

Optimal generalized quantum measurements...

PhD Thesis

Optimal Generalized Quantum Measurements and other Fantastical Beasts

Written under supervision of prof. Karol Życzkowski

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Colophon

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“You can’t see both aspects of the Necker Cube at once, so it lets you focus on one and dismiss the other. (...) You’re always better off looking at more than one side of anything.

Go on, try. Defocus. It’s the next logical step.”

– Peter Watts, *Blindsight* [1]

“A human being *should be able to:*

change a diaper, plan an invasion, butcher a hog, conn a ship, design a building, write a sonnet, balance accounts, build a wall, set a bone, comfort the dying, take orders, give orders, cooperate, act alone, solve equations, analyse a new problem, pitch manure, program a computer, cook a tasty meal, fight efficiently, die gallantly.

Specialization *is for insects.”*

– Robert A. Heinlein, *Time Enough for Love* [2]

In memory of my Grandparents.

Oświadczenie

Ja niżej podpisany Jakub Czartowski, doktorant Szkoły Doktorskiej Nauk Ścisłych i Przyrodniczych Uniwersytetu Jagiellońskiego, oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. *"Optimal Generalized Quantum Measurements and other Fantastical Beasts"* jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr hab. Karola Życzkowskiego. Pracę napisałem samodzielnie. Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami). Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia

.....

Podpis

Streszczenie

Niniejsza rozprawa doktorska przedstawia znaczący wkład w dziedzinie informacji kwantowej, podejmując szereg zróżnicowanych problemów, obejmujących takie tematy jak optymalne pomiary, metrologia kwantowa, termodynamika kwantowa oraz koncepcja designów w kontekście kanałów kwantowych.

Rozprawa rozpoczyna się pedagogicznym wprowadzeniem do mechaniki kwantowej i informacji kwantowej, napisanym tak, aby było przystępne dla studentów studiów licencjackich, a nawet dla osób bez doświadczenia w naukach ścisłych, szczególnie w dwóch pierwszych sekcjach. W tej części wyjaśniam podstawowe różnice między klasyczną a kwantową przestrzenią stanów, podkreślam probabilistyczną naturę teorii kwantowej indukowaną przez proces pomiarowy oraz opisuję, jak zamknięte i otwarte układy kwantowe ewoluują odpowiednio pod wpływem unitarnych i ogólnych kanałów kwantowych.

Pierwsza część wyników przedstawia klasyfikację dwukubitowych pomiarów von Neumanna po uwzględnieniu klas równoważności względem transformacji lokalnych. Pokazujemy, że wszystkie izosplątane pomiary należą do jednej z czterech odrębnych rodzin ciągłych; wszelkie nakładanie się tych rodzin jest miary zero i, z wyjątkiem baz stanów produktowych, żadna rodzina nie jest w pełni zawarta w innej. Analizujemy ich zastosowanie w trójkątnych sieciach kwantowych i stwierdzamy, że w takim układzie podrodzina zawierająca Elegancki Łączny Pomiar (eng. *Elegant Joint Measurement*) generuje optymalne rozkłady wyników z perspektywy nierozróżnialności wyników i uczestników. Przyszłe prace mogą rozszerzyć te wyniki na bardziej złożone systemy i pomiary, z potencjalnymi zastosowaniami w sieciach obejmujących stany wyższych wymiarów rozdzielone między węzły.

W drugiej części, metody interferometrii $SU(1,1)$ w ich interpretacji detektora przemieszczeń zostały przeformułowane na układy spinowe z wykorzystaniem jednoosiowego ściskania (eng. *one-axis twisting* lub *one-axis squeezing*). Tutaj wykazujemy, że protokół ściskania-rotacji-rozściskania wykonany z odpowiednim stopniem ściskania zapewnia optymalną skalę Heisenberga informacji kwantowej Fishera (*quantum Fisher information, QFI*) w zależności od rozmiaru spinu. W szczególności, pół-maksymalne ściskanie zapewnia stabilne skalowanie Heisenberga dla dowolnej równikowej osi rotacji. Przewaga kwantowa, mierzona skalowaniem QFI, jest utrzymywana przy rozważaniu względnych rotacji między dwoma ramionami interferometru spinowego, choć z mniejszą odpornością na zmiany stopnia ściskania.

W trzeciej części proponujemy metodę przybliżania dowolnych operacji termicznych za pomocą algorytmu składającego się z dwupoziomowych markowskich operacji termicznych przeprowadzanych na systemie z pamięcią zainicjalizowaną w równowadze termicznej. Twierdzenie jest poparte dowodami analitycznymi dla nieskończonej temperatury i, w ograniczonym zakresie, dla skończonej temperatury. W ostatnim przypadku zbieżność do operacji, które nie są objęte dowodami analitycznymi, są poparte szeroką analizą numeryczną. Na koniec rozszerzamy wyniki

przedstawione w naszym oryginalnym artykule, prezentując metodę, która powinna pozwolić na pełny dowód analityczny zbieżności proponowanego algorytmu.

Część czwarta bada problem ścisłej katalizy dla operacji termicznych, gdzie niemożliwa w inny sposób transformacja jest umożliwiona przez wprowadzenie katalizatora, który jest odzyskiwany w niezmienionym stanie po przeprowadzeniu transformacji na rozszerzonym układzie. W tej części wprowadzamy koncepcję regionu katalizowanego, zawierającego wszystkie przejścia, dla których może istnieć katalizator umożliwiający transformację. Ponadto wyznaczamy ograniczenia na strukturę i wymiarowość stanu katalizatora, ułatwiając praktyczne obliczenia; co najważniejsze, wykazujemy, że istnieją przejścia w wymiarze 3, które mogą być katalizowane w warunkach skończonej temperatury — co do których społeczność naukowa dotąd sądziła, że są niemożliwe. Na koniec przedstawiamy przykładowe zastosowanie wyznaczonych ograniczeń do generowania splątania poprzez katalizę, pokazując potencjalne korzyści wynikające z wprowadzenia pomocniczego systemu katalizatora.

W ostatniej części wprowadzamy pojęcie deseni na kanałach jako specjalizację deseni wepchniętych (*pushforward designs*), dostarczając narzędzia do uśredniania funkcji wielomianowych przez kanały kwantowe z wykorzystaniem skończonych zbiorów kanałów. Podajemy podstawowe własności takich deseni i przedstawiamy metodę konstrukcji przez indukcję z deseni unitarnych. W szczególności rozważamy desenie indukowane z grupy Clifforda, znanej jako unitarny 3-deseń, i rozszerzamy je, prezentując rodzinę 49 jednokubitowych kanałów, które poprzez wprowadzenie odpowiedniego zestawu wag zapewniają ważone 3-desenie kanałów dla dowolnej wymiarowości środowiska. Na koniec, na podstawie średnich stanów Choi, proponujemy metodę określania efektywnej wymiarowości środowiska poprzez minimalizację parametru ϵ dla deseni na kanałach. Aby zilustrować to podejście, oceniamy efektywną wymiarowość środowiska symulatorów komputerów kwantowych IBM oraz IBM Kyoto, rzeczywistym procesorze kwantowym (*Quantum Processing Unit, QPU*).

Dwa tematy przewijają się przez całą pracę: optymalność i pomiar. Te tematy wyznaczają przyszłe kierunki badań, sugerując dowody optymalności i dalsze udoskonalenia, szczególnie w termodynamice kwantowej. Niniejsza praca rozwija nasze zrozumienie procesów informacji kwantowej i stanowi podstawę do przyszłych eksploracji w tej dziedzinie, podejmując zarówno teoretyczne, jak i praktyczne aspekty pomiarów kwantowych, optymalności i katalizy.

Abstract

This thesis presents significant contribution to the field of quantum information, addressing a range of diverse problems, spanning topics such as optimal measurements, quantum metrology, quantum thermodynamics and concept of designs in the setting of quantum channels.

The dissertation begins with a pedagogical introduction to quantum mechanics and quantum information, which is meant to be accessible to undergraduate students and even to people with no experience in exact sciences when limited to the first two sections. Therein, I explain the basic difference between classical and quantum state space, exhibit the probabilistic nature of quantum theory as induced by the measurement process and, finally, present the manner in which both closed and open quantum systems evolve under unitary and general quantum channels, respectively.

The first results' part presents the classification of two-qubit von Neumann measurements based on equivalence up to local transformations. Therein we present that all isoentangled measurements fall into one of four distinct continuous parameters; all overlaps happening between the families are of measure zero, and except for the product-state bases, no family is fully contained within another. We consider their application in triangular quantum networks. In particular, we find that in such a setting a subfamily containing Elegant Joint Measurement generates optimal output distributions from output and participant indistinguishability perspectives. Future work may extend this to more complex systems and measurements, with applications to networks with higher-dimensional states distributed between the nodes.

In second part, methods from $SU(1,1)$ interferometry in their displacement-detector interpretation are translated to spin settings using single-axis squeezing are developed. Here, we demonstrate that a squeeze-rotation-unsqueeze protocol effected with proper squeezing degree provides optimal Heisenberg scaling of quantum Fisher information (QFI) with the size of the spin involved. In particular, half-maximal squeezing provides stable Heisenberg scaling for arbitrary equatorial axis of rotation. The quantum advantage, as measured by QFI scaling, is maintained when considering relative rotations between two arms of spin-interferometer, albeit with reduced robustness to the changes the squeezing degree.

In part three we propose a method for approximating arbitrary thermal operations through an algorithm consisting two-level Markovian thermal operations carried out on a system with memory initialized in thermal equilibrium. The claim is supported with analytical proofs for infinite temperature and, to a limited extent, finite temperature. Operations in latter setting which are not covered by analytical proofs are supported with extensive numerical evidence. Finally, we extend the results presented in our original manuscript by presenting a method, which should allow a complete analytical proof of convergence of the proposed algorithm.

Part four explores the problem of strict catalysis for thermal operations, where an otherwise impossible transition is facilitated by introducing a catalyst, which is

retrieved in an unperturbed state after performing a transformation on the extended system. Herein, we introduce the concept of a catalysable region, containing all transitions, for which a catalyst allowing the transformation may exist. Furthermore, bounds on catalyst state structure and dimensionality are derived, facilitating practical calculations; most importantly, we demonstrate that there exist transitions in dimension 3 which may be catalysed in finite temperature setting – existence of which was hitherto believed impossible by the community. Finally, we provide an exemplary application of the derived bounds to entanglement generation via catalysis, showing the potential advantage provided by introduction of auxiliary catalyst system.

Finally, the notion of channel designs as a specialization of pushforward design is introduced, providing a tool for averaging polynomial functions over quantum channels using finite sets of channels. We provide basic properties of such designs and present a method of construction by induction from unitary designs. In particular, we consider designs induced from Clifford group, known to be unitary 3-design, and extend them to present a family of 49 single-qubit channels, which by introducing proper set of weights provide weighted channel 3-designs for arbitrary dimension of environment. Finally, based on the average Choi states, we propose a method for determining the effective dimension of environment by minimising the ϵ -parameter for ϵ -channel designs. To demonstrate the approach, we evaluate the effective environment dimensionality of IBM quantum computer simulators as well as actual *IBM Kyoto* quantum processing unit (QPU).

Two themes recur: optimality and measurement. These themes guide future research directions, suggesting proofs of optimality and further refinements, particularly in quantum thermodynamics. This thesis advances our understanding of quantum information processes and lays the groundwork for future explorations in the field, addressing both theoretical and practical aspects of quantum measurements, optimality, and catalysis.

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Preface

*"In the beginning the Universe was created.
This has made a lot of people very angry and been widely regarded as a
bad move."*

Douglas Adams, *The Restaurant at the End of the Universe* [23]

On a personal note – the Journey begins...

Why do we do physics? To quote the words of Nobel prize winner Richard Feynman, "(...) sure, it may give some practical results, but that's not why we do it."^{*} Indeed, there is hardly anyone who could deny all the practical consequences of the very field this thesis contributes to – namely quantum mechanics and quantum information – especially when the world is experiencing the second quantum revolution, driven by such things as quantum computers, quantum encryption protocols – the list goes on.

However, we are living in the age of grandiose scientific endeavours, including scientific instruments such as Large Hadron Collider (LHC), Laser Interferometry Gravitational wave Observatory (LIGO), James Webb space telescope or International Thermonuclear Experimental Reactor (ITER); all of them consumed exorbitant amounts of money, and I find it hard to envision direct practical consequences of all but one of them – although I encourage the reader to point out my ignorance. Nevertheless, I dare say that primary motivation driving most of the people behind such projects is *to understand*, make one more step to comprehension of the inner workings of the world and uncover the principles guiding all the observable phenomena. In the end, it comes down to one quality common to some extent to all human beings – *Curiosity*, and I do believe it deserves a capital C for all the things it has brought us, for better or worse.

I am no exception. It all started for me in July 1996 in Gdańsk, but I will skip the first years as irrelevant to the subject... Ever since kindergarten I have seen myself as a curious person, with a slight twist towards the exact sciences. Some of the early books I can remember were the DIY home experiment books, children encyclopedias full of physical, chemical and biological trivia, and star atlases. I remember rather vividly

^{*} Curious reader may find the author of the quote and check its beginning easily.

that one of the latter prompted me to check what actually is a neutron star, and so it began – in for a penny, in for a pound, I was caught by the idea of understanding the world and all its quirky wonders, both the largest and the smallest, with physics.

My family provided no small influence, for which I am deeply grateful. Both my grandfathers used to teach highschool physics, my grandmother on the mothers side – mathematics. Nobody ever pushed me into this field of human endeavour, and I never limited myself only to exact science. And yet somehow I was always drifting towards it, during primary, middle high and high school, all this time getting closer to feeling that I understand something. Even if, as my father used to prove by geometry, the more I things I knew, the larger was my contact with the things I did not know.

In 2015, when my university years started, in the beginning I chose two majors – physics and japanese studies, and I intended to pursue them both. Life, however, is what it is and I was forced to choose one of them. The desire to know more of the things I did not know did the rest, and I chose physics. During my second year I attended the course of quantum information led by prof. Karol Życzkowski, who soon after became my supervisor and immediately invited me to an actual scientific project with my first collaborator, now prof. Dardo Goyeneche. Less than one year later the collaboration resulted in my first publication [3].

Taking this into account, I feel that this thesis is something more than a final step towards acquiring the title of *Doctor of Philosophy*; it is also more than a final of the three-thesis series written under supervision of prof. Życzkowski. I see it as the latest step in my journey towards understanding how little we really know about the world. My grandpa was always right – the more we know, the less we know. I truly hope that my scientific work has contributed, even if only infinitesimally, to our collective not-knowing by giving answers that lead to more, hopefully deeper, questions. To quote one of the Walking Songs of the Middle-earth, as I did 6 years ago in my Bachelor thesis,

*“Upon the hearth the fire is red,
Beneath the roof there is a bed;
But not yet weary are our feet,
Still round the corner we may meet
A sudden tree or standing stone
That none have seen but we alone.”*

J. R. R. Tolkien, *The Fellowship of the Ring* [24]

Outline of the thesis

In the remainder of this section I will complete list of my publications together with the selection of works that provide the basis for this thesis.

Following the preface, before getting in to the actual results Section 1 will introduce the reader to the very basics of quantum mechanics necessary to understand not only the chapters to follow, but the quantum physics *per se*. For this reason, it is written in as simple a manner as my ability allowed me to, hopefully providing even a lay person at least partial understanding of the mystery behind all that is quantum.

Following this, Chapters 2 through 6 are devoted to the original results. Although the sections are grouped thematically, due to diversity of the topics touched upon each consists in a separate technical introduction preceding the results proper, laid down in more detail than in the corresponding published and unpublished works.

The strictly scientific part of the thesis ends with the summary and outlook in Chapter 7, sketching the potential future of each of the subfields to which the results discussed in the thesis belong. In particular, it will focus on potential follow-up questions for each of the topics, providing a stepping stone for whoever may attempt to expand upon them.

Finally, the thesis concludes with the afterword in both Polish and English, together with acknowledgments, which are lengthy out of necessity, and therefore deferred until the very end.

List of publications

Over the course of my scientific career, dating back to 2017, I have worked on a total of 20 already available scientific papers [3–22], listed below, out of which 15 have been published in reputable scientific journals [3–11, 13–17, 19] and an additional manuscript expected to be published by the time of defense [20].

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Due to the extent of the collected work, and variety of topics covered therein, I decided to focus on a selection of the works divided into three thematical areas, namely

- Classification and application of iso-entangled measurements:
 - F. Del Santo, J. Czartowski, K. Życzkowski, and N. Gisin, ‘Iso-entangled bases and joint measurements,’ *Phys. Rev. Res.*, **6**, 2 2024 [17],
- Catalytic processes in Thermal Operations framework:
 - J. Czartowski, A. de Oliveira Junior, and K. Korzekwa, ‘Thermal recall: Memory-assisted Markovian thermal processes,’ *PRX Quantum*, **4**, 4 2023 [16],
 - J. Czartowski and A. de Oliveira Junior, ‘Catalytic transformations for thermal operations,’ *Phys. Rev. Res.*, **6**, 3 Aug. 2024 [19],
- and other Fantastical Beasts:
 - J. Czartowski, K. Życzkowski, and D. Braun, ‘Minimal-noise estimation of noncommuting rotations of a spin,’ *Quantum*, **8**, 2024 [14],
 - J. Czartowski and K. Życzkowski, ‘On channel t -designs,’ *arXiv: 24MM.XXXXX*, 2024 [25],

where among the above the manuscript [25] is not yet published anywhere[†].

Motivations and Contributions

Below I lay down details concerning the overall scope of each of the papers used for this thesis,

Iso-entangled bases and joint measurements [17]

In this work we have considered the problem of classification of iso-entangled measurements on two qubits. We have demonstrated that there exist four distinct families of bases with zero-measure overlaps between them. In particular, we have demonstrated that one of the families – the elegant family, containing Elegant Joint Measurement – contains a sub-family generating output-permutation invariant (OPI) distributions when used in the no-input triangular network with maximally entangled Bell pairs distributed along each edge.

[†] State of the affairs on August 25, 2024. This may be subject to change by the time of PhD defense.

This work has been realised in line with the goal of extending the elegant joint measurement and finding its applications, as specified in the Preludium BIS grant and the Individual Research Plan. Aside from overall involvement in the calculations necessary for proving the classification, my additional individual contribution consisted in demonstrating existence of the subfamily of elegant family generating the OPI distributions, as well as conducting considerations for extensions beyond two-qubit measurements, providing full description of bases composed of product states as well as novel construction of isoentangled bases based on robust Hadamard matrices and Latin squares.

Simultaneous minimal-noise estimation of non-commuting rotations of a spin [14]

In this work we have considered from a metrological point of view a problem of translating the displacement detector interpretation of the $SU(1,1)$ interferometer as presented by Caves in [26] to detection of spin rotations, using the ideas known from spin-coherent states and single-axis spin squeezing. We demonstrated that one may achieve Heisenberg-like scaling of quantum Fisher information (QFI), an important metric in the field of quantum metrology, for arbitrary equatorial axis of rotation. Furthermore, we have considered a situation of relative-rotation detection and demonstrated that, by proper choice of squeezing and spin sizes, one may maintain the quantum advantage as exhibited by the Heisenberg scaling of QFI. Finally, we show numerically that two-axis counter-twisting yields results worse than single-axis twisting when seen from the perspective of rotation detection.

This work has been realised as a part of a 6-months NAWA Preludium BIS international internship, carried out under supervision of prof. Daniel Braun of the University of Tübingen. Since prof. Braun and prof. Życzkowski played the roles of senior supervisors, I see all the presented results as my own, taking into account guidance from both supervisors.

Thermal recall: Memory-assisted Markovian thermal processes [16]

In this work we have considered a problem of recovering entire set of thermal operations (TOs) with effectively infinite memory using Markovian thermal operations (MTOs) acting on a system extended by memory states initialised in thermal equilibrium state. We have demonstrated analytically that there exists a specific algorithm consisting in two-level MTOs that recovers arbitrary β -swap. Furthermore, we have presented extensive numerical evidence for accessibility of arbitrary extreme transition. Most importantly, we have provided evidence for convergence to transitions previously believe to be inaccessible using two-level operations. As exemplary applications, we have considered the problem of cooling using MTOs, TOs and the manner in which the memory-extended MTOs bridge the gap between the two in terms of achievable transformation error.

My primary contribution in this work consists in putting forward the general idea for the proofs and carrying out extensive parts of the analytical calculations necessary to for them. Furthermore, I have put forward the approach to the algorithm which allows for going beyond the β -swaps towards more complicated TOs. Finally, I have put forward the idea of entire family of protocols leading to the same final distribution equivalent to the main protocol considered in the work.

For the reasons above, in this thesis I present and extend the ideas surrounding the protocol, as well as the numerical evidence for convergence of the generalized protocol. In the final section I provide a novel approach, which – if carried to its logical ending – may provide an analytical proof of convergence of the proposed protocol to arbitrary TO.

We can catalyse it for you thermally: Catalytic transformations for thermal operations [19]

In this work we have considered the problem of providing bounds on structure and dimensionality of states necessary for strict catalysis in the thermal operations' setting. By extending the tools known for majorisation-based theories to the thermomajorisation setting we provided simple bounds. Most importantly, we have shown that strict catalysis is possible in dimension 3 for finite temperatures, contrary to the common view within the field. As an application, we have considered advantage enabled by use of catalyst in the context of entanglement generation using exclusively thermal operations.

As a first author, my contribution in this work consisted in the general idea and line of reasoning behind the main results as well as majority of analytical considerations behind it. Furthermore, I have been responsible for the exemplary application to entanglement; it has, however, to be stressed that the background framework originates from a parallel work [20]. As I have not been principal investigator behind the volume considerations, I decided to omit this part in my thesis.

On channel designs [25]

In this chapter, which is not yet a part of any manuscript, we consider specialisation of the idea of designs to quantum channels and demonstrate that they can be constructed as induced designs from unitary designs. We demonstrate necessary and sufficient condition for a set of channels to be a k -channel t -design in terms of a well defined Choi states. Using the above properties we use the Clifford group, which is a known unitary 3-design, to construct single-qubit k -channel 3-designs composed of 49 channels with distinct weights, and extend it for arbitrary $k \in \mathbb{R}_+$. Finally, we propose an application of the considered distributions to estimation of the effective dimensionality of the environment of a given system.

Results of this chapter have been derived in accord with the goals put forward in the Preludium BIS grant and the individual research plan. Entirety of the results has been derived by myself, and as such are included in this thesis without omissions.

Guide to the Thesis

Given the comprehensive nature of this thesis, a structured guide is provided below to facilitate navigation through its extensive content:

- ▶ The Introduction covers fundamental concepts from quantum mechanics and information theory. Experienced readers may consider skipping this section during an initial reading.
- ▶ Each chapter includes a *Preliminaries* section designed to introduce advanced foundational concepts necessary for understanding the subsequent results. Readers may choose to bypass these sections based on their existing knowledge and treat them as a source of reference during the read-through, should such be needed.
- ▶ Chapters 2, 3, and 5 primarily comprise material that has already been presented in publications [14, 17, 19]. Specifically, Chapter 2 is a less condensed version of publication [17] and thus the published manuscript may be read as a substitute.
- ▶ Chapter 4 includes substantial revisions and extensions of the author's contributions beyond those in publication [16]. It should be read independently for a comprehensive understanding. For broader context, readers may refer to the full publication.
- ▶ Chapter 6 presents previously unpublished results from forthcoming work [25]. It is recommended to read this chapter in its entirety to gain a complete perspective.
- ▶ The Summary section serves as a complement to the Abstract, offering an overarching view of the thesis content.
- ▶ The Afterword, which does not contribute to the scientific discourse of the thesis, may be skipped. However, for those interested, it is available in both Polish and English, with the latter provided in italics following the former.

INTRODUCTION AND PRELIMINARIES



1 Quantum Introduction

*"I think I can safely say that **nobody understands quantum mechanics.**"*

Richard Feynman, *The Character of Physical Law* [27]

*"Being intelligent is not a felony. **But most societies evaluate it as at least a misdemeanor.**"*

Lazarus Long, in Robert A. Heinlein's *Time Enough for Love* [2]

For more than a century quantum mechanics has stood, side by side with Einstein's general relativity, as a cornerstone of modern physics. Over the years and decades quantum theory has passed the test of time, never failing to deliver predictions in accordance with the experiments aimed at overturning it. Roots of quantum approach to physical phenomena can be traced all the way back* to de Broglie matter-waves [28], Einstein's explanation of photoelectric effect [29] and Planck's solution to the ultraviolet catastrophe [30]. In his explanation Max Planck coined the term "**quantum**" (plural *quanta*), stemming from Latin word *quantus* for "how much", referring to any physical quantity which comes in discrete packages, in contrast to the idea of continuity of most, if not all, things that can be measured – an idea very much in line with the discretization of the matter dreamed up already in the antiquity by Democritus of Abdera in the form of indivisible chunks, called atoms from Greek *átomos*, indivisible.

Modern quantum mechanics has come a long way since matrix mechanics, first formulated by Heisenberg, Born and Jordan. It is now widely accepted as an integral part of the current scientific paradigm, despite ongoing debates concerning foundational questions such as what makes quantum quantum and not classical [31], paradoxes such as the problem of Wigner's friend [32] and its recent extensions [33], to name a few. Nowadays quantum phenomena are far more than a playground for scientists, leading to almost everyday household devices such as laser pointers used to tease cats [34], allowing us to track our position on Earth by keeping precise time with atomic clocks onboard GPS satellites [35, 36] and many others – the list could go on and on. My personal favourite goes all the way back to the original Nobel prize for Einstein for the explanation of the photoelectric effect, still used in the modern-day CMOS sensors to capture glimpses of our everyday lives [37] – more directly speaking, digital photography, examples of which are provided at the beginning of each section.

* As I am no historian of science, I do not claim accuracy of exact beginnings of the field of quantum mechanics, as we call it nowadays.

Quantum information community, standing at the intersection between quantum mechanics and information theory, is currently leading what can be seen as a prelude to the second quantum revolution. Quantum computers, which utilise the principles of quantum mechanics to operate directly on information stored in two-level quantum systems called qubits, are no longer a far-fetched futuristic concept thought up by Beniof [38], Manin [39] and Feynman [40] – thousand-qubit chips operate in the labs of IBM and other companies as I write these very words [41]. Quantum key distribution schemes, first thought up by Bennett and Brassard [43] and Ekert [44] are being implemented by Toshiba for commercial use [45] and discussed by national security agencies such as NSA [46–48]. Some of us may even have smartphones which possess a quantum random number generator module integrated into their chips [49].

2022 Nobel prize was awarded to Alain Aspect, John Clauser and Anton Zeilinger

“for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science.” [50]

This only serves to underline that quantum mechanics becomes more and more important in the modern world of science and far beyond. For this very reason, and to honour both my grandfathers and their long history of teaching physics, this introduction will be my humble attempt to describe the basics of quantum mechanics in a way that will not only give my fellow scientists tools necessary to understand the remainder of this thesis, but also give people less used to academic physics a window into the strange principles governing computers, cameras, lasers – everyday, yet mysterious, items.

1.1 Quantum versus classical state space and quantum superposition

Let us start with a simple, almost customary example – a two-sided coin, as all coins should be. If we throw such a coin and forget about finite thickness of the coin, it can fall either heads-up or tails-up[†], two perfectly distinguishable states[‡]. In this way we have described a model physical system with two-point state space $\mathcal{S} = \{H, T\}$, with letters for Heads and Tails, see Fig. 1.1. Such a system can be thought to store *a bit* – a unit of information that can assume only two different states, which are more commonly associated with off and on states of an electric circuit, or 0 and 1 binary digit, respectively.

[†] If one insists that a tossed coin can fall onto its side and such a possibility should be taken into account, we refer the reader to [51].

[‡] We disregard cases of coins with identical sides, for possible consequences see [52].



Figure 1.1: Two states of a coin: While there is nothing particularly quantum about Japanese currency, it serves just as well as any other as an exemplar of a two-faced coin. The Heads side H of the coin shows the denomination (100 yen) and the year of issue (2016, or Heisei 28), while the other side displays the issuer (Bank of Japan), the denomination in words and an impression of a cherry blossom.

Every throw of such a coin is a random event which can be described by probabilities p_H, p_T assigned to the event of falling on Heads or Tails, respectively. All of us understand intuitively an idea of a fair coin, which falls on either of its sides with equal likelihood – therefore, approximately half of the throws yield Heads, and the other half Tails. We can make it more rigorous by considering an experiment, in which we throw our coin N times, out of which N_H are Heads and N_T are Tails. The observed ratios N_H/N and N_T/N are not probabilities yet, not quite. However, as our sample size goes to infinity, $N \rightarrow \infty$, the aforementioned ratios approach the actual probabilities,

$$p_H = \lim_{N \rightarrow \infty} \frac{N_H}{N}, \quad p_T = \lim_{N \rightarrow \infty} \frac{N_T}{N},$$

a formulation in line with so called frequentist approach to probability[§]. One could easily extend this to other classical systems with more distinguishable states, such as dices with varying number of faces.

With this short explanation of a classical bit we can consider how a quantum system works. As an illustrative example, once again, we will think about a simplest system with just two distinguishable states – let us stick to the metaphor of a coin. First, we need to consider what is actually a state-space of such a system. The first assumption we will need is the following

Assumption 1.1. *For a system with d distinguishable states in total each distinguishable state j corresponds to an orthogonal normalised vector $|j\rangle \in \mathbb{C}^d$ in d -dimensional complex vector space,*

$$\langle j|k\rangle = \delta_{jk}. \quad (1.1)$$

The reason behind orthogonality, equivalent of right angle between two 3-dimensional vectors, extended to complex vector spaces. This comes with an interesting consequence – a quantum coin with distinguishable states of heads $|H\rangle$ and tail $|T\rangle$ can end up “somewhere in between” in a quantum superposition of two states

[§] This explanation would have been disputed by David Hume; to quote him, “That the sun will not rise tomorrow is no less intelligible a proposition, and implies no more contradiction, than the affirmation, that it will rise.”[53]

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm |T\rangle) \quad (1.2)$$

with the two states orthogonal, $\langle + | - \rangle = 0$ and therefore – by the assumption 1 – perfectly distinguishable (see Fig. 1.2 for not artist's impression).



Figure 1.2: Coin in a superposition(?): Although one might be tempted to think in this way, the above is not an accurate depiction of a superposition of a quantum coin.

More generally, we have superpositions $\alpha |H\rangle + \beta |T\rangle$ with $\alpha, \beta \in \mathbb{C}$. The coefficients are restricted by a relation $|\alpha|^2 + |\beta|^2 = 1$ – for now it is only a cryptic relation, which will turn out to be natural soon enough.

Note 1.1. Any state $|\psi\rangle \in \mathbb{C}^d$ can be given in terms of its components with respect to the so-called computational basis $\{|i\rangle\}_{i=1}^d$,

$$|\psi\rangle = \sum_{i=1}^d a_i |i\rangle \quad (1.3)$$

with $a_i \in \mathbb{C}$.

In the next section we will consider probabilistic aspect of quantum mechanics – a reason for the aforementioned puzzling relation between the coefficients α and β and justification for saying that the states $|\pm\rangle$ are “somewhere in between” heads and tail.

Before we move on, let us consider a situation when we have two distinguishable coins. If they were classical, the space of states would be given by a Cartesian product of the individual state spaces,

$$\mathcal{S} = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}, \quad (1.4)$$

which is intuitive – two coins can fall either both heads, both tails or heads-tails mix, see Fig. 1.3.

Taking Assumption 1.1 seriously, we find that the state-space of two quantum coins is a 4-dimensional space,

$$\mathbb{C}^4 = \text{span}(|HH\rangle, |HT\rangle, |TH\rangle, |TT\rangle) = \text{span}(|H\rangle, |T\rangle) \otimes \text{span}(|H\rangle, |T\rangle) = (\mathbb{C}^2)^{\otimes 2}, \quad (1.5)$$



Figure 1.3: States of a classical pair of coins: Four possible states of a pair of tossed coins – two heads, two tails and two ordered pairs of heads-tails.

which is, technically speaking, just a tensor product of the spaces corresponding to the individual quantum coins. This extends to arbitrary quantum systems.

Observation 1.1. Consider a quantum system consisting of n subsystems, with i -th subsystem having d_i distinguishable states. Then, the total state-space of the system is given by

$$\mathbb{C}^D = \bigotimes_{i=1}^n \mathbb{C}^{d_i} \quad (1.6)$$

with the total dimension $D = \prod_{i=1}^n d_i$.

This structure of available states allows for two quantum coins to exist in their own local states, for example

$$|\psi_{AB}\rangle = \frac{|H\rangle + |T\rangle}{\sqrt{2}} \otimes \frac{|H\rangle - |T\rangle}{\sqrt{2}} = \frac{|HH\rangle - |HT\rangle + |TH\rangle - |TT\rangle}{2} \quad (1.7)$$

but also can lead to certain peculiarities that worried founding fathers of quantum mechanics [54], namely states like

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |TT\rangle) \quad (1.8)$$

which is “somewhere in between” both coins being heads- and tails-up and therefore becoming quantumly entangled.

Note 1.2. Any bipartite state $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be written in terms of components $a_{i_A i_B}$ with respect local computational bases $\{|i_A\rangle\}_{i_A=1}^{d_A}$ and $\{|i_B\rangle\}_{i_B=1}^{d_B}$ as

$$|\psi\rangle = \sum_{i_A=1}^{d_A} \sum_{i_B=1}^{d_B} a_{i_A i_B} |i_A\rangle \otimes |i_B\rangle \equiv \sum_{i_A=1}^{d_A} \sum_{i_B=1}^{d_B} a_{i_A i_B} |i_A i_B\rangle. \quad (1.9)$$

with $a_{i_A i_B} \in \mathbb{C}$.

Note 1.3. All the states in $\mathcal{H}_{AB}$ that can be written in the form

$$|\psi_{sep}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \quad (1.10)$$

are referred to as product states and can be understood as two quantum systems in separate, local states.

Any state that does not admit such a decomposition,

$$|\psi_{ent}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \quad (1.11)$$

is called an entangled state and cannot be directly separated into two subsystems in their own local states.

How to interpret this “in-between-ness”? Everything will become clearer once we understand connection between quantum states and probabilities of measurement outcomes – an essential feature of the theory.

1.2 Measurements of distinguishable states and generalized measurements

When given a classical coin, it is straightforward to tell whether after a toss it landed heads- or tails-up. As such, measurement turns out to be a non-topic for such a classical system and others with discrete state-space – the system either is, or is not, in a certain state.

However, as it was mentioned already, by assumption 1 the states of a quantum system need to be orthogonal in order to be distinguishable. Falling back on our workhorse example of a quantum coin, a straightforward calculation of inner product between the Heads state $|H\rangle$ and the “in-between” state $|+\rangle$,

$$\langle H|+\rangle = \frac{1}{\sqrt{2}}(\langle H|H\rangle + \langle H|T\rangle) = \frac{1}{\sqrt{2}}, \quad (1.12)$$

that the two states are clearly not orthogonal and therefore not distinguishable. This prompts further questions – in what sense $|H\rangle$ not distinguishable from $|+\rangle$? To explain this, we will need to invoke the second assumption of quantum mechanics,

Assumption 1.2. Consider a pair of states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$. Probability $p(|\phi\rangle | |\psi\rangle)$ of measuring the state $|\psi\rangle$ in state $|\phi\rangle$ is given by the squared absolute value of the overlap between them,

$$p(|\phi\rangle | |\psi\rangle) = |\langle\psi|\phi\rangle|^2. \quad (1.13)$$

Note 1.4. Any operator $\hat{O} \in \mathcal{B}(H_d)$ can be given in terms of its matrix elements,

$$\hat{O} = \sum_{i,j=1}^d O_i^j |i\rangle\langle j|. \quad (1.14)$$

An example of the above is easy – probability of measuring state $|+\rangle$ in state $|H\rangle$ is given by

$$|\langle H|+\rangle|^2 = \frac{1}{2} = |\langle T|+\rangle|^2 \quad (1.15)$$

where the second equality is simple to calculate, as total probability of measuring $|+\rangle$ in either of the two available distinguishable states $|H\rangle$, $|T\rangle$ must sum up to one. Consequences of the Assumption 1.2 are in fact far-reaching, as illustrated by the following two observations:

Observation 1.2. Probability of observing a state $|\psi\rangle$ in the same state $|\psi\rangle$ should be equal to one, thus fixing the normalisation of the state,

$$|\langle \psi|\psi\rangle|^2 = 1 \Rightarrow \langle \psi|\psi\rangle = \|\psi\| = 1. \quad (1.16)$$

In a less technical talk – if we already got heads or tails laying on the table, the state of the coin will not change upon looking at it again.

Observation 1.3. Probability of observing a state $|\psi\rangle$ in a state $|\tilde{\psi}\rangle = e^{i\phi} |\psi\rangle$, differing from the original state only by a global phase $e^{i\phi}$ with $\phi \in \mathbb{R}$, is given by

$$|\langle \psi|\tilde{\psi}\rangle|^2 = |e^{i\phi}|^2 = 1, \quad (1.17)$$

and, in consequence, making the two states equivalent, $|\psi\rangle \sim e^{i\phi} |\psi\rangle$.

Collecting the two observations together – the normalisation condition and irrelevance of the global phase – we find out that Assumption 1.1 was, partially, false advertising. The actual state-space of quantum mechanics is limited to what is called **a complex projective space** $\mathbb{CP}^d$ and the full equivalence relation is given by

$$|\psi\rangle \sim z |\psi\rangle \quad (1.18)$$

with $z \in (\mathbb{C} \setminus \{0\})$ an arbitrary complex number. Of course, to keep the probabilistic interpretation in line, we keep the requirement of unit norm on representants of each equivalence class. With this, we may forget from now on such subtleties of the framework. However, we will leave it highlighted:

Observation 1.4. Quantum states of a system with d distinguishable states are belong to the complex projective space, $|\psi\rangle \in \mathbb{CP}^d$, which is also referred to as the Hilbert space $\mathcal{H}^d$.

Equipped with the probabilistic interpretation of quantum states, we may explain two further notions – **observables**, or physical quantities we measure; and **density matrices**, which describe classical mixtures of quantum states or states of subsystems within larger settings.

Let us start with the former by considering Alice – one of the two celebrated quantum characters – entering a bet. A quantum coin in a state $|\psi\rangle$ is measured; Alice gains 1¥ if it is measured in Heads state $|H\rangle$, and loses 1¥ if it ends in Tails, $|T\rangle$. Expected gain for Alice per measurement is

$$(1¥) p(|\psi\rangle, |H\rangle) + (-1¥) p(|\psi\rangle, |T\rangle) = \quad (1.19)$$

$$= (1¥) p(|\psi\rangle\langle\psi|H\rangle\langle H|) + (-1¥) p(|\psi\rangle\langle\psi|T\rangle\langle T|) = \quad (1.20)$$

$$= \langle\psi|[(1¥)|H\rangle\langle H| + (-1¥)|T\rangle\langle T|]|\psi\rangle \quad (1.21)$$

$$= \langle\psi|\hat{O}_A|\psi\rangle \quad (1.22)$$

where we implicitly defined the operator $\hat{O}_A = (1¥)|H\rangle\langle H| + (-1¥)|T\rangle\langle T|$, also in its matrix form

$$\hat{O}_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.23)$$

which captures the essence of **observables**, objects that govern all measurable quantities in quantum mechanics, summarised in (semi-independent) assumption.

Assumption 1.3. *Every measurable physical quantity can be described by a bounded Hermitian operator $\hat{O} \in \mathcal{B}(\mathcal{H}^d)$, $\hat{O} = \hat{O}^\dagger$, and its expectation value in a state $|\psi\rangle$ is given by*

$$\langle\hat{O}\rangle_{|\psi\rangle} = \langle\psi|\hat{O}|\psi\rangle. \quad (1.24)$$

The above, thanks to properties of Hermitian operators, ensures that for every state the expectation value is real, in line with the fact that we always observe real values for quantities such as position, momentum, angular momentum and all the rest.

Now let us consider Bob – the other quantum protagonist – who wants to measure whether his system is in a state $|\phi\rangle$. However, we – the all-knowing narrators of all thought experiments – know that the system he is about to measure yields quantum state $|\psi_1\rangle$ with probability p_1 , $|\psi_2\rangle$ with probability p_2 , up to p_n and $|\psi_n\rangle$. Thus, the probability of measuring state $|\phi\rangle$ is

$$p_1 |\langle\psi_1|\phi\rangle|^2 + p_2 |\langle\psi_2|\phi\rangle|^2 + \dots + p_n |\langle\psi_n|\phi\rangle|^2 = \quad (1.25)$$

$$= \sum_{i=1}^n p_i |\langle\psi_i|\phi\rangle|^2 = \langle\phi| \left(\sum_{i=1}^n p_i |\psi_i\rangle\langle\psi_i| \right) |\phi\rangle \quad (1.26)$$

$$= \langle\phi|\rho|\phi\rangle = \text{Tr}(|\phi\rangle\langle\phi|\rho) \quad (1.27)$$

where, again implicitly, we introduced the **density matrix** $\rho = \sum_{i=1}^n p_i |\psi_i\rangle\langle\psi_i|$, describing a (classically) probabilistic source of quantum states. We denote the space of d -dimensional density matrices by Ω_d

Now, in order to consider both observables and density matrices in a bipartite context, let us again consider Alice's quantum bet and extend the picture with Bob having a second quantum coin, but not taking part in the bet. From now on, for convenience, we will start using the numbering notation, assigning 0 for Heads and 1 for Tails, ie. $|H\rangle \equiv |0\rangle$, $|T\rangle \equiv |1\rangle$. In this setting the observable $\hat{O}_2$ governing the payoff is given by

$$\hat{O}_{AB} = (1\mathbb{Y}) (|00\rangle\langle 00| + |01\rangle\langle 01|) + (-1\mathbb{Y}) (|10\rangle\langle 10| + |11\rangle\langle 11|) \quad (1.28)$$

$$= [(1\mathbb{Y}) |0\rangle\langle 0| + (-1\mathbb{Y}) |1\rangle\langle 1|] \otimes [|0\rangle\langle 0|_A + |1\rangle\langle 1|] = \hat{O}_A \otimes \mathbb{I}. \quad (1.29)$$

Note 1.5. Any bipartite operator $\hat{O} \in \mathcal{B}(\mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B})$ can be given in terms of matrix elements as

$$\hat{O} = \sum_{i_A, j_A=1}^{d_A} \sum_{i_B, j_B=1}^{d_B} O_{j_A j_B}^{i_A i_B} |i_A i_B\rangle\langle j_A j_B| = \sum_{i_A, j_A=1}^{d_A} \sum_{i_B, j_B=1}^{d_B} O_{j_A j_B}^{i_A i_B} |i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B| \quad (1.30)$$

Alice's and Bob's coins are in a joint state $\mathcal{H}_{AB} \ni |\psi_{AB}\rangle = \sum_{i_A i_B} a_{i_A i_B} |i_A i_B\rangle$ and thus, by Assumption 1.3, the expected payoff for Alice is

$$\langle \psi_{AB} | \hat{O}_2 | \psi_{AB} \rangle = \sum_{i_A, j_A, k_A, l_A=0}^1 \sum_{i_B, j_B, k_B, l_B=0}^1 a_{i_A i_B}^* a_{l_A l_B} (O_1)_{k_A}^{j_A} \delta_{j_B k_B} \langle i_A i_B | j_A j_B \rangle \langle k_A k_B | l_A l_B \rangle \quad (1.31)$$

$$= \sum_{j_A, k_A, \alpha=1}^1 a_{j_A \alpha}^* a_{k_A \alpha} (O_1)_{k_A}^{j_A} = \sum_{j_A, k_A=0}^1 (O_1)_{k_A}^{j_A} \left(\sum_{\alpha=0}^1 a_{j_A \alpha}^* a_{k_A \alpha} \right) \quad (1.32)$$

$$= \sum_{j_A, k_A=0}^1 (O_1)_{k_A}^{j_A} (\rho_A)_{j_A}^{k_A} = \text{Tr}(\hat{O}_1 \rho_A) \quad (1.33)$$

In the above expression we have smuggled in an elementwise definition of the partial trace of the state $\rho_{AB} \equiv |\psi_{AB}\rangle\langle\psi_{AB}|$, namely

$$\rho_A \equiv \text{Tr}_B \rho_{AB} = \sum_{i_B} \langle i_B | \rho_{AB} | i_B \rangle \quad (1.34)$$

Note 1.6. Consider a bipartite operator $\hat{O} \in \mathcal{B}(\mathcal{H}_{AB})$ with both local dimensions equal d . It's partial trace with respect to the second subsystem is given element-wise by

$$\left(\text{Tr}_B \hat{O} \right)_{j_A}^{i_A} = \sum_{\alpha=1}^d O_{j_A \alpha}^{i_A \alpha}. \quad (1.35)$$

We can look at a specific example, when the shared state is $|\psi_{AB}\rangle = |\Psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Using the formula for partial trace we find that

$$\rho_A = \text{Tr}_B (|\psi_{AB}\rangle\langle\psi_{AB}|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbb{I}}{2} \quad (1.36)$$

and the expected gain from the bet for Alice is

$$\text{Tr}(\hat{O}_1 \rho_A) = 0. \quad (1.37)$$

Interestingly, even if we define a different local observable

$$\hat{O}_3 = |\psi\rangle\langle\psi| - |\psi_\perp\rangle\langle\psi_\perp|, \quad (1.38)$$

with the only restriction on the local states being their distinguishability, $\langle\psi|\psi_\perp\rangle = 0$, we will still find that

$$\text{Tr}(\hat{O}_3 \rho_A) = 0. \quad (1.39)$$

Unfortunately, in order to define a more general notion of a measurement, we will need to depart from the analogies used so far. Let us, however, do our best to connect it back to all that has been said up till now. At this stage we will need an extra assumption concerning the post-measurement states, which is in fact an extension of 1.3

Assumption 1.4. Consider an observable $\hat{O} \in \mathcal{B}(\mathcal{H}^d)$, defined by its eigenvalues $\lambda_i \in \mathbb{R}$ and corresponding set of d distinguishable eigenstates $|\lambda_i\rangle \in \mathcal{H}^d$. The probability of obtaining the value λ_i in a state described by a density matrix $\rho \in \mathcal{B}(\mathcal{H}^d)$ is given by

$$p_i = \text{Tr}(\rho |\lambda_i\rangle\langle\lambda_i|). \quad (1.40)$$

In such a case the post-measurement state is $\rho_i = |\lambda_i\rangle\langle\lambda_i|$.

We can distil from this a set of nonnegative operators $\{M_i = |\lambda_i\rangle\langle\lambda_i| \geq 0\}$, which provide a resolution of identity, $\sum_i |\lambda_i\rangle\langle\lambda_i| = \mathbb{I}$. Moreover, this set is actually a set of rank-1 orthogonal projectors, $M_i M_j = \delta_{ij} M_i$. A measurement defined in this manner is so call **von Neumann measurement**. One can abstract from this a notion of a generalized measurement based on positive operator-valued measure (POVM).

Definition 1.1. A set of positive operators $\{M_i \geq 0\}$ from $\mathcal{B}(\mathcal{H}^d)$, providing a resolution of identity, $\sum_i M_i = \mathbb{I}$, defines a positive operator valued measure (POVM). A quantum measurement based on a POVM, applied to a state described by a density matrix $\rho \in \mathcal{B}(\mathcal{H}^d)$, yields i -th result with probability

$$p_i = \text{Tr}(M_i \rho) \quad (1.41)$$

and results in a post-measurement state

$$\rho_i = \frac{E_i \rho E_i^\dagger}{\text{Tr}(E_i \rho E_i^\dagger)} \quad (1.42)$$

where the effects E_i are determined (not uniquely) by measurement operators and relations $M_i = E_i^\dagger E_i$.

Now we can see that the von Neumann measurement defined by an orthonormal basis $M_i = |\lambda_i\rangle\langle\lambda_i|$, $\langle\lambda_i|\lambda_j\rangle = \delta_{ij}$ fits perfectly well within the framework of measurements based on POVMs. What is more interesting is the fact, that in this case the effects $E_i = U |\lambda_i\rangle\langle\lambda_i|$ uniquely determined up to unitary post-processing U .

1.2.1 An exercise – Alice and Bob outsmarting Victor

Although this small section covers a concept which goes beyond the scope of this introduction, I encourage the reader to consider the following exercise as a mathematical riddle to play around in their spare time; experienced readers may skip it, or read it for their own enjoyment. As it exhibits one of the most famous result of the foundations of quantum information. Which? I will defer revealing this piece of quantum information until the very end of this chapter.

Let us once again think about Alice and Bob, the betting duet which we will encounter again in the course of this thesis, and place two coins in their hands, one per each. Moreover, imagine they are sitting in two distant locations. Victor got them into a game of bets, but a complicated one this time.

Classical variant

Alice and Bob are playing a game with a central party – we will call him Victor – who holds two coins of his own. He is sending the results of his own throws to Alice and Bob in sealed envelopes, each receiving the result from one coin in a form of written H or T. To the best of Alice's and Bob's knowledge, he might not be throwing coins at all, but writing the letters based on whatever he is reading currently, or even sending constantly "TT" messages – and this is what the reader should assume. The only thing that is certain is that once the envelopes are sealed and sent, the messages do not change.

Now, why did we put coins in hands of Alice and Bob? Focus on Alice, as she is flipping her coin to determine which bet out of two should she choose. The bets, which will be denoted $\mathbf{A}_H$ and $\mathbf{A}_T$, each give one point for one of the results in the envelope and minus one point for the other. In mathematical terms, we could write it as

$$\mathbf{A}_H(H) = 1, \quad \mathbf{A}_H(T) = -1, \quad (1.43)$$

or in a more verbose manner, whenever Alice gets heads on her coins, and then finds heads in the envelope from Victor, she writes one point for the round and in case of tails, she writes minus one. This is just an exemplary bet – both $\mathbf{A}_H$ and $\mathbf{A}_T$ are up to Alice to determine. Moreover, she does not add the points up – for the moment, she should keep the list of results for each round. Bob is put in the same situation, only with his own bets $\mathbf{B}_H$ and $\mathbf{B}_T$ to be determined. Furthermore, Alice and Bob keep record of which bet they happened to choose in each round.

Now we can unveil the manner in which the final score is calculated, as the three meet up to tally the points. Each time Alice and Bob chose both heads-bet, or had mixed heads-tails bets, they multiply the scores from that round and add it; however, if they both happend to have tails-bets, they multiply the scores and subtract them. Put in a mathematical formula, the joint points they receive are given by

$$\mathbf{V} = \mathbf{A}_H \mathbf{B}_H + \mathbf{A}_H \mathbf{B}_T + \mathbf{A}_T \mathbf{B}_H - \mathbf{A}_T \mathbf{B}_T. \quad (1.44)$$

The riddle consists in a simple question – what is the maximal average number of points per round that they can get? Victor is giving one yen per point – how much can they earn after a hundred-round game?

Quantum variant

Observant reader may have noticed that we introduced four coins in this game – Alice and Bob holding one each and two extra ones in the hands of Victor. For the purpose of the second part of this exercise we keep Alice's and Bob's coins classical, while changing Victor's coins to the quantum variety. His strategy changes slightly as well – he prepares them in a joint state $|\Psi_V\rangle \in \mathcal{H}_A^2 \otimes \mathcal{H}_B^2$. The notation is selected for a reason - half of the state goes to Alice, and the other half to Bob, without measuring the state.

Respective strategies of Alice and Bob also change – their bets $\mathbf{A}_H, \mathbf{A}_T, \mathbf{B}_H, \mathbf{B}_T$ get promoted to quantum observables, each of a form similar to Eq. (1.38). For clarity, we will distinguish them from their classical counterparts by attaching a hat, eg. $\hat{\mathbf{A}}_H$.

Now, if we allow Victor any state that Alice and Bob tell him to, and the observables they measure are also arbitrary, as long as they adhere to the form (1.38), what is the maximal average gain per round? The mathematical expression to be evaluated is

$$\langle \Psi_V | \hat{\mathbf{V}} | \Psi_V \rangle = \langle \Psi_V | \hat{\mathbf{A}}_H \otimes \hat{\mathbf{B}}_H + \hat{\mathbf{A}}_H \otimes \hat{\mathbf{B}}_T + \hat{\mathbf{A}}_T \otimes \hat{\mathbf{B}}_H - \hat{\mathbf{A}}_T \otimes \hat{\mathbf{B}}_T | \Psi_V \rangle. \quad (1.45)$$

This is it, the entire statement of the problem. I trust that the reader will enjoy solving the riddle!

1.3 Quantum evolution, quantum channels and channel-state isomorphism

From what has been presented so far an unacquainted reader may assume that quantum mechanics is a theory of states and measurements. Nothing further from the truth, since the most famous equation of the quantum mechanics – Schrödinger's equation, first introduced almost a century ago in 1926 [55] – is precisely what governs the time evolution. It is given as a first-order differential equation[‡],

$$i \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad (1.46)$$

where $\hat{H}$ is the Hamiltonian operator, an observable defining the energy levels of the system in question. Extending the basic result of $de^{ax}/dx = ae^{ax}$ to the matricial setting we find the solution,

$$|\psi(t)\rangle = e^{-it\hat{H}} |\psi(0)\rangle = U(t) |\psi(0)\rangle \quad (1.47)$$

where the operator $U \equiv U(t) = e^{-it\hat{H}}$ turns out to be unitary, $U^\dagger U = \mathbb{I}$, as H is hermitian.. This, in fact, ensures that normalisation of states is maintained, since unitary matrices are at the same time isometries, which is easily seen by considering

$$\begin{aligned} \|\psi(t)\rangle\| &= \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)| \underbrace{U^\dagger U}_{=\mathbb{I}} |\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = \|\psi(0)\rangle\| \end{aligned} \quad (1.48)$$

so if the starting state was properly normalized, $\|\psi(0)\rangle\| = 1$, the same is true throughout the time evolution. This property is indispensable, as it guarantees that the theory maintains its probabilistic interpretation.

As an aside, we may note that the projector $|\psi(t)\rangle\langle\psi(t)|$ can be easily shown to satisfy a differential equation called the Heisenberg equation,

$$i \frac{d}{dt} |\psi(t)\rangle\langle\psi(t)| = \hat{H} |\psi(t)\rangle\langle\psi(t)| - |\psi(t)\rangle\langle\psi(t)| \hat{H} = [\hat{H}, |\psi(t)\rangle\langle\psi(t)|] \quad (1.49)$$

with $[A, B] = AB - BA$ called the commutator of the operators A and B . By linearity it can be extended to an arbitrary density matrix,

$$i \frac{d}{dt} \rho(t) = [\hat{H}, \rho(t)]. \quad (1.50)$$

and its solution is given by $\rho(t) = U\rho(0)U^\dagger$.

[‡] Here and in all the following considerations we resort to what is called the atomic units, meaning we disregard the famous Dirac constant (also known as the reduced Planck constant) by setting the units of action in such a way that $\hbar = 1$.

Schrödinger and Heisenberg equations describe evolution of closed quantum systems, which do not interact with any external systems. Let us consider a joint evolution of a principal system described by a density matrix $\rho_A \in \mathcal{B}(\mathcal{H}^d)$ together with an ancillary two-dimensional system in a pure state $|0\rangle \in \mathcal{H}^2$. The starting joint state is given by

$$\rho_{AB}(0) = \rho_A \otimes |0\rangle\langle 0| = \begin{pmatrix} \rho_A & 0 \\ 0 & 0 \end{pmatrix} \quad (1.51)$$

where the matrix is given in a block form, that is each 0 is actually a $d \times d$ matrix filled with zeros. This state evolves according to a joint unitary U defined block-wise as

$$U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}, \quad (1.52)$$

where U_{ij} represent square blocks of order d for $i, j = 0, 1$.

Thus, the resulting state is given by

$$\rho_{AB}(t) = U \rho_{AB}(0) U^\dagger = \begin{pmatrix} U_{00} \rho U_{00}^\dagger & U_{00} \rho U_{10}^\dagger \\ U_{10} \rho U_{00}^\dagger & U_{10} \rho U_{10}^\dagger \end{pmatrix} \quad (1.53)$$

which allows us to see what is the final state of the principal system by considering its partial trace,

$$\rho_A(t) = \text{Tr}_B(\rho_{AB}(t)) = U_{00} \rho U_{00}^\dagger + U_{10} \rho U_{10}^\dagger = \sum_{i=0}^1 U_{i0} \rho U_{i0}^\dagger. \quad (1.54)$$

This clearly is not unitary evolution! Nevertheless, in order to guarantee probabilistic interpretation of the theory we need

$$\text{Tr}(\rho_A(t)) = \text{Tr} \left[\left(\sum_i U_{i0}^\dagger U_{i0} \right) \rho_A(0) \right] = \text{Tr}(\rho_A(0)) \quad (1.55)$$

which will be guaranteed for all starting $\rho_A(0)$ only if $\sum_i U_{i0}^\dagger U_{i0} = \mathbb{I}$. It is clearly true, since the matrix specified in (1.52) is unitary.

Now we may forget about the total system, which is closed and undergoing unitary evolution as prescribed by the Schrödinger equation, and consider an operation $\Phi : \mathcal{B}(\mathcal{H}^d) \mapsto \mathcal{B}(\mathcal{H}^d)$ mapping density matrices into density matrices. Such an operation, called a *quantum channel*, can be written in the form

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger. \quad (1.56)$$

where the operators K_i are called Kraus operators. The resolution of identity condition, $\sum_i K_i^\dagger K_i = \mathbb{I}$, imposed on the entire set ensures the trace preservation condition.

The above form of a quantum channel, first provided by Karl Kraus in 1983 [56], is enough to ensure another property essential for a quantum channel – *complete positivity*. Let us take a step back and look at what *positivity* of a channel is. It means that every positive semidefinite operator $\rho \geq 0$ is mapped to a positive semidefinite operator,

$$\rho \geq 0 \Rightarrow \Phi(\rho) \geq 0. \quad (1.57)$$

This ensures that probabilities in the decomposition $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ remain positive.

This, however, is not enough. A positive map may preserve positivity when acting on an entire system, but fail when applied only to a single subsystem. Thus, we consider an extension of the positivity, called *the complete positivity condition*, summarised in the following equation

$$\mathcal{B}(\mathcal{H}_A^{d_A} \otimes \mathcal{H}_B^{d_B}) \ni \rho \geq 0 \Rightarrow (\Phi \otimes \mathbb{I}_B)(\rho) \geq 0. \quad (1.58)$$

Let us translate the equation back to words. First, we take a state of a composite system consisting in parties A and B . Next, we act with the channel Φ only on the subsystem A , while maintaining B unperturbed. Complete positivity ensures that such an extended operation keeps the total system AB in a proper quantum state.

Operations which are positive, but not completely positive, are slightly counter-intuitive, and thus it is in order to provide an example of such. Consider a transposition operation $\cdot^T$, which in an obvious way preserves positivity,

$$\rho \geq 0 \Rightarrow \rho^T \geq 0. \quad (1.59)$$

Now, consider the state $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, with its non-accurate depiction in Fig. 1.4. We can write out explicitly the corresponding density matrix $|\Psi_+\rangle\langle\Psi_+|$ and its transpose in the first subsystem $|\Psi_+\rangle\langle\Psi_+|^{T_A}$, corresponding to block-wise transposition,

$$|\Psi_+\rangle\langle\Psi_+| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow |\Psi_+\rangle\langle\Psi_+|^{T_A} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1.60)$$

It is relatively simple, and we leave it to the reader, to show that for $|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ one finds that $\text{Tr}(|\Phi_-\rangle\langle\Phi_-| |\Psi_+\rangle\langle\Psi_+|^{T_A}) = -1/2$, a clearly negative value! Interestingly, the fact that partial transposition is not completely positive together with the fact that it is positive on non-entangled states adds up to a positive partial transpose (PPT) criterion, introduced independently by Horodecki family [57] and Asher Peres [58], which states that a state ρ is entangled if (but not only if!) its partial transpose ρ^{T_A} is non-positive.

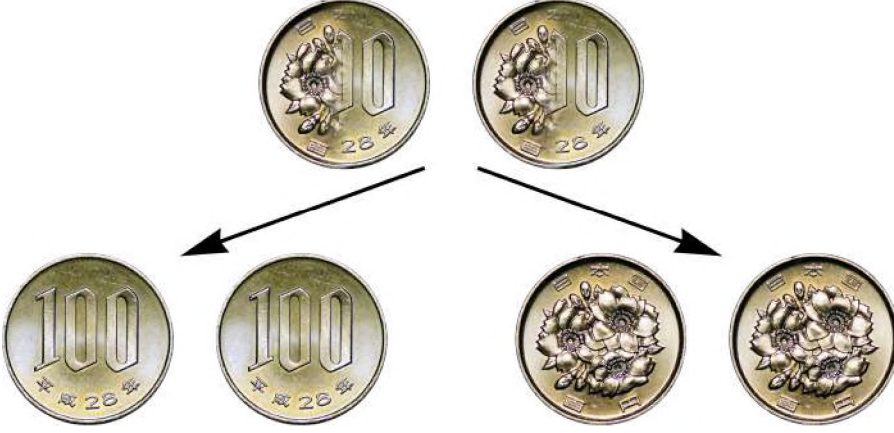


Figure 14: Non-accurate) depiction of entangled state: A maximally entangled state of two quantum coins, $|\Psi_+\rangle (|HH\rangle + |TT\rangle)/\sqrt{2}$, may be imagined as two coins in a peculiar “in-between” state, which yields only both-heads or both-tails results when flipped (or, in proper quantum-mechanical terms, measured). One should, however, keep in mind that this is not an accurate depiction – a single pair of local measurements is far from enough to determine whether a given probability distribution stems from an entangled state or not.

Last, but not least, comes the consideration of certain close kinship between channels and states, proved first by Choi [59] and Jamiołkowski [60]. Consider that all the operations we have considered so far are actually linear, in the sense that for a channel Φ we have

$$\Phi(a\rho + b\sigma) = a\Phi(\rho) + b\Phi(\sigma) \quad (1.61)$$

for any matrices ρ, σ and coefficients $a, b \in \mathbb{R}$. Thanks to this, a channel is fully characterized by its action on the matrices with only a single non-zero entry, $|i\rangle\langle j|$. This fact allows to formulate an explicit relation between a channel $\Phi : \mathcal{B}(\mathcal{H}^d) \mapsto \mathcal{B}(\mathcal{H}^d)$ and a certain state $\sigma_\Phi \in \mathcal{B}(\mathcal{H}^{d^2})$.

Consider a maximally entangled generalized Bell state in $\mathcal{H}^{d^2}$

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle. \quad (1.62)$$

The corresponding density matrix is given by

$$|\Psi\rangle\langle\Psi| = \frac{1}{d} \sum_{i,j=1}^d |i\rangle\langle j| \otimes |i\rangle\langle j|. \quad (1.63)$$

Now we consider action of the channel Φ on the second subsystem, while the first is left unperturbed. The resulting state is given as

$$\sigma_\Phi \equiv (\mathbb{I} \otimes \Phi)(|\Psi\rangle\langle\Psi|) = \frac{1}{d} \sum_{i,j=1}^d |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|). \quad (1.64)$$

When viewed as a matrix composed of $d \times d$ blocks, one sees that

$$(\sigma_\Phi)_{ij} = \Phi(|i\rangle\langle j|) \quad (1.65)$$

and the channel can be reconstructed uniquely by considering these blocks. This surprising correspondence between operations and states within the very same quantum theory leads to surprising concepts such as quantum gate teleportation [61] and will allow us in the final chapter of this thesis to connect distributions in the space of states (which are relatively easy to imagine) with distributions in the space of quantum channels.

1.4 Further reading

I do not claim this introduction to quantum mechanics to be complete by any means – a few pages would not do it no matter how smart the formulation would be. For all the less science-oriented readers I recommend the work of Leonard Susskind based on his "Theoretical Minimum" lectures, where he lays down the principles in a very reader-friendly manner [62]. For the historical perspective, I recommend looking up one of the first comprehensive works on quantum mechanics, written by Born, Heisenberg and Jordan, three among the founders of the field [63]. There are plenty modern introductions, among which I personally used handbooks by Shankar, Sakurai and Messiah [64–66]. More quantum-information oriented approaches, which will be more useful for understanding the content of this thesis, have been given in handbooks by Nielsen and Chuang [67], and Benenti, Casati and Strini [68]. Finally, a monograph on geometry of quantum states, an aspect that will be recurring in this work, has been written by Bengtsson and Życzkowski [69].

Solution to the exercise

As promised, here I put forward the way to find the answer – I once again encourage the reader to embark on their own quantum journey by searching for CHSH game, named after John Clauser, Michael Horne, Abner Shimony, and Richard Holt, Bell's theorem named after John Bell, Tsirelson's bound, named after Boris Tsirelson. At first it might seem like a deep dive, but the problem, referred to more generally as the quantum nonlocality, Bell nonlocality or *spooky action at a distance*. I refrain from giving the references – the goal is to encourage the less-knowledgeable reader to have their own try at finding a digestible source.

RESULTS OF THE RESEARCH



2 Isoentangled bases and joint measurements [17]

“Within the quantum physics three properties are considered to be emblematic of the theory: superposition, interference and uncertainty... and entanglement... four most emblematic properties...no... Amongst the most emblematic properties of quantum physics... I’ll come in again.”

Paraphrase of Monty Python’s *Spanish Inquisition* [70]

Entanglement is one of the most emblematic properties exhibited by quantum systems, referred to by Schrödinger as *“the characteristic trait of quantum mechanics,”* [71] and the first identified kind of quantum nonlocality. Einstein, who referred to as *“spooky action at a distance”* (*spukhafte Fernwirkung*) [72], believed that it points to the incompleteness of the theory, which was fully fleshed out in the now famous Einstein-Podolsky-Rosen (EPR) paper [54]. Not long after John Bell pointed out that it is not the completeness of the theory but, by considering the local hidden variable model, demonstrated that a theory can remain intact if we allow it to be nonlocal from the very beginning [73, 74].

The phenomenon of entanglement lies at the heart of some of the most celebrated protocols at the foundations of quantum information, including BB84 key distribution protocol [43], quantum teleportation [75], dense coding [76] and many many more. All of the above protocols, classic for quantum information science by now, make use of entanglement in a dual role – first, in the preparation stage of entangled states at the beginning, and second, in the measurement of the entangled states at the end. A lot of effort has been made by the quantum information community to understand the structure of entanglement and nonlocality stemming from the quantum states, producing widely renowned reviews on both nonlocality [77] and entanglement [78].

Far less has been done for the classification of the nonlocality in the measurements themselves, with several exceptions [8, 79–81], yet none of the above has gone to the length of attempting a complete classification, but rather a special subset of operations. As has been pointed out by Yeong-Cherng Liang, *“We have spent decades understanding entanglement in various contexts, and we believe the time is now ripe to do the same for entangled measurements. After all, we have learnt from the latest studies of the Wigner’s friend paradox that the nature of quantum measurements is not completely understood.”* In this chapter, based on [17], we demonstrate a small step in this direction, by providing an exhaustive classification of all possible von Neumann measurements on two qubits.

We will begin this section with preliminary notions necessary to understand the content of this section, such as entanglement measures with special focus on qubits, Schmidt rank-bound subspaces essential to simplification of the problem, and discuss input-free triangular networks and the notion of output permutation invariant distribution. After this we will move to the main results, including full classification of isoentangled two-qubit bases leading to von Neumann measurements and application of a sub-family of such bases to the triangular networks. Then we will move on to discussing partial results known for higher dimensions. We will finish the section by outlining a possible approach to extending the classification for 3-level systems.

2.1 Preliminaries

In order to understand the concepts underlying the results of this chapter, we will begin by considering measures of entanglement with particular focus on so-called *tangle*, then we will move on to the notion of Schmidt rank-bound subspaces, a tool necessary for simplification of our considerations; Finally, we will introduce the notion of quantum networks with particular focus on triangular network and output-permutation invariant distributions generated by quantum measurements.

2.1.1 Measures of entanglement

Quantum entanglement is perhaps the most emblematic property of quantum states, that distinguishes them from their classical counterparts. As has been already highlighted in Section 1.1 of the introductory chapter, in the simplest of words it is related to the impossibility of understanding a total state of a system as a product of the local states of the subsystems,

$$|\psi_{AB\dots}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes \dots, \quad (2.1)$$

when restricted to pure states or, when extended to the mixed states, not even as a probabilistic mixture of such local states,

$$\rho_{AB\dots} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi} \otimes \dots \quad (2.2)$$

with $\sum_i p_i = 1$, $p_i \geq 0$. The dots $\dots$ stand for the fact that this interpretation can be extended in a straightforward manner beyond two-party scenarios. Bipartite pure scenario, however, is in some sense the simplest and thanks to this simplicity provides some properties which are non-extendible beyond this setting. In particular, a d -dimensional bipartite pure state $|\psi_{AB}\rangle \in \mathcal{H}^{d_1} \otimes \mathcal{H}^{d_2}$ is fully described by its Schmidt decomposition,

$$|\psi_{AB}\rangle = \sum_{i=1} \sqrt{\lambda_i} |\psi_{Ai}\rangle \otimes |\psi_{Bi}\rangle, \quad (2.3)$$

which is in fact simply a simple corollary of understanding $|\psi_{AB}\rangle$ as a vectorization of a certain matrix X , where the column and row indices correspond to the first and second subsystem, respectively, and taking its singular value decomposition. Indeed, the set of Schmidt coefficient vector $\lambda \equiv \{\lambda_i\}$ provides a complete description of a state up to local transformations. Even more – relations between Schmidt coefficients of two states are enough to determine, whether one state can be transformed into another using a set of operations referred to as “local operations and classical communication” (LOCC).

In this chapter we will focus on pure states of a system composed of two qubits. Any such state can be described by a local rotation of the form $U_A \otimes U_B$ with $U_A, U_B \in \mathcal{U}(2)$ acting on a state $\sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$. Note that the normalisation condition of quantum states forces the last Schmidt coefficient to be directly related to all the others, thus leaving us with a single number λ . One may take it for granted and simply work with λ as a measure of entanglement, since they are all equivalent for pure bipartite states [82], we will be working with a well-known monotone called *tangle* ξ , equivalent to concurrence squared [83–85],

$$\xi(|\psi_{AB}\rangle) = 2[1 - \text{Tr}(\rho_A^2)] = 4\lambda(1-\lambda) = 4\lambda_1\lambda_2 = C(|\psi_{AB}\rangle)^2 \quad (2.4)$$

where ρ_A is a partial trace of a state, in our case $\rho_A = \text{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$. One reasons to choose it is that concurrence is an entanglement monotone which can be extended in a straightforward fashion to arbitrary dimensions [84, 86], including continuous setting [87].

Although the results presented in this chapter will be limited to bipartite qubit systems, it is in order to at least partially mention the problems that systems consisting in more than two parties pose. Already in the case of three qubits the matters start to become complicated. First, let us consider a state $|\psi_{ABC}\rangle$ and the problem of its Schmidt decomposition. It can be written in three ways,

$$\begin{aligned} |\psi_{ABC}\rangle &= \sum_i \sqrt{\lambda_{AB|C,i}} |\psi_{AB}\rangle \otimes |\psi_C\rangle \\ &= \sum_i \sqrt{\lambda_{AC|B,i}} |\psi_{AC}\rangle \otimes |\psi_B\rangle \\ &= \sum_i \sqrt{\lambda_{BC|A,i}} |\psi_{BC}\rangle \otimes |\psi_A\rangle \end{aligned} \quad (2.5)$$

all of them equivalent, but yielding Schmidt vectors $\lambda_{AB|C}$, $\lambda_{AC|B}$ and $\lambda_{BC|A}$, which are non-equivalent. Even more surprisingly, one may find $\rho_{AB} = \text{Tr}_C |\psi_{ABC}\rangle\langle\psi_{ABC}|$, ρ_{AC} and ρ_{BC} all separable, and yet the state itself entangled. Two extreme states with respect to 3-partite entanglement are the emblematic GHZ state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ initially introduced by Greenberger, Horne and Zeilinger in its 4-partite version and later revisited by Mermin to introduce the smaller version [88, 89] and the $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ state introduced in [90]. The former has all the bipartite reductions completely separable, while the latter possesses residual

entanglement; in fact, $|W\rangle$ state was used in [90] to demonstrate that there exist inequivalent ways to entangle 3-partite systems of qubits. The situation gets far more complicated for larger underlying dimension or a larger quantity of subsystems involved. This increase in complexity is exemplified by the fact that there are only two nonequivalent classes of entanglement for three qubits, while for four their number already goes to infinity [91]. We refer the interested reader to the extensive reviews [78, 92] and a recent systematic review of the literature [93].

The above short exhibition of properties of multipartite entanglement serves as a reason for limiting ourselves to the simplest case of two qubits. In the final section of this chapter we will outline the already known facts about bases for larger local dimensions and larger number of subsystems.

Interlude: on Bloch representation

In what follows we will be considering states in Hilbert space of complex dimension 4, which means that we need 6 real numbers to fully describe any pure state $|\psi_{AB}\rangle \in \mathcal{H}_4$. Although it is not feasible, we may resort to the reductions $\rho_A = \text{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$, $\rho_B = \text{Tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$, which are defined by just three numbers

$$\rho_A = \frac{\mathbb{I}}{2} + \frac{1}{2} \sum_{i \in \{x,y,z\}} (\vec{r}_A)_i \sigma_i \quad (2.6)$$

and similarly for ρ_B . The coefficients $(\vec{r}_A)_i = \text{Tr}(\rho_A \sigma_i)$ possess the property that $\|\vec{r}_A\| \leq 1$ with equality if and only if ρ_A is pure and, in turn, $|\psi_{AB}\rangle$ is product. This simple property is the origin of the so called Bloch ball representation of states of 2-dimensional quantum systems.

We may leverage this representation by using the fact that for a state $|\psi_{AB}\rangle = U_A \otimes U_B (\sqrt{\lambda}|00\rangle + e^{i\chi}\sqrt{1-\lambda}|11\rangle)$ the norms for both reductions agree and are given by $\|\vec{r}_A\| = \|\vec{r}_B\| = |1-2\lambda|$. Using this fact, representing a 4-dimensional state $|\psi_{AB}\rangle$ by its two Bloch vectors $\vec{r}_A, \vec{r}_B$ lacks only the information about the phase χ . This loss, however, is an acceptable price for gaining some intuition for the bases composed of 4-dimensional states we will be considering later in this chapter.

It is, however, worth mentioning at this point that a general problem of Bloch bodies for quantum systems of dimension $d \geq 2$ and methods for representing them in 3-dimensional space while preserving properties that make Bloch ball a useful tool for gaining intuition, is still an active field of investigation. We may refer interested reader to a recent work on geometry of Bloch body of qutrits as an example of such investigation [94]. However, as it is not the main topic of this chapter, we will not dwell on this topic any further.

2.1.2 Schmidt rank-bound subspaces

In our attempt to provide a complete classification of orthonormal bases on two qubits composed exclusively of states of equal entanglement it will prove useful to turn them into a canonical form, from which local rotations have been factored out. As it turns out, a useful tool to do so are the so-called *Schmidt rank-bound subspaces* considered in the work of Cubitt, Montanaro and Winter [95].

The idea behind Schmidt-rank bound subspaces is simple and can be summarised in a question – what is the minimal dimensionality $|S|$ of a vector subspace S such that it must contain at least one vector of Schmidt rank r or smaller?

The entire consideration is based on the fact that any bipartite state $|\psi_{AB}\rangle = \sum_{i,j=0}^{d_A, d_B} G_{ij} |ij\rangle$ can be represented by the matrix G of their coefficients in the computational basis and number of nonzero Schmidt coefficients is equivalent to the matrix rank of G . Then, a simple fact from matrix analysis is that a matrix rank is smaller or equal to r , $\text{rank}(G) \leq r$, if and only if determinants of all $r \times r$ submatrices vanish. In turn, the results presented in [95] are based on the analysis of intersections between the subspace S and the algebraic varieties of all solutions to the system of equations generated by vanishing minors of G , which are called *determinantal varieties*, defined below.

Definition 2.1 (Determinantal variety [95, 96]). *The affine determinantal variety $\mathcal{D}_r(d_A, d_B)$ over the field $\mathbb{F}$ in the space $\mathbb{F}^{d_A d_B}$ is the variety defined by the vanishing of all order- r minors of a $d_A \times d_B$ matrix, whose elements are considered as independent variables in $\mathbb{F}$.*

The corresponding projective determinantal varieties are determined by taking away the overall scaling of the matrix G , thus reducing the number of free parameters by one.

The important aspect of the above varieties is their dimensionality.

Lemma 2.1 (Dimension of projective determinantal varieties, Lemma 4 from [95]). *The dimension of a projective determinantal variety is given by $\dim \mathcal{D}_r(d_A, d_B) = d_A d_B - (d_A - r + 1)(d_B - r + 1) - 1$.*

From the above proceeds, as a corollary, is a Proposition crucial for us.

Proposition 2.2 (Dimensionality of Schmidt rank-bound subspace, Prop. 6 from [95]). *For any subspace $S \in \mathcal{H}^{d_A} \otimes \mathcal{H}^{d_B}$ of dimension $\dim S > (d_A - r + 1)(d_B - r + 1)$, there exists at least one state in the subspace with Schmidt rank strictly less than r .*

In our particular case we will focus on $d_A = d_B$ and will be most interested in $r = d$. This, in some sense, trivializes the statement, since in such case we need only the subspace corresponding to vanishing determinant of the entire coefficient matrix G . This, however, will prove to be sufficient for our purpose.

2.1.3 Quantum networks, Output Permutation Invariant measurements and Finner inequality

A graph is an abstract mathematical object defined by the set of its vertices V and a set of its edges $E \subset V \times V$, providing information about which vertices are connected. One may extend the concept of graphs the idea of edges to connections between 3 or more nodes – so called *hyperedges*, with their set being a subset of all possible subsets of vertices, $E \subset 2^V$. These abstract objects lead to a whole branch of mathematics, known as graph theory, which is used to model and analyse various real-world networks of interactions, ranging from less-physical contexts like social networks [97] and citation networks [98], to trophic networks in ecology [99], gene interactions [100], power grids [101] to computer networks [102].

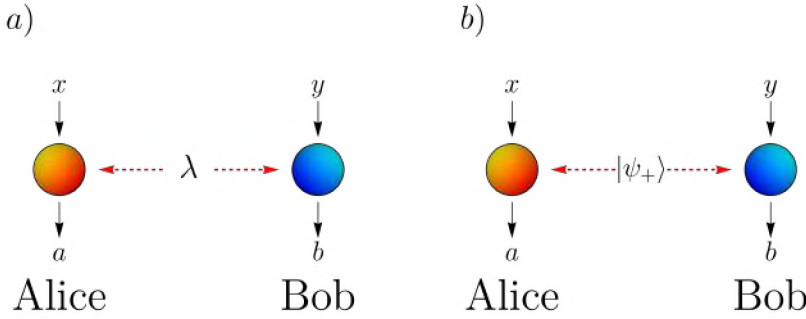


Figure 2.1: Bell nonlocality: In the simplest network, where two distant parties – Alice and Bob – share a central source of randomness. In a) it is a classical variable, and the result $p_{\text{LHVM}}(a, b | x, y)$ is a result of convolution of the local distributions $p_A(a | x; \lambda)$ and $p_B(b | y; \lambda)$ with the distribution of the hidden variable $p_{\text{LHV}}(\lambda)$. In b) Alice and Bob share an entangled state $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and perform quantum measurements to arrive at the resulting distribution $p_Q(a, b | x, y)$

In the context of quantum information graphs can be used to modelled networks, in which edges represented shared states, usually maximally entangled (with hyperedges representing usually states that exhibit multipartite entanglement), and vertices assuming the role of points in which measurements on subsystems belonging to different shared entangled states, possibly depending on external input, are deployed. The simplest scenario of such sort has been employed in the cornerstone *Gedankenexperiment* of quantum entanglement – a Bell scenario, utilising a single source of bipartite entangled states, sent over to two distant parties, Alice and Bob, performing one out of two measurements depending on external input, see Fig. 2.1. This particular scenario has been used by Bell to provide means to discern local hidden variable models (LHVM) from genuinely quantum models [73]. More specifically, the local variable leads to the distribution of outputs

$$p_{\text{LHVM}}(a, b | x, y) = \int d\lambda p_A(a | x; \lambda) p_B(b | y; \lambda) p_{\text{LHV}}(\lambda), \quad (2.7)$$

where $p_A(a | x; \lambda) p_B(b | y; \lambda)$ represent the local distributions for Alice and Bob,

respectively, conditioned on the shared variable λ with its own distribution* $p_{\text{LHV}}(\lambda)$. On the other hand, the quantum probability, assuming that the measurements the von Neumann measurements, is given by

$$p_Q(a, b | x, y) = |\langle a | x | \otimes \langle b | y | \rangle | \psi_+ \rangle|^2, \quad (2.8)$$

where $\langle a | x | a' | x \rangle = \delta_{a, a'}$ and $\langle b | y | b' | y \rangle = \delta_{b, b'}$ assure the projectivity of the measurements involved. Although different from the original expression derived by Bell, now famous CHSH inequality reads [103]

$$|p(a, b | 0, 0) - p(a, b | 0, 1) + p(a, b | 1, 0) + p(a, b | 1, 1)| \leq \begin{cases} 2 & \text{LHVM} \\ 2\sqrt{2} & \text{Quantum} \end{cases} \quad (2.9)$$

with the quantum bound derived and proved by Cirel'son [104], known as Tsirelson's bound due to different transliterations, as noted by Cirel'son himself [105].

Since then Bell nonlocality as a subfield of quantum information has expanded vastly and, for this reason, we refer the reader to an extensive review [77].

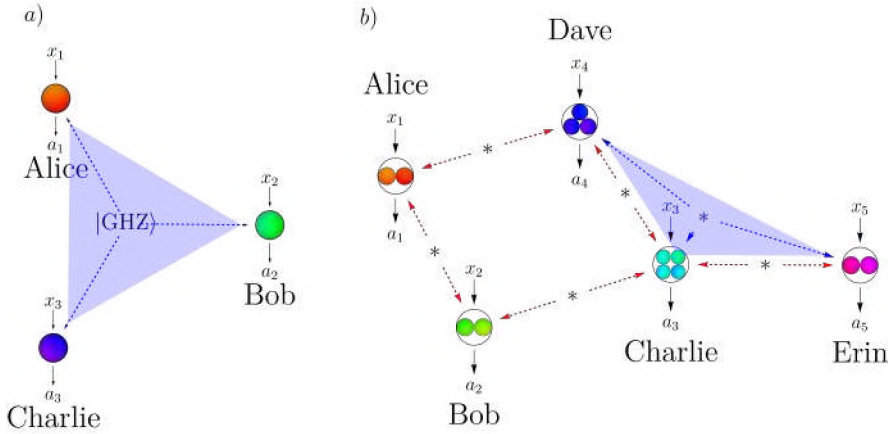


Figure 2.2: Further examples of quantum networks: a) the standard approach to tripartite scenario involves Alice, bob and Charlie sharing a single 3-qubit state $|\text{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. This setup is used in the so called quantum pseudo-telepathy game [88]. b) A generic multipartite quantum network, in which entangled states are shared between each nodes connected (hyper)-edge, thus resulting in a different number of subsystems in each node.

More generic scenarios, such as the tripartite GHZ game, representing an example of quantum pseudo-telepathy games, and other general networks can be represented in a similar setting (see Fig. 2.2). Analysing ways in which such scenarios may differ for

* Extra exercise for non-expert readers: can you see the relation between this model and the classical part of exercise given in Section 1.2.1?

local hidden variables and shared quantum states, is one of the overarching themes in the network nonlocality research – a recent review can be found in [106].

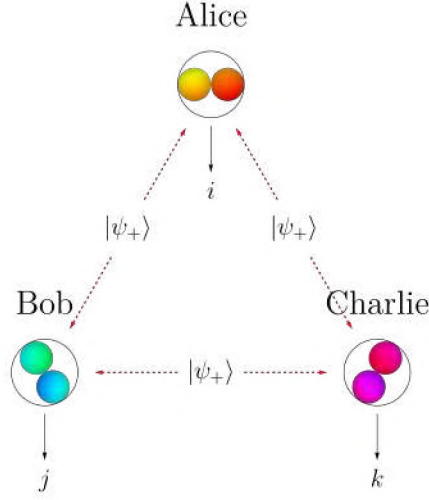


Figure 2.3: Triangular network with no inputs: Consider three parties, Alice, Bob and Charlie, with each pair sharing a maximally entangled qubit state $|\psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and each performing their local measurement $\mathbb{M}$ on a pair of qubits coming from two maximally entangled states, represented by colourful orbs. In this way, the network an output (i, j, k) from the related probability distribution $p(i, j, k)$ as defined in (2.10).

In this chapter we will focus on the case of triangular network with no inputs, in which Alice, Bob and Charlie share bipartite maximally entangled states $|\Psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$. Thus, the state of the entire network is, prior to measurement, given by $|\Psi_{ABC}\rangle = |\psi_+\rangle_{AB} \otimes |\psi_+\rangle_{BC} \otimes |\psi_+\rangle_{AC} = |\psi_+\rangle^{\otimes 3}$. Since we are considering a network without any input, at each node we will fix measurement $\mathbb{M}$ to be related to a bipartite basis $\mathbf{B} = \{|\phi_i\rangle_{AA'} = |\phi_i\rangle_{BB'} = |\phi_i\rangle_{CC'}\}_{i=1}^{d^2}$, where the double subscript with one of the letters primed denotes the fact that each basis is corresponding two parties handled by a single node. Thus, the overall probability $p(i, j, k)$ of obtaining a result $a_1 a_2 a_3 = ijk$ is given by

$$p(i, j, k) = |\langle \Phi_{ijk} | \Psi_{ABC} \rangle|^2 \quad (2.10)$$

with $|\Phi_{ijk}\rangle = |\phi_i\rangle_{AA'} \otimes |\phi_j\rangle_{BB'} \otimes |\phi_k\rangle_{CC'}$. For a generic basis $\mathbf{B}$ for qubits these distributions are already complicated, with up to 24 different amplitudes – 4 for the case of all the outputs identical, $p(i, i, i)$; 12 for $p(i, i, j) = p(i, j, i) = p(j, i, i)$, with all permutations of the indices indistinguishable; Finally, 8 amplitudes for $p(i, j, k) = p(j, k, i) = p(k, i, j)$, where equivalence is observed only up to cyclic permutation.

The most interesting case happens when the distribution $p(i, j, k)$ is invariant with respect to both permutation of the parties and relabelling of the outputs. This means

that

$$\forall_{i \neq j \neq k \neq i} : p(i, i, i) = p_1, \quad p(i, i, j) = p(i, j, i) = p(j, i, i) = p_2, \quad p(i, j, k) = p_3, \quad (2.11)$$

thus reducing the distribution to three parameters connected by normalisation constraint $4p_1 + 36p_2 + 24p_3 = 1$. We refer to such distributions as output permutation invariant (OPI). When introducing our classification, we will demonstrate that only one of the families of bases lead to OPI distribution when applied to a triangular network with maximally mixed states distributed along the edges.

Using the Finner inequality, a generalization of Hölder inequality [107], one may bound any multipartite classical or quantum distribution as [108]

$$p(a, b, c) \leq \sqrt{p(a)p(b)p(c)}. \quad (2.12)$$

In particular, when applied to OPI permutation, it simplifies to a bound for three different outputs,

$$p_3 \leq \frac{1}{8}. \quad (2.13)$$

In particular, one can give two exemplary measurement bases:

$$\mathbf{B}_{\text{OPI loc}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathbf{B}_{\text{EJM}} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & i\sqrt{2} & -i\sqrt{2} \end{pmatrix}, \quad (2.14)$$

with the states taken column-wise, which lead to OPI distributions. The former is simply a tensor product of local measurements on the Z and X eigenbases, respectively, and yields a flat distribution $(p_1, p_2, p_3)_{\text{OPI loc}} = (\frac{1}{64}, \frac{1}{64}, \frac{1}{64})$. The other, which is called the **elegant joint measurement** (EJM) gives rise to a distribution $(p_1, p_2, p_3)_{\text{EJM}} = (\frac{25}{256}, \frac{1}{256}, \frac{5}{256})$. Elegant joint measurement and the connected distributions arising in the quantum networks have been considered in numerous prior works both theoretical [8, 106, 109–112] and experimental [113, 114]. As we will show, both $\mathbf{B}_{\text{OPI loc}}$ and $\mathbf{B}_{\text{EJM}}$ fall within the same family of bases and provide endpoints of a one-parameter subfamily of OPI-generating measurements.

2.2 Classification of two-qubit isoentangled bases

Let us consider the most generic orthonormal basis on two qubits represented by a unitary matrix $B \in \mathcal{U}(4)$,

$$\mathbf{B} = \begin{pmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} \end{pmatrix} = (|\psi_1\rangle \quad |\psi_2\rangle \quad |\psi_3\rangle \quad |\psi_4\rangle), \quad (2.15)$$

with the states considered column-wise. Since the order of the states within the basis is not important, and we have freedom of local rotations, we will consider the equivalence $B \sim (V_A \otimes V_B)(\cdot)BP$, with P a permutations, $(\cdot) \in \{\mathbb{I}, \text{SWAP}\}$ and $V_A, V_B \in \mathcal{U}(2)$.

With this being said, we may first consider the subspace spanned by $|\psi_3\rangle, |\psi_4\rangle$. Using Proposition 2.2 we find that this subspace contains at least state of Schmidt rank 1, ie. product state $|\alpha\rangle \otimes |\beta\rangle$. By freedom of local rotation we may choose this state to be $|0\rangle \otimes |0\rangle$. Similarly, there also exists a state of the form $|\alpha'\rangle \otimes |\beta'\rangle \in \text{span}(|\psi_1\rangle, |\psi_2\rangle)$. Since we know that it is orthogonal to $|0\rangle \otimes |0\rangle$, by the local freedom we are able to rotate it to the form $|1\rangle \otimes |\varphi\rangle$, with $|\varphi\rangle = \cos \tau |0\rangle + \sin \tau |1\rangle$, which can be taken to be real due to freedom of local rotations of the form $\exp[i(\lambda_A \sigma_z \otimes \mathbb{I} + \lambda_B \mathbb{I} \otimes \sigma_z)]$. Using the above and introducing generalised spherical coordinates, a canonical form of U can be written as

$$B_{|\varphi\rangle} = \begin{pmatrix} 0 & 0 & -\cos \alpha \cdot e^{i\gamma} & \sin \alpha \cdot e^{i\gamma} \\ \sin \delta \cdot \cos \theta & \cos \delta \cdot \cos \theta & -\sin \alpha \cdot \sin \theta & -\cos \alpha \cdot \sin \theta \\ \sin \delta \cdot \sin \theta & \cos \delta \cdot \sin \theta & \sin \alpha \cdot \cos \theta & \cos \alpha \cdot \cos \theta \\ -\cos \delta \cdot e^{i\beta} & \sin \delta \cdot e^{i\beta} & 0 & 0 \end{pmatrix} \quad (2.16)$$

where we introduced the lower index $|\varphi\rangle$ to indicate that we are in a nonstandard basis $\{|00\rangle, |01\rangle, |1\varphi\rangle, |1\varphi^\perp\rangle\}$ with $|\varphi^\perp\rangle = \cos \tau |1\rangle - \sin \tau |0\rangle$ is defined as the orthogonal complement of $|\varphi\rangle$ (see Fig. 2.4 for Bloch spheres' representation).

With this in hand we may proceed to consider isoentanglement conditions $\xi(|\psi_i\rangle) = \xi(|\psi_j\rangle)$ for $i, j \in \{1, 2, 3, 4\}$. When considering the most general form of the equations, it turns out that all possible solutions imply $\alpha = \pm\delta$. After introducing this simplification the equations we need to consider for the remaining paramters read

$$\xi_1 - \xi_2 = -8 \cos^2 \theta \sin \theta \cos \tau (\cos \tau \sin \theta \cos(2\delta) - \sin \tau \sin(2\delta) \cos \beta) = 0, \quad (2.17)$$

$$\xi_3 - \xi_4 = -8 \cos^2 \theta \sin \theta \cos \tau (\cos \tau \sin \theta \cos(2\delta) + \sin \tau \sin(2\delta) \cos \gamma) = 0, \quad (2.18)$$

$$\xi_3 - \xi_1 = -8 \cos^2 \theta \sin \theta \cos \tau \sin(2\delta) \sin^2 \delta \sin \tau (\cos \beta + \cos \gamma) = 0, \quad (2.19)$$

where we introduced shorthand notation $\xi(|\psi_i\rangle) \equiv \xi_i$ for brevity. The remaining three equalities are linearly dependent, and thus it is sufficient to restrict to the above.

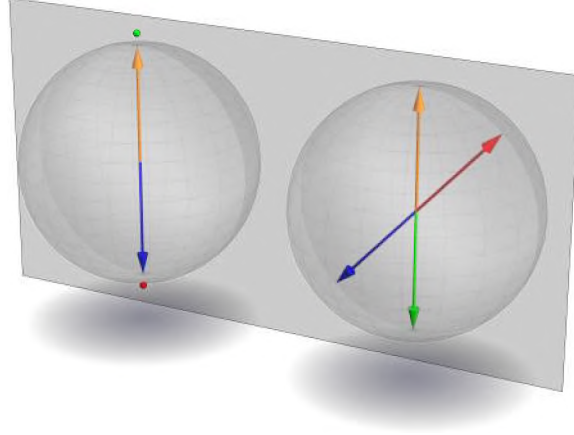


Figure 2.4: Product basis: Bloch sphere depiction of the skewed basis $\{|00\rangle, |01\rangle, |1\varphi\rangle, |1\varphi^\perp\rangle\}$, with Alice's reduction on the left hand side and Bob's reduction on the right hand side. Note that, due to degeneracy, states $|00\rangle$ and $|1\varphi\rangle$ have been depicted as small balls above the arrows.

The resulting conditions on equalities lead to following five conditions

- (i) $\cos \theta = 0$, or
- (ii) $\sin 2\theta \neq 0, \sin \tau = 0$ and $\delta = \frac{\pi}{4}$, or
- (iii.a) $\sin \theta = 0$ and $\sin \tau \neq 0$, or
- (iii.b) $\cos \tau = 0$, or
- (iv) $\cos \theta \neq 0, \sin \theta \neq 0, \cos \tau \neq 0, \sin \tau \neq 0$, and $\sin(2\delta) \neq 0$.

with (iii.a) and (iii.b) leading to solutions equivalent up to exchange of subsystems, and thus leading to equivalent solutions. Below we demonstrate the resulting four families of solutions, with short discussion about their geometric properties.

1. Product family – Starting with the condition (i), we find the resulting family of bases $\mathbf{B}^{(1)}$ to be composed of product states and, as such, locally equivalent to the basis composed of product states, $\{|00\rangle, |01\rangle, |1\varphi\rangle, |1\varphi^\perp\rangle\}$ with tangle $\xi^{(1)} = 0$, as depicted in Fig. 2.4. As such, this particular family is relatively trivial. Moreover, due to its non-full dimensionality within either Alice's or Bob's Bloch sphere, it is not possible to reconstruct complete information about either of the reductions using the results from a single measurement of this kind alone. At this point it has to be pointed out that its structure admits implementation as local measurements conditioned on previous measurements. Namely, Alice measures in $\{|0\rangle, |1\rangle\}$ basis and forwards her result bit b_A to Bob. If $b_A = 0$, Bob measures in $\{|0\rangle, |1\rangle\}$ as well; if, however, $b_A = 1$, Bob undertakes measurement with respect to $\{| \varphi \rangle, | \varphi^\perp \rangle\}$ basis.

2. Elegant family – When considering condition (ii), we find that the family can be

reduced to the form

$$\mathbf{B}_{|0\rangle}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -e^{i\zeta} & e^{i\zeta} \\ \cos \theta & \cos \theta & -\sin \theta & -\sin \theta \\ \sin \theta & \sin \theta & \cos \theta & \cos \theta \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad (2.20)$$

using local rotations of the form $\exp(i\zeta\sigma_z)$ with $\zeta = \gamma - \beta$. This particular family is noteworthy for several reasons, beginning with the fact that for $\theta = \pi/4$, $\zeta = \pi/2$ we recover the EJM as one of its extreme members (see Fig. 2.5 a)). Second, the tangle in this family depends on a single parameter θ ,

$$\xi^{(2)} = \frac{\sin^2(2\theta)}{4} \quad (2.21)$$

and, crucially, it is bounded, with maximum $\xi^{(2)} = \frac{1}{4}$ achieved at $\theta = \pi/4$. Moreover, as a generic member of this family ($\zeta \neq 0$) is full-dimensional within the Bloch balls – in both reductions we see simplices, with one being flattened to the XY plane and the other elongated along the Z axis (see , with EJM providing the only case where both simplices are regular. it allows for the reconstruction of the reduced states of subsystems, ρ_A and ρ_B , with EJM providing the optimal measurement, as demonstrated in [8]; therein, EJM is a part of a family of bipartite measurements, which are constructed using single-party symmetric informationally complete (SIC) measurement. Thus, one may conjecture that similar “Elegant family” exist in higher dimensions, with the aforementioned measurements providing their extreme points.

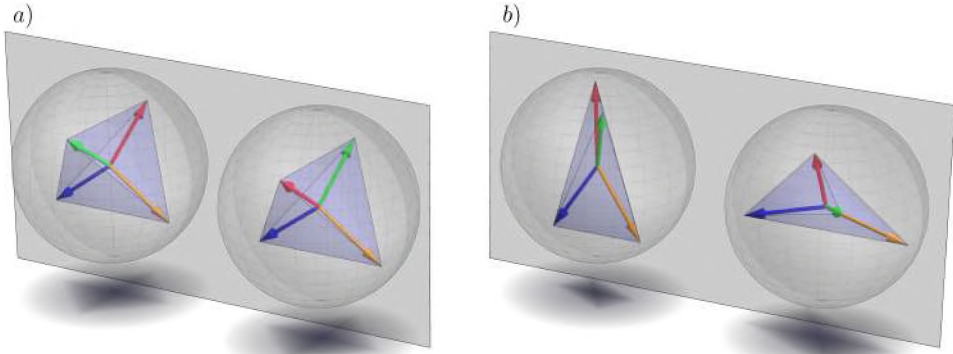


Figure 2.5: Elegant family: Bloch sphere representation of two representatives of the Elegant family (2.20). **a)** The elegant joint measurement (EJM) with two identical regular tetrahedra inscribed in a sphere of radius $\sqrt{3}/2$ in both reductions. **b)** A generic representant of the elegant family, with irregular tetrahedra, one elongated along Z axis and the other flattened towards the XY plane.

3. Bell family – Considering conditions (iii.a) and (iii.b) cumulatively due to their

equivalence up to swap, we find a 3-parameter family of the form

$$\mathbf{B}_{|0\rangle}^{(3)} = \begin{pmatrix} 0 & 0 & -\cos \delta \cdot e^{i\zeta} & \sin \delta \cdot e^{i\zeta} \\ \sin \delta & \cos \delta & 0 & 0 \\ \sin \tau \cdot \cos \delta & -\sin \tau \cdot \sin \delta & \cos \tau \cdot \sin \delta & \cos \tau \cdot \cos \delta \\ -\cos \tau \cdot \cos \delta & \cos \tau \cdot \sin \delta & \sin \tau \cdot \sin \delta & \sin \tau \cdot \cos \delta \end{pmatrix}. \quad (2.22)$$

However, one can introduce an equivalent, more informative canonical form by introducing the local vectors

$$\begin{aligned} |A_1\rangle &= \cos x |0\rangle + \sin x |1\rangle, & |A_2\rangle &= \cos x |0\rangle - \sin x |1\rangle, \\ |B_1\rangle &= \cos y |0\rangle + \sin y |1\rangle, & |B_2\rangle &= \cos y |0\rangle - \sin y |1\rangle, \end{aligned} \quad (2.23)$$

with their orthogonal complements $|A_i^\perp\rangle, |B_i^\perp\rangle$ defined implicitly. Utilising the above local states we may express the states $|\psi_j\rangle$ constituting the basis $\mathbf{B}^{(3)}$ as

$$\begin{aligned} |\psi_1\rangle &= \cos z |A_1\rangle |B_1\rangle + e^{i\phi_1} \sin z |A_1^\perp\rangle |B_1^\perp\rangle, \\ |\psi_2\rangle &= \cos z |A_1^\perp\rangle |B_2\rangle + e^{i\phi_2} \sin z |A_1\rangle |B_2^\perp\rangle, \\ |\psi_3\rangle &= \cos z |A_2\rangle |B_1^\perp\rangle + e^{i\phi_3} \sin z |A_2^\perp\rangle |B_1\rangle, \\ |\psi_4\rangle &= \cos z |A_2^\perp\rangle |B_2^\perp\rangle + e^{i\phi_4} \sin z |A_2\rangle |B_2\rangle \end{aligned} \quad (2.24)$$

with $z \in [0, \pi/4]$. Note that, from the viewpoint of the reductions, this parametrisation corresponds to the fact if $|\psi_1\rangle$ is colinear with $|\psi_2\rangle$ for Alice but not colinear in Bob's reduction; similarly for the $|\psi_3\rangle, |\psi_4\rangle$ pair and with exchanging Alice with Bob for pairs $|\psi_1\rangle, |\psi_3\rangle$ and $|\psi_2\rangle, |\psi_4\rangle$ (see Fig 2.6) The angles x, y have their geometric meaning .

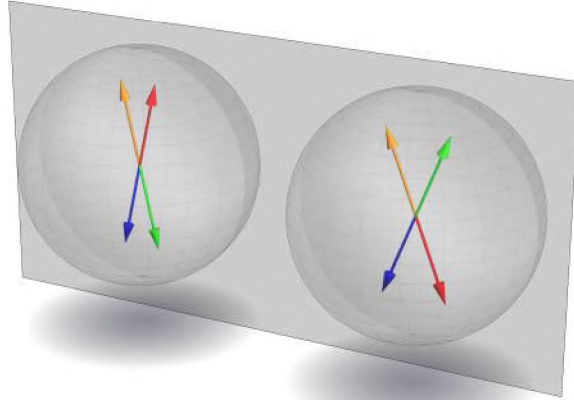


Figure 2.6: Bell family: depiction of a generic member of the Bell family in the canonical representation (2.24). Note that the states in both reductions lie in a single plane, and thus it is impossible to use measurements stemming from such bases for local reconstructions.

By definition, all the states are isoentangled thanks to equality of Schmidt coefficients,

with tangle given by

$$\xi^{(3)} = \sin^2(2\delta) \sin^2(\tau) = \sin^2(2z) \quad (2.25)$$

with respect to both representations. Solving for orthogonality we find that

$$\phi_1 = -\phi_2 = -\phi_3 = \phi_4, \quad \cos(\pm\phi_1) = -\frac{\tan 2x \tan 2y}{\sin 2z}, \quad (2.26)$$

which completes the description.

To summarise this family, we first see that its tangle spans all range of possible values, with Bell state measurement (BSM) as one extreme and product measurements as the other. Second, no basis in this family is full-dimensional within either reduction, and as such, it cannot be used for local state reconstruction.

4. General family – The final condition (iv), in essence collecting the cases in which the other conditions do not hold, yet the basis in question remains isoentangled – thus, we refer to it as the general family. it can be expressed as

$$\mathbf{B}_{|\varphi\rangle}^{(4)} = \begin{pmatrix} 0 & 0 & \cos \delta \cdot e^{\pm i\beta} & -\sin \delta \cdot e^{\pm i\beta} \\ \sin \delta \cdot \cos \theta & \cos \delta \cdot \cos \theta & -\sin \delta \cdot \sin \theta & -\cos \delta \cdot \cos \theta \\ \sin \delta \cdot \sin \theta & \cos \delta \cdot \sin \theta & \sin \delta \cdot \cos \theta & \cos \delta \cdot \cos \theta \\ -\cos \delta \cdot e^{i\beta} & \sin \delta \cdot e^{i\beta} & 0 & 0 \end{pmatrix}, \quad (2.27)$$

with an additional restriction

$$\tan(\tau) = \frac{\cos(2\delta) \sin(\theta)}{\sin(2\delta) \cos(\beta)} \quad (2.28)$$

hidden in the Bob's part $|\varphi\rangle, |\varphi^\perp\rangle$ of the skewed basis. The expression for the tangle of the states in this family reads

$$\xi^{(4)} = \frac{\sin^2(2\theta) \sin^2(2\delta)}{4} \frac{\sin^2(2\delta) \cos^2(\beta) + \cos^2(2\delta)}{\sin^2(2\delta) \cos^2(\beta) + \cos^2(2\delta) \sin^2(\theta)} \quad (2.29)$$

which allows for full range of values between 0 and 1. It is the most general of the families, as it overlaps with each of the other ones; one could also call it the **limiting family**, as the overlaps happen also when considering the singularities of both the constraint (2.28) and its tangle (2.29). In particular, we find that:

- When β approaches $\pi/2$, the General family (2.27) simplifies to a two-parameter subfamily of the Bell family (2.22). However, since the Bell family depends on the same number of parameters as the General family, it cannot be fully recovered through any limits.
- When δ approaches $\pi/4$, the General family simplifies to the Elegant family (2.20).
- When θ and δ both approach 0, the General family reduces to the skewed product basis, regardless of the direction in which these limits are approached.

- When β approaches $\pi/2$ and δ approaches $\pi/4$, the result is an interpolation between part of the Elegant family and a subfamily of the Bell family, depending on the direction of approach.
- When β approaches $\pi/2$ and θ approaches 0, the result is a subfamily of the Bell family with an upper bound on the degree of entanglement. This bound depends on the angle of approach $\phi \in (-\pi/2, \pi/2)$ as $(1 + \tan(|\phi|))^{-2}$.

From this, it is evident that the three aforementioned families represent the boundary of the General family.

2.2.1 Output permutation invariant distributions generated by a sub-family of the Elegant family

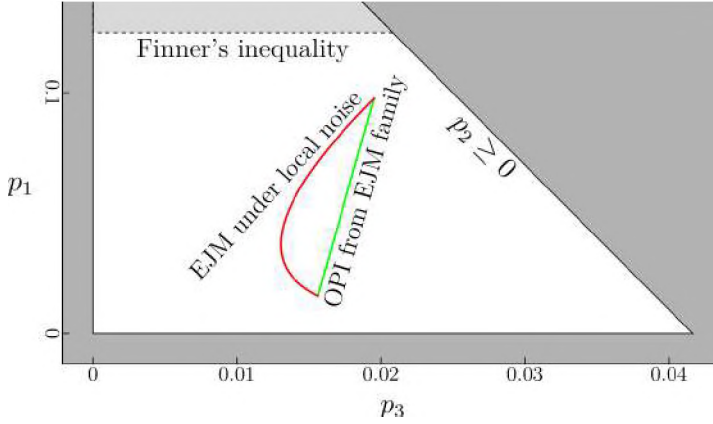


Figure 2.7: OPI from elegant subfamily: Plot of the p_3 - p_1 plane for OPI measurements. The red line depicts probabilities resulting from EJM acting on Bell states under local noise, $\Phi_{\epsilon \text{ loc}}(|\psi_+\rangle\langle\psi_+|^{\otimes 3})$, $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The green line shows the OPI (2.30) derived from the Elegant family acting on the network state composed of three copies of the maximally entangled Bell state, $|\Psi_{ABC}\rangle\langle\Psi_{ABC}| = |\psi_+\rangle\langle\psi_+|^{\otimes 3}$.

Isoentangled bases, due to high degree of symmetry between the states, are natural candidates for generation of OPI distributions as described by equation (2.11). Indeed, it turns out that by setting $\zeta = \pi/2 + \arccos(-\sin(2\theta))$, followed by a local transformation of the form $V_A \otimes V_B = \exp(-\frac{i}{4} \arccos(-\sin(2\theta)) \sigma_z)^{\otimes 2}$. In this way, we find that the probabilities for opy can be expressed as

$$p_1 = \frac{1}{512}(3 \cos(4\theta) + 5), \quad p_2 = \frac{1}{512}(13 - 5 \cos(4\theta)), \quad p_3 = \frac{1}{512}(9 - \cos(4\theta)), \quad (2.30)$$

which are depicted in Fig. 2.7. Additionally, we investigated the robustness of the resulting OPI distributions under introduction of noise. The global noise of the form $\Phi_{\epsilon \text{ glob}}(\rho_{ABC}) = (1 - \epsilon)\rho_{ABC} + \epsilon \frac{\mathbb{I}_{64}}{64}$ reproduced segments joining the original OPI distribution with the flat distribution. However, when the applied noise was changed

to local, $\Phi_{\epsilon \text{ loc}}^{\otimes 3}(\rho_{ABC})$ with $\Phi_{\epsilon \text{ loc}}(\rho) = (1 - \epsilon)\rho + \epsilon \frac{\mathbb{1}}{4}$, EJM turned out to be the only measurement producing OPI distribution consistently, with probabilities given by

$$p_1 = \frac{-3\epsilon^3 + 18\epsilon^2 - 36\epsilon + 25}{256}, \quad p_2 = \frac{\epsilon^3 - 6\epsilon^2 + 8\epsilon + 1}{256}, \quad p_3 = \frac{-\epsilon^3 + 6\epsilon^2 - 6\epsilon + 5}{256}. \quad (2.31)$$

Taking into account recent considerations output-permutation invariant distributions [115] (therein referred to as fully symmetric distributions) and conjectured bounds on the network-local OPI distributions, it may be interesting for the future research to conduct similar searches in the quantum space, taking the family of OPI-generating measurements introduced above as a starting point.

2.3 Partial results on larger dimensions and subsystems

The following paragraphs are taken directly from [17], as I highly doubt there is anything to be added, yet for completeness the considerations necessitate to sum up what is known concerning isoentangled measurements in higher dimensions and/or for larger number of parties:

The natural next step after the analysis of the simplest two-qubit bases is to shift to objects with larger local dimension, residing in spaces $\mathcal{C}^d \otimes \mathcal{C}^d$. Direct application of methods presented in this work does not seem to be realistic, thus we expect that new techniques would need to be developed to extend the current results beyond qubits. Nevertheless, there exist certain partial results and limited families in the literature.

Separable bases are the simplest case, and can be fully given in terms of *conditional measurement bases*

$$|\psi_{ij}\rangle = |i\rangle \otimes |j_i\rangle, \quad (2.32)$$

where one imposes the relations

$$\langle i|i'\rangle = \delta_{ii'}, \quad \langle j_i|j_{i'}\rangle = \delta_{jj'}. \quad (2.33)$$

and the product of the form $\langle j_i|j_{i'}\rangle$ needs not to be defined for $i \neq i'$. Using local transformations we can always set the local bases $|i\rangle$ and $|j_0\rangle$ to the computational basis, thus simplifying this family to direct sum of unitaries, $\sum_{i=1}^d U_i$.

The other extreme—bases composed of maximally entangled states—have been considered by Werner [116], where a 1-to-1 equivalence between maximally entangled bases, unitary bases, teleportation schemes and dense coding schemes has been established. Furthermore, a family of *shift-and-multiply* bases based on Hadamard matrices and Latin squares has been introduced therein.

Limited families of isoentangled bases have been considered in [8, 117, 118], without claims of being complete constructions. Below we demonstrate a method to merge

methods from [116] and [117], arriving at a new family of bases with intermediate entanglement degrees.

First we focus on a construction from [116]. Therein, a family of unitary bases based on Hadamard matrices and Latin squares is introduced. Given a $d \times d$ Latin square $\lambda(j, k) : I_d \times I_d \mapsto I_d$ and j (not necessarily distinct) Hadamard matrices, the set of unitary matrices U^{ij} of dimension d is defined by

$$U^{ij} |k\rangle = (H^j)_{ik} |\lambda(j, k)\rangle \quad (2.34)$$

and they provide an orthogonal unitary basis due to the easily verifiable property

$$\text{tr}(U^{ij\dagger} U^{i'j'}) = d \delta_{ii'} \delta_{jj'}. \quad (2.35)$$

It is also easily verified that vectorized versions

$$U \mapsto |U\rangle = \frac{1}{\sqrt{d}} \sum_{nm} U_{nm} |nm\rangle$$

provide a maximally entangled bases of the d^2 -dimensional Hilbert space.

We may now replace Hadamard matrices with robust Hadamard matrices considered in [117], such that $R^j R^{j\dagger} = \mathbb{I}$. The corresponding bistochastic matrix has the structure

$$|R^j|^2 = \begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{pmatrix} \quad (2.36)$$

with absolute value understood entry-wise. Then, a *non-unitary shift-and-multiply basis*

$$B^{ij} |k\rangle = (R^j)_{ik} |\lambda(j, k)\rangle. \quad (2.37)$$

Entire proof from [116] can be retraced to demonstrate the orthogonality of the resulting states $|B^{ij}\rangle$ and the isoentangled character of the basis down to the specific Schmidt coefficients follows directly from the form of vectorization map.

Indeed, one can extend the above construction to all unitary R^j such that all rows/-columns contain the same amplitudes up to permutation.

It is important to note that the above construction is distinct from equientangled bases given in [117], which is evident already on the level of number of robust Hadamards R^j used in the construction.

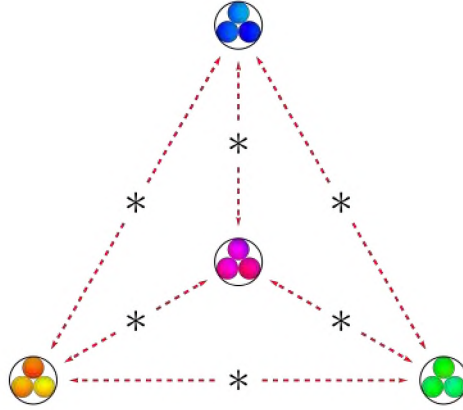


Figure 2.8: Simplex network: In a network based on a fully connected graph with 4 vertices, von Neumann measurements in the vertices would need to be related to 3-qubit bases. Thus, one may expect that a 3-qubit equivalent of EJM may lead to a proper OPI distribution within this network.

2.4 Summary and Outlook

In this chapter we have introduced a new problem of classification of nonlocal measurements up to local equivalence, and we presented the very first step by presenting a complete classification of two-qubit isoentangled bases, which are related to a subset of nonlocal von Neumann measurements. We have shown that there exist four families: product family, elegant family, Bell family and general family. Out of them, the most interesting is one-parameter subfamily of elegant family, which generates output-permutation invariant distributions in no-input triangular network.

First direction stemming from this line of research that may be pursued in the future is to classify most general two-qubit bases. However, extending the classification to larger local dimensions or more subsystems poses new problems.

For larger dimensions the first hurdle appears with the canonical form of bases – Schmidt rank-bound subspaces introduced by Cubitt *et al.* turn out to be unwieldy, as already for two qutrits smallest dimension of a subspace guaranteed to contain a product state is 5. Thus, even a pair of such subspaces overlap, and the resulting product states may not (and most likely will not) be orthogonal. However, taking one more dimension for each subspace the minimal overlap is 3-dimensional. Thus, one may conjecture that there will thanks to this there may exist a pair of orthogonal states, one for each subspace, which are orthogonal. Once the problem of canonical form is overcome, one would need to decide whether to consider “soft” isoentanglement, taking $\text{Tr}(\rho_A^2) = \text{const.}$ across the basis, or “hard” isoentanglement by requiring that the entire Schmidt vector λ is constant.

Problem of soft vs. hard isoentanglement becomes even more pronounced for systems composed of more than two parties, where the structure of entanglement becomes

much more complicated, with example of $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ representing two different classes of tripartite entanglement already for 3 qubits. Nevertheless, developing classification for such cases and, somewhere along the way, finding a multipartite generalization of EJM may prove beneficial for more complicated network structure, such as fully-connected graph on four nodes, as depicted in Fig. 2.8.



3 Simultaneous minimal-noise estimation of noncommuting rotations of a spin [14]

Interferometry, the idea of taking two coherent beams of light and overlapping them to produce oscillations in intensity, is one of the most influential concepts in physics. It was used by Michelson and Morley in a failed attempt to prove the existence of aether [119], laying the foundation for what we now recognize as the theory of relativity. Although it was initially employed to support Einstein's theories, it has evolved to serve broader purposes. Nowadays, with the US-based experiment LIGO and the Italian Virgo, which have detected 90 merger events so far [120], interferometry is used repeatedly to validate Einstein's predictions [121]. Additionally, it has shown that gravitational waves do propagate in space, thereby challenging some of Einstein's assumptions [122].

Interferometry can be also seen from the perspective of quantum mechanics, where Mach-Zender interferometer [123, 124], built using two coherent laser beams, a pair of beam-splitters and two photon counters, is interpreted from the perspective of $SU(2)$ group generated by proper combination of bosonic creation and annihilation operators [125]. When beam-splitters are replaced by nonlinear optical elements such as four-wave mixers providing the squeezing of light, the resulting experimental scheme related to non-compact $SU(1,1)$ group, can be shown to be vastly more sensitive than its $SU(2)$ counterpart, achieving Heisenberg scaling of error with the energy of the used probe [125, 126]. It is also interesting to note that squeezed states of light, squeezed vacuum more specifically, is used in Mach-Zehnder-like interferometry as well as Michelson scheme, providing orders of magnitude increase of sensitivity to the most recent LIGO experimental runs [127].

Inspired by recent reinterpretation of the $SU(1,1)$ interferometer as a displacement detector with squeeze-displace-unsqueeze protocol, provided by Caves in [26], and the idea of spin-squeezing, first considered by Kitagawa and Ueda [128], we propose a theoretical scheme for detecting spin rotations, using a squeeze-rotate-unsqueeze protocol with single-axis spin squeezed states. We show that for rotation estimation with maximal squeezing parameter the scaling of the *quantum Fisher information* (QFI) with N^2 is obtained for specific rotation axes. Furthermore, we show that for a particular value of squeezing parameter we are able to achieve QFI scaling approximately as $N^2/2$ for arbitrary equatorial axis of rotation.

We begin this chapter by introducing the preliminary notions, including quantum Fisher information and its relation to quantum metrology and estimation, $SU(1,1)$ interferometry and spin-coherent and spin-squeezed states. The results are presented

in the following sections, including the Heisenberg scaling of QFI for maximal squeezing and half-maximal squeezing with arbitrary equatorial axis of rotation. We then move on to remarks and possible limitations of using two-axis squeezing, as well as pointing out future directions of this line of research.

3.1 Preliminaries

3.1.1 SU(2) and SU(1,1) interferometry

Let us begin by recalling what a Mach-Zehnder interferometer is. In its simplest form, the device can be constructed from two mirrors, two beamsplitters and two detectors used to monitor intensity of the outgoing light. Schematic of the interferometer is presented in Fig. 3.1.

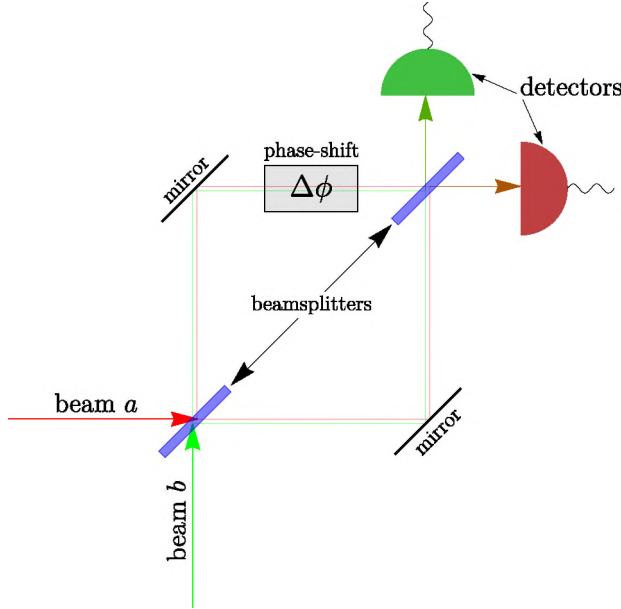


Figure 3.1: Mach-Zehnder interferometer: Two beams going into two entry ports of Mach-Zehnder interferometer are first split on the beamsplitter and travel through one of two arms, where one of them usually is assumed to contain a translucent element introducing a phase-shift ϕ . Second beamsplitter allows for interference, which in the standard case with beam b not supplied and $\phi \neq 0$ results in lowering of intensity of the outgoing beam in the red detector and appearance of nonzero intensity in the green one. Note that colours are introduced only for clarity of representation, and beams are normally generated by (almost) monochromatic lasers.

Even though the original scheme put forward by Mach and Zehnder in [123, 124] predates quantum mechanics and, as such, is based entirely on classical electromagnetic wave description for light, Fresnel equations governing the reflection and transmission for mirrors and beamsplitters, and simple intensity measurement at the

end, its transfer to the quantum grounds has proved to be fruitful, leading to different and comparably more powerful variations on the same theme [26, 125, 129].

The quantum description of the Mach-Zehnder interferometric scheme starts with describing the input beams can be described in terms of Fock states, where the basis states $|i\rangle$ correspond to an eigenstate of the photon number operator,

$$\hat{n} |i\rangle = i |i\rangle. \quad (3.1)$$

One can shift between states with different using the standard raising and lowering operators: $\hat{a}, \hat{a}^\dagger$ for the first beam and $\hat{b}, \hat{b}^\dagger$ for the second beam. Commutation relations for these operators are given by $[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0$ and $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = \mathbb{I}$. Their action is given by $\hat{a} |i\rangle = \sqrt{i} |i-1\rangle$ and $\hat{a}^\dagger |i\rangle = \sqrt{i+1} |i+1\rangle$ and analogously for $\hat{b}, \hat{b}^\dagger$; in this way, one can easily verify that $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$ is the proper number operator for the total system.

In particular, we will use the single photon states on two beams as a the simplest example to demonstrate the connection to the SU(2) group. This subspace is spanned by states $|0\rangle \otimes |1\rangle \equiv |01\rangle, |10\rangle$. In this case, there are four combination of raising and lowering operators which provide the basis for the operators:

$$\begin{aligned} \hat{a}^\dagger \hat{b} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+, & \hat{b}^\dagger \hat{a} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_-, \\ \hat{a}^\dagger \hat{a} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|, & \hat{b}^\dagger \hat{b} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|. \end{aligned} \quad (3.2)$$

The above matrix representations allow for easy construction of the three Pauli operators. Clearly, we could prove it by calculating the commutation relations between the representation in terms of commutation of the raising and lowering operators; however, let us simply say that commutation relations for larger photon numbers are preserved and thus the triplet of operators maintain their functional form, $J_i \stackrel{\text{fun.}}{=} \sigma_i$. Together they provide a representation of the $\mathfrak{su}(2)$ algebra and, in turn, connected directly to a representation of the SU(2) group with Casimir $\hat{J} = \hat{J}^2 = \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right)$. In this representation, beamsplitter, for example, is understood as an SU(2) operation $B = \exp(-i \frac{\pi}{2} J_y)$. Yurke *et al.* demonstrated [125], using the isomorphism* $\text{SU}(2) \cong \text{SO}(3)$ and the geometric interpretation proceeding from it, that the standard limit for measurements of the phase fluctuation is given by $\Delta\phi = N^{-1/2}$, where $N = \langle \hat{N} \rangle$ is the average number of photons entering the interferometer.

One may, however, construct different group as well, including noncompact groups such as SU(1,1), for which the representation of generators of the underlying algebra

* More specifically, SU(2) group is *locally* isomorphic to the rotation group SO(3). Globally SU(2) is a double cover of SO(3).

is given by

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b} \hat{b}^\dagger), \quad \hat{K}_1 = \frac{1}{2}(\hat{a}^\dagger \hat{b}^\dagger + \hat{b} \hat{a}), \quad \hat{K}_2 = -\frac{i}{2}(\hat{a}^\dagger \hat{b}^\dagger - \hat{b} \hat{a}) \quad (3.3)$$

Interestingly, when incorporated into the Mach-Zehnder interferometer and considered from a similar geometric point of view, instead of rotations generate “squeezing” of the cone when seen from the perspective of Bloch, or Poincaré sphere. This leads to improvement in the precision of measurement by reducing fluctuations of the phase shift measured down to $\Delta\phi \propto N^{-1}$.

Reinterpretation given by Caves in [26] provides us with a perspective shift that allowed us to generalize the SU(1,1) interferometry procedure to spin setting. Therein it is argued that SU(1,1) interferometry should, in fact, be understood as displacement detection scheme. The preparation scheme takes two vacuum states as inputs and proceeds to implement squeeze-displace-unsqueeze protocol, SDU for short. Such a procedure on a single beam is defined as

$$|\text{SDU}_1(\gamma; r)\rangle = \hat{S}_1(r) \hat{D}(\gamma) \hat{S}_1^\dagger(r) |0\rangle, \quad (3.4)$$

where the squeezing operators are defined as $\hat{S}_1(r) = \exp(r(\hat{a}^2 - \hat{a}^{\dagger 2})/2)$ and the displacement operator $\hat{D}(\gamma) = \exp(\gamma \hat{b}^\dagger - \gamma^* \hat{b})$ and $|0\rangle$ is a vacuum state defined as a coherent state of light with $\alpha = 0$, which incidentally coincides with the eigenstate of number operator, $\hat{N} |0\rangle = 0 |0\rangle$. Similarly, a two-mode protocol is defined by a composition

$$|\text{SDU}_2(\gamma; r)\rangle = \hat{S}_2(r) \hat{D}(\gamma) \hat{S}_2^\dagger(r) |0\rangle \otimes |0\rangle \quad (3.5)$$

with the two-mode squeezing operators $\hat{S}_2(r) = \exp(-2ir\hat{K}_2)$, and single mode displacement $\hat{D}(\gamma)$.

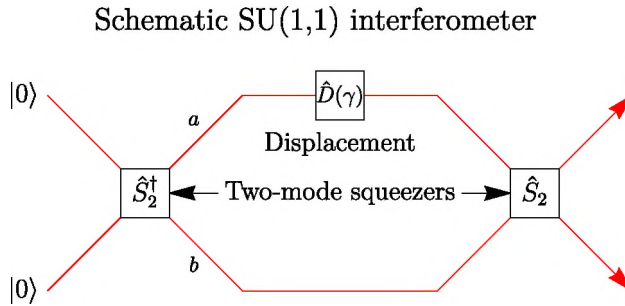


Figure 3.2: SU(1,1) interferometer: Schematic depiction of an SU(1,1) interferometer with a displacement operator $\hat{D}(\gamma)$ only on one of its arms and the two-mode squeeze operators $\hat{S}_2 = \exp(-2ir\hat{K}_2)$. Figure adapted from [14].

Introduction of squeezing enhances sensitivity of the measurement, with the resolution of the real part of displacement quantified by $\langle (\Delta \text{Re } \gamma)^2 \rangle = e^{-2r}$; heuristically, one

may say that the advantage is gained by dumping the excess uncertainty onto the complementary measurement in the spirit of uncertainty relations, which in this case is that of imaginary part of displacement, resulting in $\langle (\Delta \text{Im } \gamma)^2 \rangle = e^{2r}$. Using two-mode procedure, however, one may achieve increase in overall resolution for arbitrary direction of displacement $\langle (\Delta \gamma)^2 \rangle = e^{-2r}$, which may at first seem as though it was in violation of the uncertainty relations. However, two modes are described by 4-dimensional space spanned by x_a, x_b, p_a, p_b . In this way, enough leeway is created to squeeze two complementary directions, corresponding to Bell variables $x_a + x_b$ and $p_a - p_b$, while the remaining two are unsqueezed at the same time.

3.1.2 Spin-coherence and spin-squeezing

Coherent states, understood as highest weight states in a given representation [130], exist for arbitrary compact Lie group. Much prior to this, coherent states in $\text{SU}(2)$, or spin coherent states (SS), have been considered all the way back in 1971 by Radcliffe [131] with extended analysis following soon after in the works of Arecchi *et al.* [132] and Holts and Hanus [133]. In this case consideration is particularly simple, as representation corresponding to arbitrary half-integer total spin S can be related to the fully symmetric subspace of $2S$ spin- $\frac{1}{2}$ particles; note that the dimensionality of this subspace is $2S + 1$ and its natural basis is given by the eigenstates of J_z operator, $J_z |S, M\rangle = M |S, M\rangle$ for $-S \leq M \leq S$. From this point of view, it is simple to note that the highest weight state $|S, S\rangle_{\theta, \phi}$ corresponds to $2S$ copies of the state $|\frac{1}{2}, \frac{1}{2}\rangle_{\theta, \phi} = \cos(\frac{\theta}{2}) |0\rangle + e^{i\phi} \sin(\frac{\theta}{2}) |1\rangle$,

$$|S, S\rangle_{\theta, \phi} = \left| \frac{1}{2}, \frac{1}{2} \right\rangle_{\theta, \phi}^{\otimes 2S}. \quad (3.6)$$

Thus, if we disregard all the non-symmetric basis states of the full space, we can write

$$|S, S\rangle_{\theta, \phi} = \sum_{M=-S}^S \sqrt{\binom{2S}{S+M}} \cos^{S+M} \left(\frac{\theta}{2} \right) \left[e^{i\phi} \sin \left(\frac{\theta}{2} \right) \right]^{S-M} |S, M\rangle. \quad (3.7)$$

This representation is sufficient for all the subsequent considerations; however, we refer the reader to an extensive review on coherent states for further details [134].

Since coherent states can be squeezed in the full bosonic case, it is only natural to expect similar notions to be admissible for the generalized coherent states, including the spin coherent states. In the early attempts to formulate the notion of squeezing for spins, it was enough to have uncertainty reduced for $\langle \Delta S_x^2 \rangle$ or $\langle \Delta S_y^2 \rangle$ to call a state of a spin system squeezed [135, 136]. In the seminal work Kitagawa and Ueda noted [128] that this approach meant that even spin coherent states could be seen as squeezed. They argued that proper notion of squeezing of a spin state should take

into account minimal uncertainty along an axis perpendicular to the principal axis of the state. In order to achieved this, they proposed two squeezing Hamiltonians,

$$H_1 = \mu S_z^2, \quad H_2 = -i\mu(S_+^2 - S_-^2), \quad (3.8)$$

also referred to in the literature as one-axis squeezing (OAT) and two-axis counter-twisting (TACT), respectively. By evolving the SCS with each of them one gets one- and two-axis squeezed spin states, respectively. In particular, the latter can be related by Holstein-Primakoff transformation [137] to the standard notion of squeezing for bosonic systems. Nevertheless, analytic treatment of TACT is inaccessible. For this reason we have focused on the former, which defines the single-axis squeezed states

$$\begin{aligned} |S; \mu\rangle_{\theta, \phi} &:= e^{i\mu S_z^2} |S, S\rangle_{\theta, \phi} \\ &= \sum_{M=-S}^S \sqrt{\binom{2S}{S+M}} \cos^{S+M}\left(\frac{\theta}{2}\right) \left[e^{i\phi} \sin\left(\frac{\theta}{2}\right) \right]^{S-M} e^{i\mu M^2} |S, M\rangle. \end{aligned} \quad (3.9)$$

In particular, we will focus on squeezing coherent states oriented along an axis perpendicular to the squeezing axis; to simplify, we may choose it to be the X axis and introduce notation

$$|S; \mu\rangle \equiv |S; \mu\rangle_{\frac{\pi}{2}, 0} = \frac{1}{2^S} \sum_{M=-S}^S \sqrt{\binom{2S}{S+M}} e^{i\mu M^2} |S, M\rangle. \quad (3.10)$$

It is important to note that spin states can be oversqueezed – a fact already highlighted by Kitagawa and Ueda [128]. One finds that squeezing of spin states is periodic – for the squeezing parameter $\mu = \pi$ one gets $|n; \pi\rangle = |n, n\rangle_{\frac{\pi}{2}, \pi}$ and $|\frac{2n+1}{2}; \pi\rangle = |\frac{2n+1}{2}, \frac{2n+1}{2}\rangle_{\frac{\pi}{2}, 0}$. Thus, squeezing parameter in fact goes from $\mu = 0$ to the maximal squeezing at $\mu = \frac{\pi}{2}$ and then goes back symmetrically to the SCs for $\mu = \pi$. In the Fig 3.3 we present three examples of squeezed spin states most important in this work, ranging from SCS with $\mu = 0$ to the maximum squeezing of $\mu = \frac{\pi}{2}$. For the visualization we go to the spherical Wigner function, as defined in [138], later analysed in [139] and more recently [140] in the context of negativity of the spherical Wigner function $W(\theta, \phi)$ [141].

A two-spin counterpart for one-spin squeezing has already been studied in experimental works [142, 143], whereas the two-spin two-axis squeezing has been studied only recently in [144], where its chaotic behaviour has been demonstrated. For the above reason, similarly to the single spin case, we will mostly restrict ourselves to the one-axis twisting, which is defined as

$$|S_M, S_P; \mu\rangle_{\theta_M, \phi_M, \theta_P, \phi_P} := e^{i\mu S_{\pm} \otimes S_{\pm}} |S_M, S_M\rangle_{\theta_M, \phi_M} |S_P, S_P\rangle_{\theta_P, \phi_P}. \quad (3.11)$$

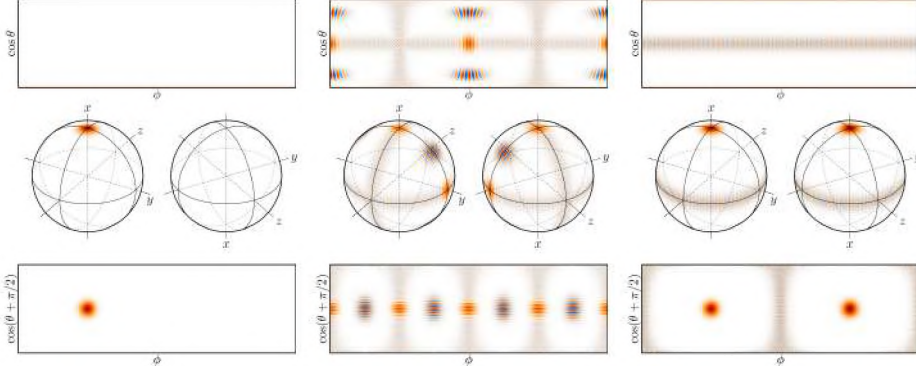


Figure 3.3: Spherical Wigner functions $W(\theta, \phi)$ for three examples of spin states with spin $S = 40$: (left) the spin coherent state (SCS); (center) a squeezed spin state with intermediate squeezing $\mu = \pi/4$; (right) maximally squeezed state with $\mu = \pi/2$. The states are shown in four different projections; top and bottom projections are cylindrical with poles along X and Z axes, respectively. The middle ones are direct projections of the sphere with two different orientations indicated by the labels next to the axes. Figure adapted from ref. [14].

3.1.3 Quantum Fisher information

Whenever we encounter a problem of obtaining a specific value for a physical quantity, the actual problem we are dealing with is **the problem of estimation**. More specifically, we may consider a random variable X and a family of distributions $f(X, \theta)$ parametrized by $\theta \in \mathbb{R}$. Then, the formal problem is to find the best unbiased estimator $\hat{\theta}$ with the smallest variance. It can be bounded from below using the classical Cramér-Rao as

$$\text{Var}[\hat{\theta}] \geq \frac{1}{\mathcal{I}(\theta)} \quad (3.12)$$

with the classical Fisher information defined as [145]

$$\mathcal{I}(\theta) = -\mathbb{E} \left[\left(\frac{\partial}{\partial \theta} \log f(X; \theta) \right)^2 \middle| \theta \right] = \mathbb{E} \left[\frac{\partial^2}{\partial \theta^2} \log f(X; \theta) \middle| \theta \right], \quad (3.13)$$

where $\mathbb{E}[\cdot]$ denotes the average value. For a more complete treatment of the classical Fisher information we refer the reader to a compendious book by Frieden [146].

In quantum metrology the role of probability distribution is taken over by quantum states and, hence, the extension of the Cramér-Rao bound, defined in the following theorem [147, 148].

Theorem 3.1. Consider a family of states $\{\rho_\theta\}$ depending on a scalar parameter $\theta \in \Theta$. The variance $\text{Var}[\hat{\theta}_{\text{est}}]$ of any locally unbiased estimator $\hat{\theta}_{\text{est}}$

$$\mathbb{E}[\hat{\theta}_{est}] \Big|_{\theta_0} = \theta_0, \quad \frac{d}{d\theta} \mathbb{E}[\hat{\theta}_{est}] \Big|_{\theta_0} = 1,$$

is bounded by the inverse of the Quantum Fisher Information

$$\text{Var}[\hat{\theta}] \geq \frac{1}{I_{\theta}(\rho_{\theta})}, \quad (3.14)$$

with $I_{\theta}(\rho_{\theta})$ the quantum Fisher information (QFI).

In full analogy to the classical Fisher information QFI can be defined in terms of a symmetric logarithmic derivative (SLD),

$$I_{\theta}(\rho_{\theta}) := \text{tr} \left[\hat{L}_{\rho_{\theta}}^2 \rho_{\theta} \right], \quad (3.15)$$

with the symmetry of SLD reflected in its implicit definition,

$$\frac{\partial \rho_{\theta}}{\partial \theta} = \frac{\hat{L}_{\rho_{\theta}} \rho_{\theta} + \rho_{\theta} \hat{L}_{\rho_{\theta}}}{2}.$$

As we will limit our considerations to pure states, the action of SLD on states of the form $\rho_{\theta} = |\psi_{\theta}\rangle\langle\psi_{\theta}|$ can be written as

$$I_{\theta}(|\psi_{\theta}\rangle) = 4 \left(\langle \partial_{\theta} \psi_{\theta} | \partial_{\theta} \psi_{\theta} \rangle - |\langle \psi_{\theta} | \partial_{\theta} \psi_{\theta} \rangle|^2 \right). \quad (3.16)$$

Moreover, the elements of the SLD operator can be given explicitly in terms of the eigenvectors $|\psi_i\rangle$ of any given state ρ_{θ} as

$$\frac{\hat{L}_{\rho_{\theta},ij}}{2} = \frac{\delta_{ij}}{p_i + p_j} \frac{p_j - p_i}{p_i + p_j} \frac{\partial p_i}{\partial \theta} \langle \psi_i | \partial_{\theta} \psi_j \rangle, \quad (3.17)$$

and the optimal measurement saturating the Cramér-Rao bound can be given in terms of projectors $M_i = |l_i\rangle\langle l_i|$ on the eigenvectors $|l_i\rangle$ of the SLD.

To gain some intuition on QFI works we may consider an equatorial spin coherent state with total spin S along an axis with angle ϕ ,

$$|S, S\rangle_{\frac{\pi}{2}, \phi} = \frac{1}{\sqrt{2^{2S}}} \sum_{M=-S}^S \sqrt{\binom{2S}{S+M}} e^{iM\phi} |S, M\rangle. \quad (3.18)$$

In order to calculate QFI with respect to the angle ϕ of the axis treated as the parameter to be estimated we need to calculate

$$\left(\langle S, S |_{\frac{\pi}{2}, \phi} \overleftarrow{\partial}_\phi \right) \left(\overrightarrow{\partial}_\phi | S, S \rangle_{\frac{\pi}{2}, \phi} \right) = S^2 + \frac{S}{2}, \quad \left| \left(\langle S, S |_{\frac{\pi}{2}, \phi} \right) \left(\partial_\phi | S, S \rangle_{\frac{\pi}{2}, \phi} \right) \right|^2 = S^2 \quad (3.19)$$

where we added the left and right arrows to the derivatives to provide additional clarity concerning their action. The above expressions together give QFI,

$$I_\phi \left(|S, S\rangle_{\frac{\pi}{2}, \phi} \right) = 2S. \quad (3.20)$$

The above shows linear scaling with the number of the systems $N = 2S$, often referred to as the classical limit, since coherent states are thought to be the most classical states.

The estimation problems encountered in practice can naturally extend beyond estimation of a single variable θ . Quantum Cramér-Rao generalizes to the case of vector estimator $\boldsymbol{\theta}_{\text{est}}$ of a vector-valued parameter $\boldsymbol{\theta} := \{\theta_i\}$ by relating its covariance matrix $\text{Cov}(\boldsymbol{\theta}_{\text{est}})$ with the Fisher information matrix by [149, 150]

$$\text{Cov}(\boldsymbol{\theta}_{\text{est}}) - \mathbf{I}_\theta(\rho_\theta)^{-1} \geq 0, \quad (3.21)$$

where the elements of the Fisher information matrix $\mathbf{I}_\theta(\rho_\theta)$ are defined as

$$(\mathbf{I}_\theta(\rho_\theta))_{ij} := \frac{1}{2} \left(\rho_\theta \{L_{\theta_i}, L_{\theta_j}\} \right), \quad (3.22)$$

with $\{A, B\} = AB + BA$ denoting the anti-commutator. Multiparameter, however, poses a unique challenge of finding a measurement, which saturates all of the bounds corresponding to the diagonalised form of the inequality. This very often leads to nonsaturability. There exists, however, a bound referred to as Holevo-Cramér-Rao bound, which can be up to a factor of two tighter than the Cramér-Rao bound [151] and can be saturated when infinite number of samples is provided; the drawback, however, is that this bound is far more difficult to evaluate. Nevertheless, if the SLD operators commute on average with the probe state, $\text{tr} \left(\rho_\theta [L_{\theta_i}, L_{\theta_j}] \right) = 0$, then the two bounds coincide and are asymptotically saturable at the same time [151–153].

3.1.3.1 Simultaneous estimability of two directions of displacement for Squeeze-Displace-Unsqueeze protocol

In order to provide a reference point for later results, we will now consider the problem of possibility of simultaneous estimation of two directions of displacement in the phase-space for SU(1,1) interferometry. We start from the expressions in [26]. It is straightforward to demonstrate, that single system in an initial vacuum state $|0\rangle$ undergoing an SDU protocol is transformed into

$$|\psi_1(\gamma, r)\rangle = S_1^\dagger(r)D(\gamma)S_1(r)|0\rangle = |\alpha e^r + i\beta e^{-r}\rangle \quad (3.23)$$

where $\gamma = \alpha + i\beta$. Since the endstate $|\alpha e^r + i\beta e^{-r}\rangle$ is coherent, it is straightforward to calculate for it the QFI matrix with respect to α and β and evaluate average commutation of SLDs,

$$\mathbf{I}(|\psi_1(\gamma, r)\rangle) = 4 \begin{pmatrix} e^{2r} & 0 \\ 0 & e^{-2r} \end{pmatrix}, \quad \text{Tr}(\rho_{\psi_1}[L_\alpha, L_\beta]) = 8i\alpha\beta. \quad (3.24)$$

where $\mathbf{I}_{11}(\cdot) = I_\alpha(\cdot)$ and $\mathbf{I}_{22}(\cdot) = I_\beta(\cdot)$. We see that the QFIs are inversely proportional to one another, as should be expected – one direction enjoys improved sensitivity, while the other, as already mentioned, serves as an "uncertainty reservoir". Furthermore, by considering the result of average commutation we see that Cramér-Rao bounds are fully saturable for simultaneous measurement only when either $\alpha = 0$ or $\beta = 0$, ie. one of the displacements is estimated around 0.

The proper SU(1,1) interferometric scheme acts on two beams and two-mode squeezing, and thus the state resulting from implementing SDU protocol on two initial vacuum modes, in accordance with Fig. 3.2, is given by

$$|\psi_2(\gamma, r)\rangle = S_2^\dagger(r)(D(\gamma) \otimes \mathbb{I})S_2(r)(|0\rangle \otimes |0\rangle) = B(|\alpha e^r + i\beta e^{-r}\rangle \otimes |\alpha e^{-r} + i\beta e^r\rangle), \quad (3.25)$$

where $B = e^{iJ_{2\pi/2}}$ is the operator representation of a beamsplitter. Again, we consider the full QFI matrix together with average commutability of SLDs,

$$\mathbf{I}(|\psi_2(\gamma, r)\rangle) = 8 \begin{pmatrix} \cosh(2r) & 0 \\ 0 & \cosh(2r) \end{pmatrix}, \quad \text{Tr}(\rho_{\psi_1}[L_\alpha, L_\beta]) = 16i. \quad (3.26)$$

From the above we find two conclusions. First, since we have the Fisher matrix proportional to identity matrix, estimation for arbitrary linear combination of α and β enjoys increased sensitivity scaling as $\cosh(r)$. However, since SLDs never commute on average, it is impossible to saturate Cramér-Rao bounds for both α and β , or displacements with respect to any pair of orthogonal directions in the phase space.

The above may seem to stand in contradiction with the results presented by Caves in [26]. Therein, he shows that both quadratures – directions of displacement in the phase space – can be estimated better than SNL sensitivity. However, it cannot be seen as simultaneous estimation in terms of the multivariate quantum Cramér-Rao bound. If we consider set of identities

$$\begin{aligned} B(S_1 \otimes S_1^\dagger)B^\dagger &= S_2, \\ BD(\gamma)^{\otimes 2}B^\dagger &= \mathbb{I} \otimes D(\sqrt{2}\gamma), \\ B(x_a + ip_b)B^\dagger &= ((x_a + x_b) + i(p_b - p_a))/\sqrt{2}. \end{aligned}$$

Caves demonstrates that observing Bell variables $x_a + x_b$ and $p_a - p_b$ is equivalent to

incorporating a beamsplitter operator B , measuring single quadratures individually, x_a on the first mode and p_b on the second, then applying the beamsplitter $B^\dagger$ on the state post-measurement. Given the setup described in equation (3.25), it becomes apparent that this procedure effectively simulates two separate local measurements on two states undergoing independent SDU protocols.

3.2 Squeeze-Rotate-Unsqueeze protocol for spin interferometry

The main contribution presented in this chapter is the combination of the squeeze-displace-unsqueeze protocol interpretation of the $SU(1,1)$ interferometry, which has been given by Caves [26] with the spin-squeezing concept introduced by Kitagawa and Ueda [128] – namely, the squeeze-**rotate**-unsqueeze (SRU) protocol, which we will show to provide QFI scaling with S^2 for arbitrary equatorial axis. In the original work the consideration starts with a simplified scenario, where a specific axis X or Y is chosen and the protocol is considered for a single-spin scenario, to which we refer reader interested in the simplest possible example [14]. In this thesis we will skip it in favour of presenting complete calculations for the proper protocol with two spins acting as analogue of two bosonic beams. The input states are taken as coherent states along X axis with the main spin S_M and probe spin S_P . Just like in the standard $SU(1,1)$ scenario, the rotation of angle ϕ around an equatorial axis along $(\cos \Phi, \sin \Phi, 0)$ is introduced only on one spin. Collecting together squeeze on two spins, rotation and unsqueeze, the final state is given by

$$|\text{SRU}_2(\gamma, \Phi; \mu)\rangle = \underbrace{e^{-i\mu S_z \otimes S_z}}_{\text{Unsqueeze}} \underbrace{e^{i\gamma S(\Phi) \otimes I}}_{\text{Rotate}} \underbrace{e^{i\mu S_z \otimes S_z}}_{\text{Squeeze}} \underbrace{|S_M, S_M\rangle_{\pi/2,0} \otimes |S_P, S_P\rangle_{\pi/2,0}}_{\text{Two SCS along X axis}}. \quad (3.27)$$

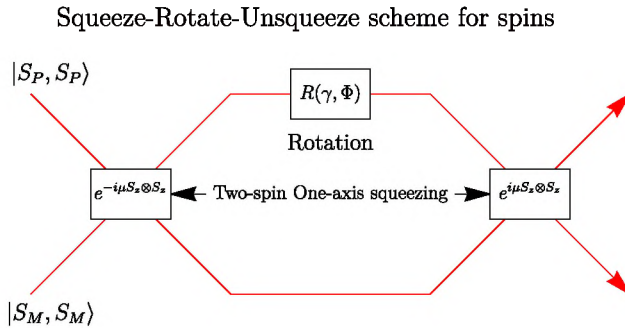


Figure 3.4: Schematic depiction of the SRU protocol described by equation (3.27) with a rotation $R(\gamma, \Phi)$ only on one of the spins and the two-spin squeeze operators $\exp(i\mu S_z \otimes S_z)$. Note the similarity between the introduced SRU scheme and the $SU(1,1)$ interferometer.

After rudimentary calculations one arrives at the explicit expression for the state after the SRU protocol in terms of the canonical basis,

$$|\text{SRU}_2(\gamma, \Phi; \mu)\rangle = \frac{1}{2^{S_M+S_P}} \sum_{j=-S_M}^{S_M} \sum_{k=-S_P}^{S_P} \sqrt{\binom{2S_M}{S_M-j} \binom{2S_P}{S_P-k}} \left(\cos \frac{\gamma}{2} + e^{i(\Phi-k\mu)} \sin \frac{\gamma}{2} \right)^{S_M+j} \times \left(\cos \frac{\gamma}{2} - e^{-i(\Phi-k\mu)} \sin \frac{\gamma}{2} \right)^{S_M-j} |j, k\rangle,$$

where for brevity we use $|j, k\rangle \equiv |S_M, j\rangle \otimes |S_P, k\rangle$. The above form allows us to derive the expression for the QFI for the rotation angle γ around any fixed equatorial axis parametrized by the angle Φ . For its evaluation, we will also require the derivative, which is given by

$$\begin{aligned} |\partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu)\rangle &= \frac{1}{2^{S_M+S_P}} \sum_{j=-S_M}^{S_M} \sum_{k=-S_P}^{S_P} \sqrt{\binom{2S_M}{S_M-j} \binom{2S_P}{S_P-k}} \\ &\times \left(\frac{(S_M-j) \cos(k\mu - \Phi) - S_M(\sin(\gamma) + i \cos(\gamma) \sin(k\mu - \Phi))}{\cos(\gamma) - i \sin(\gamma) \sin(k\mu - \Phi)} \right) \\ &\times \left(\cos \frac{\gamma}{2} + e^{i(\Phi-k\mu)} \sin \frac{\gamma}{2} \right)^{S_M+j} \left(\cos \frac{\gamma}{2} - e^{-i(\Phi-k\mu)} \sin \frac{\gamma}{2} \right)^{S_M-j} |j, k\rangle. \end{aligned} \quad (3.28)$$

Evaluation of the above expressions will be simpler once the summation is shifted by replacing $j \rightarrow S_M - j$. To further compress the subsequent expression we introduce the following three auxiliary sums

$$F_0 = \frac{1}{2^{2S_M}} \sum_{j=0}^{2S_M} \binom{2S_M}{S_M-j} (1+a)^{2S_M-j} (1-a)^j = 1, \quad (3.29a)$$

$$F_1 = \frac{1}{2^{2S_M}} \sum_{j=0}^{2S_M} j \binom{2S_M}{S_M-j} (1+a)^{2S_M-j} (1-a)^j = S_M(1-a), \quad (3.29b)$$

$$F_2 = \frac{1}{2^{2S_M}} \sum_{j=0}^{2S_M} j^2 \binom{2S_M}{S_M-j} (1+a)^{2S_M-j} (1-a)^j = S_M \left(S_M(1-a)^2 + \frac{1-a^2}{2} \right), \quad (3.29c)$$

with $a = \cos(\Phi - k\mu) \sin(\gamma)$. With these one can express the relevant inner products as

$$\begin{aligned} \langle \text{SRU}_2(\gamma, \Phi; \mu) | \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) \rangle &= \sum_{k=-S_P}^{S_P} \frac{\binom{2S_P}{S_P-k}}{2^{2S_P}} \left[(F_0 F_1) \cdot \left(\frac{S_M(i c(2\gamma) s(\Phi-k\mu) + c(\Phi-k\mu) - s(2\gamma))}{i s(2\gamma) s(\Phi-k\mu) - s^2(\gamma) + c^2(\gamma)} - \frac{c(\Phi-k\mu)}{i s(2\gamma) s(\Phi-k\mu) - s^2(\gamma) + c^2(\gamma)} \right) \right], \\ \langle \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) | \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) \rangle &= \sum_{k=-S_P}^{S_P} \frac{\binom{2S_P}{S_P-k}}{2^{2S_P}} \left[(F_0 F_1 F_2) \cdot \left(\frac{S_M^2 \left(\frac{2}{s(2\gamma) c(\Phi-k\mu) + 1} - 1 \right)}{2 S_M c(\Phi-k\mu) (s(2\gamma) - c(\Phi-k\mu))} - \frac{s^2(2\gamma) s^2(\Phi-k\mu) + c^2(2\gamma)}{c^2(\Phi-k\mu)} \right) \right], \end{aligned} \quad (3.30)$$

where for brevity we replaced $c \equiv \cos$ and $s \equiv \sin$. Once evaluated, the above

expressions are simplified to an elegant forms

$$\begin{aligned} \langle \text{SRU}_2(\gamma, \Phi; \mu) | \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) \rangle &= i S_M \cos^{2S_P} \left(\frac{\mu}{2} \right) \sin \Phi, \\ \langle \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) | \partial_\gamma \text{SRU}_2(\gamma, \Phi; \mu) \rangle &= \frac{S_M}{4} \left(1 + 2S_M - (2S_M - 1) \cos(2\Phi) \cos^{2S_P}(\mu) \right), \end{aligned} \quad (3.31)$$

which yields the expression

$$\begin{aligned} I_\gamma(|\text{SRU}_2(\gamma, \Phi; \mu)\rangle) &= S_M \left(2S_M + 1 - (2S_M - 1) \cos(2\Phi) \cos^{2S_P}(\mu) \right. \\ &\quad \left. - 4S_M \sin^2(\Phi) \cos^{4S_P} \left(\frac{\mu}{2} \right) \right). \end{aligned} \quad (3.32)$$

In order to have some qualitative feeling about the expression (3.32) in Fig. 3.5 we present plots for a fixed primary spin $S_M = 5$ and several selected values of S_P and Φ . For maximal squeezing parameter, $\mu = \pi$, QFI reaches the maximum of $I = 4S_M^2 = 100$ for each value of S_P , but it alternates between X axis for integer spins and Y axis for half-integer spins S_P . For the same value of squeezing, QFI for rotations around orthogonal axes $I = 2S = 10$. More interestingly, however, for half-maximal squeezing $\mu = \pi/2$ we find that for arbitrary axis angle Φ quantum Fisher information is well approximated by $I \approx 2S^2 + S = 55$.

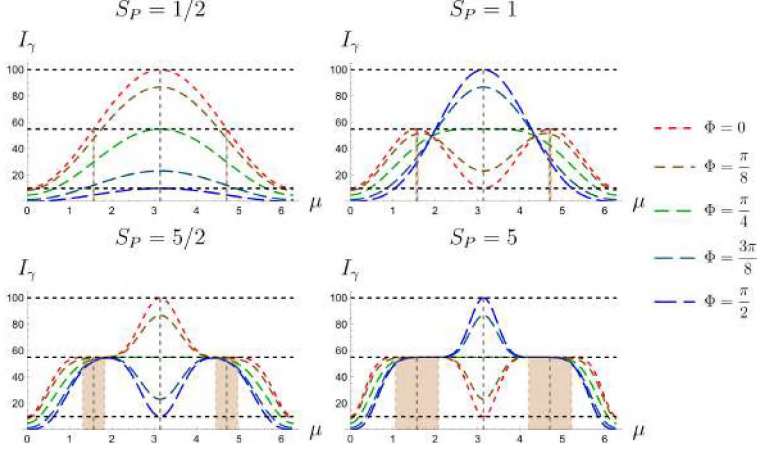


Figure 3.5: Fisher information I_γ for the SRU protocol as a function of the squeezing parameter μ and rotation axis angle Φ (Eq. (3.32)) for a fixed main spin, $S_M = 5$. Note the maximal value for $\mu = \pi$ and the plateau around $\pi/2$ (shaded region), growing with the probe spin S_P .

Considering the maximal squeezing $\mu = \pi$ we find

$$I_\gamma(|\text{SRU}_2(\gamma, \Phi; \pi)\rangle) = S_M (2S_M + 1 + (1 - 2S_M) \cos(2\Phi) (-1)^{2S_P}) \quad (3.33)$$

$$= 2S_M \sin^2(\Phi + S_P \pi) + 4S_M^2 \cos^2(\Phi + S_P \pi), \quad (3.34)$$

which can be understood as a distance to a point on an ellipse with principal semiaxes of length $(2S_M, 4S_M^2)$, and thus bounded from above and below by

$$2S_M \leq I_\gamma(|\text{SRU}_2(\gamma, \Phi; \pi)\rangle) \leq 4S_M^2. \quad (3.35)$$

More precisely, the axis X is optimal for half-spins, whereas the axis Y reaches optimality for integer spins.

For half-maximal squeezing, we find that around $\mu = \frac{\pi}{2}$ the QFI achieves a plateau at a level between the linear and quadratic scaling with the number of spins $1/2$. The observation is made rigorous by considering an approximation of the QFI with $\mu = \frac{\pi}{2} + \epsilon$ and $\epsilon \ll 1$,

$$I_\gamma\left(|\text{SRU}_2(\gamma, \Phi; \frac{\pi}{2})\rangle\right) \stackrel{(a1)}{\approx} S_M \left[2S_M + 1 + (1 - 2S_M) \cos(2\Phi) \epsilon^{2S_P} (1 + \mathcal{O}(\epsilon^2)) - 4S_M \sin^2(\Phi) 2^{-2S_P} (1 - 2S_P \epsilon + \mathcal{O}(\epsilon^2)) \right] \quad (3.36)$$

$$\stackrel{(a2)}{\approx} S_M (2S_M + 1 + (1 - 2S_M) \cos(2\Phi) \epsilon^{2S_P}) \quad (3.37)$$

$$\stackrel{(a3)}{\approx} \frac{4S_M^2 + 2S_M}{2}. \quad (3.38)$$

with the approximation (a1) achieved by Taylor expansion around $\epsilon = 0$ in the lowest order, (a2) by noting that 2^{-2S_P} vanishes in the limit of large S_P , and (a3) by noting that $\epsilon \ll 1$ justifies neglecting the term of order ϵ^{2S_P} . This provides the formal demonstration that the QFI is flat around $\mu = \frac{\pi}{2}$ and becomes progressively more flat for large probe spins. The final line demonstrates that we are exactly at the “optimal” value halfway between the Heisenberg and the classical scalings, giving a stable region within which rotation around any axis can be estimated. In particular, in the stable region, one still retains $I_\gamma(|\text{SRU}_2(\gamma, \Phi; \frac{\pi}{2})\rangle) \approx \mathcal{O}(N_M^2)$ for $S_M = N_M/2$. Note that the independence of QFI from the choice of the rotation exhibited by the SRU procedure mimics the behaviour of QFI for SU(1,1) interferometric protocol, where any direction of displacement has the same high value of QFI, allowing for estimation with enhanced sensitivity in arbitrary direction. As such the results we demonstrate for arbitrary equatorial axis can be seen as an extension of results from [154].

It is instructive to compare the expression (3.32) with a general formula for the QFI for a single spin,

$$I_\gamma(|\text{SRU}_1(\gamma, \Phi; \mu)\rangle) = S \left(2S + 1 - (2S - 1) \cos(2\Phi) \cos^{2S-2}(2\mu) - 4S \cos^2(\Phi) \cos^{4S-2}(\mu) \right) \quad (3.39)$$

where the SRU protocol acting on a single spin is defined as

$$|\text{SRU}_1(\gamma, \Phi; \mu)\rangle = e^{-i\mu S_z^2} e^{i\gamma S(\Phi)} e^{i\mu S_z^2} |S, S\rangle_{\pi/2, 0}. \quad (3.40)$$

Note that the expression for QFI is almost identical except for the replacement $\mu \rightarrow 2\mu$ and some minor changes in the powers and trigonometric functions of the angle Φ . Most importantly, the analysis concerning the plateau $|\text{SRU}_2(\gamma, \Phi; \frac{\pi}{2})\rangle$ applies without modification for $|\text{SRU}_1(\gamma, \Phi; \frac{\pi}{4})\rangle$. This suggests that the state $|S; \frac{\pi}{4}\rangle$ presented in the middle panel of Fig. 3.3, similar in the qualitative properties of its Wigner function to the compass state [155], may be of independent interest.

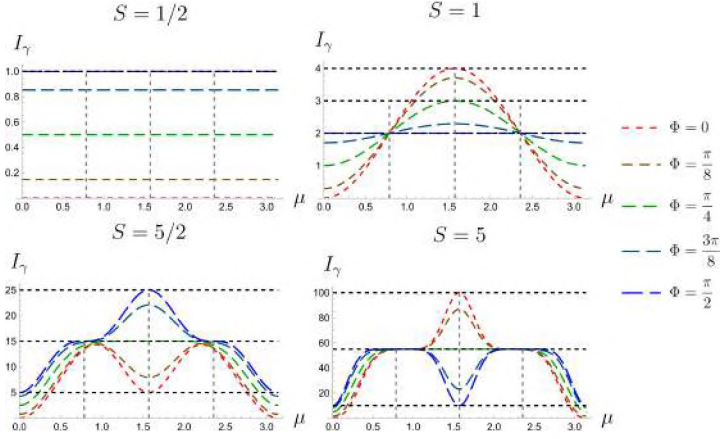


Figure 3.6: Single-spin counterpart of Fig. 3.5 for the same range of spin values. Note the reduced range of the squeezing parameter μ , from $[0, 2\pi]$ to $[0, \pi]$. The plateau around $\mu = \frac{\pi}{4}$ flattens out with growing S , in analogy with the effect of enlarging the probe spin S_P in the two-spin setting.

3.3 Relative rotation encoding and perspective for multi-variate estimation

When thinking about the standard Mach-Zehnder interferometer, as depicted in Fig. 3.1, one can think about a different scenario, in which relative phase ϕ to be estimated instead of being applied to only one of two arms is distributed, with the first arm acquiring phase $\phi/2$ and the second arm $-\phi/2$. When we consider the classical point of view, there should be no difference between the two scenarios; however, as shown in [156], in quantum setting the two different scenarios lead to different values of QFI. This suggests that there may exist measurement schemes that provide better sensitivity for one scenario than for the other.

Similar situation arises when one considers distributing displacements among the two arms of SU(1,1) interferometer [157]. In fact, the second arm of the interferometer can be considered a quantum frame of reference [158–160], which can be in principle even entangled with the principal system under consideration. In this section we

will consider a similar problem for the two-spin SRU protocol applied to the case when on a rotation $R(\gamma_1, \Phi_1)$ is effected on the first spin S_M and a rotation $R(\gamma_2, \Phi_2)$ is applied to the second spin S_P . Thus, the multiparameter problem in such a setting is to estimate a pair of rotations $\gamma = (\gamma_1, \gamma_2)$.

Squeeze-Rotate-Unsqueeze scheme with two rotations

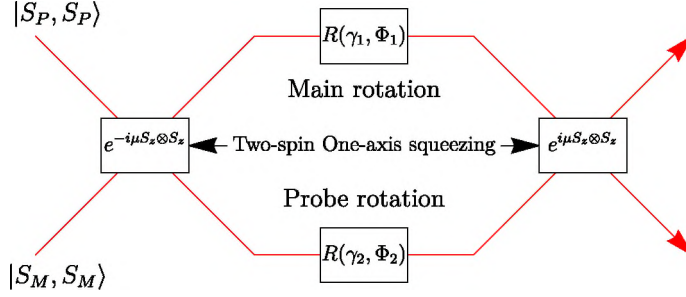


Figure 3.7: The additional rotation $R(\gamma_2, \Phi_2)$ on the probe spin is introduced in order to investigate how the coding of the rotation angle relative to the probe spin that serves as reference influences the sensitivity, in analogy to the work for the $SU(1,1)$ interferometers [157].

The quantum Fisher information matrix reads

$$I_\gamma = \begin{pmatrix} I_{\gamma_1} & (I_\gamma)_{12} \\ (I_\gamma)_{21} & I_{\gamma_2} \end{pmatrix}, \quad (3.41)$$

where for convenience we dropped the arguments. The diagonal elements are given by the already known expressions for the main spin

$$\begin{aligned} I_{\gamma_1} &\equiv I_{\gamma_1}(|\text{SRU}_2(\gamma_1, \gamma_2, \Phi_1, \Phi_2; \mu)\rangle) \\ &= S_M \left(2S_M + 1 - (2S_M - 1) \cos(2\Phi_1) \cos^{2S_P}(\mu) - 4S_M \sin^2(\Phi_1) \cos^{4S_P}\left(\frac{\mu}{2}\right) \right), \end{aligned} \quad (3.42)$$

and similarly for γ_2 , with the roles of S_M, Φ_1 and S_P, Φ_2 swapped,

$$\begin{aligned} I_{\gamma_2} &\equiv I_{\gamma_2}(|\text{SRU}_2(\gamma_1, \gamma_2, \Phi_1, \Phi_2; \mu)\rangle) \\ &= S_P \left(2S_P + 1 - (2S_P - 1) \cos(2\Phi_2) \cos^{2S_M}(\mu) - 4S_P \sin^2(\Phi_2) \cos^{4S_M}\left(\frac{\mu}{2}\right) \right). \end{aligned} \quad (3.43)$$

The off-diagonal terms can be found as the covariance between γ_1 and γ_2 ,

$$\begin{aligned} (I_\gamma)_{12} &= (I_\gamma)_{21} = 4(\langle S(\Phi_1) \otimes S(\Phi_2) \rangle - \langle S(\Phi_1) \rangle \langle S(\Phi_2) \rangle) \\ &= 4S_P S_M \cos \Phi_1 \cos \Phi_2 \cos^{2(S_P+S_M-1)}\left(\frac{\mu}{2}\right) \sin^2\left(\frac{\mu}{2}\right). \end{aligned} \quad (3.44)$$

From these the lower bounds on the variance of the estimators of γ_1 and γ_2 angles,

calculated from positivity of the primary minors of $\text{Cov}(\gamma) - \mathbf{I}_\gamma$, are given as

$$\text{Var}(\gamma_1) \geq \frac{I_{\gamma_2}}{\det(\mathbf{I}_\gamma)}, \quad \text{Var}(\gamma_2) \geq \frac{I_{\gamma_1}}{\det(\mathbf{I}_\gamma)}. \quad (3.45)$$

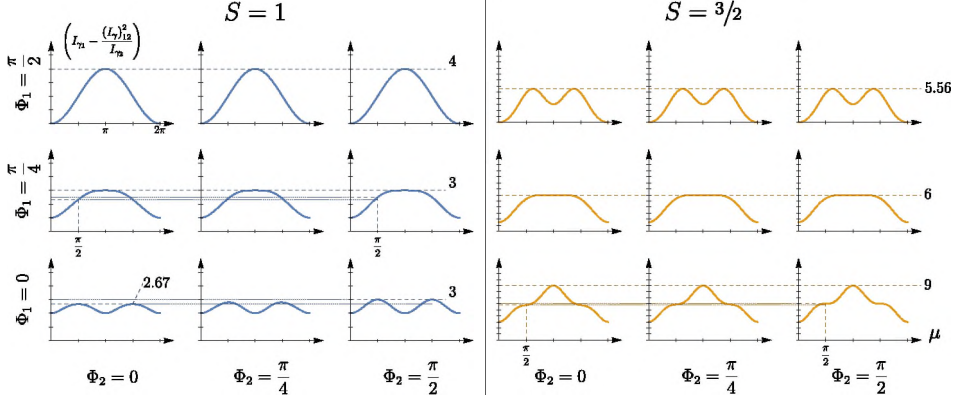


Figure 3.8: Bounds on the sensitivity for estimating γ_1 in the scheme as shown in Fig. 3.7 as function of the squeezing parameter μ , for both main and probe spins equal and for selected angles Φ_1 and Φ_2 of rotation axes. Left panel: $S = 1$ right panel: $S = 3/2$. Combinations of the same relative rotation angle $\Phi_2 - \Phi_1$ are on the lines perpendicular to the diagonal of the matrix of plots and seen to give quite different lower bounds $(I_{\gamma_1} - \frac{(I_\gamma)_{12}^2}{I_{\gamma_2}})^{-1}$ for the error of the γ_1 estimation. Note that the optimal bounds at $\mu = \pi/2$ are achieved at $\Phi_1 = \pi/2$ and $\Phi_1 = 0$ for $S = 1$ and $3/2$ respectively. Furthermore, for $S = 1$ and $\Phi_1 = 0$ we note decrease of the bound from 3 to 2.67 together with decreasing Φ_2 from $\pi/2$ to 0, marked by thin lines.

It is observed that $(I_\gamma)_{12}$ is proportional to $\cos \Phi_1 \cos \Phi_2$, hence it can be nullified by setting $\Phi_1 = \pi/2$ or $\Phi_2 = \pi/2$. Moving forward, we can analyze the denominator of the first expression (3.21) and its qualitative behavior by examining the cases of $S_P = S_M = S$ for $S = 1$ and $S = \frac{3}{2}$ (see to Fig. 3.8). For $S = 1$ and $\Phi_1 = 0$, a clear dependence on the angle Φ_2 is observed, ranging from a minimum of $8/3 \approx 2.67$ to a maximum of 3, aligning with the previously identified maximum in single-parameter estimation under similar conditions. With $\Phi_1 = \frac{\pi}{4}$, the well-known central maximum with a value of 3 is evident. However, a subtle variation is observed at half-squeezing, $\mu = \pi/2$, attributable to the fact that the correction term $\frac{(I_\gamma)_{12}^2}{I_{\gamma_2}}$ is proportionate to $\sin^2(\mu)$, which vanishes for coherent and maximally mixed states, $\mu = 0$ and $\mu = \pi$, respectively. A similar trend is noticeable for $S = \frac{3}{2}$, although the impact is diminished since the corrections are linear in both the main and auxiliary spins, thereby not significantly affecting quadratic terms. Moreover, they are contingent on $\sin^2(\frac{\mu}{2}) \cos^{2(S_M+S_P-1)}(\frac{\mu}{2})$. When $S_M = S_P = S$, the maxima of this expression are reached at $\mu_{\max} = 4 \arctan(\sqrt{2S-1} \pm \sqrt{2S})$, with $\sin^2(\frac{\mu_{\max}}{2}) \cos^{4S-2}(\frac{\mu_{\max}}{2}) = \frac{(1-2S)^{2S-1}}{(4S)^S} \approx O(S \log S)$, smaller than the leading order of S^2 , thus negligible for $2S \gg 1$.

If we want to have a proper analogy between the SRU protocol for spins and the results obtained in [156] for SDU protocol in $SU(1,1)$ interferometry we need the rotations in both spins in such a way that their difference $\Delta\gamma = \gamma_2 - \gamma_1$ is well defined; this happens when the axes of rotations agree, $\Phi_1 = \Phi_2 := \Phi$. Since the covariance between the estimators is not defined from the Cramér-Rao bounds, we may choose them to be uncorrelated, $\text{Cov}(\gamma_1, \gamma_2) = 0$. With this, we find

$$\text{Var}(\gamma_2 - \gamma_1) = \text{Var}(\gamma_2) + \text{Var}(\gamma_1) \geq \frac{I_{\gamma_1} + I_{\gamma_2}}{\det(\mathbf{I}_\gamma)} \equiv \tilde{I}_{\Delta\gamma}^{-1}. \quad (3.46)$$

Instead of providing an explicit expression for eq. (3.46) we provide qualitative analysis of the resulting QFI values in Fig. 3.9. Certain features, such as plateau at half-squeezing, still persist when considered for the estimation of $\Delta\gamma$ for any value of S_1 and S_2 . Furthermore, disappearance of the central maximum can be seen whenever $S_1 - S_2$ is half-integer.

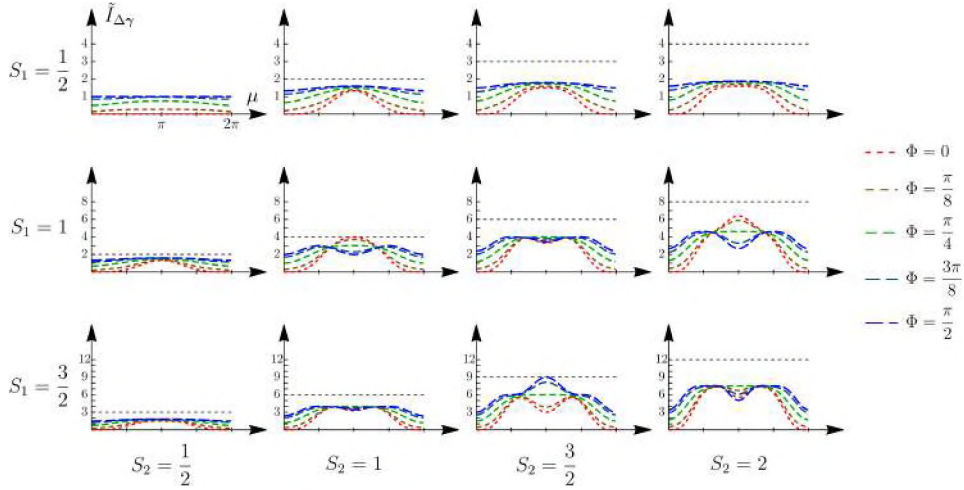


Figure 3.9: Plots of values of $\tilde{I}_{\Delta\gamma}$ defined in (3.46) for selected values of S_1 , S_2 and Φ . Note the dashed black line, corresponding to the value $\tilde{I}_{\Delta\gamma} = 4S_1S_2$, which is saturated only when $S_1 = S_2 = S$, $\mu = \pi$ and the axis of rotation is either X or Y , depending whether S is integer or half-integer.

As the other extreme case, we recall the semipositivity condition (3.21) to derive the bounds on covariances, which yields

$$\left| \text{Cov}(\gamma_1, \gamma_2) + \frac{(\mathbf{I}_\gamma)_{12}}{\det(\mathbf{I}_\gamma)} \right| \leq \sqrt{\left(\text{Var}(\gamma_1) - \frac{I_{\gamma_2}}{\det(\mathbf{I}_\gamma)} \right) \left(\text{Var}(\gamma_2) - \frac{I_{\gamma_1}}{\det(\mathbf{I}_\gamma)} \right)}. \quad (3.47)$$

We may consider an extreme case, in which one of the variances actually saturates

the bound, $\text{Var}(\gamma_1) = \frac{I_{\gamma_2}}{\det(\mathbf{I}_\gamma)}$, which in turn means that the covariance is well-defined, $\text{Cov}(\gamma_1, \gamma_2) = -\frac{(\mathbf{I}_\gamma)_{12}}{\det(\mathbf{I}_\gamma)}$. Collecting these two facts, we may write explicitly the variance of the difference of the two estimators as

$$\text{Var}(\gamma_2 - \gamma_1) = \text{Var}(\gamma_2) + \text{Var}(\gamma_1) - 2\text{Cov}(\gamma_1, \gamma_2) \geq \frac{I_{\gamma_1} + I_{\gamma_2} + 2(\mathbf{I}_\gamma)_{12}}{\det(\mathbf{I}_\gamma)} \equiv \tilde{I}_{\Delta\gamma}^{-1}, \quad (3.48)$$

which has been plotted with respect to μ in Fig. 3.10 for several values of S_1 , S_2 and Φ .

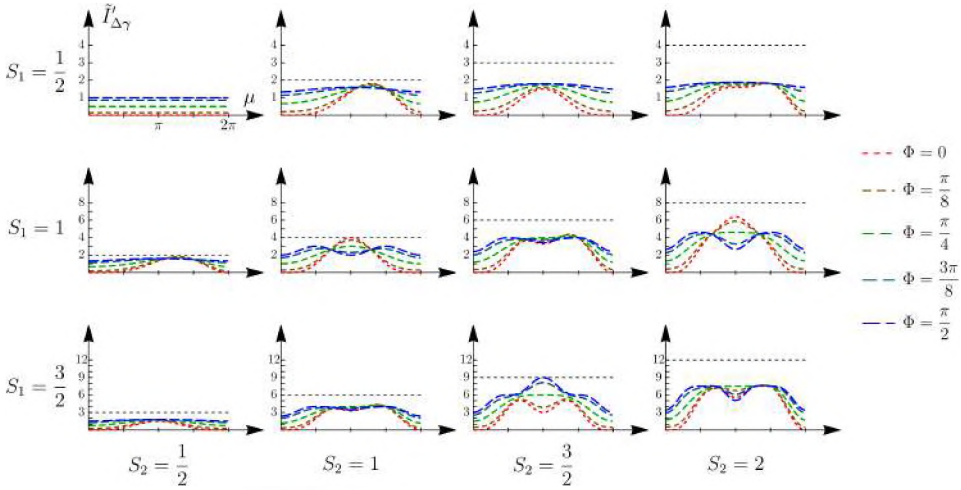


Figure 3.10: Plots of values of $\tilde{I}'_{\Delta\gamma}$ defined in (3.48) for selected values of S_1 , S_2 and Φ . Note the dashed black line, corresponding to the value $\tilde{I}'_{\Delta\gamma} = 4S_1S_2$. In particular, note the asymmetry appearing when $S_2 - S_1$ is a half-integer.

We again observe that whenever $S_1 = S_2$, plots in Fig. 3.10 resemble closely the QFI plots for standard SRU in Fig. 3.5, up to the appearance of plateaus and central maximum for Φ . However, one should note that as soon as $S_1 - S_2$ is half-integer we see asymmetry with higher values $\tilde{I}'_{\Delta\gamma}$ for $\mu \in (\pi, 2\pi)$. This behaviour is understood better once we note that in this case full period of QFI is 4π instead of 2π as before. This can be justified heuristically, since single-axis squeezing for integer spins is exactly periodic with period π , while for half-integers squeezing of $\mu = \pi$ yields a spin coherent space antipodal to the original and only for $\mu = 2\pi$ we go back to the original SCS.

3.4 Short consideration for two-axis squeezing

For completeness, we shortly consider a variant of the SRU protocol where instead of one-axis squeezing we utilize the two-axis squeezing operator. The full protocol is now defined as

$$|\text{SRU}_2^2(\gamma, \Phi; \mu)\rangle = e^{-\mu(S_-^2 - S_+^2)} e^{i\gamma S(\Phi)} e^{\mu(S_-^2 - S_+^2)} |S, S\rangle, \quad (3.49)$$

in analogy to the SRU protocol defined in (3.40). Here the squeezing Hamiltonian has been changed to $i(S_+^2 - S_-^2)$ and the input state is a spin coherent state pointing along the Z axis, following the preparation presented in [161]. Therein, the authors have demonstrated that the variance of a spin in a selected direction orthogonal to the main axis of the squeezed state for large spins approaches a constant. Later the two-axis squeezing has been studied theoretically [162] and only recently an experimental realization has been proposed [163].

In this case analytical treatment is not accessible. Therefore we present a set of plots for selected values of S which present the qualitative behaviour of two-axis squeezing. Despite clearly periodic behaviour for $S = 1$ and $S = \frac{5}{2}$, for larger spins the QFI fails to reach the value of $4S^2$ demonstrated to be attainable for one-axis squeezing. Furthermore, despite isolated points at which Fisher information agrees for different rotation axes, no stable plateaus can be found.

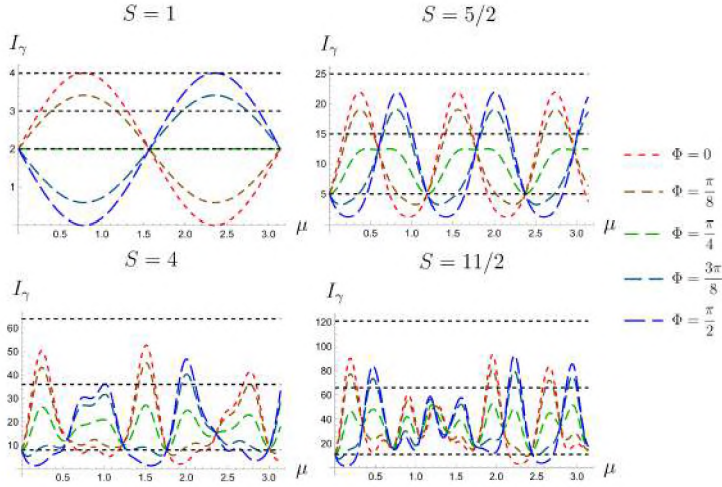


Figure 3.11: Plots of quantum Fisher information $I_\gamma(|\text{SRU}_2(\gamma, \Phi; \mu)\rangle_1)$ for state $|\text{SRU}_2(\gamma, \Phi; \mu)\rangle_1$ prepared with the two-axes squeezing protocol (3.49). Note the aperiodic behaviour and absence of stable plateaus.

3.5 Summary and Outlook

In this chapter we have considered a squeeze-rotate-unsqueeze protocol for spin systems, inspired by the squeeze-displace-unsqueeze protocol introduced in the

reinterpretation of $SU(1,1)$ interferometry as a displacement detection device [26] and the spin squeezing introduced originally by Kitagawa and Ueda [128]. We demonstrated that this protocol, when applied to a two-spin system initialised in spin coherent states along the X axis, lead to QFI scaling with the square of magnitude of primary spin S_M^2 for half-maximal squeezing and arbitrary equatorial axis of rotation; similar effect is observed for a single spin with two-spin one-axis squeezing replaced by single-spin squeezing. We also considered a scenario in which rotations are introduced both to the principal and secondary system, where we demonstrated the dependence of the resulting Cramér-Rao bounds for both angles separately on the axes of rotations Φ_1, Φ_2 . Furthermore, we studied the CRB for the difference of rotation angles when the axes of rotation agree, and demonstrated that the behaviour of the corresponding QFI depends on the covariance between the underlying single-angle estimators.

In particular, based on recent experimental efforts for generation of a four-photon state with $J = 3/2$ and tetrahedral symmetry in the polarization-related $SU(2)$ representation [164], sensible to rotations within the polarization space, one may expect that sensing of rotation of spins may find applications beyond sensing rotations of particles with actual spin to spaces which can be described effectively by spin-related structure. In particular, the scheme we propose in [14] can be scaled for arbitrary size of spin, as opposed to the tetrahedral state which is limited to the specific subspace related to four particles with spin $1/2$. In particular, we expect our method to be applicable to polarization sensing, since methods for polarization squeezing have been known for a long time already [165–168].

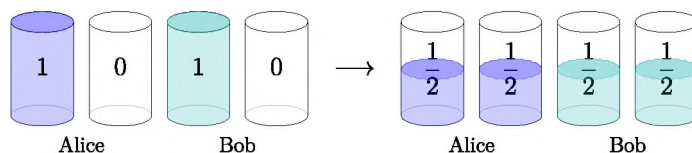
4 Thermal Recall [16]

Rather than describing Markovian Thermal Processes, Markovianity and short memory phenomena in the context of physics, let me start this particular chapter with a story containing an analogy*.

Alice and Bob come into a bar – classical bar for a change, to catch some rest from quantum games – and order drinks from Victor. Each of them is then given two glasses, a full one and an empty one. Turns out they are dragged into yet another game, this time a classical one.

The rules are simple – glasses can be connected pairwise and the liquid, in agreement with common knowledge about connected vessels, can be equalized, after which the flow stops; they can drink only the fraction of the drink that they managed to transfer to the empty glasses. Whatever is left in the other glass goes down the sink.

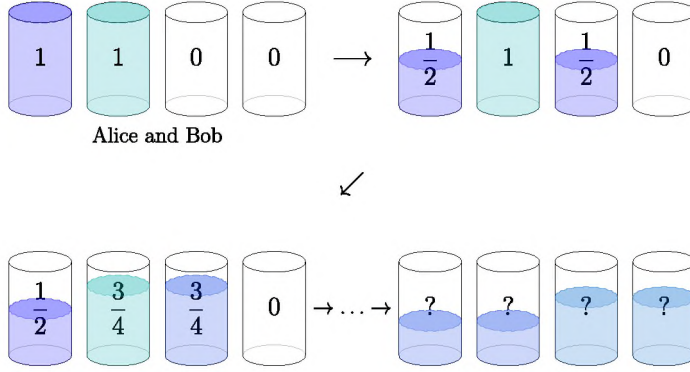
In their first attempt, Alice connected her glasses and Bob did so as well. In this manner, each has managed to drink only half of their respective drinks. Dissatisfied with their result, they decided to order another pair of drinks and join forces.



First, they connected Alice's glasses and then, Bob's full glass to the originally empty glass of Alice. Next, the same sequence of actions was repeated with Bob's glass instead of Alice's. After completing the actions and handing over the originally full glasses, to Victor's amazement they were able to drink more than a half and less than three fourths of the entire drinks (the reader is encouraged to calculate the exact fraction by themselves!). Seeing their success, Victor started to wonder...

In fact, a similar consideration set within Markovian Thermal Processes lead us to ask the question, whether a similar process extended beyond just 2-dimensional Gibbs catalyst (equivalent to Bob joining Alice instead of Alice drinking alone) to a higher-dimensional ancilla could lead to realising full swap and, more generally, realising an arbitrary Thermal Operation by Memory-extended Thermal Operations?

*The analogy itself has been mentioned to me by Kamil Korzekwa, but the story is of my own design.



We will start this chapter, as all the previous chapters, by invoking the basics necessary to understand the results, such as thermal operations, Markovian thermal processes and mathematical notions with particular focus on incomplete regularised beta function, which are necessary ingredients for later derivations. Next, we move to demonstrate that using two-level Markovian thermal operations it is possible to simulate arbitrary beta-swap as well as arbitrary extreme thermal operation, with analytical proof for the former and significant numerical evidence for the latter. We finish the chapter with considering applications of the introduced protocol.

4.1 Preliminaries

In order to fully appreciate the results obtained in [16], it is necessary to introduce several notions. Let us begin by recalling the notion of majorisation and setting it in the context of resource theories of entanglement, coherence, and noisy operations. Next, the notion of thermomajorisation, used in the context of thermal operations at finite temperatures, will be introduced. Subsequently, continuous processes will be addressed by introducing Markovian thermal operations. Finally, the discussion will cover the regularised incomplete beta function and its properties, which are indispensable for understanding and deriving the results in the following sections.

4.1.1 Majorisation and majorisation-based resource theories

Majorisation is a simple concept, which imposes a preorder on the space of probability distributions, most easily explained in the finite-dimensional setting. Let us consider a pair of probability distributions $\mathbf{p} = (p_1, \dots, p_d)$, $\mathbf{q} = (q_1, \dots, q_d) \in \Delta_d$. We say

that $\mathbf{p}$ majorises $\mathbf{q}$, which is commonly denoted as $\mathbf{p} > \mathbf{q}$, if

$$\underbrace{\sum_{i=1}^n p_i^\downarrow}_{=P_n} \geq \underbrace{\sum_{i=1}^n q_i^\downarrow}_{=Q_n}, \quad \forall 1 \leq n \leq d, \quad (4.1)$$

where we define $\mathbf{p}^\downarrow$ by reordering of elements, $p_{\sigma(i)} = p_i^\downarrow$ in such a way that they are non-increasing, $i < j \Rightarrow p_i^\downarrow \geq p_j^\downarrow$. Equivalently, one may consider piecewise-linear convex functions $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$, called majorisation functions or Lorenz curves, and defined by their elbows, $(\frac{i}{d}, P_i)$ and $(\frac{i}{d}, Q_i)$ respectively. Thus, majorization $\mathbf{p} > \mathbf{q}$ implies that the curve $f_{\mathbf{p}}$ is everywhere above $f_{\mathbf{q}}$. In yet other words, one can say that $\mathbf{p} > \mathbf{q}$ if **cummulative distribution function (CDF) of $\mathbf{p}^\downarrow$ is everywhere greater or equal to CDF of $\mathbf{q}^\downarrow$** , see for example Fig. 4.1.

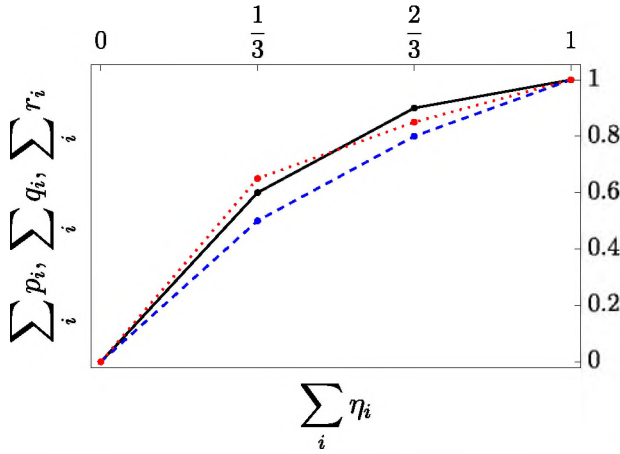


Figure 4.1: Example of three Lorenz curves corresponding to $\mathbf{p} = (0.6, 0.3, 0.1)$ (black), $\mathbf{q} = (0.5, 0.3, 0.2)$ (blue dashed) and $\mathbf{r} = (0.65, 0.15, 0.1)$ (red dotted). Note that $\mathbf{q} < \mathbf{p}$, as blue curve is everywhere less than the black one. Similarly, $\mathbf{r}$ is incomparable to $\mathbf{p}$, as the red curve is partially above and partially below the black curve. However, $\mathbf{r} > \mathbf{q}$.

One of the more important equivalent formulations of majorisation relation is due to Hardy, Littlewood and Pólya [170], where they demonstrate that $\mathbf{p}$ majorises $\mathbf{q}$ if and only if there exists bistochastic matrix B , summarised by conditions $B_{ij} \geq 0$, $\sum_i B_{ij} = \sum_j B_{ij} = 1$, transforming $\mathbf{p}$ into $\mathbf{q}$,

$$\mathbf{p} > \mathbf{q} \iff \mathbf{q} = B\mathbf{p}. \quad (4.2)$$

This, in combination with the full characterisation of the polytope of bistochastic matrices, also known as the Birkhoff[†] polytope $\mathcal{B}$ [172] provides a particularly powerful result – given an arbitrary distribution $\mathbf{p}$ one can determine its full future majorisation cone $\mathcal{T}_+(\mathbf{p})$ as the set of all distributions, into which $\mathbf{p}$ can be transformed. Indeed,

[†] The name "Birkhoff polytope" is commonly used, despite earlier derivations in terms of regular bipartite graph matchings [171].

Birkhoff polytope of bistochastic matrices is given by convex hull of all possible permutation matrices P_σ with the underlying permutations $\sigma \in \mathcal{S}_d$. Thus, the future majorization cone of $\mathbf{p}$ is the convex hull of all permutations of $\mathbf{p}$,

$$\mathcal{T}_+(\mathbf{p}) = \text{conv}(\mathbf{p}_\sigma = P_\sigma \mathbf{p}, \sigma \in \mathcal{S}_d). \quad (4.3)$$

Despite its apparent simplicity, majorisation as a concept is intricate enough to provide content for an entire monograph, which is highly recommended to the reader interested in further details concerning mathematical background of this concept [173].

The concept has been used in many contexts, ranging from economy [174, 175], where it is usually related to wealth distribution and transfers, to chemistry and classical physics, where it is related to the notion of "randomness" or entropy connected to the probability distribution.

In the context of quantum information, three well known resource theories within which the allowed transformations are governed by majorisation relation: resource theory of coherence under incoherent operations [176], entanglement resource theory under local operations and classical communication (LOCC) [78, 177] and thermal operations in infinite temperature [178, 179].

In the case of entanglement, the space of states underlying the resource theory is the space $\mathcal{H}_d^{\otimes 2}$ of bipartite states, which can be defined by their Schmidt decompositions $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |\alpha_i\rangle \otimes |\beta_i\rangle$ and, since local operations are free in this theory, we can limit ourselves to states of the form $|\Psi\rangle = \sum_i \sqrt{\lambda_i} |ii\rangle$. From this point, the state is mapped to a vector of its Schmidt coefficients, $|\psi\rangle \mapsto \mathbf{p}_\psi = (\lambda_1, \dots, \lambda_d)$. Nielsen showed in [177] that a transition $|\Psi\rangle \rightarrow |\Phi\rangle$ is possible if and only if $\mathbf{p}_\psi \prec \mathbf{p}_\phi$. One has to note, however, that this approach applies only to bipartite entangled states and does not generalize easily to situations with more subsystems.

In the case of incoherent operations, one considers the space $\mathcal{H}_d$ with distinguished basis which can be taken to be computational without loss of generality, $\{|i\rangle\}_{i=1}^d$, and considers theory where the free states are those that are incoherent with respect to that basis, $\rho = \sum p_i |i\rangle\langle i|$. Free operations in this theory are the incoherent operations, preserving the set of incoherent states. In this setting, it has been shown by Du, Bai and Guo in [180] that a state $|\psi\rangle = \sum_i \psi_i |i\rangle$ can be transformed into a state $|\phi\rangle = \sum_i \phi_i |i\rangle$ if and only if the probability vectors with respect to the computational basis, $\mathbf{p}_\psi = (|\psi_1|^2, \dots, |\psi_d|^2)$ and $\mathbf{p}_\phi = (|\phi_1|^2, \dots, |\phi_d|^2)$, are in majorization relation, $\mathbf{p}_\psi \prec \mathbf{p}_\phi$.

4.1.2 Thermal operations and thermomajorisation

The main framework used in this and the next chapter are thermal operations [179, 181, 182], which provide a minimal-assumption model for thermodynamic transformations

of finite-dimensional quantum systems in contact with thermal bath. The starting point is to consider a d -dimensional system subject to a Hamiltonian H and in contact with a thermal bath at inverse temperature[‡] $\beta = T^{-1}$ and subject to Hamiltonian H_E ; thus, the state of the bath is the Gibbs state $\gamma_E = \frac{e^{-\beta H_E}}{\text{Tr}(e^{-\beta H_E})}$. One assumes that the interaction between the primary system in a state ρ and the thermal bath is closed and energy preserving. Thus, we may introduce channel $\Phi(\rho)$ resulting from joint evolution

$$\Phi(\rho) = \text{Tr}_E \left(U(\rho \otimes \gamma_E) U^\dagger \right), \quad (4.4)$$

with an additional restriction imposed on the unitary that it commutes with the total Hamiltonian of the system-bath,

$$[U, H \otimes \mathbb{I}_E + \mathbb{I} \otimes H_E] = 0. \quad (4.5)$$

It follows from the energy preservation condition that the Gibbs state of the primary system $\gamma = e^{-\beta H} / \text{Tr}(e^{-\beta H})$ is preserved under thermal operations,

$$\Phi(\gamma) = \gamma. \quad (4.6)$$

To clarify possibility of nontrivial interactions, it is worth noting that the above restriction allows for unitary evolutions of the form

$$U = \exp \left(i \sum_{i=1}^{\infty} a_i (H \otimes \mathbb{I}_E + \mathbb{I} \otimes H_E) \right), \quad (4.7)$$

where we can set $a_1 = 0$ without loss of generality, as this term can be absorbed into local evolutions. Starting from a_2 , interactions either both alter the local dynamics by introducing terms such as $H^i \otimes \mathbb{I}_E$ and $\mathbb{I} \otimes H_E^i$ with $i > 1$, and act as actual nontrivial interactions with $H^i \otimes H_E^j$, where we have $i, j > 0$.

However, we do not intend to analyse any specific bath model, but rather put restrictions on all possible evolutions U with arbitrary bath γ_E as long as it preserves energy and is closed. Thus, this approach is able to put universal bounds on thermal processes, independent of specifics of the evolution.

We focus on state transitions and pose the following question: Given a pair of states ρ and σ , can one find a thermal operation such that $\Phi(\rho) = \sigma$? While providing a constructive answer to this question is hard in full generality, complete solutions are known (i) when $d = 2$ [183] and (ii) for any d when target state σ is energy-incoherent, $[\sigma, H] = 0$.

The latter case is interesting, as we may describe $\sigma = \sum q_i |E_i\rangle\langle E_i|$ in terms of its population vector $\mathbf{q} = (q_1, \dots, q_d)$ corresponding to energy eigenstates $H|E_i\rangle = E_i|E_i\rangle$. If we restrict ρ to be energy-incoherent as well, with population vector $\mathbf{p}$, we

[‡] Normally, this would be $\beta = (k_B T)^{-1}$ with k_B the Boltzmann constant. For the sake of simplicity, we work in natural units such that $k_B = 1$.

see that the question of attainability of a transformation between quantum states $\rho \rightarrow \sigma$ is reduced to transformation between d -dimensional probability distributions $\mathbf{p} \rightarrow \mathbf{q}$. Furthermore, in combination with the condition of Gibbs-state preservation is then reduced to a question of existence of a stochastic operation Λ such that

$$\Lambda \mathbf{p} = \mathbf{q}, \quad \Lambda \gamma = \gamma, \quad (4.8)$$

where by γ we denoted the Gibbs distribution[§] with $\gamma_i \propto \exp(-\beta E_i)$. When we go with temperature $T \rightarrow \infty$ or, equivalently, set $\beta = 0$, the Gibbs state becomes a flat distribution η with $\eta_i = 1/d$, and preservation of Gibbs state entails the operation Λ to be bistochastic.

Thus, we recover the original claim, that state transformations under noisy operations are governed by majorisation relation

$$\mathbf{p} \xrightarrow[\beta=0]{} \mathbf{q} \iff \exists \Lambda : \Lambda \mathbf{p} = \mathbf{q}, \Lambda \eta = \eta \iff \mathbf{p} > \mathbf{q}. \quad (4.9)$$

However, infinite temperature is far from generic setting. For this reason, understanding of transformations under finite β is crucial. When restricted to the energy-incoherent setting, determining whether any given transition $\mathbf{p} \xrightarrow[\beta>0]{} \mathbf{q}$ is possible is reduced to checking the thermomajorization relation as introduced by Horodecki and Oppenheim in [179]. Consider a single distribution $\mathbf{p}$ of dimension d and the corresponding Gibbs distribution γ . One defines the β -ordered probability distribution $\mathbf{p}^\beta$ as

$$p_i^\beta = p_{\pi(i)} : i < j \Rightarrow \frac{p_{\pi(i)}}{\gamma_{\pi(i)}} \geq \frac{p_{\pi(j)}}{\gamma_{\pi(j)}} \quad (4.10)$$

where permutation $\pi \equiv \pi(\mathbf{p})$ is referred to as the β -order of the probability vector $\mathbf{p}$. Based on this, the thermomajorization curve $f_{\mathbf{p}}^\beta(x)$ is constructed as a piecewise linear function joining the points $(\sum_{i=1}^j \gamma_{\pi(i)}, \sum_{i=1}^j p_{\pi(i)})$. Then, one defines thermomajorization relation between $\mathbf{p}$ and $\mathbf{q}$ as

$$\mathbf{p} >_\beta \mathbf{q} \stackrel{\text{def.}}{\iff} \forall_{x \in [0,1]} f_{\mathbf{p}}^\beta(x) \geq f_{\mathbf{q}}^\beta(x) \iff \mathbf{p} \xrightarrow[\beta]{} \mathbf{q}. \quad (4.11)$$

The last equivalence is proved in [179], but below we provide a very rough outline of the steps necessary to understand the reason why it works:

1. Consider very large bath and partition its space of states into subspaces spanned by states with degenerate energies,

$$\mathcal{H}_R = \bigoplus_E \mathcal{H}_{R,E} \quad (4.12)$$

[§] For coherent ρ and incoherent σ the framework still works, since process of decoherence $\rho \rightarrow \Phi(\rho) = \sum_i |E_i\rangle\langle E_i| \rho |E_i\rangle\langle E_i|$ is always a thermal operation.

2. Take a subset of energies $\mathcal{E}_R$ and projector $P_{\mathcal{E}_R}$ onto the corresponding subspace,

$$\mathcal{H}_{\mathcal{E}_R} = \bigoplus_{E \in \mathcal{E}_R} \mathcal{H}_{R,E}. \quad (4.13)$$

such that the Gibbs state of the bath has high overlap with this space,

$$\text{Tr}(P_{\mathcal{E}_R} \gamma_R) \geq 1 - \delta. \quad (4.14)$$

Furthermore, assume that in the above subspace the degeneracies of $g_R(E)$ and $g_R(E + \Delta E)$ can be related by standard Gibbs factor,

$$g_R(E + \Delta E) \approx g_R(E) e^{\beta \Delta E}. \quad (4.15)$$

3. One needs to show that for arbitrary energy level E_S of the system and total energy E such that $E - E_S \in \mathcal{E}_R$ the total state $\rho \otimes \gamma_R$ projected onto the subspace with energy E is well approximated by maximally mixed state η_{E-E_S} in the subspace of the bath corresponding to $E - E_S$ and the system state projected onto the subspace corresponding to E_S ,

$$\left\| \frac{1}{p_E} P_E (\rho \otimes \gamma_R) P_E - (P_{E_S} \rho P_{E_S}) \otimes \eta_{E-E_S} \right\| \leq 2\delta \quad (4.16)$$

with $p_E = \text{Tr}(P_E \rho \otimes \gamma_R)$. Furthermore, we have

$$\sum_{E: E-E_S \in \mathcal{E}_R} p_E \geq 1 - 2\delta. \quad (4.17)$$

4. Within any subspace of constant energy E for total system the diagonal part of the state can thus be approximated by probability vector $\mathbf{p}$ with entries

$$p_{E_S,i} = e^{\beta E_S} \frac{\text{Tr}(\rho P_{E_S})}{g_R(E)} \quad (4.18)$$

with $i = 1, \dots, g_R(E) e^{-\beta E_S}$.

5. Since any unitary U restricted to a subspace of constant energy commutes with the total Hamiltonian, and the same reasoning applies to arbitrary mixed unitary channel, transformations of $\mathbf{p}$ are governed by standard majorization.

Two remarks are in place. First, the fact that the assumptions about the bath are satisfied when, for instance, bath is composed of many independent copies of a single Gibbs state, $\gamma_R = \gamma^{\otimes N}$. Second, is that the entire procedure and, in turn, thermomajorization is exact only when $\delta = 0$. This can be achieved only in the thermodynamic limit for the bath, $N \rightarrow \infty$. Otherwise, the entire framework is subject to finite-size corrections, which need to be taken into account when thinking about experiments with small heat reservoirs; this kind of corrections have been considered in [184]. However, in this and next chapter we will stick with the assumptions necessary for the thermal operations' (TOs) model to work.

Among thermal operations the ones that we will be most interested in extreme operations. Among them, 2-level operations Π_{ij}^β called β -swaps,

$$\Pi_{ij}^\beta = \begin{pmatrix} 1 - e^{-\beta(E_j - E_i)} & 1 \\ e^{-\beta(E_j - E_i)} & 0 \end{pmatrix} \oplus \mathbb{I}_{\mathcal{H} \setminus \text{span}(i,j)}. \quad (4.19)$$

It is easily checked that for $\beta = 0$ transformation we refer to as the β -swap is equivalent to a full swap, and the full set of noisy operations, which is the Birkhoff polytope, is generated from composition of swaps. However, for $\beta \geq 0$ the structure of possible operations becomes highly nontrivial, with certain extreme points existing below and others above a critical temperature [185]. In particular, there exist extreme thermal operations on d level of the form

$$\hat{O} = \begin{pmatrix} 1 - \sum_{i=1}^{d-1} \gamma_i & 1 & 1 & \dots & 1 \\ \gamma_1 & 0 & 0 & \dots & 0 \\ \gamma_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{d-1} & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4.20)$$

that cannot be even approximated by any sequence of 2-level operations. Later on we will show that one needs to be more specific – such operations are impossible when 2-level operations are restricted to act on an unextended primary system.

4.1.3 Markovian Thermal Operations and evolution of Open Quantum Systems

As can be seen from the assumptions underlying thermomajorization, realization of arbitrary thermal operations require large amount of control over the joint system-bath evolution – no easy feat from the perspective of experimental realisations. In order to provide slightly more realistic model of achievable operations, one may resort to the Markovian setting, which entails weak coupling, short interaction between system and bath, and quick decay of correlations. In this case, the evolution of the state ρ of the system is governed by Gorini-Kossakowski-Lindblad-Sudarshan (GKSL) equation [186–188],

$$\frac{d\rho}{dt} = \mathcal{H}(\rho) + \mathcal{L}_t(\rho) \quad (4.21)$$

where $\mathcal{H}(\rho) = -i[\rho, H]$ is the Hamiltonian generator of evolution and $\mathcal{L}_t$ is the Lindbladian. The latter is defined by

$$\mathcal{L}_t(\rho) = \sum_i r_i(t) \left[L_i(t) \rho L_i(t)^\dagger + \frac{1}{2} \{ L_i^\dagger(t) L_i(t), \rho \} \right] \quad (4.22)$$

where $L_i(t)$ are the jump operators, which in general may be time dependent.

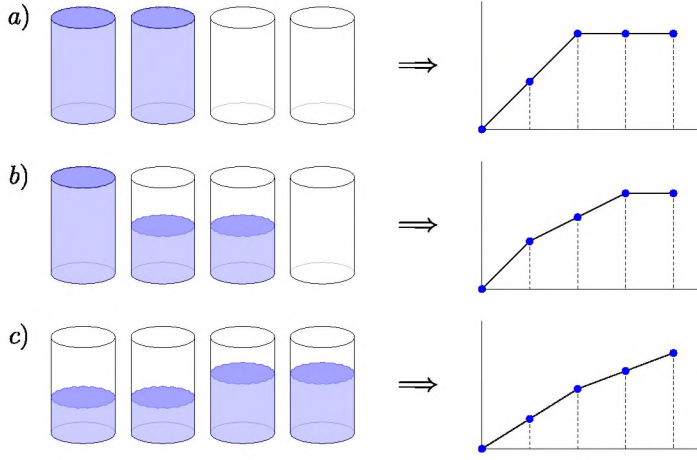


Figure 4.2: Connected vessel analogy: An extreme 2-level Markovian thermal process in the limit of infinite temperature, $\beta = 0$, is equalization of populations for two energy levels. This is reflected in the analogy directly by the fact that one shift more liquid between two connected vessels standing on the same level, than the amount necessary to equalize the heights of the columns of the liquid. Three examples include a) the initial state $(1/2, 1/2, 0, 0)$, b) first intermediate equalization $(1/2, 1/4, 1/4, 0)$ and c) final state $\frac{1}{16}(3, 3, 5, 5)$ of the system, following the procedure described in the beginning of the chapter.

In order for the above to describe evolution in contact with thermal bath, one needs to additionally require that the Gibbs state is stationary point of the evolution [189],

$$\mathcal{L}_t(\gamma) = 0 \quad (4.23)$$

and that the action of Lindbladian commutes with the Hamiltonian,

$$\mathcal{L}_t \circ \mathcal{H}(\rho) = \mathcal{H} \circ \mathcal{L}_t(\rho). \quad (4.24)$$

Markovian thermal process (MTP) is defined as integration of the equation (4.22) from $t_s = 0$ to $t_f \in [0, +\infty]$ with additional constraints (4.23, 4.24).

When restricted to energy-incoherent states, it has been shown that a process $\mathbf{p}_0 \xrightarrow{\text{MTP}} \mathbf{p}_f$ is realisable if and only if for any well-behaved convex function h the generalized entropy production inequality is satisfied [190, 191],

$$\frac{d\Sigma_h(t)}{dt} \geq 0, \quad \Sigma_h(t) = - \sum_{i=1}^d \gamma_i h\left(\frac{p_i(t)}{\gamma_i}\right). \quad (4.25)$$

This can be reduced down to a set of functions of the form

$$\frac{d\Sigma_a(t)}{dt} \geq 0, \quad \Sigma_a(t) = - \sum_{i=1}^d \left| p_i(t) - a \frac{\gamma_i}{\gamma_d} \right| \quad (4.26)$$

with $a \in [0, 1]$. Evaluation of such set of inequalities is challenging from computational

point of view. However, it turns out to be equivalent to a notion of continuous thermomajorization, understood as

$$\mathbf{p} \gg_{\beta} \mathbf{p}_f \iff \mathbf{p}(t_i) > \mathbf{p}(t_j) \quad \forall 0 \leq t_i \leq t_j \leq t_f. \quad (4.27)$$

As demonstrated by Lostaglio and Korzekwa in [191], this approach provides tools necessary for efficient verification of possibility of arbitrary transition $\mathbf{p} \xrightarrow{\text{MTP}} \mathbf{q}$ in finite number of steps, as well as algorithm allowing to determine entire future of an arbitrary state $\mathbf{p}$ with respect to Markovian thermal processes for dimensions up to $d = 7$ using standard desktop computer.

It has been shown in [191] that any extreme Markovian thermal process can be decomposed into two-level thermalization leading to the selected chamber, followed by a β -order-preserving thermal operation, which can always be realised using Markovian thermal processes, as the two sets are equivalent for states of the same β -order. As we will use them further on, we will define the two-level thermalization as

$$T_{ij} = \begin{pmatrix} \Gamma_{ij} & \Gamma_{ij} \\ \Gamma_{ji} & \Gamma_{ji} \end{pmatrix} \oplus \mathbb{I}_{\mathcal{H} \setminus \text{span}(i,j)} \quad (4.28)$$

Finally, for $\beta = 0$ limit Markovian thermal processes are modelled well by the analogy of connected vessels, as exposed in the introduction to this chapter. In Figure 4.2 we present the steps for both scenarios described in the introduction. The resulting transitions are,

$$(1, 0) \xrightarrow{\text{MTP}} (1/2, 1/2) \quad (1, 0) \otimes (1/2, 1/2) \xrightarrow{\text{MTP}} (3/8, 5/8) \otimes (1/2, 1/2), \quad (4.29)$$

thus providing the answer to the task given to the reader at the very beginning of this chapter. Once again, the driving question of this work was to extend the above approach and see whether it can provide simulation of arbitrary thermal operation (TO) with memory-extended Markovian thermal operation (MeMTP).

4.1.4 On regularised incomplete beta function

In the following sections we will make extensive use of a special function called **incomplete regularised beta function**. In order to fully appreciate the derivations it is worth to first introduce the function properly. What follows will be based mostly on [192, 193]. As it is meant primarily as a toolbox for the later derivations, **the reader is invited to skip this subsection** and come back whenever in doubts of transformations in the derivations that follow.

We will begin by taking two steps back and recalling the factorial

$$n! = n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1 \quad (4.30)$$

with the property that $(n+1)! = (n+1)n!$. It is a function that appears very often in the combinatorial contexts. Its integral representation

$$n! = \int_0^\infty t^n e^{-t} dt \equiv \Gamma(n+1) \quad (4.31)$$

is the definition of the Gamma function $\Gamma(x)$, valid for the entire complex plane without the poles at negative integers, $x \in \mathbb{C} \setminus \mathbb{Z}_-$. In what follows we will use the Legendre duplication formula

$$\Gamma(n)\Gamma\left(n + \frac{1}{2}\right) = 2^{1-2n}\sqrt{\pi}\Gamma(2n), \quad (4.32)$$

and the first-order Stirling's approximation of the Gamma function,

$$\Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n [1 + O(n^{-1})]. \quad (4.33)$$

Gamma function is a necessary ingredient for the natural definition of the beta function $B(a, b)$,

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(ab)} \underset{a, b \in \mathbb{N}}{=} \frac{a+b}{ab} \binom{a+b}{a}^{-1}. \quad (4.34)$$

In turn, we have an elegant identity, which we will find useful later on,

$$\frac{1}{nB(n, n)} = \frac{\Gamma(2n)}{n\Gamma(n)^2} = \frac{1}{2} \binom{2n}{n}. \quad (4.35)$$

Beta function may be redefined using an elegant integral formula

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt, \quad (4.36)$$

which in turns implies existence of incomplete beta function. It is defined by replacing the upper limit of integration by an additional, auxiliary variable x ,

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt. \quad (4.37)$$

For $x \in (0, 1)$ the incomplete beta function obeys inequalities

$$0 = B_0(a, b) \leq B_x(a, b) \leq B_1(a, b) = B(a, b), \quad (4.38)$$

valid for $a, b > 0$.

Dividing the above inequality by $B(a, b)$ and defining the regularised incomplete beta

function $I_x(a, b) \equiv B_x(a, b)/B(a, b)$ we find that

$$0 \leq I_x(a, b) \leq 1, \quad (4.39)$$

which shows that $I_x(a, b)$ is a proper cumulative distribution function with respect to the variable x . One may also write a simple relation for exchange between $a \leftrightarrow b$ and $x \rightarrow 1 - x$,

$$I_x(a, b) = 1 - I_{1-x}(b, a). \quad (4.40)$$

It is useful to rewrite $I_x(a, b)$ when $a, b \in \mathbb{Z}$ in terms of a sum

$$I_x(a, b) = \sum_{j=a}^{\infty} \binom{j+b-1}{j} x^j (1-x)^b, \quad (4.41)$$

which we will encounter repeatedly throughout the derivations of the main results. In fact, it will be more common to see the two following sums,

$$\sum_{j=a}^n \binom{j+b-1}{j} x^j (1-x)^b = I_x(a, b) - I_x(n+1, b), \quad (4.42a)$$

$$\sum_{j=1}^n \binom{j+b-1}{j} x^j (1-x)^b = 1 - (1-x)^b - I_x(n+1, b). \quad (4.42b)$$

Furthermore, we will at certain points need to use an identity for the following identity

$$I_x(a, a) = \frac{1}{2} I_{4x(1-x)}\left(a, \frac{1}{2}\right), \quad (4.43)$$

which holds for $x \in (0, 1/2)$.

For completeness, we provide the following recurrence formulas,

$$I_x(a, b) = I(a+1, b) + \frac{x^a(1-b)^b}{aB(a, b)} = I(a, b+1) - \frac{x^a(1-x)^b}{aB(a, b)} \quad (4.44a)$$

$$= I(a+1, b-1) + \frac{x^a(1-b)^{b-1}b}{aB(a, b)} = I(a-1, b+1) - \frac{x^{a-1}(1-x)^b}{aB(a, b)}. \quad (4.44b)$$

4.1.4.1 Asymptotic approximation

As we will be both working in the infinite limits and investigating the convergence rates of expressions related to regularised incomplete beta function, we need an asymptotic approximation. This is provided by

$$I_x(a, b) = \Gamma(a+b)x^a(1-x)^{b-1} \left[\sum_{k=0}^{n-1} \frac{1}{\Gamma(a+k-1)\Gamma(b-k)} \left(\frac{x}{1-x} \right)^k + O\left(\frac{1}{\Gamma(a+n+1)} \right) \right] \quad (4.45)$$

which is valid for fixed b, x and $a \rightarrow \infty$. In particular, to O -term vanishes only when $n \geq b$. Whenever $b, n \in \mathbb{N}$ and $n > b$. In this work, however, we will focus on the second-order expansion for $a = N$, $b = 1/2$, which yields

$$I_x\left(N, \frac{1}{2}\right) = \frac{x^N}{\sqrt{1-x}} \frac{(2N)!}{4^N (N!)^2} \left[1 - \frac{x}{2(N+1)(1-x)}\right] + O\left(\frac{x^N}{N^2}\right). \quad (4.46)$$

We will also need to consider sums of regularised beta functions of the form

$$\sum_{i=1}^a I_x(a, i+1) = \sum_{i=1}^a \frac{B_x(a, i+1) \Gamma(a+i+1)}{\Gamma(a) \Gamma(i+1)} = \int_0^x dt \sum_{i=1}^a \binom{a+i}{i} t^{a-1} (1-t)^i. \quad (4.47)$$

Using the finite-sum expressions (4.42) we may rewrite the above into

$$\int_0^x \frac{dt}{t^2} [I_x(1, a-1) - I_{1-x}(a-1, a-1)] = \int_0^x \frac{dt}{t^2} \left[\frac{1}{2} I_{4x(1-x)}\left(a-1, \frac{1}{2}\right) - I_x(a-1, 1) \right] \quad (4.48)$$

for $x \in [0, 1/2]$. Using the asymptotic expressions (4.46) we finally get

$$\begin{aligned} \sum_{i=1}^a I_x(a, i+1) &= \int_0^x dt \left\{ -t^{a-3} + \frac{1}{2x^2} \left[\frac{\Gamma(a-1/2)}{\Gamma(a+1)} \frac{[4x(1-x)]^{a-1}}{1-2x} \right. \right. \\ &\quad \left. \left. + O\left(\frac{[4x(1-x)]^{a-1} \Gamma(a-1/2)}{\Gamma(a+1)}\right) \right] \right\}. \end{aligned} \quad (4.49)$$

Since $\lim_{a \rightarrow \infty} \Gamma(a-1/2)/\Gamma(a+1) = 0$ and $\lim_{a \rightarrow \infty} [4x(1-x)]^{a-1} = 0$, we conclude that

$$\lim_{a \rightarrow \infty} \sum_{i=1}^a I_x(a, i+1) = 0. \quad (4.50)$$

Final chain of transformations that will appear in the derivation of the main result is for the following sum

$$\sum_{i=1}^N I_x(N+1-i, N) = \sum_{i=1}^N \frac{B(N+1-i, N)}{B_x(N+1-i, N)} \quad (4.51)$$

$$= \int_0^x dt \frac{(1-t)^{N-1}}{(1-t)^{N+1}} \left[\sum_{j=0}^{N-1} \binom{(N+1)+j-1}{j} t^j (1-t)^{N+1} \right] \quad (4.52)$$

$$= \int_0^x dt \frac{N}{(1-t)^2} (I_{1-x}(0, N+1) - I_{1-x}(N, N+1)) \quad (4.53)$$

$$= \int_0^x dt \frac{N}{(1-t)^2} \left(1 - I_{1-x}(N, N) - \frac{t^N (1-t)^N}{NB(N, N)} \right) \quad (4.54)$$

$$= N \int_0^x dt \frac{1}{(1-t)^2} = N \frac{x}{1-x}. \quad (4.55)$$

This is the complete set of properties for Gamma and Beta functions we will require in order to derive the primary results of this chapter.

4.2 Memory-assisted Markovian thermal operations and β -swap approximation algorithm

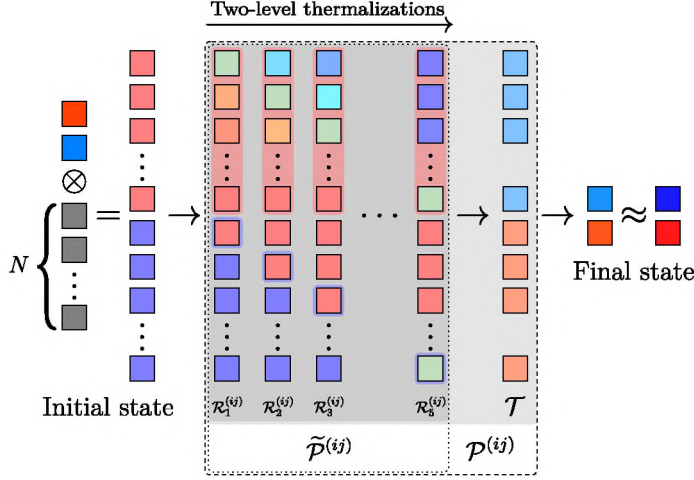


Figure 4.3: Protocol for MeMTO approximation of β -swap: Full simulation protocol, which uses a system in a starting state $\mathbf{p} = (p, 1 - p)$ extended by a flat memory state η_N can be subdivided into N subroutines $\mathcal{R}^{(ij)}_k$, where in k -th subroutine we sequentially thermalize all N "full" levels (initially red boxes on top) with the k -th "empty" level (initially blue boxes). After N such subroutines the memory system is decoupled from the primary system by final thermalization and discarded.

Now we will proceed to describe the primary result, which consists in the following: there exists an algorithm, which allows for approximation of arbitrary thermal operation by Markovian thermal operations extended by Gibbs memory. First, let us define memory-assisted Markovian thermal operations.

Definition 4.1 (Memory-assisted Markovian thermal operations (MeMTP)). *Let us take $\mathbf{p}, \mathbf{q}$ probability distributions of dimensionality d . We say that a transition $\mathbf{p} \rightarrow \mathbf{q}$ is realisable by memory-assisted Markovian thermal operations if it can be realised by a Markovian thermal operations as $\mathbf{p} \otimes \gamma_N \rightarrow \mathbf{q} \otimes \gamma_N$,*

$$\mathbf{p} \xrightarrow{\text{MeMTP}} \mathbf{q} \iff \mathbf{p} \otimes \gamma_N \xrightarrow{\text{MTP}} \mathbf{q} \otimes \gamma_N. \quad (4.56)$$

One may introduce a relatively simple algorithm consisting of N^2 steps followed by a final thermalization, which provides an approximation of a full β -swap Π_{ij}^β .

1. Extend the system ρ by an N -dimensional Gibbs state γ_N , which we assume to have a trivial Hamiltonian, thus $\gamma_N = (1/N, \dots, 1/N)$.
2. In k -th big step for connect sequentially all levels from $N(i-1) + 1$ to Ni to the $N(j-1) + k$ -th level, defining the sub-operations

$$\mathcal{R}_k^{(ij)} = \prod_{l=1}^N T_{N(i-1)+l, N(j-1)+k}. \quad (4.57)$$

3. Total operation resulting from composition of N steps is given as,

$$\tilde{\mathcal{P}}^{(ij)} = \prod_{k=1}^N \mathcal{R}_k^{(ij)}. \quad (4.58)$$

4. In the final step we perform thermalization $\mathcal{T}$ by

$$\mathcal{T}(p_{N(a-1)+1}, \dots, p_{Na}) = \left(\sum_{l=1}^N p_{N(a-1)+l} \right) (1/N, \dots, 1/N). \quad (4.59)$$

Before we move forward to the main results, let us look close at the algorithm. Visualization given in Fig. 4.3 is the original formulation as given in [16]. We can also visualize it using an $N \times N$ grid of points, where the point at position (a, b) represents thermalization $T_{a, N+b}$ between P_a and P_{N+b} levels of the state $\mathbf{P} = \mathbf{p} \otimes \gamma_N$. Note, that operations corresponding to points in different rows and columns do commute, and thus can be interchanged. Thanks to this, two grids presented in 4.4 yield equivalent procedures.

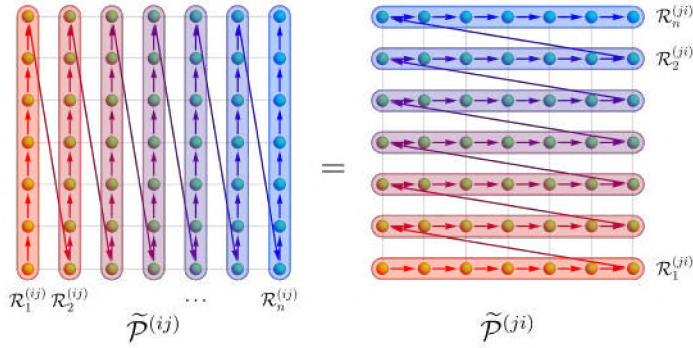


Figure 4.4: Equivalent protocols: Taking gridpoint (a, b) to represent thermalization $T_{a, N+b}$, the algorithm from Fig.4.3 can be represented as a path, seen on the left grid. As operations that share neither row nor column commute, the grid on the left is equivalent to the grid on the right.

One may take the above observation to the extreme and note, that the entire algorithm can be parallelized.

Observation 4.1. The algorithm $\widetilde{\mathcal{P}}^{(ij)}$ can be parallelized into $2N + 1$ large steps,

$$\widetilde{\mathcal{P}}^{(ij)} = \prod_{s=1}^{2N+1} \mathcal{Q}_s^{(ij)}, \quad (4.60)$$

where each parallelized step

$$\mathcal{Q}_s^{(ij)} = \prod_{k+l=s+1} T_{N(i-1)+l, N(j-1)+k} \quad (4.61)$$

consists of at most N substeps.

For visual reference of the reader, the procedure can then be represented in the grid as in Fig. 4.5.

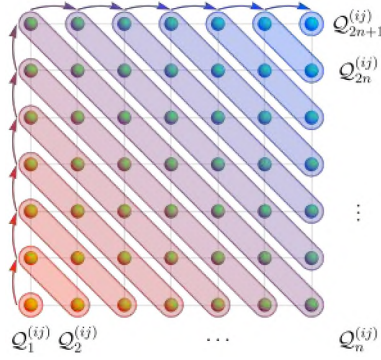


Figure 4.5: Parallelized algorithm: Since it is possible to parallelize certain two-level thermalizations, the algorithm allows for a parallelization of the form (4.60) into $2n + 1$ operations $\mathcal{Q}_s^{(ij)}$ defined by (4.61), which is most easily represented in a similar spirit as in Fig. 4.4

After giving the general feeling of the algorithm itself we can move to the main claims of this chapter, which can be summarised in the following set of theorems and conjectures:

Theorem 4.2 (Convergence to β -swaps). The operation $\mathcal{P}^{(ij)}$ consisting of N^2 elementary thermalizations $T_{x,y}$ and final thermalization, defined as

$$\mathcal{P}^{(ij)} = \prod_{k=1}^N \prod_{l=1}^N T_{N(i-1)+l, N(j-1)+k} \quad (4.62)$$

acting on the primary system extended by memory in a flat state η_N , converges to the full β -swap Π_{ij}^β on the primary system as

$$\left\| \Pi_{ij}^\beta - \Pi_{ij}^{appr} \right\|_1 = \begin{cases} \frac{1}{\sqrt{\pi N}} + o(N^{-1/2}) & \text{for } \beta = 0 \\ (4\Gamma_{ij}\Gamma_{ji})^N \left(\frac{\Gamma_{ij}}{(\Gamma_{ij}-\Gamma_{ji})^2 \sqrt{\pi N(N+1)}} + o(N^{-3/2}) \right) & \text{for } \beta > 0 \end{cases} \quad (4.63)$$

where we define $\Pi_{ij}^{appr} \mathbf{p} = \text{Tr}_E [(\mathcal{P}^{(ij)} \circ \mathcal{T})(\mathbf{p} \otimes \boldsymbol{\eta}_N)]$ and define the operator 1-norm as $\|X\|_1 \equiv \max_{\mathbf{p}: p_i \geq 0} \|X\mathbf{p}\|_1 / \|\mathbf{p}\|_1$.

Corollary 4.3 (Convergence to arbitrary permutation for infinite temperature). For $\beta = 0$ any operations Π_σ corresponding to a permutation $\sigma \in \mathcal{S}_d$ given by its decomposition into transpositions, $\sigma = (i_1 j_1) \dots (i_M j_M)$ can be approximated as

$$\Pi_\sigma^{appr} = \prod_{k=1}^M \Pi_{i_k j_k}^{appr}, \quad (4.64)$$

with convergence rate

$$\|\Pi_\sigma - \Pi_\sigma^{appr}\|_1 = \frac{M}{\sqrt{\pi N}} + o(N^{-1/2}) \leq \frac{d(d-1)}{2\sqrt{\pi N}} + o(N^{-1/2}). \quad (4.65)$$

Conjecture 4.1 (Accelerated convergence by omitting intermediate thermalizations). For $\beta = 0$ any operation Π_σ corresponding to a permutation $\sigma \in \mathcal{S}_d$ given by its decomposition into transpositions, $\sigma = (i_1 j_1) \dots (i_M j_M)$, operation defined as

$$\tilde{\mathcal{P}}_\sigma = \left(\prod_{k=1}^M \tilde{\mathcal{P}}^{(i_k j_k)} \right) \quad (4.66)$$

provides better convergence rate than protocol with intermediate thermalizations,

$$\|\Pi_\sigma - \tilde{\Pi}_\sigma^{appr}\|_1 \leq \|\Pi_\sigma - \Pi_\sigma^{appr}\|_1 \quad (4.67)$$

where $\tilde{\Pi}_\sigma^{appr} \mathbf{p} = \text{Tr}_E [\tilde{\mathcal{P}}_\sigma(\mathbf{p} \otimes \boldsymbol{\eta}_N)]$

Conjecture 4.2 (Convergence to arbitrary extreme thermal operation for $\beta > 0$). For $\beta > 0$ consider an extreme operation Π_σ^β corresponding to a permutation $\sigma \in \mathcal{S}_d$ given by its decomposition into transpositions, $\sigma = (i_1 j_1) \dots (i_M j_M)$. The operation defined as

$$\tilde{\mathcal{P}}_\sigma = \left(\prod_{k=1}^M \tilde{\mathcal{P}}^{(i_k j_k)} \right) \quad (4.68)$$

provides an approximation $\tilde{\Pi}_\sigma^{appr} \mathbf{p} = \text{Tr}_E [\tilde{\mathcal{P}}_\sigma(\mathbf{p} \otimes \boldsymbol{\eta}_N)]$ with convergence rate

$$\|\Pi_\sigma^\beta - \tilde{\Pi}_\sigma^{appr}\|_1 = O\left(\frac{e^{-A(\sigma)N}}{N^{3/2}}\right). \quad (4.69)$$

We refer to the operation Π_σ^β a β -permutation. It is defined as the extreme operation that turns state $\mathbf{p}$ with β -order π to an extreme point of the future thermal cone of $\mathbf{p}$ with β -order $\sigma \circ \pi$.

In the following subsections we will provide complete proof of Theorem 4.2, divided into the convergence proof and derivation of rates of convergence; Corollary 4.3 follows by expanding $\Pi_{ij}^{\text{appr}} = (1 - \epsilon)\Pi_{ij} + \epsilon\mathbb{I}_{ij}$, where $\epsilon = O(N^{-1/2})$ and terms of order ϵ^2 . Conjecture 4.1 will be discussed shortly based on numerics already available in [16]. Finally, Conjecture 4.2 will be discussed based on numerics as well as an approach to general proofs for any β -cyclic permutation, understood as an extreme thermal operation turning $\mathbf{p}$ with a given β -order π into an extreme point of its future thermal cone with the cyclically permuted order, $(12 \dots n) \circ \pi$, $n \leq d$.

4.2.1 Proof of convergence to β -swap

We will now proceed now to prove the convergence of the algorithm part of Theorem 4.2 without diving into considerations about the convergence rates. Algorithm is simple enough to consider it in a step-by-step manner. In particular, for full clarity of proof we will restrict ourselves to a 2-level system in a starting state $(1, 0)$. Rationale behind this is that, if we will be able to simulate a β -swap on this state, and we know that the entire transformation belongs to TOs, it must have been an extreme thermal operation, and in two dimensions β -swap is the unique extreme point.

Thus, we find that our initial state extended by an N -dimensional memory is

$$\mathbf{P} = \mathbf{p} \otimes \boldsymbol{\eta}_N = \frac{1}{N} \underbrace{(1, 1, \dots, 1)}_{N \text{ times}} \mid \underbrace{(0, 0, \dots, 0)}_{N \text{ times}} \quad (4.70)$$

and, since we will be working in the finite β setting and assume that the memory hamiltonian is trivial, we find that the joint Gibbs state is given by

$$\boldsymbol{\gamma} \otimes \boldsymbol{\eta}_N = \frac{1}{N} \underbrace{(\gamma_1, \gamma_1, \dots, \gamma_1)}_{N \text{ times}} \mid \underbrace{(\gamma_2, \gamma_2, \dots, \gamma_2)}_{N \text{ times}}, \quad (4.71)$$

where $\gamma_1 \equiv \gamma$ and $\gamma_2 = 1 - \gamma$. Considering first N thermalizations going into the big step $\mathcal{R}_1^{(1,2)}$, we find that the $N + 1$ entry evolves as

$$P_{N+1}^{(0)} = 0 \xrightarrow{T_{1,N+1}} \frac{\gamma_2}{N} \xrightarrow{T_{2,N+1}} \frac{\gamma_2(1 + \gamma_2)}{N} \rightarrow \dots \rightarrow \frac{1}{N} \sum_{i=0}^{N-1} \gamma_2^{i+1} \equiv P_{N+1}^{(N)} \quad (4.72)$$

where we have introduced the notation $P_j^{(i)}$ to denote the state of j -th component after i thermalizations it has undergone in the process. For the moment we will refrain from resolving the sum, as it will be useful to understand the general expression for $P_{N+k}^{(N)}$. In parallel, using the same expressions, it is relatively easy to find that for $j \leq N$ we have

$$P_j^{(1)} = \frac{\gamma_1}{N} \sum_{i=0}^{j-1} \gamma_2^i = \frac{1 - \gamma_2^j}{N}. \quad (4.73)$$

One more step in the algorithm will provide us with enough insight to understand the entirety of the process. We find that

$$P_{N+2}^{(0)} = 0 \xrightarrow{T_{1,N+2}} \gamma_2 P_1^{(1)} \xrightarrow{T_{2,N+2}} \gamma_2 P_2^{(1)} + \gamma_2^2 P_1^{(1)} \rightarrow \dots \rightarrow \sum_{i=0}^{N-1} P_{N-i}^{(1)} \gamma_2^{i+1} = P_{N+2}^{(N)}. \quad (4.74)$$

This, in fact, allows us already to write the recurrence formulas of the form

$$P_{N+k}^{(N)} = \sum_{i=0}^{N-1} P_{N-i}^{(k-1)}, \quad (4.75a)$$

$$P_j^{(k)} = \frac{\gamma_1}{\gamma_2} P_{N+k}^{(j)}. \quad (4.75b)$$

Let us open the expression for $P_{N+2}^{(N)}$ to find

$$P_{N+2}^{(N)} = \frac{\gamma_1}{N} \sum_{i_2=0}^{N-1} \sum_{i_1=0}^{N-i_2-1} \gamma_2^{i_1+i_2+1} = \frac{\gamma_1}{N} \sum_{i_2=0}^{N-1} \gamma_2^{i_2+1} \sum_{i_1=i_2}^{N-1} 1 = \frac{\gamma_1}{N} \sum_{i=0}^{N-1} \binom{i+1}{i} \gamma_2^{i+1}. \quad (4.76)$$

By performing similar shifts in the indices iteratively we find that

$$P_{N+k}^{(N)} = \frac{\gamma_1^{k-1}}{N} \sum_{i=0}^{N-1} \binom{i+k-1}{i} \gamma_2^{i+1} = \frac{\gamma_2}{N\gamma_1} (I_{\gamma_2}(0, k) - I_{\gamma_2}(N, k)) = \frac{\gamma_2}{N\gamma_1} I_{\gamma_1}(k, N), \quad (4.77a)$$

$$P_j^{(N)} = \frac{\gamma_1^N}{N} \sum_{i=0}^{j-1} \binom{i+N-1}{i} \gamma_2^i = \frac{1}{N} (I_{\gamma_2}(0, N) - I_{\gamma_2}(j, N)) = \frac{1}{N} I_{\gamma_1}(N, j). \quad (4.77b)$$

At this point, we may assume that $\gamma_1 \leq 1/2$ and use the sum (4.50) to find that

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N P_j^{(N)} = 0 \quad \implies \quad \lim_{N \rightarrow \infty} \sum_{j=1}^N P_{N+j}^{(N)} = 1 - \lim_{N \rightarrow \infty} \sum_{j=1}^N P_j^{(N)} = 1. \quad (4.78)$$

Using this, we know that we managed to realise a column-wise stochastic matrix

$$\mathcal{P}^{(12)} = \begin{pmatrix} 0 & a \\ 1 & 1-a \end{pmatrix}.$$

Furthermore, we know that this operation is a thermal operation, as it has been achieved by composing (possibly infinite) sequence of thermal operations. The only TO of the above form is the β -swap, and thus we conclude the proof of convergence in the limit of infinite temperature.

4.2.2 Convergence analysis

We are already in position to say that the algorithm provided converges; however, thanks to the asymptotic analysis provided in Section 4.1.4, we are able to go beyond simply saying we achieve the β -swap, but we can also quantify **how quickly** we attain it.

Let us start with explicit assumption that the Gibbs state of the primary system is given as $\gamma = (\gamma, 1 - \gamma)$ and this time we will have from the start that $\gamma \geq 1/2$. Furthermore, we will take the primary system to be in a state $\mathbf{p} = (a, b)$. Thus, by utilising the recurrence formulas (4.75) and the explicit expressions (4.77), we can write

$$\Delta_{a \rightarrow b} \equiv \sum_{i=1}^N P_i^{(N)} \Big|_{b=0, a=1} = \frac{\gamma_1^N}{N} \sum_{j=1}^N \sum_{i=1}^j \binom{j+N-1-i}{N-1} \gamma_2^{j-i} \quad (4.79a)$$

$$\Delta_{b \rightarrow a} \equiv \sum_{i=1}^N P_i^{(N)} \Big|_{b=1, a=0} = \frac{\gamma_2^N}{N} \sum_{i=1}^N \sum_{j=0}^{i-1} \binom{N+j-1}{j} \gamma_1^j \quad (4.79b)$$

and thus, the overall operation before converging is expressed by

$$\Pi_{12}^{\text{appr}} = \begin{pmatrix} \Delta_{a \rightarrow b} & 1 - \Delta_{b \rightarrow a} \\ 1 - \Delta_{a \rightarrow b} & \Delta_{b \rightarrow a} \end{pmatrix}. \quad (4.80)$$

We will start with analysing the simpler expression $\Delta_{b \rightarrow a}$ by re-expressing it in terms of regularised incomplete beta functions

$$\Delta_{b \rightarrow a} = \gamma_2^N \sum_{i=1}^N (N-i+1) \binom{N+i-2}{i-1} \gamma_1^{i-1} \quad (4.81)$$

$$= \gamma_2^N \sum_{i=0}^{N-1} \binom{N+i-1}{i} \gamma_1^i - \frac{\gamma_2^N}{N} \sum_{i=1}^{N-1} i \binom{N+i-1}{i} \gamma_1^i \quad (4.82)$$

$$= I_{\gamma_2}(N, N) - \frac{\gamma_1}{\gamma_2} I_{\gamma_2}(N+1, N-1) \quad \text{using (4.42)} \quad (4.83)$$

$$= \left(1 - \frac{\gamma_1}{\gamma_2}\right) I_{\gamma_2}(N, N) + \frac{\gamma_1}{\gamma_2} \frac{\gamma_2^N \gamma_1^{N-1}}{NB(N, N)} \quad \text{using (4.44b)} \quad (4.84)$$

$$= \frac{1}{\gamma_2} \left[(\gamma_2 - \gamma_1) \frac{1}{2} I_{4\gamma_2\gamma_1} \left(N, \frac{1}{2} \right) + \frac{(\gamma_1\gamma_2)^N}{2} \binom{2N}{N} \right] \quad \text{using (4.43).} \quad (4.85)$$

With this form we may use Stirling approximation (4.33) and the asymptotic form

(4.46) to arrive at the asymptotic expression

$$\Delta_{b \rightarrow a} = \frac{(4\gamma_2\gamma_1)^N}{2\gamma_2\sqrt{\pi N}} \left[\frac{\gamma_2 - \gamma_1}{\sqrt{1 - 4\gamma_2\gamma_1}} - \frac{4\gamma_2\gamma_1(\gamma_2 - \gamma_1)}{2(1 - 4\gamma_2\gamma_1)^{3/2}(N + 1) + 1 + o(N^{-1})} \right] \cdot [1 + O(N^{-1})] \quad (4.86)$$

Using this form we arrive at two sasymptotic expressions,

$$\Delta_{b \rightarrow a} \Big|_{\gamma_1 \geq \frac{1}{2}} = (4\gamma(1 - \gamma))^N \left[\frac{\gamma}{(2\gamma - 1)^2 \sqrt{\pi N}(N + 1)} \right] + o(N^{-3/2}), \quad (4.87a)$$

$$\Delta_{b \rightarrow a} \Big|_{\gamma_1 = \frac{1}{2}} = \frac{1}{\sqrt{N\pi}} + o(N^{-1/2}). \quad (4.87b)$$

Similar analysis for $\Delta_{a \rightarrow b}$ yields

$$\Delta_{a \rightarrow b} = \frac{\gamma_1 - \gamma_2}{\gamma_1} \left[1 - I_{4\gamma_1\gamma_2} \left(N, \frac{1}{2} \right) \right] + \frac{(4\gamma_1\gamma_2)^N}{2\gamma_1} \frac{1}{\sqrt{\pi N}}. \quad (4.88)$$

The resulting asymptotic expression, is given by

$$\begin{aligned} \Delta_{a \rightarrow b} &= \frac{\gamma_2 - \gamma_1}{2\gamma_2} \left[2 - \frac{(4\gamma_1\gamma_2)^N}{\sqrt{1 - 4\gamma_1\gamma_2}} \frac{1}{\sqrt{\pi N}} \left(1 - \frac{4\gamma_1\gamma_2}{2(N + 1)(1 - 4\gamma_1\gamma - 2)} \right) \right] \\ &+ \frac{(4\gamma_1\gamma_2)^N}{2\gamma_1} \left[\frac{1}{\sqrt{\pi N}} + o(N^{-3/2}) \right]. \end{aligned} \quad (4.89)$$

Once again, we find two different results depending on whether $\gamma_1 > \frac{1}{2}$ (finite temperature case) or $\gamma_1 = \frac{1}{2}$ (infinite temperature case), as listed below

$$\Delta_{a \rightarrow b} \Big|_{\gamma_1 \geq \frac{1}{2}} = \frac{\gamma_1 - \gamma_2}{\gamma_1} + \underbrace{\gamma_2 \frac{(4\gamma_1\gamma_2)^N}{(N + 1)\sqrt{\pi N}(\gamma - 1 - \gamma_2)^2}}_{\delta_{a \rightarrow b}} + o(N^{-3/2}), \quad (4.90a)$$

$$\Delta_{a \rightarrow b} \Big|_{\gamma_1 = \frac{1}{2}} = \frac{1}{\sqrt{N\pi}} + o(N^{-1/2}). \quad (4.90b)$$

it is clear that one needs to be careful when $\beta > 0$, since the deviation from the desired result is of the form $\delta_{a \rightarrow b} - \frac{\gamma_1 - \gamma_2}{\gamma_2}$. By taking this into account we can express the operation

$$\Pi_{12}^{\text{appr}} = \begin{pmatrix} 1 - e^{-\beta E} & 1 \\ e^{-\beta E} & 0 \end{pmatrix} + \begin{pmatrix} \delta_{a \rightarrow b} & -\Delta_{b \rightarrow a} \\ -\delta_{a \rightarrow b} & \Delta_{b \rightarrow a} \end{pmatrix}. \quad (4.91)$$

Using the above, may calculate the distance induced by 1-norm between the target

distribution and the achieved approximation by

$$\beta = 0: \quad \|(\Pi_{12} - \Pi_{12}^{\text{appr}})\mathbf{p}\|_1 = \frac{|a - b|}{\sqrt{\pi N}} + o(N^{-1/2}), \quad (4.92a)$$

$$\beta > 0: \quad \|(\Pi_{12}^\beta - \Pi_{12}^{\text{appr}})\mathbf{p}\|_1 = (4\gamma_1\gamma_2)^N \left[\frac{|a\gamma_2 - b\gamma_1|}{(\gamma_1 - \gamma_2)^2 \sqrt{\pi N(N+1)}} + o(N^{-3/2}) \right]. \quad (4.92b)$$

Equations (4.92) prove the convergence rates given in Theorem 4.2 after relabelling $\gamma_1 \rightarrow \Gamma_{ij}, \gamma_2 \rightarrow \Gamma_{ji}$ for arbitrary pair of levels i, j . Note that convergence at infinite temperature, $\beta = 0$, is significantly slower than for arbitrary finite temperature, $\beta > 0$. In the following section we will present numerical results pointing to the validity of Conjecture 4.2.

4.3 Numerical evidence for convergence without intermediate thermalizations

Concerning the numerical aspect of the work conducted, we will limit ourselves in this section to reporting what has already been presented in the published manuscript by Czartowski, de Oliveira and Korzekwa [16]. To that end, the plots provided in what follows are in terms of 1-norm, defined as

$$\delta(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{i=1}^d |p_i - q_i| \quad (4.93)$$

and shortened to $\delta(\mathbf{p}, \mathbf{q}) \equiv \delta$ whenever the states $\mathbf{p}, \mathbf{q}$ are inferred from context.

Figures 4.6 and 4.7 provide evidence for convergence of the proposed algorithms to the extreme points not only in $\beta = 0$, where it was guaranteed by the Birkhoff polytope, but also for finite temperatures, where it is far from obvious whether one can realise such extreme transformations using just two-level Markovian thermal operations.

Finally, in Fig. 4.8 we show convergence to the states resulting from the action of the extreme map $\hat{O}$ defined in (4.20) on the vector $\mathbf{s} = (1, 0, \dots, 0)$. Apparent impossibility of even approximating the state $\hat{O}\mathbf{s}$ with infinitesimal precision using two-level thermal operations, as reported by Mazurek and Horodecki in [185] turns out to be highly undermined by the numerics, unless it is more rigorously formulated to restrict operations to 2-level thermal operations on the primary system only.

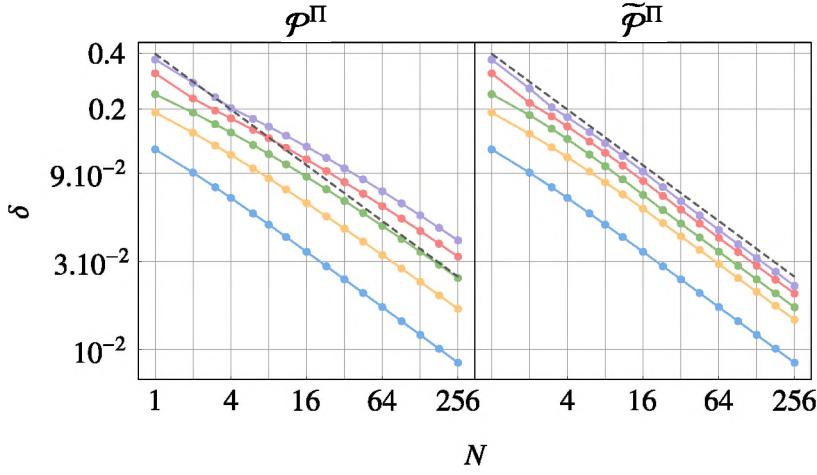


Figure 4.6: Convergence rates at infinite temperature. (Fig. 6 from [16], caption partially adjusted) Log-log plot of the total variation distance δ between the extreme points $\mathbf{p}^\pi \in \mathcal{T}_+(\mathbf{p})$ and the states obtained from $\mathbf{p}$ via the algorithm $\mathcal{P}^\Pi$ (left panel) and $\tilde{\mathcal{P}}^\Pi$ (right panel), as a function of the memory size N . Here, $\mathbf{p} = (0.37, 0.24, 0.16, 0.11, 0.07, 0.05)$, $\beta = 0$, and different colours correspond to families of extreme points $\mathbf{p}^\pi$ with different convergence rates (from bottom to top $\mathbf{p}^\pi$ is obtained from $\mathbf{p}$ via a 6-cycle, a composition of a β -6-cycle with a 5-cycle, and so on). All convergences behave as $O(N^{-1/2})$ as given in Corollary 4.3 and Conjecture 4.1, which can be seen by the comparison with the function $0.4/\sqrt{N}$ (dashed black line), with multiplicative advantage for $\tilde{\mathcal{P}}^\Pi$ over $\mathcal{P}^\Pi$.

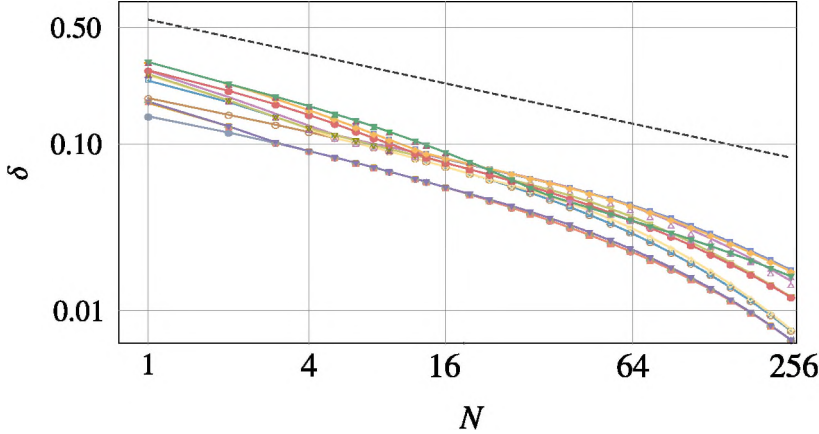


Figure 4.7: Convergence rates at finite temperature. (Fig. 7 from [16], caption partially adjusted) Log-log plot of the total variation distance δ between the states obtained from $\mathbf{p}$ via the algorithm $\tilde{\mathcal{P}}^{\Pi_{ij}}$ and the corresponding extreme points $\mathbf{p}^\pi \in \mathcal{T}_+(\mathbf{p})$, as a function of the memory size N . Here, $\mathbf{p} = (0.37, 0.24, 0.16, 0.11, 0.07, 0.05)$, $\beta = 0.1$, and different colours correspond to extreme points $\mathbf{p}^\pi$ with matrix representation of the β -order π given by $\Pi = \tilde{\Pi}_{ij}\Pi_{\mathbf{p}}$. For all curves, the convergence is better than $O(N^{-1/2})$ in agreement with Conjecture 4.2, as can be seen by the comparison with the limiting line $1/\sqrt{\pi N}$ for $\beta = 0$ (dot-dashed black line).

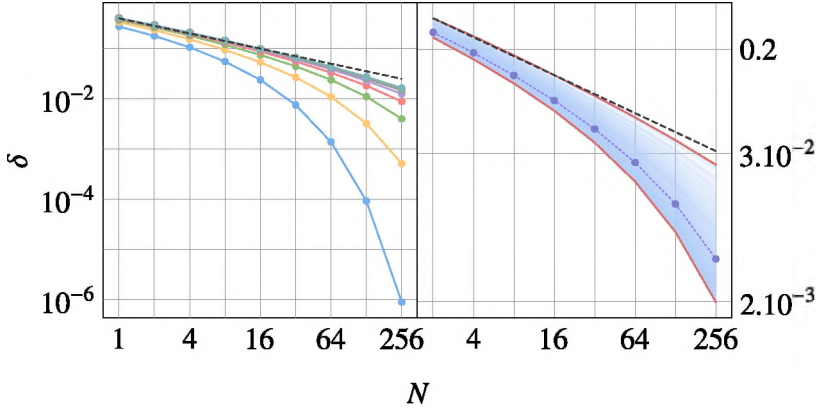


Figure 4.8: Convergence to states inaccessible via elementary thermal operations. Log-log plot of the total variation distance δ between the state $\hat{O}S$ with $\hat{O}$ from (4.20) and $s = (1, 0, \dots, 0)$, and the state obtained from $p = (1, 0, \dots, 0)$ via the algorithm $\bar{P}^{\Pi q}$, as a function of the memory size N . Left: systems with energy spectra $E_i = i$ with $d = 3, \dots, 10$ (bottom to top) and for $\beta = 1.1 \log(2) > \beta_{\text{crit}}$. For all presented dimensions the convergence is better than $1/(2\sqrt{N})$ (black dashed line). Right: systems with energy spectra $(0, E_1, E_2, 1)$ taken from a grid with interval $\Delta = 1/64$ (translucent blue lines) for $\beta = 1.1 \cdot \beta_{\text{crit}}$. The red lines represent the extreme cases of convergence, while the thick blue line is the average convergence. Note that the top red line almost agrees with $1/(2\sqrt{N})$ (black dashed line), which well approximates the expected convergence for $\beta = 0$ and agrees with the fact that it is obtained for almost completely degenerate levels.

4.4 Approach to extension beyond 3-level algorithm

In this section we will consider a d -dimensional primary system in a starting state

$$\mathbf{p} = (p_1, \dots, p_d) \quad (4.94)$$

described by a certain Hamiltonian H and in contact with the bath at inverse temperature β . Thus, the Gibbs state corresponding to the system will be given as

$$\gamma = (\gamma_1, \dots, \gamma_d) = \frac{1}{\sum_{i=1}^d e^{-\beta E_i}} (e^{-\beta E_1}, \dots, e^{-\beta E_d}). \quad (4.95)$$

The protocol in Conjecture 4.2 for approximating the extreme transition Π_o^β is based on decomposition of the underlying permutation into transpositions, which is always possible and then implementing the said transpositions as N^2 -step approximations. Thus, in total we will be taking MN^2 steps; however, these steps are taken **in sequence**.

The above observation is crucial for the general proof idea. As we take the size N of the memory to the **cardinal** infinity (think $\aleph_0$, the cardinality of countably infinite sets like integers $\mathbb{Z}$ or rationals $\mathbb{Q}$), in order to get the result of the algorithm, we need to take $N^2 \rightarrow \omega^2$ steps, where ω is the first infinite ordinal number. Although proper understanding of ordinals is beyond your truly, I believe it is in order to provide reference to the first introduction of such infinite counting numbers by Georg Cantor in 1883 [194], and some more modern introductions [195, 196]. For intuition, however, let us mention that if ω is understood as infinity of steps, then $\omega + 1$ is a valid way of saying that one took infinity of steps **and one more**. In our specific context, $\omega^2 = \omega \cdot \omega$ is in place to mean that we are having ω **steps with each consisting of ω substeps**.

The important fact from the above considerations is that we first take ω^2 steps corresponding to $\widetilde{\mathcal{P}}^{(i_1 j_1)}$ and only afterwards we take $\omega^2 + 1$ th step[¶] beginning the implementation of $\widetilde{\mathcal{P}}^{(i_2 j_2)}$ and so on, up to $M\omega^2$. Thus, the general approach to the complete proof of convergence to a β -cycle may be the following:

1. Take $\mathbf{P}_N = \mathbf{p} \otimes \boldsymbol{\eta}_N$ and implement the first protocol $\widetilde{\mathcal{P}}^{(i_1 j_1)}$. For the sake of simplicity we will assume that $\mathbf{p} = (1, 0, \dots, 0)$. Without loss of generality, we may assume it to be $\widetilde{\mathcal{P}}^{(12)}$. This gives explicit formulas for the entries as in (4.77),

$$P_{N+k}^{(N)} = \frac{\Gamma_{21}}{N\Gamma_{12}} I_{\Gamma_{12}}(k, N), \quad P_j^{(N)} = \frac{1}{N} I_{\Gamma_{12}}(N, j). \quad (4.96)$$

2. Take the limit of each entry with $N \rightarrow \infty$, thus acknowledging that indeed ω^2

[¶] I leave to the linguists the problem concerning the choice between $\omega^2 + 1$ th and $\omega^2 + 1$ st.

first steps have been taken. Numerical investigation suggests the form

$$\lim_{a+b \rightarrow \infty} I_x(a, b) \stackrel{?}{=} \Theta\left(x - \frac{a}{a+b}\right) \quad (4.97)$$

where $\Theta(x)$ is the Heaviside theta function. For the numerics themselves, see Fig. 4.9.

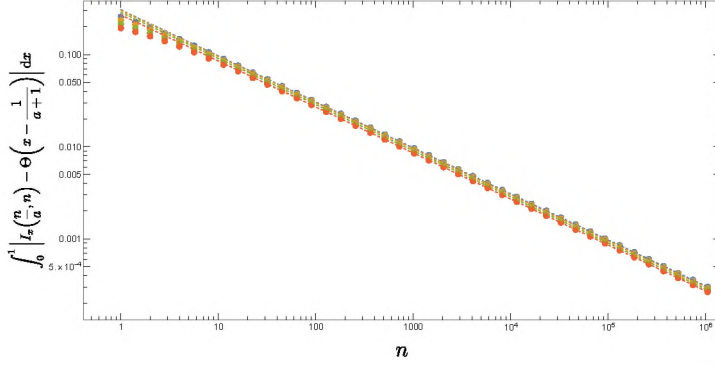


Figure 4.9: Log-log plot of convergence of 1-norm integral for the difference $I_x\left(\frac{n}{a}, n\right) - \Theta\left(x - \frac{1}{1+a}\right)$ for $a = 2, 3, 4, 5$ (blue, orange, green and red points, respectively) together with polynomial fits to the final 10 points (dashed lines of respective colours), showing convergence to 0 as $An^{-1/2}$, Thus supporting the conjecture that $\lim_{a+b \rightarrow \infty} I_x(a, b) = \Theta\left(x - \frac{a}{a+b}\right)$.

Thus, if eq. (4.97) is true, the converged forms would be

$$P_{N+k}^{(N)} \stackrel{?}{\rightarrow} \frac{\Gamma_{21}}{N\Gamma_{12}} \Theta\left(\Gamma_{12} - \frac{k}{k+N}\right), \quad (4.98)$$

$$P_j^{(N)} \stackrel{?}{\rightarrow} \frac{1}{N} \Theta\left(\Gamma_{12} - \frac{N}{N+j}\right) = \frac{1}{N} \Theta\left(\frac{j}{N+j} - \Gamma_{21}\right). \quad (4.99)$$

As a sanity check we can check that

$$\sum_{j=1}^N P_j^{(N)} \rightarrow \int_0^1 dv \Theta\left(\frac{v}{1+v} - \Gamma_{21}\right) = \int_0^1 dv \Theta\left(v - \frac{\Gamma_{21}}{\Gamma_{12}}\right) = \begin{cases} 0 & \text{for } \Gamma_{21} \geq \frac{1}{2} \\ 1 - \frac{\Gamma_{21}}{\Gamma_{12}} & \text{for } \Gamma_{21} < \frac{1}{2} \end{cases} \quad (4.100)$$

and similarly for

$$\sum_{j=1}^N P_{N+k}^{(N)} \rightarrow \frac{\Gamma_{21}}{\Gamma_{12}} \int_0^1 dv \Theta\left(\Gamma_{12} - \frac{v}{1+v}\right) = \frac{\Gamma_{21}}{\Gamma_{12}} \int_0^1 dv \Theta\left(\frac{\Gamma_{12}}{\Gamma_{21}} - v\right) = \begin{cases} 1 & \text{for } \Gamma_{21} \geq \frac{1}{2} \\ \frac{\Gamma_{21}}{\Gamma_{12}} & \text{for } \Gamma_{21} < \frac{1}{2} \end{cases} \quad (4.101)$$

The two expressions match with Π_{12} , and as such support the claim; however, it has to be stressed that the expression (4.97) is solely numerical, and needs further support in a form of a proof.

3. However, if we assume that (4.97) holds after ω^2 steps, we may continue to

consider the steps $\omega^2 + 1$ to $\omega^2 + \omega$, sticking however to N as counting variable. This results in

$$P_{2N+1}^{(N)} = \sum_{i=0}^{N-1} \Gamma_{31}^{N-i} P_{i+1}^{(N)} = \frac{1}{N} \sum_{i=0}^{N-1} \Gamma_{31}^{N-i} \Theta\left(i + 1 - N \frac{\Gamma_{21}}{\Gamma_{12}}\right) = \frac{1}{N} \sum_{i=N\Gamma_{21}/\Gamma_{12}-1}^{N-1} \Gamma_{31}^{N-i} \quad (4.102)$$

where in the last equality we assumed that since $N \rightarrow \infty$, then $\lceil N \frac{\Gamma_{21}}{\Gamma_{12}} - 1 \rceil = N \frac{\Gamma_{21}}{\Gamma_{12}} - 1$.

We could go on to describe the next steps in a systematic matter; however, the following qualitative description should suffice to understand the final result. First $N(1 - \Gamma_{21} - \Gamma_{12})$ levels are empty, and thus can be skipped, and the remaining $N\Gamma_{21}/\Gamma_{12}$ steps proceed as before; thus, the following distribution

$$P_{2N+k}^{(N)} = \frac{\Gamma_{13}^{k-1}}{N} \sum_{i=0}^{\frac{\Gamma_{21}}{\Gamma_{12}}N} \binom{i+k-1}{i} \Gamma_{31}^{i+1} = \frac{\Gamma_{31}}{N\Gamma_{13}} I_{\Gamma_{13}}\left(k, \frac{\Gamma_{21}}{\Gamma_{12}}N\right). \quad (4.103)$$

The same reasoning leads to the expression for the first N levels as

$$P_j^{(2N)} = \begin{cases} 0 & \text{for } j \leq N\left(1 - \frac{\Gamma_{21}}{\Gamma_{12}}\right) \\ \frac{\Gamma_{13}^{\frac{\Gamma_{21}}{\Gamma_{12}}N}}{N} \sum_{i=0}^{j - \left(1 - \frac{\Gamma_{21}}{\Gamma_{12}}\right)N-1} \binom{i + \frac{\Gamma_{21}}{\Gamma_{12}}N-1}{i} \Gamma_{31}^i & \text{for } j > N\left(1 - \frac{\Gamma_{21}}{\Gamma_{12}}\right), \end{cases} \quad (4.104)$$

which, effectively, is reduced to

$$P_j^{(2N)} = \frac{1}{N} \Theta\left(\frac{j}{N+j} - \Gamma_{21}\right) I_{\Gamma_{13}}\left(\frac{\Gamma_{21}}{\Gamma_{12}}N, j - \left(1 - \frac{\Gamma_{21}}{\Gamma_{12}}\right)N\right). \quad (4.105)$$

4. We repeat the limiting process as before, taking into account that now we are taking the limit connected to the steps $\omega^2 + 1$ to $2\omega^2$, which yields in this case

$$P_{2N+k}^{(N)} \rightarrow \frac{\Gamma_{31}}{N\Gamma_{13}} \Theta\left(\Gamma_{13} - \frac{k}{k + N_{21}}\right), \quad (4.106a)$$

$$P_j^{(2N)} \rightarrow \frac{1}{N} \Theta\left(\frac{j}{N+j} - \Gamma_{21}\right) \Theta\left(\frac{j - (N - N_{21})}{j - N} - \Gamma_{31}\right), \quad (4.106b)$$

where we introduced the notation $N_{ij} = \frac{\Gamma_{ij}}{\Gamma_{ji}}N$. Note that in (4.106b) only one theta will be valid, which will in fact be reexpressed as

$$P_j^{(2N)} = \frac{1}{N} \Theta\left(j - \max\left(N_{21}, \frac{\Gamma_{13}N - N_{21}}{\Gamma_{13}}\right)\right), \quad (4.107)$$

as only one Heaviside theta will survive.

5. The last step consists in taking care of the general vector, here restricted to

(p_1, p_2, p_3) . However, one can always rewrite the vector $\mathbf{p}$

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \left(1 - \frac{\gamma_3 p_2 - \gamma_2 p_3}{\gamma_2 \gamma_3} - \frac{p_3}{\gamma_3}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\frac{\gamma_3 p_2 - \gamma_2 p_3}{\gamma_2 \gamma_3} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{pmatrix}}_a + \underbrace{\frac{p_3}{\gamma_3} \boldsymbol{\gamma}}_b. \quad (4.108)$$

With this, due to Gibbs preserving character of any thermal operation, we can evaluate action of $\tilde{\Pi}_{(123)}^{\text{appr}}$ as

$$\tilde{\Pi}_{(123)}^{\text{appr}} \mathbf{p} = (1 - a - b) \tilde{\Pi}_{(123)}^{\text{appr}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a \tilde{\Pi}_{(13)}^{\text{appr}} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{pmatrix} + b \boldsymbol{\gamma}, \quad (4.109)$$

which shows that evaluation on the sharp states is sufficient to evaluate the action of the approximate operation $\tilde{\Pi}_{(123)}^{\text{appr}}$ on an arbitrary input state

Similar approach can be taken for an arbitrary permutation $\sigma = (i_1 j_1) \dots (i_M j_M)$ on a system of arbitrary dimensionality d . To provide the final piece of intuition, we may consider additional round of thermalizations by implementing $\tilde{\Pi}_{23}^{\text{appr}}$. This will entail acting on the levels

$$P_{N+k}^{(N)} = \frac{\Gamma_{21}}{N\Gamma_{12}} \Theta\left(\Gamma_{12} - \frac{k}{k+N}\right) = \frac{1}{N_{12}} \Theta(N_{12} - k), \quad (4.110a)$$

$$P_{2N+k}^{(N)} = \frac{\Gamma_{31}}{N\Gamma_{13}} \Theta\left(\Gamma_{13} - \frac{k}{k+N_{21}}\right) = \frac{1}{N_{13}} \Theta\left(\underbrace{\frac{N_{21}N_{13}}{N}}_{N_{21,13}} - k\right). \quad (4.110b)$$

We will omit intermediate steps of the calculations that lead to show that after the final ω^2 steps we arrive at

$$P_{N+k}^{(2N)} = \frac{1}{N_{12}} \Theta(N_{12} - k) \Theta\left(k - \frac{N_{12}\Gamma_{32}}{\Gamma_{23}}\right) + \frac{\Gamma_{23}}{N_{13}\Gamma_{32}} \Theta\left(\frac{\Gamma_{32}N_{21,13}}{\Gamma_{23}} - k\right), \quad (4.111a)$$

$$P_{2N+k}^{(2N)} = \frac{\Gamma_{32}}{N_{12}\Gamma_{23}} \Theta\left(\frac{N_{12}\Gamma_{23}}{\Gamma_{32}} - k\right) + \frac{1}{N_{13}} \Theta(N_{21,13} - k) \Theta\left(k - \frac{\Gamma_{23}N_{21,13}}{\Gamma_{32}}\right). \quad (4.111b)$$

Now, of course, the integration would depend on the relations between the levels γ_i of the Gibbs state. However, thanks to the fact we have not been using any specific assumptions concerning the Gibbs state, we can say that all the expressions derived in terms of Heaviside theta approximation of regularised incomplete beta function remain valid insofar as the approximation itself holds.

4.5 Summary and Outlook

In this chapter we have demonstrated that thermal operation can be approximated by Markovian thermal process effected on the system extended by a memory initialized in a flat state. The infinite-temperature case has been proved analytically in its entirety, whereas analytical proof for convergence to β -swaps has been presented together with extensive numerical evidence for convergence to arbitrary other extreme thermal operation and, by convexity, to any other.

The results concerning work extraction, cooling and the evolution of free energy under the protocol have been skipped in favour of extending the results concerning the convergence and algorithm itself, which the contributions of the author. In particular, the extensions include the short argument concerning commutativity of the elementary thermalizations constituting the algorithm and the resulting parallelization scheme, as well as the more streamlined approach to the convergence problem for arbitrary extreme thermal operations. The novel approach has been based on the observation that the size of memory is itself going to infinity as a single limit in connection with the cardinal nature of the underlying infinity, but the implementation of algorithm for approximation of permutation breaks up into independent limits related to each β -swap algorithm, as each of them needs to go through an ordinal infinity of steps before going on to the next step. Moreover, the streamlined approach is based on the conjectured approximation of the regularised incomplete beta function by Heaviside theta, which allows for quick and intuitive evaluation.

Although full proof of convergence remains to be carried out, the author believes that the arguments put forward so far, in combination with what has been demonstrated in our manuscript [16], provide sufficient grounds to believe that the statement is true. Potential future developments in connection to this work may include experimental implementations of the proposed algorithm in its finite-size versions, as well as theoretical investigation into possible extensions of this or similar procedure to the coherent setting.



5 We Can Catalyse it for You Thermally

[19]

According to the Meriam-Webster Dictionary of English, catalysis is [198]

“a modification and especially increase in the rate of a chemical reaction induced by material unchanged chemically at the end of the reaction”

and as such it has been studied extensively in chemistry* ever since the first use of this word in 1836. Cambridge Dictionary, however, attributes an additional, more indirect meaning to the verb *to catalyse* as [201]

“to make something start happening or start being successful.”

The use of the term *catalysis* in the context of quantum resource theories can be seen as something between the chemical and metaphorical definitions. We usually consider a transition $A \rightarrow B$ between states A and B , which cannot be carried out by itself. However, once we extend the system by a catalyst in a state C , the transition $AC \rightarrow BC$ becomes feasible. More importantly, just like the catalysts in chemistry, the state C remains unperturbed, and thus ready to be used for assistance in further transitions.

In the previous chapter we have seen an intriguing example of a free state of the theory – namely, the Gibbs state – being used as a catalyst in the context of Markovian thermal operations (MTO). Therein, we demonstrated it is powerful enough to bridge the gap between the transitions allowed for operations with short and arbitrarily large memory. This, however, is a somewhat peculiar example, since free states usually do not act as catalysts.

The problem of catalysis has been considered for resource theories of entanglement [202–207], coherence [183, 208–212], thermal operations [16, 19, 182, 213–224] and others [225–230][†]

The three specific theories mentioned above share one specific attribute – namely, in certain regime, transformability of states can be described by resorting to the majorisation relation between specific aspects of the states, such as Schmidt coefficients or populations with respect to the distinguished (incoherent or energy-related) basis.

* Despite it being outside of the scope of this thesis, it is worth pointing out that catalysis is one of the major subfields of chemistry, with numerous books on the topic, tutorial introductions [199] and historic reviews available [200]

[†] The citations in this paragraph have been borrowed from [231]; as I see this list as relatively complete, I see it fit to highlight its source.

This allows for certain results to be automatically transferred from one theory to another by tracing them back to the “common ancestor”. In turn, one may take the results obtained for catalysis of pure entangled states, as obtained by Grabowecky and Gour in [232], to the resource theory of coherence and thermal operations in infinite temperature. However, since thermal operations in finite temperatures make use of majorization indirectly through the notion of thermomajorization, a direct transplant is not possible. In this section we present an applicable translation of these results, thus obtaining bounds on dimensionality and structure of the catalyst states necessary for catalysing any given transition. In particular, we demonstrate that catalysis for 2-dimensional systems is impossible for arbitrary temperature; however, contrary to common wisdom, one may catalyse transitions for 3-dimensional systems as soon as inverse temperature β is not zero.

Since the majorisation and thermomajorisation has been already described in sections 4.1.1 and 4.1.2, respectively, we will limit the preliminaries of this chapter to independent presentation of the results obtained in [232]. With this in hand, we will proceed to present two approaches to translating the majorization-based results to thermomajorization. As a byproduct, we will prove that dimension 2 remains uncatalysable for arbitrary temperature and present examples of catalysable pairs for dimension 3. Then we will move on to connect the results on catalysability with possibility of generating entanglement using thermal operations, based on results in [20]. The chapter concludes with a short summary and outlook at the future research connected with catalysis.

5.1 A preliminary – catalyst limitations for entanglement resource theory

Since the starting point for the results of this chapter is the resource theory of entanglement, we will shortly recall the underlying framework and the related results obtained in [232]. Note that due to use of majorization, the following considerations automatically extend to thermal operations at zero inverse temperature $\beta = 0$.

Let us consider Alice and Bob sharing a pure bipartite state defined by its Schmidt decomposition, $|\Psi_{AB}\rangle = \sum_{i=1}^d \sqrt{p_i} |\psi_{Ai}\rangle \otimes |\psi_{Bi}\rangle$. The set of pure states $|\Phi_{AB}\rangle = \sum_{i=1}^d \sqrt{q_i} |\phi_{Ai}\rangle \otimes |\phi_{Bi}\rangle$ that they can transform their initial state into using only local operations and classical communication, $|\Psi_{AB}\rangle \xrightarrow{\text{LOCC}} |\Phi_{AB}\rangle$, is given by all the states for which the Schmidt vectors obey the majorization relation $\mathbf{q} \succ \mathbf{p}$, as proved by Nielsen in [233]. In what follows we assume, without loss of generality in this context, that the Schmidt vectors are ordered nonincreasingly, $i < j \Rightarrow p_i \geq p_j$ and similarly for $\mathbf{q}$.

In what follows we will consider catalytic extension of the LOCC by allowing use of an auxiliary entangled state, which we will define simply by its Schmidt vector $\mathbf{c} = (c_1, \dots, c_k)$; the set of such operations is referred to commonly

as the entangled assisted LOCC (ELOCC), as introduced in [234]. More formally, we find that

$$|\Psi_{AB}\rangle \xrightarrow{\text{ELOCC}} |\Phi_{AB}\rangle \iff \exists_{\mathbf{c}} : \mathbf{q} \otimes \mathbf{c} > \mathbf{p} \otimes \mathbf{c}. \quad (5.1)$$

Building up towards the main result in steps, one may first note that the states that are already in a majorization relation cannot change their relation as a result of introducing a catalyst,

$$\mathbf{q} > \mathbf{p} \implies \forall_{\mathbf{c}} \mathbf{q} \otimes \mathbf{c} > \mathbf{p} \otimes \mathbf{c}, \quad (5.2)$$

which already limits the set of the catalysable pairs $(\mathbf{p}, \mathbf{q})$ to the incomparable ones, $\mathbf{q} \not> \mathbf{p}$ and $\mathbf{p} \not> \mathbf{q}$, which we will denote as $\mathbf{p} \perp \mathbf{q}$. As a consequence, we have the following observation:

Observation 5.1. *There are no catalysable LOCC transformations for dimension $d = 2$.*

Next refinement comes by noting the following inequalities, which hold for all catalyst states $\mathbf{c}$,

$$p_1 \geq q_1 \implies (\mathbf{p} \otimes \mathbf{c})_1^\downarrow = p_1 c_1 \geq q_1 c_1 = (\mathbf{q} \otimes \mathbf{c})_1^\downarrow, \quad (5.3a)$$

$$p_d \geq q_d \implies (\mathbf{p} \otimes \mathbf{c})_{dk}^\downarrow = p_d c_k \geq q_d c_k = (\mathbf{q} \otimes \mathbf{c})_{dk}^\downarrow. \quad (5.3b)$$

The following necessary condition proceeds from the above relations as a corollary

Corollary 5.2. *A transformation $\mathbf{p} \rightarrow \mathbf{q}$ can be catalysed only if $p_1 \leq q_1$ and $p_d \geq q_d$.*

As a consequence, we have the second no-go property

Observation 5.3. *There are no catalysable LOCC transformations for dimension $d = 3$.*

It follows from the fact that for $\mathbf{p} \perp \mathbf{q}$ in $d = 3$ conditions (5.3) cannot be fulfilled.

As the final step, we will compress the main results into a single compact theorem

Theorem 5.4 (Theorems 1 and 2 from [232]). *Consider two states described by Schmidt vectors $\mathbf{p}, \mathbf{q} \in \Delta_d$ which are incomparable, $\mathbf{p} \perp \mathbf{q}$ and $p_1 < q_1$ and $p_d > q_d$. We define a set of indices $\mathcal{L}$ as:*

$$\mathcal{L} = \left\{ l \in \{1, 2, \dots, d\} \mid \sum_{i=1}^l p_i - q_i < 0 \right\} \quad (5.4)$$

and take $m = \min(\mathcal{L})$ and $n = \max(\mathcal{L})$. The catalyst $\mathbf{c} \in \Delta_k$ can catalyse the transformation, ie. $\mathbf{p} \otimes \mathbf{c} > \mathbf{q} \otimes \mathbf{c}$, if and only if its dimensionality satisfies

$$k > k^* = \frac{\log b}{\log a} + 1. \quad (5.5)$$

The coefficients a, b defined as,

$$a = \min \left(\frac{p_1}{p_m}, \frac{p_{n+1}}{p_d} \right) > \max_{v \in \{1, 2, \dots, k-1\}} \left(\frac{c_v}{c_{v+1}} \right), \quad (5.6a)$$

$$b = \max_{l \in \mathcal{L}} \left(\frac{p_l}{p_{l+1}} \right) < \frac{c_1}{c_k}, \quad (5.6b)$$

and provide bounds for the entries of the catalyst state $\mathbf{c}$.

Full proof of the above statements can be found in [232]. Here, however, we will treat this statement as a starting point for translation of the result for majorization to the setting of thermomajorization.

5.2 Two approaches to extension

In the following two subsections we will present two approaches to extending the results from [232] to the thermomajorization setting, with one being more heuristic and the other more rigorous.

Before reading on, we refer the reader again to sections 4.1.1 and 4.1.2 for proper consideration of majorization and thermomajorization within the thermal operations' setting. Furthermore, for later convenience we define the slope vector $\mathbf{s}$ associated to a population vector $\mathbf{p}$ as

$$\mathbf{s}(\mathbf{p}) := \underbrace{\left(\frac{p_1}{\gamma_1}, \dots, \frac{p_d}{\gamma_d} \right)^\downarrow}_{(s_1(\mathbf{p}), \dots, s_d(\mathbf{p}))}. \quad (5.7)$$

Let us note that the proofs contained in this section are, in essence, the same as in [19] with only minor tweaks to enhance clarity and match the notation. As such, they should be treated as extended auto-quotations.

5.2.1 Heuristic approach via derivatives

We will begin with rephrasing the results for majorization in terms of derivatives. Let us consider a state $\mathbf{p}$ and the corresponding majorisation curve $f_{\mathbf{p}}(x)$. In what follows, for brevity of notation, we will use primed notation $f'(x) \equiv df(x)/dx$ for derivative of a function and $f'(x_i) \equiv f'(x)|_{x=x_i}$. Furthermore, we will use notation $f'(x_{\pm}) \equiv \lim_{\epsilon \rightarrow 0} f'(x \pm \epsilon)$ to denote left and right derivatives, which will be necessary due to discontinuities at the elbows of (thermo)majorization curves.

For $\beta = 0$, we find a simple relation between probabilities and derivatives of the majorisation curve, namely

$$d \cdot p_i^\downarrow = f_{\mathbf{p}}'(x) \Big|_{\frac{i-1}{d} < x < \frac{i}{d}} \equiv f_{\mathbf{p}}'(x_i), \quad (5.8)$$

where we define $x_i \in [(i-1)/d, i/d]$ as an arbitrary point from the interval between the consecutive elbows of the majorisation curve. Using the relation given by Eq. (5.8) in Eqs. (5.6), we find that the coefficients a and b can be expressed in terms of derivatives as follows:

$$a = \min \left(\frac{f_{\mathbf{p}}'(x_1)}{f_{\mathbf{p}}'(x_m)}, \frac{f_{\mathbf{p}}'(x_{n+1})}{f_{\mathbf{p}}'(x_d)} \right) > \max_{v \in \{1, 2, \dots, k-1\}} \left(\frac{f_{\mathbf{c}}'(x_v)}{f_{\mathbf{c}}'(x_{v+1})} \right), \quad (5.9a)$$

$$b = \max_{l \in \mathcal{L}} \frac{f_{\mathbf{p}}'(x_l)}{f_{\mathbf{p}}'(x_{l+1})} < \frac{f_{\mathbf{c}}'(x_1)}{f_{\mathbf{c}}'(x_k)}. \quad (5.9b)$$

Now, the expressions above remain valid when we replace the subset $\mathcal{L}$, over which we maximize, with an open interval $\mathcal{L} = (m', n') = \{x : f_{\mathbf{p}}'(x) - f_{\mathbf{q}}'(x) < 0\}$; the same substitution can be applied to the right-hand side (5.9a). However, to maintain the correct interpretation, we must substitute the ratios of subsequent points with the ratios of the left and right derivatives, specifically focusing on the left-hand side of Eq. (5.9b). As a result, leads to the following

$$b = \max_{x \in \mathcal{L}} \frac{f_{\mathbf{p}}'(x_-)}{f_{\mathbf{p}}'(x_+)}. \quad (5.10)$$

Note that, whenever a function is differentiable at point x , the ratio between left and right derivative is equal to one. Therefore, any ratio value differing from one can be considered indicative of non-differentiability. This is a characteristic observed exclusively at the elbows of majorisation curves—a property we will later exploit to further simplify our calculations.

Finally, we return to the discussion of thermomajorisation curves by reintroducing $\beta > 0$, which establishes the connection between the derivatives and probabilities.

$$p_i^\beta = \gamma_i^\beta f_{\mathbf{p}}^{\beta'}(x) \Big|_{\Gamma_{i-1} < x < \Gamma_i}, \quad (5.11)$$

where we used the subsums of the Gibbs distribution $\Gamma_i = \sum_{j=1}^i \gamma_j^\beta$, which correspond to the locations of the elbows on the thermomajorisation curve $f_{\mathbf{p}}^\beta(x)$. While all inequalities remain valid, they now depend not only on the probability vectors $\mathbf{p}$ and $\mathbf{c}$ but also on the respective underlying Gibbs distributions γ and $\gamma_{\mathbf{c}}$.

Now, let us focus on the derivation of lower bound on catalyst dimensionality k_* from the derivatives' perspective. In the original approach the starting point was to note

the following chain of bounds,

$$b < \frac{c_1^\downarrow}{c_k^\downarrow} = \prod_{i=1}^{k-1} \frac{c_i^\downarrow}{c_{i+1}^\downarrow} < \left[\max_{v \in \{1,2,\dots,k-1\}} \left(\frac{c_v^\downarrow}{c_{v+1}^\downarrow} \right) \right]^{k-1} < a^{k-1}. \quad (5.12)$$

Unfortunately, the heuristic approach does not allow us to extend this result directly, as we shifted our perspective from piecewise-linear curves defined solely by their elbows to continuous functions. However, by taking the logarithm of Eq. (5.9b), one can observe that it involves the ratio between the first and last slope. This allows for the computation of the left-derivative at $x_1 = 0$ and the right-derivative at $x_k = 1$,

$$\log \left(\frac{f_c^{\beta'}(0_+)}{f_c^{\beta'}(1_-)} \right) = \log[f_c^{\beta'}(0_+)] - \log[f_c^{\beta'}(1_-)]. \quad (5.13)$$

The above can also be reformulated as the integral of a derivative,

$$\log \left(\frac{f_c^{\beta'}(0_+)}{f_c^{\beta'}(1_-)} \right) = \lim_{\delta \rightarrow 0} \int_{\delta}^{1-\delta} \left[-\log(f_c^{\beta'}(x)) \right]' dx. \quad (5.14)$$

Next, we observe that the logarithmic terms appearing in Eq. (5.14) can be explicitly expressed as

$$\log(f_c'(x)) = \sum_{i=1}^k \log(s_i(\mathbf{c}))(\Theta(x - \Gamma_{i-1}) - \Theta(x - \Gamma_i)), \quad (5.15)$$

$$\begin{aligned} [\log(f_c'(x))]' &= \sum_{i=1}^k \log(s_i(\mathbf{c}))(\delta(x - \Gamma_{i-1}) - \delta(x - \Gamma_i)) \\ &= \log(s_1(\mathbf{c}))\delta(x) - \log(s_d(\mathbf{c}))\delta(x - 1) \\ &\quad + \sum_{i=1}^{k-1} \delta(x - \Gamma_i) [\log(s_{i+1}(\mathbf{c})) - \log(s_i(\mathbf{c}))]. \end{aligned} \quad (5.16)$$

In Eqs. (5.15) and (5.16), $\Theta(x)$ represents the Heaviside step function, and $\delta(x)$ denotes the Dirac delta function, respectively.

Thus, using Eqs.(5.15) and (5.16), one can compute the integral given in Eq.(5.14) as follows:

$$\lim_{\delta \rightarrow 0} \int_{\delta}^{1-\delta} [-\log(f_c'(x))]' dx = \sum_{i=1}^{k-1} \log \left(\frac{s_i(\mathbf{c})}{s_{i+1}(\mathbf{c})} \right) \leq (k-1) \max_i \log \left(\frac{s_i(\mathbf{c})}{s_{i+1}(\mathbf{c})} \right), \quad (5.17)$$

where we recall that $\mathbf{s}(\mathbf{c})$ is the slope vector corresponding to the population of the catalyst $\mathbf{c}$. Finally, using (5.9a), we retrieve the bound on the dimensionality of the catalyst state $\mathbf{c}$,

$$k \geq \frac{\log b}{\log a} + 1 = k_*, \quad (5.18)$$

within the heuristic approach.

5.2.2 Embedding map approach

We start by recalling the embedding map [179] (see Ref. [235] for a detailed discussion), which is a restatement of the construction of thermomajorisation as presented in section 4.1.2. This allows us to draw a connection between thermomajorisation and majorisation in a space of larger dimension.

Definition 5.1 (Embedding map). *Consider a thermal distribution γ with rational entries, $\gamma_i = D_i/D$ and $D_i, D \in \mathbb{N}$, the embedding map Ξ sends a d -dimensional probability distribution $\mathbf{p}$ to a D -dimensional probability distribution $\hat{\mathbf{p}} := \Xi(\mathbf{p})$ as follows:*

$$\hat{\mathbf{p}} = \left[\underbrace{\frac{p_1}{D_1}, \dots, \frac{p_1}{D_1}}_{D_1 \text{ times}}, \underbrace{\frac{p_2}{D_2}, \dots, \frac{p_2}{D_2}}_{D_2 \text{ times}}, \dots, \underbrace{\frac{p_d}{D_d}, \dots, \frac{p_d}{D_d}}_{D_d \text{ times}} \right]. \quad (5.19)$$

A generic thermal distribution γ will not fall within the scope of Definition 5.1. Nevertheless, since the rational numbers $\mathbb{Q}$ are dense subset of the real numbers $\mathbb{R}$. Thanks to this the limit $\lim_{D \rightarrow \infty} \frac{[D\gamma_i]}{D} = \lim_{D \rightarrow \infty} \frac{[D\gamma_i]}{D} = \gamma_i$ exists. With this chain of approximations in mind, we proceed to work within the limit of dimension $D \rightarrow \infty$.

It follows from the above definition that the majorisation between embedded vectors coincides exactly with the notion of thermo-majorisation,

$$\hat{\mathbf{p}} > \hat{\mathbf{q}} \iff \mathbf{p} >_{\beta} \mathbf{q}. \quad (5.20)$$

Now, we introduce a function φ that maps elements of the embedded distribution to the corresponding elements of the original one

$$\hat{p}_j^{\downarrow} = \frac{p_{\varphi(j)}}{D_{\varphi(j)}}, \quad (5.21)$$

which, as a consequence of non-increasing ordering, yields

$$j > k \implies \frac{p_{\varphi(j)}}{D_{\varphi(j)}} \leq \frac{p_{\varphi(k)}}{D_{\varphi(k)}}. \quad (5.22)$$

The function ϕ is used in order to convert the ordering of the embedded vector $\hat{\mathbf{p}}^{\downarrow}$ to the β -ordering of $\mathbf{p}^{\beta}$. Suppose, for example, that $p_1^{\beta} = p_3$. Knowing this, we find that $\phi(j) = 3$ for all $1 \leq j \leq D_3$. More generally, it is defined to be, we find that $\phi(j) = \pi^{-1}(i)$ for $\sum_{n=1}^i D_{\pi^{-1}(n)} + 1 \leq j \leq \sum_{n=1}^i D_{\pi^{-1}(n)}$.

Focusing on the expression (5.6b) for b and applying it to the embedded vector $\hat{\mathbf{p}}$, we substitute

$$\frac{\hat{p}_l^\downarrow}{\hat{p}_{l+1}^\downarrow} = \frac{p_{\varphi(l)}}{D_{\varphi(l)}} \frac{D_{\varphi(l+1)}}{p_{\varphi(l+1)}} = \frac{p_{\varphi(l)}}{D_{\varphi(l)}/D} \frac{D_{\varphi(l+1)}/D}{p_{\varphi(l+1)}} \quad (5.23)$$

$$\xrightarrow{D \rightarrow \infty} \frac{p_{\varphi(l)}}{\gamma_{\varphi(l)}} \frac{\gamma_{\varphi(l+1)}}{p_{\varphi(l+1)}} = \frac{s_{\pi(\varphi(l))}(\mathbf{p})}{s_{\pi(\varphi(l+1))}(\mathbf{p})} \quad (5.24)$$

where in the last expression π is the permutation responsible for the beta-ordering of $\mathbf{p}^\beta$. Treating all the remaining ratios in (5.6a), (5.6b), we complete the derivation.

5.3 Bounds on thermal catalysts

5.3.1 Catalysable region from necessary condition

As we have demonstrated the methods used to extend the results of [232] to the setting of thermomajorization, we may summarise the proceeding results. Before doing so, however, let us put forward a definition of catalytic future and past thermal cone.

Definition 5.2 (Catalytic thermal cones). *For a d -dimensional energy-incoherent state $\mathbf{p}$, we define the catalytic future/past thermal cone $\mathcal{T}_{C+/-}$ as subsets of the incomparable region $\mathcal{T}_\emptyset(\mathbf{p})$ such that*

$$\mathcal{T}_{C+}(\mathbf{p}) \equiv \{\mathbf{q} \in \Delta_d : \mathbf{q} \perp_\beta \mathbf{p}, \exists_{k \geq 2, c \in \Delta_k} : \mathbf{p} \otimes \mathbf{c} >_\beta \mathbf{q} \otimes \mathbf{c}\}, \quad (5.25)$$

$$\mathcal{T}_{C-}(\mathbf{p}) \equiv \{\mathbf{q} \in \Delta_d : \mathbf{q} \perp_\beta \mathbf{p}, \exists_{k \geq 2, c \in \Delta_k} : \mathbf{q} \otimes \mathbf{c} >_\beta \mathbf{p} \otimes \mathbf{c}\}, \quad (5.26)$$

With these definitions, the first part of the results concerns a relatively straightforward necessary condition on a transformation to be catalysable.

Lemma 5.5 (Catalytic condition). *Consider a pair of incomparable states $\mathbf{q} \perp_\beta \mathbf{p}$. The state $\mathbf{q}$ is catalysed (with respect to $\mathbf{p}$), ie. $\mathbf{q} \in \mathcal{T}_{C+}(\mathbf{p})$ [or, equivalently, $\mathbf{p} \in \mathcal{T}_{C-}(\mathbf{q})$], only if it satisfies:*

$$s_1(\mathbf{p}) > s_1(\mathbf{q}) \quad \text{and} \quad s_d(\mathbf{p}) < s_d(\mathbf{q}). \quad (5.27)$$

It follows straightforwardly from the fact that the first slope of the thermomajorization curve of the composite states is simply product of the first slopes $s_1(\mathbf{p} \otimes \mathbf{c}) = s_1(\mathbf{p})s_1(\mathbf{c})$, and similarly for the last slope. It can, however, be set in a much more elegant form using the idea of *tangent vectors* introduced in our previous work [13], and recalled below.

Definition 5.3 (Thermal tangent vectors). *Given an energy-incoherent state $\mathbf{p}$ and a thermal state γ , consider distributions $\mathbf{t}^{(n, \pi)}(\mathbf{p}) \equiv \mathbf{t}^{(n, \pi)}$ in their β -ordered form, constructed*

for each permutation $\pi \in \mathcal{S}_d$ and $n \in \{1, \dots, d\}$,

$$[\mathbf{t}^{(n,\pi)}]^\beta = \left(t_{\pi(1)}^{(n,\pi)}, s_n \gamma_{\pi(2)}, \dots, s_n \gamma_{\pi(d-1)}, t_{\pi(d)}^{(n,\pi)} \right), \quad (5.28)$$

with

$$t_{\pi(1)}^{(n,\pi)} = \sum_{i=1}^n p_i^\beta - s_n \left(\sum_{i=1}^n \gamma_i^\beta - \gamma_{\pi(1)} \right), \quad (5.29a)$$

$$t_{\pi(d)}^{(n,\pi)} = 1 - t_{\pi(1)}^{(n,\pi)} - s_n \sum_{i=2}^{d-1} \gamma_{\pi(i)}, \quad (5.29b)$$

where $s_n := s_n(\mathbf{p})$.

It is in order to provide some background behind the term *tangent* appearing in the name. In the standard approach, a tangent to the function is a linear function with at most a single linear fragment common with the underlying function. Tangent vectors are essentially constructed by introducing elbow points at x coordinates $\sum_{i=1}^j \gamma_{\pi(i)}$ corresponding to the selected β -order. Finally, first and last entry are adjusted such that $f_{\mathbf{t}}^\beta(0) = 0$, $f_{\mathbf{t}}^\beta(1) = 1$, so as to guarantee that a tangent vector $\mathbf{t}$ is a (quasi-)probability distribution.

Using the above notion, an elegant formulation of catalysable regions proceeds.

Theorem 5.6 (Catalysable regions). *Consider an energy-incoherent state $\mathbf{p}$ and its associated tangent vectors $\mathbf{t}^{(n,\pi)}(\mathbf{p})$. We define two auxiliary sets*

$$T_i(\mathbf{p}) = \text{conv} \left[\left\{ \mathbf{t}^{(i,\pi)}(\mathbf{p}) \right\}_{\pi \in \mathcal{S}_d} \right], \quad (5.30)$$

where $\mathcal{S}_d$ is the set of all possible d -element permutations. The catalysable future and past regions are bounded by

$$\mathcal{C}_+(\mathbf{p}) = [T_1(\mathbf{p}) \cap T_d(\mathbf{p})] \setminus \mathcal{T}_+(\mathbf{p}), \quad (5.31)$$

$$\mathcal{C}_-(\mathbf{p}) = \mathcal{T}_\emptyset(\mathbf{p}) \setminus [T_1(\mathbf{p}) \cup T_d(\mathbf{p})]. \quad (5.32)$$

Proof of the above proceeds straightforwardly from Lemma 5.5. In fact, this relatively simple reinterpretation of the catalysable region as intersection of two simplices $T_1(\mathbf{p}) \cap T_d(\mathbf{p})$ allows for upper-bounding the number of vertices through geometric arguments.

Lemma 5.7 (Number of extreme points). *Consider a state $\mathbf{p} \in \Delta_d$. The number of extreme points of the full catalysable future thermal cone $\mathcal{CT}_+(\mathbf{p}) \equiv \mathcal{T}_+(\mathbf{p}) \cup \mathcal{C}_+(\mathbf{p})$ of any given state $\mathbf{p} \in \Delta_d$ is upper-bounded by*

$$|\text{Ext}(\mathcal{CT}_+(\mathbf{p}))| \leq \left\lceil \frac{d}{2} \right\rceil \binom{d}{\left\lceil \frac{d}{2} \right\rceil} \ll d!, \quad (5.33)$$

where $d! \geq |\text{Ext}(\mathcal{T}_+(\mathbf{p}))|$ is the upper bound on the number of vertices of the future thermal cone of $\mathbf{p}$ and $\lceil x \rceil$ denotes the ceiling of x .

The following proof is taken from our work [19] without changes aside from single words, as we believe it to be as clear as possible – thus, the following should be treated as extended auto-quotation.

Proof. We start by considering a regular d -point simplex Δ_d spanned by vertices $\mathbf{v}_i$ for $i = \{1, \dots, d\}$, with the central point $\mathbf{c} = \frac{1}{d} \sum_i \mathbf{v}_i$. In what follows, we will consider the rescaled versions of the simplex defined as

$$t\Delta_d = \text{conv} \left(\{t\mathbf{v}_i + (1-t)\mathbf{c}\}_{i=1}^d \right). \quad (5.34)$$

First, consider the intersection $\Delta_d \cap t\Delta_d$ for $t \leq 0$ which, in fact, defines an inverted or dual simplex. This will be denoted as $t_d\nabla_d$ with $t_d = -t$, a choice whose rationale will soon become apparent. For $t_d \in [0, 1/(d-1)]$, we observe that there is no intersection between the boundaries, $\partial\Delta_d \cap \partial(t_d\nabla_d) = \emptyset$. The first intersection occurs at $t = -1/(d-1)$, at which point there is only a single extreme point per d -dimensional face of Δ_d , since

$$\frac{1}{d-1}\nabla_d = \text{conv} \left(\left\{ \sum_{j=1}^d \frac{1-\delta_j}{d-1} \mathbf{v}_j \right\}_{i=1}^d \right). \quad (5.35)$$

For $t_d \geq 1/(d-1)$, the intersection points of $-t\nabla_d$ with each of the d faces of Δ_d —which are effectively $(d-1)$ -dimensional simplices—result in a $(d-1)$ -dimensional inverted simplex, denoted as $t_{d-1}\nabla_{d-1}$. Here, we define $t_{d-1} = \frac{d-1}{d} \min(0, t_d - \frac{1}{d-1})$. Consequently, there are exactly $(d-1)$ intersections per face of Δ_d , continuing until $t_{d-1} = 1/(d-2)$. The analysis then continues for the $(d-2)$ -dimensional faces of Δ_d .

Based on the reasoning above, we can identify only two types of situations:

1. The intersection $\Delta_d \cap t\nabla_d$ has a single vertex per k -dimensional face of Δ_d .
2. The intersection $\Delta_d \cap t\nabla_d$ has exactly $(d-k)$ vertices per k -dimensional face of Δ_d .

This results in either $\binom{d}{k}$ vertices or $k\binom{d}{k}$ vertices in total, with the latter being the larger one and attaining maximum at $k = \lceil d/2 \rceil$. This completes the reasoning for simplices with a common center $\mathbf{c}$.

The heuristic approach for the translations defined as

$$\Delta_d + \delta = \text{conv} \left(\{\mathbf{v}_i + \delta\}_{i=1}^d \right). \quad (5.36)$$

is based on the following argument. Let us focus on the intersection with a single face F from Δ_d and assume that $t_{d-1} < \frac{2}{d-2}$. Any shift in a direction parallel to the plane the face is contained in, $\delta \parallel F$, will result in translating the intersection points between $t\nabla_d$ and $F + \delta$. Similarly, the shift in the perpendicular direction will result in scaling

the “intersection simplex”. This observation indicates that the general structure of the intersections remains unchanged, confirming that the upper bound on the number of vertices from concentric Δ_d and $t_d \nabla_d$ holds.

5.3.2 Structure of thermal catalysts

So far we have provided only a simple necessary condition required for a given transition $\mathbf{p} \xrightarrow{\beta} \mathbf{q}$ to be catalysable. This, however, does not provide any knowledge about the structure of the catalyst state $\mathbf{c}$ necessary for a successful transition. We may now collect both lines of reasoning presented in Section 5.2 and compress them in a form of a theorem.

Theorem 5.8 (Dimensionality criteria). *For two incomparable energy-incoherent states $\mathbf{p} \perp_{\beta} \mathbf{q}$, we define a maximal interval $\mathcal{L}$ as*

$$\mathcal{L} = \{x \in (0, 1) \mid f_{\mathbf{p}}(x) - f_{\mathbf{q}}(x) < 0\}, \quad (5.37)$$

and denote $m = \min(\mathcal{L})$ and $n = \max(\mathcal{L})$. Consider a catalyst $\mathbf{c} \in \Delta_k$ with Hamiltonian $H_{\mathbf{c}}$. The transformation can be catalysed by $\mathbf{c}$, that is, $\mathbf{p} \otimes \mathbf{c} \succ_{\beta} \mathbf{q} \otimes \mathbf{c}$, only if its dimension satisfies the required conditions:

$$k > k^* = \frac{\log b}{\log a} + 1, \quad (5.38)$$

with the coefficients a, b defined as

$$a = \min \left(\frac{s_1(\mathbf{p})}{f'_{\mathbf{p}}(m_-)}, \frac{f'_{\mathbf{p}}(n_+)}{s_d(\mathbf{p})} \right) > \max_{v \in (0, 1)} \left(\frac{f'_{\mathbf{c}}(v_-)}{f'_{\mathbf{c}}(v_+)} \right), \quad (5.39)$$

$$b = \max_{l \in \mathcal{L}} \left(\frac{f'_{\mathbf{p}}(l_-)}{f'_{\mathbf{p}}(l_+)} \right) < \frac{s_1(\mathbf{c})}{s_k(\mathbf{c})}, \quad (5.40)$$

where for brevity we use $f'(x) \equiv df(x)/dx$ and $f'(x_i) \equiv f'(x)|_{x=x_i}$. Furthermore, we denote the left and right derivatives as $f'(x_{\pm}) = \lim_{y \rightarrow x_{\pm}} f'(y)$.

As remarked in the heuristic approach of Section 5.2.1, one may in fact restrict the set $\mathcal{L}$ to a much smaller, discreet set of points corresponding to the elbows of the thermomajorization curve of $\mathbf{p}$.

Lemma 5.9 (Interval to point optimisation). *For a given state $\mathbf{p}$, we define an auxiliary set of indices $\mathcal{L}' \subset \{1, \dots, d\}$ as*

$$\mathcal{L}' = \left\{ l \in \{1, \dots, d\} \mid \Gamma_l = \sum_{i=1}^l \gamma_i^{\beta}, f_{\mathbf{p}}(\Gamma_l) - f_{\mathbf{q}}(\Gamma_l) < 0 \right\}, \quad (5.41)$$

where γ^β is assumed to be ordered according to the β -order of $\mathbf{p}$. Then we find that

$$\max_{l' \in \mathcal{L}'} \frac{s_{l'}(\mathbf{p})}{s_{l'+1}(\mathbf{p})} = \max_{l \in \mathcal{L}} \left(\frac{f_{\mathbf{p}}'(l_-)}{f_{\mathbf{p}}'(l_+)} \right) = b. \quad (5.42)$$

With the above, we may formulate two observations

Observation 5.10. *For arbitrary inverse temperature $\beta \geq 0$ and probability vectors $\mathbf{p}, \mathbf{q}$ of dimension $d = 2$ there are no catalysable transitions $\mathbf{p} \rightarrow \mathbf{q}$*

Observation 5.11. *For arbitrary inverse temperature $\beta \geq 0$ and probability vectors $\mathbf{p}, \mathbf{q}$ of dimension $d \geq 3$ there exist catalysable transitions $\mathbf{p} \rightarrow \mathbf{q}$*

The former observation can be either justified by direct consideration of Theorem 5.8 or by observing the fact that 2-dimensional states under finite temperature have the same number of possible elbow positions as states with $d = 3$ for $\beta = 0$.

The latter case of $d = 3$ and $\beta > 0$ is less trivial, but it is sufficient to give a single example of a pair of states for which $\mathbf{p} \not\leq \mathbf{q}$, $\mathbf{q} \not\leq \mathbf{p}$ and $\mathbf{p} \otimes \mathbf{c} < \mathbf{q} \otimes \mathbf{c}$. Consider a pair of $\mathbf{p} = (0.42, 0.51, 0.07)$ and $\mathbf{q} = (0.52, 0.13, 0.05)$, both with energy spectrum $E_0 = 0$, $E_1 = 1$ and $E_2 = 2$ and inverse temperature $\beta = 0.2$. Then, a 2-dimensional catalyst, described by a state $\mathbf{c} = (0.55, 0.45)$ with trivial Hamiltonian, $H_{\mathbf{c}} = 0$, catalyses the transformation.

Furthermore, we may further provide specific bounds on the simplest 2-level catalysts, starting with the simplest case where the catalyst's Hamiltonian is trivial.

Corollary 5.12 (Trivial qubit catalysts). *Consider a pair of incomparable states $\mathbf{p}, \mathbf{q} \in \Delta_d$ such that $\mathbf{q} \in \mathcal{C}_+(\mathbf{p})$. A qubit catalyst in a state $\mathbf{c} = (1 - t, t)$ with $t \leq 1/2$ and described by a trivial Hamiltonian $H_{\mathbf{c}} = 0$ can catalyse the transformation, $\mathbf{p} \otimes \mathbf{c} > \mathbf{q} \otimes \mathbf{c}$ only if*

$$\frac{1}{1+a} \leq t \leq \frac{1}{1+b} \quad (5.43)$$

with a, b defined as in Theorem 5.8.

Note that the above bounds depend implicitly on the underlying Gibbs state γ via the coefficients a, b , and as such are different from the ones provided in [232]. Similar bounds can be given also for the case if catalyst has nontrivial energy level structure.

Corollary 5.13 (Nontrivial qubit catalysts). *Consider a pair of incomparable states $\mathbf{p}, \mathbf{q} \in \Delta_d$ such that $\mathbf{q} \in \mathcal{C}_+(\mathbf{p})$ and a qubit catalyst in a state $\mathbf{c} = (1 - t, t)$ with Hamiltonian such that the Gibbs state is given by $\gamma_{\mathbf{c}} = (1 - \gamma_{\mathbf{c}}, \gamma_{\mathbf{c}})$. For $t \leq \gamma_{\mathbf{c}}$ it can catalyse the process only if*

$$\frac{\gamma_{\mathbf{c}}}{a(1 - \gamma_{\mathbf{c}}) + \gamma_{\mathbf{c}}} \leq t \leq \frac{\gamma_{\mathbf{c}}}{b(1 - \gamma_{\mathbf{c}}) + \gamma_{\mathbf{c}}}; \quad (5.44)$$

for $t \geq \gamma_c$ it can catalyse the transformation only if

$$\frac{1 - \gamma_c}{b\gamma_c + 1 - \gamma_c} \leq 1 - t \leq \frac{1 - \gamma_c}{a\gamma_c + 1 - \gamma_c}, \quad (5.45)$$

where in the above we take a, b from Theorem 5.8.

Note the symmetry upon exchange $t \leftrightarrow 1 - t$ and $\gamma_c \leftrightarrow 1 - \gamma_c$, rendering the entire allowable region of 2-dimensional catalyst with nontrivial energy levels symmetric around the corresponding Gibbs state.

5.4 Application to thermal entanglement generation

The following section is taken directly from [19] to maintain clarity of the text, with the proof of Proposition 5.14 adapted from the corresponding appendix and incorporated into the main body of the thesis for clarity.

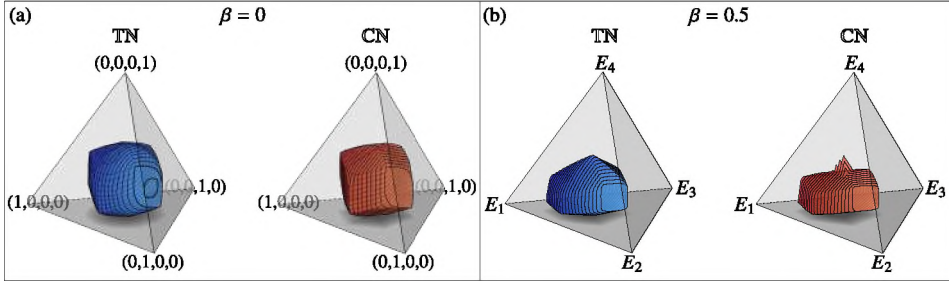


Figure 5.1: Thermally non-entangleable states with (and without) catalyst. Two-qubit energy-incoherent states which cannot become entangled under thermal operations are represented in the simplex spanned by sharp energy states (given explicitly in a)) and denoted by their corresponding energies E_i in b)) for two scenarios: without a catalyst (TN set, shown on the left in blue) and with a catalyst (CN set, shown on the right side in red), for (a) $\beta = 0$ and (b) $\beta = 0.5$. The apparent peak in CN for $\beta = 0.5$ results from the future of the state $\mathbf{p}^*$ in Proposition 5.14.

As a direct application of our results, we demonstrate the catalytic advantages of generating entanglement from separable bipartite states under thermal operations. Throughout this section, we will focus on a composite system consisting in two 2-level systems with identical energy levels $H = \text{diag}(0, \delta)$. For this reason, the total Hamiltonian of the system, $H_{AB} = H \otimes \mathbb{I} + \mathbb{I} \otimes H$ has a degenerate energy subspace corresponding to $E_{01} = E_{10} = \delta$, and without loss of generality we may set $\delta = 1$. Within this subspace any unitary operation remains Gibbs-preserving, which allows for generation of entanglement while simultaneously respecting the laws of thermodynamics.

Recently, the necessary and sufficient conditions for producing bipartite qubit entanglement from an initially separable states via thermal operations were addressed in Ref. [20]. States that cannot get entangled under thermal operations are said to

belong to the *thermally non-entangleable set*, mathematically denoted by $\mathbb{TN}$. Conversely, states that can get entangled under thermal operations lie in the *entangleable set* $\mathbb{TE}$. By using the results on the catalysable future region $\mathcal{C}_+$, we may now answer the natural question whether borrowing catalyst allow more states to get entangled. Defining the *catalytically non-entangleable set* $\mathbb{CN}$ as the set of states that cannot be entangled even with the aid of catalysis, the question can be reframed – *is the set $\mathbb{CN}$ smaller than $\mathbb{TN}$, or in terms of volumes, $\mathcal{V}(\mathbb{CN})/\mathcal{V}(\mathbb{TN}) \stackrel{?}{<} 1$ for a given inverse temperature $\beta \geq 0$?*

To begin with, we focus on the infinite temperature case $\beta = 0$. Remarkably, in this regime, the thermally non-entangleable set $\mathbb{TN}$ reaches its maximum volume and can be analytically constructed. In order to investigate the catalytically non-entangleable set $\mathbb{CN}$, we start by using Theorem 5.6, where we construct the catalysable future for two qubits and verify whether it lies within the thermally non-entangleable set. Next, by employing a numerical bisection-based algorithm inspired by vacuum-packaging, we generate an approximation of the boundary $\partial\mathbb{CN}$. This allows us to estimate the ratio of the volumes $\mathcal{V}(\mathbb{CN})/\mathcal{V}(\mathbb{TN}) \approx 0.88$ for $\beta = 0$ and $d = 4 = 2 \times 2$. In Fig. 5.1a, we show the thermally non-entangleable set without (left panel) and with (right panel) a catalyst.

Although the set $\mathbb{TN}$ remains convex as verified by numerical approximations, for $\beta > 0$ the set $\mathbb{CN}$ becomes non-convex (see Fig. 5.1b for an example with $\beta = 0.5$). In fact, by reverse-engineering the numerical approximations, we find that $\mathbb{CN}$ is a set sum of a certain convex set $\mathbb{CN}_0$ and a future of a certain distinguished subspace-thermalised state, $\mathbf{p}_*$. First, concerning the distinguished state, we have

Proposition 5.14 (Partially thermalised state & non-entangleability). *The future thermal cone $\mathcal{T}_+(\mathbf{p}^*)$ of the partially thermalised state*

$$\mathbf{p}^* = \frac{1}{4 + 2 \cosh(\beta)} (e^\beta, 1, 1, 2 + e^{-\beta}) \quad (5.46)$$

is fully contained in the catalytically non-entangleable set, $\mathcal{T}_+(\mathbf{p}^*) \subset \mathbb{CN}$.

Proof. Due to the energy-level structure the Gibbs state is given by $\gamma = (\gamma_1, \gamma_2, \gamma_2, \gamma_4)$. Furthermore, it follows that an arbitrary unitary operation in the subspace $\text{span}(|01\rangle, |10\rangle)$ is energy-preserving, and thus an admissible thermal operation (TO). Under these assumptions there exists a subset of states that can become entangled under TOs. In particular, for 2-qubit states with populations $\mathbf{p}$ a state can be entangled using energy-preserving unitaries if and only if $4p_1p_4 - (p_2 - p_3)^2 < 0$ [57, 58]. Furthermore, as shown in [19], a state $\mathbf{p}$ can be entangled by TOs if and only if the extreme point of its future thermal cone $\mathbf{p}^\pi \in \mathcal{T}_+(\mathbf{p})$ with β -order given by $\pi = (2134)$. We denote by $\mathbb{TN}$ the set of states $\mathbf{p}$ for which no point of the future cone $\mathcal{T}_+(\mathbf{p})$ can become entangled under energy-preserving unitaries; this set is called thermally non-entangleable set. Similarly, we define a set of catalytically nonentangleable set $\mathbb{CN}$ as set of states for which no state from either their future cone or the catalysable set $\mathcal{C}_+(\mathbf{p})$ can become entangled under energy-preserving unitaries.

In order to prove Proposition 5.14, we need to show that the boundary of set $\mathbb{CN}$ always contains a state thermalised within the subspace spanned by the three lowest energies, more generally referred to as a subspace-thermalised state, of the form

$$\partial\mathbb{CN} \ni \mathbf{p}^* = (1 - t')(0, 0, 0, 1) + \frac{t'}{Z_{123}}(\gamma_1, \gamma_2, \gamma_2, 0), \quad (5.47)$$

with $Z_{123} = \gamma_1 + 2\gamma_2$. This state by itself is clearly non-entangleable by using only energy-preserving unitaries, as it is proportional to identity on the subspace $\text{span}(|01\rangle, |10\rangle)$ and as such, does not admit generation of any coherences within the aforementioned subspace. However, its future thermal cone $\mathcal{T}_+(\mathbf{p}^*)$ may contain states that are entangleable.

In particular, one of the extreme points of $\mathcal{T}_+(\mathbf{p}^*)$ is a subspace-thermalised state of the form

$$\mathbf{p}^{**} = (1 - t)(0, 1, 0, 0) + \frac{t}{Z_{124}}(\gamma_1, 0, \gamma_2, \gamma_4), \quad (5.48)$$

with $Z_{123} = \gamma_1 + \gamma_2 + \gamma_4$. This admits the distinguished β -order of $\pi = (2, 1, 3, 4)$, and thus, by Theorem 2 from [20], if this is not an element of the entangleable set, $\mathbf{p}^{**} \in \mathbb{TN}$. It is easy to find t for which this state becomes non-entangleable by solving,

$$4 \frac{t^2}{Z_{124}^2} \gamma_1 \gamma_4 - \left(1 - t - \frac{t}{Z_{124}} \gamma_2\right)^2 = 0, \quad (5.49)$$

which is solved by setting

$$t = \frac{\gamma_1 + \gamma_2 + \gamma_4}{2\sqrt{\gamma_1}\sqrt{\gamma_4 + 1}}. \quad (5.50)$$

Now we would like to find a state $\mathbf{p}^*$ which tightly majorises the state $\mathbf{p}^{**}$, denoted as $\mathbf{p}^* \succcurlyeq \mathbf{p}^{**}$. It is achieved by setting $t' = \frac{Z_{123}}{Z_{124}}$. By reinstating $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = e^{-\beta}$, $\gamma_4 = e^{-2\beta}$, we retrieve Proposition 5.14.

According to the above derivation, we can say that $\mathcal{T}_+(\mathbf{p}^*) \subset \mathbb{TN}$, but also $\mathcal{T}_+(\mathbf{p}^*) \subset \mathbb{CN}$, as $\mathcal{C}_+(\mathbf{p}^*) = \emptyset$ and for all $\mathbf{q} < \mathbf{p}^*$ we find that $\mathcal{C}_+(\mathbf{q}) \subset \mathcal{T}_+(\mathbf{p}^*)$. However, by explicit calculation one can verify that for $\mathbf{q}' \in \mathcal{B}(\mathbf{p}^*, \epsilon) \cap \mathcal{T}_+(\mathbf{p}^*)$ and $\epsilon \leq \epsilon_*$, where $\mathcal{B}(\mathbf{p}^*, \epsilon)$ is a ball of radius ϵ centered at $\mathbf{p}^*$ and $\epsilon_* \ll 1$ a certain critical value, we find $\mathbf{q}' \in \mathbb{CE}$.

Furthermore, based on numerics we put forward the following

Conjecture 5.1 (Non-entangleable decomposition). *The catalytically non-entangleable set can be decomposed into*

$$\mathbb{CT} = \mathbb{CT}_0 \cup \mathcal{T}_+(\mathbf{p}^*), \quad (5.51)$$

where

$$\mathbb{CT}_0 = \mathbb{CT} \setminus [\mathcal{T}_+(\mathbf{p}^*) \setminus \mathcal{T}_+(\mathbf{p}^{**})], \quad (5.52)$$

with $\mathbf{p}^{**} = \frac{1}{4+2\cosh(\beta)}(e^\beta, 3, 1, e^{-\beta})$.

The conjecture is based on numerical approximations with high precision, and we believe it to hold true. Furthermore, it can be partially supported by noting that for a ball $\mathcal{B}(\mathbf{p}^*, \delta)$ of radius δ around the state $\mathbf{p}^*$ any state $\mathbf{q} \in [\mathcal{B}(\mathbf{p}^*, \delta) \setminus \mathcal{T}_+(\mathbf{p}^*)]$ does not belong to $\mathbb{CN}$ by simple arguments of either not belonging to $\mathbb{TN}$ or by the fact that $\mathcal{C}_+(\mathbf{q}) \setminus \mathbb{TN} \neq \emptyset$.

Using the introduced numerical approximations we derived the dependence of the volumes $\mathcal{V}(\mathbb{TN})$ and $\mathcal{V}(\mathbb{CN})$ for both thermal and catalytically non-entangleable sets and their ratio as a function of β , demonstrated in Fig 5.2.

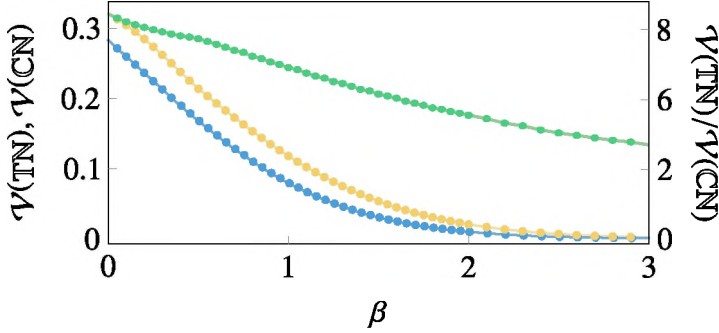


Figure 5.2: Thermally non-entangleable volume with(out) catalyst as a function of inverse temperature β . Numerical estimates for the non-entangleable volumes with catalyst [$\mathcal{V}(\mathbb{CN})$, blue line] and without catalyst [$\mathcal{V}(\mathbb{TN})$, orange line] and their ratio (green line), with the axis on the left for volumes and on the right for the ration. The character of the curves shows that, even though the volumes of both thermally and catalytically non-entangleable sets both go to zero together with β going to infinity, the ratio of the volumes also diminishes, pointing to an increasing catalytic advantage in entanglement generation.

5.5 Summary and Outlook

In this chapter, we have considered achievability of state transitions under thermal operations with help of strict catalysts. The proposed method utilises exclusively information coming from the thermomajorization curve corresponding to the initial and final state. We demonstrated a simple necessary (but not sufficient) condition on any transition to be catalysable, thus providing a simple tool for discarding the transitions which cannot benefit from the strong catalysts. Furthermore, provided bounds on both structure of the catalyst states and their dimensionality. Most importantly, the aforementioned bounds show that a) catalysis is impossible for principal systems of dimension $d = 2$ and for all temperatures, and b) for $d = 3$ one may find transitions, that are catalysable as soon as the inverse temperature β is positive. As an exemplary application we considered entanglement generation under thermal operations and catalytic thermal operations, showing that there exists a subset of transitions, which may yield non-zero entanglement using methods from [20] extended to catalytic thermal operations as opposed to zero entanglement for thermal operations alone.

Our results suggest numerous extensions, such as studying catalysis within thermal operations restricted to 2-level operations, known as elementary thermal operations.

The characterization of catalysable past and future depends on tangent vectors that tightly thermomajorise a given state, and extreme points achievable under elementary thermal operations are known to be β -swaps that tightly thermomajorise the initial distribution. This opens the possibility of adapting our technical findings to this context. Additionally, relaxing the condition of an unperturbed catalyst to allow for minor errors, quantified by trace distance, presents another avenue for exploration.

Finally, one might be inclined to propose that our findings, building on the prior work of Grabowecky and Gour [232], could lead to more comprehensive statements. Since majorisation and thermomajorisation curves are complete, ordered sets of monotones, we could conjecture the following: if the first and last monotones of state X are greater than those of state Y , then the transformation is catalysable. This would offer a way to determine the catalysable past and future for broader resource theories, in which the complete set of resource monotones comes with total order, or at least lattice structure. However, this exploration lies beyond the scope of [19], on which this chapter is based.



6 On channel designs [25]

Sampling directly from a continuous distribution over any given space of states enough times in order to obtain an estimate of average may be costly in terms of both computation and memory, depending on the desired level of error. However, the problem may be simplified by choosing a finite set of points in a particular way and then averaging the desired quantity using this set. In this way one can ensure that the approximation is accurate up to a polynomial of a certain order.

Such sets of points, closely related to the quadratures known from numerical integration, are referred to as designs. In the context of quantum information, two most widely celebrated notions are complex projective t -designs [236] and unitary t -designs [237], used for averaging over the space of pure quantum states and over the unitary operations, respectively. Since their introduction they have been used for many applications, including both state [238] and process tomography [239], derandomization of probabilistic constructions [240], entanglement detection [241] and others.

Recently, several similar concepts have been introduced, such as quantum conic designs [242], mixed state designs [5], continuous variable designs [243] or projective toric designs [244]; however, to the best of the author's knowledge, no similar concept has been developed for the case of quantum channels.

In what follows we propose a method of filling the gap by introducing a notion of channel designs. We show that such designs may be induced by taking partial trace of existing unitary designs, thus providing an example of a novel notion of pushforward designs. Furthermore, by relaxing the requirement of the induced measure on the space of quantum channels to be uniform, we introduce a whole family of designs corresponding to different measures depending on the dimensionality of the environment. Based on the above, we propose a related method for estimating the effective dimensionality of the environment. The concept is demonstrated by evaluating the effective dimensionality for a selected pair of qubits in **IBM Kyoto** quantum computer, thus demonstrating viability of the concept.

6.1 Preliminaries

6.1.1 General concept of a t -design and its specific realizations

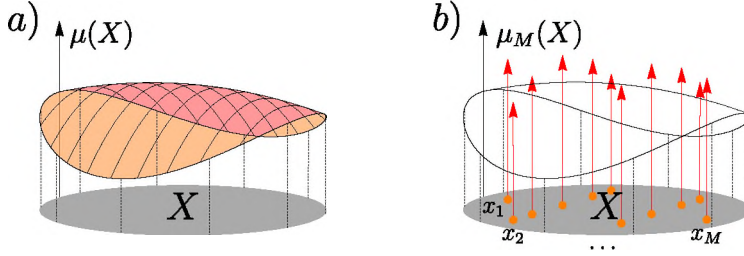


Figure 6.1: Design as approximation of a measure: Consider a space X and a probability measure $\mu(X)$ defined on this set [panel a)]. A t -design is a set of M points $\{x_i\}_{i=1}^M$, which define a discrete measure $\mu_M(X)$ mimicking the continuous measure $\mu(X)$, so that an average of a polynomial functions up to certain degree t over $\mu_M(X)$ is equal to that over $\mu(X)$.

The central idea of this chapter is what we call a t -design, and can be summarised qualitatively as answering the following question: “How to approximate a continuous distribution over a space of states using a finite set of points taken from it?” – a concept which is represented visually in Fig. 6.1. More formally, we consider a set $S = \{s_i \in \mathcal{S}\}_{i=1}^N$ of states from the space $\mathcal{S}$ endowed with a certain measure μ , which in turn implies a certain distribution of states. The definition that works for arbitrary space of states is the multi-copy approach. The set S is called a t -design if the averages over t copies of states are equal,

$$\frac{1}{|S|} \sum_{s_i \in S} s_i^{\otimes t} = \int_S s^{\otimes t} d\mu(s). \quad (6.1)$$

If the states $s \in \mathcal{S}$ can be depicted as a set of components, $s \mapsto \mathbf{v}(s) = (v_1(s), \dots, v_D(s)) \in \mathbb{C}^D$, the above component-agnostic approach is equivalent to considering an agreement between averages of arbitrary balanced polynomial function $f_{t,t}(s)$ in the components $v_i(s)$ and their conjugates $v_i^*(s)$,

$$\frac{1}{|S|} \sum_{s_i \in S} f_{t,t}(s_i) = \int_S f_{t,t}(s) d\mu(s). \quad (6.2)$$

The equivalence of the two definitions is straightforward, as the tensor powers $(\mathbf{v}(s) \otimes \mathbf{v}^*(s))^{\otimes t}$ contains all the balanced monomials of degree (t, t) , which constitute the natural basis for balanced polynomials $f_{t,t}$. Both definitions can be extended to what is called *weighted t -design*, which is defined by replacing the LHS average with a weighted average. A more mathematically rigorous treatment of designs on generalized spaces has been taken by Levehnstein in several works [245–247], to which I refer any reader interested in learning further details.

We can summarise the entire concept by saying that a **t -design is an approximation of a continuous distribution using a finite number of points, reproducing it up to t -copies or degree- t polynomials.** With this in hand, we may move on to present a number of specific applications of this definition relevant to quantum information, starting with a small warm-up.

6.1.1.1 Interval designs, spherical designs and why should we care

We begin our stroll through the designs by setting our space of states to be a finite real interval, $S = [-1, 1] \subset \mathbb{R}$ with a flat measure.

Definition 6.1 (Interval t -design). *Consider a set $X = \{x_i \in [0, 1] \subset \mathbb{R}\}$ and a polynomial function $f_t(x) = \sum_{i=0}^t a_i x^i$ with $a_i \in \mathbb{R}$. We call such a set an interval t -design if the average over the interval $[0, 1]$ and over the set X agree,*

$$\frac{1}{|X|} \sum_{x_i \in X} f_t(x_i) = \int_{-1}^1 f_t(x) dx. \quad (6.3)$$

Similar task should be recognisable to anyone who performed numerical integration as the problem of finding a set of points which minimize the errors associated with the approximation. In particular, one solution to this problem to it can be found in the form of Gaussian quadratures, sets of points together with associated weights which allow to approximate any integral up to a polynomial of degree $2n - 1$, first derived by Carl Friedrich Gauss in 1815 [248], later formulated in their modern form by Jacobi in 1826 [249].

For our purpose, however, we request all the points in the quadrature to be equally weighted, as opposed to what happens in the aforementioned case, and we refer to such sets X as *interval t -designs*. It can be easily shown that the minimal arrangement for $t = 2n$ is given by $2n$ points and that the same arrangement works for $t = 2n + 1$ as well [5]*.

Another easily understood concept is that of a *spherical t -design*, which is an arrangement of the points on the sphere, and for our purposes we will limit ourselves to a 2-dimensional sphere S^2 which, incidentally, is isomorphic to the complex projective space $\mathbb{CP}^1$. Such a design is defined, formally, as a set R of N points on the sphere such that the average over the points is equivalent to the average over the entire sphere, which is yet another instantiation of the definition (6.2),

Definition 6.2 (Spherical t -design). *Consider a set of points $R = \{\mathbf{r}_i \in S^2\}$ from a 2-dimensional sphere and a polynomial function $f_t(\mathbf{r})$ defined on the sphere. We call such a set of points a spherical t design if the average over the set R of the function f_t agrees with*

* Curious as it stands, I am not aware of any other reference, yet I would be deeply interested in knowing of any prior work to derive these equal-weight arrangements.

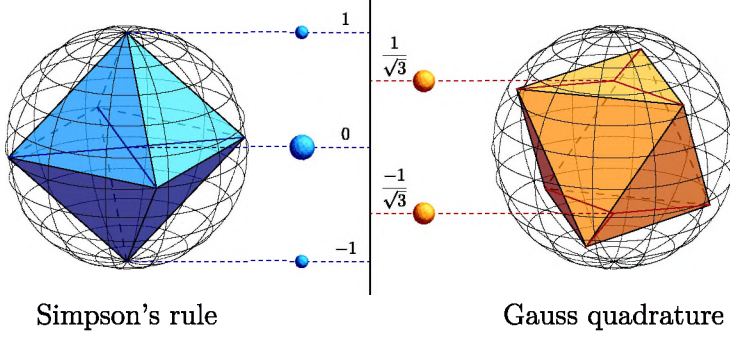


Figure 6.2: Interval designs as projections of spherical designs: Let us consider an octahedron, which defines a spherical 3-design and take two different orientations of it. By projecting the vertices on the Z axis we find two of well-known rules for numerical integration – the Simpson rule and 2-point Gauss quadrature.

the average taken over the Haar measure invariant with respect to the representation of the orthogonal group $O(3)$ on the sphere,

$$\frac{1}{|R|} \sum_{\mathbf{r}_i \in R} f_t(\mathbf{r}_i) = \int_{S^2} f_t(\mathbf{r}) d\Omega \quad (6.4)$$

An extensive online catalogue of spherical designs, together with their higher-dimensional counterparts, has been provided online by Hardin and Sloane [250].

Spherical and interval designs are invoked in the same subsection, because they are closely related by a simple relation of projection

Observation 6.1. Consider a spherical t -design $R = \{\mathbf{r}_i\}$. For any axis going through the center of the sphere, represented by a unit vector $\mathbf{n}$, the set of projections $\mathbf{n} \cdot R = \{\mathbf{n} \cdot \mathbf{r}_i\}$ defines an interval t -design.

Proof. It is relatively simple observation stemming from a fact that flat distribution over the sphere projected on an arbitrary axis $\mathbf{n}$ yields a flat distribution on an interval. Due to the linearity of such a projection, the t -design property of the set R is then reproduced after projection, making $\mathbf{n} \cdot R$ a t -design as well.

This claim is best understood by looking at it – in Fig. 6.2 we present two famous rules used for numerical integration – Simpson’s rule and 2-point Gauss quadrature – and their origin from two different projections of an octahedron, which is a spherical 3-design.

With this warm-up we may move on to transplanting the notion of designs onto the quantum ground, starting with the designs of states.

6.1.1.2 Complex projective designs, simplex designs and stochastic designs

The stage on which the quantum theory plays out, as long as pure states are concerned, is the complex projective space $\mathbb{CP}^d \equiv \mathcal{H}_d$. The complex projective t -designs have been first considered by Neumeier [251] with later extensive insights into their properties developed by Hoggar in a series of papers [252–255]. Let us proceed to provide a definition

Definition 6.3 (Complex projective t -design). *Consider a set $V = \{|\psi_i\rangle \in \mathcal{H}_d\}_{i=1}^m$ of pure states of dimension d and a balanced polynomial function $f_{t,t}(|\psi\rangle)$ in the components of the state $c_j = \langle j|\psi\rangle$ and their conjugates. We call such a set a complex projective t if and only if for any $f_{t,t}$ the average over the whole space of states with respect to the Haar measure is equal to the average over the set V ,*

$$\int_{\mathcal{H}_d} f_{t,t}(|\psi\rangle) = \frac{1}{m} \sum_{i=1}^m f_{t,t}(|\psi_i\rangle). \quad (6.5)$$

Alternatively, one can write that the function we consider is a polynomial function $g_t(|\psi\rangle\langle\psi|)$ in the elements of the projection operator $|\psi\rangle\langle\psi|$. Another interesting reformulation is in this case the statement concerning equality of t -copy averages. In the particular case of $t = 2$,

$$\frac{1}{m} \sum_{i=1}^m |\psi_i\rangle\langle\psi_i|^{\otimes t} = \int_{\mathcal{H}_d} |\psi\rangle\langle\psi|^{\otimes t} = \binom{d+t-1}{t}^{-1} \Pi_{sym}^{(t)}, \quad (6.6)$$

where $\Pi_{sym}^{(t)}$ is the projection operator onto the symmetric subspace in the t -copy space $\mathcal{H}_d^{\otimes t}$.

The canonic examples of $\mathbb{CP}$ 2-designs include symmetric informationally complete (SIC) POVMs, defined as sets of d^2 vectors $|\psi_i\rangle \in \mathcal{H}_d$ satisfying equality of overlaps condition

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{d+1}. \quad (6.7)$$

For the simplest case of $N = 2$, this construction involves four quantum states corresponding to the vertices of a regular tetrahedron within the Bloch sphere. Zauner hypothesized [236, 256] that such ensembles exist for any dimension d . Analytical methods have been developed [257] for all dimensions up to $d = 53$, and numerical methods have found solutions [258–260] for dimensions up to $d = 193$. Additionally, some solutions have been identified for specific dimensions [261–264] reaching up to $d = 39\,604$. Moreover, recent results on relation between SIC-POVMs and quadratic fields point to the fact that there may exist only a finite number of inequivalent SIC-POVMs per dimension [265].

The second widely known example is that of full sets of mutually unbiased bases (MUBs). Two orthonormal bases B_1, B_2 defined by operators of dimension d , are said

to be unbiased if the overlap between any pair of states taken from different bases is constant; in other words, the entries of the product matrix $B_1 B_2^\dagger$ are given by

$$|(B_1 B_2^\dagger)_{ij}|^2 = \frac{1}{d}. \quad (6.8)$$

The above condition is equivalent to saying that the matrix $B_1 B_2^\dagger$ is a complex Hadamard matrix.

It is known that in dimension d there exist no more than $(d + 1)$ MUBs and that such a full set provides a 2-design. Construction of MUBs is known for prime and prime power dimensions [266]. The smallest composite dimension $d = 6 = 2 \cdot 3$ is at the same time the first unresolved example, with no more than 3 MUBs found, and multiple no-go theorems for extending certain sets with a fourth basis already in place.

One of the most interesting and useful properties of $\mathbb{C}P$ t -designs is their relation to Welch bounds, which are satisfied by any set of vectors from $\mathbb{C}P^d$ and first proved by Welch in 1974 [267],

Theorem 6.2. *Consider a set of states $V = \{|\psi_i\rangle \in \mathcal{H}_d\}_{i=1}^m$. The following bound, called Welch bound*

$$\sum_{i,j=1}^m |\langle \psi_i | \psi_j \rangle|^2 \geq m^2 \binom{d+t-1}{t}^{-1} \quad (6.9)$$

is satisfied for any set V and degree $t \in \mathbb{N}$ and is saturated if and only if V is a t -design.

This property is particularly useful when attempting to prove that a given set of states is a t -design or for numeric generation of approximate t -designs. It is, for example, relatively straightforward using Welch bounds that SIC-POVMs and full sets of MUBs are indeed 2-designs. For a more in-depth considerations of this and related bounds, I refer the reader to [268].

Complex projective t -designs for $t = 2$ have found their use, among other applications mentioned in the beginning of this section, in linear state tomography and have been proved optimal for this task [238]. The reconstruction formula for a density matrix ρ takes a particularly simple, elegant form

$$\rho = \frac{d(d+1)}{m} \sum_{i=1}^m p_i |\psi_i\rangle\langle\psi_i| - \mathbb{I}, \quad (6.10)$$

where $p_i = \langle \psi_i | \rho | \psi_i \rangle$ is the probability of measuring ρ in the state $|\psi_i\rangle$. A particularly elegant proof of this fact, presented in terms of operator frames and their relation to complex projective 2-designs, has been presented in [238]. Below, for completeness, we give a proof based on the projector on 2-copy symmetric subspace is given by

$$\Pi_{sym}^{(2)} = \sum_{a,b} (|ab\rangle\langle ab| + |ab\rangle\langle ba|).$$

We may act with this projection operator on a state $\rho \otimes \mathbb{I}$ and trace out the first subsystem to find that

$$\mathrm{Tr}_A \left((\rho \otimes \mathbb{I}) \Pi_{sym}^{(2)} \right) = \rho + \mathbb{I} \mathrm{Tr}(\rho) = \rho + \mathbb{I}. \quad (6.11)$$

On the other hand, we have the sum over the design,

$$\sum_i \mathrm{Tr}_A \left(\rho |\psi_i\rangle\langle\psi_i| \otimes |\psi_i\rangle\langle\psi_i| \right) = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \quad (6.12)$$

Putting the two parts together we retrieve the reconstruction formula (6.10).

Final bit of this subsection is the notion of designs in the set of d -point probability distributions, denoted as Δ_d .

Definition 6.4 (Simplex t -design). *Consider a set $P = \{\mathbf{p}_i \in \Delta_d\}$ of probability distribution and a polynomial function $f_t(\mathbf{p})$ of order t in the elements of the probability distribution. We call such a set a simplex t -design if the averages*

$$\frac{1}{N} \sum_{\mathbf{p}_i \in P} f_t(\mathbf{p}_i) = \int_{\Delta_d} f_t(\mathbf{p}) d\mathbf{p} \quad (6.13)$$

are equal, with the integral over Δ_d taken with a flat measure.

We introduce this notion because of a curious connection between the simplex t -designs and complex projective t -designs with the same dimension d .

Observation 6.3. *Consider a complex projective t -design $S = \{|\psi_i\rangle \in \mathcal{H}_d\}$ and an orthonormal basis represented by a unitary matrix V . The set of probability distributions comprised of projections of the states onto a selected basis $\mathrm{proj}_V(S) = \{\mathbf{p}_i = \mathrm{diag}(V |\psi_i\rangle\langle\psi_i| V^\dagger)\}$ provides a t -design in the simplex Δ_d .*

Proof. It follows by simple observation that the measure on the probability simplex Δ_d induced by projection from the Haar measure on $\mathbb{C}P^d$ is the flat measure.

The above is the first, somewhat trivial, example of induced measures and the way they relate designs from different space with one another. We will return to this notion later on, but first we will consider one more fundamental notion connected to quantum operations.

Let us shortly recall the definition of a *stochastic matrix*. We call matrix $S \in \mathbb{R}_+^{d \times d}$ stochastic if all its entries are positive and each column is normalized to identity, $\sum_i S_{ij} = 1$. Since all of the columns of such a matrix are bound by independent conditions, one may generate random stochastic matrices by generating a set of d independent probability vectors p_i , $i = 1, \dots, d$ and arranging them column-wise to form a matrix S . Combining this and the induction of flat measure over a simplex Δ_d from the complex projective space $\mathbb{C}P^d$, we have the following statement.

Observation 6.4. Consider a complex projective t -design $V = \{|\psi_i\rangle \in \mathcal{H}^d\}_{i=1}^M$. Its d -fold Cartesian product $V^{\times d}$ provides via decoherence a t -design over the set of stochastic matrices. Moreover, note that decoherence for each copy may be taken with respect to a different basis $U \in \mathcal{U}(d)$.

6.1.1.3 Unitary designs

Since the unitary group $\mathcal{U}(d)$ is the set of natural operations governing evolution of pure states of d -dimensional quantum systems, it is natural to consider structures allowing effective approximation of this group. This, again, can be handled by introducing a concept of designs.

Definition 6.5 (Unitary t -design). Consider a set $\mathbb{U} = \{U_i \in \mathcal{U}(d)\}$ and a balanced polynomial function $f_{t,t}(U)$ in the components of U and its conjugate U^* . We call the set $\mathbb{U}$ a unitary t -design if the average of $f_{t,t}$ over the set is equal to the average over the entire group $\mathcal{U}_d$ with the natural Haar measure,

$$\frac{1}{|\mathbb{U}|} \sum_{U_i \in \mathbb{U}} f_{t,t}(U) = \int_{\mathcal{U}(d)} f_{t,t}(U) dU. \quad (6.14)$$

This is the natural definition, and in this case it is worth rewriting it in the t -copy form as well,

$$\frac{1}{|\mathbb{U}|} \sum_{U_i \in \mathbb{U}} (U_i \otimes U_i^\dagger)^{\otimes t} = \int_{\mathcal{U}(d)} (U \otimes U^\dagger)^{\otimes t} dU. \quad (6.15)$$

In this formulation, it is clear that the problem is equivalent to finding the I matrix known from the Weingarten calculus and solutions can be found analytically for $1 \leq t \leq d$ and have to be calculated numerically for $t > d$ [269].

In particular, it is interesting to note that one can equivalently define the particular $t = 2$ case in terms of so called twirling supermap Λ_X with respect to a subset X of unitary operations, which maps channels into channels based on the following recipe

$$\Lambda_X[\Phi](\rho) = \frac{1}{|X|} \sum_{U_i \in X} U_i^\dagger \Phi(U_i \rho U_i^\dagger) U_i. \quad (6.16)$$

Using this operation, one can say that X is a unitary 2-design if and only if the following equality

$$\Lambda_X[\Phi] = \Lambda_{\mathcal{U}}[\Phi] \quad (6.17)$$

holds for every channel Φ . In particular, it works for a channel defined by two operators

A and B ,

$$\frac{1}{|\mathbb{U}|} \sum_{U_i \in \mathbb{U}} U_i^\dagger A U_i \rho U_i^\dagger B U_i = \int_{\mathbb{U}} U^\dagger A U \rho U^\dagger B U \, dU \quad (6.18)$$

$$= \frac{\text{Tr}(AB) \text{Tr}(\rho)}{d^2} + \left(\frac{d \text{Tr}(A) \text{Tr}(B) - \text{Tr}(AB)}{d(d^2 - 1)} \right) \times \\ \times \left(\rho - \text{Tr}(\rho) \frac{\mathbb{I}}{d} \right) \quad (6.19)$$

with the RHS evaluated explicitly using group-theoretic approach in [270].

A canonical example of unitary 3-design is the Clifford group. In order to define it let us first consider Pauli group on n qubits, given by operators of the form

$$\mathcal{P}_n \equiv \left\{ P = i^l \bigotimes_{a=1}^n \sigma_x^{j_a} \sigma_z^{k_a} \right\}, \quad (6.20)$$

with $l \in \{0, 1, 2, 3\}$, $j_a, k_a \in \{0, 1\}$. It can be generated from Pauli gates σ_x and σ_z acting locally on each qubit. The number of elements in the group is straightforward to calculate, equal to $|\mathcal{P}_n| = 4^{n+1}$. With this in hand, we may define the Clifford group on n qubits as the group of unitary operations which map Pauli group onto itself under adjoint action,

$$\mathcal{C}_n = \{ C : C \in SU(2^n), \forall P \in \mathcal{P}_n C P C^\dagger \in \mathcal{P}_n \}. \quad (6.21)$$

Clifford group is generated by local Hadamard operations,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (6.22)$$

local phase-shifts $S = \text{diag}(1, I)$ and all pairwise CNOT _{ij} operators. The order of Clifford group grows far more quickly than that of the underlying Pauli group [271],

$$|\mathcal{C}_n| = 2^{n^2+2n} \prod_{j=1}^n (4^j - 1). \quad (6.23)$$

In the table below we provide comparison of sizes for small n

Table 6.1: Orders of the Pauli group $\mathcal{P}_n$ and Clifford group $\mathcal{C}_n$ for small numbers of qubit n . Notice rapid growth of the Clifford group, compared to the Pauli group [271].

n	1	2	3	4	5
$ \mathcal{P}_n $	16	64	256	1024	4096
$ \mathcal{C}_n $	24	11 520	92 897 280	12 128 668 876 800	$\sim 2.54 \cdot 10^{19}$

An interesting thing here is to consider $A = B = |\psi\rangle\langle\psi|$ as a projector on an arbitrary

pure state $|\psi\rangle \in \mathcal{H}_d$. By using this we may show that

$$\frac{1}{|X|} \sum_{U_i \in X} U_i^\dagger |\psi\rangle\langle\psi| U_i \rho U_i^\dagger |\psi\rangle\langle\psi| U_i = \frac{1}{|X|} \sum_{U_i \in X} \left(\langle\psi| U_i \rho U_i^\dagger |\psi\rangle \right) \left(U_i |\psi\rangle\langle\psi| U_i^\dagger \right) \quad (6.24)$$

$$= \frac{\mathbb{I} \operatorname{Tr}(\rho)}{d^2} + \frac{1}{d(d+1)} \left(\rho - \frac{\mathbb{I} \operatorname{Tr}(\rho)}{d} \right) \quad (6.25)$$

$$= \frac{1}{d(d+1)} [\rho + \mathbb{I} \operatorname{Tr}(\rho)] \quad (6.26)$$

and by simple transformation, one arrives at the reconstruction formula

$$\rho = \frac{d(d+1)}{|X|} \sum_{i=1}^{|X|} p_i |\psi_i\rangle\langle\psi_i| - \mathbb{I} \operatorname{Tr}(\rho) \quad (6.27)$$

where we denote $|\psi_i\rangle = U_i |\psi\rangle$ and $p_i = \langle\psi_i|\rho|\psi_i\rangle$. This formula coincides with the reconstruction formula (6.10) for the $\mathbb{CP}_d$ 2-design. This points to a general observation, already known in the literature [237, 272, 273].

Observation 6.5. Consider a unitary t -design $X = \{U_i \in \mathcal{U}(d)\}$ comprised of unitary matrices of dimension d , and an arbitrary pure state $|\psi\rangle \in \mathcal{H}_d$. Then, the set $X|\psi\rangle = \{U_i |\psi\rangle\}$ is a complex projective t -design.

The proof proceeds straightforwardly straightforward by demonstrating saturation of the Welch bound for the given t . This, in fact, to a bound for unitary t -designs of a given t , equivalent to Welch bound, which to the best of my knowledge does not appear so far in the literature.

Corollary 6.6. Consider a set of unitary matrices $X = \{U_i \in \mathcal{U}(d)\}$. The following bound

$$\sum_{a=1}^d \sum_{U_i, U_j \in X} |\langle a|U_i^\dagger U_j|a\rangle|^{2t} = \sum_{a=1}^d \sum_{U_i, U_j \in X} \left| \sum_{b=1}^d \overline{(U_i)_{ab}} (U_j)_{ba} \right|^{2t} \geq d|X|^2 \binom{d+t-1}{t} \quad (6.28)$$

is satisfied by any set X and is saturated if and only if X is a unitary t -design.

This, in fact, provides an amenable way to generate, numerically, unitary designs and for checking whether a given set of unitary operations indeed provides a unitary t -design.

Indeed, the induction of $\mathbb{CP}_d$ t -designs from unitary t -designs leads us to yet another induced structure – what is called *mixed states designs*.

6.1.1.4 Mixed states designs

Consider the space $B(\mathcal{H}_d)$ of mixed quantum states of dimension d , which arises naturally in the setting of open quantum systems. Thus, it is natural to consider notion of designs which allow averaging over such spaces [5].

Definition 6.6 (Mixed state t -design). *Consider a set $X = \{\rho_i \in \Omega_d\}$ of density matrices of dimension d and a polynomial $f_t(\rho)$ of degree t in the elements of ρ . We call the set X a mixed state t -design if the average over X for every function f_t is equal to its average over the full state of density matrices taken with flat measure,*

$$\frac{1}{|X|} \sum_{\rho \in X} f_t(\rho_i) = \int_{B(\mathcal{H}_d)} f_t(\rho) d\rho. \quad (6.29)$$

We introduce this concept as it provides another example of induced designs implied by induced measures, in this by partial trace from a larger Hilbert space.

Observation 6.7 ([274]). *The Haar measure with respect to $\mathcal{U}(d)$ on the Hilbert space $\mathcal{H}_d^{\otimes 2}$ induces, by partial trace, a flat measure on the space of density matrices Ω_d .*

Corollary 6.8 ([5]). *A complex projective t -design $X = \{|\psi_i\rangle\}$ on $\mathcal{H}_d^{\otimes 2}$ induces, by partial trace, a mixed state t -design $\text{Tr}_A X = \{\text{Tr}_A |\psi_i\rangle\langle\psi_i|\}$.*

A visual depiction of the above induction is given in Fig. 6.3[†].

The above corollary does not necessitate that all mixed state designs can be produced as partial traces of complex projective designs. Although no formula akin to Welch bound is known, a t -copy average over the space of mixed states has been calculated in a closed form [5]:

$$\int_{B(\mathcal{H}_d)} \rho^{\otimes t} d\rho = \frac{\sum_{\sigma \in S_t} \text{Tr}(O_\sigma) O_\sigma}{\sum_{\sigma \in S_t} \text{Tr}(O_\sigma)^2} \equiv \omega_{d,t} \quad (6.30)$$

where O_σ is the matrix representation of a permutation σ of t subsystems of dimension d each. Thus, in order to verify whether a given set X is a mixed state t -design, it is sufficient to show that

$$\left\| \omega_{d,t} - \frac{1}{|X|} \sum_{\rho \in X} \rho_i^{\otimes t} \right\| = 0. \quad (6.31)$$

Although more computationally costly, it allows for numerical generation of mixed state designs.

This short introduction completes our stroll through different quantum-related designs, which allow for averaging over spaces of pure states, unitary operations and

[†] Disclaimer: The upper right panel, depicting bipartite space Ω_{AB} of mixed states is artists depiction and should not be taken as rigorous representation.

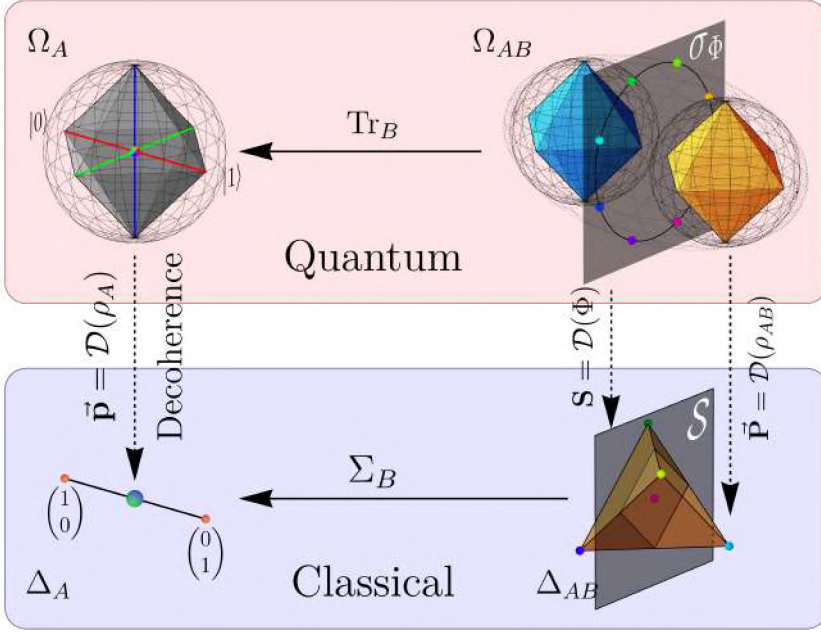


Figure 6.3: Relations between designs by induction: Starting from a complex-projective t -design on a bipartite space $\mathcal{H}_{AB}$ (right top) one may induce by partial trace Tr_B a mixed-state t -design in a unipartite space Ω_A (left top) or a simplex t -design by decoherence $\mathcal{D}$ to the bipartite probability distribution space Δ_{AB} (bottom right). Both yield t -design in a unipartite probability space Δ_A ; the former does so by decoherence $\mathcal{D}$, while the latter by summing over the subsystem B to obtain the marginal distribution. Furthermore, given a design on the subspace of quantum channels (dark plane in upper right denoted by σ_Φ) one induces, also by decoherence, a design over the set of stochastic matrices (dark plane in lower right denoted by S).

others. As we will extend this family to the quantum channels, we need to consider the problem of measures on the space of quantum channels.

6.1.2 k -measures on the space of quantum channels

As has been shown in Section 1.3, the most general quantum operations $\Phi : \Omega_d \mapsto \Omega_d$ are described by completely positive trace preserving (CPTP) maps. They can be described in three equivalent manners:

1. By a set of d_E Kraus operators K_i , with the action defined as

$$\Phi(\rho) = \sum_{i=1}^{d_E} K_i \rho K_i^\dagger \quad (6.32)$$

with the identity resolution condition $\sum_i K_i^\dagger K_i = \mathbb{I}$ ensuring the trace preservation condition.

2. By a unitary operation $U \in \mathcal{U}(d \times d_E)$ with the action of the channel defined by

$$\Phi(\rho) = \text{Tr}_E [U(\rho \otimes |1\rangle\langle 1|)U^\dagger]. \quad (6.33)$$

3. By the corresponding Choi-Jamiołkowski state σ_Φ defined as

$$\sigma_\Phi = (\Phi \otimes \mathbb{I})[|\psi_+\rangle\langle\psi_+|] \quad (6.34)$$

where $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ is the standard maximally entangled state.

From this follow three natural ways for generating random quantum channels – by random Kraus operators, random unitary operations and random Choi-Jamiołkowski states [275]. In order to understand the all the methods, we first need to shortly define a handful of ensembles of random matrices.

Definition 6.7 (Ginibre ensembles). *A real Ginibre matrix $G_{\mathbb{R}} \in \mathbb{R}^{d_1 \times d_2}$ that each element is taken from independent normal distribution $\mathcal{N}(1, 0)$ of unit variance and zero mean. Complex Ginibre matrix $G_{\mathbb{C}} \equiv G = (G'_{\mathbb{R}} + iG''_{\mathbb{R}})/\sqrt{2}$.*

Definition 6.8 (Gaussian Unitary ensemble). *Gaussian Unitary ensemble (GUE) of $d \times d$ Hermitian matrices, which is invariant under unitary operations, contains matrices of the form $H = (G + G^\dagger)/2$, where G is a matrix taken from complex Ginibre ensemble.*

Definition 6.9 (Wishart ensemble). *Complex Wishart ensemble with parameters (d, s) consists of matrices of the form $W = GG^\dagger$, where G is a rectangular $d \times s$ matrix taken from the complex Ginibre ensemble.*

Note 6.1. *Wishart ensemble can be extended to allow for real parameter $s \in \{1, \dots, d^2 - 1\} \cup [d^2, \infty) \equiv \mathcal{S}$ [276].*

Definition 6.10 (Circular Unitary ensemble). *Circular Unitary ensemble (CUE) consists of unitary matrices $U \in \mathcal{U}(d)$ distributed according to the Haar measure.*

Equipped with the above ingredients, we are ready to provide three recipes for generating random quantum channels, corresponding to the three aforementioned (equivalent) definitions, taken from [275] and adapted for channels that preserve dimensionality of the system, ie. $\Phi : \mathcal{H}_d \mapsto \mathcal{H}_d$.

Construction 6.1 (Random Quantum Channels via Kraus operators). *Let $s \geq 1$ be an arbitrary natural number. We define $\mu_{d,s}^{\text{Kraus}}$ to be the probability measure of the random quantum channel Φ defined in the following way:*

1. Take s random matrices $G_1, \dots, G_s$ of size $d \times d$, drawn independently from the complex Ginibre ensemble.
2. Compute the matrix $H = \sum_{i=1}^s G_i^\dagger G_i$.
3. Define Kraus operators as $K_i = G_i H^{-1/2}$, $i \in \{1, \dots, s\}$.

Construction 6.2 (Random Quantum Channels via Choi states). *Let $s \in \mathcal{S}$ be a real number. We define $\mu_{d,s}^{\text{Choi}}$ to be the probability measure of the random quantum channel Φ defined in the following way:*

1. *Take a random complex Wishart matrix W with parameters (d^2, s) ;*
2. *Find the positive semidefinite matrix defined by partial trace $H \equiv \text{Tr}_B(W)$;*
3. *Construct the sampled Choi-Jamiołkowski state as*

$$\sigma_\Phi \equiv \left(H^{-1/2} \otimes \mathbb{I} \right) W \left(H^{-1/2} \otimes \mathbb{I} \right). \quad (6.35)$$

Construction 6.3 (Random Quantum Channels via environmental form). *Let s be an integer and $s \geq 1$. We define $\mu_{d,s}^{\text{Stinespring}}$ to be the probability measure of the random quantum channel Φ , defined as follows:*

1. *Consider a random unitary matrix $U \in \mathcal{U}(Md)$, acting on the principal system and the environment E of dimension M .*
2. *The channel Φ is defined as*

$$\Phi(\rho) = \text{Tr}_E \left[U (\rho \otimes |0\rangle\langle 0|_E) U^\dagger \right]. \quad (6.36)$$

For Constructions 6.2 and 6.1 one needs to take note of the zero-measure sets, in which the matrices H are non-invertible and, therefore, do not adhere to the scheme.

All the above constructions should be complemented with the natural (flat) Lebesgue measure μ_d^{Lebesgue} on the space of channels, induced by normalising the Hilbert-Schmidt volume to 1. In [275] the following equality of measures has been proven:

$$\mu_d^{\text{Lebesgue}} = \mu_{d,d^2}^{\text{Stinespring}} = \mu_{d,d^2}^{\text{Choi}} = \mu_{d,d^2}^{\text{Kraus}}, \quad (6.37)$$

with the first two equalities being crucial to our later considerations.

6.2 Pushforward designs, channel designs and their properties

Thinking back to the induction of mixed state designs and simplex designs by partial trace and decoherence, respectively, we may take this line of reasoning one step further by generalising it to arbitrary pair of spaces imbued with a measure.

Definition 6.11 (Pushforward design). *Consider state spaces X and Y , a function $f : X \rightarrow Y$, and a measure μ on X . The function f induces a measurable function F that maps $\mu(x)$ to $F[\mu(x)] = \nu$ on Y . Let $\{x_i\}_{i=1}^M$ be a t -design. We say that the set $\{y_j\}_{j=1}^N = f(\{x_i\})$, with associated weights w_j corresponding to the multiplicities of each y_j , is an induced t -design if it provides a t -design with respect to the pushforward measure ν on Y .*

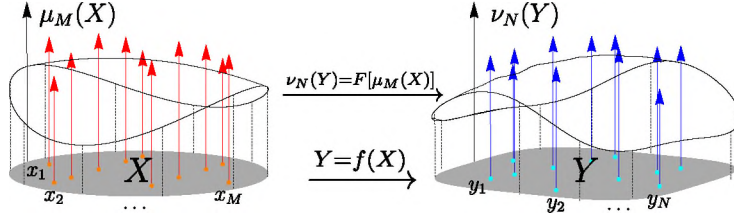


Figure 6.4: Pushforward design: This figure illustrates the transformation of a t -design from one state space X to another state space Y via a mapping function $f : X \rightarrow Y$. On the left, the state space X with points $\{x_i\}_{i=1}^M$ and their associated measure $\mu_M(X)$ is depicted using red arrows. The function f maps these points to the state space Y on the right, resulting in points $\{y_j\}_{j=1}^N$ with the measure $\nu_N(Y)$, represented by blue arrows. The transformation of the measure $\mu_M(X)$ to $\nu_N(Y)$ is denoted as $\nu_N(Y) = F[\mu_M(X)]$, where F is the induced measurable map from f . This illustrates how a t -design in X is converted into a pushforward t -design in Y .

The concept of pushforward design has been depicted in an abstract manner in Fig. 6.4. One may instantly formulate the following observation:

Observation 6.9. *The Definition 6.11 is realised whenever $F(\cdot)$ is linear and $\dim(X) \leq \dim(Y)$.*

The observation is proved by triviality of the statement – linear maps take polynomial bases of degree t to polynomial bases with the same degree. Then, the relation between dimensionalities of the spaces X and Y guarantees completeness of the aforementioned bases.

Let us consider a set of channels $\{\Phi_i\}_{i=1}^m$ acting on a d -dimensional space $\mathcal{H}^d$. We define a k -channel t -design in a natural way, based on the environmental form of a quantum channel, in relation to the averaging the action of t copies of a unitary operation U over the set of unitary operations $\mathcal{U}_{dk}$ of dimension $d \cdot k$ with k -dimensional environment initialised in a pure state $|0\rangle_E \in \mathcal{H}^k$,

$$\begin{aligned} \forall_{\rho \in \mathcal{B}(\mathcal{H}^{dt})} \frac{1}{m} \sum_{i=1}^m \Phi_i^{\otimes t}(\rho) &= \int_{\mathcal{U}_{dk}} \text{Tr}_E \left[U^{\otimes t} \left(\rho \otimes |0\rangle\langle 0|_E^{\otimes t} \right) (U^\dagger)^{\otimes t} \right] dU \\ &\equiv \left\langle \text{Tr}_E \left[U^{\otimes t} \left(\rho \otimes |0\rangle\langle 0|_E^{\otimes t} \right) (U^\dagger)^{\otimes t} \right] \right\rangle_{\mathcal{U}_{dk}} \end{aligned} \quad (6.38)$$

where in the second line we introduce, for brevity, notation for averaging over the unitary group $\mathcal{U}(dk)$. A direct consequence of the above is that a unitary t -design in dimension $d \cdot k$ induces, by partial trace, a k -channel t -design. In particular, let us note that for $k = d^2$ we obtain channel designs corresponding to flat measure (6.37) over the full set of quantum channels.

One can also reformulate the statement in terms of corresponding Choi states and the

measure induced on the space of Choi states,

$$\frac{1}{m} \sum_{i=1}^m \sigma_{\Phi_i^{\otimes t}} = \left\langle \text{Tr}_E \left[(U_{\mathbf{AE}}^{\otimes t} \otimes \mathbb{I}_{\mathbf{B}}^{\otimes t}) (|\psi_+\rangle\langle\psi_+|_{\mathbf{AB}}^{\otimes t} \otimes |0\rangle\langle 0|_{\mathbf{E}}^{\otimes t}) \left((U_{\mathbf{AE}}^\dagger)^{\otimes t} \otimes \mathbb{I}_{\mathbf{B}}^{\otimes t} \right) \right] \right\rangle_{\mathcal{U}_{dk}} \quad (6.39)$$

where the bold subscripts $\mathbf{A} \equiv A_1 A_2 \dots A_t$ and similarly to $\mathbf{B}$ and $\mathbf{E}$ correspond to t copies of the principal systems A, B and the environment E .

Now, let us take a closer look at the RHS of (6.39) and first evaluate it explicitly for $t = 1$. We may consider the block elements of the averaged Choi-Jamiołkowski state $\langle \sigma_\Phi \rangle_{\mathcal{U}(dk)} \equiv \langle \sigma_\Phi \rangle$, which are defined by

$$\langle \sigma_\Phi \rangle_b^a = \frac{1}{d} \Phi(|a\rangle\langle b|) = \frac{1}{d} \sum_{l=0}^{k-1} K^{(l)} |a\rangle\langle b| K^{(l)\dagger}. \quad (6.40)$$

Shifting to the Stinespring dilation, we know that the elements of Kraus operators are defined explicitly as $(K^{(l)})_j^i = U_{0j}^{li}$, which then allows us to consider the elements of the Choi state as

$$\langle \sigma_\Phi \rangle_{bj}^{ai} = \frac{1}{d} \left\langle \sum_{l=0}^{k-1} U_{0n}^{li} \delta_a^n \delta_m^b (U^\dagger)_{lj}^{0m} \right\rangle = \frac{1}{d} \sum_{l=0}^{k-1} \langle U_{0a}^{li} \bar{U}_{0b}^{lj} \rangle \quad (6.41)$$

where we use the Einstein summation convention, with sum over l index kept explicitly to highlight it goes over a different range of values; we also shortened $\langle \cdot \rangle \equiv \langle \cdot \rangle_{\mathcal{U}(dk)}$. In this case, due to symmetry we may conclude without resorting to more sophisticated methods that

$$\langle \sigma_\Phi \rangle_{bj}^{ai} = \frac{1}{d} \sum_{l=0}^{k-1} \frac{\delta_b^a \delta_j^i}{D} = \frac{\delta_{bj}^{ai}}{d^2} \quad (6.42)$$

where for later convenience we introduced $D = dk$. This expression is in accordance with intuition that no specific direction should be distinguished in the space of states and thus, the averaged channel symmetric with respect to the adjoint action of the local unitary group $\mathcal{U}(d)$ should contract to the only point that does not change under this action, ie. the maximally mixed state. Thus, it cannot be anything else than the maximally depolarising channel. This points in fact to a following theorem

Observation 6.10. k -channel 1-design $\{\Phi_i\}_{i=1}^m$ is also k' -channel 1-design for any k' .

Expressions for $t = 2$ become more involved and read

$$\langle \sigma_{\Phi^{\otimes 2}} \rangle_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} = \frac{1}{d^2} \left\langle \sum_{l_1, l_2=0}^{k-1} U_{0 n_1}^{l_1 i_1} U_{0 n_2}^{l_2 i_2} \delta_{a_1}^{n_1} \delta_{a_2}^{n_2} \delta_{m_1}^{b_1} \delta_{m_2}^{b_2} (U^\dagger)_{l_1 j_1}^{0 m_1} (U^\dagger)_{l_2 j_2}^{0 m_2} \right\rangle \quad (6.43)$$

$$= \frac{1}{d^2} \sum_{l_1, l_2=0}^{k-1} \left\langle U_{0 a_1}^{l_1 i_1} U_{0 a_2}^{l_2 i_2} \bar{U}_{0 b_1}^{l_1 j_1} \bar{U}_{0 b_2}^{l_2 j_2} \right\rangle \quad (6.44)$$

$$\stackrel{(a)}{=} \frac{1}{d^2} \sum_{l_1, l_2=0}^{k-1} \left[\frac{1}{D^2 - 1} \left(\delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} + \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) - \frac{1}{D(D^2 - 1)} \left(\delta_{b_2 b_1 j_1 j_2}^{a_1 a_2 i_1 i_2} + \delta_{b_1 b_2 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) \right] \quad (6.45)$$

$$= \frac{1}{d^2(D^2 - 1)} \left(k^2 \delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} + k \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} - \frac{k^2}{D} \delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} - \frac{k}{D} \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) \quad (6.46)$$

with the factors after (a) connected to the Weingarten functions $\text{Wg}(\sigma, D)$, connected to averaging over unitary group by

$$\int_{\mathcal{U}(D)} dU \prod_{n=1}^t U_{j_n}^{i_n} \bar{U}_{j'_n}^{i'_n} = \sum_{\sigma, \tau \in S_t} \text{Wg}(\sigma \tau^{-1}, D) \left(\prod_{n=1}^t \delta_{i'_n}^{i_n} \delta_{j'_{\tau(n)}}^{j_n} \right). \quad (6.47)$$

In particular, Weingarten functions depend only on the cycle structure of permutation.

By inspecting the above calculation we note that permutation between block indices a_n, b_n is not equivalent to permutation for in-block indices i_n, j_n – a factor of k^2 appears when no permutation is effected, while only k appears, due to evaluation of $\delta_{l_2}^{l_1}$, when there is a swap. Taking this into account, we give below the general formula

$$\langle \sigma_{\Phi^{\otimes t}} \rangle_{b_1 \dots b_t j_1 \dots j_t}^{a_1 \dots a_t i_1 \dots i_t} = \sum_{\sigma, \tau \in S_t} \text{Wg}(\sigma \tau^{-1}, D) k^{\text{Cl}(\tau)} \left(\prod_{n=1}^t \delta_{b_{\sigma(n)}}^{a_n} \delta_{j'_{\tau(n)}}^{i_n} \right) \quad (6.48)$$

with $\text{Cl}(\tau)$ is the total number of independent cycles in the permutation τ .

Using the above, one may put forward the following theorem

Proposition 6.11. *Consider a set of channels $X = \{\Phi_i\}$ acting on d -dimensional Hilbert space $\mathcal{H}^d$ and any $t \geq 1$. For any set X it holds that*

$$\left\| \frac{1}{|X|} \sum_{i=1}^{|X|} \sigma_{\Phi_i^{\otimes t}} - \langle \sigma_{\Phi^{\otimes t}} \rangle_{\mathcal{U}(dk)} \right\| \geq 0 \quad (6.49)$$

with $\langle \sigma_{\Phi^{\otimes t}} \rangle$ given explicitly in (6.48). The equality is achieved if and only if the set X is a k -channel t -design.

6.2.1 Unistochastic designs

Furthermore, one may extend the concept of channel designs to unistochastic channels, defined by action of a unitary operation $U \in \mathcal{U}(d^2)$ on a d -dimensional state ρ extended by a maximally mixed state,

$$\Xi_U(\rho) = \text{Tr}_E \left[U \left(\rho \otimes \frac{\mathbb{I}_E}{d} \right) U^\dagger \right]. \quad (6.50)$$

Thus, a unitary t -design X on dimension d^2 induces a unistochastic t -design,

$$\frac{1}{|X|} \sum_{U_i \in X} \Xi_{U_i}^{\otimes t}(\rho) = \int_{\mathcal{U}(d^2)} \text{Tr}_E \left[U^{\otimes t} \left(\rho \otimes \frac{\mathbb{I}_E}{d^t} \right) U^{\dagger \otimes t} \right] dU. \quad (6.51)$$

To the best of the author's knowledge, even though the measure on unistochastic channels induced by Haar measure on the unitary channels is well defined, it is not clear whether this measure is flat in the Lebesgue sense. Nevertheless, it is interesting to see the structures that may approximate such natural measure.

We may carry out similar calculations in this case as for the channel designs,

$$\langle \sigma_{\Xi^{\otimes 2}} \rangle_{b_1 a_2 i_1 i_2}^{a_1 a_2 i_1 i_2} = \frac{1}{d^4} \left\langle \sum_{l_1, l_2, o_1, o_2=0}^{d-1} U_{o_1 n_1}^{l_1 i_1} U_{o_2 n_2}^{l_2 i_2} \delta_{a_1}^{n_1} \delta_{a_2}^{n_2} \delta_{m_1}^{b_1} \delta_{m_2}^{b_2} (U^\dagger)_{l_1 j_1}^{o_1 m_1} (U^\dagger)_{l_2 j_2}^{o_2 m_2} \right\rangle \quad (6.52)$$

$$= \frac{1}{d^4} \sum_{l_1, l_2, o_1, o_2=0}^{d-1} \left\langle U_{o_1 a_1}^{l_1 i_1} U_{o_2 a_2}^{l_2 i_2} \bar{U}_{o_1 b_1}^{l_1 j_1} \bar{U}_{o_2 b_2}^{l_2 j_2} \right\rangle \quad (6.53)$$

$$\stackrel{(a)}{=} \frac{1}{d^4} \sum_{l_1, l_2, o_1, o_2=0}^{d-1} \left[\frac{1}{D^2 - 1} \left(\delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} + \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) - \frac{1}{D(D^2 - 1)} \left(\delta_{b_2 b_1 j_1 j_2}^{a_1 a_2 i_1 i_2} + \delta_{b_1 b_2 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) \right] \quad (6.54)$$

$$= \frac{1}{d^4 (D^2 - 1)} \left(d^4 \delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} + d^2 \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} - \frac{d^4}{D} \delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} - \frac{d^2}{D} \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) \quad (6.55)$$

$$= \frac{1}{d^4 - 1} \left(\delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} + \frac{1}{d^2} \left(\delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} - \delta_{b_1 b_2 j_1 j_2}^{a_1 a_2 i_1 i_2} \right) - \frac{1}{d^4} \delta_{b_2 b_1 j_2 j_1}^{a_1 a_2 i_1 i_2} \right) \quad (6.56)$$

where we kept $D = d^2$ in (6.54) and (6.55) to facilitate comparison between the expressions obtained here and for channel designs.

Note that the resulting average is different from the formula for k -channel designs, and generalizing it for arbitrary t can yields

$$\langle \sigma_{\Xi^{\otimes t}} \rangle_{b_1 \dots b_t j_1 \dots j_t}^{a_1 \dots a_t i_1 \dots i_t} = \sum_{\sigma, \tau \in S_t} \text{Wg}(\sigma \tau^{-1}, d^2) d^{2 \text{Cl}(\tau)} \left(\prod_{n=1}^t \delta_{b_{\sigma(n)}}^{a_n} \delta_{j_{\tau(n)}}^{i_n} \right). \quad (6.57)$$

which can be used to verify whether any given set of operations yields a proper design on unistochastic channels induced by unitary operations of dimension d^2 .

6.2.2 Simple example from Clifford group

For 2-channel 2-design on qubits, induced by the Clifford group on 2 qubits, we find 24 distinct unitary channels with multiplicity 96 each and 24 channels with Kraus rank 2 and multiplicity 384. These lead to a unit weight for each unitary and weight 4 for the rank-2 operations.

First, we note that the 24 unitary operations forming the set $\mathcal{R}_1$ correspond naturally to the elements of the Clifford group on a single qubit, $\mathcal{R}_1 = \mathcal{C}_1$.

The remaining 24 channels of rank 2, collected in the set $\mathcal{R}_2$, are given by Choi states with eigenvalues $\{1/2, 1/2, 0, 0\}$. They can be easily summarised by pairs of product states constituting the support of the operators,

$$\mathcal{R}_2 = \left\{ \sigma_\Phi = \frac{1}{2} \sum_{i=0}^1 |\psi_i\rangle\langle\psi_i| : |\psi_i\rangle = (A \otimes B) \text{CNOT}_{2 \rightarrow 1}^k \left(\sigma_x^j \otimes \mathbb{I} \right) |0i\rangle \right\} \quad (6.58)$$

with $j, k \in \{0, 1\}$ and $A, B \in \{\mathbb{I}, H, SH\}$, after discarding duplicate channels. In particular, the subset for $k = 1$ consists only of complete contraction channels towards 6 vertices of an octahedron inscribed in the Bloch sphere. The other 18 operations are equal combinations of pairs of Clifford unitaries.

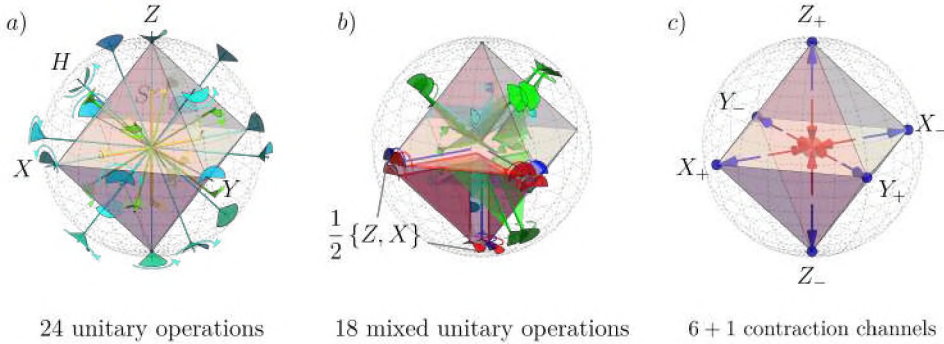


Figure 6.5: Channel families for k -channel 2-designs. k -channel 2-design for arbitrary $k \in [1, \infty)$ can be realised using 24 unitary operations comprising the Clifford group $\mathcal{C}_1$ (a), each operation represented by the related axis and sector of a circle representing the angle of the rotation) and 24 operations $\mathcal{R}_2$ of Kraus rank 2: 18 mixed unitary operations with equal-weight between different Clifford unitaries (b), each operation represented as a pair of semiaxes, with colours representing different mixtures; blue semiaxes represent mixing between half-rotations and identity transformation) and six depolarising channels, taking all the states into one of six vertices of an octahedron c), each blue arrow represents an operation). Taken with proper weights and complemented by the maximally depolarising channel (red arrows in c)), can be used to construct channel 2-designs related to environment of arbitrary dimensionality k .

An example of 4-channel 3-design is hard to calculate explicitly due to the order of Clifford group on 3 qubits, which is equal to 92 897 280, or on the order of 10^8 . Due to the above it is necessary to resort to sampling approach, using up to $5 \cdot 10^6$ sample channels sampled from a flat distribution over the Clifford group using approach demonstrated in [277]. This allows to formulate the following observation:

1. The induced design is composed of the same channels as the 2-channel 3-design induced by Clifford group on 2 qubits, extended by maximally depolarising channel, corresponding to the maximally mixed Choi state, $\sigma_\Phi = \frac{\mathbb{I}_4}{4}$.
2. The weights are estimated to be 1 for each rank-1 channel, 12 for each rank-2 channel and 192 for the maximally depolarising channel. Weights are confirmed by using Proposition 6.11.
3. No rank-3 channels are necessary to construct a 4-channel 2-design.

The above lead to an assumption that any k -channel 3-design can be realised by taking the channels from Clifford group $\mathcal{C}_1$, the rank-2 channels $\mathcal{R}_2$ and the maximally depolarising channel.

First, we may verify that it is possible to realise any 2-design using only the aforementioned channels. By considering

$$\frac{1}{a+b+c} \left(a \sum_{\Phi \in \mathcal{C}_1} \sigma_{\Phi^{\otimes 2}} + b \sum_{\Phi \in \mathcal{R}_2} \sigma_{\Phi^{\otimes 2}} + c \frac{\mathbb{I}_{16}}{16} \right) = \langle \sigma_{\Phi^{\otimes 2}} \rangle_{\mathcal{U}_{2k}}. \quad (6.59)$$

We may assume $a = 1$ and by solving for b and c we find the weight corresponding to the rank-2 channels as $b = 4(k-1)$, and the weight of the maximally depolarising channel as $c = 32(k^2 - 3k + 2)$. Furthermore, we know that for $k = 2^l$ the expressions must match the Clifford group on $l+1$ qubits, and thus a, b, c are also proper weights for 3 designs in such cases.

Finally, Clifford group on two qubits $\mathcal{C}_2$ gives rise to a 3-design in the space of unistochastic channels. Such a design is in fact composed of 43 channels, divided into 24 unitary channels from the single-qubit Clifford group $\mathcal{C}_1$ with weight 1 each, 18 mixed unitary channels $\mathcal{R}'_2 = \mathcal{R}_2|_{\text{mixed unitary}}$ with a weight 12 assigned to each and, finally, the maximally depolarising channel of weight 240.

6.3 Effective environment dimension estimation

An interesting consequence of the fact that the elements of the average Choi state, given in (6.48), are polynomial in k is that one may consider non-integer values of k . Thus, we put forward a concept of **effective environment dimension** k^* , defined as

$$k^* = \arg \min_{k > 1} \left\| \frac{1}{|X|} \sum_{i=1}^{|X|} \sigma_{\Phi_i^{\otimes t}} - \langle \sigma_{\Phi^{\otimes t}} \rangle_{\mathcal{U}(dk)} \right\| \quad (6.60)$$

where X is representative set of channels in a given physical system. One possibility of applying this metric is in quantum computers. Assuming first that gate implementations are low-noise, so that quantum process tomography can be implemented with fidelity high enough, one may consider allowing free evolution for time T and then evaluating the effective dimension of environment for the resulting noise channels Θ_T . Such method may give insight into the effective number of levels $k^*(T)$ that actually interact with the primary system after a given time.

One may also argue that the minimum distance itself,

$$\epsilon^* = \min_{k \geq 1} \left\| \frac{1}{|X|} \sum_{i=1}^{|X|} \sigma_{\Phi_i^{\otimes t}} - \langle \sigma_{\Phi^{\otimes t}} \rangle_{\mathcal{U}(dk)} \right\| \quad (6.61)$$

is an informative measure; if this norm reaches 0 for any value of k , it might signify that the interaction with the environment is not uniform and, as such, may require further investigation.

Unfortunately, estimation of $\frac{1}{|X|} \sum_{i=1}^{|X|} \sigma_{\Phi_i^{\otimes t}}$ turns out to be problematic, as it cannot be done using the standard ancilla-assisted process tomography [278]. Let us recall that in order to prepare the Choi-Jamiołkowski state $\sigma_{\mathcal{E}}$ for a channel $\mathcal{E}$, one needs to first prepare a maximally entangled state $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$. This, of course, can be implemented in a relatively straightforward manner within quantum computers, with a limited error involved. However, the next step, involving implementation of $\mathcal{E}$ on the first subsystem and identity channel $\mathcal{I}$ on the other. If we were to make a tomography of an arbitrary channel $\mathcal{E}$, it would be doable up to errors. In the specific case when we are attempting to make tomography of the very channel $\mathcal{N}_T$ of noise, such an approach is impossible, since by keeping the working qubits idle to accumulate the noise over time T , the ancillary systems are also accumulating similar noise.

Therefore, we need to implement a tomography scheme which does not rely on an unperturbed ancillary system, which involves state tomography for a set of at least d^2 linearly independent quantum states, such that reconstruction of the blocks $\mathcal{N}_T(|a\rangle\langle b|)$ of the Choi state can be done in post-processing [279, 280]. In our approach we will take the full set of MUBs on two qubits as the set of input states, keep the device in question in an idle state for a fixed amount of time T and afterwards perform state tomography using measurements based on the same set of MUBs.

This procedure, depicted schematically in Fig. 6.6, consists in three steps: 1) state preparation, 2) physical delay, acting as noise and 3) final measurement.

In the first step we select a state from the computational basis and act on it with a unitary U_i preparing the i -th MUB, and thus we have the state

$$|\psi_1\rangle = U_i |\psi_0\rangle, \quad |\psi_0\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \quad (6.62)$$

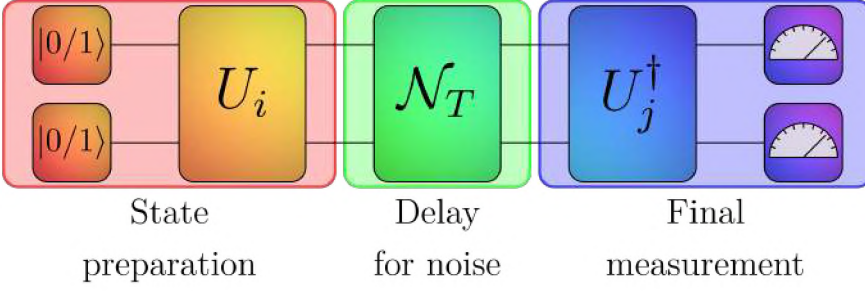


Figure 6.6: No-ancilla noise tomography scheme: The circuit used for tomography of the noise channel $\mathcal{N}_T$ consists in three basic steps. First, one prepares the input state by preparing qubits in one of the computational basis states and acting on them with unitary matrix $U_i, i \in \{0, \dots, 4\}$, corresponding to one of 5 mutually unbiased bases (MUBs). In second step, the noise channel is implemented by forcing the quantum computer to delay the next operation by time T . Finally, in the third step, an inverse $U_j^\dagger$, $j \in \{0, \dots, 4\}$ to the MUB preparation is implemented and both qubits are measured. The measurement results from the resulting 100 circuits are used to reconstruct the Choi state $\sigma_{\mathcal{N}_T}$ corresponding to the noise.

and the matrix U_i coming explicitly from the set

$$\begin{aligned}
 U_0 &= \mathbb{I} & U_1 &= H^{\otimes 2} & U_2 &= S^{\otimes 2} U_1 \\
 U_3 &= \text{CNOT}_{1 \rightarrow 2} \text{CNOT}_{2 \rightarrow 1} U_2 & U_4 &= \text{CNOT}_{2 \rightarrow 1} \text{CNOT}_{1 \rightarrow 2} U_2
 \end{aligned}$$

with

$$\begin{aligned}
 H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & \text{CNOT}_{1 \rightarrow 2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 S &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, & \text{CNOT}_{1 \rightarrow 2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
 \end{aligned}$$

After the second step the state becomes, in principle, mixed,

$$\rho_2 = \mathcal{N}_T(|\psi_1\rangle\langle\psi_1|). \quad (6.63)$$

The final step uses inverses $U_j^\dagger$ of MUB preparations, so that the measured state is

$$\rho_3 = U_j^\dagger \rho_2 U_j. \quad (6.64)$$

Data obtained from this procedure is then fed back to the formula (6.10) to reconstruct the states ρ_2 for each preparation, yielding a total of $20 \cdot 5 = 100$ circuits per time T .

Finally, we may consider the full set of states ρ_2 , which for no noise ($T = 0$) should be

given by all the MUB states, $\left\{ \left| e_i^j \right\rangle \left\langle e_i^j \right| \right\}_{i,j=1}^{d,d+1}$ with $d = 4$ for the case of two qubits. We may consider a matrix $\mathbf{A}$ defined row-wise by vectorizations of the MUB states. We may use its Moore-Penrose pseudo-inverse $\mathbf{A}^+$ to obtain

$$\left(\mathbf{A}^+ \mathbf{A} \right) \Big|_{T=0} = \mathbb{I}, \quad (6.65)$$

which, under the same interpretation, consists of row-wise vectorizations of all the projectors $|a\rangle\langle b|$ with $a, b \in \{1, \dots, d\}$. Naturally, the states ρ_2 are affected by noise acting over nonzero time $T > 0$. Thus, evaluating the same expression for the reconstructed matrix $\mathbf{A}$,

$$\left(\mathbf{A}^+ \Big|_{T=0} \right) \left(\mathbf{A} \Big|_{T>0} \right) \quad (6.66)$$

gives row-wise vectorizations of the images of the projectors under noise, $\mathcal{N}_T(|a\rangle\langle b|)$, from which one may reconstruct the Choi state

$$\sigma_{\mathcal{N}_T} = \sum_{a,b=1}^4 \mathcal{N}_T(|a\rangle\langle b|) \otimes |a\rangle\langle b|.$$

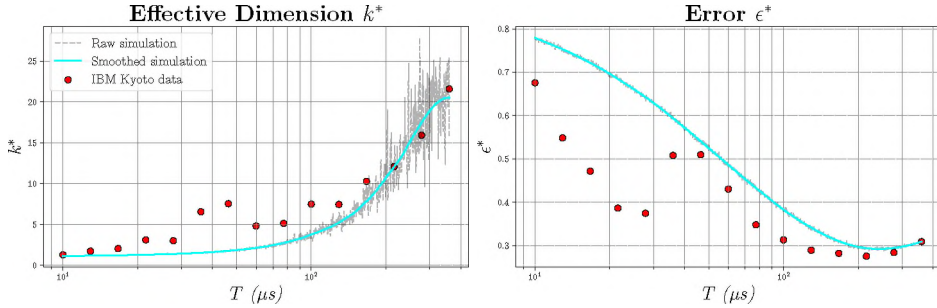


Figure 6.7: Effective environment dimensionality for IBM Kyoto quantum computer: The effective dimension k^* and corresponding error ϵ^* for qubits 9 and 10 were estimated over varying noise times T . Simulations based on the computer’s noise model (translucent black for raw data, cyan for smoothed) and data from the physical machine (red points) were utilized. In the left plot, k^* rises from approximately 1 for short times to around 20 for longer times, indicating the effective dimensionality over time. The right plot illustrates ϵ^* , showing a better fit for the actual machine data compared to simulations, especially noticeable at shorter times. Error saturation is observed for longer timescales.

Using the above scheme, we have evaluated the effective dimension k^* and the corresponding ϵ^* for 127-qubit quantum computer *IBM Kyoto*, using 15 points[‡] spread evenly from $10 \mu s$ to approximately $500 \mu s$ with respect to the logarithmic scale. Each point corresponds to 100 circuits with 1000 shots each, thus amounting to a total of 10^5 measurements per delay time. The results of this experiment are depicted in Fig. 6.7.

[‡] At the time of writing of this thesis, IBM allows 10 minutes of free computational, out of which the aforementioned 15 points took approximately 9 minutes 45 seconds. Due to the nature of the experiment, the measurements should be taken during a single calibration window, which are below 1 hour [281].

As can be seen in the left panel, effective dimension k^* starts low for times on the order of $10 \mu s$, which are short in relation to the energy relaxation time t_1 and coherence time t_2 , which are both on the order of $100 \mu s$ for *IBM Kyoto*. For times two orders of magnitude larger we see slow saturation of effective dimensionality at around $k^* \approx 20$. Most importantly, however, the dimensionality obtained from simulated noise matches closely the one obtained from actual quantum machine for small and large times, while at intermediate times we see discrepancies.

The error value ϵ^* shows, however, that the uniform interaction model performs better for actual machine than the simulated noise, with a minimum unpredicted by the simulation. For large times the error ϵ^* seems to saturate at nonzero value.

The above non-vanishing of ϵ^* suggests that uniform interaction with environment may not be the proper way of modelling the environment. One phenomenon, that is not taken into account, is that qubits are more prone to de-excitation, eg. by emitting a photon to the environment, rather than excitation by photon absorption. To take this into account, we modified the weight $4(k-1)$ from (6.59) assigned specifically to the complete emission channel to $4(k-1) + w$, adjusting the overall normalization accordingly. Results of fitting such model are presented in Fig. 6.8.

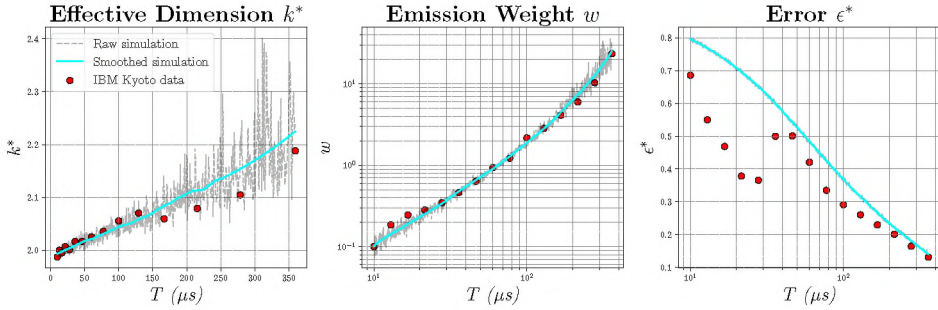


Figure 6.8: Effective environment dimensionality with enhanced emission for IBM Kyoto quantum computer: The data from Fig. 6.7 is utilized to estimate the dimensionality k^* of the environment under enhanced emission conditions, characterized by an additional weight w . In the middle plot, the added weight w is depicted, while the right plot shows the estimated dimension k^* , which linearly transitions from 2 to 2.2 over time T . The right plot demonstrates that the model with enhanced emission provides a better fit to actual machine data compared to simulations, particularly noticeable as time T increases. The error appears to saturate for longer timescales.

As can be seen, the noise simulation leads to the effective dimensionality of the environment at around $k^* = 2$ for short times up to $k^* \approx 2.2$ for long times, with agreement between simulation and experiment. The additional weight t assigned to the emission channel grows polynomially, as can be seen from linearity of the middle plot in the log-log scale. Finally, the ϵ^* value also exhibits unpredicted minimum for relatively short times, and in the emission-related model, seems to converge to zero for long times, pointing to validity of the model in this regime.

There are several takeaways from the above analysis. First, uniform interaction of qubits in quantum computers is of limited application, with best performance for

times on the order of the energy relaxation and decoherence times. Second, the emission model seems to provide better fit, especially for longer times, with low effective dimensionality of environment corresponding to the fact that IBM quantum computers are based on superconducting elements cooled to temperatures in the range of tens of milli-Kelvins. Final remark is that both models can be implemented relatively easily with the 49 channels contributing to Eq. (6.59), thus making for an easy manner of simulating noise at a relatively low cost.

6.4 Summary and Outlook

In this chapter, we introduced the concept of channel t -designs, which mimic the measure over the entire space of quantum channels up to t -copy samples or, equivalently, up to polynomials of degree t . We provided an analytical method to verify whether a given set of quantum channels forms such a design. Additionally, we derived closed expressions for averages of t -copy Choi states over natural measures induced on the space of quantum channels by unitary operations on the primary system extended by an environment of any given dimension k .

We also presented a method for constructing channel t -designs by partially tracing known unitary t -designs, similar to how mixed state designs are constructed from complex projective designs, as introduced in our earlier work [5]. These constructions, along with the induction of simplex designs by decoherence of complex projective designs (see Fig. 6.3), illustrate the novel concept of pushforward designs.

A key insight is that a pushforward design can be constructed in the image space whenever the function acting on measures is linear. As a future research direction, we suggest exploring nonlinear transformations and the potential introduction of non-polynomial designs. Linear functions on a space X form a vector subspace of all possible functions. Under nonlinear transformations, this subspace maps onto a non-linear but continuous surface within the space of all functions on Y . A design mimicking averages of linear functions on X over continuous measures would yield a set of points recovering averages of functions on this surface. A particularly interesting possibility is the concept of exponential designs, where 2-designs in the pre-image space yield Gaussian designs as the pushforward structures. However, this idea extends beyond the scope of this chapter.

To illustrate the construction we used the Clifford group on n qubits, a known unitary 3-design, as a starting point for k -channel 3-designs on a single qubit. This approach allows the construction of a channel 3-design for any k using only 49 channels: 24 original Clifford gates on a single qubit, 18 mixed-unitary channels, and 7 complete contraction channels related to the vertices of an octahedron and its center.

We also proposed a method to estimate the effective environment dimensionality k^* via noise reconstruction. This method compares distributions induced by the unitary group acting on the system extended by an environment of dimension k^* . Importantly,

since all such measures are polynomial with respect to the environment's dimension, non-integer values of k^* are possible. For a single qubit, any dimension k^* can be emulated up to polynomials of degree 3 using the 49 channels mentioned earlier.

We demonstrated the concept of effective dimensionality through noise reconstruction for various time intervals on a pair of qubits in the *IBM Kyoto* quantum computer. For times longer than decoherence and energy relaxation time scales, the noise was well emulated assuming an effective environment dimension of $k^* \approx 20$. By assigning additional weight to the full emission channel, we found that a better fit occurred when the effective environment dimensionality was $k^* = 2$ for short times, growing linearly over time with an increasing weight on the emission channel.

In summary, our work has introduced channel t -designs, provided methods for their construction, and demonstrated their practical application in estimating effective environment dimensionality. These contributions advance the understanding and practical application of quantum channel designs, providing a foundation for future research in the field.

SUMMARY AND AFTERWORD



7 Summary

In this thesis, I have presented results that contribute to several facets of the field of quantum information, showcasing the diversity of problems tackled by its community.

In Chapter 2, I demonstrated the first step towards the classification of measurements up to local equivalence by focusing on two-qubit von Neumann measurements connected to bases composed of states with a constant degree of entanglement. Among the four identified families, a subfamily can be used to generate output-permutation invariant distributions in triangular quantum networks, considered optimal from the standpoint of indistinguishability of different parties and outputs. Future research may extend this classification to distributed measurements involving a greater number of parties, higher-dimensional local systems, and a larger number of measurement outcomes.

Chapter 3 translates methods from $SU(1,1)$ interferometry, which uses squeezed states of light to detect displacement in phase space, to a spin setting, replacing displacement with rotation and bosonic squeezing with single-axis squeezing. I demonstrated that the squeeze-rotation-unsqueeze protocol provides optimal scaling of the quantum Fisher information, given the correct degree of squeezing. This optimality is preserved when considering relative rotations, although it is less robust in this context.

In Chapter 4, I presented a method to approximate thermal operations using a specific algorithm of two-level Markovian thermal operations acting on a system extended by memory initialized in thermal equilibrium. Analytically, I showed that the procedure converges to arbitrary permutations at infinite temperature and to extremal two-level operations at finite temperatures, given infinite memory. Extensive numerical evidence supports convergence towards arbitrary extremal transitions. I also extended the analytical proofs beyond two-level operations for finite temperatures, suggesting that the proposed algorithm optimally converges in terms of the number of operations involved.

Chapter 5 addresses strict catalysis for thermal operations, where the catalyst state remains unaltered and does not obtain coherence with the principal system. I introduced the concept of a catalysable region, where transformations fulfill necessary, but not sufficient, conditions for the existence of a catalyst state. I demonstrated bounds on both the structure and dimensionality of this catalyst state, calculated using the thermomajorization curves of the initial and target states. Although not optimal, these

bounds are easily calculable. I provided an example application to entanglement generation, showing that certain states, which cannot generate entanglement via unassisted thermal operations, can do so with catalyst assistance.

In the final Chapter 6, I introduced the notion of channel designs, a specialization of the general idea of designs. Channel designs provide a tool for averaging polynomial functions over the space of quantum channels using a well-defined finite set of points, as opposed to random sampling approaches. I also proposed measuring the effective dimension of the environment by optimizing the distance to the averaged Choi state concerning the dimension of the underlying total system.

Throughout these chapters, two recurring themes emerge: optimality, although understood differently for various problems, and the concept of measurement, whether direct or as estimation. These themes, though not present in all topics, can guide future research directions. I envision proofs of optimality and further refinements, particularly in the chapters focused on quantum thermodynamics.

In conclusion, this thesis not only advances our understanding of specific quantum information processes but also lays the groundwork for future explorations in the field. By addressing both theoretical and practical aspects of quantum measurements, optimality, and catalysis, this work opens new avenues for research and applications in quantum information science.



Posłowie (Afterword)

„Nie ma końca. To tylko kres dawnych czasów, Loki. **I początek nowych.**”

*“There is no end. It is simply the end of the old times, Loki, and the beginning of the new times.” **

Neil Gaiman, *Mitologia Nordycka (Norse Mythology)*,
w tłumaczeniu Pauliny Braiter [282, 283]

Setka z ogonem stron później; dziewięć lat później; tysiące godzin nauki, dyskusji i pracy później... Wiele wody musiało upłynąć w rzekach tego świata nim dotarłem do tego punktu mojej kariery naukowej. Nie tak wyglądał pierwotny plan; życie jednak, jak powszechnie wiadomo, pisze własne scenariusze, niezależne od naszych planów i widzimisiów. Gdyby jednak istniała możliwość, sprzeczna ze współczesną wiedzą fizyczną, by cofnąć się te dziewięć lat w czasie i porozmawiać z samym sobą, mam poczucie że nie zmieniałbym swojego własnego zdania. Młody i ambitny wówczas licealista u progu dorosłości, czy tego co szumnie się dorosłością nazywa, stanął przed wyborem: fizyka, japonistyka i zdrowie – wybierz dwa. Po latach trudów, wyężonej pracy, gór i dolinek naukowej wyprawy po wiedzę, mogę śmiało stwierdzić – nie żałuję podjętej wówczas, i podtrzymanej aż do teraz, decyzji.

Innym punktem zwrotnym w całej tej historii był kurs Informatyki Kwantowej, prowadzony do teraz przez mojego promotora, prof. Karola Życzkowskiego. W pamięć najbardziej zapadł mi egzamin, na którym z braku czasu nie dokończyłem formalnie jednego zadania – choć teraz z pewnością zrobiłbym je z jedną ręką związaną za plecami – i zamiast tego podałem wyłącznie jakościowe rozwiązanie, które przychodziło jako naturalna kontynuacja przeprowadzonych obliczeń. To mieszane podejście do dziś wspomaga mnie na co dzień, pozwalając przeskakiwać nad zasiekami obliczeń bezpośrednio do pożądanego wyniku. Odbiegam jednak od tematu... Po zakończonym kursie prof. Życzkowski wyszedł z propozycją współpracy naukowej z nim i z jego ówczesnym *postdokiem*, dr Dardo Goyeneche. W tamtym czasie nie uwierzyłbym jeszcze, że przyjęcie tej oferty zaprowadzi mnie do wizyt w Chile i znajomości na lata, jeśli nie na całe życie.

* This final part is written primarily in Polish, with translation prepared in tandem with an AI tool. After the Polish text the English translation is provided *in italic text*.

W trakcie lat moich studiów pojawiały się różnego rodzaju kryzysy, trudności i wyzwania różnorodnej natury. Jednym z największych kryzysów, który dotknął cały świat w taki czy inny sposób, była pandemia COVID-19, która rozpoczęła się w styczniu 2020 roku, zaś sam stan zagrożenia został odwołany przez WHO w maju 2023 roku [284]. Choć te trzy lata przyniosły wiele ciężkich sytuacji, chciałbym tu przytoczyć i unieśmiertelnić sytuację z początku lockdownów, jeszcze z ostatniego semestru moich studiów magisterskich. Miałem wtedy zaszczyt i (wątpliwą) przyjemność poprowadzić pierwsze zdalne seminarium magisterskie. Wszystkie testy przeprowadzone godzinę przed rozpoczęciem wraz z koordynującymi seminarium profesorami Maciej Nowakiem i Mariuszem Sadzikowskim, poszły dobrze. W momencie jednak, kiedy miałem już rozpoczynać, padł prąd. Żadne gniazdko nie działało w salonie, bezpieczniki nie padły. Pierwszą osobą, do której zwróciłem się wówczas o pomoc telefonicznie, był mój tata, ten jednak mimo wykształcenia elektronicznego nie był w stanie wymyślić, co poszło nie tak. Drugą osobą, do której zwróciłem się o poradę, był mój dziadek Piotr. Posłuchał sytuacji – że prądu w gniazdkach w salonie nie ma, że bezpiecznik nie poszedł w dół, że w innych pokojach nie ma problemu – i od razu stwierdził, że prawdopodobnie bezpiecznik się poluzował, trzeba go dokręcić, i nie, nic się nie stanie, jeżeli zrobisz to zwykłym śrubokrętem. Dzięki temu – i zdecydowanie zbyt dużej ilości przedłużaczy – seminarium zostało uratowane i odbyło się, choć z opóźnieniem blisko godzinnym.

Historię tę przytaczam, bo dziadek Piotr – obok dziadka Kazimierza – był jednym z dwóch nauczycieli fizyki w mojej rodzinie, a także jednym z trójki nauczycieli przedmiotów ścisłych, wliczając uczącą matematyki babcię Mariannę. Choć nie pamiętam, żeby ktokolwiek z moich dziadków naciskał, bym poszedł w stronę nauk ścisłych, wspierali mnie jak tylko mogli na każdym kroku mojej ścieżki. Co jednak może ważniejsze, wszyscy oni pokazywali mi, że w życiu jest więcej niż tylko nauka. W szczególności, dziadek Kazik zaprosił mnie do świata pszczół, ale też wędkarstwa, zaś dziadek Piotr nauczył grać w ping-ponga i w siatkówkę (w które sam grał nawet na emeryturze).

Teraz, dziewięć lat, setkę stron i dwadzieścia artykułów później, powstała praca doktorska, pierwsza w mojej rodzinie. Poza nauką, biorąc do serca podejście moich dziadków, studiowałem japoński, zwiedzałem Kraków i okolice na rowerze, pielęgnowałem swoje pasje do gitary i fotografii, i zleciałem pół świata. Gdyby dziadek Piotr i dziadek Kazik byli wciąż z nami, nie przeczytaliby tej pracy – żaden z nich nie znał angielskiego na tyle dobrze – jednak posadziłbym ich obok siebie i wszystko opowiedział własnymi słowami. Oni zaś pewnie dopytaliby tu i tam, i uśmiechnęli się. Niestety, nie ma ich już z nami – dlatego też pracę w całości im dedykuję, nauczycielom fizyki starej daty, najdroższymi dziadkom.

Over a hundred pages later; nine years later; thousands of hours of study, discussions, and work later... A lot of water has flowed in the rivers of this world before I reached this point in my academic career. This was not the original plan; life, as we know, writes its own scenarios, independent of our plans and whims. However, if there were a possibility, contrary to modern

physical knowledge, to go back those nine years and talk to myself, I feel I would not change my own mind. Young and ambitious then, a high school student on the threshold of adulthood, or what is grandly called adulthood, stood before a choice: physics, Japanese studies, and health – choose two. After years of hardships, intense work, ups and downs of the academic journey for knowledge, I can boldly state – I do not regret the decision I made then and upheld until now.

Another turning point in this whole story was the Quantum Computing course, still conducted by my supervisor, Prof. Karol Życzkowski. The most memorable part was the exam, where, due to lack of time, I didn't formally complete one task – though now I would certainly do it with one hand tied behind my back – and instead, I provided only a qualitative solution, which came naturally as a continuation of the performed calculations. This mixed approach still assists me daily, allowing me to leap over the hurdles of calculations directly to the desired results. But I digress... After completing the course, Prof. Życzkowski proposed scientific cooperation with him and his then postdoc, Dr. Dardo Goyeneche. At that time, I wouldn't have believed that accepting this offer would lead me to visits in Chile and acquaintances for years, if not for life.

During my years of study, various crises, difficulties, and challenges of different natures arose. One of the biggest crises, which affected the entire world in one way or another, was the COVID-19 pandemic, which began in January 2020, and the state of emergency was lifted by WHO in May 2023. Although these three years brought many tough situations, I want to recount and immortalize an event from the early lockdowns, still in the last semester of my master's studies. I had the honor and (dubious) pleasure of conducting the first remote master's seminar. All tests conducted an hour before the start, along with the professors coordinating the seminar, Maciej Nowak and Mariusz Sadzikowski, went well. However, when I was about to start, the power went out. No outlet worked in the living room, the fuses didn't blow. The first person I called for help was my dad, who, despite his electronic education, couldn't figure out what went wrong. The second person I turned to was my grandfather Piotr. He listened to the situation – that there was no power in the living room outlets, that the fuse didn't go down, that there was no problem in other rooms – and immediately stated that the fuse probably got loose, it needed to be tightened, and no, nothing would happen if you did it with a regular screwdriver. Thanks to this – and an excessive amount of extension cords – the seminar was saved and took place, although with almost an hour delay.

I recount this story because my grandfather Piotr – along with my grandfather Kazimierz – was one of two physics teachers in my family, and also one of three science teachers, including my grandmother Marianna, who taught mathematics. Although I don't remember any of my grandparents pressuring me to pursue the sciences, they supported me as much as they could at every step of my path. What may be even more important, they all showed me that there is more to life than just science. In particular, Grandpa Kazik introduced me to the world of bees and fishing, while Grandpa Piotr taught me to play ping-pong and volleyball (which he played even in retirement).

Now, nine years, a hundred pages, and twenty articles later, this doctoral thesis, the first in my family, has come to be. Besides science, taking to heart my grandparents' approach, I studied Japanese, explored Krakow and its surroundings by bike, nurtured my passions for guitar and photography, and traveled half the world. If Grandpa Piotr and Grandpa Kazik were still with

us, they wouldn't read this thesis – neither of them knew English well enough – but I would sit them beside me and tell them everything in my own words. They would probably ask a few questions here and there and smile. Unfortunately, they are no longer with us – therefore, I dedicate this work entirely to them, old-school physics teachers, my dearest grandparents.

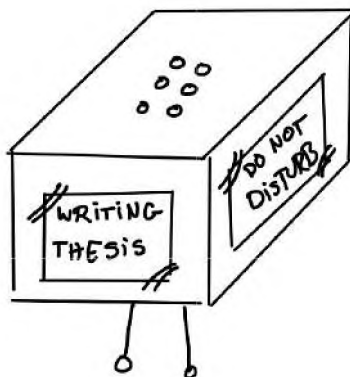


Figure 1: „Proszę cię, narysuj mi doktoranta.” [285] “If you please – draw me a doctoral student!” [286]
Rysunek autorstwa Aleksandry Morawiec. Drawing by Aleksandra Morawiec.

Podziękowania (Acknowledgements)

Czas potrzebny na dojście do tego miejsca – niemalże dekada, lub jedna trzecia mojego dotychczasowego życia – sprawia, że liczba osób, którym warto byłoby podziękować, jest już znacząca. Zaryzykuję stwierdzenie, że na tyle znacząca, że z pewnością kogoś pominię – więc jeśli jest ktoś, kto poczuwa się do bycia wspomnianym, a nie został, jest to pominięcie najpewniej niecelowe!

W pierwszej kolejności chciałbym podziękować swoim rodzicom, wszystkim dziadkom i całej rodzinie, która wspierała mnie na całej mojej naukowej ścieżce. Po ukończeniu licencjatu moja mama, wręczając mi pisma zebrane Einsteina, stwierdziła że nigdy nie spodziewali się, że aż tak zakotwicze się w nauce. Z ich wsparciem zaszedłem dużo dalej.

Skoro już mowa o wsparciu, podziękowania należą się wszystkim moim znajomym, kolegom i przyjaciołom, zarówno tym, którzy są nadal przy mnie, jak i tym, których ścieżki odeszły od mojej. Nie będę wymieniał was z imienia, ani wspominał szczegółów, jednak jeżeli to czytacie – wasza obecność była mi oparciem, nawet jeżeli nie dałem temu w pełni wyrazu. Niektórzy z was będą pamiętać te momenty, w których bez waszego wsparcia nie poradziłbym sobie tak dobrze, jak to wyszło na koniec.

Grzechem byłoby nie wspomnieć o moich dwóch kołach naukowych, które dawały mi schronienie i bezpieczną przystań na Wydziale Fizyki, Astronomii i Informatyki Stosowanej przez te wszystkie lata. Podejrzewam, że duża część moich prac powstawała w jednym z kołowych pokoi. Tak więc – kołom NKF i SMP, i wszystkim

ich członkom których poznałem na przestrzeni lat, składam najserdeczniejsze podziękowania – moi najdrożsi, walczyć u waszego boku, to był zaszczyt!

Szczególne uznanie należy się Hannie Went – przyjaciółce-artystce, która przygotowała okładkę dla tej rozprawy.

Dodatkowo, podziękowania należą się Aleksandrze Morawiec, która narysowała mi baranka (patrz rysunek 1).

Podziękowania *stricte* naukowe należą się w pierwszej kolejności mojemu wieloletniemu promotorowi, prof. Karolowi Życzkowskiemu, bez którego moje portfolio prac i współpracowników byłoby dużo uboższe. Nawet jeżeli czasami czułem się zalany poprawkami, ciągnącymi się zawijasami po marginesach prac i manuskryptów, trudno o lepsze wsparcie w karierze naukowej.

Skoro już mowa o współpracy naukowej, podziękować chciałbym wszystkim moim współautorom, zarówno tym z którymi mam już opublikowane prace, jak i wszystkim tym, z którymi współpraca trwa (kolejność alfabetyczna): Davide Bacco, Rafał Bistróń, Tanmoy Biswas, Daniel Braun, Adam Burchardt, Flavio Del Santo, Fabio Delan Cunden, Paolo Facchi, Noel Farrugia, Yan V. Fyodorov, Bartłomiej Gardas, Nicolas Gisin, Victor Gonzalez Avella, Dardo Goyeneche, Giovanni Gramegna, Markus Grassl, Paweł Horodecki, Kamil Korzekwa, Gianmarco Lemmi, Marilena Ligabo, Patryk Lipka-Bartosik, Tomas Agustin Merlo Vergara, Paweł Mazurek, Ronny Müller, Nelly H. Y. Ng, Alexssandre de Oliveira Junior, Kevin de la Ossa, Antoine Petitjean, Grzegorz Rajchel-Mieldzioć, Ricard Ravell Rodriguez, Domenico Ribezzo, Roberto Salazar, Adam Sawicki, Flora Segur, Timo Simnacher, Jeongrak Son, Konrad Szymański, Abraham Vega Vargas, Andre Xuereb, Alessandro Zavatta. Wierzę, że nauka jest wysiłkiem zbiorowym, a wszystkie te osoby pomogły mi przez te lata utrzymać tę wiarę.

Szczególne podziękowania należą się prof. Danielowi Braunowi i jego grupie (członkowie, których miałem przyjemność poznać osobiście: Alessio Belenchia, Riccardo Bellesse, Lukas Fiderer, Francis Headley, Matthias Huls, Martin Korzeczek, Emre Kose, Nadia Milazzo, Ronny Muller, Mahdi Rouhbakhshnabati, Jirawat Saiphet, Felix Spengler) – na Uniwersytecie w Tybindze, z którymi miałem przyjemność przebywać podczas półrocznego pobytu w roku 2022 w ramach stażu NAWA Preludium BIS, jak również w trakcie poprzednich wizyt w latach 2018 i 2019. Z naukowej perspektywy był to dla mnie czas niezwykle wartościowy, który przyniósł długoterminowe korzyści. Bardziej osobiście, przejażdżki w łodziach Stocherkahn, wieczorne wyjścia z przyjaciółmi z grupy, jak i pożegnalne kręgle i kolacja to wspomnienia, które pozostaną ze mną na zawsze.

Podobne podziękowania należą się prof. Dardo Goyeneche, który dwukrotnie zaprosił mnie do Chile, raz do Antofagasty, gdzie poznałem Victora Avelę, a później do nowopowstałej grupy QUDIT na Chilijskim Papieskim Uniwersytecie Katolickim w Santiago de Chile. W obu przypadkach zostałem przyjęty ciepło, i miałem przyjemność poznać wiele niezwykłych osób (których listy nie załączę, bo obawiam się, że mógłbym kogoś pominąć). Zbyt wiele pisać o wszystkim, co miałem przyjemność tam zobaczyć

– starczy tylko wspomnieć, że wycieczka do teleskopów i Mano del Desierto na Atacamie i piknik nad rzeką niedaleko Santiago wryły mi się głęboko w pamięć.

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Pozwolę sobie zamknąć te podziękowania pewnym przekształceniem cytatu z Bilbo Bagginsa, który tak by rzekł, gdyby miał podziękowania napisać [287]:

Ani połowie spośród was nie podziękowałem nawet do połowy tak dobrze, jak bym pragnął; (...).

Drugą połowę wypowiedzi opuszczam – nie ma w tych podziękowaniach ani jednej osoby, której podziękować bym nie chciał.

The time needed to reach this point – almost a decade, or one-third of my life so far – makes the number of people worth thanking already significant. I risk saying that it is significant enough that I will certainly forget someone – so if there is someone who feels they should be mentioned but is not, it is most likely an unintentional omission!

First and foremost, I would like to thank my parents, all my grandparents, and my entire family who supported me throughout my academic journey. After completing my bachelor's degree, my mom, handing me the collected papers of Einstein, said they never expected I would anchor myself so deeply in science. With their support, I have come much further.

Speaking of support, thanks are due to all my acquaintances, colleagues, and friends, both those who are still with me and those whose paths have diverged from mine. I will not name you or mention details, but if you are reading this – your presence was a support to me, even if

I did not fully express it. Some of you will remember the moments when, without your support, I would not have managed as well as I did in the end.

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Special recognition goes to Hanna Went — a friend and artist who designed the cover for this dissertation.

Additionally, thanks are due to Aleksandra Morawiec, who drew the lamb for me (see Figure 1).

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I will close these acknowledgments with a paraphrased quote from Bilbo Baggins, who would say, if he were to write acknowledgments [288]:

I didn't thank half of you half as well as I should like; (...).

I omit the second half of the statement – there is not a single person in these acknowledgments to whom I would not want to express my gratitude.

O zdjęciach (About photos)

Należy się słowo o zdjęciach, które znalazły się na kartach tej pracy, otwierając każdy kolejny rozdział. Wszystkie zdjęcia zostały wykonane i obrobione osobiście przeze mnie, i wszędzie tam, gdzie byłem w stanie, wykorzystałem kadr z miejsca związanego z pracą lub tematyką danego rozdziału:

A word about the photos that appear in this thesis, opening each subsequent chapter. All the photos were taken and processed personally by me, and wherever possible, I used a shot from a place related to the work or the theme of the given chapter:

Preface: Stare Pobrzeże i Żuraw Gdański, zdjęcie robione od strony Wyspy Spichrzów, Sołdka i Ołowianki; Gdańsk, jako moje miasto rodzinne, stanowi naturalny wybór na zdjęcie do przedmowy.

The Old Waterfront and the Gdańsk Crane, photographed from the side of the Granary Island, Soldek, and Ołowianka; Gdańsk, as my hometown, is a natural choice for the preface photo.

Chapter 1: Kładka ojca Bernatka nad Wisłą w Krakowie, widok od strony Kościoła na Skałce; Moje pierwsze spotkanie z mechaniką i informacją kwantową, jak i całe dalsze studia, miały miejsce w tym mieście.

Father Bernatek Footbridge over the Vistula River in Kraków, viewed from the side of the Church on the Rock; My first encounter with mechanics and quantum information, as well as my entire further studies, took place in this city.

Chapter 2: Portret rudzika, uchwycony na brzegu Jeziora Genewskiego w Genewie; praca powstawała m.in. podczas wizyty u prof. Nicolas Gisin, w trakcie której uchwyciłem ten niezwykły kadr.

Portrait of a robin, captured on the shores of Lake Geneva in Geneva; the work was developed, among others, during a visit to Prof. Nicolas Gisin, during which I captured this remarkable shot.

Chapter 3: Żaba i kwiat lotosu, uchwycone na terenie Ogrodu Botanicznego Uniwersytetu w Tybindze; wszystkie wyniki pracy opisanej w tej części zostały opracowane w trakcie pobytu w Tybindze.

A frog and a lotus flower, captured in the Botanical Garden of the University of Tübingen; all the results of the work described in this section were developed during my stay in Tübingen.

Chapter 4: Wnętrze Ruin Bars w Budapeszcie; zdjęcie wybrane z uwagi na skojarzenie z fantastyczno-naukową twórczością Philipa Kendreda Dicka, na podstawie której powstał film *Total Recall*, będący inspiracją dla tytułu tej pracy.

Interior of Ruin Bars in Budapest; the photo was chosen due to its association with the science fiction works of Philip K. Dick, on which the movie Total Recall was based, serving as an inspiration for the title of this chapter.

Chapter 5: Zachód słońca, odbity w oknach budynku na terenie Jægersborg Dyrehave w Lyngby; istotna część prac nad manuskrypcem opisanym w tym rozdziale odbyła się na Technicznym Uniwersytecie Danii, znajdującym się w Lyngby.

Sunset reflected in the windows of a building in Jægersborg Deer Park in Lyngby; a significant part of the work on the manuscript described in this chapter was carried out at the Technical University of Denmark, located in Lyngby.

Chapter 6: Portret jaskółki siedzącej na molo w Brzeźnie, Gdańsk; pełni ona tutaj rolę wyłącznie artystyczną, bez powiązania z trwającym projektem.

Portrait of a swallow sitting on the pier in Brzeźno, Gdańsk; it serves here solely an artistic role, without connection to the ongoing project.

Chapter 7: Panorama wzdłuż Wisły w kierunku Tatr, uchwycona ze Skałek Piekarskich niedaleko Tyńca; początek mojej pracy naukowej miał miejsce w Krakowie jednak podsumowanie mojej pracy doktorskiej powinno stanowić perspektywę na szerszy świat.

Panorama along the Vistula River towards the Tatra Mountains, captured from the Piekary Rocks near Tyniec; the beginning of my scientific work took place in Kraków, but the summary of my doctoral work should present a perspective on the wider world.

Postowie: Śpiący wróbel, uchwycony na gałęzi krzaka na brzegu Dunaju w Budapeszcie; przedstawia doktoranta (w domyśle mnie), który po dobrze wykonanej pracy może nareszcie odpocząć.

A sleeping sparrow, captured on a branch of a bush on the banks of the Danube in Budapest; it represents the doctoral candidate (implicitly me) who can finally rest after a job well done.

Bibliography: Wschód słońca nad górami leżącymi na wschód od Santiago de Chile; Zdjęcie bez związku z bibliografią, zamieszczone w celach wyłącznie estetycznych.

Sunrise over the mountains to the east of Santiago de Chile; This photo is unrelated to the bibliography and is included solely for aesthetic purposes.

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