 + k^2 + k \sum_{i=1}^{n-1} \|\mathbf{w}_i - \mathbf{w}_i^{hk}\|_V^2 \right). \end{aligned}$$

By an application of the Gronwall inequality,

$$\begin{aligned} \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 & \leq c [\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + (\sigma_n, \boldsymbol{\varepsilon}(\mathbf{v}_n^h - \mathbf{w}_n))_Q - (\mathbf{f}_n, \mathbf{v}_n^h - \mathbf{w}_n)_V] \\ & \quad + c \int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da + c (h^2 + k^2). \end{aligned} \quad (3.39)$$

We multiply (3.17) by an arbitrary $\mathbf{v} \in V$, integrate over Ω and perform an integration by parts,

$$\int_{\Gamma} \sigma \mathbf{v} \cdot \mathbf{v} da - \int_{\Omega} \sigma \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx + \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} dx = 0.$$

Split the integral over Γ to three sub-integrals: the sub-integral over Γ_1 is zero since $\mathbf{v} = \mathbf{0}$ on Γ_1 ; for the sub-integral over Γ_2 , we apply the relation (3.18); for the sub-integral over Γ_3 , we use the relation (3.19). As a result,

$$\int_{\Omega} \sigma \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx - \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} dx - \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} da = \int_{\Gamma_3} \sigma_{\tau} \cdot \mathbf{v}_{\tau} da.$$

Thus, (3.39) can be reduced to

$$\|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \leq c (\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + h^2 + k^2) + c \int_{\Gamma_3} [\sigma_{n\tau} \cdot (\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) + j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau})] da. \quad (3.40)$$

Since $\sigma \in C([0, T]; H^1(\Omega)^{d \times d})$ by (3.15), we have

$$\sigma_\tau \in C([0, T]; L^2(\Gamma_3)^d).$$

Thus,

$$\int_{\Gamma_3} \sigma_{n\tau} \cdot (\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da \leq c \|\sigma_\tau\|_{C([0, T]; L^2(\Gamma_3)^d)} \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|_{L^2(\Gamma_3)^d}.$$

By Lemma 3.2, $\|\mathbf{w}_n^{hk}\|_V$ is uniformly bounded. Then, from (2.18)(c),

$$j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) \leq c (1 + \|\mathbf{w}_{n\tau}^{hk}\|) \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|,$$

we have a constant c depending on the upper bound M from Lemma 3.2 that

$$\int_{\Gamma_3} j^0(\mathbf{w}_{n\tau}^{hk}; \mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}) da \leq c \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|_{L^2(\Gamma_3)^d}.$$

Therefore, from (3.40), we can derive the inequality

$$\|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V^2 \leq c (\|\mathbf{w}_n - \mathbf{v}_n^h\|_V^2 + \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|_{L^2(\Gamma_3)^d}^2 + h^2 + k^2) \quad \forall \mathbf{v}^h \in V^h. \quad (3.41)$$

Based on (3.41), (3.27) and (3.28), we have proved the following Céa inequality for error estimation.

Theorem 3.3. Assume the conditions stated in Theorem 2.3. Let $(\mathbf{u}, \zeta)$ be the solution of Problem 2.2, $\mathbf{w} = \dot{\mathbf{u}}$, and let $(\mathbf{u}^{hk}, \mathbf{w}^{hk}, \zeta^{hk})$ be defined by Problem 3.1. Then under the solution regularity assumptions (3.14)–(3.16), and (3.6) on the initial values for the discrete problem, we have

$$\begin{aligned} & \max_n \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V + \max_n \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \max_n \|\zeta_n - \zeta_n^{hk}\|_{Z_0} + \left(k \sum_{n=1}^N |\zeta_n - \zeta_n^{hk}|_Z^2 \right)^{1/2} \\ & \leq c(h + k) + c \max_n \inf_{\mathbf{v}_n^h \in V^h} \left(\|\mathbf{w}_n - \mathbf{v}_n^h\|_V + \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|_{L^2(\Gamma_3)^d}^{1/2} \right). \end{aligned} \quad (3.42)$$

We can apply the standard finite element approximation theory (cf. [26–28]) to bound the error

$$\max_n \inf_{\mathbf{v}_n^h \in V^h} \left(\|\mathbf{w}_n - \mathbf{v}_n^h\|_V + \|\mathbf{v}_{n\tau}^h - \mathbf{w}_{n\tau}\|_{L^2(\Gamma_3)^d}^{1/2} \right)$$

in (3.42), and derive the next result from Theorem 3.3.

Corollary 3.4. Keep the assumptions stated in Theorem 3.3. Under the additional solution regularity assumptions

$$\mathbf{w} \in C([0, T]; H^2(\Omega)^d), \quad \mathbf{w}_\tau|_{\Gamma_{3,i}} \in C([0, T]; H^2(\Gamma_{3,i})^d), \quad 1 \leq i \leq i_3, \quad (3.43)$$

we have the following error bound:

$$\max_n \|\mathbf{w}_n - \mathbf{w}_n^{hk}\|_V + \max_n \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V + \max_n \|\zeta_n - \zeta_n^{hk}\|_{Z_0} + \left(k \sum_{n=1}^N |\zeta_n - \zeta_n^{hk}|_Z^2 \right)^{1/2} \leq c(h + k). \quad (3.44)$$

4. Numerical results

In this section, we report some computer simulation results. The spatial domain is a rectangle and we use the finite element spaces specified at the beginning of Section 3 corresponding to a family of uniform triangulations of the rectangle. To solve the discrete hemivariational inequality from (3.8)–(3.9), we reformulate it as an optimization problem and use built-in constrained minimization algorithm from the Python library SciPy. A version of this approach was presented in [29]. Other algorithms specialized for nonsmooth optimizations could also be adapted. The coupled variational inequality for damage (3.10) is solved alongside in turns using the Uzawa approach. We also incorporate the Schur complement method to decrease the dimension of the problem.

We employ the Kelvin–Voigt type short memory viscoelastic law for the isotropic body, modified to reflect the damage effect on elastic properties of the body. The viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{B}$ are defined by

$$\begin{aligned} \mathcal{A}(\boldsymbol{\tau}) &= 2\eta \boldsymbol{\tau} + \xi \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}, \quad \boldsymbol{\tau} \in \mathbb{S}^2, \\ \mathcal{B}(\boldsymbol{\tau}, \zeta) &= \zeta(2\mu \boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau})\mathbf{I}), \quad \boldsymbol{\tau} \in \mathbb{S}^2, \zeta \in [0, 1], \end{aligned} \quad (4.1)$$

where I is the identity matrix, tr is the trace operator on a matrix, μ and λ are the Lamé coefficients, whereas η and ξ represent the viscosity coefficients, $\mu, \lambda, \eta, \xi > 0$. In all our simulations we take the following data

$$\begin{aligned} T &= 1, \\ \eta &= \xi = 2, \quad \mu = \lambda = 4, \\ \mathbf{u}_0(\mathbf{x}) &= (0, 0), \quad \mathbf{x} \in \Omega, \\ j(\xi) &= 20 \|\xi\|, \quad \xi \in \mathbb{R}^2, \\ \phi(\boldsymbol{\tau}, \zeta) &= \begin{cases} 2 \frac{1-\zeta}{\zeta} - 20 \|\boldsymbol{\tau}\|^2, & \zeta \in [0.2, 1], \\ 8 - 20 \|\boldsymbol{\tau}\|^2, & \zeta \in [0, 0.2], \end{cases} \quad \boldsymbol{\tau} \in \mathbb{S}^2, \\ \kappa &= 0.5. \end{aligned} \tag{4.2}$$

We first demonstrate the effect of different partitions of the boundary and applied forces on the deformation of the body. In all cases, we show the initial configuration along with the shape of the body and damage field at the final time $t = 1$. Choose a rectangular-shaped domain $\Omega = (0, 2) \times (0, 1) \subset \mathbb{R}^2$. For spatial discretization, we use uniform triangular partitions of Ω and the corresponding linear finite element spaces $\{V^h\}_h$; here h represents the mesh-size such that the unit length part $\{0\} \times [0, 1]$ of the boundary is divided into $1/h$ equal size sub-intervals. For the temporal discretization, we use the uniform partitions of the time interval $[0, 1]$ with the time step size $k = 1/N$ for a positive integer N . The numerical solutions correspond to the time step size $k = 1/32$ and the mesh-size $h = 1/32$.

Experiment 1: We take the following data

$$\begin{aligned} \Gamma_1 &= \{0\} \times [0, 1], \\ \Gamma_2 &= ([0, 2] \times \{1\}) \cup (\{2\} \times [0, 1]) \cup ([0, 2] \times \{0\}), \\ \Gamma_3 &= \emptyset, \\ \mathbf{f}_0(\mathbf{x}, t) &= (0, -0.2), \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \\ \mathbf{f}_2(\mathbf{x}, t) &= (0, 0), \quad \mathbf{x} \in \Gamma_2, \quad t \in [0, T]. \end{aligned}$$

In this experiment we push the body down using a force with density $\mathbf{f}_0$. In Fig. 1 we observe that the body is curved downward. As a result of twisting forces, the damage inside the body gradually increases when the spatial point moves closer to Γ_1 .

Experiment 2: We change the data to

$$\begin{aligned} \Gamma_1 &= \{0\} \times [0, 1], \\ \Gamma_2 &= ([0, 2] \times \{1\}) \cup (\{2\} \times [0, 1]) \cup ([1, 2] \times \{0\}), \\ \Gamma_3 &= [0, 1] \times \{0\}, \\ \mathbf{f}_0(\mathbf{x}, t) &= (0, -0.8), \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \\ \mathbf{f}_2(\mathbf{x}, t) &= (0, 0), \quad \mathbf{x} \in \Gamma_2, \quad t \in [0, T]. \end{aligned}$$

We once again push the body down, but in this case we introduce a rigid obstacle in contact with part of the body. In Fig. 2 we see that severe damage occurs in an area near the point $(1, 0)$, which is a corner point of the rigid foundation. There is damage also in the upper part of the body, as a result of expansion of the material.

Experiment 3: In the final experiment we take

$$\begin{aligned} \Gamma_1 &= \{0\} \times [0, 1], \\ \Gamma_2 &= ([0, 2] \times \{1\}) \cup (\{2\} \times [0, 1]), \\ \Gamma_3 &= [0, 2] \times \{0\}, \\ \mathbf{f}_0(\mathbf{x}, t) &= (0, 0), \quad \mathbf{x} \in \Omega, \quad t \in [0, T], \\ \mathbf{f}_2(\mathbf{x}, t) &= (-1, -1), \quad \mathbf{x} \in \Gamma_2, \quad t \in [0, T]. \end{aligned}$$

In this case the entire bottom part of the body is in contact with a rigid obstacle, and we push the body down and to the left using a force with density $\mathbf{f}_2$. In Fig. 3 we see that as a result of the action of the boundary force, an increased amount of damage is observed towards the upper and right side of the body. It is also interesting to examine the effect of the frictional force on the interface Γ_3 between the foundation and the body; the frictional force prevents the body from moving further to the left.

We find that simulation results from these experiments agree with our physical intuition.

We now turn to explore the convergence orders of the numerical solutions of a model problem. We take a square-shaped domain $\Omega = (0, 1) \times (0, 1)$, use the same data as in (4.1) and (4.2) and

$$\Gamma_1 = \{0\} \times [0, 1],$$

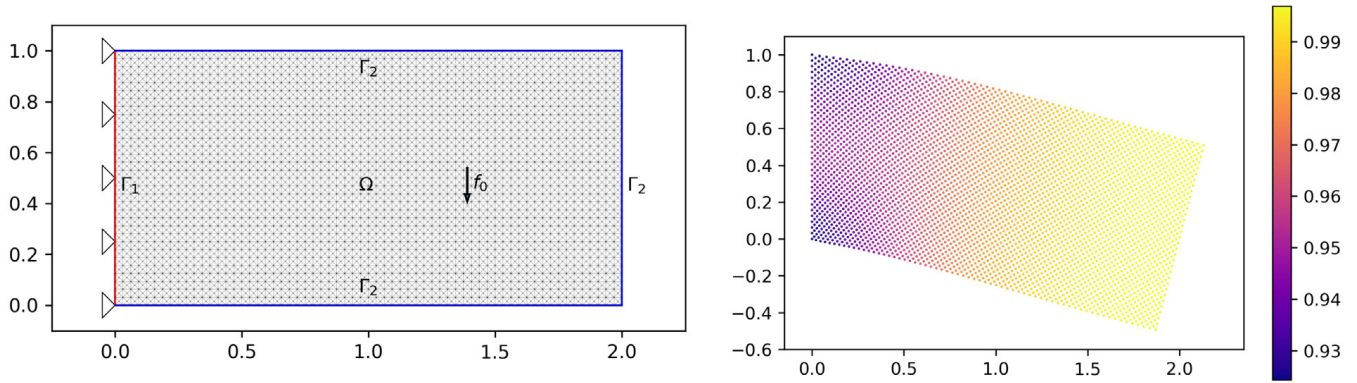


Fig. 1. Initial configuration, body position and damage in first experiment.

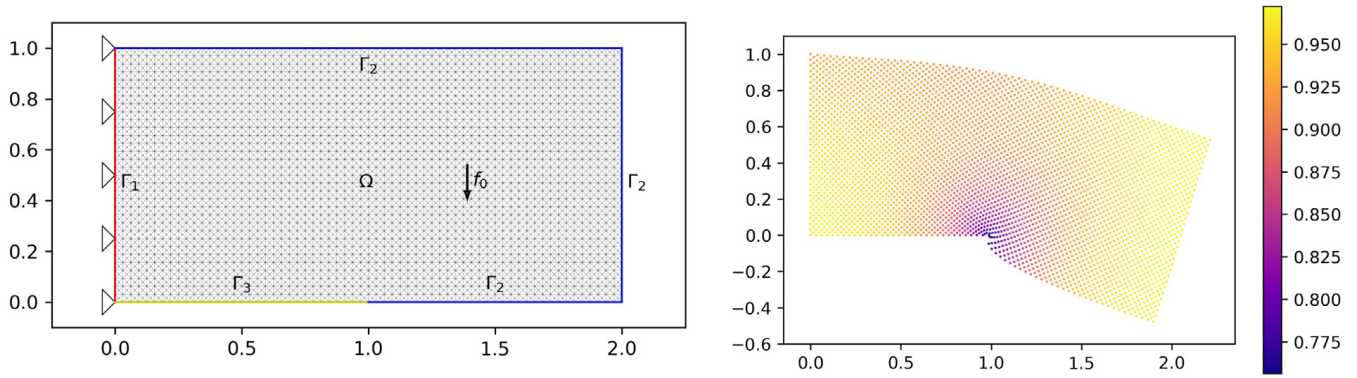


Fig. 2. Initial configuration, body position and damage in second experiment.

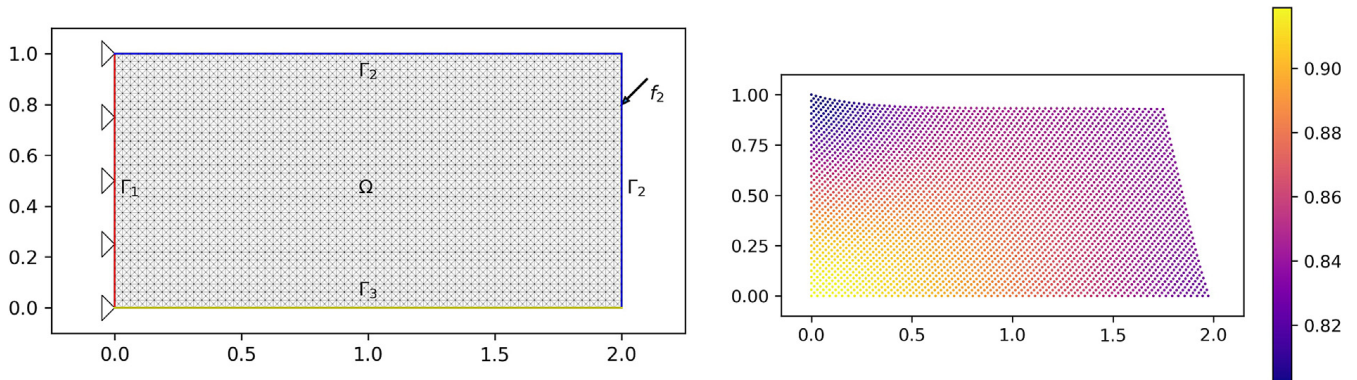


Fig. 3. Initial configuration, body position and damage in third experiment.

$$\Gamma_2 = ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1]),$$

$$\Gamma_3 = [0, 1] \times \{0\},$$

$$\mathbf{f}_0(\mathbf{x}, t) = (0, 0), \quad \mathbf{x} \in \Omega, \quad t \in [0, T],$$

$$\mathbf{f}_2(\mathbf{x}, t) = (-1.4, -0.2), \quad \mathbf{x} \in \Gamma_2, \quad t \in [0, T].$$

We use uniform triangulations of the spatial domain and uniform partitions of the time interval, and let h and k be the spatial mesh-size and time step-size as defined above. We present a comparison of numerical errors $\|\mathbf{w} - \mathbf{w}^{hk}\|_V$ and $\|\zeta - \zeta^{hk}\|_{Z_0}$ computed for a sequence of numerical solutions. The numerical solution corresponding to $h = 1/128$ and $k = 1/128$ is taken as the “true” solution $\mathbf{w}$ and ζ in computing the numerical errors; $\|\mathbf{w}\|_V \doteq 0.23525$ and $\|\zeta\|_{Z_0} \doteq 0.75375$.

First, we fix $k = 1/128$ and start with $h = 1/2$, which is successively halved. The results are presented in Table 1 and Fig. 4, where the dependence of the relative error estimates $\|\mathbf{w} - \mathbf{w}^{hk}\|_V$ and $\|\zeta - \zeta^{hk}\|_{Z_0}$ with respect to h are plotted on

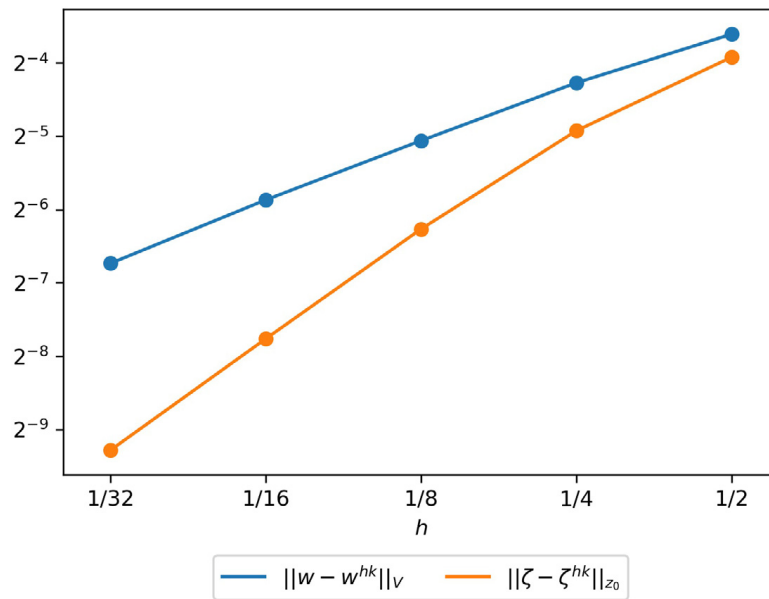


Fig. 4. Errors vs. h with fixed $k = 1/128$.

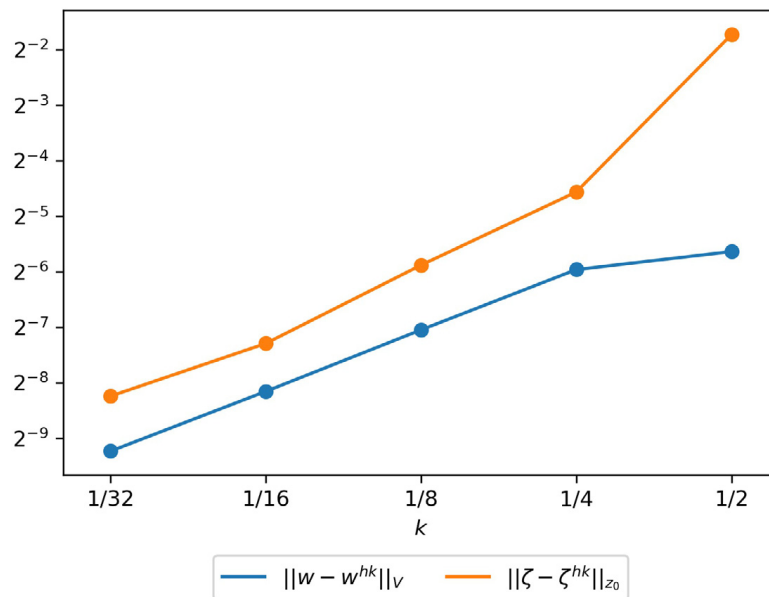


Fig. 5. Errors vs. k with fixed $h = 1/128$.

a log-log scale. Asymptotic convergence orders close to one for the velocity variable and slightly higher for the damage variable can be observed for the numerical solutions.

Then, we fix $h = 1/128$ and start with $k = 1/2$, which is successively halved. The results are presented in Table 2 and Fig. 5. Asymptotic convergence orders close to one for both unknowns can be observed.

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Table 1Numerical errors vs. h with fixed $k = 1/128$.

h	1/2	1/4	1/8	1/16	1/32
$\ w - w^{hk}\ _V / \ w\ _V$	3.4889e-1	2.2027e-1	1.2745e-1	7.2811e-2	3.9967e-2
Convergence order		0.6635	0.7893	0.8077	0.8653
$\ \zeta - \zeta^{hk}\ _{Z_0} / \ \zeta\ _{Z_0}$	8.7595e-2	4.3672e-2	1.7248e-2	6.1124e-3	2.1316e-3
Convergence order		1.0041	1.3402	1.4966	1.5198

Table 2Numerical errors vs. k with fixed $h = 1/128$.

k	1/2	1/4	1/8	1/16	1/32
$\ w - w^{hk}\ _V / \ w\ _V$	8.5393e-2	6.8195e-2	3.2105e-2	1.4883e-2	7.0551e-3
Convergence order		0.3244	1.0868	1.1091	1.0766
$\ \zeta - \zeta^{hk}\ _{Z_0} / \ \zeta\ _{Z_0}$	4.0107e-1	5.6124e-2	2.2501e-2	8.4583e-3	4.3778e-3
Convergence order		2.8372	1.3186	1.4115	0.9502

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III

Michał Jureczka, Anna Ochal

**A nonsmooth optimization approach for
hemivariational inequalities with applications
to contact mechanics**



A Nonsmooth Optimization Approach for Hemivariational Inequalities with Applications to Contact Mechanics

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Abstract

In this paper we introduce an abstract nonsmooth optimization problem and prove existence and uniqueness of its solution. We present a numerical scheme to approximate this solution. The theory is later applied to a sample static contact problem describing an elastic body in frictional contact with a foundation. This contact is governed by a nonmonotone friction law with dependence on normal and tangential components of displacement. Finally, computational simulations are performed to illustrate obtained results.

Keywords Nonmonotone friction · Optimization problem · Error estimate · Finite element method · Numerical simulations

Mathematics Subject Classification 35Q74 · 49J40 · 65K10 · 65M60 · 74S05 · 74M15 · 74M10 · 74G15

1 Introduction

In the literature we can find examples of many models describing deformation of a body that is partly in contact with another object, the so-called foundation. In various contact models boundary conditions enforced on the part of the body contacting the foundation appear. Functions that occur in these conditions model response of the foundation in the normal direction to the contact boundary and in tangential direction to the boundary (friction law). In many cases these functions are monotone, such as when Coulomb's law of dry friction is considered, but in applications this may not

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always be the case. What is more, the friction bound may change as the penetration of the foundation by body increases. Nonmonotonicity of functions describing contact laws and influence of normal displacement of the body on friction law cause some difficulties in analytical and numerical treatment of considered problems.

In this paper we introduce an abstract framework that can be used to numerically approximate a solution to a class of mechanical contact problems. We present a nonsmooth optimization problem and prove existence and uniqueness of a solution to this problem. Next we present a numerical scheme approximating this solution and provide numerical error estimate. We apply this theory to a static contact problem describing an elastic body in contact with a foundation. This contact is governed by a nonmonotone friction law with dependence on normal and tangential components of displacement. Weak formulation of introduced contact problem is presented in the form of hemivariational inequality. In the end we show results of computational simulations and describe the numerical algorithm that was used to obtain these results.

Let us now briefly present references in the literature. The definition and properties of Clarke subdifferential and tools used to solve optimization problems were introduced in [6]. Comparison of nonsmooth and nonconvex optimization methods can be found in [1], and details on computational contact mechanics is presented in [19]. The theory of hemivariational inequalities was developed in [17], and the idea to use Finite Element Method to solve these inequalities was presented in [13]. Another early study of vector-valued hemivariational problems in the context of FEM can be found in [14]. More recent analysis of hemivariational and variational–hemivariational inequalities was presented in [10,15,16], whereas numerical analysis of such problems can be found for example in papers [2–4,8,9,11,12].

A similar mechanical model to the one described in the paper was already considered in [15], where the authors prove only existence of a solution using surjectivity result for pseudomonotone, coercive multifunction without requiring any smallness assumption.

An error estimate concerning stationary variational–hemivariational inequalities was presented in [8]. In our case variational part of inequality is not present and the inequality is not constrained, however error estimate had to be generalized to reflect dependence of friction law on normal component of the displacement.

A numerical treatment of mechanical problem leading to hemivariational inequality using two approaches—nonsmooth and nonconvex optimization and quasi-augmented Lagrangian method is presented in [2]. As the smallness assumption is not required, this once again does not guarantee uniqueness and leads to a nonconvex optimization problem. There, the authors assume contact to be bilateral and consider friction law which does not depend on normal component of the displacement.

This paper is organized as follows. Section 2 contains a general differential inclusion problem and an optimization problem. We show that under introduced assumptions both problems are equivalent and have a unique solution. In Sect. 3 we proceed with a discrete scheme that approximates solution to introduced optimization problem and we prove theorem concerning numerical error estimate. An application of presented theory in the form of mechanical contact model is indicated in Sect. 4, along with its weak formulation. Finally, in Sect. 5, we describe computational algorithm used to solve mechanical contact problem and present simulations for a set of sample data.

2 A General Optimization Problem

Let us start with basic notation used in this paper. For a normed space X , we denote by $\|\cdot\|_X$ its norm, by X^* its dual space and by $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing of X^* and X . By $c > 0$ we denote a generic constant (value of c may differ in different equations).

Let us now assume that $j: X \rightarrow \mathbb{R}$ is locally Lipschitz continuous. The generalized directional derivative of j at $x \in X$ in the direction $v \in X$ is defined by

$$j^0(x; v) := \limsup_{y \rightarrow x, \lambda \searrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized subdifferential of j at x is a subset of the dual space X^* given by

$$\partial j(x) := \{\xi \in X^* \mid \langle \xi, v \rangle_{X^* \times X} \leq j^0(x; v) \text{ for all } v \in X\}.$$

If $j: X^n \rightarrow \mathbb{R}$ is a locally Lipschitz function of n variables, then we denote by $\partial_i j$ and j_i^0 the Clarke subdifferential and generalized directional derivative with respect to i -th variable of j , respectively.

Let now V be a reflexive Banach space and X be a Banach space. Let $\gamma \in \mathcal{L}(V, X)$ be linear and continuous operator from V to X , and $c_\gamma := \|\gamma\|_{\mathcal{L}(V, X)}$. We denote by $\gamma^*: X^* \rightarrow V^*$ the adjoint operator to γ . Let $A: V \rightarrow V^*$, $J: X \times X \rightarrow \mathbb{R}$ and $f \in V^*$. We formulate the operator inclusion problem as follows.

Problem P_{incl} Find $u \in V$ such that

$$Au + \gamma^* \partial_2 J(\gamma u, \gamma u) \ni f.$$

In the study of Problem P_{incl} we make the following assumptions.

$H(A)$: The operator $A: V \rightarrow V^*$ is such that

- (a) A is linear and bounded,
- (b) A is symmetric, i.e. $\langle Au, v \rangle_{V^* \times V} = \langle Av, u \rangle_{V^* \times V}$ for all $u, v \in V$,
- (c) there exists $m_A > 0$ such that $\langle Au, u \rangle_{V^* \times V} \geq m_A \|u\|_V^2$ for all $u \in V$.

$H(J)$: The functional $J: X \times X \rightarrow \mathbb{R}$ satisfies

- (a) J is locally Lipschitz continuous with respect to its second variable,
- (b) there exist $c_0, c_1, c_2 \geq 0$ such that

$$\|\partial_2 J(w, v)\|_{X^*} \leq c_0 + c_1 \|v\|_X + c_2 \|w\|_X \quad \text{for all } w, v \in X,$$

- (c) there exist $m_\alpha, m_L \geq 0$ such that

$$\begin{aligned} & J_2^0(w_1, v_1; v_2 - v_1) + J_2^0(w_2, v_2; v_1 - v_2) \\ & \leq m_\alpha \|v_1 - v_2\|_X^2 + m_L \|w_1 - w_2\|_X \|v_1 - v_2\|_X \end{aligned}$$

for all $w_1, w_2, v_1, v_2 \in X$.

$$H(f) : f \in V^*.$$

$$(H_s) : m_A > (2m_\alpha + m_L)c_\gamma^2.$$

We remark that $H(J)(c)$ is equivalent to the following condition

$$\begin{aligned} & \langle \partial_2 J(w_1, v_1) - \partial_2 J(w_2, v_2), v_1 - v_2 \rangle_{X^* \times X} \\ & \geq -m_\alpha \|v_1 - v_2\|_X^2 - m_L \|w_1 - w_2\|_X \|v_1 - v_2\|_X \end{aligned} \quad (2.1)$$

for all $w_1, w_2, v_1, v_2 \in X$. Moreover, in a special case when J does not depend on its first variable, condition $H(J)(c)$ holds with $m_L = 0$ and we obtain the well known relaxed monotonicity condition, i.e. for all $v_1, v_2 \in X$

$$\langle \partial J(v_1) - \partial J(v_2), v_1 - v_2 \rangle_{X^* \times X} \geq -m_\alpha \|v_1 - v_2\|_X^2. \quad (2.2)$$

So, condition $H(J)(c)$ is more general than (2.2).

We start with a uniqueness result for Problem P_{incl} , provided that a solution exists.

Lemma 1 Assume that $H(A)$, $H(J)$, $H(f)$ and (H_s) hold. If Problem P_{incl} has a solution $u \in V$, then it is unique and satisfies

$$\|u\|_V \leq c(1 + \|f\|_{V^*}) \quad (2.3)$$

with a positive constant c .

Proof Let $u \in V$ be a solution to Problem P_{incl} . This means that there exists $z \in \partial_2 J(\gamma u, \gamma u)$ such that

$$Au + \gamma^* z = f.$$

From the definition of generalized directional derivative of $J(\gamma u, \cdot)$ we have for all $v \in V$

$$\langle f - Au, v \rangle_{V^* \times V} = \langle \gamma^* z, v \rangle_{V^* \times V} = \langle z, \gamma v \rangle_{X^* \times X} \leq J_2^0(\gamma u, \gamma u; \gamma v). \quad (2.4)$$

Let us now assume that Problem P_{incl} has two different solutions u_1 and u_2 . For a solution u_1 we set $v = u_2 - u_1$ in (2.4) to get

$$\langle f, u_2 - u_1 \rangle_{V^* \times V} - \langle Au_1, u_2 - u_1 \rangle_{V^* \times V} \leq J_2^0(\gamma u_1, \gamma u_1; \gamma u_2 - \gamma u_1).$$

For a solution u_2 we set $v = u_1 - u_2$ in (2.4) to get

$$\langle f, u_1 - u_2 \rangle_{V^* \times V} - \langle Au_2, u_1 - u_2 \rangle_{V^* \times V} \leq J_2^0(\gamma u_2, \gamma u_2; \gamma u_1 - \gamma u_2).$$

Adding the above inequalities, we obtain

$$\begin{aligned} & \langle Au_1 - Au_2, u_1 - u_2 \rangle_{V^* \times V} \\ & \leq J_2^0(\gamma u_1, \gamma u_1; \gamma u_2 - \gamma u_1) + J_2^0(\gamma u_2, \gamma u_2; \gamma u_1 - \gamma u_2). \end{aligned}$$

Hence, $H(A)(c)$ and $H(J)(c)$ yield

$$m_A \|u_1 - u_2\|_V^2 \leq (m_\alpha + m_L) \|\gamma u_1 - \gamma u_2\|_X^2,$$

and finally

$$(m_A - (m_\alpha + m_L)c_\gamma^2) \|u_1 - u_2\|_V^2 \leq 0.$$

Under assumption (H_s) , we obtain that if Problem P_{incl} has a solution, it is unique.

Now, in order to prove (2.3), we set $v = -u$ in (2.4) to obtain

$$\langle Au, u \rangle_{V^* \times V} \leq J_2^0(\gamma u, \gamma u; -\gamma u) + \langle f, u \rangle_{V^* \times V}. \quad (2.5)$$

Using $H(J)(b)$ and (c), we get

$$\begin{aligned} J_2^0(\gamma u, \gamma u; -\gamma u) &\leq (m_\alpha + m_L) \|\gamma u\|_X^2 - J_2^0(0, 0; \gamma u) \\ &\leq (m_\alpha + m_L) \|\gamma u\|_X^2 + c_0 \|\gamma u\|_X. \end{aligned} \quad (2.6)$$

Combining (2.5) and (2.6), we have

$$m_A \|u\|_V^2 \leq (m_\alpha + m_L) \|\gamma u\|_X^2 + c_0 \|\gamma u\|_X + \|f\|_{V^*} \|u\|_V$$

and

$$(m_A - (m_\alpha + m_L)c_\gamma^2) \|u\|_V \leq c(1 + \|f\|_{V^*}).$$

From (H_s) we obtain required estimate. $\square$

We now consider an optimization problem, which will be equivalent to Problem P_{incl} under introduced assumptions. To this end, let the functional $\mathcal{L} : V \times V \rightarrow \mathbb{R}$ be defined for all $w, v \in V$ as follows

$$\mathcal{L}(w, v) = \frac{1}{2} \langle Av, v \rangle_{V^* \times V} - \langle f, v \rangle_{V^* \times V} + J(\gamma w, \gamma v). \quad (2.7)$$

The next lemma collects some properties of the functional $\mathcal{L}$.

Lemma 2 *Under assumptions $H(A)$, $H(J)$, $H(f)$ and (H_s) , the functional $\mathcal{L} : V \times V \rightarrow \mathbb{R}$ defined by (2.7) satisfies*

- (i) $\mathcal{L}(w, \cdot)$ is locally Lipschitz continuous for all $w \in V$,
- (ii) $\partial_2 \mathcal{L}(w, v) \subseteq Av - f + \gamma^* \partial_2 J(\gamma w, \gamma v)$ for all $w, v \in V$,
- (iii) $\mathcal{L}(w, \cdot)$ is strictly convex for all $w \in V$,
- (iv) $\mathcal{L}(w, \cdot)$ is coercive for all $w \in V$.

Proof The proof of (i) is immediate since for a fixed $w \in V$ the functional $\mathcal{L}(w, \cdot)$ is locally Lipschitz continuous as a sum of locally Lipschitz continuous functions with respect to v .

For the proof of (ii), we observe that from $H(A)$ and $H(f)$, the functions

$$f_1: V \ni v \mapsto \frac{1}{2} \langle Av, v \rangle_{V^* \times V} \in \mathbb{R}, \quad f_2: V \ni v \mapsto \langle f, v \rangle_{V^* \times V} \in \mathbb{R}$$

are strictly differentiable and

$$f'_1(v) = Av, \quad f'_2(v) = f.$$

Now, using the chain rule for generalized subgradient (cf. Propositions 3.35 and 3.37 in [15]), we obtain

$$\begin{aligned} \partial_2 \mathcal{L}(w, v) &= f'_1(v) - f'_2(v) + \partial_2(J \circ \gamma)(\gamma w, v) \\ &\subseteq Av - f + \gamma^* \partial_2 J(\gamma w, \gamma v), \end{aligned}$$

which concludes (ii).

To prove (iii), first, we will show that for any fixed $w \in V$, the operator $v \mapsto \partial_2 \mathcal{L}(w, v)$ is strongly monotone. Then, we will use Theorem 3.4 in [7] to deduce that $\mathcal{L}(w, \cdot)$ is strongly convex for all $w \in V$. Finally, the latter implies condition (iii). Hence, it remains to show that $v \mapsto \partial_2 \mathcal{L}(w, v)$ is strongly monotone. To this end, let us fix $w, v_i \in V$ with $i = 1, 2$. We take $\zeta_i \in \partial_2 \mathcal{L}(w, v_i)$. From (ii) there exist $z_i \in \partial_2 J(\gamma w, \gamma v_i)$ such that

$$\zeta_i = Av_i - f + \gamma^* z_i.$$

Hence, using $H(A)(c)$ and (2.2), we obtain

$$\begin{aligned} &\langle \zeta_1 - \zeta_2, v_1 - v_2 \rangle_{V^* \times V} \\ &= \langle Av_1 - Av_2, v_1 - v_2 \rangle_{V^* \times V} + \langle \gamma^* z_1 - \gamma^* z_2, v_1 - v_2 \rangle_{V^* \times V} \\ &\geq m_A \|v_1 - v_2\|_V^2 + \langle z_1 - z_2, \gamma v_1 - \gamma v_2 \rangle_{X^* \times X} \\ &\geq m_A \|v_1 - v_2\|_V^2 - m_\alpha \|\gamma v_1 - \gamma v_2\|_X^2 \\ &\geq (m_A - m_\alpha c_\gamma^2) \|v_1 - v_2\|_V^2. \end{aligned}$$

From (H_s) we see that $\partial_2 \mathcal{L}(w, \cdot)$ is strongly monotone for every $w \in V$.

For the proof of (iv), let us fix $w, v \in V$. From $H(A)(c)$ and $H(f)$ we obtain

$$\mathcal{L}(w, v) \geq \frac{1}{2} m_A \|v\|_V^2 - \|f\|_{V^*} \|v\|_V + J(\gamma w, \gamma v). \quad (2.8)$$

Now, using the Lebourg mean value theorem (cf. Proposition 3.36 in [15]), we get that there exists $\lambda \in (0, 1)$ and $\eta \in \partial_2(J \circ \gamma)(\gamma w, \lambda v)$ such that

$$J(\gamma w, \gamma v) = \langle \eta, v \rangle_{V^* \times V} + J(\gamma w, 0). \quad (2.9)$$

Since $\partial_2(J \circ \gamma)(\gamma w, \lambda v) \subseteq \gamma^* \partial_2 J(\gamma w, \lambda \gamma v)$ we have $\eta \in \gamma^* \partial_2 J(\gamma w, \lambda \gamma v)$. Then there exists $z_1 \in \partial_2 J(\gamma w, \lambda \gamma v)$ such that $\eta = \gamma^* z_1$. Taking $z_2 \in \partial_2 J(\gamma w, 0)$ and by (2.1), we obtain

$$\lambda \langle \gamma^* z_1 - \gamma^* z_2, v \rangle_{V^* \times V} = \langle z_1 - z_2, \lambda \gamma v \rangle_{X^* \times X} \geq -m_\alpha \|\lambda \gamma v\|_X^2 \geq -m_\alpha \lambda^2 c_\gamma^2 \|v\|_V^2,$$

and this, along with the fact that $\lambda \in (0, 1)$, leads to

$$\langle \eta, v \rangle_{V^* \times V} \geq -m_\alpha c_\gamma^2 \|v\|_V^2 + \langle z_2, \gamma v \rangle_{X^* \times X}. \quad (2.10)$$

Using $H(J)(b)$, we get

$$\langle z_2, \gamma v \rangle_{X^* \times X} \geq -|\langle z_2, \gamma v \rangle_{X^* \times X}| \geq -\|\partial_2 J(\gamma w, 0)\|_{X^*} \|\gamma v\|_X \geq -c(1 + \|w\|_V) \|v\|_V. \quad (2.11)$$

Combining (2.8)–(2.11) and because $J(\gamma w, 0)$ is bounded from below for fixed w , we get

$$\mathcal{L}(w, v) \geq \left(\frac{1}{2} m_A - m_\alpha c_\gamma^2 \right) \|v\|_V^2 - c \|v\|_V - c.$$

From (H_s) we see that $\mathcal{L}(w, \cdot)$ is coercive for every $w \in V$. □

The problem under consideration reads as follows.

Problem P_{opt} Find $u \in V$ such that

$$0 \in \partial_2 \mathcal{L}(u, u).$$

We are now in a position to prove the existence and uniqueness result for the above optimization problem.

Lemma 3 Assume that $H(A)$, $H(J)$, $H(f)$ and (H_s) hold. Then Problem P_{opt} has a unique solution $u \in V$.

Proof We introduce operator $\Lambda: V \rightarrow V$ defined for all $w \in V$ as follows

$$\Lambda w = \arg \min_{v \in V} \mathcal{L}(w, v).$$

From Lemma 2(i), (iv) we see that for fixed w functional $\mathcal{L}(w, \cdot)$ is proper, lower semicontinuous and coercive. This implies that it attains a global minimum. Uniqueness of that minimum is guaranteed by Lemma 2(iii). We can conclude that operator

Λ is well defined. Now we prove that it is a contraction. Take $u_i \in V$ for $i = 1, 2$ and let $\widehat{u}_i = \Lambda u_i$. Because of strict convexity of $\mathcal{L}(w, \cdot)$, we have

$$\widehat{u}_i = \arg \min_{v \in V} \mathcal{L}(u_i, v) \quad \text{if and only if} \quad 0 \in \partial_2 \mathcal{L}(u_i, \widehat{u}_i)$$

(see Theorem 1.23 in [13]). From similar arguments to those used in proofs of Lemmata 1 and 2 with fixed first argument of functional $\mathcal{L}$, we have for all $v \in V$

$$\langle f - A\widehat{u}_i, v \rangle_{V^* \times V} \leq J_2^0(\gamma u_i, \gamma \widehat{u}_i; \gamma v).$$

Taking for $i = 1$ value $v = \widehat{u}_2 - \widehat{u}_1$, for $i = 2$ value $v = \widehat{u}_1 - \widehat{u}_2$ and adding these inequalities, we obtain

$$\begin{aligned} & \langle A\widehat{u}_1 - A\widehat{u}_2, \widehat{u}_1 - \widehat{u}_2 \rangle_{V^* \times V} \\ & \leq J_2^0(\gamma u_1, \gamma \widehat{u}_1; \gamma \widehat{u}_2 - \gamma \widehat{u}_1) + J_2^0(\gamma u_2, \gamma \widehat{u}_2; \gamma \widehat{u}_1 - \gamma \widehat{u}_2). \end{aligned}$$

From assumptions $H(A)(c)$ and $H(J)(c)$, we get

$$m_A \|\widehat{u}_1 - \widehat{u}_2\|_V^2 \leq m_\alpha \|\gamma \widehat{u}_1 - \gamma \widehat{u}_2\|_X^2 + m_L \|\gamma u_1 - \gamma u_2\|_X \|\gamma \widehat{u}_1 - \gamma \widehat{u}_2\|_X.$$

Using the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we obtain

$$m_A \|\widehat{u}_1 - \widehat{u}_2\|_V^2 \leq m_\alpha c_\gamma^2 \|\widehat{u}_1 - \widehat{u}_2\|_V^2 + \frac{m_L c_\gamma^2}{2} (\|u_1 - u_2\|_V^2 + \|\widehat{u}_1 - \widehat{u}_2\|_V^2).$$

Because of (H_s) , we can rearrange these terms to get

$$\|\widehat{u}_1 - \widehat{u}_2\|_V^2 \leq \frac{m_L c_\gamma^2}{2m_A - 2m_\alpha c_\gamma^2 - m_L c_\gamma^2} \|u_1 - u_2\|_V^2.$$

Using assumption (H_s) once more, we obtain that the operator Λ is a contraction. From the Banach fixed point theorem we know that there exists a unique $u^* \in V$ such that $\Lambda u^* = u^*$, so $0 \in \partial_2 \mathcal{L}(u^*, u^*)$. $\square$

Let us conclude the results from Lemmata 1, 2 and 3 in the following theorem.

Theorem 4 Assume that $H(A)$, $H(J)$, $H(f)$ and (H_s) hold. Then Problems P_{incl} and P_{opt} are equivalent, they have a unique solution $u \in V$ and this solution satisfies

$$\|u\|_V \leq c(1 + \|f\|_{V^*})$$

with a positive constant c .

Proof Lemma 2(ii) implies that every solution to Problem P_{opt} solves Problem P_{incl} . Using this fact, Lemmata 1 and 3 we see that a unique solution to Problem P_{opt} is also a unique solution to Problem P_{incl} . Because of the uniqueness of the solution to Problem P_{incl} we get that Problems P_{incl} and P_{opt} are equivalent. The estimate in the statement of the theorem follows from Lemma 1. $\square$

3 Numerical Scheme

Let $V^h \subset V$ be a finite dimensional subspace with a discretization parameter $h > 0$. We present the following discrete scheme of Problem P_{opt} .

Problem P_{opt}^h Find $u^h \in V^h$ such that

$$0 \in \partial_2 \mathcal{L}(u^h, u^h).$$

We remark that existence of a unique solution to Problem P_{opt}^h and equivalence to the discrete version of Problem P_{incl} follow from application of Theorem 4 in this new setting. Now let us present the following main theorem concerning error estimatie of introduced numerical scheme.

Theorem 5 Assume that $H(A)$, $H(J)$, $H(f)$ and (H_s) hold. Then for the unique solutions u and u^h to Problems P_{opt} and P_{opt}^h , respectively, there exists a constant $c > 0$ such that

$$\|u - u^h\|_V^2 \leq c \inf_{v^h \in V^h} \left\{ \|u - v^h\|_V^2 + \|\gamma u - \gamma v^h\|_X + R(u, v^h) \right\}, \quad (3.1)$$

where a residual quantity is given by

$$R(u, v^h) = \langle Au, v^h - u \rangle_{V^* \times V} + \langle f, u - v^h \rangle_{V^* \times V}. \quad (3.2)$$

Proof Let u be a solution to Problem P_{opt} and u^h be a solution to Problem P_{opt}^h . Then they are solutions to corresponding inclusion problems and satisfy respectively

$$\langle f - Au, v \rangle_{V^* \times V} \leq J_2^0(\gamma u, \gamma u; \gamma v) \quad \text{for all } v \in V, \quad (3.3)$$

$$\langle f - Au^h, v \rangle_{V^* \times V} \leq J_2^0(\gamma u^h, \gamma u^h; \gamma v) \quad \text{for all } v \in V^h. \quad (3.4)$$

Taking (3.3) with $v = u^h - u$, and (3.4) with $v = v^h - u^h$, then adding these inequalities, we obtain for all $v^h \in V^h$

$$\begin{aligned} & \langle f, v^h - u \rangle_{V^* \times V} + \langle Au^h - Au, u^h - u \rangle_{V^* \times V} - \langle Au^h, v^h - u \rangle_{V^* \times V} \\ & \leq J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h). \end{aligned} \quad (3.5)$$

We observe that by subadditivity of generalized directional derivative (cf. [15], Proposition 3.23(i)) and $H(J)(c)$, we have

$$\begin{aligned} & J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \\ & \leq J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma u - \gamma u^h) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u) \\ & \leq (m_\alpha + m_L) \|\gamma u^h - \gamma u\|_X^2 + \left(c_0 + (c_1 + c_2) \|\gamma u^h\|_X \right) \|\gamma v^h - \gamma u\|_X. \end{aligned} \quad (3.6)$$

From the statement of Lemma 1 applied to discrete version of Problem P_{incl} we get that $\|\gamma u^h\|_X \leq c_\gamma \|u^h\|_V \leq c(1 + \|f\|_{V^*})$ is uniformly bounded with respect to h . Hence, returning to (3.5) and using (3.6), we obtain for all $v^h \in V^h$

$$\begin{aligned} \langle Au^h - Au, u^h - u \rangle_{V^* \times V} & \leq \langle Au^h - Au, v^h - u \rangle_{V^* \times V} + \langle Au, v^h - u \rangle_{V^* \times V} \\ & + \langle f, u - v^h \rangle_{V^* \times V} + (m_\alpha + m_L) c_\gamma^2 \|u^h - u\|_V^2 + c \|\gamma v^h - \gamma u\|_X. \end{aligned}$$

By assumption $H(A)$ and definition (3.2), we get for all $v^h \in V^h$

$$\begin{aligned} m_A \|u^h - u\|_V^2 & \leq c \|u^h - u\|_V \|v^h - u\|_V + R(u, v^h) \\ & + (m_\alpha + m_L) c_\gamma^2 \|u - u^h\|_V^2 + c \|\gamma u - \gamma v^h\|_X. \end{aligned}$$

Finally, the elementary inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ with $\varepsilon > 0$ yields

$$\begin{aligned} m_A \|u - u^h\|_V^2 & \leq \varepsilon \|u - u^h\|_V^2 + \frac{c^2}{4\varepsilon} \|u - v^h\|_V^2 + R(u, v^h) \\ & + (m_\alpha + m_L) c_\gamma^2 \|u - u^h\|_V^2 + c \|\gamma u - \gamma v^h\|_X. \end{aligned}$$

This is equivalent for all $v^h \in V^h$ to

$$\left(m_A - (m_\alpha + m_L) c_\gamma^2 - \varepsilon \right) \|u - u^h\|_V^2 \leq \frac{c}{\varepsilon} \|u - v^h\|_V^2 + R(u, v^h) + c \|\gamma u - \gamma v^h\|_X.$$

Taking sufficiently small ε and using (H_s) we obtain the desired conclusion. $\square$

4 Application to Contact Mechanics

In this section we apply the results of previous sections to a sample mechanical contact problem. Let us start by introducing the physical setting and notation.

An elastic body occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$ in application. We assume that its boundary Γ is divided into three disjoint measurable parts: $\Gamma_D, \Gamma_C, \Gamma_N$, where the part Γ_D has a positive measure. Additionally Γ is Lipschitz continuous, and therefore the outward normal vector $\mathbf{v}$ to Γ exists a.e. on the boundary. The body is clamped on Γ_D , i.e. its displacement is equal to $\mathbf{0}$ on this part of boundary. A surface force of density $\mathbf{f}_N$ acts on the boundary Γ_N and a body force of density $\mathbf{f}_0$ acts in Ω .

The contact phenomenon on Γ_C is modeled using general subdifferential inclusions. We are interested in finding the displacement of the body in a static state.

Let us denote by “ $\cdot$ ” and $\|\cdot\|$ the scalar product and the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{S}^d$, respectively, where $\mathbb{S}^d = \mathbb{R}_{sym}^{d \times d}$. Indices i and j run from 1 to d and the index after a comma represents the partial derivative with respect to the corresponding component of the independent variable. Summation over repeated indices is implied. We denote the divergence operator by $\text{Div } \sigma = (\sigma_{ij,j})$. The standard Lebesgue and Sobolev spaces $L^2(\Omega)^d = L^2(\Omega; \mathbb{R}^d)$ and $H^1(\Omega)^d = H^1(\Omega; \mathbb{R}^d)$ are used. The linearized (small) strain tensor for displacement $\mathbf{u} \in H^1(\Omega)^d$ is defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

Let $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\sigma_\nu = \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\nu}$ be the normal components of $\mathbf{u}$ and $\boldsymbol{\sigma}$, respectively, and let $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ be their tangential components, respectively. In what follows, for simplicity, we sometimes do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x}$.

Now let us introduce the classical formulation of considered mechanical contact problem.

Problem P: Find a displacement field $\mathbf{u}: \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}: \Omega \rightarrow \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{A}(\varepsilon(\mathbf{u})) \quad \text{in } \Omega \quad (4.1)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \quad (4.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad (4.3)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_N \quad \text{on } \Gamma_N \quad (4.4)$$

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_C \quad (4.5)$$

$$-\boldsymbol{\sigma}_\tau \in h_\tau(u_\nu) \partial j_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_C \quad (4.6)$$

Here, Eq. (4.1) represents an elastic constitutive law and $\mathcal{A}$ is an elasticity operator. Equilibrium equation (4.2) reflects the fact that problem is static. Equation (4.3) represents clamped boundary condition on Γ_D and (4.4) represents the action of the traction on Γ_N . Inclusion (4.5) describes the response of the foundation in normal direction, whereas the friction is modeled by inclusion (4.6), where j_ν and j_τ are given superpotentials, and h_τ is a given friction bound.

We consider the following Hilbert spaces

$$\mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad V = \{\mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\},$$

endowed with the inner scalar products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \quad (\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}},$$

respectively. The fact that space V equipped with the norm $\|\cdot\|_V$ is complete follows from Korn's inequality, and its application is allowed because we assume that $\text{meas}(\Gamma_D) > 0$. We consider the trace operator $\gamma: V \rightarrow L^2(\Gamma_C)^d = X$. By the Sobolev trace theorem we know that $\gamma \in \mathcal{L}(V, X)$ with the norm equal to c_γ .

Now we present the hypotheses on data of Problem P .

$H(\mathcal{A})$: $\mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies

- (a) $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}) = (a_{ijkh}(\mathbf{x})\tau_{kh})$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$, $a_{ijkh} \in L^\infty(\Omega)$,
- (b) $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}_1) \cdot \boldsymbol{\tau}_2 = \boldsymbol{\tau}_1 \cdot \mathcal{A}(\mathbf{x}, \boldsymbol{\tau}_2)$ for all $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$,
- (c) there exists $m_{\mathcal{A}} > 0$ such that $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}) \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}}\|\boldsymbol{\tau}\|^2$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.

$H(j_\nu)$: $j_\nu: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $j_\nu(\cdot, \xi)$ is measurable on Γ_C for all $\xi \in \mathbb{R}$ and there exists $e \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, e(\cdot)) \in L^1(\Gamma_C)$,
- (b) $j_\nu(\mathbf{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}$ for a.e. $\mathbf{x} \in \Gamma_C$,
- (c) there exist $c_{\nu 0}, c_{\nu 1} \geq 0$ such that

$$|\partial_2 j_\nu(\mathbf{x}, \xi)| \leq c_{\nu 0} + c_{\nu 1}|\xi| \quad \text{for all } \xi \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C,$$

- (d) there exists $\alpha_\nu \geq 0$ such that

$$(j_\nu)_2^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + (j_\nu)_2^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq \alpha_\nu |\xi_1 - \xi_2|^2$$

for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(j_\tau)$: $j_\tau: \Gamma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

- (a) $j_\tau(\cdot, \boldsymbol{\xi})$ is measurable on Γ_C for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and there exists $\mathbf{e} \in L^2(\Gamma_C)^d$ such that $j_\tau(\cdot, \mathbf{e}(\cdot)) \in L^1(\Gamma_C)$,
- (b) there exists $c_\tau > 0$ such that

$$|j_\tau(\mathbf{x}, \boldsymbol{\xi}_1) - j_\tau(\mathbf{x}, \boldsymbol{\xi}_2)| \leq c_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \quad \text{for all } \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_C,$$

- (c) there exists $\alpha_\tau \geq 0$ such that

$$(j_\tau)_2^0(\mathbf{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + (j_\tau)_2^0(\mathbf{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq \alpha_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$$

for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(h)$: $h_\tau: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $h_\tau(\cdot, \eta)$ is measurable on Γ_C for all $\eta \in \mathbb{R}$,
- (b) there exists $\overline{h_\tau} > 0$ such that $0 \leq h_\tau(\mathbf{x}, \eta) \leq \overline{h_\tau}$ for all $\eta \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$,
- (c) there exists $L_{h_\tau} > 0$ such that

$$|h_\tau(\mathbf{x}, \eta_1) - h_\tau(\mathbf{x}, \eta_2)| \leq L_{h_\tau} |\eta_1 - \eta_2| \quad \text{for all } \eta_1, \eta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_C.$$

$$(H_0) : \quad \mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_N \in L^2(\Gamma_N)^d.$$

We remark that condition $H(j_\tau)(b)$ is equivalent to the fact that $j_\tau(\mathbf{x}, \cdot)$ is locally Lipschitz continuous and there exists $c_\tau \geq 0$ such that $\|\partial_2 j_\tau(\mathbf{x}, \xi)\| \leq c_\tau$ for all $\xi \in \mathbb{R}^d$ and a.e. $\mathbf{x} \in \Gamma_C$. Moreover, condition $H(h)(b)$ is sufficient to obtain presented mathematical results, but from mechanical point of view we should additionally assume that $h(r) = 0$ for $r \leq 0$. This corresponds to the situation when body is separated from the foundation and friction force vanishes.

Using the standard procedure, the Green formula and the definition of generalized sub-differential, we obtain a weak formulation of Problem P in the form of hemivariational inequality.

Problem P_{hvi} Find a displacement $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V^* \times V} + \int_{\Gamma_C} j_3^0(\mathbf{x}, \gamma \mathbf{u}(\mathbf{x}), \gamma \mathbf{u}(\mathbf{x}); \gamma \mathbf{v}(\mathbf{x})) da \geq \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V}. \quad (4.7)$$

Here, the operator $A: V \rightarrow V^*$ and $\mathbf{f} \in V^*$ are defined for all $\mathbf{w}, \mathbf{v} \in V$ as follows

$$\begin{aligned} \langle A\mathbf{w}, \mathbf{v} \rangle_{V^* \times V} &= (\mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{w})), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \\ \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V} &= \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \gamma \mathbf{v} da \end{aligned}$$

and $j: \Gamma_C \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is defined for all $\eta, \xi \in \mathbb{R}^d$ and $\mathbf{x} \in \Gamma_C$ by

$$j(\mathbf{x}, \eta, \xi) = j_\nu(\mathbf{x}, \xi_\nu) + h_\tau(\mathbf{x}, \eta_\nu) j_\tau(\mathbf{x}, \xi_\tau). \quad (4.8)$$

It is easy to check that under assumptions $H(\mathcal{A})$ and (H_0) , the operator A and the functional $\mathbf{f}$ satisfy $H(A)$ and $H(\mathbf{f})$, respectively. We also define the functional $J: L^2(\Gamma_C)^d \times L^2(\Gamma_C)^d \rightarrow \mathbb{R}$ for all $\mathbf{w}, \mathbf{v} \in L^2(\Gamma_C)^d$ by

$$J(\mathbf{w}, \mathbf{v}) = \int_{\Gamma_C} j(\mathbf{x}, \mathbf{w}(\mathbf{x}), \mathbf{v}(\mathbf{x})) da. \quad (4.9)$$

Below we present some properties of the functional J .

Lemma 6 Assumptions $H(j_\nu)$, $H(j_\tau)$ and $H(h)$ imply that functional J defined by (4.8)–(4.9) satisfies $H(J)$.

Proof We first observe that from $H(j_\nu)(a),(b)$, $H(j_\tau)(a),(b)$ and $H(h)(a),(c)$ the function $j(\cdot, \eta, \xi)$ is measurable on Γ_C , there exist $\mathbf{e}_1, \mathbf{e}_2 \in L^2(\Gamma_C)^d$ such that $j(\cdot, \mathbf{e}_1(\cdot), \mathbf{e}_2(\cdot)) \in L^1(\Gamma_C)$, $j(\mathbf{x}, \cdot, \xi)$ is continuous and $j(\mathbf{x}, \eta, \cdot)$ is locally Lipschitz. Moreover, by $H(j_\nu)(c)$, $H(j_\tau)(c)$ and $H(h)(b)$ we easily conclude

$$\|\partial_3 j(\mathbf{x}, \eta, \xi)\| \leq |\partial_2 j_\nu(\mathbf{x}, \xi_\nu)| + h_\tau(\mathbf{x}, \eta_\nu) \|\partial_2 j_\tau(\mathbf{x}, \xi_\tau)\| \leq c_{\nu 0} + (c_{\nu 1} + \overline{h_\tau} c_\tau) \|\xi\|.$$

Applying Corollary 4.15 in [15], we obtain that functional J is well defined, locally Lipschitz with respect to the second variable and the growth condition $H(J)(b)$ holds with $c_0 = \sqrt{2 \operatorname{meas}(\Gamma_C)} c_{v0}$, $c_1 = \sqrt{2} (c_{v1} + \overline{h_\tau} c_\tau)$ and $c_2 = 0$.

To prove $H(J)(c)$, we take $\eta_i, \xi_i \in \mathbb{R}^d$, $i = 1, 2$, and by the sum rules (cf. Proposition 3.35 in [15]) and from $H(j_v)(d)$, $H(j_\tau)(b), (c)$ and $H(h)(b), (c)$, we obtain

$$\begin{aligned} & j_3^0(x, \eta_1, \xi_1; \xi_2 - \xi_1) + j_3^0(x, \eta_2, \xi_2; \xi_1 - \xi_2) \\ & \leq (j_v)_2^0(x, \xi_{1v}; \xi_{2v} - \xi_{1v}) + (j_v)_2^0(x, \xi_{2v}; \xi_{1v} - \xi_{2v}) \\ & \quad + h_\tau(x, \eta_{1v}) \left((j_\tau)_2^0(x, \xi_{1\tau}; \xi_{2\tau} - \xi_{1\tau}) + (j_\tau)_2^0(x, \xi_{2\tau}; \xi_{1\tau} - \xi_{2\tau}) \right) \\ & \quad + (h_\tau(x, \eta_{2v}) - h_\tau(x, \eta_{1v})) (j_\tau)_2^0(x, \xi_{2\tau}; \xi_{1\tau} - \xi_{2\tau}) \\ & \leq (\alpha_v + \overline{h_\tau} \alpha_\tau) \|\xi_1 - \xi_2\|^2 + L_{h_\tau} c_\tau \|\eta_1 - \eta_2\| \|\xi_1 - \xi_2\|. \end{aligned}$$

And consequently, since

$$J_2^0(w, v; z) \leq \int_{\Gamma_C} j_3^0(x, w(x), v(x); z(x)) da$$

(cf. Corollary 4.15(iii) in [15]), we have

$$\begin{aligned} & J_2^0(w_1, v_1; v_2 - v_1) + J_2^0(w_2, v_2; v_1 - v_2) \\ & \leq \int_{\Gamma_C} \left((\alpha_v + \overline{h_\tau} \alpha_\tau) \|v_1(x) - v_2(x)\|^2 + L_{h_\tau} c_\tau \|w_1(x) - w_2(x)\| \|v_1(x) - v_2(x)\| \right) da. \end{aligned}$$

Hence, by the Hölder inequality, we obtain $H(J)(c)$ with $m_\alpha = \alpha_v + \overline{h_\tau} \alpha_\tau$ and $m_L = L_{h_\tau} c_\tau$. $\square$

With the above properties, we have the following existence and uniqueness result for Problem P_{hvi} .

Theorem 7 *If assumptions $H(A)$, $H(j_v)$, $H(j_\tau)$, $H(h)$, (H_0) and (H_s) hold, then Problems P_{hvi} and P_{incl} are equivalent. Moreover, they have a unique solution $u \in V$ and this solution satisfies*

$$\|u\|_V \leq c (1 + \|f\|_{V^*})$$

with a positive constant c .

Proof We notice that the assumptions of Theorem 4 are satisfied. This implies that Problem P_{incl} has a unique solution. By (2.4) and Corollary 4.15(iii) in [15] we get that every solution to Problem P_{incl} solves Problem P_{hvi} . Using similar technique as in the proof of Lemma 1, we can show that if Problem P_{hvi} has a solution, it is unique. Combining these facts we obtain our assertion.

We conclude this section by providing a sample error estimate under additional assumptions on the solution regularity. We consider a polygonal domain Ω and a space

of continuous piecewise affine functions V^h . We introduce the following discretized version of Problem P_{hvi} .

Problem P_{hvi}^h Find a displacement $\mathbf{u}^h \in V^h$ such that for all $\mathbf{v}^h \in V^h$

$$\langle A\mathbf{u}^h, \mathbf{v}^h \rangle_{V^* \times V} + \int_{\Gamma_C} j_3^0(\mathbf{x}, \gamma \mathbf{u}^h(\mathbf{x}), \gamma \mathbf{u}^h(\mathbf{x}); \gamma \mathbf{v}^h(\mathbf{x})) da \geq \langle \mathbf{f}, \mathbf{v}^h \rangle_{V^* \times V}. \quad (4.10)$$

Theorem 8 Assume $H(\mathcal{A})$, $H(j_v)$, $H(j_\tau)$, $H(h)$, (H_0) and (H_s) and assume the solution regularity $\mathbf{u} \in H^2(\Omega)^d$, $\gamma \mathbf{u} \in H^2(\Gamma_C)^d$, $\boldsymbol{\sigma} \mathbf{v} \in L^2(\Gamma_C)^d$. Additionally, assume that Γ_C is a flat component of the boundary Γ . Then, for the solution $\mathbf{u}$ to Problem P_{hvi} and the solution $\mathbf{u}^h$ to Problem P_{hvi}^h there exists a constant $c > 0$ such that

$$\|\mathbf{u} - \mathbf{u}^h\|_V \leq c h.$$

Proof We denote by $\Pi^h \mathbf{u} \in V^h$ the finite element interpolant of $\mathbf{u}$. By the standard finite element interpolation error bounds (see [5]) we have for all $\boldsymbol{\eta} \in H^2(\Omega)^d$ such that $\gamma \boldsymbol{\eta} \in H^2(\Gamma_C)^d$

$$\|\boldsymbol{\eta} - \Pi^h \boldsymbol{\eta}\|_V \leq c h \|\boldsymbol{\eta}\|_{H^2(\Omega)^d}, \quad (4.11)$$

$$\|\gamma \boldsymbol{\eta} - \gamma \Pi^h \boldsymbol{\eta}\|_{L^2(\Gamma_C)^d} \leq c h^2 \|\gamma \boldsymbol{\eta}\|_{H^2(\Gamma_C)^d}. \quad (4.12)$$

We now bound the residual term defined by (3.2) using similar procedure to that one described in [8]. Let $\mathbf{v} = \pm \mathbf{w}$ in inequality (4.7), where the arbitrary function $\mathbf{w} \in V$ is such that $\mathbf{w} \in C^\infty(\overline{\Omega})^d$ and $\mathbf{w} = \mathbf{0}$ on $\Gamma_D \cup \Gamma_C$. Then we obtain the identity

$$\langle A\mathbf{u}, \mathbf{w} \rangle_{V^* \times V} = \langle \mathbf{f}, \mathbf{w} \rangle_{V^* \times V}.$$

From this identity, using fundamental lemma of calculus of variations, we can deduce that

$$\text{Div } \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})) + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \quad (4.13)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_N \quad \text{on } \Gamma_N. \quad (4.14)$$

We multiply equation (4.13) by $\mathbf{v}^h - \mathbf{u}$ and obtain

$$\int_{\Gamma} \boldsymbol{\sigma} \mathbf{v} \cdot (\gamma \mathbf{v}^h - \gamma \mathbf{u}) da - \int_{\Omega} \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}) dx + \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v}^h - \mathbf{u}) dx = 0. \quad (4.15)$$

Using the homogenous Dirichlet boundary condition of $\mathbf{v}^h - \mathbf{u}$ on Γ_D and the traction boundary condition given by (4.14) we have

$$\langle A\mathbf{u}, \mathbf{v}^h - \mathbf{u} \rangle_{V^* \times V} = \int_{\Gamma_C} \boldsymbol{\sigma} \mathbf{v} \cdot (\gamma \mathbf{v}^h - \gamma \mathbf{u}) da + \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u} \rangle_{V^* \times V}. \quad (4.16)$$

Using this and (3.2) we obtain

$$R(\mathbf{u}, \mathbf{v}^h) = \int_{\Gamma_C} \boldsymbol{\sigma} \mathbf{v} \cdot (\gamma \mathbf{v}^h - \gamma \mathbf{u}) \, da \leq c \|\gamma \mathbf{u} - \gamma \mathbf{v}^h\|_{L^2(\Gamma_C)^d}. \quad (4.17)$$

From inequalities (3.1), (4.11), (4.12) and (4.17) we get

$$\|\mathbf{u} - \mathbf{u}^h\|_V^2 \leq c \left(\|\mathbf{u} - \Pi^h \mathbf{u}\|_V^2 + \|\gamma \mathbf{u} - \gamma \Pi^h \mathbf{u}\|_{L^2(\Gamma_C)^d} \right) \leq c h^2,$$

and we obtain required estimate. $\square$

5 Simulations

In this section we present results of our computational simulations. From Theorems 4 and 7 we know that Problems P_{hvi} and P_{opt} are equivalent. Hence, we can apply numerical scheme P_{opt}^h and use Theorem 5 to approximate solution of P_{hvi} . We employ Finite Element Method and use space V^h of continuous piecewise affine functions as a family of approximating subspaces. The idea for algorithm used to calculate solution of discretized problem is based on the proof of Lemma 3 and is described by Algorithm 1.

Algorithm 1 Iterative optimization algorithm

```

Let  $\varepsilon > 0$  and  $\mathbf{u}_0^h$  be given
 $k \leftarrow 0$ 
repeat
   $k \leftarrow k + 1$ 
   $\mathbf{u}_k^h = \arg \min_{\mathbf{v}^h \in V^h} \mathcal{L}(\mathbf{u}_{k-1}^h, \mathbf{v}^h)$ 
until  $\|\mathbf{u}_k^h - \mathbf{u}_{k-1}^h\|_V \leq \varepsilon$ 
return  $\mathbf{u}_k^h$ 

```

In order to minimize not necessarily differentiable function $\mathcal{L}(\mathbf{w}^h, \cdot)$ we use Powell's conjugate direction method. This method was introduced in [18] and does not require the assumption on differentiability of optimized function. Other, more refined nonsmooth optimization algorithms described for example in [1], could also be adapted. For a starting point $\mathbf{u}_0^h$ we take a solution to problem with $\boldsymbol{\sigma} \mathbf{v} = \mathbf{0}$ on Γ_C , although it can be chosen arbitrarily.

We set $d = 2$ and consider a rectangular set $\Omega = [0, 2] \times [0, 1]$ with following parts of the boundary

$$\Gamma_D = \{0\} \times [0, 1], \quad \Gamma_N = ([0, 2] \times \{1\}) \cup (\{2\} \times [0, 1]), \quad \Gamma_C = [0, 2] \times \{0\}.$$

The elasticity operator $\mathcal{A}$ is defined by

$$\mathcal{A}(\boldsymbol{\tau}) = 2\eta \boldsymbol{\tau} + \lambda \operatorname{tr}(\boldsymbol{\tau}) I, \quad \boldsymbol{\tau} \in \mathbb{S}^2.$$

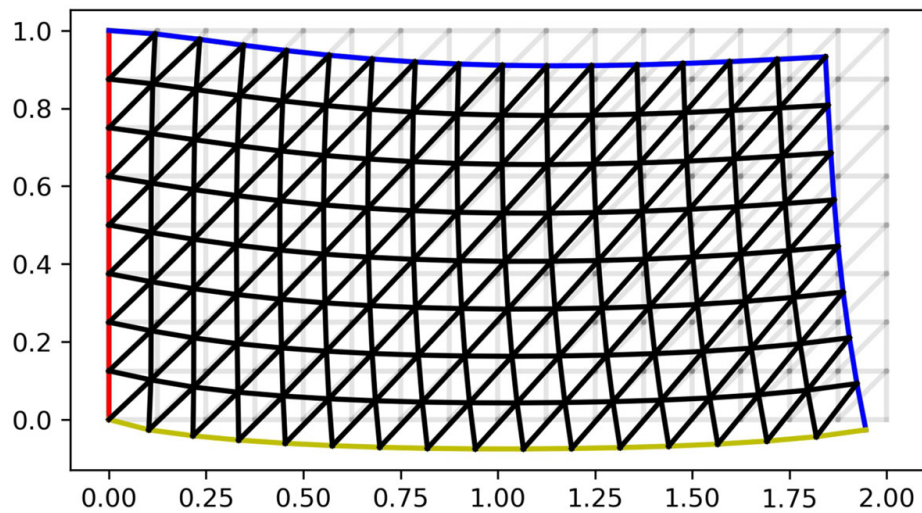


Fig. 1 Initial data

Here I denotes the identity matrix, tr denotes the trace of the matrix, λ and η are the Lamé coefficients, $\lambda, \eta > 0$. In our simulations we take the following data

$$\lambda = \eta = 4,$$

$$\mathbf{u}_0(\mathbf{x}) = (0, 0), \quad \mathbf{x} \in \Omega,$$

$$j_v(\mathbf{x}, \xi) = \begin{cases} 0, & \xi \in (-\infty, 0), \\ 10\xi^2, & \xi \in [0, 0.1), \\ 0.1, & \xi \in [0.1, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C,$$

$$j_\tau(\mathbf{x}, \xi) = \ln(\|\xi\| + 1), \quad \xi \in \mathbb{R}^2, \quad \mathbf{x} \in \Gamma_C,$$

$$h_\tau(\mathbf{x}, \eta) = \begin{cases} 0, & \eta \in (-\infty, 0), \\ 8\eta, & \eta \in [0, 0.1), \\ 0.8, & \eta \in [0.1, \infty) \end{cases} \quad \mathbf{x} \in \Gamma_C,$$

$$\mathbf{f}_0(\mathbf{x}) = (-1.2, -0.9), \quad \mathbf{x} \in \Omega,$$

$$\mathbf{f}_N(\mathbf{x}) = (0, 0), \quad \mathbf{x} \in \Omega.$$

Both functions j_v and j_τ are nondifferentiable and nonconvex. Our aim is to investigate reaction of the body to various modifications of input data.

In Fig. 1 we present output obtained without any modifications. We push the body down and to the left with a force $\mathbf{f}_0$. As a result the body penetrates the foundation, but because of frictional forces it is squeezed to the left more in the higher part than in the lower part. In Fig. 2 we modify the function h_τ to be given by

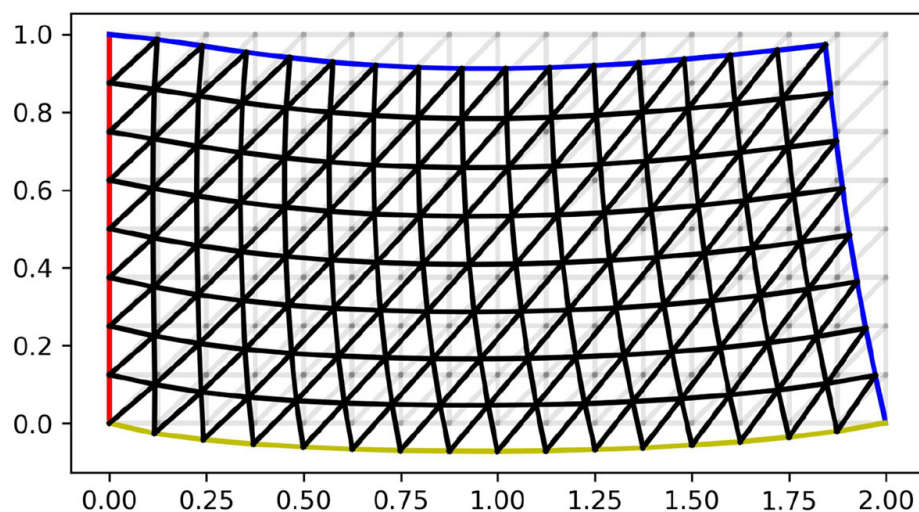


Fig. 2 Modified function h_τ

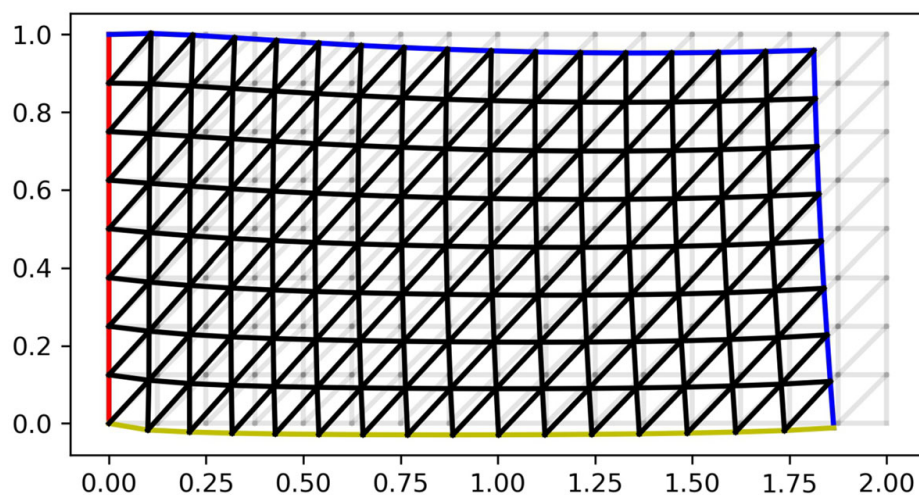


Fig. 3 Modified function j_v

$$h_\tau(\mathbf{x}, \eta) = \begin{cases} 0, & \eta \in (-\infty, 0), \\ 16\eta, & \eta \in [0, 0.1), \\ 1.6 & \eta \in [0.1, \infty) \end{cases} \quad \mathbf{x} \in \Gamma_C.$$

As a result we see that the penetration of the foundation does not change, but increased friction prevents the body from sliding to the left on Γ_C . In Fig. 3 we return to original data and modify only the function j_v to the following

$$j_v(\mathbf{x}, \xi) = \begin{cases} 0, & \xi \in (-\infty, 0), \\ 30\xi^2, & \xi \in [0, 0.1), \\ 0.3, & \xi \in [0.1, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C.$$

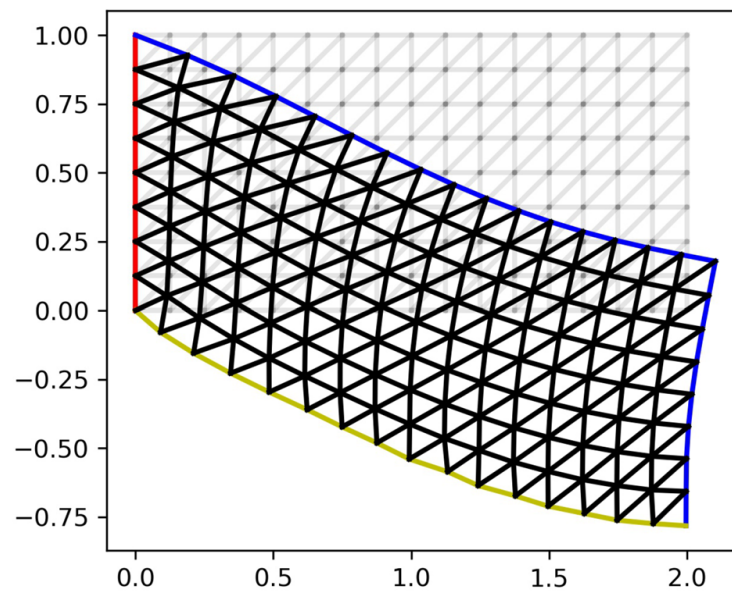


Fig. 4 Increased force f_0

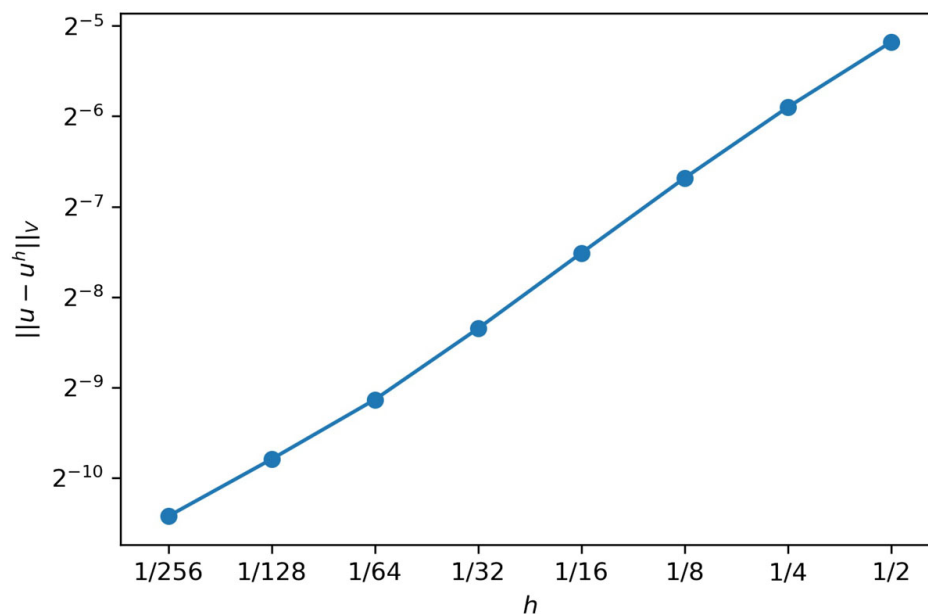


Fig. 5 Numerical errors

We can observe that the response of the foundation is more significant and the body moves downward only slightly. At the same time friction decreases due to influence of function h_τ that depends on normal component of body displacement. In Fig. 4 we once more return to original data and slightly increase force f_0 to be equal to

$$f_0(x) = (-1.2, -1.0), \quad x \in \Omega.$$

As a result, the body breaks through the threshold of quadratic response of the function j_ν into the part where this function is constant. This reflects the situation when there

is no response of the foundation in normal direction (e.g. the foundation broke) and causes the penetration to increase drastically.

In order to illustrate the error estimate obtained in Sect. 4, we present a comparison of numerical errors $\|\mathbf{u} - \mathbf{u}^h\|_V$ computed for a sequence of solutions to discretized problems. We use a uniform discretization of the problem domain according to the spatial discretization parameter h . The boundary Γ_C of Ω is divided into $1/h$ equal parts. We start with $h = 1$, which is successively halved. The numerical solution corresponding to $h = 1/512$ was taken as the “exact” solution $\mathbf{u}$. The numerical results are presented in Fig. 5, where the dependence of the error estimate $\|\mathbf{u} - \mathbf{u}^h\|_V$ with respect to h is plotted on a log–log scale. A first order convergence can be observed, providing numerical evidence of the theoretical optimal order error estimate obtained at the end of Sect. 4.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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IV

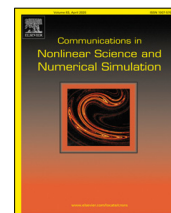
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A survey of numerical methods for hemivariational inequalities with applications to contact mechanics



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A survey of numerical methods for hemivariational inequalities with applications to contact mechanics

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ABSTRACT

In this paper we present an abstract nonsmooth optimization problem for which we recall existence and uniqueness results. We show a numerical scheme to approximate its solution. The theory is later applied to a sample static contact problem that describes an elastic body in frictional contact with a foundation. This problem leads to a hemivariational inequality which we solve numerically. Finally, we compare three computational methods of solving contact mechanical problems: the direct optimization method, the augmented Lagrangian method and the primal–dual active set strategy.

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1. Introduction

Mathematical models that describe the contact between a deformable body and a foundation have various applications. Because of that, many of them have already been analyzed in the literature. In such models, the behavior of the body on the contact boundary is usually governed by monotone functions responsible for foundation resistance in the normal direction and friction in the tangent plane. Considering general nonmonotone functions instead requires a specialized analytical and numerical treatment and may be needed for practical applications. For example, let us consider a foundation made of multiple layers of different materials with varying properties. This can lead to a contact mechanical problem that involves nonmonotone functions.

In this paper, we consider a model with nonmonotone friction law that depends on both normal and tangential components of displacement and its weak formulation, which leads to hemivariational inequality. To find an approximate solution, we formulate an abstract scheme for the chosen class of contact mechanical problems. We start with an introduction of a general nonsmooth optimization problem together with required assumptions followed by existence and uniqueness results. Numerical approximation of optimization problem with the obtained error estimate allows us then to use this abstract scheme to a static contact problem. Considered mechanical model is similar to the one described in [1], but with a more general operator A being a potential operator. The novelty of the paper is the comparison of three popular methods of solving the introduced problem: the direct optimization method, the augmented Lagrangian method and the primal–dual active set strategy. However, these are not the only methods used in the literature. Other methods include the nonconvex bundle method [2], the boundary element method [3], and more recently the virtual element method [4].

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For introduction to finite element method (FEM) in context of problems considered in this paper we refer to [5]. The definition and properties of the Clarke subdifferential and tools used to solve optimization problems can be found in [6], differences between nonsmooth and nonconvex optimization methods are presented in [7] and different notions of monotonicity are compared in [8]. An introduction to the theory of hemivariational inequalities is presented in [9], and the first use of FEM to solve such inequalities can be found in [10]. For an early study of vector-valued hemivariational problems related to FEM we refer to [11] and recent analysis of hemivariational and variational-hemivariational inequalities is presented in [12,13]. Numerical analysis of such problems can be found, for example, in papers [14–19]. Details on computational contact mechanics are presented in [20].

An error estimate of stationary variational-hemivariational inequalities is presented in [17]. In our paper the variational part of inequality is not present and the inequality is not constrained, nevertheless, to reflect the dependence of friction law on the normal component of the displacement error estimate had to be generalized.

Early ideas about optimization in Contact Mechanics were presented in [9] and further details can be found in [1,9]. The augmented Lagrangian method was reviewed and applied in [15,20–22]. The direct optimization method was previously compared to the augmented Lagrangian method in [14]. However, usually in papers containing contact mechanics simulations, only one method is selected and presented. In this paper, we also include third method, called the primal-dual active set strategy, for comparison of implementation and obtained results. Details on this third method can be found in [23–26].

The paper is organized as follows. In Section 2, we formulate a general differential inclusion problem and the equivalent optimization problem. We present the results of existence and uniqueness under the usual assumptions. Next, we introduce the discrete formulation of the optimization problem together with the numerical error estimate theorem. The abstract scheme is then used for a weak formulation of the mechanical contact problem and is presented in Section 3. In Section 4, we briefly describe three alternative methods of solving the problem from the previous section: the direct optimization method, the augmented Lagrangian method and the primal-dual active set strategy. Finally, we compare the results of the error estimate obtained for each method.

2. A general optimization problem

In this section, we recall notation, definitions and preliminary material (we refer to [6,10,27]), and provide analysis of a general optimization problem. For a normed space X , we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual and by $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing of X^* and X . Given two normed spaces X and Y , $\mathcal{L}(X, Y)$ is the space of all linear continuous operators from X to Y with the norm $\|\cdot\|_{\mathcal{L}(X, Y)}$. Let $L \in \mathcal{L}(X, Y)$, then the adjoint operator to L is denoted by $L^*: Y^* \rightarrow X^*$. Let X be a real Banach space, and let $j: X \rightarrow \mathbb{R}$ be locally Lipschitz continuous. Then the generalized (Clarke) directional derivative of j at $x \in X$ in the direction $v \in X$ is

$$j^0(x; v) := \limsup_{y \rightarrow x, \lambda \searrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized subdifferential of j at x is

$$\partial j(x) := \{\xi \in X^* \mid \langle \xi, v \rangle_{X^* \times X} \leq j^0(x; v) \text{ for all } v \in X\}.$$

If $\partial j(x)$ is nonempty, then any element $\xi \in \partial j(x)$ is called a subgradient of j at x (cf. [6]). If $j: X^n \rightarrow \mathbb{R}$ is a locally Lipschitz function of n variables, then we use $\partial_j j$ and j_i^0 to denote the Clarke subdifferential and generalized directional derivative with respect to i th variable of j , respectively.

Recall that an operator $A: X \rightarrow X^*$ is called a potential operator if there exists a Gâteaux differentiable functional $F_A: X \rightarrow \mathbb{R}$ such that $A = F'_A$. The functional F_A is called a potential of A . The basic properties of potential operators can be found in [27]. We recall that $A \in \mathcal{L}(X, X^*)$ is a potential operator if and only if it is symmetric. Moreover, under this symmetry condition, a potential functional is given by

$$F_A(v) = \frac{1}{2} \langle Av, v \rangle_{X^* \times X} \text{ for all } v \in X.$$

Throughout the paper, by $c > 0$ we denote a generic constant whose value may change from one place to another, but it is independent of other quantities of concern.

Let now V be a reflexive Banach space and X be a Banach space. Given an operator $A: V \rightarrow V^*$, a locally Lipschitz function $J: X \times X \rightarrow \mathbb{R}$, a linear operator $\gamma: V \rightarrow X$, and a linear functional $f: V \rightarrow \mathbb{R}$, we consider the following operator inclusion problem

Problem P_{incl} : Find $u \in V$ such that

$$Au + \gamma^* \partial_2 J(\gamma u, \gamma u) \ni f.$$

We say that $u \in V$ is a solution to Problem P_{incl} if there exists $z \in \partial_2 J(\gamma u, \gamma u)$ such that $Au + \gamma^* z = f$. In the study of Problem P_{incl} we adopt the following hypotheses

$H(A)$: The operator $A: V \rightarrow V^*$ is such that

- (a) A is Lipschitz continuous, i.e., $\|Au - Av\|_{V^*} \leq L_A \|u - v\|_V$ for all $u, v \in V$ with $L_A > 0$,
- (b) A is a potential operator with a potential F_A ,
- (c) A is strongly monotone, i.e., $\langle Au - Av, u - v \rangle_{V^* \times V} \geq m_A \|u - v\|_V^2$ for all $u, v \in V$ with $m_A > 0$.

$H(J)$: The functional $J: X \times X \rightarrow \mathbb{R}$ satisfies

- (a) J is locally Lipschitz continuous with respect to its second variable,
- (b) there exist $c_0, c_1, c_2 \geq 0$ such that $\|\partial_2 J(w, v)\|_{X^*} \leq c_0 + c_1 \|v\|_X + c_2 \|w\|_X$ for all $w, v \in X$,
- (c) there exist $m_\alpha, m_L \geq 0$ such that $J_2^0(w_1, v_1; v_2 - v_1) + J_2^0(w_2, v_2; v_1 - v_2) \leq m_\alpha \|v_1 - v_2\|_X^2 + m_L \|w_1 - w_2\|_X \|v_1 - v_2\|_X$ for all $w_1, w_2, v_1, v_2 \in X$.

$H(\gamma, f)$: $\gamma \in \mathcal{L}(V, X)$, $f \in V^*$.

(H_s) : $m_A > (m_\alpha + m_L) c_\gamma^2$, where $c_\gamma := \|\gamma\|_{\mathcal{L}(V, X)}$.

It is easy to see that $H(J)(c)$ is equivalent to the following condition

$$\langle \partial_2 J(w_1, v_1) - \partial_2 J(w_2, v_2), v_1 - v_2 \rangle_{X^* \times X} \geq -m_\alpha \|v_1 - v_2\|_X^2 - m_L \|w_1 - w_2\|_X \|v_1 - v_2\|_X$$

for all $w_1, w_2, v_1, v_2 \in X$. We remark that this condition generates the relaxed monotonicity condition which holds in a case of J independent of its first variable, i.e.,

$$\langle \partial J(v_1) - \partial J(v_2), v_1 - v_2 \rangle_{X^* \times X} \geq -m_\alpha \|v_1 - v_2\|_X^2. \quad (2.1)$$

for all $v_1, v_2 \in X$. Under the assumptions introduced, we consider the following optimization problem

Problem P_{opt} : Find $u \in V$ such that

$$0 \in \partial_2 \mathcal{L}(u, u).$$

Here, $\mathcal{L}: V \times V \rightarrow \mathbb{R}$ is defined by

$$\mathcal{L}(w, v) = F_A(v) - \langle f, v \rangle_{V^* \times V} + J(\gamma w, \gamma v). \quad (2.2)$$

for all $w, v \in V$.

We start by recalling some properties of the functional $\mathcal{L}$. This result is followed by an argument similar to the ones used in [1, Lemma 2] but with a more general operator A being a potential operator (see also [28, Proposition 2.5]).

Lemma 1. *If the hypotheses $H(A)$, $H(J)$, $H(\gamma, f)$ and (H_s) hold, then for a fixed $w \in V$ the functional $\mathcal{L}(w, \cdot): V \rightarrow \mathbb{R}$, defined by (2.2), is locally Lipschitz continuous and strictly convex, and hence also coercive, and*

$$\partial_2 \mathcal{L}(w, v) \subseteq Av - f + \gamma^* \partial_2 J(\gamma w, \gamma v).$$

The following result shows the relation between Problems P_{incl} and P_{opt} as well the existence of their unique solution.

Theorem 2. *If the hypotheses $H(A)$, $H(J)$, $H(\gamma, f)$ and (H_s) hold, then Problems P_{incl} and P_{opt} are equivalent, they have a unique solution $u \in V$ and this solution satisfies estimation*

$$\|u\|_V \leq c(1 + \|f\|_{V^*}) \quad (2.3)$$

with a positive constant c .

Detailed arguments can be found in [1] and are omitted here. We only mention the main steps of the proof. We first observe that Lemma 1 implies that every solution to Problem P_{opt} solves Problem P_{incl} . Moreover, if Problem P_{incl} has a solution, then it is unique. Then, it can be shown that Problem P_{opt} has a unique solution. This follows from the Banach fixed point theorem applied to an operator $\Lambda: V \rightarrow V$ given by

$$\Lambda w := \arg \min_{v \in V} \mathcal{L}(w, v) \text{ for all } w \in V.$$

Using the above facts, we see that a unique solution to Problem P_{opt} is also a unique solution to Problem P_{incl} . And due to the uniqueness of the solution to Problem P_{incl} we deduce that Problem P_{incl} and Problem P_{opt} are equivalent. Finally, the inequality (2.3) is a consequence of $H(J)(b)-(c)$, $H(A)(c)$ and (H_s) .

We are now in a position to present numerical methods for solving the optimization problem. We keep the assumptions $H(A)$, $H(J)$, $H(\gamma, f)$ and (H_s) so that Problem P_{opt} has a unique solution $u \in V$. Let $V^h \subset V$ be a finite dimensional subspace with a discretization parameter $h > 0$. We consider the following discrete scheme of Problem P_{opt} .

Problem P_{opt}^h : Find $u^h \in V^h$ such that

$$0 \in \partial_2 \mathcal{L}(u^h, u^h).$$

We can apply the arguments of the proof of [Theorem 2](#) in the setting of the finite dimensional space V^h , to conclude the existence of a unique solution to Problem P_{opt}^h and equivalence to the discrete version of Problem P_{incl} . We now present the theorem concerning the error estimate of the introduced numerical scheme.

Theorem 3. *If the hypotheses $H(A)$, $H(J)$, $H(\gamma, f)$ and (H_s) hold, then for the unique solutions u and u^h to Problems P_{opt} and P_{opt}^h , respectively, there exists a constant $c > 0$ such that*

$$\|u - u^h\|_V^2 \leq c \inf_{v^h \in V^h} \left\{ \|u - v^h\|_V^2 + \|\gamma u - \gamma v^h\|_X + R(u, v^h) \right\}, \quad (2.4)$$

where a residual quantity is defined by

$$R(u, v^h) = \langle Au, v^h - u \rangle_{V^* \times V} + \langle f, u - v^h \rangle_{V^* \times V}. \quad (2.5)$$

Proof. Let u and u^h be solutions to Problems P_{opt} and P_{opt}^h , respectively. Hence, they satisfy the corresponding inclusion problems and the following inequalities, respectively

$$\begin{aligned} \langle f - Au, v \rangle_{V^* \times V} &\leq J_2^0(\gamma u, \gamma u; \gamma v) \quad \text{for all } v \in V, \\ \langle f - Au^h, v \rangle_{V^* \times V} &\leq J_2^0(\gamma u^h, \gamma u^h; \gamma v) \quad \text{for all } v \in V^h. \end{aligned}$$

Setting $v = u^h - u$ in the first inequality, and $v = v^h - u^h$ with $v^h \in V^h$ in the second one, then adding the resulting inequalities, we deduce for all $v^h \in V^h$

$$\begin{aligned} \langle f, v^h - u \rangle_{V^* \times V} + \langle Au^h - Au, u^h - u \rangle_{V^* \times V} - \langle Au^h, v^h - u \rangle_{V^* \times V} \\ \leq J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h). \end{aligned}$$

The subadditivity of the generalized directional derivative (cf. [13]) and $H(J)(c)$, gives the following

$$\begin{aligned} J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u^h) \\ \leq J_2^0(\gamma u, \gamma u; \gamma u^h - \gamma u) + J_2^0(\gamma u^h, \gamma u^h; \gamma u - \gamma u^h) + J_2^0(\gamma u^h, \gamma u^h; \gamma v^h - \gamma u) \\ \leq (m_\alpha + m_L) \|\gamma u^h - \gamma u\|_X^2 + (c_0 + (c_1 + c_2) \|\gamma u^h\|_X) \|\gamma v^h - \gamma u\|_X. \end{aligned}$$

From [Theorem 2](#) applied to the discrete version of Problem P_{incl} we obtain the uniform boundedness property with respect to h

$$\|\gamma u^h\|_X \leq c_\gamma \|u^h\|_V \leq c(1 + \|f\|_{V^*}).$$

Hence, combining the above inequalities, we deduce for all $v^h \in V^h$

$$\begin{aligned} \langle Au^h - Au, u^h - u \rangle_{V^* \times V} &\leq \langle Au^h - Au, v^h - u \rangle_{V^* \times V} + \langle Au, v^h - u \rangle_{V^* \times V} \\ &\quad + \langle f, u - v^h \rangle_{V^* \times V} + (m_\alpha + m_L) c_\gamma^2 \|u^h - u\|_V^2 + c \|\gamma v^h - \gamma u\|_X. \end{aligned}$$

Using definition (2.5) and assumption $H(A)$, we have for all $v^h \in V^h$

$$\begin{aligned} m_A \|u^h - u\|_V^2 &\leq L_A \|u^h - u\|_V \|v^h - u\|_V + R(u, v^h) \\ &\quad + (m_\alpha + m_L) c_\gamma^2 \|u - u^h\|_V^2 + c \|\gamma u - \gamma v^h\|_X. \end{aligned}$$

Finally, the Cauchy inequality with $\varepsilon > 0$ yields

$$\begin{aligned} m_A \|u - u^h\|_V^2 &\leq \varepsilon \|u - u^h\|_V^2 + \frac{L_A^2}{4\varepsilon} \|u - v^h\|_V^2 + R(u, v^h) \\ &\quad + (m_\alpha + m_L) c_\gamma^2 \|u - u^h\|_V^2 + c \|\gamma u - \gamma v^h\|_X \end{aligned}$$

which implies for all $v^h \in V^h$

$$\left(m_A - (m_\alpha + m_L) c_\gamma^2 - \varepsilon \right) \|u - u^h\|_V^2 \leq \frac{c}{\varepsilon} \|u - v^h\|_V^2 + R(u, v^h) + c \|\gamma u - \gamma v^h\|_X.$$

For sufficiently small ε and by (H_s) , we obtain the desired Céa type inequality. ■

3. Application to contact mechanics

This section presents a sample mechanical contact problem where results of the previous section are applied. We want to find the body displacement in a static state. At the beginning we introduce the physical setting and notation.

Let us consider an elastic body in a domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$ in application. Boundary of Ω is denoted as Γ and is divided into three disjoint measurable parts: $\Gamma_C, \Gamma_N, \Gamma_D$, where the measure of part Γ_D is positive. Moreover Γ is Lipschitz continuous, so the outward normal vector $\mathbf{v}$ to Γ exists a.e. on the boundary. To model contact with the foundation on boundary Γ_C we use general subdifferential inclusions. Displacement of the body is equal $\mathbf{0}$ on Γ_D , a surface force of density $\mathbf{f}_N$ acts on the boundary Γ_N and a body force of density $\mathbf{f}_0$ acts in Ω .

Let us denote by “ $\cdot$ ” and $\|\cdot\|$ the scalar product and the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{S}^d$, respectively, where $\mathbb{S}^d = \mathbb{R}_{sym}^{d \times d}$. Indices i and j run from 1 to d and summation over repeated indices is implied. We denote the divergence operator by $\text{Div } \boldsymbol{\sigma} = \left(\frac{\partial \sigma_{ij}}{\partial x_j} \right)$. The linearized (small) strain tensor for displacement $\mathbf{u} \in H^1(\Omega)^d$ is defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Let $u_\nu = \mathbf{u} \cdot \mathbf{v}$ and $\sigma_\nu = \boldsymbol{\sigma} \mathbf{v} \cdot \mathbf{v}$ be the normal components of $\mathbf{u}$ and $\boldsymbol{\sigma}$, respectively, and let $\mathbf{u}_\tau = \mathbf{u} - u_\nu \mathbf{v}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}$ be their tangential components, respectively. In what follows, for simplicity, we sometimes do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x}$. Let us now introduce the classical formulation of the considered mechanical contact problem.

Problem P: Find a displacement field $\mathbf{u}: \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}: \Omega \rightarrow \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega \quad (3.1)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega \quad (3.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad (3.3)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_N \quad \text{on } \Gamma_N \quad (3.4)$$

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_C \quad (3.5)$$

$$-\boldsymbol{\sigma}_\tau \in h_\tau \partial j_\tau(\mathbf{u}_\tau) \quad \text{on } \Gamma_C \quad (3.6)$$

Here, Eq. (3.1) represents an elastic constitutive law and $\mathcal{A}$ is an elasticity operator. Equilibrium equation (3.2) reflects the fact that the problem is static. Eq. (3.3) represents the clamped boundary condition on Γ_D and (3.4) represents tractions applied on Γ_N . Inclusion (3.5) describes the response of the foundation in normal direction, whereas the friction is modeled by inclusion (3.6), where j_ν and j_τ are given superpotentials, and h_τ is a given friction bound. Note that to simplify simulation, the function h_τ does not depend on u_ν . We now present the hypotheses on data of Problem P.

$H(\mathcal{A}) : \mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies

- (a) $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}) = (a_{ijkh}(\mathbf{x}) \tau_{kh})$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$, $a_{ijkh} \in L^\infty(\Omega)$,
- (b) $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}_1) \cdot \boldsymbol{\tau}_2 = \boldsymbol{\tau}_1 \cdot \mathcal{A}(\mathbf{x}, \boldsymbol{\tau}_2)$ for all $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$,
- (c) there exists $m_{\mathcal{A}} > 0$ such that $\mathcal{A}(\mathbf{x}, \boldsymbol{\tau}) \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}} \|\boldsymbol{\tau}\|^2$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.

$H(j_\nu) : j_\nu: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $j_\nu(\cdot, \xi)$ is measurable on Γ_C for all $\xi \in \mathbb{R}$ and there exists $e \in L^2(\Gamma_C)$ such that $j_\nu(\cdot, e(\cdot)) \in L^1(\Gamma_C)$,
- (b) $j_\nu(\mathbf{x}, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}$ for a.e. $\mathbf{x} \in \Gamma_C$,
- (c) there exist $c_{\nu 0}, c_{\nu 1} \geq 0$ such that $|\partial_2 j_\nu(\mathbf{x}, \xi)| \leq c_{\nu 0} + c_{\nu 1} |\xi|$ for all $\xi \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$,
- (d) there exists $\alpha_\nu \geq 0$ such that $(j_\nu)_2^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + (j_\nu)_2^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq \alpha_\nu |\xi_1 - \xi_2|^2$ for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(j_\tau) : j_\tau: \Gamma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

- (a) $j_\tau(\cdot, \boldsymbol{\xi})$ is measurable on Γ_C for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and there exists $\mathbf{e} \in L^2(\Gamma_C)^d$ such that $j_\tau(\cdot, \mathbf{e}(\cdot)) \in L^1(\Gamma_C)$,
- (b) there exists $c_\tau > 0$ such that $|j_\tau(\mathbf{x}, \boldsymbol{\xi}_1) - j_\tau(\mathbf{x}, \boldsymbol{\xi}_2)| \leq c_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$, a.e. $\mathbf{x} \in \Gamma_C$,
- (c) there exists $\alpha_\tau \geq 0$ such that $(j_\tau)_2^0(\mathbf{x}, \boldsymbol{\xi}_1; \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + (j_\tau)_2^0(\mathbf{x}, \boldsymbol{\xi}_2; \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq \alpha_\tau \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$, a.e. $\mathbf{x} \in \Gamma_C$.

$H(h) : h_\tau: \Gamma_C \rightarrow \mathbb{R}$ satisfies

- (a) $h_\tau(\cdot)$ is measurable on Γ_C
- (b) there exists $\bar{h}_\tau > 0$ such that $0 \leq h_\tau(\mathbf{x}) \leq \bar{h}_\tau$ a.e. $\mathbf{x} \in \Gamma_C$,

$$(H_0) : \mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_N \in L^2(\Gamma_N)^d.$$

Note that condition $H(j_\tau)(b)$ is equivalent to the fact that $j_\tau(\mathbf{x}, \cdot)$ is locally Lipschitz continuous and there exists $c_\tau \geq 0$ such that $\|\partial_2 j_\tau(\mathbf{x}, \xi)\| \leq c_\tau$ for all $\xi \in \mathbb{R}^d$ and a.e. $\mathbf{x} \in \Gamma_C$. To obtain a weak formulation of Problem P we consider the following Hilbert spaces

$$\mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad V = \{v \in H^1(\Omega)^d \mid v = \mathbf{0} \text{ on } \Gamma_D\},$$

endowed with the inner scalar products

$$(\sigma, \tau)_{\mathcal{H}} = \int_{\Omega} \sigma_{ij} \tau_{ij} dx, \quad (\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}},$$

respectively. The fact that the space V equipped with the corresponding norm $\|\cdot\|_V$ is complete follows from Korn's inequality, and its application is allowed because we assume that $\text{meas}(\Gamma_D) > 0$. We consider the trace operator $\gamma: V \rightarrow L^2(\Gamma_C)^d = X$. Using standard procedure, the Green formula and the definition of generalized subdifferential, we obtain a weak formulation of Problem P in the form of hemivariational inequality.

Problem P_{hvi} : Find a displacement $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V^* \times V} + \int_{\Gamma_C} j_2^0(\mathbf{x}, \gamma \mathbf{u}(\mathbf{x}); \gamma \mathbf{v}(\mathbf{x})) da \geq \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V}. \quad (3.7)$$

Here, the operators $A: V \rightarrow V^*$ and $\mathbf{f} \in V^*$ are defined for all $\mathbf{w}, \mathbf{v} \in V$ as follows

$$\langle A\mathbf{w}, \mathbf{v} \rangle_{V^* \times V} = (\mathcal{A}(\varepsilon(\mathbf{w})), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad (3.8)$$

$$\langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V} = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \gamma \mathbf{v} da \quad (3.9)$$

and $j: \Gamma_C \times \mathbb{R}^d \rightarrow \mathbb{R}$ is defined for all $\xi \in \mathbb{R}^d$ and $\mathbf{x} \in \Gamma_C$ by

$$j(\mathbf{x}, \xi) = j_v(\mathbf{x}, \xi_v) + h_\tau(\mathbf{x}) j_\tau(\mathbf{x}, \xi_\tau). \quad (3.10)$$

It is easy to check that under assumptions $H(\mathcal{A})$ and (H_0) and by the Sobolev trace theorem the operator A , the functional $\mathbf{f}$ and $\gamma \in \mathcal{L}(V, X)$ satisfy $H(A)$ and $H(\gamma, \mathbf{f})$, respectively. We also define the functional $J: L^2(\Gamma_C)^d \rightarrow \mathbb{R}$ for all $\mathbf{v} \in L^2(\Gamma_C)^d$ by

$$J(\mathbf{v}) = \int_{\Gamma_C} j(\mathbf{x}, \mathbf{v}(\mathbf{x})) da, \quad (3.11)$$

We note that the functional J defined by (3.10)–(3.11) under assumptions $H(j_v)$, $H(j_\tau)$ and $H(h)$ satisfies $H(J)$ (cf. [1, Lemma 4]). With the above properties, we have the following existence and uniqueness result for Problem P_{hvi} .

Theorem 4. If assumptions $H(\mathcal{A})$, $H(j_v)$, $H(j_\tau)$, $H(h)$, (H_0) and (H_s) hold, then Problems P_{hvi} and P_{incl} (with functional J dependent only on one variable) are equivalent. Moreover, they have a unique solution $\mathbf{u} \in V$ and this solution satisfies

$$\|\mathbf{u}\|_V \leq c(1 + \|\mathbf{f}\|_{V^*})$$

with a positive constant c .

Proof. We notice that the assumptions of Theorem 2 are satisfied. This implies that Problem P_{incl} has a unique solution. If $\mathbf{u} \in V$ is a solution to Problem P_{incl} then it satisfies $\langle \mathbf{f} - A\mathbf{u}, \mathbf{v} \rangle_{V^* \times V} \leq J^0(\gamma \mathbf{u}; \gamma \mathbf{v})$ for all $\mathbf{v} \in V$. Hence, by Corollary 4.15 (iii) in [13] we find that every solution to Problem P_{incl} solves Problem P_{hvi} . Using a similar technique as in the proof of Theorem 2, we can show that if Problem P_{hvi} has a solution, it is unique. Combining these facts, we obtain our assertion. ■

4. Methods overview

In this section, we present a brief overview of three established algorithms for solving contact problems, namely, the direct optimization method, the augmented Lagrangian method and the primal–dual active set strategy. References are also provided for a more detailed treatment of each method. In the versions presented here, all listed algorithms employ the finite element method (FEM). Let us start with the direct optimization method.

4.1. Direct optimization method

The idea behind the direct optimization method is to replace a weak formulation of the contact problem with an equivalent minimization problem. In the case of Problem P_{hvi} it takes the form

Problem $\widehat{P}_{opt}^h$: Find $\mathbf{u}^h \in V^h$ such that

$$0 \in \partial \widehat{\mathcal{L}}(\mathbf{u}^h),$$

where a functional $\widehat{\mathcal{L}} : V \rightarrow \mathbb{R}$ is defined for all $\mathbf{v} \in V$ as follows

$$\widehat{\mathcal{L}}(\mathbf{v}) = \frac{1}{2} \langle A\mathbf{v}, \mathbf{v} \rangle_{V^* \times V} - \langle \mathbf{f}, \mathbf{v} \rangle_{V^* \times V} + J(\gamma \mathbf{v}),$$

and the operators $A, \mathbf{f}$ and J are defined by (3.8), (3.9) and (3.11), respectively.

Even though functions j_v and j_τ can be nonmonotone and nonconvex, because of the relaxed monotonicity condition on their subdifferentials combined with the smallness assumption, $\widehat{\mathcal{L}}$ is still convex. The direct optimization method in a more complex setting (with the function h_τ dependent on $u_v(\mathbf{x})$) using the Uzawa algorithm is presented in [1].

4.2. Augmented Lagrangian method

The augmented Lagrangian method is a technique which regularizes nondifferentiable terms governing a body's behavior on a contact boundary by addition of auxiliary Lagrange multipliers. These multipliers, represented by $\boldsymbol{\lambda}^h$, can be interpreted as normal and tangential forces acting on the body. The augmented Lagrangian approach expresses Problem P_{hvi} as a system of nonlinear equations.

Let us introduce Γ_C^h as a discretization of a contact interface based on FEM mesh, such that it consists of those nodes and edges of triangles that represent the boundary Γ_C . Let n_C be the number of independent points on the boundary Γ_C^h and n_{tot} be the total number of independent nodes (outside of the boundary Γ_D) on the FEM mesh.

In order to introduce the space of Lagrange multipliers, we use a contact element composed by one edge of Γ_C^h and one Lagrange multiplier node. In our case FEM with affine polynomials is used for the displacement and FEM with constant polynomials is used for multipliers. This leads to the space $H_{\Gamma_C}^h$, containing linear combinations of piecewise constant functions equal to 1 on one edge of Γ_C^h and 0 everywhere else.

Let us now introduce matrices $W^h = (w_{ij}) \in \mathbb{R}^{d \times n_{tot}}$ and $C^h = (c_{ij}) \in \mathbb{R}^{d \times n_C}$ with w_{ij} and c_{ij} being i th coordinates of functions $\mathbf{v}^h \in V^h$ and $\boldsymbol{\gamma}^h \in H_{\Gamma_C}^h$ at node j of FEM mesh, respectively (recall that d is the dimension of considered body). We now reshape those matrices to obtain vectors $\widehat{\mathbf{v}}^h \in \mathbb{R}^{d \cdot n_{tot}}$ and $\widehat{\boldsymbol{\gamma}}^h \in \mathbb{R}^{d \cdot n_C}$, containing consecutive rows of respective matrices stacked in sequence. The generalized elastic term $\widetilde{G}(\widehat{\mathbf{u}}^h) \in \mathbb{R}^{d \cdot n_{tot}} \times \mathbb{R}^{d \cdot n_C}$ is defined by $\widetilde{G}(\widehat{\mathbf{u}}^h) = (G(\widehat{\mathbf{u}}^h), \mathbf{0}_{\mathbb{R}^{d \cdot n_C}})$, where $\mathbf{0}_{\mathbb{R}^{d \cdot n_C}}$ is the zero element of $\mathbb{R}^{d \cdot n_C}$ and $G(\widehat{\mathbf{u}}^h)$ denotes the term given for all $\mathbf{u}^h, \mathbf{v}^h \in V^h$ by

$$(G(\widehat{\mathbf{u}}^h) \cdot \widehat{\mathbf{v}}^h)_{\mathbb{R}^{d \cdot n_{tot}}} = \langle A\mathbf{u}^h - \mathbf{f}, \mathbf{v}^h \rangle_{V^* \times V}.$$

Here, operators A and $\mathbf{f}$ are defined by (3.8), (3.9), respectively. One possibility of contact operator $\mathcal{F}(\widehat{\mathbf{u}}^h, \widehat{\boldsymbol{\xi}}^h)$, that deals with contact effects, can be defined for all $\widehat{\mathbf{u}}^h, \widehat{\mathbf{v}}^h \in \mathbb{R}^{d \cdot n_{tot}}$, $\widehat{\boldsymbol{\lambda}}^h, \widehat{\boldsymbol{\gamma}}^h \in \mathbb{R}^{d \cdot n_C}$, $\mathbf{u}^h, \mathbf{v}^h \in V^h$ and $\boldsymbol{\lambda}^h, \boldsymbol{\gamma}^h \in H_{\Gamma_C}^h$ by

$$\begin{aligned} & (\mathcal{F}(\widehat{\mathbf{u}}^h, \widehat{\boldsymbol{\lambda}}^h) \cdot (\widehat{\mathbf{v}}^h, \widehat{\boldsymbol{\gamma}}^h))_{\mathbb{R}^{d \cdot n_{tot}} + d \cdot n_C} \\ & + \int_{\Gamma_C} p(u_v^h) \cdot \mathbf{v}^h da \\ & = \int_{\Gamma_C} \nabla_{\mathbf{w}} [l_v(\mathbf{u}^h, \boldsymbol{\lambda}^h) + l_\tau(\mathbf{u}^h, \boldsymbol{\lambda}^h)] \cdot \mathbf{v}^h da \\ & + \int_{\Gamma_C} \nabla_{\boldsymbol{\lambda}} [l_v(\mathbf{u}^h, \boldsymbol{\lambda}^h) + l_\tau(\mathbf{u}^h, \boldsymbol{\lambda}^h)] \cdot \boldsymbol{\gamma}^h da, \end{aligned}$$

where $\nabla_{\mathbf{y}}$ represents the gradient operator with respect to the variable $\mathbf{y}$. Functions p , l_v and l_τ depend on the specific problem and are defined in Section 5. The augmented Lagrangian approach is now expressed by the following system of equations.

Problem P_{Lag}^h : Find a displacement $\widehat{\mathbf{u}}^h \in \mathbb{R}^{d \cdot n_{tot}}$ and a stress multiplier field $\widehat{\boldsymbol{\lambda}}^h \in \mathbb{R}^{d \cdot n_C}$ such that

$$\widetilde{G}(\widehat{\mathbf{u}}^h) + \mathcal{F}(\widehat{\mathbf{u}}^h, \widehat{\boldsymbol{\lambda}}^h) = \mathbf{0}.$$

Note that using this method we approximate not only value of $\mathbf{u}$, but also values σ_v and σ_τ by $\widehat{\lambda}_v^h$ and $\widehat{\lambda}_\tau^h$, respectively. More complete description of the presented approach can be found in [14,15]. For further details about the discretization of a contact surface and the augmented Lagrangian method in general we refer to [20–22].

4.3. Primal–dual active set strategy

In the primal–dual active set strategy we keep track of all points on the discretized boundary Γ_C^h , and assign them to sets that reflect different “parts” of the boundary laws ∂j_v and ∂j_τ . For example, every active set for ∂j_τ corresponds to a single or multivalued part of this function and can be interpreted as the physical state of the point assigned to this

set (e.g. a stick vs. a slip zone). The division into these sets is specific to each chosen function and is reflected in the implementation.

The main idea behind this strategy is to simplify integral over contact boundary in Problem P_{hvi} . We can do that using conditions that are implied by assignment of any point $\mathbf{x} \in \Gamma_C$ to a specific set (under the assumption that this assignment is correct). Initially all points are assigned to sets corresponding to $u_v = 0$ and $\mathbf{u}_\tau = \mathbf{0}$. In the iterative procedure we successively solve the simplified problem with points assigned to selection of sets, and reassign them after each iteration. This reassignment is conducted according to specific rules, so that we can move only between adjacent “parts” of the graph of functions ∂j_v and ∂j_τ . We repeat this procedure until convergence to finally obtain a solution with all points in correct sets with respect both to ∂j_v and ∂j_τ . Further details of the primal–dual active set strategy can be found in [23], and applications of this algorithm are presented for example in [24–26].

5. Simulations

We now consider Problem P and its weak formulation Problem P_{hvi} . We use data within previously presented theoretical framework, so the resulting model satisfies required assumptions and can be easily implemented using all three introduced methods.

5.1. Data

We set $d = 2$ and consider a rectangular set $\Omega = [0, 1] \times [0, 1]$ with following partition of the boundary

$$\Gamma_D = \{0\} \times [0, 1], \quad \Gamma_N = ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1]), \quad \Gamma_C = [0, 1] \times \{0\}.$$

The elasticity operator $\mathcal{A}$ is defined by

$$\mathcal{A}(\boldsymbol{\tau}) = 2\eta\boldsymbol{\tau} + \lambda \text{tr}(\boldsymbol{\tau})I, \quad \boldsymbol{\tau} \in \mathbb{S}^2.$$

Here, I is the identity matrix, tr denotes the trace of the matrix, λ and η are the Lamé coefficients, $\lambda, \eta > 0$. In our simulations we take the following data

$$\begin{aligned} \lambda &= \eta = 4, \\ \mathbf{u}_0(\mathbf{x}) &= (0, 0), \quad \mathbf{x} \in \Omega, \\ \mathbf{f}_N(\mathbf{x}) &= (0, 0), \quad \mathbf{x} \in \Gamma_N. \end{aligned}$$

We take the nondifferentiable and nonconvex function j_v and the nondifferentiable function j_τ such that

$$\begin{aligned} j_v(\mathbf{x}, \xi) &= \begin{cases} 0, & \xi \in (-\infty, 0), \\ \frac{p_{\text{const}}}{2} \xi^2 + q_{\text{max}} \xi, & \xi \in [0, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C, \\ j_\tau(\mathbf{x}, \xi) &= \|\xi\|, \quad \xi \in \mathbb{R}^2, \quad \mathbf{x} \in \Gamma_C, \end{aligned}$$

where $q_{\text{max}}, p_{\text{const}} > 0$. This choice corresponds to (3.5) and (3.6) with

$$\begin{aligned} \partial j_v(\mathbf{x}, \xi) &= p(\mathbf{x}, \xi) + \partial q(\mathbf{x}, \xi), \quad \xi \in \mathbb{R}, \quad \mathbf{x} \in \Gamma_C, \\ \partial j_\tau(\mathbf{x}, \xi) &= \begin{cases} -1, & \xi \in (-\infty, 0), \\ [-1, 1], & \xi = 0, \\ 1, & \xi \in (0, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C, \end{aligned}$$

where

$$\begin{aligned} p(\mathbf{x}, \xi) &= \begin{cases} 0, & \xi \in (-\infty, 0), \\ p_{\text{const}} \xi, & \xi \in [0, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C, \\ \partial q(\mathbf{x}, \xi) &= \begin{cases} 0, & \xi \in (-\infty, 0), \\ [0, q_{\text{max}}], & \xi = 0, \\ q_{\text{max}}, & \xi \in (0, \infty), \end{cases} \quad \mathbf{x} \in \Gamma_C. \end{aligned}$$

Remaining data are introduced for each example later, so that by applying various modifications of input parameters we can observe change in reaction of the body. It can be checked that all selected functions and data satisfy corresponding assumptions $H(\mathcal{A})$, $H(j_v)$, $H(j_\tau)$, $H(h)$, (H_0) . Hence, we know that Problem P_{hvi} has a unique solution that can be estimated numerically.

5.2. Implementation details

We employ FEM and use the space V^h of continuous piecewise affine functions as a family of approximating subspaces. The uniform discretization of the problem domain according to the spatial discretization parameter h is used. The contact boundary Γ_C is divided into $1/h$ equal parts. For each example we start simulations with $h = 1/2$, which is successively halved. For all presented algorithms we can choose starting point $\mathbf{u}_0^h$ arbitrarily, but this choice affects convergence. To speed it up, we use “warm start” procedure: for mesh size $h = 1/2$ we take $\mathbf{u}_0^h = \mathbf{0}$, and for every other h we use a solution to problem with mesh size $h/2$.

To decrease the dimension of the considered discrete problem, we use the Schur complement method described in [9]. This method reformulates a discrete scheme, so that we have to solve only for vertices located on the contact boundary Γ_C . Then we can retrieve the solution for all other vertices using simple matrix inversion and multiplication. These operations still consume a lot of resources, but are feasible for a much bigger mesh size h .

The code is written in Python, partly using Cython to obtain better performance. The solution is calculated by using implementations of solvers from SciPy library. For the augmented Lagrangian method we use *fsolve* to solve $\hat{\mathbf{P}}_{Lag}^h$, and for the primal–dual active set strategy we use *newton_krylov* to solve discretized and simplified version of $\hat{\mathbf{P}}_{opt}^h$. For the direct optimization method we use *minimize* function with Powell’s conjugate direction method to solve the $\hat{\mathbf{P}}_{opt}^h$, because the minimized functional $\hat{\mathcal{L}}$ is not necessarily differentiable. We can also employ other nonsmooth optimization algorithms such as the proximal bundle method (see [7]) or methods empirically proven to work for nondifferentiable functions, such as the Broyden–Fletcher–Goldfarb–Shanno algorithm. Each presented implementation is chosen as one that gives the best performance for each method. In all cases, we used default stopping criteria, based on change of the argument value and change of the function value with a maximal limit of iterations.

Let us now present how chosen data correspond to implementation details for the augmented Lagrangian method and the primal–dual active set strategy.

For the augmented Lagrangian method, we introduce positive penalty coefficients $\epsilon_v, \epsilon_\tau > 0$. Definitions of functions j_v and j_τ translate to auxiliary operators l_v and l_τ in the form of

$$l_v(\mathbf{u}^h, \lambda^h) = \begin{cases} q_{max} u_v^h, & \lambda_v^h + \epsilon_v u_v^h \in (-\infty, -q_{max}], \\ \lambda_v^h u_v^h + \frac{\epsilon_v}{2} (u_v^h)^2, & \lambda_v^h + \epsilon_v u_v^h \in (-q_{max}, 0], \\ -\frac{1}{2\epsilon_v} (\lambda_v^h)^2, & \lambda_v^h + \epsilon_v u_v^h \in (0, \infty), \end{cases}$$

$$l_\tau(\mathbf{u}^h, \lambda^h) = \begin{cases} \lambda_\tau^h \cdot \mathbf{u}_\tau^h + \frac{\epsilon_\tau}{2} \|\mathbf{u}_\tau^h\|^2, & \|\lambda_\tau^h + \epsilon_\tau \mathbf{u}_\tau^h\| \leq h_\tau, \\ -\frac{1}{2\epsilon_\tau} [\|\lambda_\tau^h\|^2 - 2h_\tau \|\lambda_\tau^h\| + (h_\tau)^2], & \|\lambda_\tau^h + \epsilon_\tau \mathbf{u}_\tau^h\| > h_\tau. \end{cases}$$

In the iterative procedure we solve a system of equations present in $\hat{\mathbf{P}}_{Lag}^h$ for fixed values of penalty parameters $\epsilon_v, \epsilon_\tau$, decrease them, and, taking previously obtained solution as a starting point, repeat until convergence.

In the case of the primal–dual active set strategy, the function ∂j_v divides points of Γ_C into three exclusive sets N^1, N^2, N^3 and the function ∂j_τ into two exclusive sets T^1, T^2 . For any point $\mathbf{x} \in \Gamma_C^h$ we have the following possibilities

- if $\mathbf{x} \in N^1$ then $u_v^h(\mathbf{x}) < 0$, which implies $\sigma_v^h(\mathbf{x}) = 0$ (points on the boundary lifted from the foundation),
- if $\mathbf{x} \in N^2$ then $u_v^h(\mathbf{x}) = 0$, which implies $-\sigma_v^h(\mathbf{x}) \in \partial q(\mathbf{x}, 0)$ (points in contact with the foundation experiencing force in normal direction below or equal to specified threshold q_{max} , i.e. in the rigid state),
- if $\mathbf{x} \in N^3$ then $u_v^h(\mathbf{x}) > 0$, which implies $-\sigma_v^h(\mathbf{x}) = p_{const} \xi + q_{max}$ (points as described above, but with the force over specified threshold q_{max} , i.e. in the flexible state),
- if $\mathbf{x} \in T^1$ then $\|\mathbf{u}_\tau^h(\mathbf{x})\| = 0$, which implies $-\sigma_\tau^h(\mathbf{x}) \in [-h_\tau, h_\tau]$ (points on the boundary experiencing the force in tangential direction with norm below or equal to the friction bound h_τ , i.e. in the stick zone),
- if $\mathbf{x} \in T^2$ then $\|\mathbf{u}_\tau^h(\mathbf{x})\| > 0$, which implies $-\sigma_\tau^h(\mathbf{x}) \in \{-h_\tau, h_\tau\}$ (points as described above, but with the norm of force over the friction bound h_τ , i.e. in the slip zone).

Initially all points are assigned to N_0^1 and T_0^1 , with subscript denoting current iteration. Then the following rules (with $\epsilon > 0$ being a small value added for numerical stability) are applied

- if $\mathbf{x} \in N_i^1$ and $u_v^h(\mathbf{x}) \in (-\infty, -\epsilon)$, then $\mathbf{x} \in N_{i+1}^1$,
- if $\mathbf{x} \in N_i^1$ and $u_v^h(\mathbf{x}) \in [-\epsilon, \infty)$, then $\mathbf{x} \in N_{i+1}^2$,
- if $\mathbf{x} \in N_i^2$ and $\sigma_v^h(\mathbf{x}) \in (-\infty, -\epsilon)$, then $\mathbf{x} \in N_{i+1}^1$,
- if $\mathbf{x} \in N_i^2$ and $\sigma_v^h(\mathbf{x}) \in [-\epsilon, q_{max} + \epsilon)$, then $\mathbf{x} \in N_{i+1}^2$,
- if $\mathbf{x} \in N_i^2$ and $\sigma_v^h(\mathbf{x}) \in [q_{max} + \epsilon, \infty)$, then $\mathbf{x} \in N_{i+1}^3$,
- if $\mathbf{x} \in N_i^3$ and $u_v^h(\mathbf{x}) \in (-\infty, -\epsilon)$, then $\mathbf{x} \in N_{i+1}^2$,
- if $\mathbf{x} \in N_i^3$ and $u_v^h(\mathbf{x}) \in [-\epsilon, \infty)$, then $\mathbf{x} \in N_{i+1}^3$,
- if $\mathbf{x} \in T_i^1 \cup T_i^2$ and $\|\sigma_\tau^h(\mathbf{x})\| < h_\tau + \epsilon$, then $\mathbf{x} \in T_{i+1}^1$,
- if $\mathbf{x} \in T_i^1 \cup T_i^2$, and $\|\sigma_\tau^h(\mathbf{x})\| \geq h_\tau + \epsilon$, then $\mathbf{x} \in T_{i+1}^2$.

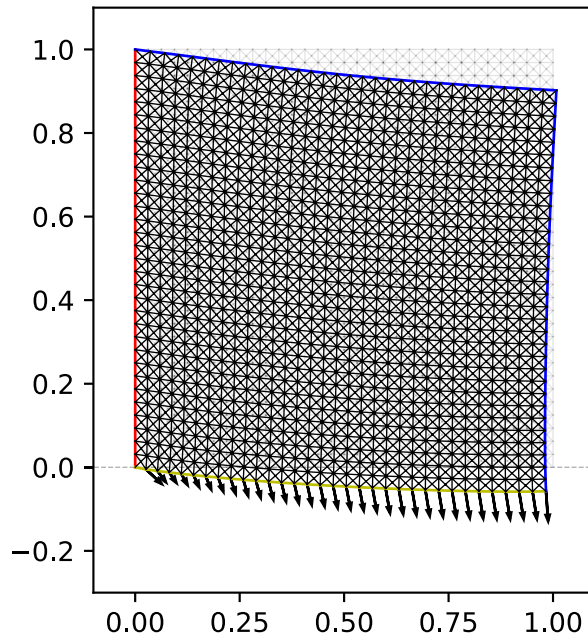


Fig. 1. Output of the augmented Lagrangian method.

Estimation of $\sigma^h(\mathbf{x})$ required for this step can be calculated from discretization of constitutive law, using values of displacement $\mathbf{u}^h$ of neighbors of $\mathbf{x}$ on the FEM mesh.

5.3. Final results

Finally, we present outputs obtained in our simulations and report an empirical estimation of the numerical errors. In examples we choose a mesh corresponding to $h = 1/32$ for better visibility. As expected, all the considered algorithms give similar solutions, so for each example we select the output of one algorithm for the illustration. We plot the deformation of the body and the forces acting on the contact surface (mirrored with respect to the boundary for visibility).

Fig. 1 presents result for data

$$\begin{aligned} \mathbf{f}_0(\mathbf{x}) &= (-0.5, -1.0), \quad \mathbf{x} \in \Omega, \\ h_\tau &= 0.1, \quad q_{\max} = 0.1, \quad p_{\text{const}} = 10. \end{aligned}$$

We push the body down and to the left with force $\mathbf{f}_0$. In this case the coefficient p_{const} has the highest influence on response of the foundation in normal direction. It causes forces to increase gradually with penetration and models a foundation made of a soft material. A small influence of friction can also be observed.

Fig. 2 presents result for data

$$\begin{aligned} \mathbf{f}_0(\mathbf{x}) &= (-0.5, -1.0), \quad \mathbf{x} \in \Omega, \\ h_\tau &= 0.1, \quad q_{\max} = 0.7, \quad p_{\text{const}} = 0. \end{aligned}$$

Here the forces in the normal direction increase up to a factor $q_{\max}$ and, because $p_{\text{const}} = 0$, stop increasing any further. The foundation response is therefore limited by this factor.

Fig. 3 presents result for data

$$\begin{aligned} \mathbf{f}_0(\mathbf{x}) &= (-0.5, -1.0), \quad \mathbf{x} \in \Omega, \\ h_\tau &= 0.5, \quad q_{\max} = 0.5, \quad p_{\text{const}} = 0. \end{aligned}$$

In this example we increase the friction bound h_τ . As we push the body to the left, points on the left side of the boundary move to the slip zone while points on the right side cannot overcome friction bound and stay in the stick zone.

Fig. 4 presents result for data

$$\begin{aligned} \mathbf{f}_0(\mathbf{x}) &= (0.5, -1.0), \quad \mathbf{x} \in \Omega, \\ h_\tau &= 0.1, \quad q_{\max} = 10, \quad p_{\text{const}} = 0. \end{aligned}$$

We change force $\mathbf{f}_0$ and, as a result, the body is pushed to the right. We also set $q_{\max} = 10$, and this effectively enforces the Signorini condition. We can see that the body cannot penetrate the foundation and we can also observe the influence of friction forces.

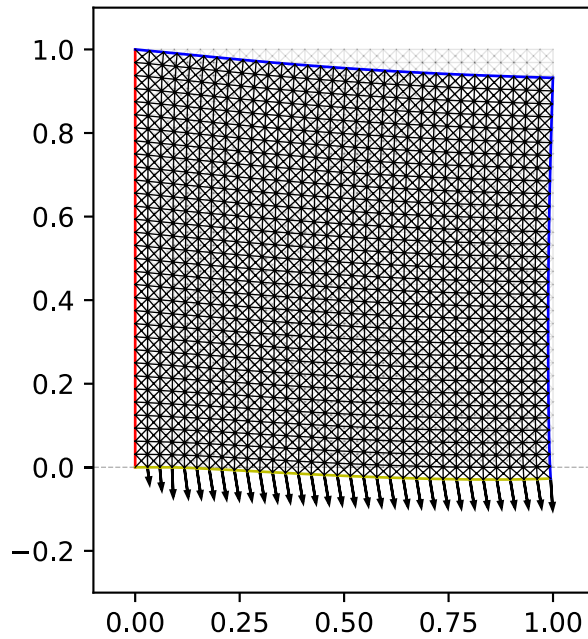


Fig. 2. Output of the direct optimization method.

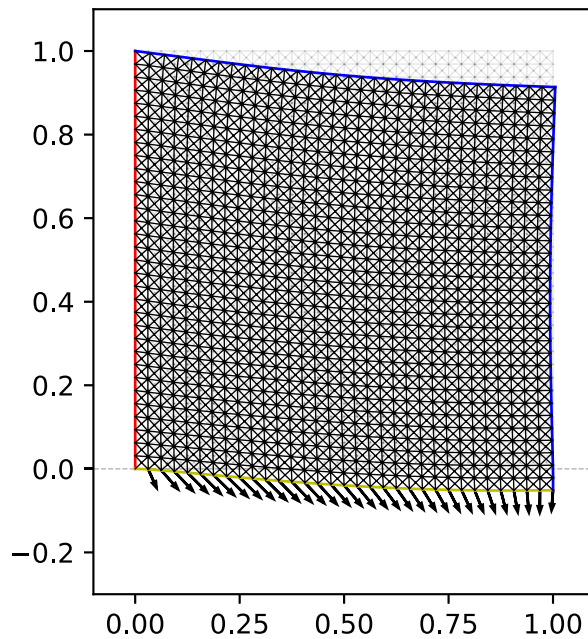


Fig. 3. Output of the primal-dual active set strategy.

A comparison of numerical errors $\|\mathbf{u} - \mathbf{u}^h\|_V$ calculated for a sequence of solutions to discretized problems on a model problem with data

$$\begin{aligned} \mathbf{f}_0(\mathbf{x}) &= (-0.8, -0.8), \quad \mathbf{x} \in \Omega, \\ h_\tau &= 0.5, \quad q_{\max} = 0.3, \quad p_{\text{const}} = 2 \end{aligned} \tag{5.1}$$

is presented in Fig. 5, where the dependence of the error estimate $\|\mathbf{u} - \mathbf{u}^h\|_V$ with respect to h is plotted on a log-log scale. Because no analytical solution can be obtained, we took three numerical estimations with $h = 1/256$ and corresponding to each presented method as such “exact” solutions. All sequences of numerical solutions with varying h were cross examined against each of “exact” solutions, giving 9 plots. We denote by u the “exact” solutions (depending on the chosen method), by u^h the sequence of numerical approximations (also for each method) and use abbreviations of the names of the presented methods. As can be seen, in this case, the primal-dual active set strategy and the direct optimization method gave similar final estimations, closer to the reference solutions than the augmented Lagrangian method.

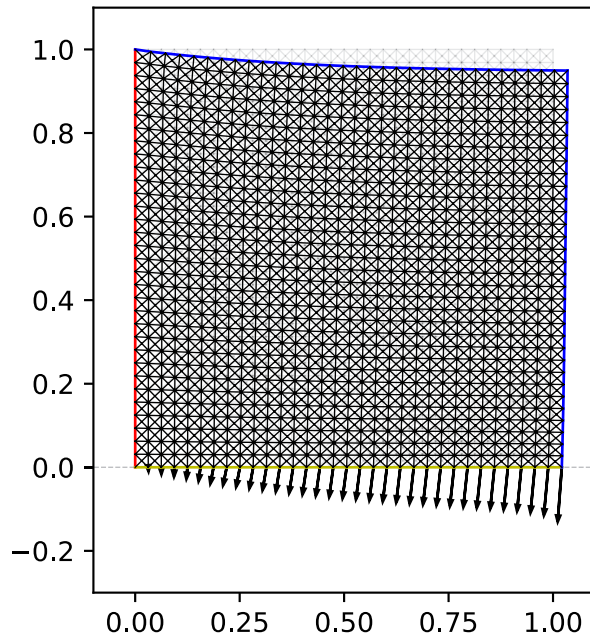


Fig. 4. Output of the direct optimization method.

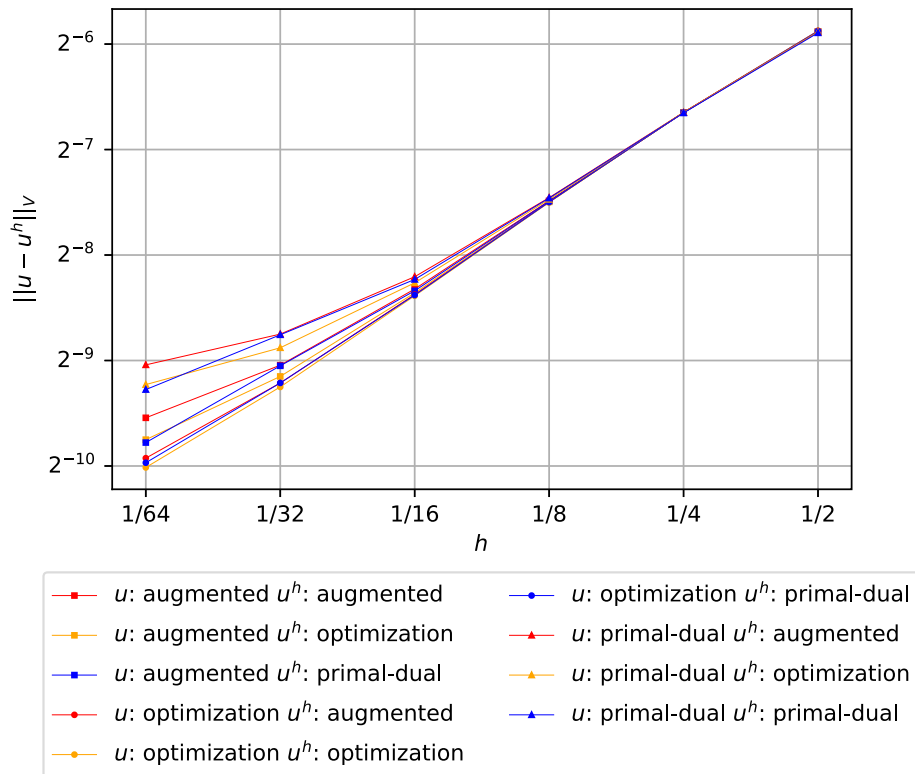


Fig. 5. Numerical errors.

In Table 1 we summarize the computation time and the number of iterations for each algorithm. The presented results do not include the time for the computation of a stiffness matrix, which is calculated beforehand and is the same for all methods. An additional metric “function evaluations” for the direct optimization denotes how many times functional $\widehat{\mathcal{L}}$ was evaluated. “Set changes” for the primal–dual method denote how many iterations of assignments to sets N and T were performed. The fastest method for finer meshes in this case was the augmented Lagrangian method, closely followed by the direct optimization method. Our implementation of the primal–dual active set strategy was fastest for the coarse meshes, but slowest for the fine mesh sizes.

Table 1

Computation time and number of iterations for each algorithm.

h		1/8	1/16	1/32	1/64	1/128
Direct optimization	Time	0.14 s	0.36 s	1.33 s	3.97 s	40.40 s
	Functional evaluations	1027	1397	2832	2748	5329
Augmented Lagrangian	Time	0.26 s	0.74 s	1.38 s	3.79 s	37.60 s
Primal–Dual	Time	0.07 s	0.19 s	0.76 s	6.96 s	178.07 s
	Set iterations	4	4	4	5	6

We remark that the easiest method to implement was the direct optimization method, followed by the primal–dual active set strategy and the augmented Lagrangian method, as it has the most complicated interpretation. As stated above, the augmented Lagrangian method simultaneously with u on Ω calculates values of σ on Γ_C , which for other methods had to be additionally estimated from the value of u . We also point out that all the results presented may vary depending on the details of specific implementations.

CRedit authorship contribution statement

Anna Ochal: supervision, Writing – review & editing. **Michał Jureczka:** Writing – original draft, software, visualization. **Piotr Bartman:** software, validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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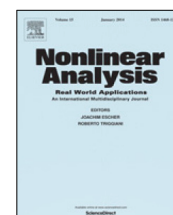
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Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem



Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem



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ABSTRACT

In this work, we analyze a non-clamped dynamic viscoelastic contact problem involving thermal effect. The friction law is described by a non-monotone relation between the tangential stress and the tangential velocity. This leads to a system of second-order inclusion for displacement and a parabolic equation for temperature. We provide a fully discrete approximation of the problem and find optimal error estimates without any smallness assumption on the data. The theoretical result is illustrated by numerical simulations.

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1. Introduction

Physical contact processes appear in many fields of industry and everyday life. Hence, it is reasonable to study mathematical models concerning various properties and parameters of physical bodies in contact with each other. In particular, thermal effects play a special role in such models. A physical body may change its properties and shape when heated. On the other hand, contact between two physical bodies is directly related to a heat exchange. Additionally, frictional contact usually leads to heat generation. It is clear that a well-formulated mathematical model should take into account all of the above phenomena.

There are numerous publications concerning mathematical analysis of contact processes with thermal effect, which essentially deal with the existence and uniqueness of a solution for the stated problems. We recommend [1–7] as representative references in this field.

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Apart from the theoretical aspect of mathematical modeling, numerical analysis also plays a significant role. Typically, it includes approximation schemes based on temporal and spatial discretization, an estimate of the error between the approximate and exact solutions, and numerical simulations. This kind of approach carried out for various mechanical contact problems with or without thermal effect is presented, for instance, in [8–17].

In this paper we deal with the dynamic contact of a viscoelastic body with a foundation involving a thermal effect. The contact is frictional, and the friction law is modeled by a non-monotone relation between the tangential velocity and the tangential stress. More precisely, the relation has the form of an inclusion involving Clarke subdifferential of a locally Lipschitz non-convex potential. It is worth mentioning that such inclusions are closely related to the concept of hemivariational inequalities (HVIs), which are used as a variational formulation of various non-monotone contact problems of mechanics. The theory and applications of HVIs have been broadly developed in the last few decades. For example, we refer to [18] for a comprehensive review of recent achievements in this field. The numerical approach to stationary HVIs based on the Finite Element Method (FEM) was first investigated in [19]. In recent years, similar techniques supported by numerical simulations have been applied to the numerical analysis of various contact processes in mechanics modeled by Clarke subdifferential inclusions or HVIs. We refer to [20–26] for further details.

The non-monotone character of the friction law introduces one of the main difficulties in our model. Despite this issue, we provide a numerical analysis of the problem based on the fully discrete scheme. The main result consists of the estimate of the error between the discrete and the exact solution. We show that the error estimate is linear with respect to the temporal and spatial discretization parameters in the case where the spatial discretization is based on the first-order FEM. We also present results from numerical simulations that illustrate and validate the linear rate of convergence with respect to the discretization parameters.

This paper provides a generalization of the result obtained in [23]. In contrast to the model studied there, we deal with the non-clamped body, which introduces additional difficulty to the problem. Furthermore, our results are obtained without smallness assumptions, which are required in [23]. These advances are possible due to the arguments contained in Corollary 3.

The remainder of the paper is organized as follows. In Section 2 we introduce some preliminary material. In Section 3 we formulate the mathematical model of the dynamic thermoviscoelastic contact problem and present its variational formulation. The main results on the error estimates for the fully discrete numerical scheme are presented in Section 4. Finally, in Section 5, we present numerical simulations of a two- and three-dimensional contact problem and provide numerical evidence of optimal order convergence for the linear finite elements.

2. Mathematical preliminaries

In this section, we briefly present the definitions and notation used in the paper. In what follows, we denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between a Banach space X and its dual X^* . Throughout the paper, we denote by C a generic constant whose value may change from line to line. The symbol $C(0, T; X)$ represents the space of continuous functions on $[0, T]$ with values in X . For a function $u: [0, T] \rightarrow X$ we denote by $\dot{u}$ and $\ddot{u}$ its first and second time derivatives, respectively.

We recall the definitions of the generalized directional derivative and the generalized Clarke subgradient for a locally Lipschitz function $\varphi: X \rightarrow \mathbb{R}$, where X is a Banach space (see [27]). The generalized directional derivative of φ at the point $x \in X$ in direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

The generalized subgradient of φ at x , denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by

$$\partial\varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X \}.$$

Let $\mathbb{R}^d$ denote an Euclidean space of dimension d , with the scalar product and the norm given by

$$u \cdot v = u_i v_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^d.$$

Hereafter, all indices i, j, k, l run between 1 and d . Moreover, we apply the so-called summation convention over repeated indices, e.g. the notation $a_i b_i$ means $\sum_{i=1}^d a_i b_i$ and $a_{ij} b_{ij}$ means $\sum_{i,j=1}^d a_{ij} b_{ij}$. We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors in $\mathbb{R}^d$, or equivalently the space of symmetric matrices $d \times d$. The scalar product and the norm in $\mathbb{S}^d$ are given by

$$\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$. For any function $u: \Omega \rightarrow \mathbb{R}^d$ such that $u = (u_i)$, $u_i: \Omega \rightarrow \mathbb{R}$, we use the symbol $u_{i,j}$ for its partial derivative $\frac{\partial u_i}{\partial x_j}$. Furthermore, we use symbols $\operatorname{div} u$ and $\varepsilon(u)$ to denote its divergence operator and the deformation operator, respectively, defined by

$$\operatorname{div} u = u_{i,i}, \quad \varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

For a function $\sigma: \Omega \rightarrow \mathbb{S}^d$, such that $\sigma = (\sigma_{ij})$, $\sigma_{ij}: \Omega \rightarrow \mathbb{R}$, we use the symbol $\operatorname{Div} \sigma$ to denote its divergence operator given by

$$\operatorname{Div} \sigma = (\sigma_{ij,j}).$$

In the sequel, we will use the following spaces of functions defined in Ω .

$$H = L^2(\Omega; \mathbb{R}^d), \quad Q = \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \}.$$

The spaces H and Q are real Hilbert spaces with inner products given by

$$(u, v)_H = \int_{\Omega} u_i v_i \, dx, \quad (\sigma, \tau)_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx.$$

The norms induced by the scalar products mentioned above are denoted by $\|\cdot\|_H$ and $\|\cdot\|_Q$, respectively.

Let $\nu = (\nu_i)$ be the outward unit normal vector at the boundary $\partial\Omega$. We denote by ν_ν and ν_τ the normal and tangential parts of the vector field $v: \partial\Omega \rightarrow \mathbb{R}^d$, defined by $\nu_\nu = v \cdot \nu$ and $\nu_\tau = v - \nu_\nu \nu$, respectively. Similarly, σ_ν and σ_τ represent the normal and tangential parts of the tensor field $\sigma: \partial\Omega \rightarrow \mathbb{S}^d$ defined by $\sigma_\nu = (\sigma \nu) \cdot \nu$ and $\sigma_\tau = \sigma - \sigma_\nu \nu$, respectively.

We now recall two Green formulas (cf. [18], (2.6) and (2.7), respectively) needed to obtain a variational formulation of the problem of contact mechanics.

$$\int_{\Omega} (u \operatorname{div} v + \nabla u \cdot v) \, dx = \int_{\Gamma} u (v \cdot \nu) \, d\Gamma \quad \text{for all } u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega; \mathbb{R}^d), \quad (1)$$

where $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(\sigma, \varepsilon(v))_Q + (\operatorname{Div} \sigma, v)_H = \int_{\Gamma} \sigma_\nu \cdot v \, d\Gamma \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^d), \sigma \in C^1(\Omega; \mathbb{S}^d). \quad (2)$$

We complete this section with the following two lemmas, which will be used throughout the paper.

Lemma 1 ([28], Lemma 8.4.12). *Let X, Y and Z be Banach spaces such that X is compactly embedded in Y , and Y is continuously embedded in Z . Then, for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\|x\|_Y \leq \varepsilon \|x\|_X + C(\varepsilon) \|x\|_Z \quad \text{for all } x \in X.$$

Lemma 2 ([29], Lemma 7.25). *Let $T > 0$ be fixed, $N > 1$ and $k = T/N$. Assume $\{e_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ be two sequences of non-negative numbers satisfying*

$$e_n \leq cg_n + c \sum_{j=1}^n ke_j \quad \text{for } n = 1, \dots, N,$$

with a positive constant c independent of N . Then there exists a positive constant $\hat{c}$, independent of N , such that

$$\max_{1 \leq n \leq N} e_n \leq \hat{c} \max_{1 \leq n \leq N} g_n.$$

3. Mechanical problem and variational formulation

In this section, we describe the mathematical model of the thermoviscoelastic dynamic contact problem in its classical and variational formulations.

We consider a body that, in its reference configuration, occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a Lipschitz boundary Γ , consisting of two parts Γ_N and Γ_C , such that $\Gamma_N \cap \Gamma_C = \emptyset$. The part of the boundary Γ_C is in contact with a foundation. We are interested in a mathematical model that describes the evolution of body displacement, stress, and temperature. Let $T > 0$ be given, and let $[0, T]$ represent the time interval considered. In what follows, we denote by $x \in \Omega$ and $t \in [0, T]$ the spatial and temporal variables, respectively. We also denote by $u = (u_i(x, t))$, $\sigma = (\sigma_{ij}(x, t))$ and $\theta = \theta(x, t)$ the displacement field, the stress field, and the temperature, respectively. These quantities are assumed to satisfy the following constitutive law (for simplicity, we omit the symbols x and t below)

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + \mathcal{B}\varepsilon(u) + C\theta \quad \text{in } \Omega \times (0, T). \quad (3)$$

The symbols $\mathcal{A}$, $\mathcal{B}$ and C in (3) represent the viscosity operator, the elasticity operator, and the heat expansion tensor, respectively. For specific examples of these operators, we refer to Section 5. The process is assumed to be dynamic, and hence the equation of motion takes the form of

$$\rho \ddot{u} = \text{Div } \sigma + f_0 \quad \text{in } \Omega \times (0, T), \quad (4)$$

where ρ is the density of the material and f_0 is the density of the volume forces. For simplicity of notation, we assume that $\rho = 1$. We assume that the traction force is applied on Γ_N , that is,

$$\sigma \nu = f_2 \quad \text{on } \Gamma_N \times (0, T). \quad (5)$$

The normal compliance condition is described by relation

$$-\sigma_\nu = p(u_\nu) \quad \text{on } \Gamma_C \times (0, T), \quad (6)$$

where p_ν is a given non-negative function that vanishes when its argument is negative. Moreover, we assume that the frictional law takes the following multivalued, possibly non-monotone form

$$-\sigma_\tau \in \partial j(\dot{u}_\tau) \quad \text{on } \Gamma_C \times (0, T), \quad (7)$$

where j is a locally Lipschitz function, and ∂j stands for its Clarke subgradient; see Section 2. In our model, we assume a simplified and linearized version of the energy law that takes the form

$$\rho c_p \dot{\theta} - \text{div}(K \nabla \theta) = c_{ij} \dot{u}_{i,j} + g \quad \text{on } \Omega \times (0, T), \quad (8)$$

where c_p is the heat capacity, K is the thermal conductivity and g represents the heat sources. We assume that the heat transfer between the body and the foundation is described by the law

$$-K\nabla\theta \cdot \nu = -r(\theta) - h(\|\dot{u}_\tau\|_{\mathbb{R}^d}) \quad \text{on } \Gamma_C \times (0, T). \quad (9)$$

Finally, we impose the initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (10)$$

The mechanical problem reads as follows

Problem. $\mathcal{P}_{\mathcal{M}}$. Find the displacement $u: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, the stress $\sigma: \Omega \times [0, T] \rightarrow \mathbb{S}^d$ and the temperature $\theta: \Omega \times [0, T] \rightarrow \mathbb{R}$ that satisfy conditions (3)–(10).

In the study of Problem $\mathcal{P}_{\mathcal{M}}$ we need the following assumptions on its data.

$H(\mathcal{A})$: The viscosity operator $\mathcal{A}: \Omega \times [0, T] \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfies

- (a) $\mathcal{A}(\cdot, \cdot, \varepsilon)$ is continuous on $\Omega \times [0, T]$ for all $\varepsilon \in \mathbb{S}^d$;
- (b) $(\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}^2$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega \times (0, T)$ with $m_{\mathcal{A}} > 0$;
- (c) $\|\mathcal{A}(x, t, \varepsilon_1) - \mathcal{A}(x, t, \varepsilon_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, a.e. $(x, t) \in \Omega \times (0, T)$ with $L_{\mathcal{A}} > 0$.

$H(\mathcal{B})$: The elasticity operator $\mathcal{B}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is bounded, symmetric, positive fourth-order tensor, i.e.

- (a) $\mathcal{B}_{ijkl} \in L^\infty(\Omega)$, $1 \leq i, j, k, l \leq d$;
- (b) $\mathcal{B}\sigma : \tau = \sigma : \mathcal{B}\tau$ for all $\sigma, \tau \in \mathbb{S}^d$, a.e. in Ω ;
- (c) $\mathcal{B}\tau : \tau \geq 0$ for all $\tau \in \mathbb{S}^d$, a.e. in Ω .

$H(C)$: The thermal expansion tensor $C = (c_{ij})$ satisfies

- (a) $c_{ij} \in L^\infty(\Omega)$, $1 \leq i, j \leq d$;
- (b) $c_{ij} = c_{ji}$, $1 \leq i, j \leq d$.

$H(K)$: The thermal conductivity operator $K: \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

- (a) $K(\cdot, \cdot, \xi)$ is continuous on $\Omega \times [0, T]$ for all $\xi \in \mathbb{R}^d$;
- (b) $(K(x, t, \xi_1) - K(x, t, \xi_2)) \cdot (\xi_1 - \xi_2) \geq m_K \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2$ for all $\xi_1, \xi_2 \in \mathbb{R}^d$, a.e. $(x, t) \in \Omega \times (0, T)$ with $m_K > 0$;
- (c) $\|K(x, t, \xi_1) - K(x, t, \xi_2)\|_{\mathbb{R}^d} \leq L_K \|\xi_1 - \xi_2\|_{\mathbb{R}^d}$ for all $\xi_1, \xi_2 \in \mathbb{R}^d$, a.e. $(x, t) \in \Omega \times (0, T)$ with $L_K > 0$.

$H(p)$: The function $p: \Gamma_C \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $p(\cdot, \cdot, \xi)$ is measurable on $\Gamma_C \times (0, T)$ for all $\xi \in \mathbb{R}^d$;
- (b) $|p(x, t, \xi_1) - p(x, t, \xi_2)| \leq L_p |\xi_1 - \xi_2|$ for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $(x, t) \in \Gamma_C \times (0, T)$ with $L_p > 0$.

$H(j)$: The potential $j: \Gamma_C \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies

- (a) $j(\cdot, \cdot, \xi)$ is measurable on $\Gamma_C \times (0, T)$ for all $\xi \in \mathbb{R}^d$;
- (b) $j(x, t, \cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $(x, t) \in \Gamma_C \times (0, T)$;
- (c) $\|\partial j(x, t, \xi)\|_{\mathbb{R}^d} \leq c_j$ for all $\xi \in \mathbb{R}^d$, a.e. $(x, t) \in \Gamma_C \times (0, T)$ with $c_j \geq 0$;
- (d) $(\zeta_1 - \zeta_2) \cdot (\xi_1 - \xi_2) \geq -m_j \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2$ for all $\zeta_i \in \partial j(x, t, \xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(x, t) \in \Gamma_C \times (0, T)$ with $m_j \geq 0$.

$H(r)$: The function $r: \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

- (a) $r(\cdot, \zeta) \in L^2(\Gamma_C)$ for all $\zeta \in \mathbb{R}$;

(b) $|r(x, \zeta_1) - r(x, \zeta_2)| \leq L_r |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $L_r > 0$.

$H(h)$: The function $h: \Gamma_C \times [0, \infty) \rightarrow [0, \infty)$ satisfies

(a) $h(\cdot, \zeta) \in L^2(\Gamma_C)$ for all $\zeta \in [0, \infty)$;

(b) $|h(x, \zeta_1) - h(x, \zeta_2)| \leq L_h |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in [0, \infty)$, a.e. $x \in \Gamma_C$ with $L_h > 0$.

$H(f)$: The volume forces and traction densities satisfy

$$f_0 \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad f_2 \in C(0, T; L^2(\Gamma_N; \mathbb{R}^d)).$$

$H(g)$: The heat source satisfies

$$g \in C(0, T; L^2(\Omega)).$$

We remark that the result presented in the sequel can be obtained under a weaker assumption than $H(j)(c)$. Namely it is enough to assume a sublinear growth of the subgradient ∂j with respect to the last variable. However, for the sake of simplicity we assume that it is bounded.

In order to provide a variational formulation of the problem, we define spaces

$$E = H^1(\Omega), \quad V = H^1(\Omega; \mathbb{R}^d),$$

where E is equipped with the classical H^1 norm, and in V we define the inner product

$$(u, v)_V = (u, v)_H + (\varepsilon(u), \varepsilon(v))_Q \quad \text{for all } u, v \in V$$

and the corresponding norm $\|u\|^2 := (u, u)_V$ for all $u \in V$. It follows that the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega; \mathbb{R}^d)}$ are equivalent. We know that the embedding $i: E \rightarrow H^{1-\delta}(\Omega)$ is compact for $\delta \in (0, \frac{1}{2})$, and the trace operator $\bar{\gamma}: H^{1-\delta}(\Omega) \rightarrow L^2(\Gamma)$ is continuous. Hence, we see that $\gamma = \bar{\gamma}i: E \rightarrow L^2(\Gamma)$ is compact. We omit the notation of the embedding and the trace operator, and write simply v instead of γv for $v \in E$. We proceed similarly for the space V . We introduce the following spaces of vector-valued functions

$$\mathbb{V} = \{v \in L^2(0, T; V) \mid \dot{v} \in L^2(0, T; V^*)\}, \quad \mathbb{E} = \{v \in L^2(0, T; E) \mid \dot{v} \in L^2(0, T; E^*)\},$$

where $\dot{v}$ denotes time derivative of v in the sense of distributions.

Next, we define operators $A: (0, T) \times V \rightarrow V^*$, $B: V \rightarrow V^*$, $C_1: E \rightarrow V^*$, $C_2: (0, T) \times E \rightarrow E^*$, $C_3: V \rightarrow E^*$ by

$$\langle A(t, u), v \rangle_V = (\mathcal{A}(t, \varepsilon(u)), \varepsilon(v))_Q \quad \text{for all } u, v \in V, \quad t \in (0, T),$$

$$\langle Bu, v \rangle_V = (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q \quad \text{for all } u, v \in V,$$

$$\langle C_1\theta, v \rangle_V = \int_{\Omega} c_{ij} v_{i,j} \theta \, dx \quad \text{for all } v \in V, \theta \in E,$$

$$\langle C_2(t, \theta), \eta \rangle_E = (K(t, \nabla\theta), \nabla\eta)_{L^2(\Omega)} \quad \text{for all } \theta, \eta \in E,$$

$$\langle C_3v, \eta \rangle_E = - \int_{\Omega} c_{ij} v_{i,j} \eta \, dx \quad \text{for all } v \in V, \eta \in E.$$

We define the functions $f: (0, T) \rightarrow V^*$ and $\tilde{g}: (0, T) \rightarrow E^*$ for all $v \in V$, $\eta \in E$, a.e. $t \in (0, T)$ by

$$\langle f(t), v \rangle_{V^* \times V} = \langle f_0(t), v \rangle_H + \langle f_2(t), v \rangle_{L^2(\Gamma_N; \mathbb{R}^d)},$$

$$\langle \tilde{g}(t), \eta \rangle_E = \langle g(t), \eta \rangle_{L^2(\Omega)}.$$

We also introduce an additional hypothesis on the data

$$\underline{H_0} : u_0 \in V, v_0 \in H, \theta_0 \in E.$$

Now we are in a position to present a weak formulation of Problem $\mathcal{P}_{\mathcal{M}}$.

Problem. $\mathcal{P}_{\mathcal{E}}$. Find a displacement $u \in L^2(0, T; V)$ with $\dot{u} \in \mathbb{V}$, and a temperature $\theta \in \mathbb{E}$ such that

$$\langle \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) + C_1\theta(t) - f(t), w \rangle_V = - \int_{\Gamma_C} p(u_\nu)w_\nu d\Gamma \quad (11)$$

$$- \int_{\Gamma_C} \xi(t) \cdot w_\tau d\Gamma \quad \text{for all } w \in V, \text{ a.e. } t \in (0, T),$$

$$\langle \dot{\theta}(t) + C_2(t, \theta(t)) + C_3\dot{u}(t) - \tilde{g}(t), \eta \rangle_E = \int_{\Gamma_C} r(\theta)\eta d\Gamma \quad (12)$$

$$+ \int_{\Gamma_C} h(\|\dot{u}_\tau(t)\|_{\mathbb{R}^d})\eta d\Gamma \quad \text{for all } \eta \in E, \text{ a.e. } t \in (0, T),$$

$$\xi \in \partial j(\dot{u}_\tau), \text{ a.e. } (x, t) \in \Gamma_C \times (0, T), \quad (13)$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \theta(0) = \theta_0.$$

Problem $\mathcal{P}_{\mathcal{E}}$ can be obtained from Eqs. (4) and (8) by multiplying them by test functions $w \in V$ and $\eta \in E$, respectively, performing the Green formulas (1) and (2), applying conditions (3)–(10), using definitions of operators above and denoting $\xi = \sigma_\tau$. Every solution of Problem $\mathcal{P}_{\mathcal{E}}$ is called a weak solution of Problem $\mathcal{P}_{\mathcal{M}}$.

Existence and uniqueness of the solution to Problem $\mathcal{P}_{\mathcal{E}}$ can be proved following the proof of Theorem 9 in [30] and using Corollary 3. Since this paper is devoted to numerical analysis, we omit this proof. For more details on non-clamped problems see e.g. [31] and for problems with no smallness assumption see e.g. [32].

Corollary 3. *Lemma 1 holds for spaces $X = H^1(\Omega)$, $Y = H^{1-\delta}(\Omega)$, $Z = L^2(\Omega)$ for $\delta \in (0, 1/2)$. Moreover, from the trace theorem, we have $\|v\|_{L^2(\Gamma)} \leq C\|v\|_{H^{1-\delta}(\Omega)}$, hence for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\|v\|_{L^2(\Gamma)} \leq \varepsilon\|v\|_E + C(\varepsilon)\|v\|_{L^2(\Omega)} \quad \text{for all } v \in E.$$

Similarly, we obtain

$$\|v\|_{L^2(\Gamma; \mathbb{R}^d)} \leq \varepsilon\|v\|_V + C(\varepsilon)\|v\|_{L^2(\Omega; \mathbb{R}^d)} \quad \text{for all } v \in V.$$

4. Discretization and error estimates

In this section we consider a fully discrete approximation of Problem $\mathcal{P}_{\mathcal{E}}$ and examine error estimates for the approximate solution and the exact one.

Let E^h and V^h be finite-dimensional subspaces of E and V , respectively, with $h > 0$ denoting the spatial discretization parameter. Let $u_0^h, v_0^h \in V^h$ and $\theta_0^h \in E^h$ be suitable approximations of u_0, v_0, θ_0 . We define a uniform partition of $[0, T]$ denoted by $0 = t_0 < t_1 < \dots < t_N = T$, where $N \in \mathbb{N}$, $t_j = jk$ for $j = 1, \dots, N$ and $k = \frac{T}{N}$ is the time step. For a continuous function φ we use the notation $\varphi_n = \varphi(t_n)$. For a sequence $\{z_n\}_{n=0}^N$ we denote by $\delta z_n = (z_n - z_{n-1})/k$ for $n = 1, \dots, N$ its backward divided difference.

Let (u, θ) be a solution of Problem $\mathcal{P}_{\mathcal{E}}$ and $v = \dot{u}$. The fully discrete approximation of Problem $\mathcal{P}_{\mathcal{E}}$ is the following.

Problem. $\mathcal{P}_{\mathcal{E}}^{hk}$. Find sequences of velocity $\{v_n^{hk}\}_{n=1}^N \subset V^h$ and temperature $\{\theta_n^{hk}\}_{n=1}^N \subset E^h$ such that there exists $\{\xi_n^{hk}\}_{n=1}^N \subset L^2(\Gamma_C; \mathbb{R}^d)$ satisfying for $j = 1, \dots, N$

$$\langle \delta v_j^{hk} + A_j(v_j^{hk}) + Bu_j^{hk} + C_1\theta_j^{hk} - f_j, w^h \rangle_V = - \int_{\Gamma_C} p(u_{j\nu}^{hk}) \cdot w_\nu^h d\Gamma \quad (14)$$

$$- \int_{\Gamma_C} \xi_j^{hk} \cdot w_\tau^h d\Gamma \quad \text{for all } w^h \in V^h, \text{ a.e. } t \in (0, T),$$

$$\langle \delta \theta_j^{hk} + C_2(\theta_j^{hk}) + C_3 v_j^{hk} - \tilde{g}_j, \eta^h \rangle_E = \int_{\Gamma_C} r(\theta_j^{hk}) \eta^h d\Gamma \quad (15)$$

$$+ \int_{\Gamma_C} h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) \eta^h d\Gamma \quad \text{for all } \eta^h \in E^h, \\ \xi_j^{hk} \in \partial j(v_{j\tau}^{hk}), \quad (16)$$

where $v_0^{hk} = v_0^h$, $\theta_0^{hk} = \theta_0^h$ and $\{u_n^{hk}\}_{n=1}^N \subset V^h$ is given by

$$u_n^{hk} = u_0^{hk} + \sum_{j=1}^n k v_j^{hk} \quad \text{for } n = 1, \dots, N \quad (17)$$

with $u_0^{hk} = u_0^h$.

Now we deal with the error analysis for the discrete scheme, Problem $\mathcal{P}_{\mathcal{E}}^{hk}$.

Let $\{u_n^{hk}\}_{n=1}^N$, $\{v_n^{hk}\}_{n=1}^N$ and $\{\theta_n^{hk}\}_{n=1}^N$ denote the discrete approximation of displacement u , velocity v and temperature θ , respectively, derived by the numerical scheme in Problem $\mathcal{P}_{\mathcal{E}}^{hk}$. We prove the following estimate.

Theorem 4. *Let the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$, $H(C)$, $H(K)$, $H(j)$, $H(r)$, $H(h)$, $H(f)$, $H(g)$, and H_0 hold. Moreover, we impose the following additional regularity conditions $u \in C^2(0, T; H) \cap C^1(0, T; V)$, $\dot{u}_\tau \in C(0, T; L^2(\Gamma_C; \mathbb{R}^d))$, $\theta \in C(0, T; L^2(\Gamma_C))$. Then, there exists $C > 0$ such that for all $\{w_n^h\}_{n=1}^N \subset V^h$ and $\{\eta_n^h\}_{n=1}^N \subset E^h$ we have*

$$\begin{aligned} & \max_{1 \leq n \leq N} \{ \|v_n - v_n^{hk}\|_H^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 \} + k \sum_{j=1}^N \|v_j - v_j^{hk}\|_V^2 + k \sum_{j=1}^N \|\theta_j - \theta_j^{hk}\|_E^2 \\ & \leq C \left[k \sum_{j=1}^N (\|\dot{v}_j - \delta v_j\|_H^2 + \|v_j - w_j^h\|_V^2) + \max_{1 \leq n \leq N} \|v_{n\tau} - w_{n\tau}^h\|_{L^2(\Gamma_C; \mathbb{R}^d)} \right. \\ & + \frac{1}{k} \sum_{j=1}^{N-1} \|(v_j - w_j^h) - (v_{j+1} - w_{j+1}^h)\|_H^2 + \max_{1 \leq n \leq N} \|v_n - w_n^h\|_H^2 \\ & + \|u_0 - u_0^h\|_V^2 + \|v_0 - v_0^h\|_H^2 + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}^2 + k^2 \|u\|_{H^2(0, T; V)}^2 \\ & + k \sum_{j=1}^N (\|\dot{\theta}_j - \delta \theta_j\|_{L^2(\Omega)}^2 + \|\theta_j - \eta_j^h\|_E^2) \\ & \left. + \frac{1}{k} \sum_{j=1}^{N-1} \|(\theta_j - \eta_j^h) - (\theta_{j+1} - \eta_{j+1}^h)\|_{L^2(\Omega)}^2 + \max_{1 \leq n \leq N} \|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2 \right]. \quad (18) \end{aligned}$$

Proof. Take $w^h \in V^h$ and $\eta^h \in E^h$ as test functions in both Problem $\mathcal{P}_{\mathcal{E}}$ and Problem $\mathcal{P}_{\mathcal{E}}^{hk}$, respectively, and subtract relevant equations to obtain

$$\begin{aligned} & (\dot{v}_j - \delta v_j^{hk}, w^h)_H + \langle A_j(v_j) - A_j(v_j^{hk}), w^h \rangle_V + \langle B(u_j - u_j^{hk}), w^h \rangle_V \\ & + \langle C_1(\theta_j - \theta_j^{hk}), w^h \rangle_V + \int_{\Gamma_C} (p(u_{j\nu}) - p(u_{j\nu}^{hk})) \cdot w_\nu^h d\Gamma + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot w_\tau^h d\Gamma \\ & + (\dot{\theta}_j - \delta \theta_j^{hk}, \eta^h)_{L^2(\Omega)} + \langle C_2 \theta_j - C_2 \theta_j^{hk}, \eta^h \rangle_E + \langle C_3(v_j - v_j^{hk}), \eta^h \rangle_E \\ & + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) \eta^h d\Gamma + \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) \eta^h d\Gamma = 0 \quad (19) \end{aligned}$$

for $j = 1, \dots, N$. Let $\{w_j^h\}_{j=1}^N \subset V$ and $\{\eta_j^h\}_{j=1}^N \subset E$ be arbitrary. Applying (19) with $w^h = v_j^{hk}$, $\eta^h = \theta_j^{hk}$ and then with $w^h = w_j^h$, $\eta^h = \eta_j^h$, we find

$$\begin{aligned} & (\dot{v}_j - \delta v_j^{hk}, v_j - v_j^{hk})_H + \langle A_j(v_j) - A_j(v_j^{hk}), v_j - v_j^{hk} \rangle_V \\ & + \langle B(u_j - u_j^{hk}), v_j - v_j^{hk} \rangle_V + \langle C_1(\theta_j - \theta_j^{hk}), v_j - v_j^{hk} \rangle_V + \langle \dot{\theta}_j - \delta \theta_j^{hk}, \theta_j - \theta_j^{hk} \rangle_E \\ & + \langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \theta_j^{hk} \rangle_E + \langle C_3(v_j - v_j^{hk}), \theta_j - \theta_j^{hk} \rangle_E \\ & + \int_{\Gamma_C} (p(u_{j\nu}) - p(u_{j\nu}^{hk})) \cdot (v_{j\nu} - v_{j\nu}^{hk}) d\Gamma + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot (v_{j\tau} - v_{j\tau}^{hk}) d\Gamma \\ & + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \theta_j^{hk}) d\Gamma + \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) (\theta_j - \theta_j^{hk}) d\Gamma \\ & = (\dot{v}_j - \delta v_j^{hk}, v_j - w_j^h)_H + \langle A_j(v_j) - A_j(v_j^{hk}), v_j - w_j^h \rangle_V \\ & + \langle B(u_j - u_j^{hk}), v_j - w_j^h \rangle_V + \langle C_1(\theta_j - \theta_j^{hk}), v_j - w_j^h \rangle_V + \langle \dot{\theta}_j - \delta \theta_j^{hk}, \theta_j - \eta_j^h \rangle_E \\ & + \langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \eta_j^h \rangle_E + \langle C_3(v_j - v_j^{hk}), \theta_j - \eta_j^h \rangle_E \\ & + \int_{\Gamma_C} (p(u_{j\nu}) - p(u_{j\nu}^{hk})) \cdot (v_{j\nu} - w_{j\nu}^h) d\Gamma + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot (v_{j\tau} - w_{j\tau}^h) d\Gamma \\ & + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \eta_j^h) d\Gamma + \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) (\theta_j - \eta_j^h) d\Gamma. \end{aligned}$$

Using the fact that $\langle C_1 \eta, w \rangle_V + \langle C_3 w, \eta \rangle_E = 0$ for all $w \in V$, $\eta \in E$, we get

$$\begin{aligned} & (\delta v_j - \delta v_j^{hk}, v_j - v_j^{hk})_H + \langle A_j(v_j) - A_j(v_j^{hk}), v_j - v_j^{hk} \rangle_V \\ & + (\delta \theta_j - \delta \theta_j^{hk}, \theta_j - \theta_j^{hk})_{L^2(\Omega)} + \langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \theta_j^{hk} \rangle_E \\ & + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot (v_{j\tau} - v_{j\tau}^{hk}) d\Gamma + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \theta_j^{hk}) d\Gamma \\ & + \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) (\theta_j - \theta_j^{hk}) d\Gamma \\ & = (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_H + (\dot{v}_j - \delta v_j, (v_j - w_j^h) + (v_j^{hk} - v_j))_H \\ & + \langle A_j(v_j) - A_j(v_j^{hk}), v_j - w_j^h \rangle_V + \langle B(u_j - u_j^{hk}), (v_j - w_j^h) + (v_j^{hk} - v_j) \rangle_V \\ & + \langle C_1(\theta_j - \theta_j^{hk}), v_j - w_j^h \rangle_V + (\delta \theta_j - \delta \theta_j^{hk}, \theta_j - \eta_j^h)_{L^2(\Omega)} \\ & + (\dot{\theta}_j - \delta \theta_j, (\theta_j - \eta_j^h) + (\theta_j^{hk} - \theta_j))_{L^2(\Omega)} + \langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \eta_j^h \rangle_E \\ & + \langle C_3(v_j - v_j^{hk}), \theta_j - \eta_j^h \rangle_E + \int_{\Gamma_C} (p(u_{j\nu}) - p(u_{j\nu}^{hk})) \cdot ((v_{j\nu} - w_{j\nu}^h) + (v_{j\nu}^{hk} - v_{j\nu})) d\Gamma \\ & + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot (v_{j\tau} - w_{j\tau}^h) d\Gamma + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \eta_j^h) d\Gamma \\ & + \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) (\theta_j - \eta_j^h) d\Gamma. \end{aligned} \tag{20}$$

From the formula $2\langle a - b, a \rangle_H = \|a - b\|_H^2 + \|a\|_H^2 - \|b\|_H^2$ for $a, b \in H$, we get

$$(\delta v_j - \delta v_j^{hk}, v_j - v_j^{hk})_H \geq \frac{1}{2k} (\|v_j - v_j^{hk}\|_H^2 - \|v_{j-1} - v_{j-1}^{hk}\|_H^2), \tag{21}$$

$$(\delta \theta_j - \delta \theta_j^{hk}, \theta_j - \theta_j^{hk})_{L^2(\Omega)} \geq \frac{1}{2k} (\|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 - \|\theta_{j-1} - \theta_{j-1}^{hk}\|_{L^2(\Omega)}^2). \tag{22}$$

By $H(\mathcal{A})(b)$, $H(j)(d)$ and Corollary 3, we get

$$\begin{aligned} & \langle A_j(v_j) - A_j(v_j^{hk}), v_j - v_j^{hk} \rangle_V + \int_{\Gamma_C} (\xi_j - \xi_j^{hk}) \cdot (v_{j\tau} - v_{j\tau}^{hk}) d\Gamma \\ & \geq m_{\mathcal{A}} \|v_j - v_j^{hk}\|_V^2 - m_{\mathcal{A}} \|v_j - v_j^{hk}\|_H^2 - m_j \|v_j - v_j^{hk}\|_{L^2(\Gamma_C; \mathbb{R}^d)}^2 \\ & \geq (m_{\mathcal{A}} - \varepsilon) \|v_j - v_j^{hk}\|_V^2 - (m_{\mathcal{A}} + C(\varepsilon)) \|v_j - v_j^{hk}\|_H^2. \end{aligned} \tag{23}$$

Similarly, we obtain

$$\begin{aligned}
 & \langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \theta_j^{hk} \rangle_E + \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \theta_j^{hk}) d\Gamma \\
 & + \int_{\Gamma_C} (h(\|v_j^{hk}\|_{\mathbb{R}^d}) - h(\|v_j\|_{\mathbb{R}^d})) (\theta_j - \theta_j^{hk}) d\Gamma \\
 & \geq m_K \|\theta_j - \theta_j^{hk}\|_E^2 - m_K \|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 \\
 & - (L_r + \frac{1}{2}) \|\theta_j - \theta_j^{hk}\|_{L^2(\Gamma)}^2 - \frac{L_h^2}{2} \|v_j - v_j^{hk}\|_{L^2(\Gamma_C)}^2 \\
 & \geq (m_K - \varepsilon) \|\theta_j - \theta_j^{hk}\|_E^2 - (m_K + C(\varepsilon)) \|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 \\
 & - \varepsilon \|v_j - v_j^{hk}\|_V^2 - C(\varepsilon) \|v_j - v_j^{hk}\|_H^2.
 \end{aligned} \tag{24}$$

Next, from the Lipschitz continuity of $\mathcal{A}$ and K , we have

$$\langle A_j(v_j) - A_j(v_j^{hk}), v_j - w_j^h \rangle_V \leq \varepsilon \|v_j - v_j^{hk}\|_V^2 + C(\varepsilon) \|v_j - w_j^h\|_V^2, \tag{25}$$

$$\langle C_2 \theta_j - C_2 \theta_j^{hk}, \theta_j - \eta_j^h \rangle_E \leq \varepsilon \|\theta_j - \theta_j^{hk}\|_E^2 + C(\varepsilon) \|\theta_j - \eta_j^h\|_E^2. \tag{26}$$

By the linearity of B , C_1 and C_3 , we obtain

$$\begin{aligned}
 & \langle B(u_j - u_j^{hk}), (v_j - w_j^h) + (v_j^{hk} - v_j) \rangle_V \leq \varepsilon \|v_j - v_j^{hk}\|_V^2 \\
 & + C(\varepsilon) \|u_j - u_j^{hk}\|_V^2 + C \|v_j - w_j^h\|_V^2,
 \end{aligned} \tag{27}$$

$$\langle C_1(\theta_j - \theta_j^{hk}), v_j - w_j^h \rangle_V \leq \varepsilon \|\theta_j - \theta_j^{hk}\|_E^2 + C(\varepsilon) \|v_j - w_j^h\|_V^2, \tag{28}$$

$$\langle C_3(v_j - v_j^{hk}), \theta_j - \eta_j^h \rangle_E \leq \varepsilon \|v_j - v_j^{hk}\|_V^2 + C(\varepsilon) \|\theta_j - \eta_j^h\|_E^2. \tag{29}$$

From $H(p)$, and the continuity of the trace operator $\gamma: V \rightarrow L^2(\Gamma; \mathbb{R}^d)$, we deduce

$$\begin{aligned}
 & \int_{\Gamma_C} (p(u_{j\nu}) - p(u_{j\nu}^{hk})) \cdot ((v_{j\nu} - w_{j\nu}^h) + (v_{j\nu}^{hk} - v_{j\nu})) d\Gamma \\
 & \leq \varepsilon \|v_{j\nu} - v_{j\nu}^{hk}\|_{L^2(\Gamma_C)}^2 + C(\varepsilon) \|u_{j\nu} - u_{j\nu}^{hk}\|_{L^2(\Gamma_C)}^2 + C \|v_{j\nu} - w_{j\nu}^h\|_{L^2(\Gamma_C)}^2
 \end{aligned} \tag{30}$$

$$\leq \varepsilon \|v_j - v_j^{hk}\|_V^2 + C(\varepsilon) \|u_j - u_j^{hk}\|_V^2 + C \|v_j - w_j^h\|_V^2. \tag{31}$$

By $H(j)(c)$, $H(r)(b)$, $H(h)(b)$, and the continuity of the trace operator $\gamma: E \rightarrow L^2(\Gamma)$, we get

$$\begin{aligned}
 & \int_{\Gamma_C} (\xi_j^{hk} - \xi_j) \cdot (v_{j\tau} - w_{j\tau}^h) d\Gamma \leq 2c_j \sqrt{\text{meas}(\Gamma_C)} \|v_{j\tau} - w_{j\tau}^h\|_{L^2(\Gamma_C; \mathbb{R}^d)} \\
 & \leq C \|v_j - w_j^h\|_{L^2(\Gamma_C; \mathbb{R}^d)},
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 & \int_{\Gamma_C} (r(\theta_j^{hk}) - r(\theta_j)) (\theta_j - \eta_j^h) d\Gamma \leq L_r \|\theta_j^{hk} - \theta_j\|_{L^2(\Gamma)} \|\theta_j - \eta_j^h\|_{L^2(\Gamma_C)} \\
 & \leq \varepsilon \|\theta_j^{hk} - \theta_j\|_E^2 + C(\varepsilon) \|\theta_j - \eta_j^h\|_E^2,
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 & \int_{\Gamma_C} (h(\|v_{j\tau}^{hk}\|_{\mathbb{R}^d}) - h(\|v_{j\tau}\|_{\mathbb{R}^d})) (\theta_j - \eta_j^h) d\Gamma \\
 & \leq \varepsilon \|v_j^{hk} - v_j\|_V^2 + C(\varepsilon) \|\theta_j - \eta_j^h\|_E^2.
 \end{aligned} \tag{34}$$

The symbol $\text{meas}(\Gamma_C)$ in (32) represents the Lebesgue surface measure of Γ_C . Finally, we estimate

$$\begin{aligned}
 & (\dot{v}_j - \delta v_j, (v_j - w_j^h) + (v_j^{hk} - v_j))_H \\
 & \leq \|\dot{v}_j - \delta v_j\|_H^2 + \frac{1}{2} \|v_j - w_j^h\|_V^2 + \frac{1}{2} \|v_j^{hk} - v_j\|_H^2,
 \end{aligned} \tag{35}$$

$$\begin{aligned} & (\dot{\theta}_j - \delta\theta_j, (\theta_j - \eta_j^h) + (\theta_j^{hk} - \theta_j))_{L^2(\Omega)} \\ & \leq \|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\theta_j - \eta_j^h\|_E^2 + \frac{1}{2}\|\theta_j^{hk} - \theta_j\|_{L^2(\Omega)}^2. \end{aligned} \quad (36)$$

Applying (21)–(36) in (20), we deduce

$$\begin{aligned} & \frac{1}{2k}(\|v_j - v_j^{hk}\|_H^2 - \|v_{j-1} - v_{j-1}^{hk}\|_H^2) + (m_A - 7\varepsilon)\|v_j - v_j^{hk}\|_V^2 \\ & + \frac{1}{2k}(\|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 - \|\theta_{j-1} - \theta_{j-1}^{hk}\|_{L^2(\Omega)}^2) + (m_K - 4\varepsilon)\|\theta_j - \theta_j^{hk}\|_E^2 \\ & \leq C\left(\|\dot{v}_j - \delta v_j\|_H^2 + \|v_j - w_j^h\|_V^2 + \|u_j - u_j^{hk}\|_V^2 + \|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)}^2\right. \\ & \quad \left. + \|\theta_j - \eta_j^h\|_E^2 + \|v_j - w_j^h\|_{L^2(\Gamma_C; \mathbb{R}^d)} + \|v_j - v_j^{hk}\|_H^2\right. \\ & \quad \left. + \|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2\right) + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_H + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h)_{L^2(\Omega)}. \end{aligned}$$

Summing up the last inequality from $j = 1$ to $j = n$, and taking into account that $v_0 = \dot{u}(0) = v_0$, we obtain the following

$$\begin{aligned} & \|v_n - v_n^{hk}\|_H^2 + 2k(m_A - 7\varepsilon)\sum_{j=1}^n\|v_j - v_j^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 \\ & + 2k(m_K - 4\varepsilon)\sum_{j=1}^n\|\theta_j - \theta_j^{hk}\|_E^2 \leq \|v_0 - v_0^{hk}\|_V^2 + \|\theta_0 - \theta_0^{hk}\|_E^2 \\ & + Ck\sum_{j=1}^n\left(\|\dot{v}_j - \delta v_j\|_H^2 + \|v_j - w_j^h\|_V^2 + \|u_j - u_j^{hk}\|_V^2 + \|\dot{\theta}_j - \delta\theta_j\|_{L^2(\Omega)}^2\right) \\ & + Ck\sum_{j=1}^n\left(\|\theta_j - \eta_j^h\|_E^2 + \|v_j - w_j^h\|_{L^2(\Gamma_C; \mathbb{R}^d)} + \|v_j - v_j^{hk}\|_H^2 + \|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2\right) \\ & + 2k\sum_{j=1}^n(\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_H + 2k\sum_{j=1}^n(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h)_{L^2(\Omega)} \end{aligned}$$

for all $\{w_j^h\}_{j=1}^n \subset V^h$ and $\{\eta_j^h\}_{j=1}^n \subset E^h$. We have

$$\begin{aligned} & \sum_{j=1}^n k(\delta v_j - \delta v_j^{hk}, v_j - w_j^h)_H \leq \varepsilon\|v_n - v_n^{hk}\|_H^2 + C\|v_n - w_n^h\|_H^2 + C\|v_0 - v_0^{hk}\|_H^2 \\ & + C\|v_1 - w_1^h\|_H^2 + \sum_{j=1}^{n-1} 4k\|v_j - v_j^{hk}\|_H^2 + \frac{1}{k}\sum_{j=1}^{n-1}\|(v_j - w_j^h) - (v_{j+1} - w_{j+1}^h)\|_H^2 \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=1}^n k(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h)_{L^2(\Omega)} \leq \varepsilon\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + C\|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2 \\ & + C\|\theta_0 - \theta_0^{hk}\|_{L^2(\Omega)}^2 + C\|\theta_1 - \eta_1^h\|_{L^2(\Omega)}^2 + \sum_{j=1}^{n-1} 4k\|\theta_j - \theta_j^{hk}\|_{L^2(\Omega)}^2 \\ & + \frac{1}{k}\sum_{j=1}^{n-1}\|(\theta_j - \eta_j^h) - (\theta_{j+1} - \eta_{j+1}^h)\|_{L^2(\Omega)}^2. \end{aligned}$$

Next, we estimate

$$\|u_j - u_j^{hk}\|_V \leq \|u_0 - u_0^{hk}\|_V + \sum_{l=1}^j k\|v_l - v_l^{hk}\|_V + I_j, \quad (37)$$

where I_j is the integration error

$$I_j = \left\| \int_0^{t_j} v(s) ds - \sum_{l=1}^j k v_l \right\|_V.$$

Since $I_j \leq k \|u\|_{H^2(0,T;V)}$, we get

$$\|u_j - u_j^{hk}\|_V^2 \leq C \left(\|u_0 - u_0^{hk}\|_V^2 + j \sum_{l=1}^j k^2 \|v_l - v_l^{hk}\|_V^2 + k^2 \|u\|_{H^2(0,T;V)}^2 \right)$$

Using the fact that $Nk = T$, we have the estimate

$$\begin{aligned} \sum_{j=1}^n k \|u_j - u_j^{hk}\|_V^2 &\leq CT \left(\|u_0 - u_0^{hk}\|_V^2 + k^2 \|u\|_{H^2(0,T;V)}^2 \right) \\ &+ T \sum_{j=1}^n k \sum_{l=1}^j k \|v_l - v_l^{hk}\|_V^2. \end{aligned} \quad (38)$$

We denote

$$e_n = \|v_n - v_n^{hk}\|_H^2 + \sum_{j=1}^n k \|v_j - v_j^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + \sum_{j=1}^n k \|\theta_j - \theta_j^{hk}\|_E^2,$$

and

$$\begin{aligned} g_n &= k \sum_{j=1}^n \left(\|\dot{v}_j - \delta v_j\|_H^2 + \|v_j - w_j^h\|_V^2 + \|\dot{\theta}_j - \delta \theta_j\|_{L^2(\Omega)}^2 + \|\theta_j - \eta_j^h\|_E^2 \right) \\ &+ \frac{1}{k} \sum_{j=1}^{n-1} \left(\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|_H^2 + \|\theta_j - \eta_j^h - (\theta_{j+1} - \eta_{j+1}^h)\|_{L^2(\Omega)}^2 \right) \\ &+ k \sum_{j=1}^n \|v_j - w_j^h\|_{L^2(\Gamma_C; \mathbb{R}^d)}^2 + k^2 \|u\|_{H^2(0,T;V)}^2 + \|v_0 - v_0^{hk}\|_V^2 \\ &+ \|v_0 - v_0^{hk}\|_H^2 + \|v_1 - u_0^h\|_V^2 + \|\theta_0 - \theta_0^{hk}\|_{L^2(\Omega)}^2 + \|\theta_0 - \theta_0^{hk}\|_E^2 \\ &+ \|\theta_1 - \eta_1^h\|_{L^2(\Omega)}^2 + \|v_1 - w_1^h\|_H^2 + \|v_n - w_n^h\|_H^2 + \|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2. \end{aligned}$$

Then, we find that

$$e_n \leq C g_n + C \sum_{j=1}^n k e_j \quad \text{for } n = 1, \dots, N \quad (39)$$

with $C > 0$. Using Lemma 2, we obtain the desired estimate (18). $\square$

Now we consider a special situation in which the spaces E^h and V^h are based on the piecewise affine finite element approximation. Assume that Ω is a polygonal domain in $\mathbb{R}^2$ or a polyhedral domain in $\mathbb{R}^3$. Then each of parts of the boundary Γ_N and Γ_C is a sum of segments (in the case $\Omega \subset \mathbb{R}^2$) or polygons (if $\Omega \subset \mathbb{R}^3$), having mutually disjoint interiors i.e.

$$\bar{\Gamma}_j = \bigcup_{i=1}^{N_j} \Gamma_{j,i}, \quad j \in \{N, C\}.$$

Let $\{\mathcal{T}^h\}$ be a family of regular triangulations of the set $\bar{\Omega}$ of diameter h (cf. [33], Chapter 3, Section 3.1) which introduces a division of $\bar{\Omega}$ into triangles/tetrahedrons compatible with the division of the boundary $\partial\Omega$ into parts $\Gamma_{j,i}$, $1 \leq i \leq N_j$, $j \in \{N, C\}$, in that sense that if any edge of a triangle/any face of tetrahedron that belongs to $\mathcal{T}^h$ has an intersection of positive boundary measure with any of sets $\Gamma_{j,i}$, then the whole

edge/face is contained in $\Gamma_{j,i}$. For an element $T \in \mathcal{T}^h$, let $P_1(T)$ denote the space of first-order polynomials on T . Then E^h and V^h are defined as the spaces of continuous piecewise affine functions on elements $T \in \mathcal{T}^h$, that is,

$$E^h = \{w^h \in C(\overline{\Omega}) \mid w^h|_T \in [P_1(T)] \text{ for all } T \in \mathcal{T}^h\} \subset E. \quad (40)$$

$$V^h = \{w^h \in C(\overline{\Omega}; \mathbb{R}^d) \mid w^h|_T \in [P_1(T)]^d \text{ for all } T \in \mathcal{T}^h\} \subset V. \quad (41)$$

Corollary 5. *Suppose that the solution (u, θ) of Problem $\mathcal{P}_\varepsilon$ satisfies the following regularity assumptions*

$$\begin{aligned} u &\in C^1(0, T; H^2(\Omega; \mathbb{R}^d)) \cap H^3(0, T; H) \cap H^2(0, T; V), \quad \dot{u}_\tau \in C(0, T; H^2(\Gamma_C; \mathbb{R}^d)), \\ \theta &\in C(0, T; H^2(\Omega)) \cap H^2(0, T; L^2(\Omega)) \cap H^1(0, T; E). \end{aligned}$$

Moreover, we assume that the initial data satisfy

$$u_0 \in H^2(\Omega; \mathbb{R}^d), \quad v_0 \in H^1(\Omega; \mathbb{R}^d), \quad \theta_0 \in H^2(\Omega).$$

Then, for $h < 1$, we have the optimal order error estimate

$$\begin{aligned} &\max_{1 \leq n \leq N} \{ \|u_n - u_n^{hk}\|_V^2 + \|v_n - v_n^{hk}\|_H^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 \} \\ &+ k \sum_{j=1}^N \|v_j - v_j^{hk}\|_V^2 + k \sum_{j=1}^N \|\theta_j - \theta_j^{hk}\|_E^2 \leq C(h + k). \end{aligned} \quad (42)$$

Proof. To prove (42), we will use Theorem 4 and estimate each of the terms on the right-hand side of (18). We start with the initial conditions. Let us define elements $u_0^h = \Pi^h u_0$, $v_0^h = \Pi^h v_0$, $\theta_0^h = \Pi^h \theta_0$ as finite element interpolants of u_0 , v_0 and θ_0 , respectively (see [33], (2.3.29)). By the standard finite element interpolation error estimates [33–35] we have the following

$$\begin{aligned} \|u_0 - u_0^h\|_V &\leq ch \|u_0\|_{H^2(\Omega; \mathbb{R}^d)}, \\ \|v_0 - v_0^h\|_H &\leq ch \|v_0\|_{H^1(\Omega; \mathbb{R}^d)}, \\ \|\theta_0 - \theta_0^h\|_E &\leq ch \|\theta_0\|_{H^2(\Omega)}. \end{aligned}$$

Let $w^h(t) = \Pi^h \dot{u}(t) \in V^h$ be the finite element interpolant of $\dot{u}(t)$, for $t \in (0, T]$. Note that $w_\tau^h(t) = (\Pi^h \dot{u}(t))_\tau$ is the continuous piecewise-linear interpolant of $\dot{u}_\tau(t)$ on Γ_C . Similarly, let $\eta^h(t) = \Pi^h \theta(t) \in E^h$ be the finite element interpolant of $\theta(t)$, a.e. $t \in [0, T]$. Then we have

$$\begin{aligned} \|\dot{u}(t) - w^h(t)\|_V &\leq ch \|\dot{u}(t)\|_{H^2(\Omega; \mathbb{R}^d)}, \\ \|\dot{u}(t) - w^h(t)\|_H &\leq ch^2 \|\dot{u}(t)\|_{H^2(\Omega; \mathbb{R}^d)}, \\ \|\dot{u}_\tau(t) - w_\tau^h(t)\|_{L^2(\Gamma_C; \mathbb{R}^d)} &\leq ch^2 \|\dot{u}_\tau(t)\|_{H^2(\Gamma_C; \mathbb{R}^d)}, \\ \|\theta(t) - \eta^h(t)\|_E &\leq ch \|\theta(t)\|_{H^2(\Omega)}, \\ \|\theta(t) - \eta^h(t)\|_{L^2(\Omega)} &\leq ch^2 \|\theta(t)\|_{H^2(\Omega)}. \end{aligned}$$

Applying (18) with $w_n^h := w^h(t_n)$ and $\eta_n^h := \eta^h(t_n)$, $n = 1, \dots, N$, we get the following estimates

$$\begin{aligned} k \sum_{j=1}^N \|v_j - w_j^h\|_V^2 &\leq ch^2 \|\dot{u}\|_{C(0, T; H^2(\Omega; \mathbb{R}^d))}^2, \\ \max_{1 \leq n \leq N} \|v_{n\tau} - w_{n\tau}^h\|_{L^2(\Gamma_C; \mathbb{R}^d)} &\leq ch^2 \|\dot{u}_\tau\|_{C(0, T; H^2(\Gamma_C; \mathbb{R}^d))}, \end{aligned}$$

$$\begin{aligned} \max_{1 \leq n \leq N} \|v_n - w_n^h\|_H^2 &\leq ch^4 \|\dot{u}\|_{C(0,T;H^2(\Omega;\mathbb{R}^d))}^2, \\ k \sum_{j=1}^N \|\theta_j - \eta_j^h\|_E^2 &\leq ch^2 \|\theta\|_{C(0,T;H^2(\Omega))}^2, \\ \max_{1 \leq n \leq N} \|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2 &\leq ch^4 \|\theta\|_{C(0,T;H^2(\Omega))}^2. \end{aligned}$$

Moreover, similarly to Section 5 of [14], we get

$$\begin{aligned} k \sum_{j=1}^N \|\dot{v}_j - \delta v_j\|_H^2 &\leq ck^2 \|u\|_{H^3(0,T;H)}^2, \\ k \sum_{j=1}^N \|\dot{\theta}_j - \delta \theta_j\|_{L^2(\Omega)}^2 &\leq ck^2 \|\theta\|_{H^2(0,T;L^2(\Omega))}^2, \\ \frac{1}{k} \sum_{j=1}^{N-1} \|(v_j - w_j^h) - (v_{j+1} - w_{j+1}^h)\|_H^2 &\leq ch^2 \|u\|_{H^2(0,T;V)}^2, \\ \frac{1}{k} \sum_{j=1}^{N-1} \|(\theta_j - \eta_j^h) - (\theta_{j+1} - \eta_{j+1}^h)\|_{L^2(\Omega)}^2 &\leq ch^2 \|\theta\|_{H^1(0,T;E)}^2. \end{aligned}$$

Then the error bound (42) follows from (18) combined with (37). The proof of the corollary is complete. $\square$

5. Numerical simulations

In this section, we present the results of our numerical simulations. Such simulations are conducted to compare the empirical and theoretical convergence results of the proposed fully discrete scheme and visualize the role of various factors in the considered model. In Section 5.1, an academic two-dimensional sample problem is solved to obtain empirical error estimates and compare them with the theoretical estimates presented in Corollary 5. Next, in Section 5.2 we consider the three-dimensional problem with a more complex shape of the body.

We employ FEM and use the definition of spaces E^h and W^h presented in (40) and (41), respectively. For the sake of clarity, we assume dimensionless models with mass density and heat capacity $\rho = c_p = 1$. In our simulations, we use the original open source package *conmech* (<https://github.com/KOS-UJ/conmech>). The code is written in Python and uses just-in-time compilation mechanisms provided by packages *numba* [36] and *JAX* [37].

For two-dimensional simulations, we use the direct optimization method with the Schur complement method to decrease the dimension of the system [38]. We use Powell's conjugate direction method from the *SciPy* [39] package since the minimized functional is not necessarily differentiable.

In the case of three-dimensional simulations, to speed up the calculations, we perform computations on GPU using the L-BFGS solver implemented using *JAX*. We use *TetWild* [40] to generate the volumetric mesh from the surface mesh obtained from the Stanford 3D Scanning Repository. The results are visualized using *Blender* [41].

5.1. Two-dimensional numerical example

We consider an isotropic homogeneous domain $\Omega = [0, 1.5] \times [0, 1]$, set $T = 1$, and assume the following boundary conditions

$$\Gamma_N = (\{0\} \times (0, 1]) \cup ([0, 1.5] \times \{1\}) \cup (\{1\} \times (0, 1]), \quad \Gamma_C = [0, 1.5] \times \{0\}.$$

The domain Ω represents the cross-section of a three-dimensional viscoelastic body. We use the Kelvin–Voigt type short-memory viscoelastic law. The viscosity operator $\mathcal{A}$, elasticity operator $\mathcal{B}$, heat expansion tensor C and the thermal conductivity operator K are defined by

$$\begin{aligned}\mathcal{A}(t, \tau) &= 2\phi\tau + \xi \text{tr}(\tau)I, \quad \tau \in \mathbb{S}^3, t \in [0, T], \\ \mathcal{B}(\tau) &= 2\mu\tau + \lambda \text{tr}(\tau)I, \quad \tau \in \mathbb{S}^3, \\ C &= cI, \\ K(t, \zeta) &= \kappa\zeta, \quad \zeta \in \mathbb{R}^3, t \in [0, T].\end{aligned}$$

Here, $I \in \mathbb{R}^{2 \times 2}$ is the identity matrix, $\text{tr}(\cdot)$ is the trace of a square matrix, λ and μ are the Lamé coefficients, while ξ and ϕ represent the viscosity coefficients, $\lambda, \mu, \xi, \phi, c, k > 0$. In the initial state, the body rests on the foundation, that is, the initial displacement u_0 and the initial velocity v_0 are equal to zero at any point in Ω , and the gap between the body and the foundation $g_0 = 0$ at any point on the contact boundary Γ_C . The viscoelastic properties of the body are characterized by $\phi = 4.5, \xi = 10.5, \nu = 45, \lambda = 105$. The thermal effect is parameterized by the heat expansion coefficient $c = 0.5$ and the thermal conductivity coefficient $\kappa = 0.033$. The internal heat source g and the heat exchange between the body and the foundation r are modeled by the functions

$$\begin{aligned}g(x, t) &= 0, \quad x \in \Omega, t \in [0, T], \\ r(x, \theta) &= r_F(\theta - \theta_F), \quad x \in \Gamma_C, \theta \in \mathbb{R},\end{aligned}$$

where $r_F = 0.1$ and $\theta_F = 0.27$ denote the heat exchange coefficient and the constant foundation temperature, respectively. We assume the initial temperature $\theta_0 = 0.25$, constant density of the volume forces $f_0 = (0, -1)^T$ inside the body, and non-zero traction force applying squeezing from both sides of the body, namely

$$f_N(x_1, x_2) = \begin{cases} (48(0.25 - (x_2 - 0.5)^2), 0)^T, & \text{for } (x_1, x_2) \in (\{0\} \times (0, 1]), \\ (-44(0.25 - (x_2 - 0.5)^2), 0)^T, & \text{for } (x_1, x_2) \in (\{1\} \times (0, 1]), \\ (0, 0)^T, & \text{for } (x_1, x_2) \in ([0, 1.5] \times \{1\}). \end{cases}$$

On the contact boundary Γ_C we set

$$\begin{aligned}p_\nu(x, \zeta) &= \begin{cases} 0, & \text{for } \zeta \in (-\infty, 0), \quad x \in \Gamma_C, \\ 10^3 \zeta, & \text{for } \zeta \in [0, \infty), \quad x \in \Gamma_C, \end{cases} \\ j(\zeta) &= \log(\|\zeta\| + 1) \quad \text{for } \zeta \in \mathbb{R}_+, \\ h(x, \eta) &= \begin{cases} 0, & \text{for } \eta \in (-\infty, 0), \quad x \in \Gamma_C, \\ 0.1\eta, & \text{for } \eta \in [0, \infty), \quad x \in \Gamma_C. \end{cases}\end{aligned}$$

as conditions governing the response in the normal direction, friction, and heat generated by friction. Note that $j(\cdot)$ is in the form of a non-differentiable and non-convex function.

Fig. 1 demonstrates the solution of the sample problem in which we squeeze the body that lies on the foundation. The traction force from the left side is greater than from the right side, so we can observe slipping of the body on the foundation and therefore friction and heat generated by friction. On the right side, we can see the corresponding relative temperature error computed in comparison with the reference solution with $h = 2^{-7}$, $k = 2^{-9}$ and with 148,419 degrees of freedom.

Fig. 2 depicts the empirical error estimation for the sample problem considered plotted on a log–log scale. The color of the plot corresponds to the time step used to compute the solution. As we can see, for any fixed spatial step h we can observe that the numerical error goes to zero as the time step goes to zero. The gray line shows the linear order of convergence for comparison with the theoretical results.

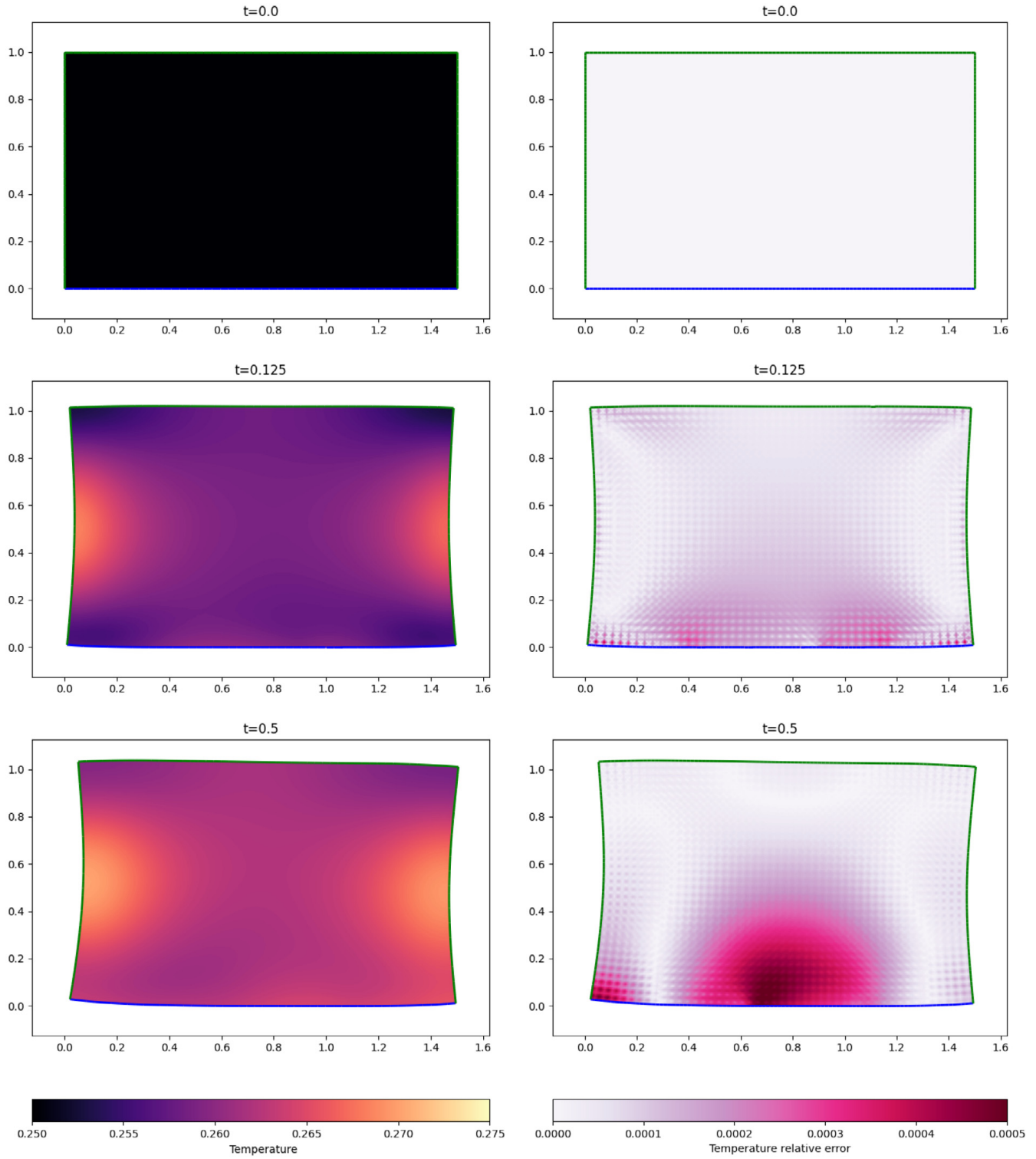


Fig. 1. Solution at chosen time moments t . Plots present the temperature on the left and relative temperature error on the right for $h = 2^{-5}$ and $k = 2^{-9}$.

5.2. Three-dimensional numerical example

To visualize the influence of various modeled factors, we set $d = 3$ and consider a volumetric mesh consisting of 42,684 nodes and 230,213 tetrahedrons. We set $\Gamma_N = \emptyset$ and $\Gamma_C = \Gamma$. The final time $T = 2.5$, the time step $k = 0.01$, and the results are presented at times $t \in \{0.5, 1, 1.5, 2, 2.5\}$. The position and shape of

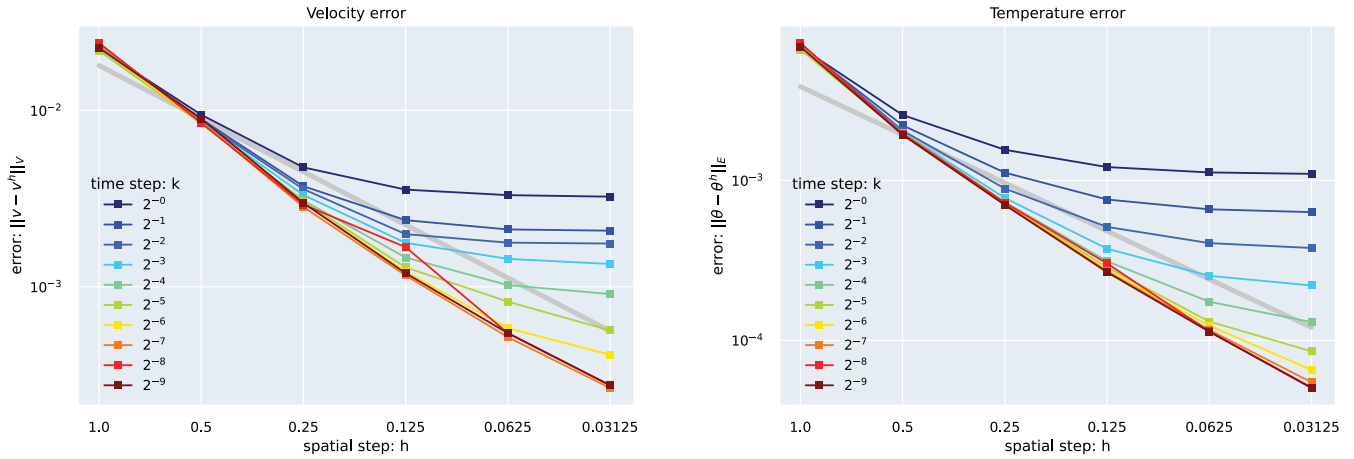


Fig. 2. Numerical error estimate for velocity (left) and temperature (right).

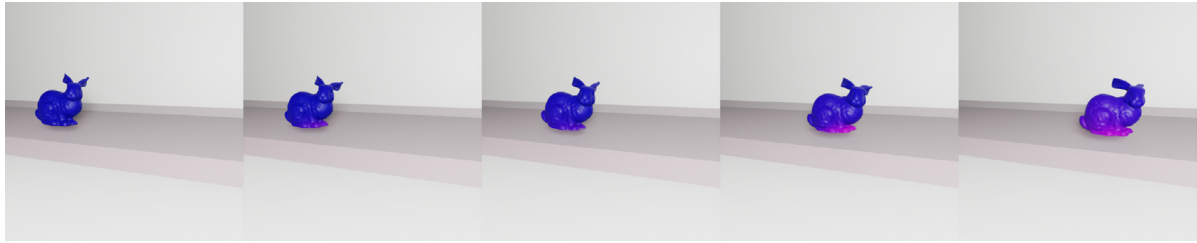


Fig. 3. First simulation — base data.

the foundation can be deduced from the attached illustrations. Body temperature is once again represented by its color, where blue is the coldest region and yellow is the hottest. We apply analogous constitutive laws as in a two-dimensional example and set

$$\begin{aligned} \lambda = \mu = 10, \quad \xi = \phi = 20 \\ c = 1, \quad \kappa = 0.1 \end{aligned}$$

The initial displacement u_0 and the initial velocity v_0 are equal to zero at any point in Ω . The internal force, the internal heat source and the heat exchange between the body and the foundation are modeled by

$$\begin{aligned} f_0(x, t) &= (-0.7, 0, 2)^T, \quad x \in \Omega, \quad t \in [0, T], \\ g(x, t) &= 0, \quad x \in \Omega, \quad t \in [0, T], \\ r(x, \theta) &= 0, \quad x \in \Gamma_C, \quad \theta \in \mathbb{R}. \end{aligned}$$

On the contact boundary Γ_C we take the following data

$$\begin{aligned} p_\nu(x, \zeta) &= \begin{cases} 0, & \text{for } \zeta \in (-\infty, 0), \\ \hat{p}\zeta, & \text{for } \zeta \in [0, \infty), \end{cases} \quad x \in \Gamma_C, \\ j(\zeta) &= \hat{j} \log(\|\zeta\| + 1) \quad \text{for } \zeta \in \mathbb{R}_+, \\ h(x, \eta) &= \begin{cases} 0, & \text{for } \eta \in (-\infty, 0), \\ \hat{h}\eta, & \text{for } \eta \in [0, \infty), \end{cases} \quad x \in \Gamma_C. \end{aligned}$$

The values of the hardness of the foundation $\hat{p}$, the friction coefficient $\hat{j}$ and the coefficient of heat generated by friction $\hat{h}$ vary between examples.

We now provide a brief overview of the results. The first simulation shows the data $\hat{p} = 200$, $\hat{j} = 0.1$ and $\hat{h} = 0.2$ and is presented in Fig. 3. This is the base configuration that will be modified in the following

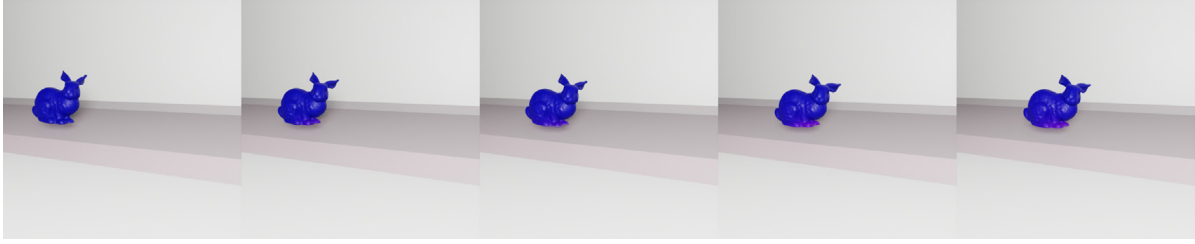


Fig. 4. Second simulation — increased friction.

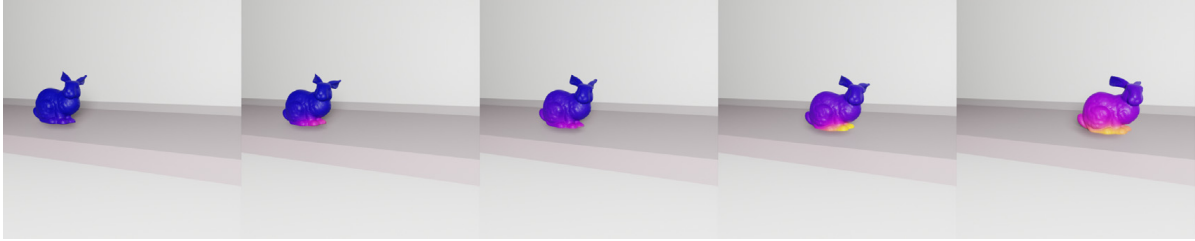


Fig. 5. Third simulation — increased heat generated by friction.

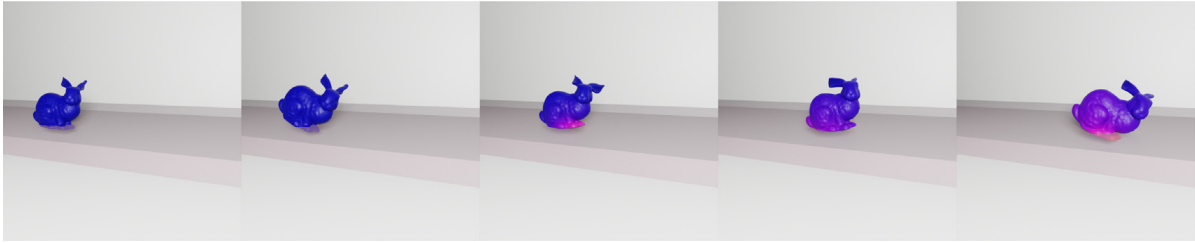


Fig. 6. Fourth simulation — softer foundation.

examples. We push the body down and to the right with force f_0 . The body falls to the obstacle, rebounds, and slides to the right. We can also observe the dissipation of heat generated by friction.

The second simulation shows the data $\hat{p} = 200$, $\hat{j} = 1.0$ and $\hat{h} = 0.2$ and is presented in Fig. 4. The only change compared to the base configuration is the increase in friction coefficient. This causes the body to cover a shorter distance and, as a result of the lower speed of the nodes at the contact boundary, the heat generated by friction is decreased.

The third simulation shows the data $\hat{p} = 200$, $\hat{j} = 0.1$ and $\hat{h} = 0.6$ and is presented in Fig. 5. In this case, the coefficient of heat generated by friction is increased compared to the base configuration. This increases the heat generated at the contact boundary, which is then diffused inside the body.

The fourth simulation shows the data $\hat{p} = 50$, $\hat{j} = 0.1$ and $\hat{h} = 0.2$ and is presented in Fig. 6. In this case, the boundary hardness coefficient is decreased. The softer foundation causes the body to initially dive further, and the resulting rebound produces a less stable trajectory and body rotation. We remark that the infinitesimal strain tensor considered in this paper results in a constitutive law that is not rotationally invariant. As a result, its application in the case of large body rotation would cause the volume of the body to increase unnaturally.

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Appendix A

Statements of the authors

Authorship statement

I hereby declare that I have contributed to the following papers my original work:

- Anna Ochal, Michał Jureczka, **Numerical treatment of contact problems with thermal effect**, *Discrete and Continuous Dynamical Systems - Series B*, 23(1), 2018, 387–400,
doi:10.3934/dcdsb.2018027
I estimate my contribution at 60%.
- Michał Jureczka, Anna Ochal, **Numerical analysis and simulations of contact problem with wear**, *Computers and Mathematics with Applications*, 77(11), 2018, 2980–2988,
doi:10.1016/j.camwa.2018.08.044
I estimate my contribution at 70%.
- Danfu Han, Weimin Han, Michał Jureczka, Anna Ochal, **Numerical analysis of a contact problem with wear**, *Computers and Mathematics with Applications*, 79(10), 2020, 2942–2951,
doi:10.1016/j.camwa.2019.12.027
I estimate my contribution at 25%.
- Weimin Han, Michał Jureczka, Anna Ochal, **Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage**, *Journal of Computational and Applied Mathematics*, 377, 2020, 112886,
doi:10.1016/j.cam.2020.11288
I estimate my contribution at 45%.
- Michał Jureczka, Anna Ochal, **A nonsmooth optimization approach for hemivariational inequalities with applications to Contact Mechanics**, *Applied Mathematics and Optimization*, 83, 2021, 1465–1485,
doi:10.1007/s00245-019-09593-y
I estimate my contribution at 80%.
- Anna Ochal, Michał Jureczka, Piotr Bartman, **A survey of numerical methods for hemivariational inequalities with applications to Contact Mechanics**, *Communications in Nonlinear Science and Numerical Simulation*, 114, 2022, 106563,
doi.org/10.1016/j.cnsns.2022.106563
I estimate my contribution at 65%.
- Michał Jureczka, Anna Ochal, Piotr Bartman, **A nonsmooth optimization approach for time-dependent hemivariational inequalities**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103871,
doi:10.1016/j.nonrwa.2023.103871
I estimate my contribution at 40%.
- Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafranec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870,
doi.org/10.1016/j.nonrwa.2023.103870
I estimate my contribution at 25%.

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Authorship statement

I hereby declare that following papers are the result of common scientific research of mgr Michał Jureczka and myself:

- Anna Ochal, Michał Jureczka, **Numerical treatment of contact problems with thermal effect**, *Discrete and Continuous Dynamical Systems - Series B*, 23(1), 2018, 387–400, doi:10.3934/dcdsb.2018027

I estimate my contribution at 40%.

- Michał Jureczka, Anna Ochal, **Numerical analysis and simulations of contact problem with wear**, *Computers and Mathematics with Applications*, 77(11), 2018, 2980–2988, doi:10.1016/j.camwa.2018.08.044

I estimate my contribution at 30%.

- Danfu Han, Weimin Han, Michał Jureczka, Anna Ochal, **Numerical analysis of a contact problem with wear**, *Computers and Mathematics with Applications*, 79(10), 2020, 2942–2951, doi:10.1016/j.camwa.2019.12.027

I estimate my contribution at 15%.

- Weimin Han, Michał Jureczka, Anna Ochal, **Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage**, *Journal of Computational and Applied Mathematics*, 377, 2020, 112886, doi:10.1016/j.cam.2020.11288

I estimate my contribution at 20%.

- Michał Jureczka, Anna Ochal, **A nonsmooth optimization approach for hemivariational inequalities with applications to Contact Mechanics**, *Applied Mathematics and Optimization*, 83, 2021, 1465–1485, doi:10.1007/s00245-019-09593-y

I estimate my contribution at 20%.

- Anna Ochal, Michał Jureczka, Piotr Bartman, **A survey of numerical methods for hemivariational inequalities with applications to Contact Mechanics**, *Communications in Nonlinear Science and Numerical Simulation*, 114, 2022, 106563, doi.org/10.1016/j.cnsns.2022.106563

I estimate my contribution at 20%.

- Michał Jureczka, Anna Ochal, Piotr Bartman, **A nonsmooth optimization approach for time-dependent hemivariational inequalities**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103871, doi:10.1016/j.nonrwa.2023.103871

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Authorship statement

I hereby declare that following papers are the result of common scientific research of mgr Michał Jureczka and myself:

- Anna Ochal, Michał Jureczka, Piotr Bartman, **A survey of numerical methods for hemivariational inequalities with applications to Contact Mechanics**, *Communications in Nonlinear Science and Numerical Simulation*, 114, 2022, 106563, doi.org/10.1016/j.cnsns.2022.106563

I estimate my contribution at 15%.

- Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafraniec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870, doi.org/10.1016/j.nonrwa.2023.103870

I estimate my contribution at 25%.

- Michał Jureczka, Anna Ochal, Piotr Bartman, **A nonsmooth optimization approach for time-dependent hemivariational inequalities**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103871, doi:10.1016/j.nonrwa.2023.103871

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Authorship statement

I hereby declare that following papers are the result of common scientific research of mgr Michal Jureczka and myself:

- Danfu Han, Weimin Han, Michal Jureczka, Anna Ochal, Numerical analysis of a contact problem with wear, *Computers and Mathematics with Applications*, 2019, doi:10.1016/j.camwa.2019.12.027

I estimate my contribution at 30%.

- Michal Jureczka, Anna Ochal, Weimin Han, Numerical studies of a hemivariational inequality for a viscoelastic contact problem with damage, *Journal of Computational and Applied Mathematics*, 377, 2020, doi.org/10.1016/j.cam.2020.112886

I estimate my contribution at 35%.

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October 21, 2020

Authorship statement

I hereby declare that the following paper is the result of common scientific research of mgr Michal Jureczka and myself:

- Danfu Han, Weimin Han, Michal Jureczka, Anna Ochal, Numerical analysis of a contact problem with wear, *Computers and Mathematics with Applications*, 2019, doi:10.1016/j.camwa.2019.12.027

I estimate my contribution at 30%.

A handwritten signature in black ink that reads "Danfu Han". The script is cursive and fluid, with the first letters of each name being capitalized and prominent.

Danfu Han

Hangzhou Normal University

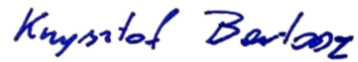
May 23, 2023

Authorship statement

I hereby declare that following papers are the result of common scientific research of mgr Michał Jureczka and myself:

- Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafraniec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870, doi.org/10.1016/j.nonrwa.2023.103870

I estimate my contribution at 25%.



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Authorship statement

I hereby declare that following papers are the result of common scientific research of mgr Michał Jureczka and myself:

- Piotr Bartman, Krzysztof Bartosz, Michał Jureczka, Paweł Szafraniec, **Numerical analysis of a non-clamped dynamic thermoviscoelastic contact problem**, *Nonlinear Analysis: Real World Applications*, 73, 2023, 103870, doi.org/10.1016/j.nonrwa.2023.103870

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Appendix B

Theorems and lemmas

For the convenience of the reader, in this section, we collect several theorems, lemmas, and definitions commonly used throughout this dissertation. Let $(X, \|\cdot\|_X)$ be a Banach space, X^* represent its dual, and $\langle \cdot, \cdot \rangle_{X^* \times X}$ denote the duality pairing between X^* and X .

Definition 1. (*Lipschitz function; [43], Definition 3.19*) A function $\varphi: U \subseteq X \rightarrow \mathbb{R}$ is said to be Lipschitz on U , if there exists $K > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq K \|y - z\|_X \quad \text{for all } y, z \in U. \quad (\text{B.1})$$

The constant K is called the Lipschitz constant. The inequality (B.1) is also known as a Lipschitz condition.

Definition 2. (*Locally Lipschitz function; [43], Definition 3.20*) A function $\varphi: U \subseteq X \rightarrow \mathbb{R}$ is said to be locally Lipschitz on U , if for all $x \in U$ there exists a neighborhood $\mathcal{N}(x)$ and $K_x > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq K_x \|y - z\|_X \quad \text{for all } y, z \in \mathcal{N}(x). \quad (\text{B.2})$$

Definition 3. (*Generalized directional derivative; [43], Definition 3.21*). The generalized directional derivative (in the sense of Clarke) of the locally Lipschitz function $\varphi: U \subseteq X \rightarrow \mathbb{R}$ at the point $x \in U$ in direction $v \in X$, denoted $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \rightarrow x} \sup_{\lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}. \quad (\text{B.3})$$

We note that, in contrast to the usual directional derivative, the generalized directional derivative φ^0 is always defined.

Definition 4. (*Generalized gradient; [43], Definition 3.22*) Let $\varphi: U \subseteq X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The (Clarke) subdifferential or the generalized gradient in the sense of Clarke of φ at $x \in U$, denoted $\partial\varphi(x)$, is the subset of a dual space X^* defined by

$$\partial\varphi(x) = \{\zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X} \text{ for all } v \in X\}. \quad (\text{B.4})$$

Lemma 5. ([43], Lemma 3.23) If $\varphi: U \rightarrow \mathbb{R}$ is a locally Lipschitz function on a subset U of X , then

i For every $x \in U$, the function $X \ni v \mapsto \varphi^0(x; v) \in \mathbb{R}$ is positively homogeneous (that is, $\varphi^0(x; \lambda v) = \lambda \varphi^0(x; v)$ for all $\lambda \geq 0$), subadditive (that is, $\varphi^0(x; v_1 + v_2) \leq \varphi^0(x; v_1) + \varphi^0(x; v_2)$ for all $v_1, v_2 \in X$, and satisfies the inequality $|\varphi^0(x; v)| \leq K_x \|x\|_X$ with $K_x > 0$ being the Lipschitz constant of φ near x . Moreover, it is Lipschitz continuous and $\varphi^0(x; -v) = (-\varphi)^0(x; v)$ for all $v \in X$.

ii For every $v \in X$ we have $\varphi^0(x; v) = \max\{\langle \zeta, v \rangle_{X^* \times X} \mid \zeta \in \partial\varphi(x)\}$.

Definition 6. (Convex subdifferential; [43], Definition 3.31) Let U be an open convex subset of X and let $\varphi: U \rightarrow \mathbb{R}$ be a convex function. An element $x^* \in X^*$ is called the subgradient of φ at $x \in U$ if the following inequality holds

$$\varphi(v) \geq \varphi(x) + \langle x^*, v - x \rangle_{X^* \times X} \text{ for all } v \in U. \quad (\text{B.5})$$

The set of all $x^* \in X^*$ that satisfies this inequality is called the (convex) subdifferential of φ at x , and is denoted by $\partial\varphi(x)$. Sometimes, we refer to $\partial\varphi$ as the subdifferential of the convex function φ or the subdifferential of φ in the sense of convex analysis.

Definition 7. (Regular function; [43], Definition 3.25) A locally Lipschitz function $\varphi: U \rightarrow \mathbb{R}$ on a subset U of X is said to be regular (in the sense of Clarke) at $x \in U$, if

i for all $v \in X$ the directional derivative $\varphi'(x; v)$ exists.

ii for all $v \in X$, $\varphi'(x; v) = \varphi^0(x; v)$

The function is regular (in the sense of Clarke) on U if it is regular at every point $x \in U$.

Lemma 8. ([43], Lemma 3.32) Let $\varphi: U \rightarrow \mathbb{R}$ be defined on a subset U of X . Then

i The function φ is strictly differentiable at $x \in U$ if and only if φ is locally Lipschitz near x and $\partial\varphi(x)$ is a singleton (which is necessarily the strict derivative of φ at x). In particular, if φ is continuously differentiable at $x \in U$, then $\varphi^0(x; v) = \varphi'(x; v) = \langle \varphi'(x), v \rangle_{X^* \times X}$ for all $v \in X$ and $\partial\varphi(x) = \{\varphi'(x)\}$.

ii If φ is regular at $x \in U$, $\varphi'(x)$ exists, and $\psi: U \rightarrow \mathbb{R}$ is locally Lipschitz near x , then $\partial(\varphi + \psi)(x) = \{\varphi'(x)\} + \partial\psi(x)$.

iii If U is an open convex set and φ is convex, then the Clarke subdifferential $\partial\varphi(x)$ at any $x \in U$ coincides with the subdifferential of φ at x in the sense of convex analysis.

Lemma 9. ([43], Lemma 3.35) Let $\varphi, \varphi_1, \varphi_2: U \rightarrow \mathbb{R}$ be locally Lipschitz functions on an open subset U of X . Then

i (Scalar multipliers) the equality $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ holds, for all $\lambda \in \mathbb{R}$ and all $x \in U$.

ii (Sum rules) the inclusion

$$\partial(\varphi_1 + \varphi_2)(x) \subseteq \partial\varphi_1(x) + \partial\varphi_2(x) \quad (\text{B.6})$$

holds for all $x \in U$ or, equivalently,

$$(\varphi_1 + \varphi_2)^0(x; v) \leq \varphi_1^0(x; v) + \varphi_2^0(x; v) \quad (\text{B.7})$$

for all $x \in U, v \in X$.

iii If one of φ_1, φ_2 is strictly differentiable at $x \in U$, then (i) and (ii) hold with equalities.

iv If φ_1, φ_2 are regular at $x \in U$, then $\varphi_1 + \varphi_2$ is regular at $x \in U$ and we also have equalities in (i) and (ii).

Lemma 10. (Lebourg mean value theorem; [43], Lemma 3.36). Let $x, y \in X$ and $\varphi: U \subseteq X \rightarrow \mathbb{R}$ be a locally Lipschitz function defined on an open subset U that contains the line segment $[x, y]$ (defined by $[x, y] = \{z = \lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\}$). Then there exist $z \in [x, y]$ and $\zeta \in \partial\varphi(z)$ such that $\varphi(y) - \varphi(x) = \langle \zeta, y - x \rangle_{X^* \times X}$.

Lemma 11. ([43], Lemma 3.37) Let X and Y be Banach spaces, $\psi: Y \rightarrow \mathbb{R}$ be locally Lipschitz, and $T: X \rightarrow Y$ be given by $Tx = Ax + y$ for $x \in X$, where $A \in \mathcal{L}(X, Y)$ and $y \in Y$ are fixed. Then the function $\varphi: X \rightarrow \mathbb{R}$ defined by $\varphi(x) = \psi(Tx)$ is locally Lipschitz and

$$i \quad \varphi^0(x; v) \leq \psi^0(Tx; Av) \text{ for all } x, v \in X.$$

$$ii \quad \partial\varphi(x) \subseteq A^* \partial\psi(Tx) \text{ for all } x \in X,$$

where $A^* \in \mathcal{L}(Y^*, X^*)$ denotes the adjoint operator to A . Moreover, if ψ (or $-\psi$) is regular, then φ (or $-\varphi$) is regular and in (i) and (ii) we have equalities. These equalities are also true if instead of the regularity condition we assume that A is surjective.

Lemma 12. (Green-type formula; [43], Lemma 2.25) Let Ω be an open, bounded and connected set of $\mathbb{R}^d$ with a Lipschitz boundary Γ . Then

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \text{Div } \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\gamma} \mathbf{v} \, da \quad (\text{B.8})$$

for all $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ and $\boldsymbol{\sigma} \in C^1(\overline{\Omega}; \mathbb{S}^d)$.

Lemma 13. (Young inequality; [43], Lemma 2.6) Let $1 \leq p < \infty, 1/p + 1/q = 1$, and $\varepsilon > 0$. Then

$$ab \leq \frac{\varepsilon^p}{p} |a|^p + \frac{1}{\varepsilon^q q} |b|^q \quad \text{for all } a, b \in \mathbb{R}. \quad (\text{B.9})$$

In a particular case,

$$ab \leq \varepsilon |a|^2 + c_{\varepsilon} |b|^2 \quad \text{for all } a, b \in \mathbb{R} \quad (\text{B.10})$$

where c_{ε} is a constant that depends on ε .

Lemma 14. (*Jensen inequality; particular case of [43], Lemma 2.5*) Let $I \subset \mathbb{R}$ be a set of positive measure and let $f: I \rightarrow \mathbb{R}$ be an integrable function. Then

$$\left(\frac{1}{|I|} \int_I f(x) dx \right)^2 \leq \frac{1}{|I|} \int_I (f(x))^2 dx.$$

In a particular case, for $t_m, t_{m+1} \in I$, $t_m < t_{m+1}$, $t_{m+1} - t_m = k$, we have

$$\left(\int_{t_m}^{t_{m+1}} f(x) dx \right)^2 \leq k \int_{t_m}^{t_{m+1}} (f(x))^2 dx. \quad (\text{B.11})$$

Lemma 15. (*Discrete version of Gronwall inequality; [47], Lemma 7.26*) Let T be given. For $N > 0$ we define $k = T/N$. Let $\{g_n\}_{n=1}^N$, $\{e_n\}_{n=1}^N$ be two nonnegative sequences that satisfy for $c > 0$ and for all $n \in \{1, \dots, N\}$

$$e_n \leq cg_n + ck \sum_{j=1}^n e_j.$$

Then

$$\max_{1 \leq n \leq N} e_n \leq \hat{c} \max_{1 \leq n \leq N} g_n \quad \text{with } \hat{c} > 0.$$

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