

Report on the thesis

Characterizing quantum behaviors

by Moisés Bermejo Morán

This Dissertation deals with the interesting problem of bounding the set of observable measurement statistics in different quantum communication scenarios. This includes Bell scenarios, but also more exotic situations, e.g.: allowing some parties to conduct joint measurements.

At the technical level, the author heavily relies on the framework of non-commutative polynomial optimization (NPO) theory. NPO theory is a subfield of operations research concerned with optimization problems over quantum states and operators satisfying a number of polynomial relations, and over the Hilbert spaces where such state and operators respectively live and act. For many such problems, NPO theory provides complete hierarchies of SDP relaxations. Characterizing the measurement statistics generated in quantum Bell-type experiments requires making claims over the set of all states and measurement operators; it can therefore be cast as an NPO problem.

Unfortunately, while the SDP hierarchies in NPO theory are promised to converge to the solution of each NPO problem, sometimes they do so very slowly. It is thus crucial to develop methods to simplify or compress the SDPs, in order to reach higher orders of relaxation with the same computational power.

The first contribution of the Dissertation goes along these lines. In NPO problems with equality constraints, deciding whether two polynomials are equal modulo equality constraints leads to great simplifications of the associated SDP relaxations. In the commutative case, this problem is solved by constructing a Gröbner basis, i.e., a finite set of polynomials that, by division, allow reducing any polynomial to a canonical form in a bounded number of steps. Two polynomials are thus the same iff their canonical forms coincide. Regretfully, although in the commutative case all polynomial ideals have a finite Gröbner basis, this is not true in the non-commutative framework. In fact, the characterization of non-commutative polynomial ideals is impossible in the general case.

In this regard, the author notes that most practical problems in NPO contain equality constraints between pairs of monomials. The author of the Dissertation finds that, if every clique of “independent” (non-commuting) variables admits a Gröbner basis, then there exists an algorithm to solve the ideal membership problem. This first result of the author’s has immediate application in contextuality scenarios, where a set of measurements is defined and commutation relations between the POVM elements corresponding to some pairs of measurements are postulated.

Throughout the rest of the thesis, the author explores applications of NPO in different correlation scenarios.

In Chapter 3, the author introduces “correlation settings”: quantum experiments involving several measurements, some of which are compatible, i.e., they can be simultaneously conducted. The author considers the problem of characterizing the set of behaviors or probability distributions that the measurement outcomes might follow. Some scenarios further assume bounds on the dimension of the local Hilbert spaces associated to the different “factors” or “parties” carrying out each measurement.

The author introduces a hierarchy of SDP relaxations to tackle the general problem, complete under certain assumptions on how to model the factors. This hierarchy allows him to bound the limits of the set of quantum correlations in the following scenarios:

- (a) Bell scenarios under local dimension constraints. In a tripartite scenario, he investigates to what extent parties 1 and 2 and 1 and 3 can simultaneously violate the Clauser-Horne-Shimony-Holt (CHSH) inequality. He finds a trade-off between each violation that depends on the dimensionality d of the Hilbert space at site 1, ranging from monogamy at $d=2$ to maximum violation at $d=4$.
- (b) Extended Bell scenarios, where a subset of the parties can conduct collective measurements. Again, the assumption of a local dimension is optional.
- (c) Contextuality scenarios, with or without dimension constraints, defined by a hypergraph whose vertices represent projectors and whose edges represent which subsets of those projectors add up to the identity.

Finally, the author considers, in Chapter 4, the problem of generating uncertainty relations, namely, proving that a polynomial of average values of a given number of observables must be non-negative. E.g.: $1 - \sum_{i=1,2,3} \langle \sigma_i \rangle^2 \geq 0$, where $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli matrices. Inspired by this example, the author considers uncertainty relations of the form

$$K \geq \sum_j |\langle U_j \rangle|^2,$$

for a given set of unitaries $(U_j)_j$. NPO provides some SDP hierarchies to tackle these problems, but they all require great computational power.

To tackle this issue, the author introduces a hierarchy of SDPs that he dubs “Lovász hierarchy”, as its first level corresponds to the Lovász number of a weighted graph that summarizes the (anti-)commutation relations between the unitaries. While it is unknown if the Lovász hierarchy is complete, the author shows that it typically returns reasonably good upper bounds for K , given reasonable computational resources.

The above results are fairly good. The author has studied so far unexplored correlation scenarios through extensions of standard NPO methods. This has led to new effective hierarchies of SDP relaxations and also to new tools to simplify NPO.

I have a concern, however, with the presentation of the thesis. Although I enjoyed a lot reading Chapter 1, I found Chapter 2 a mess. The chapter introduces the reader to Gröbner bases, many different correlation scenarios and methods to tackle large scale NPO problems. Not only are these topics loosely connected, but most of the material in the chapter was not derived by the author. Therefore, I would like the author to extend the introductory Chapter 1 with the non-original material of Chapter 2 and devote Chapter 2 to just explaining the way he treats partially commuting monoids. Original results and literature review must be clearly separated in a PhD thesis.

In addition, I have several comments, which I would like the author to address:

- 1) Page 19, definition 21: “associative algebra”. But, by definition 12, an algebra is a ring; by definition 9, rings contain monoids and, by definition 7, the monoid product is associative. So, aren’t all algebras associative?
- 2) Page 20. The strong operator topology makes $\phi_x(T) = Tx$ continuous, not $\langle x, Tx \rangle$.

- 3) Page 32, claim 47: “since $M(y) \geq 0$, there are vectors $\bar{u}$ for each monomial u such that $M(\bar{y})_{\{u,v\}} = \langle \bar{u}, \bar{v} \rangle$ ”. That is clear if $M(y)$ is a finite dimensional matrix; for the infinite dimensional case, it needs to be proven.
- 4) Page 35: “these rules define a congruence relation”. Since the relation is not symmetric, it is not a congruence.
- 5) The part on “Polynomial replacement rules” is not very well written: unless the reader has spent some time thinking about the problem, some parts are almost incomprehensible. Some examples:
 - a) Page 37: “when a monomial ordering is available, the only obstacle to constructing a complete set of replacement rules is confluence. To overcome this, we have to make sure that $h-g \rightarrow_H 0$ whenever $h, g \rightarrow_H 0$.” After some time, I got why this was the case, but it is not evident at all.
 - b) Page 37: “Obstructions of this form are redundant”. Again, this is not difficult to see, but it is not obvious either; a short explanation is in order.
 - c) Page 38: “in such a case, either $S = \text{fwlt}(g) - \text{lt}(f)wg$ or ...”.
- 6) Definition 56: please specify that l, r, λ, ρ are monomials.
- 7) Page 40: “we have reduced the existence of a weak Gröbner representation for $S(a)$ to the existence of such a representation for each $S(a-b)$ with $1 < b \leq a$. Thus, the problem inductively reduces to the existence of such a representation for $S(0)$ ”. I would rather say that the problem inductively reduces to the existence of such a representation for $S(a-m+1)$, the last obstruction in eq. (2.18).
- 8) Page 44, eq. 2.24: Specify whether $\{x_i\}_i$ are letters or words.
- 9) Pages 44-45: The symbol “ I ” means sometimes an ideal; some other times, a set of pairs of variables.
- 10) Page 45: The proof of Lemma 65.
 - (a) The proof starts by making reference to a word w that is not yet shown to exist.
 - (b) “ $(a, b) \notin \text{mathcal{C}}$ ” $\rightarrow$ “ $(a_i, b_j) \notin I$ ”.
 - (c) The proof is not complete: it remains to show that the word w recovers the local data.
- 11) Definition 69: what is the monomial ordering in “ $w_C^* < v_C^*$ ”?
- 12) Page 49: The definition of graph product should appear after the section “commutations and local operations” (page 50), where V is defined.
- 13) Page 50: the definition of local algebras is introduced without any motivation. What are these algebras supposed to represent? Are they related to the factors H_k in page 52?
- 14) Page 57: “a unitary action of a finite group on $k < M >$ ”. What is a unitary transformation in $k < M >$?
- 15) Page 58: “when V is an Euclidean Jordan algebra”. With respect to which product?
- 16) Page 63: ref. [112] does not introduce any device-independent physical principle, so I would not write that “the goal was achieved”.
- 17) Page 64: “the choice for effects in the definition is so that the sets Q_d are convex”. The set of d -dimensional quantum correlations is not convex, even if one defines it through POVMS rather than projective measurements. Think of the bipartite Bell scenario and consider two feasible d -dimensional points $P_1(a, b|x, y)$, $P_2(a, b|x, y)$. Then it is known that, in general, the point $(1/2) * P_1(a, b|x, y) + (1/2) * P_2(a, b|x, y)$ is not d -dimensional, see <https://journals.aps.org/prabstract/10.1103/PhysRevA.80.042114>. Throughout the thesis, the author is working with the convexification of Q_d , which is fine: it is the relevant set to consider in non-i.i.d. scenarios. However, this must be acknowledged in the thesis.
- 18) Page 68: Explain why $p_{\{123\}} \in Q$, $p_{\{14\}} \in Q$ implies $p \in Q$.

- 19) Page 69: the author's notation obfuscates the fact that measurements 0 and 1 of Alice's in $p_{\{12\}}$ are not the same as in $p_{\{13\}}$. Does this answer the previous point?
- 20) Eq. (3.14): the first line of the constraints is wrong; also, I miss the constraints $T_{\{a|x\}}^2 = T_{\{a|x\}}$, $\sum_a T_{\{a|x\}} = 1$.
- 21) Section 3.5: "a contextuality scenario is specified by a graph". The author means "hypergraph".

Corrections aside, the author's thesis is good when compared to international standards. Hence, I recommend that he be awarded a PhD.

Dr. Miguel Navascués



Vienna, 11.12.2024