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Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems

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Doctoral dissertation



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Oświadczenie

Ja niżej podpisany Adam Cieřlik (nr indeksu: 1113522), doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „Monte Carlo methods for stationary solutions of general-relativistic Vlasov system” jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr hab. Patryka Macha, prof. UJ. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia

Podpis doktoranta

Podziękowania

Pragnę wyrazić głęboką wdzięczność mojemu promotorowi Patrykowi Machowi za jego nieocenioną pomoc, wsparcie i inspirację podczas całego procesu tworzenia tej rozprawy doktorskiej. Praca z nim była najbardziej rozwijającym okresem mojego życia. Zawdzięczam mu zrozumienie co to znaczy być Naukowcem.

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Jestem wdzięczny za łut szczęścia dzięki któremu znalazłem się tu gdzie jestem.

Non scholae, sed vitae discimus

Streszczenie

Celem pracy jest przedstawienie nowej metody symulacji Monte Carlo, pozwalającej na analizę rozwiązań ogólnorelatywistycznego równania Własowa, opisującego gaz nie zderzających się cząstek w silnych polach grawitacyjnych. W pracy przedstawiono podstawy formalizmu relatywistycznej przestrzeni fazowej, metody klasyfikacji trajektorii oraz ogólną konstrukcję metody Monte Carlo. Na podstawie opracowanej metody stworzono symulację numeryczną akrecji gazu Własowa na statyczną czarną dziurę Schwarzschilda oraz na czarną dziurę poruszającą się przez dwuwymiarowy rozkład gazu w płaszczyźnie równikowej. W obu przypadkach gaz rozciąga się do nieskończoności, gdzie spełnia rozkład Maxwella-Jüttnera. Uzyskane wyniki pokazują, że stworzona metoda może być stosowana z powodzeniem do analizy przypadków wykraczających poza sferyczną symetrię lub osiową symetrię w przypadku płaskim. Porównanie wyników symulacji z dokładnymi wyrażeniami na gęstość prądu cząstek wykazuje doskonałą zgodność.

Drugim ważnym wynikiem pracy jest przedstawienie nowej metody rozwiązywania równań geodezyjnych w czasoprzestrzeniach Schwarzschilda i Kerra. Bazując na mało znanych wynikach Biermanna i Weierstrassa, dla każdej z czasoprzestrzeni otrzymano pojedynczy układ wzorów opisujących wszystkie nieradialne, czasopodobne i zerowe geodezyjne, wyrażonych za pomocą funkcji Weierstrassa. Rozwiązania są parametryzowane przez stałe ruchu oraz współrzędne początkowe. Dodatkowo w przypadku czasoprzestrzeni Kerra, po raz pierwszy przedstawiono jawnie rzeczywiste wyrażenia opisujące geodezyjne bez radialnych punktów zwrotnych.

Rozprawa składa się z pięciu publikacji, jednego artykułu opublikowanego w formie preprintu oraz dodatkowego wprowadzenia.

Abstract

This thesis presents a new Monte Carlo simulation method enabling the analysis of solutions to the general-relativistic Vlasov equation describing a gas of non-colliding particles in strong gravitational fields. We describe essential elements of the general-relativistic phase-space formalism, a classification of relevant trajectories, and the general construction of our Monte Carlo scheme. Using this method, we performed numerical simulations of the accretion of the Vlasov gas onto a static Schwarzschild black hole and onto a black hole moving through a planar distribution of a gas. In both cases, the gas extends to infinity, where it obeys the Maxwell-Jüttner distribution. Our results demonstrate that this method can be successfully applied to cases that do not assume spherical symmetry or axial symmetry in the planar case. A comparison of our simulation results with exact expressions for the particle current density shows an excellent agreement.

The second significant result of this thesis is a new method of solving geodesic equations in the Schwarzschild and the Kerr spacetimes. Basing on a little-known result of Biermann and Weierstrass, we derived, separately for the Schwarzschild and the Kerr spacetimes, a single set of formulas describing all non-radial, timelike and null trajectories, expressed in terms of Weierstrass functions. The solutions are explicitly parameterized by constants of motion and the initial coordinates. Additionally, in the case of the Kerr spacetime, real expressions describing orbits without radial turning points are explicitly given.

The dissertation consists of five publications, one preprint and an additional introduction.

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List of publications

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- II Adam Cieřlik, Patryk Mach, “Timelike and Null Geodesics in the Schwarzschild Space-time: Analytical Solutions,” *Acta Phys. Pol. B Proc. Suppl.* 16, 6-A10 (June 2023). <https://www.actaphys.uj.edu.pl/S/16/6-A10/pdf>
- III Patryk Mach, Adam Cieřlik, and Andrzej Odrzywołek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes,” *Phys. Rev. D* 108, No. 12 (December 2023): 124057. <https://link.aps.org/doi/10.1103/PhysRevD.108.124057>
- IV Adam Cieřlik, Patryk Mach, and Andrzej Odrzywołek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole,” *arXiv*: 2406.04471 (June 2024). <https://arxiv.org/abs/2406.04471>
- V Adam Cieřlik, Eva Hackamnn, and Patryk Mach, “Kerr geodesics in terms of Weierstrass elliptic functions,” *Phys. Rev. D* 108, No. 2 (July 2023): 024056. <https://link.aps.org/doi/10.1103/PhysRevD.108.024056>
- VI Adam Cieřlik, Patryk Mach, and Andrzej Odrzywołek, “Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole,” *Phys. Rev. D* 106, No. 10 (November 2022): 104056. <https://link.aps.org/doi/10.1103/PhysRevD.106.104056>

Preface

The general-relativistic kinetic theory provides a description of matter complementary (or alternative) to the general-relativistic hydrodynamics or magnetohydrodynamics. If collisions between individual particles of the gas are rare, the assumption of the local thermodynamic equilibrium may be not satisfied and the hydrodynamical approximation may no longer be valid. Traditionally, the general-relativistic kinetic theory was used to study stellar systems (i.e., systems in which the stars are treated as individual particles of the gas) [39,40], collisionless plasma [13,19,22,34], or more recently, the dynamics of dark matter [20,28,29,41]. In mathematical relativity, the kinetic theory provides a reasonable matter model—the resulting system of equations is usually referred to as the Vlasov-Einstein, Boltzmann-Einstein, or Vlasov-Maxwell-Einstein system, depending on the assumed details [1–5,9–11,25]. Modern introductions to the general-relativistic kinetic theory can be found in [14]. Classic textbooks on the special-relativistic kinetic theory include [15,24].

In this thesis we deal with accretion of the collisionless matter onto black holes. Such models were first developed in 1970's in [21,27,38,43], but a rigorous treatment in the framework of the General Relativity did not appear until 2017, when Rioseco and Sarbach investigated the spherical stationary accretion model of the collisionless gas in the Schwarzschild spacetime [37]. Such models were later developed in a series of works, including the accretion in the Reissner-Nordström spacetime, the Kerr spacetime, or stationary models of the accretion onto moving Schwarzschild black holes [28,30,31]. The latter were especially investigated in the context of the dark matter accretion [29].

The accretion of magnetised collisionless plasma onto Kerr black holes was recently investigated numerically in a series of works employing the so-called Particle-In-Cell (PIC) simulation method [13,19,22,34]. Such models—closely related to a rigorous general-relativistic kinetic approach—provide a useful alternative to much more popular magnetohydrodynamical simulations. I should however emphasise a spectacular success of magnetohydrodynamical models in explaining the results of the Event Horizon Telescope Collaboration (EHT) and interpreting the radio images of matter in the vicinity of the supermassive black holes SgrA* [8] and M87* [6]. This was made possible by emulating the properties of a collisionless plasma in the general-relativistic magnetohydrodynamical simulations by appropriately adjusting electron and proton temperatures [7].

The main aim of this thesis is to provide a general-relativistic Monte Carlo simulation method capable of deriving stationary solutions of kinetic systems in strong gravitational fields. The existing Particle-In-Cell methods [13,19,22,34] are not well suited to this aim, as they are rather adapted to an initial value formulation. On the other hand, we view the Monte Carlo approach as corresponding to the natural interpretation of the kinetic description of gas as a collection of individual particles. If collisions between the particles in the gas are absent, the particles of the gas move along geodesics. Consequently,

coarse-grained quantities characterising the gas can be computed by averaging over a sample of selected geodesic trajectories. Our method consists of three steps: i) Selecting an appropriate sample of trajectories of individual particles, ii) Solving geodesic equations, iii) Computing suitable averages to obtain macroscopic quantities such as the particle current density or the components of the energy momentum tensor.

Our Monte Carlo method is described in papers III and IV. Its key novel feature is that it offers a possibility to compute stationary solutions of the general-relativistic kinetic theory. We demonstrate the robustness of this method by recovering spherically symmetric accretion solutions in the Schwarzschild spacetime obtained in [37], but also solutions of a planar accretion onto a moving Schwarzschild black hole, similar to [28, 29].

The problem of an efficient description of the geodesic motion arises naturally in our construction of the Monte Carlo method for kinetic systems. We address this problem in papers I, II, and V, devoted to the analysis of timelike and null geodesics in the Schwarzschild and Kerr spacetimes. We use Weierstrass elliptic functions, but our approach differs from existing methods. Both in the Schwarzschild and the Kerr case we were able to provide a single set of formulas describing all generic types of timelike and null geodesics, regardless of their radial type. Our formulas depend explicitly on initial data, and no a priori knowledge of the turning points is required. In addition, our analysis remains valid for a special family of geodesics in the Kerr spacetime, for which no radial turning points occur. Our method provides explicitly real formulas describing such geodesics. Up to our knowledge, this is the only existing formulation with this property. All these results are based on a little-known formula derived by Biermann and Weierstrass and published in Biermann’s dissertation [12]. Since the original Biermann’s sketch of the proof as well as the later proof given in [23] may be difficult to understand to modern readers, we provide a separate proof in Appendix A of Publication I.

This doctoral thesis is based on a series of six articles I co-authored. They are listed in the “List of Publications” as Publications I–VI. Five of these papers were already published in scientific journals. Publication IV is currently available as a preprint. The introduction of this dissertation aims to systematise the content of Publications I–VI into a logical sequence and to briefly guide the reader through these papers. To achieve this aim, these papers are not discussed in a chronological order of their publication date.

The introductory part is organised as follows. Section 1.1 introduces main elements found in all publications of this thesis. It covers some aspects of the Hamiltonian formalism, defines the relativistic phase space and introduces the Vlasov equation. It also defines some of the observable quantities which we deal with in this dissertation. Section 1.2 guides the reader through four Publications I–IV, describing some aspects of our Monte Carlo simulations. Its first two parts are devoted to the classification of geodesic orbits, relevant in our work. The following subsections discuss main results of Publications III–IV. Section 1.3 presents main results of Publications V and VI. It is not as extensive as Section 1.2, because to some degree Publications V and VI constitute generalisations of ideas discussed in the previous section to the Kerr spacetime. Our aim was to show that the method developed for the Schwarzschild spacetime can also be applied to the Kerr case in the future.

The second part of this thesis consists of Publications I–IV.

Additionally, as a part of the work on the results contained in this thesis, I prepared two *Wolfram Mathematica* packages [17] and [18]. The first one implements our solutions of timelike and null geodesics in the Kerr spacetime derived in Publication V. The second one contains an implementation of the Monte Carlo simulation of the accretion of the Vlasov gas onto a moving black hole and is based on Publication IV. Both packages are publicly available.

Notation

In Publications I–VI, spacetime dimensions are labelled with Greek indices, $\mu = 0, 1, 2, 3$, while spatial dimensions are labelled with Latin ones, $i = 1, 2, 3$. The signature of the metric is assumed to be $(-, +, +, +)$. Geometric units with $c = G = 1$ are used, where c is the speed of light, and G denotes the gravitational constant.

Our notation differs slightly across Publications I–VI. Alterations include, e.g., different symbols used to denote the proper and the coordinate time (also in their dimensionless versions) or different symbols denoting distribution functions. On the other hand, I tried to keep a consistent notation in the entire introductory part of this thesis. To ensure clarity in the text, the symbols that may be easily confused are listed below, with page references:

- s – proper time (p. 12),
- τ – affine parameter (p. 12),
- $\bar{\tau}$ – Mino time (p. 21),
- t – time coordinate (p. 15),
- $\bar{t}$ – time in general coordinates that can be adapted to different spacetime foliations (p. 15),
- T – asymptotic temperature of gas (p. 23),
- $\bar{T}$ – dimensionless time coordinate (p. 29),
- m – test particle rest mass (p. 16),
- m_0 – a fixed rest mass of a single particle (p. 23)
- M – black hole mass (p. 15),
- φ – azimuthal longitude (p. 15),
- ψ – true anomaly (p. 19),
- r – radial distance (p. 15),
- $\bar{r}$ – radius in general coordinates that can be adapted to different spacetime foliations (p. 15),
- ξ – dimensionless radial coordinate (p. 16),
- $\xi_{\max}$ – location of the maximum of the effective potential (p. 17),
- E – energy of the test particle (p. 16),
- ε – dimensionless energy of the test particle (p. 16),
- l_z – azimuthal angular momentum of the test particle (p. 16),
- λ_z – dimensionless azimuthal angular momentum of the test particle (p. 16),
- l – total angular momentum of the test particle (p. 16),
- λ – dimensionless total angular momentum of the test particle (p. 16),
- λ_c – critical angular momentum for unbound trajectories (p. 17),
- $\lambda_{\max}$ – maximum angular momentum for unbound trajectories (p. 17)

- $U_{l,m}$ – effective potential (p. 16),
- U_λ – dimensionless effective potential (p. 17),
- η_S – induced volume element on S (p. 14),
- η – a function used to control the time foliation (p. 15),
- $\mathcal{N}[S]$ – number of particles in S (p. 14),
- N – the g^{rr} element of the Schwarzschild metric (p. 15),
- n – particle number density (p. 15),
- n_s – particle surface density (p. 32),
- $\mathcal{F}$ – distribution function (p. 13),
- $\mathcal{F}^{(N)}$ – fine-grained distribution function (p. 22),
- $\tilde{F}(\phi, k)$ – Legendre elliptic integral (p. 20),
- $\mathcal{J}_\mu$ – particle current density (p. 14),
- J_μ – surface particle current density (p. 32),
- $\mathcal{T}_{\mu\nu}$ – energy-momentum tensor (p. 15),
- $T_{\mu\nu}$ – surface energy-momentum tensor (p. 32)

1. Introduction

1.1. Preliminaries

1.1.1. Hamiltonian description

In the spacetime $(\mathcal{M}, g)$, where $\mathcal{M}$ is a three dimensional differential manifold and g denotes the metric, a freely falling test particle moves along a geodesic $\tau \mapsto x^\alpha(\tau)$, described by geodesic equations

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \quad (1.1)$$

Here $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol associated with the metric g . The normalisation of the parameter τ is a matter of convention. In this case, we assume that

$$\tau = \frac{s}{m}, \quad (1.2)$$

where s is the particle proper time and m is its rest mass. The four-velocity is defined by $u^\mu = dx^\mu/ds$, where $g^{\mu\nu} u_\mu u_\nu = -1$.

An equivalent method of describing the motion of a test particle is to use the Hamiltonian formalism. In this approach, the geodesic equations (1.1) correspond to Hamilton's equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}, \quad (1.3)$$

where the Hamiltonian is given by

$$H = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu \quad (1.4)$$

and (x^μ, p_ν) are canonical variables. Assuming that

$$p^\mu = \frac{dx^\mu}{d\tau}, \quad (1.5)$$

one can write the Hamiltonian in a particularly simple form

$$H = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu = -\frac{1}{2} m^2. \quad (1.6)$$

1.1.2. Relativistic phase space

The relativistic kinetic theory is formulated in terms of the so-called distribution function defined on a phase space Γ . The latter is understood as a set of all possible particle states, represented by pairs

(x^μ, p_ν) , forming a subset of the cotangent bundle $T^*\mathcal{M}$, defined as [26, 37, 42]

$$T^*\mathcal{M} = \{(x, p) : x \in M, p \in T_x^*\mathcal{M}\}. \quad (1.7)$$

In this thesis I restrict myself to the so-called “simple gas,” i.e., the case in which the masses of all particles are the same. This limits the discussion to the mass shell Γ_m^+ , defined as follows:

$$\Gamma_m^+ = \{(x, p) \in T^*\mathcal{M} : g^{\mu\nu} p_\mu p_\nu = -m^2, p \text{ is future-directed}\}. \quad (1.8)$$

The mass shell condition can be also imposed by defining the distribution function $\mathcal{F}$ on $T^*\mathcal{M}$ and assuming that

$$\mathcal{F} \sim \delta(\sqrt{-p_\mu p^\mu} - m). \quad (1.9)$$

1.1.3. Vlasov equation

For particles that do not collide, exactly one trajectory passes through each point of phase space Γ . The set of these trajectories defines a congruence of curves on Γ . The vector field on Γ associated with this phase flow is given by [26, 37]

$$X_H = \frac{\partial H}{\partial p_\mu} \frac{\partial}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial}{\partial p_\mu} \quad (1.10)$$

and the integral curves $\gamma(\tau)$ of X_H (geodesics in the phase space) are described by Hamilton’s equations (1.3). For the Hamiltonian of the form (1.4), the associated vector field is called the Liouville vector field $L := X_H$ and can be written as

$$L = g^{\mu\nu} p_\nu \frac{\partial}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial x^\mu} \frac{\partial}{\partial p_\mu}. \quad (1.11)$$

By applying the Liouville operator to the distribution function $\mathcal{F}$, one obtains

$$L[\mathcal{F}] = \{H, \mathcal{F}\} = \frac{d}{d\tau} \mathcal{F}(x^\mu(\tau), p_\nu(\tau)), \quad (1.12)$$

where $\{\cdot, \cdot\}$ are Poisson brackets. The lack of collisions between particles is understood mathematically as a requirement that the distribution function $\mathcal{F}$ is invariant along the geodesic, i.e.,

$$\frac{d}{d\tau} \mathcal{F}(x^\mu(\tau), p_\nu(\tau)) = 0. \quad (1.13)$$

This yields, in explicit terms,

$$g^{\mu\nu} p_\nu \frac{\partial \mathcal{F}}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial x^\mu} \frac{\partial \mathcal{F}}{\partial p_\mu} = 0, \quad (1.14)$$

which is known as the Vlasov equation (also referred to as the Liouville or the collisionless Boltzmann equation). In the literature there exist at least two other popular forms of this equation [1, 15, 35], both consistent with an alternative tangent bundle formulation. Instead of viewing the phase space Γ as a part of $T^*\mathcal{M}$, one can also define it as part of the tangent bundle $T\mathcal{M}$. The relation between the coordinates

on the cotangent bundle (x^μ, p_ν) and the coordinates on the tangent bundle $(\tilde{x}^\mu, \tilde{p}^\mu)$ is given by

$$\tilde{x}^\mu = x^\mu, \quad \tilde{p}^\mu = g^{\mu\nu}(x^\alpha)p_\nu, \quad (1.15)$$

which leads to the following transformations

$$\frac{\partial \mathcal{F}}{\partial x^\mu} = \frac{\partial \mathcal{F}}{\partial \tilde{x}^\mu} + \frac{\partial g^{\mu\alpha}}{\partial x^\nu} p_\alpha \frac{\partial \mathcal{F}}{\partial \tilde{p}^\mu}, \quad \frac{\partial \mathcal{F}}{\partial p_\nu} = g^{\mu\nu} \frac{\partial \mathcal{F}}{\partial \tilde{p}^\mu}. \quad (1.16)$$

After short calculations, it can be shown that the analogue of the Vlasov equation (1.14) on a tangent bundle reads

$$\tilde{p}^\mu \frac{\partial \mathcal{F}}{\partial \tilde{x}^\mu} - \Gamma^\mu_{\alpha\beta} \tilde{p}^\alpha \tilde{p}^\beta \frac{\partial \mathcal{F}}{\partial \tilde{p}^\mu} = 0. \quad (1.17)$$

By assuming the mass shell condition $g_{\mu\nu}p^\mu p^\nu = -m^2$, the above equation can be transformed into another equally popular form [1, 16]

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{\tilde{p}^i}{\tilde{p}^0} \frac{\partial \mathcal{F}}{\partial \tilde{x}^i} - \frac{1}{\tilde{p}^0} \Gamma^i_{\alpha\beta} \tilde{p}^\alpha \tilde{p}^\beta \frac{\partial \mathcal{F}}{\partial \tilde{p}^i} = 0, \quad (1.18)$$

where $\tilde{x}^0 = t$.

1.1.4. Physical observables

The distribution function $\mathcal{F}$ is a scalar invariant quantity, but it is not an observable. In order to determine the observable quantities, one needs to look at the statistical interpretation of the distribution function. Since, as mentioned earlier, precisely one trajectory passes through each point of the phase space Γ , the average number of particles can be expressed as a suitable integral of the distribution function over the occupied volume in Γ . Strictly speaking, for a certain spacelike hypersurface S in $\mathcal{M}$, the average number of particle trajectories whose projections on $\mathcal{M}$ intersect S can be expressed as [14]

$$\mathcal{N}[S] = - \int_S \left[\int_{P_x^+} \mathcal{F}(x, p) p_\mu s^\mu \text{dvol}_x(p) \right] \eta_S, \quad (1.19)$$

where

$$P_x^+ = \{p \in T_x^* \mathcal{M} : g^{\mu\nu} p_\mu p_\nu < 0, p \text{ is future-directed}\}, \quad (1.20)$$

and s is a future-directed unit vector normal to S , η_S denotes the induced volume element on S , and $\text{dvol}_x(p)$ is the volume element on P_x^+ . In local adapted coordinates $\text{dvol}_x(p)$ is given as

$$\text{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)} dp_0 dp_1 dp_2 dp_3. \quad (1.21)$$

By defining the particle current density as

$$\mathcal{J}_\mu(x) = \int_{P_x^+} \mathcal{F}(x, p) p_\mu \text{dvol}_x(p), \quad (1.22)$$

equation (1.19) can be expressed in the form

$$\mathcal{N}[S] = - \int_S \mathcal{J}_\mu s^\mu \eta_S. \quad (1.23)$$

Using Eq. (1.14), it can be demonstrated that the particle current density obeys the conservation law $\nabla_\mu \mathcal{J}^\mu = 0$ [36], which further supports the expression (1.23). The particle number density can be covariantly defined as

$$n = \sqrt{-\mathcal{J}_\mu \mathcal{J}^\mu}. \quad (1.24)$$

Defining the particle number density as a scalar quantity helps to maintain a connection with the general-relativistic hydrodynamics. Similarly to (1.22), the energy-momentum tensor components can be defined as

$$\mathcal{T}_{\mu\nu}(x) = \int_{P_x^+} \mathcal{F}(x, p) p_\mu p_\nu d\text{vol}_x(p). \quad (1.25)$$

As before, the Vlasov equation implies that $\nabla_\mu \mathcal{T}^{\mu\nu} = 0$.

1.2. Accretion in the Schwarzschild spacetime: Monte Carlo simulations

This chapter begins with a discussion of timelike geodesics in the Schwarzschild spacetime. An efficient description of the Schwarzschild geodesics constitutes a foundation of all four Publications I–IV. The next part presents a summary of the results of these works.

1.2.1. Conserved quantities

In Publications II, III, and IV we work in standard Schwarzschild coordinates (t, r, θ, φ) and with the metric of the form

$$g = -N dt^2 + \frac{1}{N} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1.26)$$

where

$$N = 1 - \frac{2M}{r} \quad (1.27)$$

and $M > 0$ is the black hole mass. This form of the metric was selected due its simplicity, which is crucial in preparing Monte Carlo simulations. However, one can also work in horizon-penetrating coordinates. There is an elegant way to define the metric that combines both cases. Consider the following transformation

$$\bar{t} = t + \int^r \left[\frac{1}{N(\tau)} - \eta(\tau) \right] d\tau, \quad \bar{r} = r, \quad (1.28)$$

where $\eta = \eta(\bar{r})$ specifies the time foliation. In coordinates $(\bar{t}, \bar{r})$ the metric has the form

$$g_{\mu\nu} = -N d\bar{t}^2 + 2(1 - N\eta) d\bar{t} d\bar{r} + \eta(2 - N\eta) d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\varphi^2. \quad (1.29)$$

The choice $\eta \equiv 1$ corresponds to the so-called Eddington-Finkelstein coordinates, regular at the horizon. For the parameter $\eta \equiv 1/N$, one obtains the form (1.26).

The form of (1.29) was adopted in Publication I, as this paper aims to develop a general analytic description of geodesics in the Schwarzschild spacetime. In the context of the general-relativistic kinetic theory, metric (1.29) was also used in the works [28, 29, 37], devoted to analytic solutions.

The conserved quantities arise naturally from the spherical symmetry and stationarity of the Schwarzschild spacetime. To show that the choice of η does not affect their forms, one can write the geodesic Hamiltonian associated with metric (1.29),

$$H = \frac{1}{2} \left[g^{\bar{t}\bar{t}}(\bar{r}) p_{\bar{t}}^2 + 2g^{\bar{t}\bar{r}}(\bar{r}) p_{\bar{t}} p_{\bar{r}} + g^{\bar{r}\bar{r}}(\bar{r}) p_{\bar{r}}^2 + \frac{1}{\bar{r}^2} \left(p_{\theta}^2 + \frac{p_{\varphi}^2}{\sin^2 \theta} \right) \right], \quad (1.30)$$

where the contravariant metric components are given by

$$g^{\bar{t}\bar{t}} = \eta(-2 + N\eta), \quad g^{\bar{t}\bar{r}} = (1 - N\eta), \quad g^{\bar{r}\bar{r}} = N. \quad (1.31)$$

Clearly, the Hamiltonian H depends neither on $\bar{t}$ nor on φ , so the associated momenta $p_{\bar{t}} \equiv -E$ (the energy) and $p_{\varphi} \equiv l_z$ are preserved. Additionally, H does not explicitly depend on the affine parameter τ , and consequently it is also conserved. Due to the adopted normalisation of τ , this fact is equivalent to the conservation of the particle rest mass $m = \sqrt{-2H}$. It can also be shown that total angular momentum

$$l = \sqrt{p_{\theta}^2 + \frac{p_{\varphi}^2}{\sin^2 \theta}} \quad (1.32)$$

is conserved.

1.2.2. Classification of trajectories

By rewriting equation (1.30) using the conserved quantities E , l , l_z , m and solving for $p_{\bar{r}}$, one obtains

$$p_{\bar{r}} = \frac{g^{\bar{t}\bar{r}} E + \epsilon_r \sqrt{[(g^{\bar{t}\bar{r}})^2 - g^{\bar{t}\bar{t}} g^{\bar{r}\bar{r}}] E^2 - g^{\bar{r}\bar{r}} (m^2 + \frac{l^2}{\bar{r}^2})}}{g^{\bar{r}\bar{r}}} = \frac{[1 - \eta(1 - \frac{2M}{r})] E + \epsilon_r \sqrt{E^2 - U_{l,m}(\bar{r})}}{N}, \quad (1.33)$$

where $U_{l,m}(\bar{r})$ denotes the effective potential

$$U_{l,m}(\bar{r}) = \left(1 - \frac{2M}{\bar{r}}\right) \left(m^2 + \frac{l^2}{\bar{r}^2}\right). \quad (1.34)$$

The parameter $\epsilon_r = \pm 1$ determines the radial direction of the motion. For $\epsilon_r = -1$, the particle moves towards the centre of the coordinate system; when $\epsilon_r = +1$, it moves in the opposite direction. For simplicity and because $\bar{r} = r$, the dash symbol above r is omitted in the analysis below.

To facilitate further study, it is useful to introduce dimensionless variables

$$r = M\xi, \quad p_r = m\pi\xi, \quad E = m\varepsilon, \quad l = Mm\lambda, \quad l_z = Mm\lambda_z. \quad (1.35)$$

This (or similar) parametrisation is adopted in all Publications I–VI.

It is clear from equation (1.33) that the motion of a massive particle must be confined to the region in which the following inequality holds:

$$\varepsilon^2 - U_\lambda(\xi) \geq 0, \quad (1.36)$$

where

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi^2}\right). \quad (1.37)$$

Appendix B of Publication I provides a complete description of all possible trajectories in the Schwarzschild spacetime, both for timelike and null geodesics. However, in accretion problems investigated in this thesis, only unbound timelike trajectories are relevant, i.e., those for which inequality (1.36) is satisfied at asymptotic infinity. This requirement can be strictly expressed as a condition on the energy value of the particles $\varepsilon \geq 1$. Such trajectories can be further categorised as absorbed or scattered by the black hole. An incoming particle on a scattered trajectory encounters a centrifugal barrier (a turning point) in its radial motion. Let

$$\xi_{\max} = \frac{\lambda^2}{2} \left(1 - \sqrt{1 - \frac{12}{\lambda^2}}\right) \quad (1.38)$$

be a location of the maximum of effective potential $U_\lambda(\xi)$. It corresponds to the smallest possible radius at which this radial turning point can occur. Consider the equation

$$U_\lambda(\xi_{\max}) = \varepsilon^2, \quad (1.39)$$

and solve it for λ . The resulting total angular momentum λ_c is called critical and can be expressed by

$$\lambda_c(\varepsilon) = \frac{12}{1 - \frac{4}{\left(\frac{3\varepsilon}{\sqrt{9\varepsilon^2 - 8} + 1}\right)^2}}. \quad (1.40)$$

It determines the limiting value of λ for which the particle is able to bounce off the potential barrier. If the particle total angular momentum is too low, i.e., $\lambda < \lambda_c(\varepsilon)$, such a particle will be absorbed. For scattered particles, $\lambda_c(\varepsilon) < \lambda$, but the total angular momentum is limited from above. The maximum angular momentum $\lambda_{\max}(\varepsilon, \xi)$ is obtained directly from equation (1.36) and is given as

$$\lambda_{\max}(\varepsilon, \xi) = \xi \sqrt{\frac{\varepsilon^2}{1 - \frac{2}{\xi}} - 1}. \quad (1.41)$$

Consequently for scattered particles $\lambda_c(\varepsilon) < \lambda \leq \lambda_{\max}(\varepsilon, \xi)$. Note that $\xi_{\max}$ given by (1.38) satisfies $\xi_{\max} > 3$. Consequently scattered trajectories can only occur in the region with $\xi > 3$. Moreover (1.38) yields $\lambda > \xi/\sqrt{\xi - 3}$ for scattered trajectories. This gives a lower bound on the energy of scattered particles:

$$\varepsilon^2 \geq U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{1}{\xi - 3}\right). \quad (1.42)$$

Combining this result with the fact that $\varepsilon \geq 1$ for unbound trajectories, we get the bound in the form

$$\varepsilon_{\min}(\xi) = \begin{cases} \infty, & \xi \leq 3, \\ \sqrt{\left(1 - \frac{2}{\xi}\right) \left(1 + \frac{1}{\xi-3}\right)}, & 3 < \xi < 4, \\ 1, & \xi \geq 4, \end{cases} \quad (1.43)$$

which was used in Publications I, III, and IV.

1.2.3. Exact solution of the equations of motion in Schwarzschild spacetime

This section briefly discusses the main results of Publications:

- I Adam Cieřlik, Patryk Mach, “Revisiting timelike and null geodesics in the Schwarzschild spacetime: general expressions in terms of Weierstrass elliptic functions,” *Class. Quantum Grav.* 39, no. 22 (October 2022): 225003. <https://dx.doi.org/10.1088/1361-6382/ac95f2>
- II Adam Cieřlik, Patryk Mach, “Timelike and Null Geodesics in the Schwarzschild Space-time: Analytical Solutions,” *Acta Phys. Pol. B Proc. Suppl.* 16, 6-A10 (June 2023). <https://www.actaphys.uj.edu.pl/S/16/6-A10/pdf>

They were essential in preparing Monte Carlo simulations discussed in Publications III and IV. To maintain clarity, this short overview will cover only a few technical details.

We focus on finding the general solution of the geodesic equations corresponding to metric (1.29). They have the following form:

$$\frac{dr}{d\tau} = \frac{\partial H}{\partial p_r} = \epsilon_r \sqrt{E^2 - U_{l,m}(r)}, \quad (1.44a)$$

$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi} = \frac{l_z}{r^2 \sin^2 \theta}, \quad (1.44b)$$

$$\frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta} = \frac{\epsilon_\theta}{r^2} \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}, \quad (1.44c)$$

$$\frac{dt}{d\tau} = \frac{\partial H}{\partial p_t} = \frac{E}{N} + \epsilon_r \frac{1 - N\eta}{N} \sqrt{E^2 - U_{l,m}(r)}, \quad (1.44d)$$

This is the form used in Publication I. Note that in Publication II we make a slightly different choice of the geodesic parameter, leading to different equations and forms of solutions. However, the general idea and the procedure used in both publications are the same. Additionally, Publications III and IV utilise results from Publication I. Therefore, this introduction focuses on Publication I and briefly discusses the differences with respect to Publication II at the end.

Publication I takes advantage of the fact that in the Schwarzschild spacetime each trajectory moves in a certain plane. This allows for parameterising the equations of motion using the true anomaly ψ . Consequently, one needs to solve a system of three equations that only depend on the radial coordinate. Finding the solution of the radial equation allows us to solve the remaining ones by quadratures. The equation that determines the change in the radial coordinate is

$$\frac{d\xi}{d\psi} = \epsilon_r \frac{\xi^2}{\lambda} \sqrt{\varepsilon^2 - U_\lambda(\xi)} = \epsilon_r \sqrt{\frac{\varepsilon^2 - 1}{\lambda^2} \xi^4 + \frac{2}{\lambda^2} \xi^3 - \xi^2 + 2\xi}, \quad (1.45)$$

where dimensionless variables (1.35) were used. In the next steps it is convenient to adopt the following notation. Defining

$$f(\xi) = a_0 \xi^4 + 4a_1 \xi^3 + 6a_2 \xi^2 + 4a_3 \xi + a_4, \quad (1.46)$$

where

$$a_0 = \frac{\varepsilon^2 - 1}{\lambda^2}, \quad 4a_1 = \frac{2}{\lambda^2}, \quad 6a_2 = -1, \quad 4a_3 = 2, \quad a_4 = 0, \quad (1.47)$$

one can write equation (1.45) in the form

$$\frac{d\xi}{d\psi} = \epsilon_r \sqrt{f(\xi)}. \quad (1.48)$$

For a segment of a trajectory for which ϵ_r is constant, one gets

$$\psi = \epsilon_r \int_{\xi_0}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}}, \quad (1.49)$$

where ξ_0 is an arbitrarily chosen radius corresponding to the angle $\psi = 0$. Equation (1.49) involves an elliptic integral which can be computed in several ways. Two cases are important in the context of the further course of this thesis:

$$\psi(\xi)^{(\text{scat})} = \frac{-\epsilon_r}{\sqrt{y_3 - y_1}} \left[\tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi}}{y_2 - y_1}}, k \right) - \tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi_0}}{y_2 - y_1}}, k \right) \right] \quad (1.50)$$

$$\psi(\xi)^{(\text{abs})} = \frac{1}{2\sqrt{\tilde{\mu}}} \left[\tilde{F} \left(2 \arctan \sqrt{\frac{\frac{1}{2\xi} - \frac{1}{12} - \tilde{y}_1}{\tilde{\mu}}}, \tilde{k} \right) - \tilde{F} \left(2 \arctan \sqrt{\frac{\frac{1}{2\xi_0} - \frac{1}{12} - \tilde{y}_1}{\tilde{\mu}}}, \tilde{k} \right) \right], \quad (1.51)$$

corresponding to the scattered and absorbed trajectories respectively. In equation (1.50), y_1 , y_2 , and y_3 denote the roots of the polynomial $4y^3 - g_2y - g_3$, satisfying $y_1 < y_2 < y_3$. In equation (1.51) $\tilde{y}_1$ denotes a real zero of the polynomial $4y^3 - g_2y - g_3 = 4(y - y_1)(y^2 + py + q)$, where $p^2 - 4q < 0$ and thus $y^2 + py + q > 0$. The constants g_2 and g_3 are given by

$$g_2 = \frac{1}{12} - \frac{1}{\lambda^2}, \quad (1.52)$$

$$g_3 = \frac{1}{6^3} - \frac{1}{12\lambda^2} - \frac{\varepsilon^2 - 1}{4\lambda^2}. \quad (1.53)$$

They are referred to as Weierstrass invariants. The remaining parameters are defined as follows:

$$k^2 = \frac{y_2 - y_1}{y_3 - y_1}, \quad (1.54)$$

$$\tilde{k}^2 = \frac{1}{2} \left(1 - \frac{\tilde{y}_1 + p/2}{\tilde{\mu}} \right), \quad (1.55)$$

where $\tilde{\mu} = \sqrt{\tilde{y}_1^2 + p\tilde{y}_1 + q}$. A function $\tilde{F}(\phi, k)$ is the Legendre elliptic integral defined by

$$\tilde{F}(\phi, k) = \int_0^\phi \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (1.56)$$

Equations (1.50) and (1.51) are discussed in Appendix C of Publication I. However, the primary outcome of Publication I is an application of a little-known result by Biermann and Weierstrass [12] to equation (1.45), allowing one to derive a new form of the solution for $\xi = \xi(\psi)$. It reads

$$\xi(\psi) = \xi_0 + \frac{-\epsilon_r \sqrt{f(\xi_0)} \wp'(\psi) + \frac{1}{2} f'(\xi_0) \left[\wp(\psi) - \frac{1}{24} f''(\xi_0) \right] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2 \left[\wp(\psi) - \frac{1}{24} f''(\xi_0) \right]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)}. \quad (1.57)$$

Here $\wp$ is a Weierstrass elliptic function, defined by the invariants g_2 , and g_3 given by equations (1.52), i.e., $\wp(z) = \wp(z; g_2, g_3)$, f is defined in equation (1.46), and ξ_0 is the initial position.

In Publication I, it is demonstrated that formula (1.57) is valid for all types of trajectories in the Schwarzschild spacetime, even when the sign ϵ_r changes. In fact the value of ϵ_r appearing in (1.57) corresponds to the initial direction of the radial motion. Additionally, a similar equation is derived for null geodesics. Basing on the equation (1.57), the dependence of the affine parameter and the time coordinate on the true anomaly is calculated. As a result, we obtained a complete set of equations that fully describe the motion of the test particle, regardless of the type of the trajectory.

The main difference between Publications I and II is the use of different coordinate systems. In Publication II, the classical Schwarzschild coordinates (1.26) were adopted. Also, in order to be able to apply the Biermann-Weierstrass theorem to the resulting geodesic equations, the transformation of the affine parameter τ was carried out as follows:

$$\frac{dx^\mu}{d\bar{\tau}} = r^2 \frac{dx^\mu}{d\tau}. \quad (1.58)$$

The new parameter $\bar{\tau}$ is called the Mino time. This parametrisation is typically used to solve geodesic equations in the Kerr spacetime. Publication V is devoted to this issue. The solutions obtained in II constitute a limiting case of solutions obtained in V for the vanishing spin of the black hole.

1.2.4. Monte Carlo simulation methods for stationary solutions of general-relativistic Vlasov systems

In this section we give a brief overview of the main ideas and results of the following two publications:

- III Patryk Mach, Adam Cieřlik, and Andrzej Odrzywoćek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes.” *Phys. Rev. D* 108, no. 12 (December 2023): 124057. <https://link.aps.org/doi/10.1103/PhysRevD.108.124057>
- IV Adam Cieřlik, Patryk Mach, and Andrzej Odrzywoćek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole.” *arXiv*: 2406.04471 (Jun 2024). <https://arxiv.org/abs/2406.04471>

The primary outcome of these two works is a construction of a new Monte Carlo method allowing for computing stationary solutions of the general-relativistic Vlasov equation. This method was described on particular examples related to accretion problems in the Schwarzschild spacetime, but it is in fact a general one.

Since ultimately one is not interested in the distribution function but in observable quantities, such as the particle current density or the components of the energy momentum tensor, our method focuses directly on such observable quantities.

Consider the problem of computing the particle current density $\mathcal{J}_\mu(x)$ defined by (1.22). Our method estimates the components of $\mathcal{J}_\mu(x)$ corresponding to a continuous distribution function $\mathcal{F}$ by considering

a discrete distribution function

$$\mathcal{F}^{(N)}(x^\mu, p_\nu) = \sum_{i=1}^N \int \delta^{(4)}(x^\mu - x_{(i)}^\mu(\tau)) \delta^{(4)}(p_\nu - p_{\nu}^{(i)}(\tau)) d\tau \quad (1.59)$$

representing a sample of N particles moving along given trajectories $\tau \mapsto (x_{(i)}^\mu(\tau), p_{\nu}^{(i)}(\tau))$, $i = 1, \dots, N$.

In the next step, we compute the particle current density

$$\mathcal{J}_\mu^{(N)}(x) = \int_{P_x^+} \mathcal{F}^{(N)}(x, p) p_\mu \sqrt{-\det g^{\alpha\beta}(x)} dp_0 dp_1 dp_2 dp_3 \quad (1.60)$$

associated with $\mathcal{F}^{(N)}$. The resulting expression contains several Dirac delta terms. The particle current density associated with the smooth distribution function $\mathcal{F}$ is approximated by computing averages over numerical cells—subsets of suitably chosen hypersurfaces in $\mathcal{M}$. Let $\Sigma \in \mathcal{M}$ be a hypersurface, not necessarily space-like. We choose a small region $\sigma \in \Sigma$ (called a numerical cell) such that $x \in \sigma$. The components of $\mathcal{J}_\mu$ are approximated by the averages

$$\langle \mathcal{J}_\mu(x) \rangle = \frac{\int_\sigma \mathcal{J}_\mu^{(N)} \eta_\Sigma}{\int_\sigma \eta_\Sigma}, \quad (1.61)$$

where η_Σ denotes the volume element on Σ . In practice, equation (1.61) reduces to counting intersections of particle trajectories crossing σ with suitable weights, which depend on the choice of Σ . A more formal explanation justifying equation (1.61) can be found in Publication IV. It is worth emphasising that the method constructed in this way can be applied not only to the Schwarzschild spacetime but also to more complex ones, such as the Kerr spacetime. Below we provide a brief discussion of the main differences between Publications III and IV. A sample of results demonstrating the accuracy of the developed method is provided at the end.

Dealing with stationary solutions requires a suitable choice of Σ as a timelike hypersurface. Such a choice is made in both Publications III and IV. Also, in both Publications III and IV these hypersurfaces are adapted to a spherical grid of coordinates, but apart from that our choices are different. In Publication III we chose hypersurfaces of a constant angle $\varphi = \bar{\varphi}$:

$$\Sigma = \{(t, r, \theta, \varphi) : t \in \mathbb{R}, r \in \mathbb{R}, \theta \in [0, \pi), \varphi = \bar{\varphi}\}. \quad (1.62)$$

In Publication IV, the hypersurfaces of constant $r = \bar{r}$ were chosen, but this paper only deals with the planar accretion of particles confined to a single (equatorial) plane, therefore

$$\tilde{\Sigma} = \{(t, r, \theta, \varphi) : t \in \mathbb{R}, r = \bar{r}, \theta = \pi/2, \varphi \in [0, 2\pi)\}. \quad (1.63)$$

Since both publications use two different hypersurfaces to perform the counts of trajectories, they use two separate formulas to determine the particle position. Publication III uses the $\xi = \xi(\psi)$ relationship given by (1.57). Publication IV uses the relations defined by formulas (1.50) and (1.51).

In both Publications III and IV we test our method on particular Bondi-type accretion solutions. In

all cases the gas is stationary and extends to infinity. We consider either spherically symmetric or planar models, in which the motion of the gas is confined to a single plane. Simple solutions of this type can be restricted to monoenergetic particles. In more complicated ones, we assume that the gas at infinity can be described by the Maxwell-Jüttner distribution. In Publication III we only take into account models in which the gas at infinity is uniform and at rest. In Publication IV we investigate a planar model of the accretion onto a moving black hole. Such an accretion process is sometimes referred to as the Bondi-Hoyle-Lyttleton problem. It is customary, to describe it in the reference frame associated with the black hole. In this case, one considers the gas which is asymptotically homogeneous and moves with a uniform and constant velocity in a given direction. In all cases, exact solutions were known or they could have been obtained analytically.

To be slightly more precise, in Publication III we use the Maxwell-Jüttner distribution given by

$$\begin{aligned}\mathcal{F}(x, p) &= A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m_0} k^\mu p_\mu\right) = A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m_0} p_t\right) \\ &= A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp(-\beta\varepsilon),\end{aligned}\quad (1.64)$$

where $\beta = m_0/(k_B T)$ denotes the Boltzmann constant, T is the asymptotic temperature of gas, A is a proportionality constant, $k^\mu = (1, 0, 0, 0)$ is a timelike Killing vector, and m_0 denotes the rest mass of a single particle. Publication IV examines a system in which asymptotically the gas is boosted at a speed v in the direction of the x -axis. This is equivalent to the motion of the black hole in the negative direction. In the Cartesian coordinates (t, x, y, z) vector k^μ transforms as

$$k'^\mu = B^\mu{}_\nu k^\nu = (\gamma, \gamma v, 0, 0), \quad \gamma = \frac{1}{\sqrt{1-v^2}}. \quad (1.65)$$

Consequently, the boosted Maxwell-Jüttner distribution becomes

$$\mathcal{F}'(x, p) = A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left[\frac{\beta}{m_0} \gamma(p_t + v p_x)\right]. \quad (1.66)$$

In spherical coordinates, for the motion confined to the equatorial plane one obtains

$$\begin{aligned}\mathcal{F}'(x, p) &= A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left[\frac{\beta}{m_0} \gamma(p_t + v \cos\varphi p_r)\right] \\ &= A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left[-\beta\gamma(\varepsilon - \epsilon_r v \sqrt{\varepsilon^2 - 1} \cos\varphi)\right].\end{aligned}\quad (1.67)$$

These asymptotic distributions serve as a basis for selecting parameters of individual trajectories appearing in our simulation method. We have to admit that this selection procedure remains an actual challenge in all examples investigated in Publications III–IV.

Since distribution (1.64) depends effectively only on ε , selecting geodesic parameters according to this distribution function we were able to employ a classic von Neumann rejection method [33]. This is no longer the case of distribution (1.67), which depends effectively on two parameters, ε and φ . Hence, in Publication IV we used a more advanced Markov Chain Monte Carlo (MCMC) method.

We believe that the best confirmation of the quality of our method can be provided by a sample of

graphs from Publication IV, which we reproduce in Figures 1.1–1.4. They show consistency between the results obtained using the exact analytical expressions and the corresponding Monte Carlo estimators.

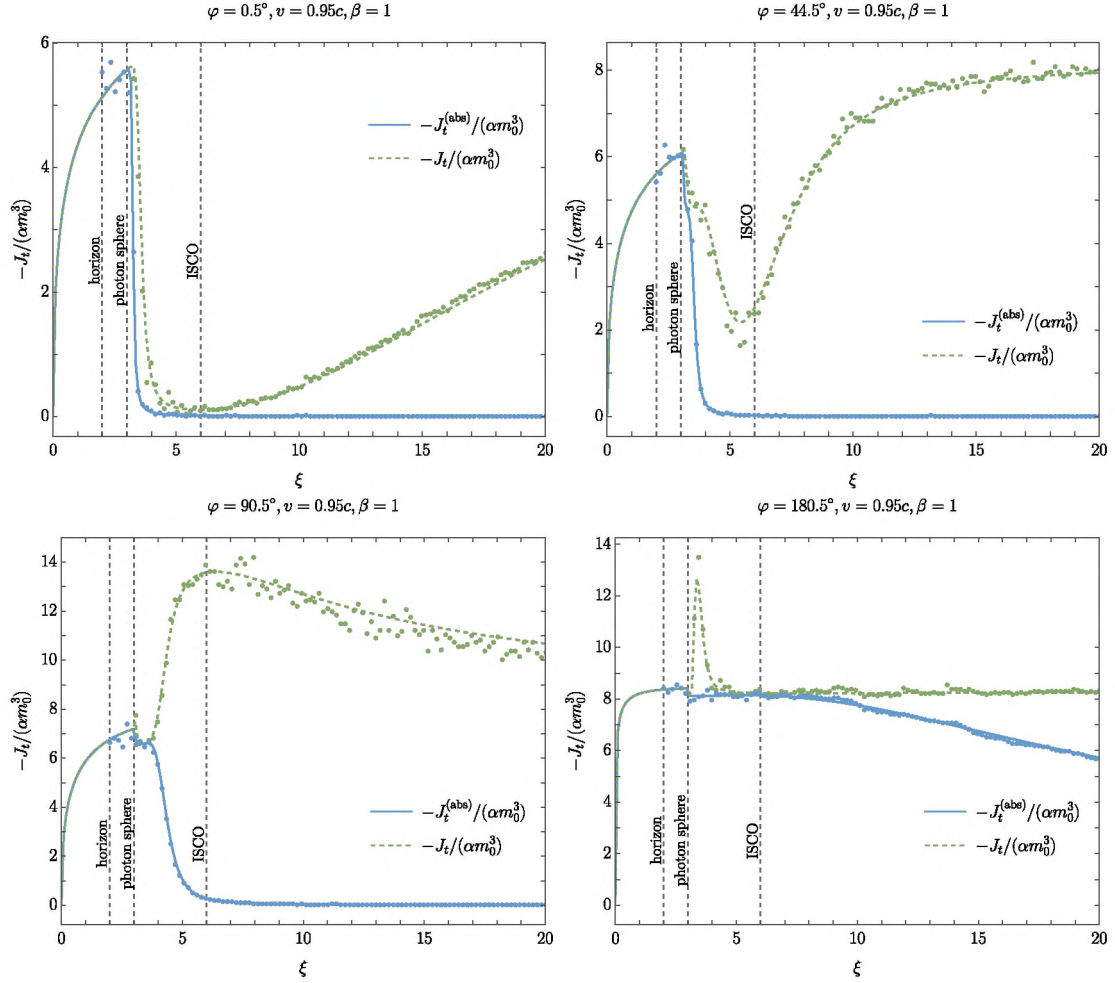


Figure 1.1: Time components of the particle current surface density J_t in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation. There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

1.3. Accretion in the Kerr spacetime

This section is devoted to the discussion of main results of Publications V and VI:

- V Adam Cieřlik, Eva Hackamnn, and Patryk Mach, “Kerr geodesics in terms of Weierstrass elliptic functions,” *Phys. Rev. D* 108, no. 2 (July 2023): 024056. <https://link.aps.org/doi/10.1103/PhysRevD.108.024056>
- VI Adam Cieřlik, Patryk Mach, and Andrzej Odrzywółek, “Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole,” *Phys. Rev. D* 106, no. 10 (November 2022): 104056. <https://link.aps.org/doi/10.1103/PhysRevD.106.104056>

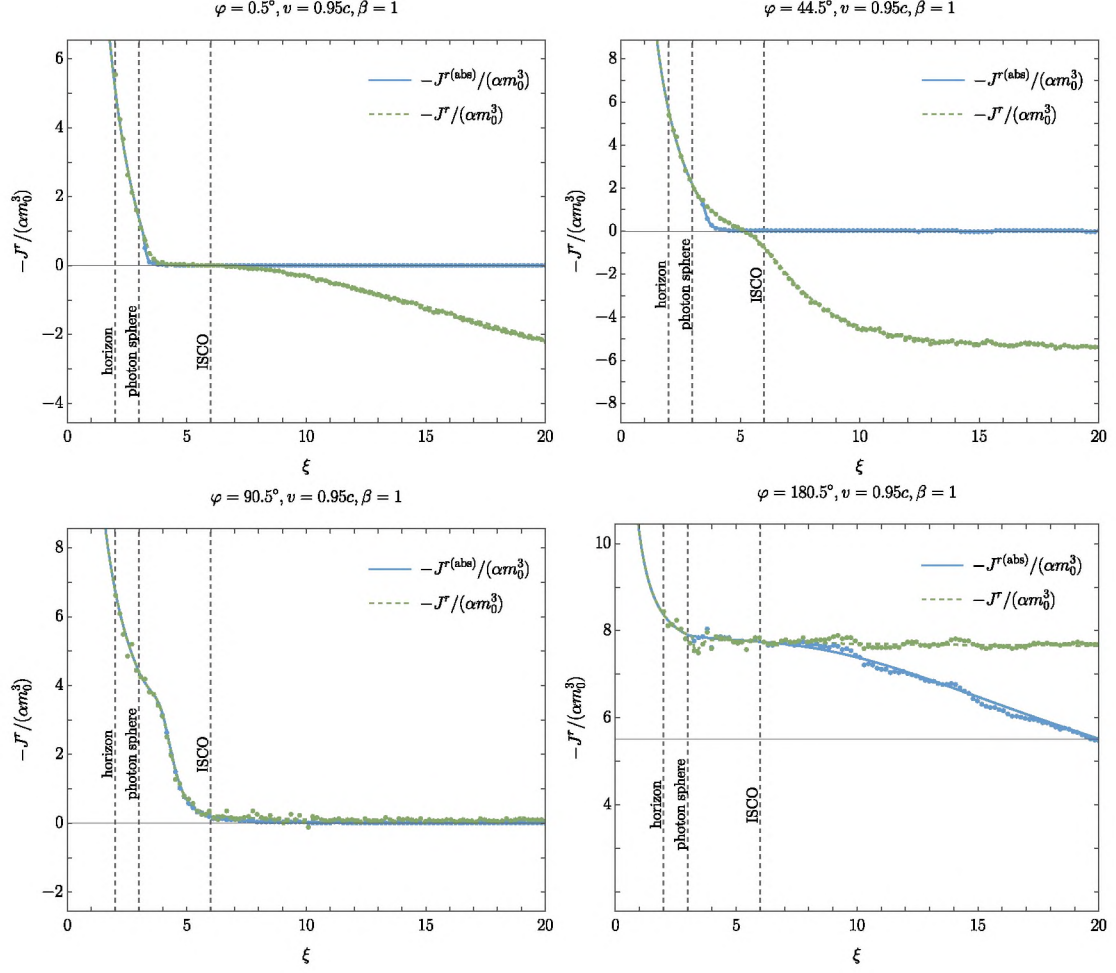


Figure 1.2: Radial components of the particle current surface density J^r in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation. There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

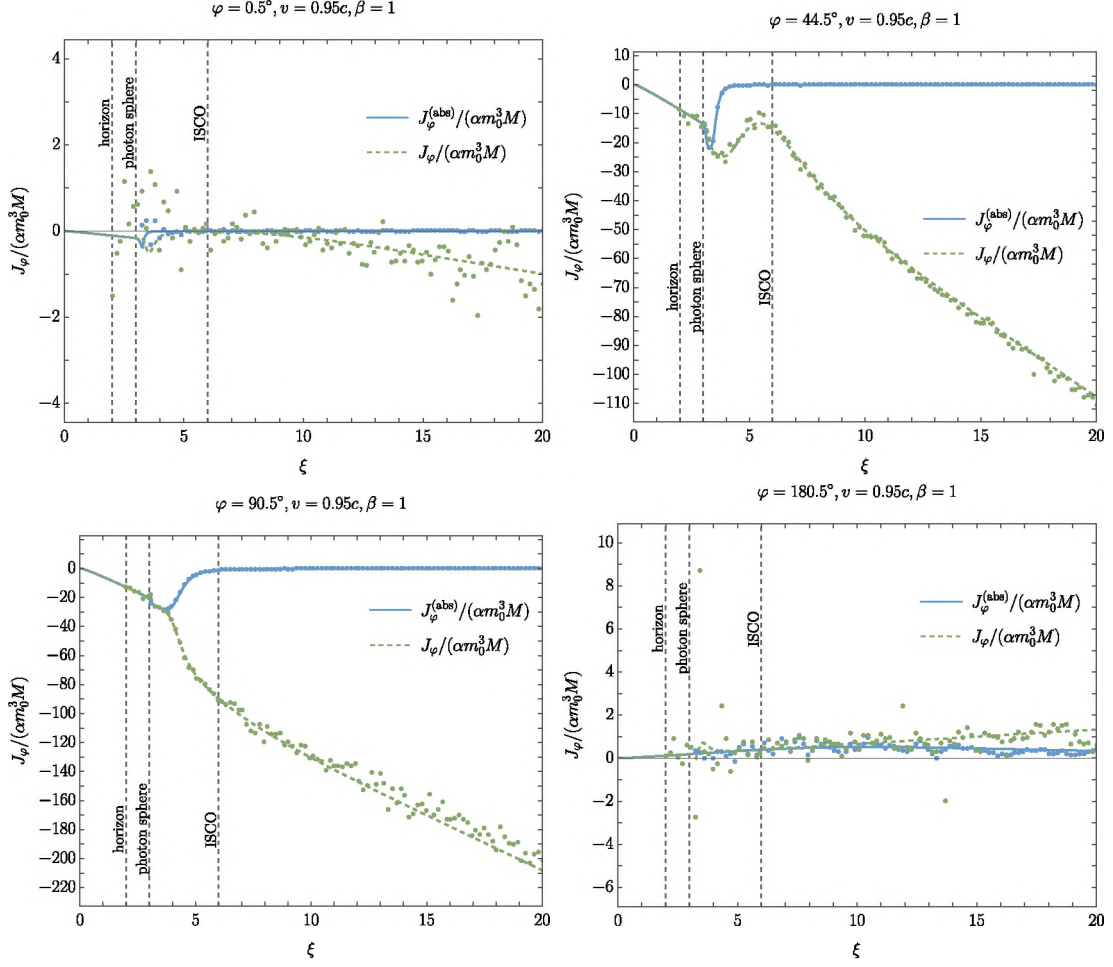


Figure 1.3: Angular components of the particle current surface density J^r in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation. There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

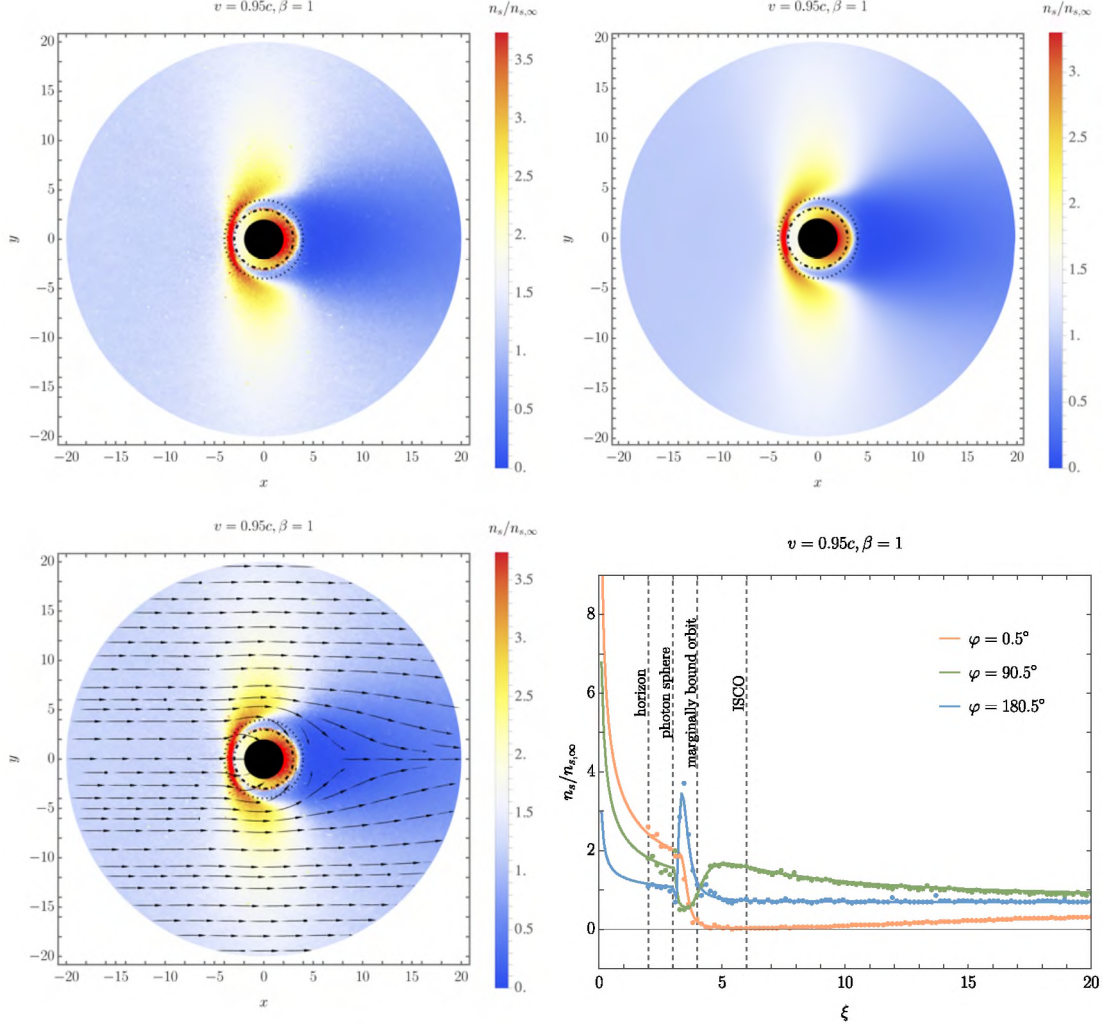


Figure 1.4: The particle surface density ratio $n_s/n_{s,\infty}$ in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. The black hole moves in the negative direction of the x axis. The first three plots show the particle number surface density in the equatorial plane. Plots on the left show the results obtained in the Monte Carlo simulation. The upper right plot shows the exact result. The remaining graph depicts the radial density profiles for three selected values of the angle φ . The simulation was prepared for 168 486 945 particle trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

These publications extend the concepts discussed so far to the Kerr spacetime. As a result, this introductory section has a similar structure to the previous one, but it no longer provides detailed explanations of the elements presented earlier.

Both publications use the same Kerr metric in Boyer-Lindquist coordinates (t, r, θ, φ) :

$$g = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + 2g_{t\varphi} dt d\varphi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \quad (1.68)$$

where

$$g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad g_{t\varphi} = -\frac{2Mar \sin^2 \theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}, \quad (1.69)$$

$$g_{\theta\theta} = \rho^2, \quad g_{\varphi\varphi} = \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2} \right) \sin^2 \theta. \quad (1.70)$$

and a denotes the spin of the black hole.

In both publications dimensionless variables are defined in a way similar to the Schwarzschild case [cf. (1.35)], with the exception that the role played previously by the conserved total angular momentum is taken over by the Carter constant. We denote the Carter constant by $\mathcal{K}$ or use an equivalent quantity l , which in the Schwarzschild limit tends to the total angular momentum. This introduction adopts the notation from Publication VI, and uses the quantity l , instead of $\mathcal{K}$.

Using the Hamiltonian formalism (1.4) and (1.3), one can express the geodesic equations for the metric (1.68) in the following form:

$$\frac{dr}{d\tau} = \frac{\partial H}{\partial p_r} = \frac{1}{\rho^2} \epsilon_r \sqrt{R(r)}, \quad (1.71a)$$

$$\frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta} = \frac{1}{\rho^2} \epsilon_\theta \sqrt{\Theta(\theta)}, \quad (1.71b)$$

$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi} = \frac{1}{\rho^2} \left[\frac{a [(r^2 + a^2) E - al_z]}{\Delta} + \frac{1}{\sin^2 \theta} (l_z - aE \sin^2 \theta) \right], \quad (1.71c)$$

$$\frac{dt}{d\tau} = \frac{\partial H}{\partial p_t} = \frac{1}{\rho^2} \left[\frac{(r^2 + a^2) [(r^2 + a^2) E - al_z]}{\Delta} + a (l_z - aE \sin^2 \theta) \right], \quad (1.71d)$$

where

$$R(r) \equiv [(r^2 + a^2)E - al_z]^2 - \Delta(m^2 r^2 + l) \quad (1.72)$$

and

$$\Theta(\theta) \equiv l - m^2 a^2 \cos^2 \theta - \left(\frac{l_z}{\sin \theta} - a \sin \theta E \right)^2. \quad (1.73)$$

1.3.1. Exact solution of geodesic equations in the Kerr spacetime

The main result of Publication V is the discovery of a new form of the general solution to the system of equations (1.71). It has been obtained by applying and developing the method introduced in Publication I for Schwarzschild geodesics. The advantage of our solution is that it is given by a single set of formulas, valid for all generic types of timelike and null geodesics. The main steps involved in the process of finding these solutions are briefly outlined below.

The solution starts by applying a reparametrisation of the affine parameter τ , introduced by Mino in [32]:

$$\frac{dx^\mu}{d\bar{\tau}} = \rho^2 \frac{dx^\mu}{d\tau}. \quad (1.74)$$

It is easy to see that for $a \rightarrow 0$, one gets the transformation (1.58). Denoting $\mu = \cos \theta$, one obtains

$$\frac{d\xi}{d\bar{\tau}} = \epsilon_r \sqrt{\bar{R}}, \quad (1.75a)$$

$$\frac{d\mu}{d\bar{\tau}} = \epsilon_\theta \sqrt{\bar{\Theta}}, \quad (1.75b)$$

$$\frac{d\varphi}{d\bar{\tau}} = \frac{\alpha [(\xi^2 + \alpha^2) \varepsilon - \alpha \lambda_z]}{\xi^2 - 2\xi + \alpha^2} + \frac{1}{\sin^2 \theta} (\lambda_z - \alpha \varepsilon + \alpha \varepsilon \mu^2), \quad (1.75c)$$

$$\frac{d\bar{T}}{d\bar{\tau}} = \frac{(\xi^2 + \alpha^2) [(\xi^2 + \alpha^2) \varepsilon - \alpha \lambda_z]}{\xi^2 - 2\xi + \alpha^2} + \alpha (\lambda_z - \alpha \varepsilon + \alpha \varepsilon \mu^2), \quad (1.75d)$$

where dimensionless variables were used, and $a = M\alpha$, $t = M\bar{T}$. Additionally

$$\bar{R} \equiv [(\xi^2 + \alpha^2) \varepsilon - \alpha \lambda_z]^2 - (\xi^2 - 2\xi + \alpha^2)(\delta_1 \xi^2 + \lambda), \quad (1.76)$$

$$\bar{\Theta} \equiv \lambda - \delta_1 \alpha^2 \mu^2 (1 - \mu^2) - (\lambda_z - \alpha \varepsilon + \alpha \varepsilon \mu^2)^2. \quad (1.77)$$

Here $\delta_1 = -g_{\mu\nu} u^\mu u^\nu$ is a parameter specifying the type of trajectory; it is defined as

$$\delta_1 = \begin{cases} 1 & \text{for timelike geodesics,} \\ 0 & \text{for null geodesics.} \end{cases} \quad (1.78)$$

Note a slight difference in the notation in equations (1.76) and (1.77) with respect to the one used in Publication V. In Publication V the polynomial $\bar{R}$ is denoted by $\tilde{R}$, a polynomial $\bar{\Theta}(\mu)$ by $g(\mu)$.

Equations (1.76) and (1.77) constitute two separate polynomial equations of the 4th degree. Thus equations (1.75a) and (1.75b) can be written in the form involving elliptic integrals analogously to (1.49). Just like in I, Publication V uses the Biermann-Weierstrass theorem to solve them, which leads to new solutions for equations (1.75a) and (1.75b). The radial motion is given by

$$\xi(\bar{\tau}) = \xi_0 + \frac{-\epsilon_r \sqrt{\bar{R}(\xi_0)} \wp'_R(\bar{\tau}) + \frac{1}{2} \bar{R}'(\xi_0) [\wp_R(\bar{\tau}) - \frac{1}{24} \bar{R}''(\xi_0)] + \frac{1}{24} \bar{R}(\xi_0) \bar{R}'''(\xi_0)}{2 [\wp_R(\bar{\tau}) - \frac{1}{24} \bar{R}''(\xi_0)]^2 - \frac{1}{48} \bar{R}(\xi_0) \bar{R}^{(4)}(\xi_0)}, \quad (1.79)$$

where $\wp_R(\bar{\tau}) = \wp(\bar{\tau}; g_{\bar{R},2}, g_{\bar{R},3})$, $g_{\bar{R},2}$ and $g_{\bar{R},3}$ are Weierstrass invariants corresponding to polynomial (1.76). As in the case of Schwarzschild geodesics, the sign ϵ_r refers to the initial direction of motion.

Similarly, solution for $\mu = \mu(\bar{\tau})$ can be given as

$$\mu(\bar{\tau}) = \mu_0 + \frac{\epsilon_\theta \sqrt{\bar{\Theta}(\mu_0)} \wp'_\Theta(\bar{\tau}) + \frac{1}{2} \bar{\Theta}'(\mu_0) [\wp_\Theta(\bar{\tau}) - \frac{1}{24} \bar{\Theta}''(\mu_0)] + \frac{1}{24} \bar{\Theta}(\mu_0) \bar{\Theta}'''(\mu_0)}{2 [\wp_\Theta(\bar{\tau}) - \frac{1}{24} \bar{\Theta}''(\mu_0)]^2 - \frac{1}{48} \bar{\Theta}(\mu_0) \bar{\Theta}^{(4)}(\mu_0)}, \quad (1.80)$$

where $\wp_\Theta(\bar{\tau}) = \wp(\bar{\tau}; g_{\bar{\Theta},2}, g_{\bar{\Theta},3})$, $g_{\bar{\Theta},2}$ and $g_{\bar{\Theta},3}$ are Weierstrass invariants corresponding to polynomial (1.77). The initial value μ_0 is given by $\mu_0 = \mu(0) = \cos \theta_0 = \cos \theta(0)$. Here, as in the radial case, the sign of ϵ_θ is to be understood as referring to the polar momentum component at the initial location θ_0 .

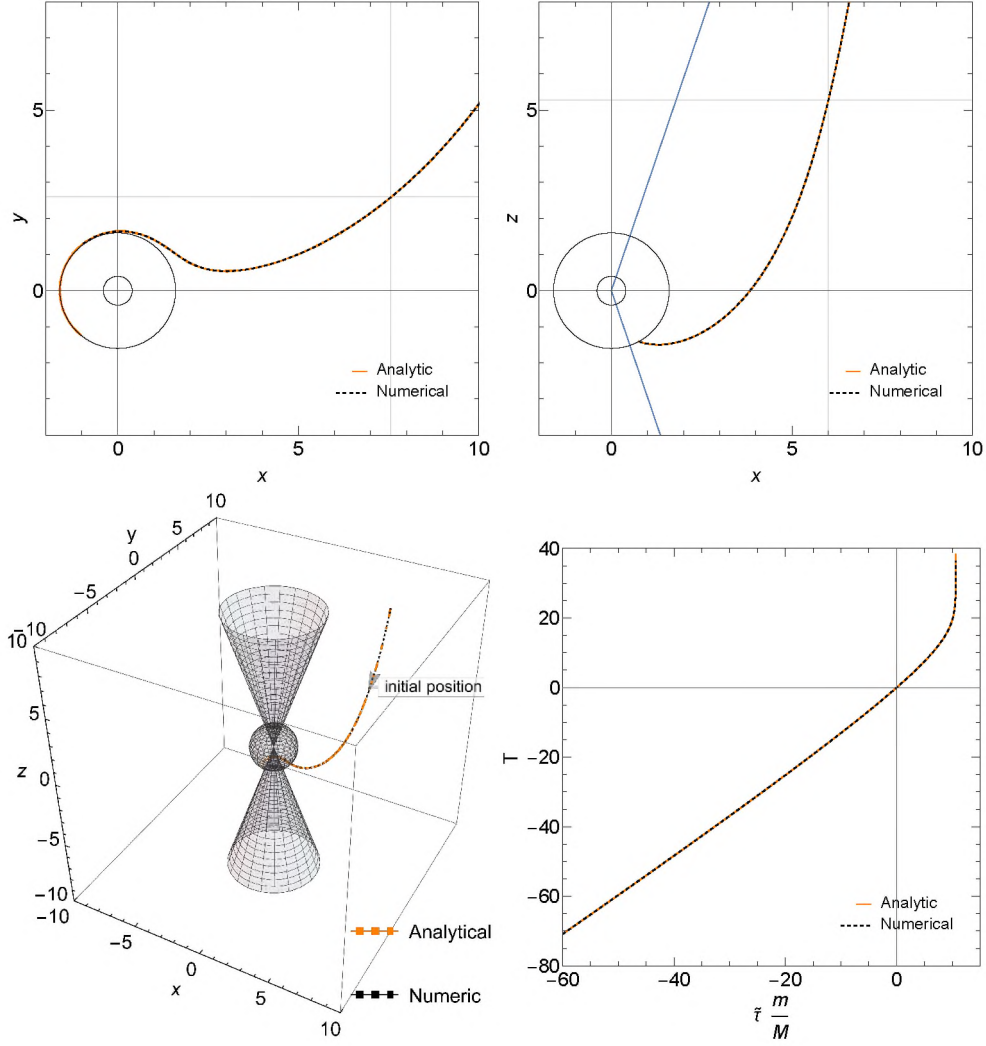


Figure 1.5: An example of a timelike unbound absorbed orbit with $\varepsilon^2 = 1.1$, $\lambda_z = -1$, $\alpha = 0.8$, $\lambda = 12$ and initial position at $\xi_0 = 8$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$. Solid color lines correspond to solutions obtained with Eqs. (1.79), (1.80). Dotted lines depict corresponding numerical solutions. The motion is depicted in two planes: xy on the right and xz on the left. The intersection of thin black lines marks the initial position. Blue lines in the xz plane and cones in the three-dimensional plot correspond to extremal values of θ . Gray spheres and dark circles depict black hole horizons. The figure in the lower right corner shows to the coordinate time T versus the proper time $\bar{\tau}$.

Equations (1.75c) and (1.75d) can be solved by substituting solutions (1.79) and (1.80) and performing integration over the Mino time $\bar{\tau}$. The forms of solutions for $\varphi(\bar{\tau})$ and $T(\bar{\tau})$ are involved, and consequently they are omitted in this summary.

Figure 1.5 illustrates a sample solution. The solid line represents our analytic results, while dashed lines represent numerical solutions.

1.3.2. Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole

The main goal of Publication VI is to investigate the properties of the accretion of the collisionless Vlasov gas onto the Kerr black hole in the equatorial plane. To achieve this, we employ the method used previously to analyse the accretion around the Schwarzschild black hole in [37] and the

Reissner-Nordström black hole in [16]. The key elements of this analysis and its outcomes are outlined below.

As before, we consider a Bondi-type stationary accretion process. The gas extends to infinity, although it is confined to the equatorial plane only. At infinity, the gas is assumed to be at rest and it is described by the (two dimensional) Maxwell-Jüttner distribution. Also as before, we only take into account gas particles moving along unbound orbits.

The first step is to classify the orbits. As in the Schwarzschild case, unbound orbits are characterised by $\varepsilon \geq 1$. Additionally, for the polynomial (1.72) a new dimensionless notation is adopted

$$R(r) = M^4 m^2 \tilde{R}(\xi), \quad (1.81)$$

where

$$\begin{aligned} \tilde{R}(\xi) &= (\xi^2 \varepsilon - \epsilon_\sigma \alpha \lambda)^2 - (\xi^2 - 2\xi + \alpha^2)(\xi^2 + \lambda^2) \\ &= \xi^4 \left[\varepsilon^2 - \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi^2}\right) - \frac{2\epsilon_\sigma \alpha \varepsilon \lambda + \alpha^2}{\xi^2} \right], \end{aligned} \quad (1.82)$$

and the sign $\epsilon_\sigma = \pm 1$ is related with the direction of motion relative to the black hole rotation. As previously, unbound trajectories are divided into two subgroups:

- absorbed orbits—with one end at infinity and the other behind the horizon. These trajectories satisfy $\varepsilon \geq 1$ and $\lambda \leq \lambda_c(\varepsilon, \epsilon_\sigma)$,
- scattered orbits—with both end points at infinity. For these orbits $\varepsilon_{\min}(\xi, \epsilon_\sigma) \leq \varepsilon$ and $\lambda_c(\varepsilon, \epsilon_\sigma) \leq \lambda \leq \lambda_{\max}(\xi, \varepsilon, \epsilon_\sigma)$.

Here $\lambda_c(\varepsilon, \epsilon_\sigma)$ is a solution (with respect to λ) of the set of equations

$$\tilde{R}(\xi) = 0, \quad (1.83a)$$

$$\frac{d\tilde{R}(\xi)}{d\xi} = 0. \quad (1.83b)$$

The formula for the upper limit $\lambda_{\max}(\xi, \varepsilon, \epsilon_\sigma)$ follows from the requirement that $\tilde{R} \geq 0$. The symbol $\varepsilon_{\min}(\xi, \epsilon_\sigma)$ denotes the minimal energy allowed for the scattered trajectory.

Next, similarly as in the Schwarzschild case (1.64), we assume the following Maxwell-Jüttner distribution at infinity

$$\mathcal{F} = A \delta\left(\theta - \frac{\pi}{2}\right) \delta(p_\theta) \delta\left(\sqrt{-p_\mu p^\mu} - m_0\right) \exp\left(\frac{\beta}{m_0} p_t\right), \quad (1.84)$$

where the terms $\delta(\theta - \pi/2)$ and $\delta(p_\theta)$ correspond to the confinement of the gas to the equatorial plane.

In order to describe the behaviour of gas in the vicinity of a black hole, we perform a suitable canonical transformation $(t, r, \varphi, p_t, p_r, p_\varphi) \mapsto (Q^\mu, P_\nu)$. It is then shown that the distribution (1.84) satisfies the Vlasov equation (1.12) in the entire phase space.

For a given distribution, it is possible to calculate the surface particle current density J_μ , the surface energy-momentum tensor $T_{\mu\nu}$ and the surface particle density $n_s = \sqrt{J^\mu J_\mu}$. The relationship between the surface particle current density and the particle current density $\mathcal{J}_\mu$ defined in (1.22) is given by

$$\mathcal{J}_\mu(t, r, \theta, \varphi) = \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) J_\mu(t, r, \varphi). \quad (1.85)$$

Similarly for the energy-momentum tensor defined in (1.25) we write

$$\mathcal{T}_{\mu\nu}(t, r, \theta, \varphi) = \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) T_{\mu\nu}(t, r, \varphi), \quad (1.86)$$

where $T_{\mu\nu}$ can be referred to as the components of the energy-momentum tensor surface density.

Ultimately, it is possible to estimate the rest-mass accretion rate $\dot{M}$, the energy accretion rate $\dot{\mathcal{E}}$ and the angular momentum accretion rate $\dot{\mathcal{L}}$:

$$\dot{M} = -2\pi M m_0 \xi J^r = 2\pi A M m_0^4 \sum_{\epsilon_\sigma=\pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \lambda_c(\varepsilon, \epsilon_\sigma), \quad (1.87)$$

$$\dot{\mathcal{E}} = 2\pi M \xi T^r_t = 2\pi A m_0^4 M \sum_{\epsilon_\sigma=\pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma), \quad (1.88)$$

$$\dot{\mathcal{L}} = -2\pi M \xi T^r_\varphi = 2\pi A m_0^4 M^2 \sum_{\epsilon_\sigma=\pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \left[\frac{\epsilon_\sigma}{2} \lambda_c^2(\varepsilon, \epsilon_\sigma) + \alpha \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma) \right]. \quad (1.89)$$

Publication VI contains a detailed analysis of the behaviour of quantities (1.87)–(1.89) for different values of the black hole spin α and asymptotic gas temperatures T , related to the parameter β by $\beta = m_0/(k_B T)$.

1.4. Concluding remarks

This thesis introduces a Monte Carlo method of solving the general-relativistic Vlasov equation. A novel feature of our method, which sets it in contrast to existing particle-in-cell schemes, is that it allows for dealing with stationary solutions. We demonstrate the robustness of our method on a collection of examples related to Bondi-type accretion problems in the Schwarzschild spacetime, for which exact solutions can be obtained within the framework of the general-relativistic kinetic theory. In particular, we were able to go beyond the spherical symmetry (or axial symmetry for planar problems) and recover exact solutions representing the planar accretion of the collisionless gas onto a moving Schwarzschild black hole.

We believe that our Monte Carlo scheme opens a new perspective for dealing with kinetic systems in which controlling the underlying phase-space geometry is hard, and consequently finding exact solutions remain beyond the reach of current methods. As an example of possible future applications one might think of the accretion of the collisionless gas onto a moving Kerr black hole or dealing with a magnetised gas. On the other hand, even simpler problems are still left open. We believe that our method could be used to confirm the results on the accretion of the collisionless gas onto a moving Schwarzschild black hole, obtained recently in [28, 29].

Our results concerning the description of geodesics in the Schwarzschild and Kerr spacetimes appeared as a by-product of the development of Monte Carlo simulations presented in this thesis. We were quite surprised to discover the possibility of a consistent and uniform description of the Schwarzschild and the Kerr geodesics in terms of Weierstrass functions, made possible by the application of the Biermann-Weierstrass theorem. The most remarkable feature of this formulation is that a single set of formulas can be used to describe all generic timelike and null geodesics, irrespective of their type (e.g., the number of radial turning points). In the Kerr spacetime there exists a class of geodesics with no radial turning points. This stands in contrast to the Schwarzschild case, in which $r = 0$ can always be regarded as a formal radial turning point. Unlike other solution methods available in the literature, our formalism provides explicitly real analytic expressions for the geodesics belonging to this class.

We also believe that our formalism, used so far for the Schwarzschild and Kerr geodesics, can be generalised to other important cases. A generalisation to the geodesic motion in the Reissner-Nordström or Kerr-Newman spacetimes seems to be nearly straightforward. Well-known spacetimes with cosmological constant would probably require the use of hyperelliptic functions.

Bibliography

- [1] Håkan Andréasson, “The Einstein-Vlasov System/Kinetic Theory”, *Living Rev. Relativ.* 14 (May 2011): 4. <https://doi.org/10.12942/lrr-2011-4>
- [2] Håkan Andréasson, “Existence of steady states of the massless Einstein–Vlasov system surrounding a Schwarzschild black hole”, *Ann. Henri Poincaré* 22 (September 2021): 4271–4297. <https://doi.org/10.1007/s00023-021-01104-6>
- [3] Håkan Andréasson, Markus Kunze, and Gerhard Rein, “Existence of axially symmetric static solutions of the Einstein-Vlasov system”, *Commun. Math. Phys.* 308 (August 2010): 23–47. <https://doi.org/10.1007/s00220-011-1324-8>
- [4] Håkan Andréasson, Markus Kunze, and Gerhard Rein, “Rotating, Stationary, Axially Symmetric Spacetimes with Collisionless Matter”, *Commun. Math. Phys.* 329 (July 2014): 787–808 (2014). <https://doi.org/10.1007/s00220-014-1904-5>
- [5] Håkan Andréasson and Gerhard Rein, “On the steady states of the spherically symmetric Einstein-Vlasov system”, *Class. Quantum Grav.* 24, No. 7 (March 2007): 1809. <https://iopscience.iop.org/article/10.1088/0264-9381/24/7/008>
- [6] Kazunori Akiyama et al., “First M87 Event Horizon Telescope results. I. The shadow of the supermassive black hole”, *Astrophys. J.* 875, No. 1 (April 2019): L1. <https://iopscience.iop.org/article/10.3847/2041-8213/ab0ec7>
- [7] Kazunori Akiyama et al., “First M87 Event Horizon Telescope results. VIII. Magnetic field structure near the event horizon”, *Astrophys. J.* 910, No. 1 (March 2021): L13. <https://iopscience.iop.org/article/10.3847/2041-8213/abe4de>
- [8] Kazunori Akiyama et al., “First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way”, *Astrophys. J.* 930, No. 2 (May 2022): L12. <https://iopscience.iop.org/article/10.3847/2041-8213/ac6674>
- [9] Ellery Ames, Håkan Andréasson, “Stationary solutions of the axially symmetric Einstein–Vlasov system: present status and open problems”, *Class. Quantum Grav.* 41, No. 7 (March 2024): 073001. <https://iopscience.iop.org/article/10.1088/1361-6382/ad29e7/meta>
- [10] Ellery Ames, Håkan Andréasson, and Anders Logg, “On axisymmetric and stationary solutions of the self-gravitating Vlasov system”, *Class. Quantum Grav.* 33, No. 15 (July 2016): 155008. <https://dx.doi.org/10.1088/0264-9381/33/15/155008>
- [11] Ellery Ames, Håkan Andréasson, and Oliver Rinne, “Dynamics of gravitational collapse in the axisymmetric Einstein–Vlasov system”, *Class. Quantum Grav.* 38, No. 10 (April 2021): 105003. <https://dx.doi.org/10.1088/1361-6382/abdd0c>
- [12] Wilhelm G. A. Biermann, “Problemata quaedam mechanica functionum ellipticarum ope soluta”, *PhD diss.*, Humboldt-Universität (Berlin, 1865). <http://dx.doi.org/10.18452/108>
- [13] Ashley Bransgrove, Bart Ripperda, and Alexander Philippov, “Magnetic Hair and Reconnection in Black

- Hole Magnetospheres”, *Phys. Rev. Lett.* 127, No.5 (July 2021): 055101. <https://link.aps.org/doi/10.1103/PhysRevLett.127.055101>
- [14] Rubén O. Acuña-Cárdenas, Carlos Gabarrete, and Olivier Sarbach, “An introduction to the relativistic kinetic theory on curved spacetimes”, *Gen Relativ Gravit* 54, No. 23 (February 2022). <https://doi.org/10.1007/s10714-022-02908-5>
- [15] Carlo Cercignani and Gilberto M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Basel 2002). <https://doi.org/10.1007/978-3-0348-8165-4>
- [16] Adam Cieřlik and Patryk Mach, “Accretion of the Vlasov gas on Reissner-Nordström black holes”, *Phys. Rev. D* 102, No. 2 (July 2022): 024032. <https://link.aps.org/doi/10.1103/PhysRevD.102.024032>
- [17] Adam Cieřlik, Wolfram Mathematica packages supplementing the article: Adam Cieřlik, Eva Hackamnn and Patryk Mach, “Kerr geodesics in terms of Weierstrass elliptic functions”, PANGAEA, 2023, <https://doi.org/10.1594/PANGAEA.960855>.
- [18] Adam Cieřlik, Wolfram Mathematica packages supplementing the article: Adam Cieřlik, Patryk Mach, and Andrzej Odrzywółek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole”, Jagiellonian University in Kraków, 2024, To appear in Cracow Open Research Data Repository (RODBUK). Available at <https://github.com/AdamCieslik/Monte-Carlo-for-stationary-sol-of-GR-Vlasov-systems-planar-accretion-onto-a-moving-Schwarzschild-BH>
- [19] Benjamin Crinquand et al., “Synthetic Images of Magnetospheric Reconnection-Powered Radiation around Supermassive Black Holes”, *Phys. Rev. Lett.* 129, No. 20 (November 2022):205101. <https://link.aps.org/doi/10.1103/PhysRevLett.129.205101>
- [20] Paola Domínguez-Fernández et al., “Description of the evolution of inhomogeneities on a dark matter halo with the Vlasov equation”, *Gen. Relativ. Gravit.* 49, No. 19 (April 2017): 123. <https://doi.org/10.1007/s10714-017-2286-8>
- [21] Juhan Frank and Martin J. Rees, “Effects of Massive Central Black Holes on Dense Stellar Systems”, *Mon. Not. R. Astron. Soc.* 176, No. 633 (September 1976): 633-647. <https://doi.org/10.1093/mnras/176.3.633>
- [22] Alisa Galishnikova et al., “Collisionless Accretion onto Black Holes: Dynamics and Flares”, *Phys. Rev. Lett.* 130, No. 6 (March 2023): 115201. <https://link.aps.org/doi/10.1103/PhysRevLett.130.115201>
- [23] George A. Greenhill, *The applications of elliptic functions* (London, New York, Macmillan, 1892). <https://archive.org/details/applicationselli00greerich>
- [24] Sybren R. de Groot, Willem A. van Leeuwen, and Christiann G. van Weert, *Relativistic Kinetic Theory* (North-Holland Publishing Company, Amsterdam, 1980).
- [25] Mahir Hadžić, Zhiwu Lin, and Gerhard Rein, “Stability and Instability of Self-Gravitating Relativistic Matter Distributions”, *Archive for Rational Mechanics and Analysis* 241 (May 2021): 1–89. <https://doi.org/10.1007/s00205-021-01647-2>
- [26] Werner Israel, “Relativistic kinetic theory of a simple gas”, *J. Math. Phys.* 4, No. 9 (September 1963): 1163–1181. <https://doi.org/10.1063/1.1704047>
- [27] Alan P. Lightman and Stuart L. Shapiro, “The distribution and consumption rate of stars around a massive, collapsed object”, *Astrophys. J.* 211 (January 1977): 244-262. <https://ui.adsabs.harvard.edu/abs/1977ApJ...211..244L/abstract>
- [28] Patryk Mach and Andrzej Odrzywółek, “Accretion of the relativistic Vlasov gas onto a moving Schwarzschild

- black hole: Exact solutions”, *Phys. Rev. D* 103, No. 2 (January 2021): 024044. <https://link.aps.org/doi/10.1103/PhysRevD.103.024044>
- [29] Patryk Mach and Andrzej Odrzywolek, “Accretion of Dark Matter onto a Moving Schwarzschild Black Hole: An Exact Solution”, *Phys. Rev. Lett.* 126, No. 10 (March 2021): 101104. <https://link.aps.org/doi/10.1103/PhysRevLett.126.101104>
- [30] Patryk Mach and Andrzej Odrzywolek, “Accretion of the Relativistic Vlasov Gas onto a Moving Schwarzschild Black Hole: Low-temperature Limit and Numerical Aspects”, *Acta Phys. Pol. B Proc. Suppl.* 15, 1-A7 (January 2022). <https://www.actaphys.uj.edu.pl/S/15/1-A7/pdf>
- [31] Patryk Mach, Andrzej Odrzywolek, “Equatorial accretion on the Kerr black hole”, *Acta Phys. Pol. B Proc. Suppl.* 16, 6-A13 (June 2023). <https://www.actaphys.uj.edu.pl/S/16/6-A13/pdf>
- [32] Yasushi Mino, “Perturbative approach to an orbital evolution around a supermassive black hole”, *Phys. Rev. D* 67, No. 8 (April 2003): 084027. <https://doi.org/10.1103/PhysRevD.67.084027>
- [33] John von Neumann, “Various techniques used in connection with random digits, in Monte Carlo Method”, *Appl. Math. Series* 12 (1951): 36–38. https://mcnp.lanl.gov/pdf_files/InBook_Computing_1961_Neumann_JohnVonNeumannCollectedWorks_VariousTechniquesUsedinConnectionwithRandomDigits.pdf
- [34] Kyle Parfrey, Alexander Philippov, and Benoît Cerutti, “First-Principles Plasma Simulations of Black-Hole Jet Launching”, *Phys. Rev. Lett.* 122, No. 3 (January 2019): 035101. <https://link.aps.org/doi/10.1103/PhysRevLett.122.035101>
- [35] Alan D. Rendal, “An introduction to the Einstein-Vlasov system”, *Banach Center Publications* 41, No. 1 (1997): 35–68. <http://eudml.org/doc/252216>
- [36] Luciano Rezzolla and Olindo Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, 2013).
- [37] Paola Rioseco and Olivier Sarbach, “Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole”, *Class. Quantum Grav.* 34, No. 9 (April 2017): 095007. <https://dx.doi.org/10.1088/1361-6382/aa65fa>
- [38] Stuart L. Shapiro and Alan B. Marchant, “Star clusters containing massive, central black holes: Monte Carlo simulations in two-dimensional phase space”, *Astrophys. J.* 225 (October 1978): 603-624. <https://ui.adsabs.harvard.edu/abs/1978ApJ...225..603S/abstract>
- [39] Stuart L. Shapiro and Saul A. Teukolsky, “Relativistic stellar systems with spindle singularities”, *Astrophys. J.* 419 (December 1993): 622. <https://ui.adsabs.harvard.edu/abs/1993ApJ...419..622S/abstract>
- [40] Stuart L. Shapiro and Saul A. Teukolsky, “Relativistic stellar systems with rotation”, *Astrophys. J.* 419 (December 1993): 636. <https://ui.adsabs.harvard.edu/abs/1993ApJ...419..636S/abstract>
- [41] Stuart L. Shapiro, “Spikes and accretion of unbound, collisionless matter around black holes”, *Phys. Rev. D* 108, No. 8 (October 2023): 083037. <https://journals.aps.org/prd/abstract/10.1103/PhysRevD.108.083037>
- [42] Gerald E. Tauber and James W. Weinberg, “Internal State of a Gravitating Gas”, *Phys. Rev.* 122, No. 4 (May 1961): 1342–1365. <https://link.aps.org/doi/10.1103/PhysRev.122.1342>.
- [43] Peter J. Young, “Stellar density cusp around a massive black hole”, *Astrophys. J.* 217, No. 1 (October 1977): 287-295. <https://ui.adsabs.harvard.edu/abs/1977ApJ...217..287Y/abstract>

2. Publications

- 2.1. Revisiting timelike and null geodesics in the Schwarzschild spacetime:
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Revisiting timelike and null geodesics in the Schwarzschild spacetime: general expressions in terms of Weierstrass elliptic functions

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Abstract

The theory of Schwarzschild geodesics is revisited. Basing on a result by Weierstrass and Biermann, we derive a formula describing all non radial, timelike and null trajectories in terms of Weierstrass elliptic functions. Quite remarkably, a single formula works for an entire geodesic trajectory, even if it passes through turning points. Using this formula, we derive expressions for the proper and coordinate time along the geodesic.

Keywords: Weierstrass elliptic functions, Schwarzschild geodesic, analytic methods

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we revisit some elements of the theory of geodesics in the Schwarzschild spacetime. The motivation for repeating this classic calculation in a new form stems from the works on the kinetic description of the Vlasov gas on the Schwarzschild spacetime and the accretion of the Vlasov gas onto Schwarzschild black holes [13, 23, 25, 43–45, 53, 54]. In particular, we were motivated by an ongoing project aiming at constructing Monte Carlo type simulations of the gas consisting of collisionless particles moving around the Schwarzschild black hole.

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Having those applications in mind, we put special emphasis on unbounded trajectories—they are neglected in many discussions, but they play a crucial role in the description of Bondi-type accretion processes.

Existing descriptions of Schwarzschild geodesics differ in the parametrization and also in types elliptic functions used to express the solutions. As usual, different prescriptions appear to be more or less convenient, depending on the actual problem at hand. Our goal was to specify the constants of motion (in particular the energy and the angular momentum) together with the initial location of the particle, and obtain the corresponding trajectory in an exact and reliable manner. We achieve this aim using Weierstrass elliptic functions, but our prescription turns out to be different than existing ones (or at least the ones we are aware of). The main result presented in this paper is a concise formula describing all types of timelike and null trajectories in the Schwarzschild metric (except for the purely radial ones), based on a theorem due to Biermann and Weierstrass.

The problem of an analytic description of the motion of test particles in the Schwarzschild spacetime is nearly as old as the Schwarzschild solution itself. The first attempt to solve geodesic equations in the Schwarzschild spacetime was published in 1917 by Droste, together with a derivation of the Schwarzschild metric [18] (this paper is now also available as a ‘Golden Oldie’ reprint [19]). Droste expressed his solution in terms of the Weierstrass elliptic function. Thirteen years later Hagihara gave a full description of the motion of test particles around a Schwarzschild black hole [34]. His work contains a complete characterization of all types of allowed orbits and is now a classic position in the theory of Schwarzschild geodesics.

Simultaneously to the development of solutions based on Weierstrass functions, Forsyth, Greenhill, Morton, Darwin, Mielnik, and Plebański succeeded in expressing Schwarzschild geodesics in terms of Jacobi elliptic functions and Legendre integrals [16, 17, 21, 30, 47, 49]. Publication years of these papers span across several decades. As a historical remark, let us note that the authors of references [16, 47], published in 1959 and 1962, already admitted that their calculations could have been made almost forty years earlier, as all required mathematical tools were already known at that time.

In subsequent years, researchers were mostly aware of the existence of two general ways of solving Schwarzschild geodesic equations, choosing between them according to their preferences and needs [8, 10, 15, 24, 26, 31, 32, 37–39, 42, 46, 50, 55, 56, 58–60, 62]. Similar methods were also applied to an analysis of the geodesic motion in other spherically symmetric spacetimes, for which geodesic equations are solvable [12, 14, 51, 61]. In some of such cases equations of motion contain effectively a fifth degree polynomial expression, leading to hyperelliptic abelian integrals [33, 40, 61].

A slightly different approach to the geodesic motion in the Schwarzschild metric was recently proposed by Scharf [57]. Scharf’s analysis is based on a simplified version of a result which we will refer to as the Biermann–Weierstrass formula.

According to references [29, 52, 65], around 1860 Weierstrass obtained a general solution of an ordinary differential equation

$$\frac{d^2x(t)}{dt^2} = \alpha x(t)^3 + \beta x(t)^2 + \gamma x(t) + \delta, \quad (1)$$

where α , β , γ , and δ are constant coefficients, which is roughly equivalent to an integral problem

$$t = \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{f(x')}}, \quad (2)$$

where f is a quartic polynomial, and $x_0 = x(0)$. Weierstrass' solution was later published by his student Biermann, as a part of his inaugural dissertation [6]. The Weierstrass–Biermann formula is derived in Greenhill's textbook [29]; it appears also in the classic textbook by Whittaker and Watson [65]. On the other hand, it is absent in other textbooks dealing with elliptic functions from that time [3, 9, 20, 35, 36], nor does it appear in standard textbooks and tables used by physicists [1, 2, 5, 7, 11, 27].

The Biermann–Weierstrass formula for $x = x(t)$ simplifies, if x_0 is a zero of the polynomial f , and this version is used by Scharf. In the context of the geodesic motion this requirement restricts the choice of the starting (reference) point of the trajectory to turning points. In our work, we start with the general Biermann–Weierstrass expression, and hence this restriction is removed. A single formula (equation (36) of this article) describes all time-like and null geodesic trajectories, except purely radial ones. Moreover, it is valid along the entire trajectory, even if it passes through turning points. The latter fact is not immediately obvious from the derivation of equation (36), but it can be shown using addition theorems for elliptic functions.

The affine parameter and the coordinate time along a geodesic can be obtained as integrals involving the Biermann–Weierstrass expression. While, in principle, they can be evaluated assuming the general form of equation (36), the resulting formulas are lengthy and thus of little practical use. For completeness, we decided to provide appropriate expressions for the affine parameter and the coordinate time, assuming the simplified version of the Biermann–Weierstrass formula.

A proof of the Biermann–Weierstrass formula is given in appendix A; we decided to provide this material, since existing, known to us proofs of the Biermann–Weierstrass formula are rather hard to follow in detail. Apart from a sketch of the proof given in Biermann's dissertation [6], there is a proof in Greenhill's textbook [29], and its more modern account in [52]. We fill some minor gaps missing in [52].

The order of this paper is as follows. The next section specifies horizon-penetrating coordinates used in this paper. Equations of motion are derived in section 3. The main result of this paper—a single formula describing non-radial, timelike and null trajectories—is given in section 4. In section 5 we discuss the range of the true anomaly parameter. The proper time and the coordinate time along a geodesic are computed in sections 6 and 7, respectively. The proof of the Biermann–Weierstrass formula is given in appendix A. Appendix B contains a brief classification of timelike and null geodesics. In appendix C we evaluate certain elliptic integrals, required to control the range of the true anomaly parameter for unbounded orbits.

Throughout the paper we use geometric units with $c = G = 1$, where c is the speed of light, and G denotes the gravitational constant. The signature of the metric is assumed to be $(-, +, +, +)$. Spacetime dimensions are labeled with Greek indices, $\mu = 0, 1, 2, 3$.

2. Horizon-penetrating coordinates

We will work in spherical coordinates (t, r, θ, φ) . In its simplest form (in the so-called Schwarzschild coordinates) the Schwarzschild metric can be written as

$$g = -N d\bar{t}^2 + \frac{d\bar{r}^2}{N} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\varphi^2, \quad (3)$$

where

$$N = 1 - \frac{2M}{\bar{r}}, \quad (4)$$

and M denotes the black hole mass. Since ultimately we envisage applications of geodesic solutions to accretion problems, we will also need coordinates in which the metric is explicitly regular at the horizon. Such coordinates can be easily obtained by a choice of the time foliation. The transformation

$$t = \bar{t} + \int^{\bar{r}} \left[\frac{1}{N(s)} - \eta(s) \right] ds, \quad r = \bar{r}, \quad (5)$$

where $\eta = \eta(\bar{r})$ is a function of radius $\bar{r}$, yields the metric in the form

$$g = -Ndt^2 + 2(1 - N\eta)dt dr + \eta(2 - N\eta)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (6)$$

The function η defines the time foliation. A popular choice $\eta \equiv 1$ leads to coordinates which are manifestly regular at the horizon $r = 2M$, and which are sometimes referred to as Eddington–Finkelstein coordinates. Note that since we only change the time foliation, the radial coordinate r retains its interpretation as the areal radius.

Contravariant components of the metric (6) are given by

$$g^{tt} = \eta(-2 + N\eta), \quad g^{tr} = 1 - N\eta, \quad g^{rr} = N, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad g^{\varphi\varphi} = \frac{1}{r^2 \sin^2 \theta}. \quad (7)$$

Moreover,

$$(g^{tr})^2 - g^{rr}g^{tt} = 1. \quad (8)$$

3. Geodesic motion

There are many well-known descriptions of the geodesic motion. In order to keep a connection with the works [44, 45, 53, 54], we will work in the Hamiltonian framework. The Hamiltonian H describing the geodesic motion of a free particle can be chosen as

$$H = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu. \quad (9)$$

Here (x^μ, p_μ) are treated as canonical variables, and H depends on x^α through $g^{\mu\nu}(x^\alpha)$. It is easy to verify that the Hamilton equations

$$\frac{dx^\mu}{d\tilde{s}} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tilde{s}} = -\frac{\partial H}{\partial x^\mu} \quad (10)$$

lead to standard geodesic equations of the form

$$\frac{d^2 x^\mu}{d\tilde{s}^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}} = 0. \quad (11)$$

For timelike geodesics we choose the affine parameter $\tilde{s}$ as a rescaled proper time $\tilde{\tau}$, i.e., $\tilde{s} = \tilde{\tau}/m$, where m is the particle rest mass. The four velocity $u^\mu = dx^\mu/d\tilde{\tau}$ is normalized as $g_{\mu\nu}u^\mu u^\nu = -1$. We require that $p^\mu = dx^\mu/d\tilde{s}$, and that $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = -\frac{1}{2} m^2$.

For null geodesics $m = 0$ and $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = 0$. In this case the affine parameter $\tilde{s}$ is unique up to a transformation

$$\tilde{s} \rightarrow \tilde{s}' = \alpha\tilde{s} + \beta, \quad (12)$$

with $\alpha > 0$ and an arbitrary β . Such an affine reparametrization implies a rescaling of the tangent vector

$$p^\mu \rightarrow p'^\mu = \frac{1}{\alpha} p^\mu. \quad (13)$$

The explicit form of the Hamiltonian H , assuming metric (6), reads

$$H = \frac{1}{2} \left[g^{tt}(r) p_t^2 + 2g^{tr}(r) p_t p_r + g^{rr}(r) p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right) \right]. \quad (14)$$

Since H depends neither on t nor on φ , the momentum components $E \equiv -p_t$ (the energy) and $l_z \equiv p_\varphi$ are constants of motion. The Hamiltonian H is also independent of $\bar{s}$, and hence it is also conserved. A simple calculation allows one to check that the total angular momentum

$$l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}} \quad (15)$$

is another constant of motion.

The remaining momentum components p_θ and p_r can be expressed as

$$p_\theta = \epsilon_\theta \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}} \quad (16)$$

and

$$p_r = \frac{g^{tr} E + \epsilon_r \sqrt{[(g^{tr})^2 - g^{tt} g^{rr}] E^2 - g^{rr} \left(m^2 + \frac{l^2}{r^2} \right)}}{g^{rr}} = \frac{(1 - N\eta) E + \epsilon_r \sqrt{E^2 - \tilde{U}_{l,m}(r)}}{N}, \quad (17)$$

where

$$\tilde{U}_{l,m}(r) = \left(1 - \frac{2M}{r} \right) \left(m^2 + \frac{l^2}{r^2} \right) \quad (18)$$

is the radial effective potential, and where we have introduced the signs $\epsilon_\theta = \pm 1$, and $\epsilon_r = \pm 1$, corresponding to the directions of motion. Here and in what follows, the symbol $\sqrt{}$ denotes the non-negative branch of the square root. Equation (16) follows directly from equation (15). Equation (17) can be obtained from the equality $H = -\frac{1}{2}m^2$. Note also that, contrary to the formula for p_r , the expression for p' ,

$$p' = \epsilon_r \sqrt{E^2 - \tilde{U}_{l,m}(r)}, \quad (19)$$

does not depend on η . On the other hand

$$p' = \frac{E}{N} + \epsilon_r \frac{1 - N\eta}{N} \sqrt{E^2 - \tilde{U}_{l,m}(r)}, \quad (20)$$

and this expression enters the equations of motion listed below. Also note that, while for $\epsilon_r = +1$ (outgoing motion) the expression for p_r diverges at the horizon, the corresponding

expression for $\epsilon_r = -1$ is perfectly regular (particles are allowed to fall into the black hole). An expression for p_r with $\epsilon_r = -1$, manifestly regular at $r = 2M$, reads

$$p_r = -\eta E + \frac{m^2 + \frac{l_z^2}{r^2}}{E + \sqrt{E^2 - \tilde{U}_{l,m}(r)}}. \quad (21)$$

Equations $dx^\mu/d\tilde{s} = \partial H/\partial p_\mu$ can be written as

$$\frac{dr}{d\tilde{s}} = \frac{\partial H}{\partial p_r} = \epsilon_r \sqrt{E^2 - \tilde{U}_{l,m}(r)}, \quad (22a)$$

$$\frac{d\varphi}{d\tilde{s}} = \frac{\partial H}{\partial p_\varphi} = \frac{l_z}{r^2 \sin^2 \theta}, \quad (22b)$$

$$\frac{d\theta}{d\tilde{s}} = \frac{\partial H}{\partial p_\theta} = \frac{\epsilon_\theta}{r^2} \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}, \quad (22c)$$

$$\frac{dt}{d\tilde{s}} = \frac{\partial H}{\partial p_t} = \frac{E}{N} + \epsilon_r \frac{1 - N\eta}{N} \sqrt{E^2 - \tilde{U}_{l,m}(r)}. \quad (22d)$$

Note that the dependence on η appears only in equation (22d). In standard Schwarzschild coordinates $N\eta \equiv 1$ and $dt/d\tilde{s} = E/N$.

It is convenient to work in dimensionless rescaled variables. For timelike geodesics we define such variables as in [54], i.e., by

$$t = M\tau, \quad r = M\xi, \quad p_r = m\pi_\xi, \quad p_\theta = Mm\pi_\theta, \quad E = m\varepsilon, \quad l = Mm\lambda, \quad l_z = Mm\lambda_z. \quad (23)$$

In addition, a new affine parameter s is defined by

$$\tilde{s} = \frac{M}{m}s. \quad (24)$$

For null geodesics $m = 0$. We introduce an arbitrary mass parameter $\tilde{m} > 0$, and define

$$t = M\tau, \quad r = M\xi, \quad p_r = \tilde{m}\pi_\xi, \quad p_\theta = M\tilde{m}\pi_\theta, \quad E = \tilde{m}\varepsilon, \quad l = M\tilde{m}\lambda, \quad l_z = M\tilde{m}\lambda_z, \quad (25)$$

and $\tilde{s} = (M/\tilde{m})s$.

In terms of these dimensionless variables, the equations of motion (22) can be written as

$$\frac{d\xi}{ds} = \epsilon_r \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \quad (26a)$$

$$\frac{d\varphi}{ds} = \frac{\lambda_z}{\xi^2 \sin^2 \theta}, \quad (26b)$$

$$\frac{d\theta}{ds} = \epsilon_\theta \frac{1}{\xi^2} \sqrt{\lambda^2 - \frac{\lambda_z^2}{\sin^2 \theta}}, \quad (26c)$$

$$\frac{d\tau}{ds} = \frac{\varepsilon}{N(\xi)} + \epsilon_r \frac{1 - N(\xi)\eta(\xi)}{N(\xi)} \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \quad (26d)$$

where $N(\xi) = 1 - 2/\xi$. The dimensionless radial potential reads

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi^2}\right) = 1 - \frac{2}{\xi} + \frac{\lambda^2}{\xi^2} - \frac{2\lambda^2}{\xi^3} \quad (27)$$

for timelike geodesics, and

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \frac{\lambda^2}{\xi^2} \quad (28)$$

for null ones.

It is well known that geodesic motion in the Schwarzschild spacetime is confined to a plane. Choosing the coordinate system so that $\theta \equiv \pi/2$, and $d\theta/ds \equiv 0$, we get $\lambda^2 = \lambda_z^2$, and thus $\lambda_z = \pm\lambda$. We will adopt a convention with $\lambda \geq 0$ and define the angle in the orbital plane (the so-called true anomaly) $\psi = \text{sgn}(\lambda_z)\varphi$. The relevant equations of motion can be written as

$$\frac{d\xi}{ds} = \epsilon_r \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \quad (29a)$$

$$\frac{d\psi}{ds} = \frac{\lambda}{\xi^2}, \quad (29b)$$

$$\frac{d\tau}{ds} = \frac{\varepsilon}{N(\xi)} + \epsilon_r \frac{1 - N(\xi)\eta(\xi)}{N(\xi)} \sqrt{\varepsilon^2 - U_\lambda(\xi)}. \quad (29c)$$

System (29) can also be obtained by introducing standard orbital elements such as the orbital inclination, the argument of periapsis, the argument of latitude, and the true anomaly (see, e.g. [39]). Another possibility to (partially) decouple the equations of motion (26) is to introduce the so-called Mino time [48].

A qualitative analysis of the effective radial potential allows for a general classification of different types of orbits. This is done, to some extent, in appendix B, both for timelike and null orbits. In general, we divide trajectories into bound and unbound ones. Unbound trajectories can either start at infinity and plunge into the black hole (we refer to such trajectories as absorbed ones). The second large class of unbound trajectories consists of orbits characterized by sufficiently large angular momentum. In this case the particles arriving from infinity are scattered off the centrifugal barrier (these trajectories are referred to as scattered ones).

4. Solution of equations of motion

4.1. Timelike geodesics

We will start our analysis with timelike geodesics. Given the form of equation (29), it is natural to treat ψ as a parameter and search for a solution of the form $\xi = \xi(\psi)$. From (29a) and (29b) we get immediately

$$\frac{d\xi}{d\psi} = \epsilon_r \frac{\xi^2}{\lambda} \sqrt{\varepsilon^2 - U_\lambda(\xi)} = \epsilon_r \sqrt{\frac{\varepsilon^2 - 1}{\lambda^2} \xi^4 + \frac{2}{\lambda^2} \xi^3 - \xi^2 + 2\xi}. \quad (30)$$

Defining

$$f(\xi) = a_0 \xi^4 + 4a_1 \xi^3 + 6a_2 \xi^2 + 4a_3 \xi + a_4, \quad (31)$$

where

$$a_0 = \frac{\varepsilon^2 - 1}{\lambda^2}, \quad 4a_1 = \frac{2}{\lambda^2}, \quad 6a_2 = -1, \quad 4a_3 = 2, \quad a_4 = 0, \quad (32)$$

one can write equation (30) as

$$\frac{d\xi}{d\psi} = \epsilon_r \sqrt{f(\xi)}. \quad (33)$$

For a segment of the trajectory for which ϵ_r is constant, we get

$$\psi = \epsilon_r \int_{\xi_0}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}}, \quad (34)$$

where ξ_0 is an arbitrarily chosen radius corresponding to the angle $\psi = 0$. We emphasize that $\sqrt{\cdot}$ is assumed to be non-negative. Weierstrass invariants of the polynomial f read (see appendix A)

$$g_2 = \frac{1}{12} - \frac{1}{\lambda^2}, \quad (35a)$$

$$g_3 = \frac{1}{6^3} - \frac{1}{12\lambda^2} - \frac{\varepsilon^2 - 1}{4\lambda^2}. \quad (35b)$$

Therefore, thanks to the Biermann–Weierstrass theorem (see appendix A for a statement of this theorem and the proof), we can write the formula for $\xi = \xi(\psi)$ as

$$\xi(\psi) = \xi_0 + \frac{-\epsilon_r \sqrt{f(\xi_0)} \wp'(\psi) + \frac{1}{2} f'(\xi_0) [\wp(\psi) - \frac{1}{24} f''(\xi_0)] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2 [\wp(\psi) - \frac{1}{24} f''(\xi_0)]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)}. \quad (36)$$

Here $\wp$ is understood to be defined by the invariants g_2 , and g_3 given by equation (35), i.e., $\wp(z) = \wp(z; g_2, g_3)$, and f is defined in equations (31) and (32).

We emphasize that formula (36) works in a much more general setting than described above. It turns out to be valid also for trajectories along which the sign ϵ_r changes. This can be checked numerically, but there is also a way to demonstrate this fact analytically. The argument can be sketched as follows.

Denote the functions defined by equation (36) and corresponding to two different signs ϵ_r as

$$\xi_{-}(\psi; \xi_0) = \xi_0 + \frac{+\sqrt{f(\xi_0)} \wp'(\psi) + \frac{1}{2} f'(\xi_0) [\wp(\psi) - \frac{1}{24} f''(\xi_0)] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2 [\wp(\psi) - \frac{1}{24} f''(\xi_0)]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)} \quad (37)$$

and

$$\xi_{+}(\psi; \xi_0) = \xi_0 + \frac{-\sqrt{f(\xi_0)} \wp'(\psi) + \frac{1}{2} f'(\xi_0) [\wp(\psi) - \frac{1}{24} f''(\xi_0)] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2 [\wp(\psi) - \frac{1}{24} f''(\xi_0)]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)}. \quad (38)$$

It follows from equation (34) that $\xi_{-}(\psi; \xi_0) = \xi_{+}(-\psi; \xi_0)$.

Consider a particle moving initially inwards (i.e., with $\epsilon_r = -1$) from a starting position ξ_0 to the turning point ξ_1 , for which $f(\xi_1) = 0$, and then moving outwards (with $\epsilon_r = +1$) up

to a point with the radius ξ . The angle ψ corresponding to this motion can be expressed as $\psi = \psi_1 + \psi_2$, where

$$\psi_1 = -\int_{\xi_0}^{\xi_1} \frac{d\xi'}{\sqrt{f(\xi')}} = \int_{\xi_1}^{\xi_0} \frac{d\xi'}{\sqrt{f(\xi')}}, \quad \psi_2 = \int_{\xi_1}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}}. \quad (39)$$

For both angles ψ_1 and ψ_2 we have, according to the Biermann–Weierstrass theorem (equation (A.19)):

$$\wp(\psi_1) = \frac{f'(\xi_1)}{4(\xi_0 - \xi_1)} + \frac{f''(\xi_1)}{24}, \quad (40a)$$

$$\wp(\psi_2) = \frac{f'(\xi_1)}{4(\xi - \xi_1)} + \frac{f''(\xi_1)}{24}, \quad (40b)$$

$$\wp'(\psi_1) = -\frac{f'(\xi_1)\sqrt{f(\xi_0)}}{4(\xi_0 - \xi_1)^2}, \quad (40c)$$

$$\wp'(\psi_2) = -\frac{f'(\xi_1)\sqrt{f(\xi)}}{4(\xi - \xi_1)^2}. \quad (40d)$$

The simplicity of the above formulas is, of course, due to the fact that $f(\xi_1) = 0$. Using expression (37) we get $\xi_1 = \xi_-(\psi_1; \xi_0)$. The fact that the formula (37) describes the continuation of the trajectory in the segment from ξ_1 to ξ means that

$$\xi = \xi_-(\psi_1 + \psi_2; \xi_0) = \xi_+(\psi_2; \xi_1) = \xi_-(-\psi_2; \xi_1). \quad (41)$$

While the above expression could, in principle, be checked directly, it is much easier to check the corresponding relations involving Weierstrass $\wp$ functions. According to the addition theorem for the Weierstrass elliptic function $\wp$, we have

$$\wp(\psi_1 - \psi_2) = \frac{1}{4} \left[\frac{\wp'(\psi_1) + \wp'(\psi_2)}{\wp(\psi_1) - \wp(\psi_2)} \right]^2 - \wp(\psi_1) - \wp(\psi_2). \quad (42)$$

Inserting in the above formula the expressions for $\wp(\psi_1)$, $\wp(\psi_2)$, $\wp'(\psi_1)$, and $\wp'(\psi_2)$ given by equation (40), we get, after some algebra,

$$\wp(\psi_1 - \psi_2) = \frac{\sqrt{f(\xi)f(\xi_0)} + f(\xi_0)}{2(\xi - \xi_0)^2} + \frac{f'(\xi_0)}{4(\xi - \xi_0)} + \frac{f''(\xi_0)}{24}, \quad (43)$$

as predicted by the Biermann–Weierstrass formula (A.19). Deriving equation (43), we have to remember that f is a fourth order polynomial given by equation (31), and $f(\xi_1) = 0$. The reason for considering the difference $\psi_1 - \psi_2$, instead of the sum $\psi_1 + \psi_2$, can be understood in the light of equation (41) and the fact that $\xi_+(\psi_2; \xi_1) = \xi_-(-\psi_2; \xi_1)$.

In summary, equation (36) can be used to describe any orbit with $\psi = 0$ for $\xi = \xi_0$. The sign ϵ_r in equation (36) can be understood as referring to the direction of motion at $\psi = 0$, and it need not be changed as the trajectory passes through a turning point. This stays in a clear contrast to the approaches based on Jacobi and Legendre elliptic functions, where one has to deal with different types of orbits separately. Apart from this universality, the main practical advantage of formula (36) is the fact that it does not require finding zeros of the polynomial f . Of course, there are applications in which the knowledge about zeros of the polynomial f

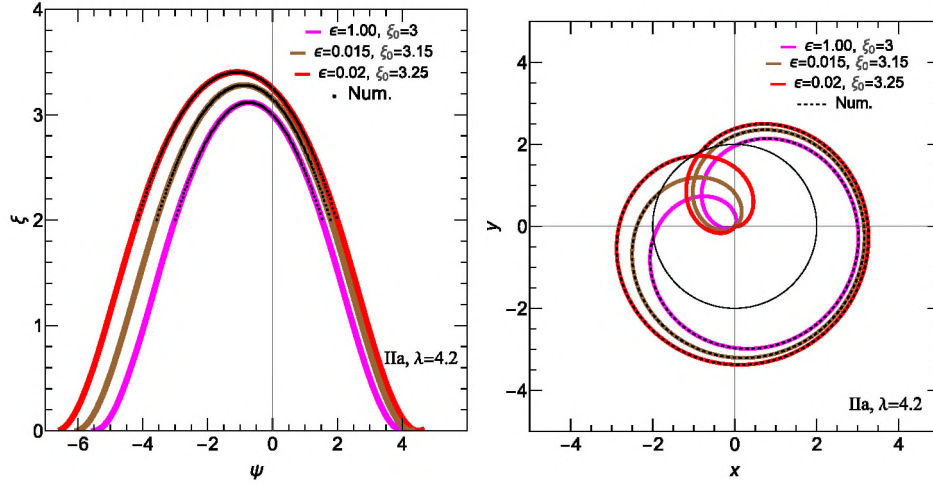


Figure 1. Sample of timelike inner bound orbits (type IIa) for $\lambda = 4.2$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

is required—we require such knowledge indirectly in sections 5–7, dealing with the allowed range of ψ , the proper and coordinate time s and τ , respectively. Note that

$$f(\xi) = \frac{\xi^4}{\lambda^2} [\varepsilon^2 - U_\lambda(\xi)], \quad (44)$$

and consequently zeros of the polynomial f are related to zeros of the expression $\varepsilon^2 - U_\lambda(\xi)$, corresponding to turning points and discussed in appendix B. The Biermann–Weierstrass expression is based on a transformation of the integral appearing on the right-hand side of equation (34) to the Weierstrass form, i.e.,

$$\int_{\xi_0}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}} = \pm \int_{w(x)}^{\infty} \frac{dw'}{\sqrt{4w'^3 - g_2w' - g_3}} \quad (45)$$

(see appendix A). Zeros of the polynomial $W = 4w^3 - g_2w - g_3$ depend on the sign of the discriminant $\Delta = g_2^3 - 27g_3^2$. The case with $\Delta = 0$ corresponds to $\lambda = \lambda_c(\varepsilon)$, defined by equation (B.6).

Figures 1–4 show various kinds of orbits obtained with the help of equation (36). Figure 1 depicts examples of bound inner orbits. Figure 2 shows a sample outer bound orbit. Unbound absorbed orbits are shown in figure 3. Finally, a family of unbound scattered orbits is plotted in figure 4. In all figures, the left panel depicts the radius ξ versus the angle ψ . Right panels show the orbits in the orbital plane with Cartesian coordinates x, y . For comparison, in all cases we draw the same orbits obtained by integrating geodesic equations numerically. These numerical results are depicted with dotted or dashed lines.

4.2. Null geodesics

The reasoning for null geodesics is analogous. The equation defining the trajectory reads

$$\frac{d\xi}{d\psi} = \epsilon_r \sqrt{\frac{\varepsilon^2}{\lambda^2} \xi^4 - \xi^2 + 2\xi}. \quad (46)$$

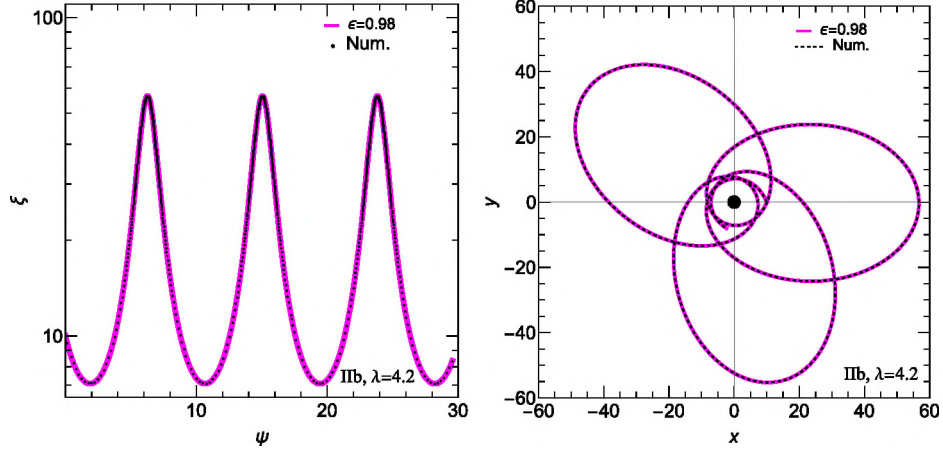


Figure 2. Sample of timelike outer bound orbits (type IIb) for $\lambda = 4.2$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

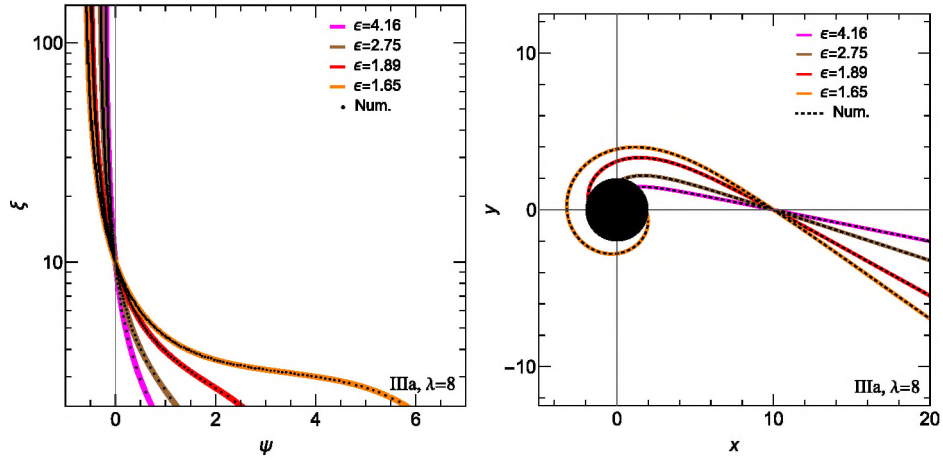


Figure 3. Sample of timelike unbound absorbed orbits (type IIIa) for $\lambda = 8$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

Adhering to the same notation as for timelike orbits, we set

$$f(\xi) = a_0\xi^4 + 4a_1\xi^3 + 6a_2\xi^2 + 4a_3\xi + a_4 = \frac{\varepsilon^2}{\lambda^2}\xi^4 - \xi^2 + 2\xi, \quad (47)$$

i.e.,

$$a_0 = \frac{\varepsilon^2}{\lambda^2}, \quad a_2 = -\frac{1}{6}, \quad a_3 = \frac{1}{2}, \quad (48)$$

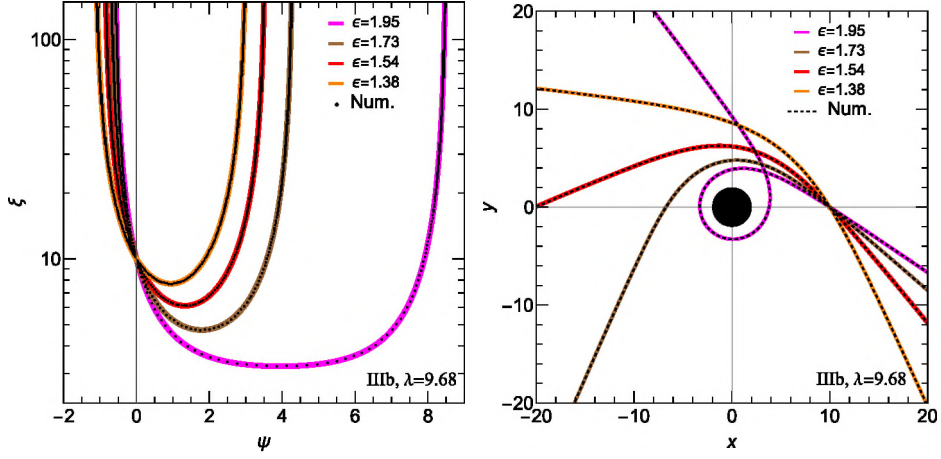


Figure 4. Sample of timelike unbound scattered orbits (type IIIb) for $\lambda = 9.68$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

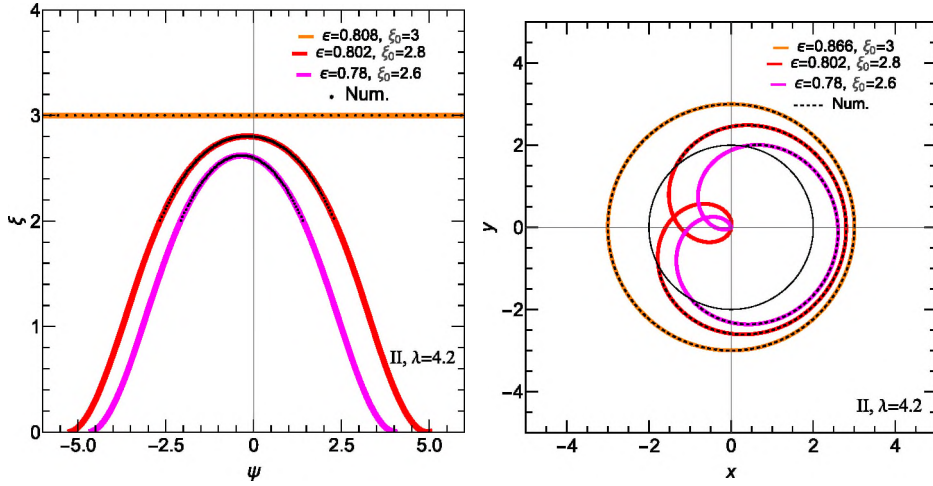


Figure 5. Sample of null bound orbits (type II) for $\lambda = 4.2$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

and $a_1 = a_4 = 0$. The Weierstrass invariants can be written as

$$g_2 = \frac{1}{12}, \quad (49a)$$

$$g_3 = \frac{1}{216} - \frac{\varepsilon^2}{4\lambda^2}. \quad (49b)$$

With these modifications, remaining equations of the previous subsection hold for null geodesics as well. In particular, equation (36), with $f(\xi)$ and the Weierstrass invariants given by equation (47) and (49), is valid also for null geodesics.

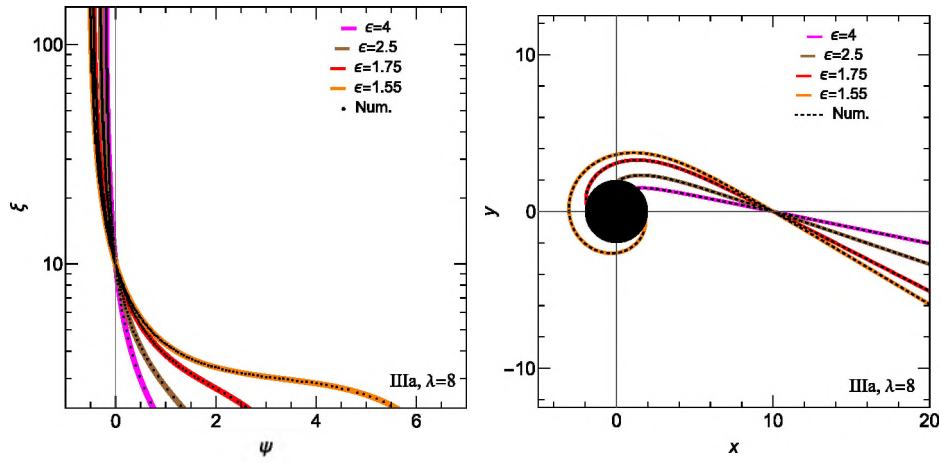


Figure 6. Sample of null unbound absorbed orbits (type IIIa) for $\lambda = 8$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

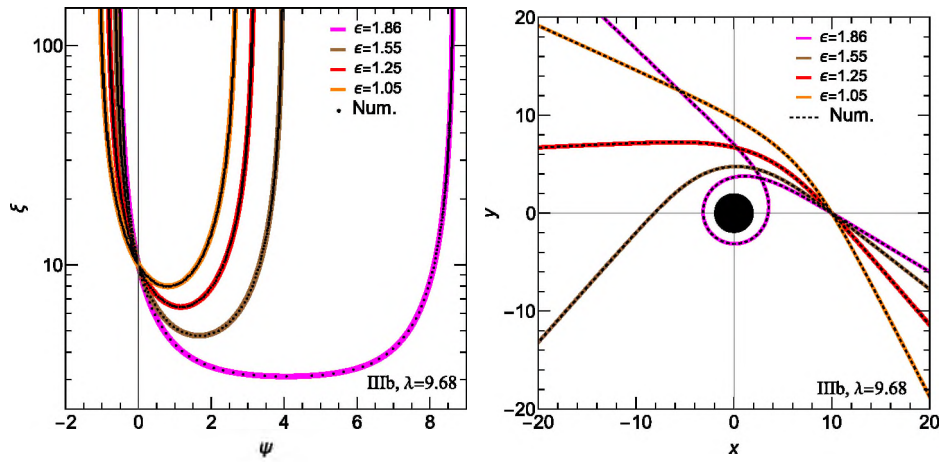


Figure 7. Sample of null unbound absorbed orbits (type IIIa) for $\lambda = 9.68$. Solid color lines correspond to solutions obtained with equation (36). Dotted lines depict corresponding numerical solutions.

Figures 5–7 show a sample of null trajectories obtained with equation (36). As for timelike geodesics, dotted lines depict solutions obtained numerically.

5. The range of ψ

In practical applications, one may need to control the allowed range of the parameter ψ or to compute the values of ψ referring to certain points at the trajectory (pericenter, apocenter).

In particular, for unbound scattered orbits the angles $\psi_{\infty\pm}$ corresponding to the asymptotics $\xi \rightarrow \infty$ could be obtained from the equation

$$\wp(\psi_{\infty\pm}) = \frac{1}{24}f''(\xi_0) \pm \sqrt{\frac{1}{96}f(\xi_0)f^{(4)}(\xi_0)}, \quad (50)$$

i.e., from the requirement that the denominator in equation (36) vanishes.

For unbound trajectories of particles that fall into the black hole, the sign ϵ_r is constant along the trajectory. As a consequence, one can use equation (A.19) of appendix A applied directly to the integral in equation (34). This yields

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \wp(\psi) &= \lim_{\xi \rightarrow \infty} \left[\frac{\sqrt{f(\xi)f(\xi_0)} + f(\xi_0)}{2(\xi - \xi_0)^2} + \frac{f'(\xi_0)}{4(\xi - \xi_0)} + \frac{f''(\xi_0)}{24} \right] \\ &= \frac{f''(\xi_0)}{24} + \frac{\sqrt{a_0 f(\xi_0)}}{2} = \frac{1}{24}f''(\xi_0) + \sqrt{\frac{1}{96}f(\xi_0)f^{(4)}(\xi_0)} \\ &= \wp(\psi_{\infty+}), \end{aligned} \quad (51)$$

meaning that $\psi_{\infty+}$ is the relevant angle in this case.

Note that in order to get $\psi_{\infty\pm}$ directly from equation (50), one would have to invert (locally) the Weierstrass function $\wp$, which is troublesome in practical applications, as $\wp$ is not a one to one map.

In appendix C, we express the function

$$X(\xi_0) = \int_{\xi_0}^{\infty} \frac{d\xi}{\sqrt{f(\xi)}} \quad (52)$$

for an unbound scattered timelike or null trajectory in terms of the Legendre elliptic integrals. The result reads

$$X(\xi) = \frac{1}{\sqrt{y_3 - y_1}} \left[F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi}}{y_2 - y_1}}, k\right) - F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12}}{y_2 - y_1}}, k\right) \right], \quad (53)$$

where $y_1 < y_2 < y_3$ are real zeros of the polynomial $4y^3 - g_2y - g_3$, and $k^2 = (y_2 - y_1)/(y_3 - y_1)$.

For a particle arriving from infinity at a scattered trajectory, the angles $\psi_{\infty\pm}$ can be obtained in a way illustrated in figure 8. Let ξ_0 be a location of an incoming particle, with $\epsilon_r = -1$, corresponding to $\psi = 0$. The angle $\psi_{\infty+} < 0$ can be expressed as $\psi_{\infty+} = -X(\xi_0)$. Denote the location of the pericenter as ξ_{per} ; suppose it corresponds to $\psi = \tilde{\psi}$. We have

$$\tilde{\psi} + |\psi_{\infty+}| = \tilde{\psi} - \psi_{\infty+} = X(\xi_{\text{per}}). \quad (54)$$

Since the orbit is symmetric with respect to ξ_{per} , one can express $\psi_{\infty-}$ as

$$\psi_{\infty-} = 2\tilde{\psi} + |\psi_{\infty+}| = 2X(\xi_{\text{per}}) - |\psi_{\infty+}| = 2X(\xi_{\text{per}}) - X(\xi_0). \quad (55)$$

For an unbound absorbed trajectory the parameter $\psi_{\infty+}$ can be expressed as before by $\psi_{\infty+} = -X(\xi_0)$ (we take $\epsilon_r = -1$), but an explicitly real expression for $X(\xi_0)$ is different, and it is given by equations (C.15) and (C.17) of appendix C.

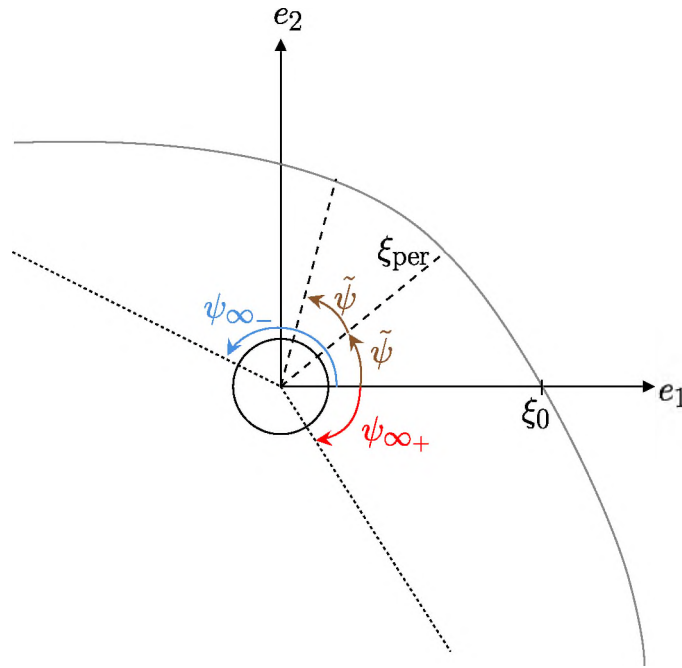


Figure 8. Trajectory of a scattered particle in the motion plane. The radius ξ_{per} and the angle ψ correspond to the pericenter.

For all types of trajectories—bounded and unbounded ones—one can define the integral

$$Y(\xi_0; \tilde{\xi}) = \int_{\xi_0}^{\tilde{\xi}} \frac{d\xi}{\sqrt{f(\xi)}}, \quad (56)$$

which we also compute in appendix C.

Finally note that for scattered trajectories the values $\psi_{\infty\pm}$ yield the bending (or deflection) angle in the Schwarzschild spacetime, which has been studied both for timelike and null geodesics [22, 41, 63, 64].

6. Affine parameter (proper time)

In this section we compute the affine parameter s associated with a given angle ψ . For timelike geodesics the value of s is related to the proper time $\bar{\tau}$ by $\bar{\tau} = Ms$.

Given an expression for $\xi = \xi(\psi)$, the corresponding affine parameter can be computed by integrating equation (29b), i.e., as

$$s(\psi) = \frac{1}{\lambda} \int_0^\psi \xi^2(\psi') d\psi'. \quad (57)$$

Integrating the square of expression (36) is, in principle, possible, but it is tedious, and the result seems to be too complicated to be useful in practical applications. Much simpler formulas can be obtained using equation (29b) with the reference position taken at a zero of the polynomial $f(\xi)$.

Let ξ_1 denote a radius such that $f(\xi_1) = 0$ (usually a periapsis or an apoapsis); assume that it corresponds to $\psi = 0$. The radius ξ corresponding to an angle ψ reads

$$\xi(\psi) = \xi_1 + \frac{\frac{1}{4}f'(\xi_1)}{\wp(\psi) - \frac{1}{24}f''(\xi_1)} \quad (58)$$

(irrespectively of the radial direction of motion, i.e., the value of ϵ_r). The affine parameter elapsed during the motion from $\psi = 0$ to some $\psi = \psi_2$ can be written as

$$s_*(\psi_2; \xi_1) = \frac{1}{\lambda} \int_0^{\psi_2} \left\{ \xi_1^2 + \frac{\frac{1}{2}f'(\xi_1)\xi_1}{\wp(\psi) - \frac{1}{24}f''(\xi_1)} + \frac{\frac{1}{16}[f'(\xi_1)]^2}{[\wp(\psi) - \frac{1}{24}f''(\xi_1)]^2} \right\} d\psi. \quad (59)$$

The above integral can be computed with the help of the following two integral formulas ([7], p 312 and [27], p 626):

$$I_1(x; y) = \int \frac{dx}{\wp(x) - \wp(y)} = \frac{1}{\wp'(y)} \left(2\zeta(y)x + \ln \frac{\sigma(x-y)}{\sigma(x+y)} \right), \quad (60)$$

$$I_2(x; y) = \int \frac{dx}{(\wp(x) - \wp(y))^2} = -\frac{\wp''(y)}{\wp'^3(y)} \ln \frac{\sigma(x-y)}{\sigma(x+y)} - \frac{1}{\wp'^2(y)} \left(\zeta(x+y) + \zeta(x-y) + \left(2\wp(y) + \frac{2\wp''(y)\zeta(y)}{\wp'(y)} \right) x \right), \quad (61)$$

where $\zeta(x)$ and $\sigma(x)$ denote the Weierstrass functions $\zeta(x; g_2, g_3)$ and $\sigma(x; g_2, g_3)$, respectively. We have

$$s_*(\psi_2; \xi_1) = \frac{1}{\lambda} \left\{ \xi_1^2 \psi_2 + \frac{1}{2} f'(\xi_1) \xi_1 [I_1(\psi_2; y) - I_1(0; y)] + \frac{1}{16} [f'(\xi_1)]^2 [I_2(\psi_2; y) - I_2(0; y)] \right\}, \quad (62)$$

where $\wp(y) = \frac{1}{24}f''(\xi_1)$ or $y = \wp^{-1}(\frac{1}{24}f''(\xi_1))$. Usually, using the inverse of the Weierstrass function is troublesome, since $\wp$ is not a one-to-one function. Fortunately, in formula (62), one is permitted to choose any y satisfying the above condition.

We now invoke to the reasoning from the end of the previous section. Consider a motion of a particle starting from an arbitrary location ξ_0 and moving inwards, up to a periapsis with the radius ξ_1 (thus $f(\xi_1) = 0$). Next the particle moves outwards, up to a location with a radius ξ . Define the angles ψ_1 and ψ_2 by equation (39). Both angles satisfy $\psi_1 \geq 0$ and $\psi_2 \geq 0$. Let $\psi = \psi_1 + \psi_2$. Because of symmetry, the proper time of the entire motion can be written as

$$s(\psi) = s_*(\psi_1; \xi_1) + s_*(\psi_2; \xi_2) = s_*(\psi_1; \xi_1) + s_*(\psi - \psi_1; \xi_1). \quad (63)$$

Formula (63) can be understood as a replacement for integral (57) with $\xi(\psi)$ given by equation (36). Note that, since $s_*(\psi_2; \xi_1)$ is an odd function of ψ_2 , we get $s(\psi = 0) = 0$, as expected. It can also be checked that the same formula holds for ξ_1 corresponding to an apoapsis, provided that definitions (39) are changed accordingly, so that $\psi_1 \geq 0$ and $\psi_2 \geq 0$.

7. Coordinate time

The coordinate time τ can be obtained in a way similar to the calculation of the affine parameter s . Consider a trajectory originating at $\xi = \xi_0$, $\psi = 0$, $\tau = 0$. The coordinate time corresponding to the lapse of the parameter ψ can be computed by integrating equation (29). Combining equations (29b) and (29c), one gets

$$\tau(\psi) = T_1(\psi) + T_2(\psi), \quad (64)$$

where

$$T_1(\psi) = \frac{\varepsilon}{\lambda} \int_0^\psi \frac{\xi^2(\psi')}{N(\xi(\psi'))} d\psi' = \frac{\varepsilon}{\lambda} \int_0^\psi \frac{\xi^2(\psi')}{1 - \frac{2}{\xi(\psi')}} d\psi' \quad (65)$$

and

$$\begin{aligned} T_2(\psi) &= \frac{1}{\lambda} \int_0^\psi \frac{\epsilon_r \xi^2(\psi') [1 - N(\xi(\psi')) \eta(\xi(\psi'))] \sqrt{\varepsilon^2 - U_\lambda(\xi(\psi'))}}{N(\xi(\psi'))} d\psi' \\ &= \int_{\xi_0}^{\xi(\psi)} \left[\frac{1}{1 - \frac{2}{\xi}} - \eta(\xi') \right] d\xi'. \end{aligned} \quad (66)$$

The second equality in equation (66) follows from equation (29a).

The integral T_2 is clearly gauge-dependent. In the standard Schwarzschild coordinates $\eta(\xi) = 1/N(\xi)$, and $T_2 \equiv 0$. Of course,

$$T_2(\psi) = \int_{\xi_0}^{\xi(\psi)} \left[\frac{1}{1 - \frac{2}{\xi}} - \eta(\xi') \right] d\xi' = \xi(\psi) - \xi_0 + 2 \ln \frac{\xi(\psi) - 2}{\xi_0 - 2} - \int_{\xi_0}^{\xi(\psi)} \eta(\xi') d\xi'. \quad (67)$$

With the help of the identity

$$\frac{\xi^2}{1 - \frac{2}{\xi}} = \xi^2 + 2\xi + 4 + \frac{8}{\xi - 2}, \quad (68)$$

integral (65) can be written as

$$T_1(\psi) = \frac{\varepsilon}{\lambda} \left[\int_0^\psi \xi^2(\psi') d\psi' + 2 \int_0^\psi \xi(\psi') d\psi' + 4\psi + 8 \int_0^\psi \frac{1}{\xi(\psi') - 2} d\psi' \right]. \quad (69)$$

In analogy to the discussion of the previous section, we will start the computation of T_1 considering at first the special case of a trajectory originating at a turning point $\xi = \xi_1$, $\psi = 0$, such that $f(\xi_1) = 0$. In this case the radius $\xi = \xi(\psi)$ is given by equation (58). The lapse of the function T_1 during the motion from $\psi = 0$ to $\psi = \psi_2$ can be expressed as

$$\begin{aligned} T_{1*}(\psi_2) &= \varepsilon s_*(\psi_2; \xi_1) + \frac{\varepsilon}{\lambda} \left\{ 2\xi_1 \psi_2 + \frac{1}{2} f'(\xi_1) [I_1(\psi_2; y) - I_1(0; y)] + 4\psi_2 \right\} \\ &\quad + \frac{8\varepsilon}{\lambda} \int_0^{\psi_2} \frac{1}{\xi_1 - 2 + \frac{(1/4)f'(\xi_1)}{\varphi(\psi') - (1/24)f''(\xi_1)}} d\psi', \end{aligned} \quad (70)$$

where $\wp(y) = \frac{1}{24}f''(\xi_1)$ and the function I_1 is defined by equation (60). The last integral can be written in the form

$$\begin{aligned} & \int_0^{\psi_2} \frac{1}{\xi_1 - 2 + \frac{(1/4)f'(\xi_1)}{\wp(\psi') - (1/24)f''(\xi_1)}} d\psi' \\ &= \frac{1}{\xi_1 - 2} \int_0^{\psi_2} 1 - \frac{\frac{f'(\xi_1)}{4(\xi_1 - 2)}}{\wp(\psi) - \wp(z)} d\psi' \\ &= \frac{1}{\xi_1 - 2} \left\{ \psi_2 - \frac{f'(\xi_1)}{4(\xi_1 - 2)} [I_1(\psi_2; z) - I_1(0; z)] \right\}, \end{aligned} \quad (71)$$

where $\wp(z) = \frac{1}{24}f''(\xi_1) - \frac{f'(\xi_1)}{4(\xi_1 - 2)}$. Equation (70) can now be written as

$$\begin{aligned} T_{1*}(\psi_2) &= \varepsilon s_*(\psi_2; \xi_1) + \frac{\varepsilon}{\lambda} \left\{ \frac{2\xi_1^2}{\xi_1 - 2} \psi_2 + \frac{1}{2} f'(\xi_1) [I_1(\psi_2; y) - I_1(0; y)] \right. \\ &\quad \left. - \frac{2f'(\xi_1)}{(\xi_1 - 2)^2} [I_1(\psi_2; z) - I_1(0; z)] \right\}. \end{aligned} \quad (72)$$

The next step proceeds as in the previous section. Consider a particle on a trajectory originating at the radius ξ_0 and $\psi = 0$, moving inwards to the turning point $\xi = \xi_1$, $\psi = \psi_1$, and then continuing outwards, up to a location with an arbitrary radius $\xi = \xi(\psi)$. Let $\psi = \psi_1 + \psi_2$, $\psi_1 \geq 0$, $\psi_2 \geq 0$, where ψ_1 and ψ_2 are given by (39). Thanks to symmetry

$$T_1(\psi) = T_{1*}(\psi_1; \xi_1) + T_{1*}(\psi_2; \xi_2) = T_{1*}(\psi_1; \xi_1) + T_{1*}(\psi - \psi_1; \xi_1). \quad (73)$$

Again, the same formula holds for a particle moving initially outwards, provided that the signs in the definitions of ψ_1 and ψ_2 are suitably adjusted.

8. Summary

We have revisited the theory of timelike and null geodesics in the Schwarzschild spacetime. The novel aspect of our work is the application of the Biermann–Weierstrass theorem to the description of Schwarzschild geodesics. A single formula (36) describes all types of timelike or null geodesic orbits, except for purely radial ones. Working with a single formula gives an advantage in those applications, in which one is forced to deal with many different orbits at the same time. We should emphasize that, in contrast to standard numerical methods, equation (36) yields exact solutions for arbitrary evolution times, even in the case of dynamically unstable orbits.

Our motivation comes from works on kinetic description of relativistic gases. Hence, we parameterize geodesics with conserved quantities (the energy and the angular momentum of the particle) and the particle initial location. Although such a parametrization is natural (and perhaps also the most popular), it might not be optimal in some applications, especially in the context of null geodesics, in which case specifying the locations of the emitter and the observer could be more convenient (cf [28]).

The Biermann–Weierstrass method of this paper is fairly general, and it is deliberately presented as such in this paper. We choose as our example the Schwarzschild spacetime, but a generalization to a large class of spherically symmetric metrics is straightforward, the Reissner–Nordström spacetime being one of natural possibilities. This fact opens up a variety of

applications, including astrophysical ones, related to testing the nature of astrophysical black holes, both in the context of light propagation and the motion of massive particles (see, e.g. [4]).

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Biermann–Weierstrass theorem

The proofs of lemma A.1 and theorem A.2 given below are adapted from references [6, 29, 52].

For any quartic polynomial

$$f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4, \quad (\text{A.1})$$

we express its Weierstrass invariants ([65], p 453) as

$$g_2 \equiv a_0a_4 - 4a_1a_3 + 3a_2^2, \quad (\text{A.2a})$$

$$g_3 \equiv a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \quad (\text{A.2b})$$

The Weierstrass elliptic function $\wp$ satisfies the integral formula

$$z \equiv \int_{\wp(z; g_2, g_3)}^{\infty} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}; \quad (\text{A.3})$$

the derivative of $\wp$ satisfies the relation

$$\left[\frac{d\wp(z; g_2, g_3)}{dz} \right]^2 = 4\wp(z; g_2, g_3)^3 - g_2\wp(z; g_2, g_3) - g_3. \quad (\text{A.4})$$

In what follows, we will use an abbreviated notation: $\wp(z) = \wp(z; g_2, g_3)$, $\wp'(z) = d\wp(z; g_2, g_3)/dz$.

Lemma A.1. (Euler 1761). *Let $f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$. The differential equation*

$$\frac{dy}{dx} = \frac{\sqrt{f(y(x))}}{\sqrt{f(x)}} \quad (\text{A.5})$$

has an integral of the form

$$\left[\frac{\sqrt{f(x)} + \sqrt{f(y(x))}}{x - y(x)} \right]^2 = a_0[x + y(x)]^2 + 4a_1[x + y(x)] + w', \quad (\text{A.6})$$

where w' is an integration constant. Similarly, equation

$$\frac{dy}{dx} = -\frac{\sqrt{f(y(x))}}{\sqrt{f(x)}} \quad (\text{A.7})$$

has an integral of the form

$$\left[\frac{\sqrt{f(x)} - \sqrt{f(y(x))}}{x - y(x)} \right]^2 = a_0[x + y(x)]^2 + 4a_1[x + y(x)] + w', \quad (\text{A.8})$$

Proof. The following proof is due to Lagrange [29]. For simplicity, we only give the proof of equation (A.6). Equation (A.8) can be proved in an analogous way.

Let us introduce a new independent variable Λ such that $dx/d\Lambda = \sqrt{f(x)}$. With a slight abuse of notation we write $y(\Lambda) = y(x(\Lambda))$. It follows that $dy/d\Lambda = \sqrt{f(y(\Lambda))}$. Define $p(\Lambda) = x(\Lambda) + y(\Lambda)$ and $q(\Lambda) = x(\Lambda) - y(\Lambda)$, so that

$$\frac{dp}{d\Lambda} = \sqrt{f(x(\Lambda))} + \sqrt{f(y(\Lambda))}, \quad \frac{dq}{d\Lambda} = \sqrt{f(x(\Lambda))} - \sqrt{f(y(\Lambda))}. \quad (\text{A.9})$$

Differentiating further with respect to Λ , one gets

$$\frac{d^2 p}{d\Lambda^2} = \frac{1}{2} [f'(x(\Lambda)) + f'(y(\Lambda))] = \frac{1}{2} a_0(p^3 + 3pq^2) + 3a_1(p^2 + q^2) + 6a_2p + 4a_3 \quad (\text{A.10})$$

and

$$\frac{dp}{d\Lambda} \frac{dq}{d\Lambda} = f(x(\Lambda)) - f(y(\Lambda)) = \frac{1}{2} a_0 pq(p^2 + q^2) + a_1 q(3p^2 + q^2) + 6a_2 pq + 4a_3 q. \quad (\text{A.11})$$

Hence

$$\frac{2}{q^2} \frac{dp}{d\Lambda} \frac{d^2 p}{d\Lambda^2} - \frac{2}{q^3} \frac{dq}{d\Lambda} \left(\frac{dp}{d\Lambda} \right)^2 = 2a_0 p \frac{dp}{d\Lambda} + 4a_1 \frac{dp}{d\Lambda}. \quad (\text{A.12})$$

The above equation can be readily integrated, yielding

$$\left(\frac{1}{q} \frac{dp}{d\Lambda} \right)^2 = a_0 p^2 + 4a_1 p + w \quad (\text{A.13})$$

or, equivalently,

$$\left[\frac{\sqrt{f(x)} + \sqrt{f(y(x))}}{x - y(x)} \right]^2 = a_0(x + y)^2 + 4a_1(x + y) + w', \quad (\text{A.14})$$

where w' is an integration constant. $\square$

Theorem A.2. (Biermann–Weierstrass). *Let*

$$f(x) = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4, \quad (\text{A.15})$$

be a quartic polynomial. Denote the Weierstrass invariants of f by g_2 and g_3 , i.e.,

$$g_2 \equiv a_0 a_4 - 4a_1 a_3 + 3a_2^2, \quad (\text{A.16a})$$

$$g_3 \equiv a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4. \quad (\text{A.16b})$$

Let

$$z(x) = \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}}, \quad (\text{A.17})$$

where x_0 is any constant, not necessarily a zero of $f(x)$. Then

$$x = x_0 + \frac{-\sqrt{f(x_0)}\wp'(z) + \frac{1}{2}f'(x_0)\left(\wp(z) - \frac{1}{24}f''(x_0)\right) + \frac{1}{24}f(x_0)f'''(x_0)}{2\left(\wp(z) - \frac{1}{24}f''(x_0)\right)^2 - \frac{1}{48}f(x_0)f^{(4)}(x_0)}, \quad (\text{A.18})$$

and

$$\wp(z) = \frac{\sqrt{f(x)f(x_0)} + f(x_0)}{2(x - x_0)^2} + \frac{f'(x_0)}{4(x - x_0)} + \frac{f''(x_0)}{24}, \quad (\text{A.19a})$$

$$\wp'(z) = -\left[\frac{f(x)}{(x - x_0)^3} - \frac{f'(x)}{4(x - x_0)^2}\right]\sqrt{f(x_0)} - \left[\frac{f(x_0)}{(x - x_0)^3} + \frac{f'(x_0)}{4(x - x_0)^2}\right]\sqrt{f(x)}, \quad (\text{A.19b})$$

where $\wp(z) = \wp(z; g_2, g_3)$ is the Weierstrass function corresponding to invariants (A.16).

Proof. In what follows, we assume that z , x , and x_0 are real. We also assume that $f(x) \geq 0$ in the interval (x_0, x) . Hence $z > 0$ for $x > x_0$, and conversely $z < 0$ for $x < x_0$. In the first step of the proof, we show that the integral (A.17) can be transformed to the Weierstrass form, i.e., there exists a transformation $w = w(x)$ such that

$$z(x) = \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}} = \pm \int_{w(x_0)}^{w(x)} \frac{dw'}{\sqrt{4w'^3 - g_2 w' - g_3}}. \quad (\text{A.20})$$

Quite remarkably, such a transformation is related to formula (A.6) of lemma A.1. Let us take

$$w(x) = \frac{1}{4} \left[\frac{\sqrt{f(x)} + \sqrt{f(y)}}{x - y} \right]^2 - \frac{1}{4} a_0 (x + y)^2 - a_1 (x + y) - a_2, \quad (\text{A.21})$$

where y is treated as a parameter. Note that $w = \frac{1}{4} w' - a_2$, where w' is the constant appearing in equation (A.6). A straightforward computation yields

$$\frac{dw}{dx} = -\frac{A(x, y)}{\sqrt{f(x)}}, \quad (\text{A.22})$$

where

$$A(x, y) = \left[\frac{f(x)}{(x-y)^3} - \frac{f'(x)}{4(x-y)^2} \right] \sqrt{f(y)} + \left[\frac{f(y)}{(x-y)^3} + \frac{f'(y)}{4(x-y)^2} \right] \sqrt{f(x)}. \quad (\text{A.23})$$

In the following, we restrict ourselves to the range in which dw/dx (and hence also $A(x, y)$) has a definite sign, so that the map $x \rightarrow w(x)$ constitutes a valid change of the integration variable. One can show that $A^2(x, y) = W(w(x))$, where $W = 4w^3 - g_2w - g_3$, and the invariants g_2 and g_3 are given by equation (A.16). Consequently, $A(x, y) = \epsilon \sqrt{W(w(x))}$, where $\epsilon = \pm 1$. This proves equation (A.20). More precisely,

$$\begin{aligned} z(x) &= \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}} = -\epsilon \int_{w(x_0)}^{w(x)} \frac{dw'}{\sqrt{4w'^3 - g_2w' - g_3}} \\ &= \epsilon \int_{w(x)}^{w(x_0)} \frac{dw'}{\sqrt{4w'^3 - g_2w' - g_3}}. \end{aligned} \quad (\text{A.24})$$

Setting $y = x_0$ in equation (A.21), we get $w \rightarrow +\infty$ for $x \rightarrow x_0$. As a consequence, one obtains

$$z(x) = \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}} = \epsilon \int_{w(x)}^{\infty} \frac{dw'}{\sqrt{4w'^3 - g_2w' - g_3}}. \quad (\text{A.25})$$

Note that $\epsilon = +1$ for $x > x_0$, and $\epsilon = -1$ for $x < x_0$. It follows that $w(x)$ can be written as $w(x) = \wp(\epsilon z(x); g_2, g_3) = \wp(z(x); g_2, g_3)$ and $\sqrt{W(w(x))} = -\epsilon \wp'(z(x); g_2, g_3)$ (the last relation can be obtained directly by differentiating equation (A.25) with respect to x).

It is easy to check that w defined by equation (A.21) can be also written as

$$w = \frac{F_1(x, y) + \sqrt{f(x)f(y)}}{2(x-y)^2}, \quad (\text{A.26})$$

where $F_1(x, y) = f(y) + \frac{1}{2}f'(y)(x-y) + \frac{1}{12}f''(y)(x-y)^2$. It is a positive root of the quadratic equation

$$(x-y)^2w^2 - F_1(x, y)w + F_2(x, y) = 0, \quad (\text{A.27})$$

where

$$F_2(x, y) = \frac{F_1(x, y)^2 - f(x)f(y)}{4(x-y)^2}. \quad (\text{A.28})$$

Using the relation

$$f(x) = f(y) + f'(y)(x-y) + \frac{1}{2}f''(y)(x-y)^2 + \frac{1}{6}f'''(y)(x-y)^3 + \frac{1}{24}f^{(4)}(y)(x-y)^4, \quad (\text{A.29})$$

one can transform equation (A.27) into the form

$$\begin{aligned}
& \left[w^2 - \frac{1}{12} w f''(y) + \frac{1}{576} f''(y)^2 - \frac{1}{96} f(y) f^4(y) \right] (x-y)^2 \\
& + \left[-\frac{1}{2} w f'(y) + \frac{1}{48} f'(y) f''(y) - \frac{1}{24} f(y) f'''(y) \right] (x-y) \\
& - w f(y) + \frac{1}{16} f'(y)^2 - \frac{1}{12} f(y) f''(y) = 0,
\end{aligned} \tag{A.30}$$

which is a quadratic equation with respect to $x - y$. Solutions of this equation can be written as

$$x - y = \frac{\pm \sqrt{f(y)} \sqrt{W} + \frac{1}{2} f'(y) \left[w - \frac{1}{24} f''(y) \right] + \frac{1}{24} f(y) f'''(y)}{2 \left[w - \frac{1}{24} f''(y) \right]^2 - \frac{1}{48} f(y) f^{(4)}(y)}. \tag{A.31}$$

A close inspection shows that the plus and minus sign in the above expression is correlated with the sign of $x - y$. We have

$$x - y = \frac{+\sqrt{f(y)} \sqrt{W} + \frac{1}{2} f'(y) \left[w - \frac{1}{24} f''(y) \right] + \frac{1}{24} f(y) f'''(y)}{2 \left[w - \frac{1}{24} f''(y) \right]^2 - \frac{1}{48} f(y) f^{(4)}(y)} \tag{A.32}$$

for $x > y$, and

$$x - y = \frac{-\sqrt{f(y)} \sqrt{W} + \frac{1}{2} f'(y) \left[w - \frac{1}{24} f''(y) \right] + \frac{1}{24} f(y) f'''(y)}{2 \left[w - \frac{1}{24} f''(y) \right]^2 - \frac{1}{48} f(y) f^{(4)}(y)} \tag{A.33}$$

for $x < y$. This observation follows from noticing that $w(x)$ is a decreasing function of x for $x > y$ and an increasing function of x for $x < y$ (cf equation (A.22)), and from inspecting the limits of the above expressions for $w \rightarrow +\infty$. Given that $\wp'(z(x); g_2, g_3) = -\epsilon \sqrt{W(w(x))} = -A(x, x_0)$, and returning to our choice $y = x_0$, we write the expression for x as

$$x = x_0 + \frac{-\sqrt{f(x_0)} \wp'(z) + \frac{1}{2} f'(x_0) \left[\wp(z) - \frac{1}{24} f'''(x_0) \right] + \frac{1}{24} f(x_0) f'''(x_0)}{2 \left[\wp(z) - \frac{1}{24} f'''(x_0) \right]^2 - \frac{1}{48} f(x_0) f^{(4)}(x_0)}, \tag{A.34}$$

i.e., in the form of equation (A.18). Equation (A.19) follow directly from equations (A.23) and (A.26). $\square$

Appendix B. Classification of trajectories

B.1. Timelike trajectories

Qualitative behavior of timelike trajectories depend on the properties of the dimensionless effective radial potential (27). The motion of a massive particle is only possible in regions where

$$\varepsilon^2 - U_\lambda(\xi) \geq 0. \tag{B.1}$$

For $0 \leq \lambda^2 \leq 12$, $U_\lambda(\xi)$ is a monotonically increasing function of ξ , growing from 0 at $\xi = 2$ to $U_\lambda(\xi) \rightarrow 1$ for $\xi \rightarrow \infty$. For $\lambda^2 > 12$, $U_\lambda(\xi)$ has two local extrema: a local minimum at

$$\xi_{\min} = \frac{\lambda^2}{2} \left(1 + \sqrt{1 - \frac{12}{\lambda^2}} \right), \quad (\text{B.2})$$

and a local maximum at

$$\xi_{\max} = \frac{\lambda^2}{2} \left(1 - \sqrt{1 - \frac{12}{\lambda^2}} \right) \quad (\text{B.3})$$

(see e.g. [54]). The location of the minimum $\xi_{\min}$ grows monotonically from 6 to infinity, as λ^2 increases from 12 to infinity. At the same time, the radius $\xi_{\max}$ decreases monotonically from 6 to 3. We have

$$U_\lambda(\xi_{\min}) = \frac{8}{9} + \frac{\lambda^2 - 12}{9\xi_{\min}}, \quad U_\lambda(\xi_{\max}) = \frac{8}{9} + \frac{\lambda^2 - 12}{9\xi_{\max}}. \quad (\text{B.4})$$

The value $U_\lambda(\xi_{\min})$ grows from $8/9$ to 1, as λ^2 increases from 12 to infinity. Simultaneously, the value $U_\lambda(\xi_{\max})$ grows from $8/9$ to infinity. For $\lambda^2 \geq 16$, the value of the potential at the maximum is always greater than or equal to one; otherwise, it is smaller.

Consider an equation

$$U_\lambda(\xi_{\max}) = \varepsilon^2, \quad (\text{B.5})$$

i.e., a limiting case of inequality (B.1), where $\xi_{\max}$ is given by equation (B.3). A solution of equation (B.5) with respect to λ^2 reads

$$\lambda_c(\varepsilon)^2 = \frac{12}{1 - \frac{4}{\left(\frac{3\varepsilon}{\sqrt{9\varepsilon^2 - 8}} + 1 \right)^2}}. \quad (\text{B.6})$$

In other words, $\lambda_c(\varepsilon)$ denotes the value of the angular momentum for which the radial potential at the local maximum is equal to ε^2 . It turns out to be particularly useful in classifying different types of timelike trajectories.

There are three main types of orbits: radial, bound, and unbound. They can be characterized as follows.

- Type I (*radial orbits*). This class consists of trajectories for which $\lambda^2 = 0$. Test particles move radially.
- Type II (*bound orbits*). Bound orbits never reach $\xi = \infty$. They can be divided into the following sub-types:
 - * *Inner orbits*. This is a class of bound orbits with at least one of the endpoints beneath the black hole horizon. For $\lambda^2 < 12$, this is the only type of bound orbits. The energy associated with such orbits is limited by $\varepsilon^2 < 1$ for $\lambda^2 < 16$ and $\varepsilon^2 \leq U_\lambda(\xi_{\max})$ for $16 \leq \lambda^2$. For $\lambda^2 \geq 12$, there are limiting cases with $\lambda = \lambda_c(\varepsilon)$, in which the orbits can spiral asymptotically towards $\xi = \xi_{\max}$.
 - * *Outer orbits*. These are trajectories trapped in a potential well, which can exist for $3 < \xi_{\max} \leq \xi$. In the generic case of outer bound orbits, the equation $\varepsilon^2 = U_\lambda(\xi)$ has three positive roots ξ_1, ξ_2, ξ_3 , satisfying $\xi_1 < \xi_{\max} < \xi_2 < \xi_{\min} < \xi_3$, and the particle oscillates between ξ_2 and ξ_3 . Thus outer bound orbits exist for $12 \leq \lambda^2$.

For $12 \leq \lambda^2 < 16$, the energy ε is bounded by $U_\lambda(\xi_{\min}) \leq \varepsilon^2 \leq U_\lambda(\xi_{\max}) < 1$.
 For $16 \leq \lambda^2$, the energy ε satisfies $U_\lambda(\xi_{\min}) \leq \varepsilon^2 < 1$.

Alternatively, the phase space occupied by outer bound orbits can be characterized by

$$\tilde{\varepsilon}_{\min} \leq \varepsilon < 1, \quad \lambda_c(\varepsilon) \leq \lambda \leq \lambda_{\max}(\varepsilon, \xi), \quad (\text{B.7})$$

where

$$\tilde{\varepsilon}_{\min} = \begin{cases} \infty & \xi \leq 3, \\ \sqrt{\left(1 - \frac{2}{\xi}\right)\left(1 + \frac{1}{\xi - 3}\right)} & 3 < \xi, \end{cases} \quad (\text{B.8})$$

and

$$\lambda_{\max}(\varepsilon, \xi) = \xi \sqrt{\frac{\varepsilon^2}{1 - \frac{2}{\xi}} - 1} \quad (\text{B.9})$$

(see, e.g. [23]). Note that circular orbits with either $\xi = \xi_{\max}$ (stable) or $\xi = \xi_{\min}$ (unstable) belong to this class. There are also limiting cases with $\lambda = \lambda_c(\varepsilon)$, similar to the limiting cases of type IIa and type IIIc.

- Type III (*unbound orbits*). In this case $\varepsilon \geq 1$. Unbound trajectories are divided into the following two sub-types:

* *Absorbed orbits*. These trajectories originate at $\xi = \infty$ and end beneath the black hole horizon. The angular momentum associated with absorbed trajectories satisfies $\lambda < \lambda_c(\varepsilon)$.

* *Scattered orbits*. Both endpoints of scattered trajectories reach infinity. The particles never reach below $\xi = 3$, i.e., below the photon sphere. Their energy is bounded from below by

$$\varepsilon_{\min} = \begin{cases} \infty & \xi \leq 3, \\ \sqrt{\left(1 - \frac{2}{\xi}\right)\left(1 + \frac{1}{\xi - 3}\right)} & 3 < \xi < 4, \\ 1 & \xi \geq 4. \end{cases} \quad (\text{B.10})$$

The total angular momentum of a scattered particle is limited from above, i.e., $\lambda_c(\varepsilon) < \lambda \leq \lambda_{\max}(\varepsilon, \xi)$.

- * A limiting case with $\lambda = \lambda_c(\varepsilon)$. The particle travels from infinity and spirals asymptotically to $\xi = \xi_{\max}$.

A comprehensive discussion of the classification of orbits can be found in [10, 39, 54]. Figure B1 shows the radial effective potential $U_\lambda(\xi)$ corresponding to different types of orbits listed above.

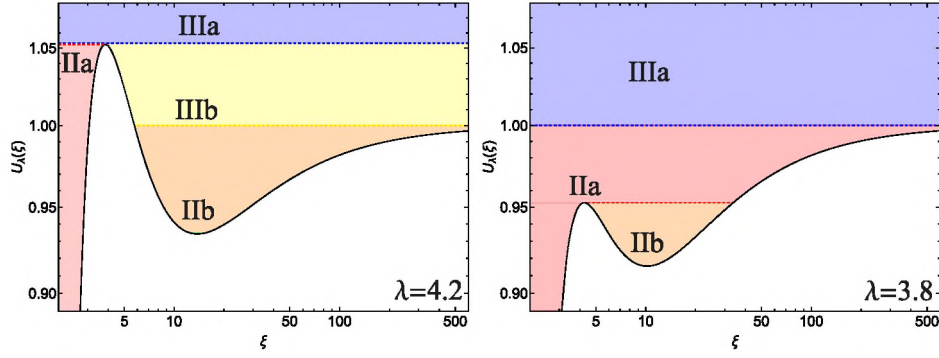


Figure B1. The effective potential $U_\lambda(\xi)$ (equation (27)) for $\lambda = 4.2$ (left) and $\lambda = 3.8$ (right). The properties of the orbit depend on the energy of the particle and the location with respect to the local maximum of $U_\lambda(\xi)$. Different types of orbits (IIa, IIb, IIIa, IIIb) are marked with separate colors.

B.2. Null trajectories

The classification of null trajectories is similar to that of timelike orbits, but it is in many respects much simpler. The dimensionless radial potential $U_\lambda(\xi)$, defined by equation (28), has a single maximum at $\xi_{\max} = 3$. The value of the potential at the maximum reads

$$U_\lambda(\xi_{\max}) = \frac{\lambda^2}{27}. \quad (\text{B.11})$$

Consequently, the equivalent of the function $\lambda_c(\varepsilon)$, defined in equation (B.5), reads

$$\lambda_c(\varepsilon) = \sqrt{27\varepsilon}. \quad (\text{B.12})$$

The orbits are divided into the following classes.

- Type I (*radial orbits*). As for timelike geodesics, this class consists of orbits with $\lambda = 0$.
- Type II (*bound orbits*). Null bound orbits exist for $\xi \leq 3$ and $\varepsilon^2 \leq \lambda^2/27$. This type includes (as a limiting case) the circular photon orbit with the radius $\xi = 3$.
- Type III (*unbound orbits*). As for timelike trajectories, unbound orbits can be divided into two following types.

* *Absorbed orbits*. In this case $\varepsilon^2 > 0$ and $\lambda < \lambda_c(\varepsilon)$.

* *Scattered orbits*. For scattered orbits $\lambda_c(\varepsilon) < \lambda \leq \lambda_{\max}(\varepsilon, \xi)$, where

$$\lambda_{\max}(\varepsilon, \xi) = \frac{\varepsilon\xi}{\sqrt{1 - \frac{2}{\xi}}}. \quad (\text{B.13})$$

These orbits exist only for $\xi > 3$.

* A limiting case with $\lambda = \lambda_c(\varepsilon)$.

Appendix C. Elliptic expressions for $\psi(\xi)$

In this appendix we derive expressions for

$$X(\xi_0) = \int_{\xi_0}^{\infty} \frac{d\xi}{\sqrt{f(\xi)}} \quad (\text{C.1})$$

and

$$Y(\xi_0; \tilde{\xi}) = \int_{\xi_0}^{\tilde{\xi}} \frac{d\xi}{\sqrt{f(\xi)}} \quad (\text{C.2})$$

in terms of Legendre elliptic integrals. Of course, $X(\xi_0) = Y(\xi_0; \infty)$. Substitutions given in this appendix are known, and they are used e.g. in [39]. They can be introduced quite generally, both for timelike and null orbits.

Let us start with a more general elliptic integral

$$I = \int_a^b \frac{d\xi}{\sqrt{f(\xi)}}, \quad (\text{C.3})$$

where $f(\xi) = a_0\xi^4 + 4a_1\xi^3 + 6a_2\xi^2 + 4a_3\xi + a_4$. Let e be a zero of f . Substituting $\xi = e + \frac{1}{x}$, we get

$$I = - \int_{\frac{1}{a-e}}^{\frac{1}{b-e}} \frac{dx}{\sqrt{A_0 + 4A_1x + 6A_2x^2 + 4A_3x^3}}, \quad (\text{C.4})$$

where $A_0 = a_0$, $A_1 = a_1 + a_0e$, $A_2 = a_2 + 2a_1e + a_0e^2$, $A_3 = a_3 + 3a_2e + 3a_1e^2 + a_0e^3$. The transformation $\xi = e + \frac{1}{x}$ maps the zero $\xi = e$ to infinity, removing one factor $(\xi - e)$ from the factorization of $f(\xi)$. The remaining zeros of $f(\xi)$ are mapped into zeros of $A_0 + 4A_1x + 6A_2x^2 + 4A_3x^3$. Next, another substitution $x = \frac{y - \frac{2}{3}A_2}{A_3}$, brings the above integral to the Weierstrass form

$$I = - \text{sgn}(A_3) \int_{\frac{A_2}{2} + \frac{A_3}{a-e}}^{\frac{A_2}{2} + \frac{A_3}{b-e}} \frac{dy}{\sqrt{4y^3 - g_2y - g_3}}, \quad (\text{C.5})$$

where g_2 and g_3 are given by (A.16).

Further reduction to Legendre integrals requires a control of the integration range with respect to zeros of the polynomial $4y^3 - g_2y - g_3$, provided that one wants to have explicitly real expressions. We start by computing the integral $X(\xi_0)$, assuming a scattered unbound orbit. In this case, $f(\xi)$ has four real zeros, one of which is simply $\xi = 0$. This can be seen as follows. We have $f(\xi) = \xi^4(\varepsilon^2 - U_\lambda(\xi))$. The expression $\varepsilon^2 - U_\lambda(\xi)$ has three real zeros: two of them are positive, as follows from the discussion concerning the centrifugal barrier. The third one is negative. For timelike trajectories this fact can be seen by noting that

$$\varepsilon^2 - U_\lambda(\xi) = \varepsilon^2 - 1 + \frac{1}{\xi^3}(2\xi^2 - \lambda^2\xi + 2\lambda^2). \quad (\text{C.6})$$

Consequently, $\varepsilon^2 - U_\lambda(\xi)$ tends to $-\infty$ for $\xi \rightarrow 0_-$, and $\varepsilon^2 - U_\lambda(\xi) \rightarrow \varepsilon^2 - 1 > 0$, for $\xi \rightarrow -\infty$. Hence, $\varepsilon^2 - U_\lambda(\xi)$ changes its sign for $\xi < 0$. For null trajectories the reasoning is analogous, but this time

$$\varepsilon^2 - U_\lambda(\xi) = \varepsilon^2 - \left(1 - \frac{2}{\xi}\right) \frac{\lambda^2}{\xi^2}. \quad (\text{C.7})$$

Consequently, $\varepsilon^2 - U_\lambda(\xi) \rightarrow \varepsilon^2 > 0$, for $\xi \rightarrow -\infty$. In both cases (timelike and null) we will denote the zeros of $f(\xi)$ as $\xi_1 < 0 < \xi_3 < \xi_2$.

In the substitution $\xi = e + \frac{1}{x}$ leading to equation (C.4), we now choose $e = 0$. This yields $A_0 = a_0$, $A_1 = a_1$, $A_2 = a_2$, $A_3 = a_3$. Note that $a_2 = -1/6$, $a_3 = 1/2$, both for timelike and null trajectories. Hence, $X(\xi_0)$ can be written as

$$X(\xi_0) = \int_{-\frac{1}{12}}^{-\frac{1}{12} + \frac{1}{2\xi_0}} \frac{dy}{\sqrt{4y^3 - g_2y - g_3}}. \quad (\text{C.8})$$

The substitution $y = -\frac{1}{12} + \frac{1}{2\xi}$ maps the zeros ξ_1, ξ_2, ξ_3 of $f(\xi)$ to y_1, y_2, y_3 , respectively, but this time $y_1 < y_2 < y_3$. The original integration range of ξ , $\xi_2 \leq \xi_0 \leq \xi < \infty$, is mapped into the segment: $y_1 < -1/12 < y \leq y_2$.

We now make a substitution

$$y = y_2 - \mu^2 \cos^2 \chi, \quad \mu^2 = y_2 - y_1, \quad k^2 = \frac{y_2 - y_1}{y_3 - y_1}, \quad 0 \leq \chi \leq \frac{\pi}{2}. \quad (\text{C.9})$$

Thus $y = y_1$ for $\chi = 0$, and $y = y_2$ for $\chi = \pi/2$. This yields

$$\int \frac{dy}{\sqrt{4(y - y_1)(y - y_2)(y - y_3)}} = \frac{k}{\mu} \int \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad (\text{C.10})$$

where $k/\mu = 1/\sqrt{y_3 - y_1}$. Consequently,

$$X(\xi_0) = \frac{1}{\sqrt{y_3 - y_1}} \left[F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi_0}}{y_2 - y_1}}, k\right) - F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12}}{y_2 - y_1}}, k\right) \right], \quad (\text{C.11})$$

where

$$F(\phi, k) = \int_0^\phi \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (\text{C.12})$$

For generic unbound absorbed orbits, the situation is slightly different. In this case the polynomial $4y^3 - g_2y - g_3$ has only one real zero $y_1 < -1/12$. We write: $4y^3 - g_2y - g_3 = 4(y - y_1)(y^2 + py + q)$, where $p^2 - 4q < 0$ and thus $y^2 + py + q > 0$. The substitution which turns equation (C.8) into the Legendre form reads now

$$y = y_1 + \mu \tan^2 \frac{\chi}{2}, \quad \mu = \sqrt{y_1^2 + py_1 + q}, \quad 0 \leq \chi < \frac{\pi}{2}. \quad (\text{C.13})$$

We have $y = y_1$ for $\chi = 0$ and $y \rightarrow \infty$ for $\chi \rightarrow \pi/2$. A straightforward algebraic calculation yields now

$$\int \frac{dy}{\sqrt{4(y - y_1)(y^2 + py + q)}} = \frac{1}{2\sqrt{\mu}} \int \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad (\text{C.14})$$

where

$$k^2 = \frac{1}{2} \left(1 - \frac{y_1 + p/2}{\mu}\right). \quad (\text{C.15})$$

Note that k^2 is real and positive. This fact follows directly from the inequality $p^2 - 4q < 0$, which ensures that

$$\left(\frac{y_1 + p/2}{\mu}\right)^2 = \frac{y_1^2 + py_1 + \frac{p^2}{4}}{y_1^2 + py_1 + q} < 1. \quad (\text{C.16})$$

As a consequence, $X(\xi_0)$ can be written as

$$X(\xi_0) = \frac{1}{2\sqrt{\mu}} \left[F\left(2 \arctan \sqrt{\frac{-\frac{1}{12} + \frac{1}{2\xi_0} - y_1}{\mu}}, k\right) - F\left(2 \arctan \sqrt{\frac{-\frac{1}{12} - y_1}{\mu}}, k\right) \right]. \quad (\text{C.17})$$

For generic timelike outer bound orbits the expression $\varepsilon^2 - U_\lambda(\xi)$ has three real positive roots $\xi_3 < \xi_2 < \xi_1$, which are also the zeros of $f(\xi)$ (the fourth root being $\xi_4 = 0$). The motion is allowed in the range $\xi_2 \leq \xi \leq \xi_1$. The transformation $y = -\frac{1}{12} + \frac{1}{2\xi}$ maps the zeros ξ_3, ξ_2, ξ_1 into $y_1 < y_2 < y_3$ (the zeros of $4y^3 - g_2y - g_3$). Explicitly real expressions for $Y(\xi_0, \tilde{\xi})$ can be obtained with substitutions (C.9). We get

$$Y(\xi_0, \tilde{\xi}) = \frac{1}{\sqrt{y_3 - y_1}} \left[F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi_0}}{y_2 - y_1}}, k\right) - F\left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\tilde{\xi}}}{y_2 - y_1}}, k\right) \right], \quad (\text{C.18})$$

The case of inner bound orbits is more complex, since, depending on the values of ε and λ , they correspond either to a case with three real zeros of $4y^3 - g_2y - g_3$ or to a case in which this polynomial has just one real zero and two complex ones. Here again, substitutions (C.9) and (C.13) work, but one has to adjust the details (carefully select the roots y_1, y_2 , and y_3).

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References

- [1] Abramowitz M and Stegun I 1964 *Handbook of Mathematical Functions* 10 edn (Washington, D.C.: United States Department of Commerce, National Bureau of Standards)
- [2] Akhiezer N 1990 *Elements of the Theory of Elliptic Functions* vol 79 (Providence, RI: American Mathematical Society)
- [3] Baker A 1890 *Elliptic Functions: An Elementary Text-Book for Students of Mathematics* (New York: Wiley)
- [4] Bambhaniya P, Joshi A B, Dey D and Joshi P S 2019 Timelike geodesics in naked singularity and black hole spacetimes *Phys. Rev. D* **100** 124020
- [5] Bateman H 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)
- [6] Biermann G G A 1865 *Problemata quaedam mechanica functionum ellipticarum ope soluta* *PhD Thesis* Humboldt University of Berlin 5–7
- [7] Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* (Berlin: Springer)

- [8] Čadež A and Kostić U 2005 Optics in the Schwarzschild spacetime *Phys. Rev. D* **72** 104024
- [9] Cayley A 1876 *An Elementary Treatise on Elliptic Functions* (Cambridge: Cambridge University Press)
- [10] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* (New York: Oxford University Press)
- [11] Chandrasekharan K 1985 *Elliptic Functions* vol 281 (Berlin: Springer)
- [12] Chatterjee A K, Flathmann K, Nandan H and Rudra A 2019 Analytic solutions of the geodesic equation for Reissner–Nordström–(anti–)de Sitter black holes surrounded by different kinds of regular and exotic matter fields *Phys. Rev. D* **100** 024044
- [13] Cieřlik A and Mach P 2020 Accretion of the Vlasov gas on Reissner–Nordström black holes *Phys. Rev. D* **102** 024032
- [14] Cruz N, Olivares M and Villanueva J R 2005 The geodesic structure of the Schwarzschild anti-de Sitter black hole *Class. Quantum Grav.* **22** 1167–90
- [15] Dabrowski M 1995 Light curves of relativistic charged neutron star *Astrophys. Space Sci.* **229** 139–55
- [16] Darwin C G 1959 The gravity field of a particle *Proc. R. Soc. A* **249** 180–94
- [17] Darwin C G 1961 The gravity field of a particle. II *Proc. R. Soc. A* **263** 39–50
- [18] Droste J 1917 The field of a single centre in Einstein’s theory of gravitation and the motion of a particle in that field (Huygens Institute–Royal Netherlands Academy of Arts and Sciences (KNAW)) pp 197–215
- [19] Droste J 2002 ‘Golden oldie’: the field of a single centre in Einstein’s theory of gravitation, and the motion of a particle in that field *Gen. Relativ. Gravit.* **34** 1545–63
- [20] Forsyth A R 1918 *Theory of Functions of a Complex Variable* (Cambridge: Cambridge University Press)
- [21] Forsyth A R 1920 Note on the central differential equation in the relativity theory of gravitation *Proc. R. Soc. A* **97** 145–51
- [22] Frittelli S, Kling T P and Newman E T 2000 Spacetime perspective of Schwarzschild lensing *Phys. Rev. D* **61** 064021
- [23] Gabarrete C and Sarbach O 2022 Kinetic gas disks surrounding Schwarzschild black holes *Acta Phys. Pol. B* **15** 1–A10
- [24] Gackstatter F 1983 Über Planetenbewegung und Lichtbahnen im Schwarzschild- und im Reissner–Nordström–Raum *Ann. Phys., Lpz.* **495** 352–74
- [25] Gamboa A, Gabarrete C, Domínguez-Fernández P, Núñez D and Sarbach O 2021 Accretion of a Vlasov gas onto a black hole from a sphere of finite radius and the role of angular momentum *Phys. Rev. D* **104** 083001
- [26] Gibbons G W and Vyska M 2012 The application of Weierstrass elliptic functions to Schwarzschild null geodesics *Class. Quantum Grav.* **29** 065016
- [27] Gradshteyn I S, Ryzhik I M and Jeffrey A 2007 *Table of Integrals, Series, and Products* 7 edn (New York: Academic Press)
- [28] Grasso M, Korzyński M and Serbenta J 2019 Geometric optics in general relativity using bilocal operators *Phys. Rev. D* **99** 064038
- [29] Greenhill A G 1892 *The Applications of Elliptic Functions* (London: Macmillan and Co.)
- [30] Greenhill G 1921 X. Newton–Einstein planetary orbit *London, Edinburgh Dublin Phil. Mag. J. Sci.* **41** 143–8
- [31] Grunau S and Kagramanova V 2011 Geodesics of electrically and magnetically charged test particles in the Reissner–Nordström space-time: analytical solutions *Phys. Rev. D* **83** 044009
- [32] Hackmann E and Lämmerzahl C 2014 Analytical solution methods for geodesic motion *Recent Developments on Physics in Strong Gravitational Fields: V Leopoldo García-Colín Mexican Meeting on Mathematical and Experimental Physics* (México City, México) pp 78–88
- [33] Hackmann E and Lämmerzahl C 2008 Geodesic equation in Schwarzschild–(anti–)de Sitter spacetimes: analytical solutions and applications *Phys. Rev. D* **78** 024035
- [34] Hagihara Y 1930 Theory of the relativistic trajectories in a gravitational field of Schwarzschild *Japan. J. Astron. Geophys.* **8** 67–176
- [35] Hancock H 1917 *Elliptic Integrals* (New York: Wiley)
- [36] Hardy G 1916 *The Integration of Functions of a Single Variable* (Cambridge: Cambridge University Press)
- [37] Hioe F T and Kuebel D 2010 Characterizing planetary orbits and the trajectories of light in the Schwarzschild metric *Phys. Rev. D* **81** 084017

- [38] Hioe F T 2009 Exact expressions for the precession of a planet and the deflection of light *Phys. Lett. A* **373** 1506–9
- [39] Kostić U 2012 Analytical time-like geodesics in Schwarzschild space-time *Gen. Relativ. Gravit.* **44** 1057–72
- [40] Kraniotis G V and Whitehouse S B 2003 Compact calculation of the perihelion precession of Mercury in general relativity, the cosmological constant and Jacobi’s inversion problem *Class. Quantum Grav.* **20** 4817–35
- [41] Liu X, Yang N and Jia J 2016 Gravitational lensing of massive particles in Schwarzschild gravity *Class. Quantum Grav.* **33** 175014
- [42] Luninet J 1979 Image of a spherical black hole with thin accretion disk *Astron. Astrophys.* **75** 228–35
- [43] Mach P and Odrzywołek A 2022 Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: low-temperature limit and numerical aspects *Acta Phys. Pol. B* **15** 1–A7
- [44] Mach P and Odrzywołek A 2021 Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: exact solutions *Phys. Rev. D* **103** 024044
- [45] Mach P and Odrzywołek A 2021 Accretion of dark matter onto a moving Schwarzschild black hole: an exact solution *Phys. Rev. Lett.* **126** 101104
- [46] Metzner A W K 1963 Observable properties of large relativistic masses *J. Math. Phys.* **4** 1194–205
- [47] Mielnik B and Plebański J 1962 A study of geodesic motion in the field of Schwarzschild’s solution *Acta Phys. Pol.* **21** 239–68
- [48] Mino Y 2003 Perturbative approach to an orbital evolution around a supermassive black hole *Phys. Rev. D* **67** 084027
- [49] Morton W B 1921 LXI the forms of planetary orbits on the theory of relativity *London, Edinburgh Dublin Phil. Mag. J. Sci.* **42** 511–22
- [50] Muñoz G 2014 Orbits of massless particles in the Schwarzschild metric: exact solutions *Am. J. Phys.* **82** 564–73
- [51] Panotopoulos G, Rincón Á and Lopes I 2021 Orbits of light rays in scale-dependent gravity: exact analytical solutions to the null geodesic equations *Phys. Rev. D* **103** 104040
- [52] Reynolds M J 1989 An exact solution in non-linear oscillations *J. Phys. A: Math. Gen.* **22** L723–6
- [53] Rioseco P and Sarbach O 2017 Spherical steady-state accretion of a relativistic collisionless gas into a Schwarzschild black hole *J. Phys.: Conf. Ser.* **831** 012009
- [54] Rioseco P and Sarbach O 2017 Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole *Class. Quantum Grav.* **34** 095007
- [55] Rodríguez C M 1987 Orbits in general relativity: the Jacobian elliptic functions *Nuovo Cimento B* **98** 87–96
- [56] Rosales-Vera M 2018 Asymptotic description of a test particle around a Schwarzschild black hole *Eur. J. Phys.* **39** 025602
- [57] Scharf G 2011 Schwarzschild geodesics in terms of elliptic functions and the related red shift *J. Mod. Phys.* **02** 274–83
- [58] Semerák O 2015 Approximating light rays in the Schwarzschild field *Astrophys. J.* **800** 77
- [59] Sharp N A 1979 Geodesics in black hole space-times *Gen. Relativ. Gravit.* **10** 659–70
- [60] Slezakova G 2006 *Geodesic Geometry of Black Holes* (Hamilton: The University of Waikato)
- [61] Soroushfar S, Saffari R, Kunz J and Lämmerzahl C 2015 Analytical solutions of the geodesic equation in the spacetime of a black hole in $f(R)$ gravity *Phys. Rev. D* **92** 044010
- [62] Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland)
- [63] Tsupko O Y 2014 Unbound motion of massive particles in the Schwarzschild metric: analytical description in case of strong deflection *Phys. Rev. D* **89** 084075
- [64] Virbhadra K S and Ellis G F R 2000 Schwarzschild black hole lensing *Phys. Rev. D* **62** 084003
- [65] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)

2.2. Timelike and Null Geodesics in the Schwarzschild Space-time: Analytical Solutions

TIMELIKE AND NULL GEODESICS IN THE SCHWARZSCHILD SPACE-TIME: ANALYTICAL SOLUTIONS*

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The theory of Schwarzschild geodesics is revisited. Using a theorem due to Weierstrass and Biermann, we derive concise formulas describing all timelike and null trajectories in terms of Weierstrass elliptic functions. The formulation given in this note uses an analogue of the so-called Mino time.

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1. Introduction and motivation

Since the discovery of the Schwarzschild metric, different ways of expressing solutions to the Schwarzschild geodesic equations have been developed [1–6]. In most cases, they differ in the assumed parametrization and in the types of elliptic functions used to write the solution.

Our motivation for revisiting the theory of Schwarzschild geodesics is related to works on the kinetic description of the Vlasov gas in the Schwarzschild spacetime and the accretion of the Vlasov gas onto Schwarzschild black holes [7–13], Reissner–Nordström black holes [14], and Kerr black holes [15–18]. In particular, aiming at constructing Monte Carlo type simulations of the gas consisting of non-colliding particles moving around the Schwarzschild black hole, we were searching for a concise description of all types of timelike and null trajectories in the Schwarzschild metric, which would depend explicitly on standard constants of motion (in particular, the energy and the angular momentum) together with the initial location of the particle. It turns out that such a description can be written in terms of Weierstrass

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elliptic functions, basing on a theorem due to Weierstrass and Biermann [19]. A detailed account of these results can be found in our recent paper [6]. This note is intended to give a short summary, but, in comparison to [6], we use a different parametrization. In [6], solutions are expressed in terms of the so-called true anomaly; in this note, we use an analogue of the Mino time, introduced originally for the Kerr geodesics in [20].

2. Geodesic equations

We work in standard Schwarzschild coordinates. The metric reads

$$g = -N dt^2 + \frac{dr^2}{N} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad (1)$$

where

$$N = 1 - \frac{2M}{r}, \quad (2)$$

and M is the Schwarzschild mass. Geodesic equations can be written in the form [6]

$$\frac{dr}{d\tilde{s}} = \epsilon_r \sqrt{E^2 - U_{l,m}(r)}, \quad (3a)$$

$$\frac{d\theta}{d\tilde{s}} = \frac{\epsilon_\theta}{r^2} \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}, \quad (3b)$$

$$\frac{d\varphi}{d\tilde{s}} = \frac{l_z}{r^2 \sin^2 \theta}, \quad (3c)$$

$$\frac{dt}{d\tilde{s}} = \frac{E}{N}, \quad (3d)$$

where

$$U_{l,m}(r) = \left(1 - \frac{2M}{r}\right) \left(\delta_m + \frac{l^2}{r^2}\right) \quad (4)$$

is the radial effective potential and $E, l \geq 0, l_z$ denote constant values of the particle energy, the total angular momentum, and the azimuthal component of the angular momentum, respectively. The constant δ_m is defined as

$$\delta_m = \begin{cases} m^2 & \text{for timelike geodesics,} \\ 0 & \text{for null geodesics,} \end{cases} \quad (5)$$

and the signs $\epsilon_\theta = \pm 1$ and $\epsilon_r = \pm 1$ correspond to the directions of motion.

There is a convenient way to partially decouple the above equations, which also works for the geodesic motion in the Kerr spacetime, and which for the latter case was introduced by Mino in [20]. The trick is to reparam-

trize geodesics so that

$$\frac{dx^\mu}{d\bar{s}} = r^2 \frac{dx^\mu}{d\tilde{s}} \quad \text{or} \quad \tilde{s} = \int_0^{\bar{s}} r^2 ds. \quad (6)$$

We will refer to the parameter $\bar{s}$ as the Mino time. Additionally, to simplify further calculations, we will work in dimensionless rescaled variables, defined as in [7], *i.e.*,

$$\begin{aligned} t &= M\tau, & r &= M\xi, & p_r &= m\pi_\xi, \\ p_\theta &= Mm\pi_\theta, & E &= m\varepsilon, & l &= Mm\lambda, & l_z &= Mm\lambda_z. \end{aligned} \quad (7)$$

A new Mino time s is defined by

$$\bar{s} = \frac{s}{Mm}. \quad (8)$$

In terms of these dimensionless variables, geodesic equations (3) can be written as

$$\frac{d\xi}{ds} = \epsilon_r \xi^2 \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \quad (9a)$$

$$\frac{d\theta}{ds} = \epsilon_\theta \sqrt{\lambda^2 - \frac{\lambda_z^2}{\sin^2 \theta}}, \quad (9b)$$

$$\frac{d\varphi}{ds} = \frac{\lambda_z}{\sin^2 \theta}, \quad (9c)$$

$$\frac{d\tau}{ds} = \frac{\varepsilon \xi^2}{N(\xi)}, \quad (9d)$$

where $N(\xi) = 1 - 2/\xi$. The dimensionless radial potential reads

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(\delta + \frac{\lambda^2}{\xi^2}\right) = \delta - \frac{2}{\xi}\delta + \frac{\lambda^2}{\xi^2} - \frac{2\lambda^2}{\xi^3}, \quad (10)$$

where

$$\delta = \begin{cases} 1 & \text{for time-like geodesics,} \\ 0 & \text{for null geodesics.} \end{cases} \quad (11)$$

3. Biermann–Weierstrass theorem

Solutions given in this work are based on the following result due to Biermann and Weierstrass. The original formulation of this theorem has been published in [19]. More detailed proofs can be found in [6, 21, 22].

Theorem 3.1 *Let*

$$f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \quad (12)$$

be a quartic polynomial. Denote the invariants of f by g_2 and g_3 , i.e.,

$$g_2 \equiv a_0a_4 - 4a_1a_3 + 3a_2^2, \quad (13a)$$

$$g_3 \equiv a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \quad (13b)$$

Let

$$z(x) = \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}}, \quad (14)$$

where x_0 is any constant, not necessarily a zero of $f(x)$. Then

$$x = x_0 + \frac{-\sqrt{f(x_0)}\wp'(z) + \frac{1}{2}f'(x_0) [\wp(z) - \frac{1}{24}f''(x_0)] + \frac{1}{24}f(x_0)f'''(x_0)}{2 [\wp(z) - \frac{1}{24}f''(x_0)]^2 - \frac{1}{48}f(x_0)f^{(4)}(x_0)}, \quad (15)$$

and

$$\wp(z) = \frac{\sqrt{f(x)f(x_0)} + f(x_0)}{2(x - x_0)^2} + \frac{f'(x_0)}{4(x - x_0)} + \frac{f''(x_0)}{24}, \quad (16a)$$

$$\wp'(z) = - \left[\frac{f(x)}{(x-x_0)^3} - \frac{f'(x)}{4(x-x_0)^2} \right] \sqrt{f(x_0)} - \left[\frac{f(x_0)}{(x-x_0)^3} + \frac{f'(x_0)}{4(x-x_0)^2} \right] \sqrt{f(x)}, \quad (16b)$$

where $\wp(z) = \wp(z; g_2, g_3)$ is the Weierstrass function corresponding to invariants (13).

4. Solutions

4.1. Solution for $\xi(s)$

Equation (9a) can be rewritten in the Weierstrass form

$$\frac{d\xi}{ds} = \epsilon_r \sqrt{f(\xi)}, \quad (17)$$

where

$$f(\xi) = a_0\xi^4 + 4a_1\xi^3 + 6a_2\xi^2 + 4a_3\xi + a_4, \quad (18)$$

and

$$a_0 = \varepsilon^2 - \delta, \quad 4a_1 = 2\delta, \quad 6a_2 = -\lambda^2, \quad 4a_3 = 2\lambda^2, \quad a_4 = 0. \quad (19)$$

For a segment of the trajectory from ξ_0 to ξ , we get

$$s = \epsilon_r \int_{\xi_0}^{\xi} \frac{d\xi'}{\sqrt{f(\xi')}} , \quad (20)$$

where ξ_0 is a reference radius corresponding to $s = 0$. Weierstrass invariants of the polynomial f read

$$g_2 = \frac{1}{12} \lambda^4 - \delta \lambda^2 , \quad (21a)$$

$$g_3 = \frac{1}{6^3} \lambda^6 - \frac{\delta}{12} \lambda^4 - \frac{1}{4} (\varepsilon^2 - \delta) \lambda^4 . \quad (21b)$$

Therefore, thanks to the Biermann–Weierstrass theorem, we can write the formula for $\xi = \xi(s)$ as

$$\xi(s) = \xi_0 + \frac{-\epsilon_r \sqrt{f(\xi_0)} \wp'(s) + \frac{1}{2} f'(\xi_0) [\wp(s) - \frac{1}{24} f''(\xi_0)] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2 [\wp(s) - \frac{1}{24} f''(\xi_0)]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)} . \quad (22)$$

Here, $\wp$ is understood to be defined by the invariants g_2 , and g_3 given by Eq. (21), *i.e.*, $\wp(z) = \wp(z; g_2, g_3)$, and f is defined in Eqs. (18) and (19).

The above equation is a general solution to Eq. (17), valid for all types of allowed trajectories. One can moreover show that Eq. (22) gives a correct solution also in the presence of turning points. In this case, the sign ϵ_r in Eq. (22) has to be interpreted as referring to the initial position ξ_0 (in other words, ϵ_r can be treated as a part of initial data).

4.2. Solution for $\theta(s)$ and $\varphi(s)$

Equation (9b) can be integrated in a straightforward way. Substituting $\mu = \cos \theta$, we get

$$\begin{aligned} s &= \epsilon_\theta \int_{\theta_0}^{\theta} \frac{1}{\sqrt{\lambda^2 - \frac{\lambda_z^2}{\sin^2 \bar{\theta}}}} d\bar{\theta} = -\frac{\epsilon_\theta}{\lambda} \int_{\mu_0}^{\mu} \frac{1}{\sqrt{1 - \frac{\lambda_z^2}{\lambda^2} - \tilde{\mu}^2}} d\tilde{\mu} \\ &= -\frac{\epsilon_\theta}{\lambda} \left(\arcsin \frac{\mu}{\sqrt{1 - \frac{\lambda_z^2}{\lambda^2}}} - \arcsin \frac{\mu_0}{\sqrt{1 - \frac{\lambda_z^2}{\lambda^2}}} \right) , \end{aligned}$$

where $\mu_0 = \cos \theta_0$ correspond to a reference point. Setting

$$\beta_0 = \frac{\epsilon_\theta}{\lambda} \arcsin \frac{\mu_0}{\sqrt{1 - \frac{\lambda_z^2}{\lambda^2}}} ,$$

one gets

$$\mu(s) = -\epsilon_\theta \sqrt{1 - \frac{\lambda_z^2}{\lambda^2}} \sin[\lambda(s - \beta_0)] . \quad (23)$$

Equation (9c) can be now integrated as

$$\begin{aligned} \varphi(s) - \varphi_0 &= \lambda_z \int_0^s \frac{1}{\sin^2 \theta(\tilde{s})} d\tilde{s} = \lambda_z \int_0^s \frac{1}{1 - \mu(\tilde{s})^2} d\tilde{s} \\ &= \arctan \left[\frac{\lambda_z}{\lambda} \tan[\lambda(s - \beta_0)] \right] + \arctan \left[\frac{\lambda_z}{\lambda} \tan(\lambda\beta_0) \right] , \end{aligned}$$

where we have used Eq. (23), and $\varphi_0 = \varphi(0)$. The above expression can be corrected for discontinuities due to the tangent function. The final formula has the form

$$\varphi(s) = \varphi_0 + \arctan \left[\frac{\lambda_z}{\lambda} \tan[\lambda(s - \beta_0)] \right] + \arctan \left[\frac{\lambda_z}{\lambda} \tan(\lambda\beta_0) \right] + n\pi \operatorname{sgn} \lambda_z , \quad (24)$$

where n is an integer part of $\frac{\lambda}{\pi}(s - \beta_0) \pm \frac{1}{2}$. Here, the plus corresponds to $s > 0$ and the minus to $s < 0$.

4.3. Solution for $\tau(s)$

A direct integration of Eq. (9d) yields

$$\tau(s) - \tau(0) = \varepsilon \int_0^s \frac{\xi(\tilde{s})^2}{1 - \frac{2}{\xi(\tilde{s})}} d\tilde{s} = \varepsilon \int_0^s \left(\xi(\tilde{s})^2 + 2\xi(\tilde{s}) + 4 + \frac{8}{\xi(\tilde{s}) - 2} \right) d\tilde{s} .$$

These integrals can be computed by substituting expression (22) for $\xi(s)$, however the result may be too complicated and impractical. Instead, we will focus on a simpler case, in which a turning point can be chosen as a reference point.

Let ξ_1 denote the radius of the turning point, so that $f(\xi_1) = 0$. Choosing ξ_1 as a reference radius corresponding to $s = 0$, one can rewrite Eq. (22) as

$$\xi(s) = \xi_1 + \frac{\frac{1}{4}f'(\xi_1)}{\wp(s) - \frac{1}{24}f''(\xi_1)} \quad (25)$$

irrespectively of the radial direction of motion, *i.e.*, the value of ϵ_r . The coordinate time elapsed during the motion from $s = 0$ to $s = s_2$ can be

written as

$$\begin{aligned} \tau_*(s_2; \xi_1) = \varepsilon \int_0^{s_2} & \left\{ \frac{\xi_1^3}{\xi_1 - 2} + \frac{\frac{\xi_1 + 1}{2} f'(\xi_1)}{\wp(\tilde{s}) - \wp(y)} - \frac{\frac{2}{(\xi_1 - 2)^2} f'(\xi_1)}{\wp(\tilde{s}) - \wp(z)} \right. \\ & \left. + \frac{\frac{1}{16} f'(\xi_1)^2}{[\wp(\tilde{s}) - \wp(y)]^2} \right\} d\tilde{s}, \end{aligned} \quad (26)$$

where $\wp(y) = \frac{1}{24} f''(\xi_1)$ and $\wp(z) = \frac{1}{24} f''(\xi_1) - \frac{f'(\xi_1)}{4(\xi_1 - 2)}$. One can choose any values y and z satisfying these conditions. Integral (26) can be computed with the help of the following two integral formulas ([23], p. 312 and [24], p. 626):

$$I_1(x; y) = \int \frac{dx}{\wp(x) - \wp(y)} = \frac{1}{\wp'(y)} \left[2\zeta(y)x + \ln \frac{\sigma(x - y)}{\sigma(x + y)} \right], \quad (27)$$

$$\begin{aligned} I_2(x; y) = \int \frac{dx}{(\wp(x) - \wp(y))^2} = & -\frac{\wp''(y)}{\wp'^3(y)} \ln \frac{\sigma(x - y)}{\sigma(x + y)} \\ & - \frac{1}{\wp'^2(y)} \left\{ \zeta(x + y) + \zeta(x - y) + \left[2\wp(y) + \frac{2\wp''(y)\zeta(y)}{\wp'(y)} \right] x \right\}, \end{aligned} \quad (28)$$

where $\zeta(x)$ and $\sigma(x)$ denote the Weierstrass functions $\zeta(x; g_2, g_3)$ and $\sigma(x; g_2, g_3)$, respectively. Hence, we obtain

$$\begin{aligned} \tau_*(s_2; \xi_1) = \varepsilon \left\{ \frac{\xi_1^3}{\xi_1 - 2} s_2 + \frac{\xi_1 + 1}{2} f'(\xi_1) [I_1(s_2; y) - I_1(0; y)] \right. \\ \left. - \frac{2f'(\xi_1)}{(\xi_1 - 2)^2} [I_1(s_2; z) - I_1(0; z)] + \frac{f'(\xi_1)^2}{16} [I_2(s_2; y) - I_2(0; y)] \right\}. \end{aligned} \quad (29)$$

In general, to obtain a continuous expression for τ_* , one has to be careful in selecting appropriate branches of the logarithms appearing in Eqs. (27) and (28).

As an example, consider a motion of a particle starting from an arbitrary location ξ_0 and moving inwards until it reaches a periapsis with the radius ξ_1 (thus $f(\xi_1) = 0$). Next, the particle moves outwards up to a location with a radius ξ . Denote the Mino time needed for the motion from ξ_0 to ξ_1 by s_1 and the Mino time needed for the motion from ξ_1 to ξ by s_2 . Thanks to symmetry, the coordinate time of the entire motion can be written as

$$\tau(s) = \tau_*(s_1; \xi_1) + \tau_*(s_2; \xi_1) = \tau_*(s_1; \xi_1) + \tau_*(s - s_1; \xi_1), \quad (30)$$

where $s = s_1 + s_2$.

The value s_1 can be computed from Eq. (20). The integral in Eq. (20) can be expressed in terms of the Legendre elliptic integrals, but the exact form of the result depends on a type of particle trajectory. For an unbound scattered trajectory

$$\begin{aligned}
 s(\xi_1) &= \epsilon_r \int_{\xi_1}^{\infty} \frac{d\xi}{\sqrt{f(\xi)}} \\
 &= \frac{\epsilon_r}{\sqrt{y_3 - y_1}} \left[F \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi}}{y_2 - y_1}}, k \right) - F \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12}}{y_2 - y_1}}, k \right) \right],
 \end{aligned} \tag{31}$$

where $y_1 < y_2 < y_3$ are real zeros of the polynomial $4y^3 - g_2y - g_3$, and $k^2 = (y_2 - y_1)/(y_3 - y_1)$. A longer discussion of this result can be found in [6, 9], where it is referred to as the elliptic X function. A comprehensive discussion of the classification of orbits and different expressions for the integral in (20) can be found in [3, 5].

5. Conclusions

We have revisited the theory of timelike and null geodesics in the Schwarzschild spacetime. A novel aspect of our study is an application of the Biermann–Weierstrass theorem, providing a single formula describing bound and unbound geodesic orbits. More details on this approach can be found in our recent article [6]. In [6], geodesics are parametrized using the so-called true anomaly — an angle measured within the orbital plane. Here, we introduce a parametrization by an analogue of the Mino time, commonly used in the theory of the Kerr geodesics. The solutions formulated in this way can serve as limiting expressions for Kerr geodesics in the limit of non-spinning black holes.

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REFERENCES

- [1] Y. Hagihara, *Jpn. J. Astron. Geophys.* **8**, 67 (1930),
<https://ui.adsabs.harvard.edu/abs/1930JaJAG...8...67H/abstract>
- [2] B. Mielnik, J. Plebański, *Acta Phys. Pol.* **21**, 239 (1962).
- [3] S. Chandrasekhar, «The Mathematical Theory of Black Holes», *Oxford University Press*, New York, NY 1983.

- [4] G. Scharf, *J. Mod. Phys.* **2**, 274 (2011).
- [5] U. Kostić, *Gen. Relativ. Gravit.* **44**, 1057 (2012).
- [6] A. Cieřlik, P. Mach, *Class. Quantum Grav.* **39**, 225003 (2022).
- [7] P. Rioseco, O. Sarbach, *Class. Quantum Grav.* **34**, 095007 (2017).
- [8] P. Rioseco, O. Sarbach, *J. Phys.: Conf. Ser.* **831**, 012009 (2017).
- [9] P. Mach, A. Odrzywółek, *Phys. Rev. D* **103**, 024044 (2021).
- [10] P. Mach, A. Odrzywółek, *Phys. Rev. Lett.* **126**, 101104 (2021).
- [11] P. Mach, A. Odrzywółek, *Acta Phys. Pol. B Proc. Suppl.* **15**, 1-A7 (2022).
- [12] C. Gabarrete, O. Sarbach, *Acta Phys. Pol. B Proc. Suppl.* **15**, 1-A10 (2022).
- [13] A. Gamboa *et al.*, *Phys. Rev. D* **104**, 083001 (2021).
- [14] A. Cieřlik, P. Mach, *Phys. Rev. D* **102**, 024032 (2020).
- [15] P. Rioseco, O. Sarbach, *Phys. Rev. D* **98**, 124024 (2018).
- [16] L. Andersson, P. Blue, J. Joudioux, *Commun. Part. Diff. Eq.* **43**, 47 (2018).
- [17] A. Cieřlik, P. Mach, A. Odrzywółek, *Phys. Rev. D* **106**, 104056 (2022).
- [18] P. Mach, A. Odrzywółek, *Acta Phys. Pol. B Proc. Suppl.* **16**, 6-AX.X (2023), this issue.
- [19] W. Biermann, «Problemata quaedam mechanica functionum ellipticarum ope soluta», Ph.D. Thesis, Berlin 1865,
<http://resolver.sub.uni-goettingen.de/purl?PPN313151385>
- [20] Y. Mino, *Phys. Rev. D* **67**, 084027 (2003).
- [21] A.G. Greenhill, «The applications of elliptic functions», Macmillan, London 1892, <https://archive.org/details/applicationselli00greerich>
- [22] M.J. Reynolds, *J. Phys. A: Math. Gen.* **22**, L723 (1989).
- [23] P.F. Byrd, M.D. Friedman, «Handbook of Elliptic Integrals for Engineers and Scientists», Springer, Berlin, Heidelberg 1971.
- [24] I.S. Gradshteyn, I.M. Ryzhik, A. Jeffrey, «Table of integrals, series, and products», Academic Press, Amsterdam, Boston 2007.

2.3. Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes

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Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes

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We develop a Monte Carlo simulation method for computing stationary solutions of the general-relativistic Vlasov equation describing a gas of noncolliding particles. As specific examples, we select planar or spherically symmetric accretion models on the Schwarzschild background spacetime. In all cases the gas extends to infinity, which poses an additional difficulty in the Monte Carlo approach. We discuss models with monoenergetic particles as well as solutions obeying the Maxwell-Jüttner distribution at infinity. For all models, exact expressions for the particle current density are known or can be computed analytically. We demonstrate perfect agreement between exact expressions for the particle current density and the results of our Monte Carlo simulations.

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I. INTRODUCTION

In the general-relativistic kinetic theory of gases, observable quantities (the particle current density, energy-momentum tensor, energy density, pressure, etc.) are computed as suitable integrals over momentum space [1–9]. With sufficient control of the phase-space structure (e.g., a good description of the regions in the phase space available for the motion of gas particles), one can, in many cases, compute such integrals directly, providing a solution to the problem at hand (see, e.g., [10–18]). For complex problems, a good description of the phase-space structure may not be available, and one has to resort to other methods. In this work we deal with a simple (but important) case in which collisions between individual particles of the gas are neglected. Since a collisionless gas consists of particles moving along geodesic trajectories, a possible way would be to construct a Monte Carlo simulation, in which one would select a sample of geodesic trajectories and then compute averages over this sample, in order to obtain desired observables. Although this idea sounds simple, its actual implementation poses several difficulties related to the selection of geodesics and the averaging procedure.

Another option is to use the so-called particle-in-cell method, which has been employed successfully in general-relativistic simulations of kinetic systems with magnetic fields [19–22].

In this article we present a technical implementation of a Monte Carlo approach, designed to work with stationary solutions. While stationary solutions seem to be natural and even simple, they require a somewhat counterintuitive approach to the averaging process, as will become clear in the remainder of this paper. In short, instead of

counting intersections of geodesic trajectories with regions of spacelike hypersurfaces, we count intersections with suitable timelike hypersurfaces.

For clarity, we will focus on several easy accretion-type problems in the Schwarzschild spacetime, for which analytic solutions are available. Specifically, we will consider stationary Bondi-type solutions, in which the gas extends to infinity, where it is assumed to be homogeneous and at rest [10,11,23]. We will work with planar models, assuming that the gas is confined to a single plane (similarly to the accretion model in the Kerr spacetime described in [17]), and with spherically symmetric configurations. In the simplest case, we restrict ourselves to monoenergetic particles, but we also give examples in which the gas obeys the Maxwell-Jüttner distribution at infinity [24,25].

As a supplement to this paper, we provide sample Wolfram *Mathematica* [26] codes performing our simulations. They will be publicly available at [27].

This paper is organized as follows. In Sec. II we introduce our notation and define the one-particle distribution function. Section III is devoted to planar models, in which the motion of particles is confined to a common plane. We start with an elementary model of a planar uniform distribution of monoenergetic particles in the Minkowski spacetime. Two planar accretion models for the Schwarzschild spacetime are given in Sec. III B. In Sec. III C we discuss Monte Carlo integration in the context of these two models. This section provides a link between the two approaches: an analytic one and an approach based on the actual Monte Carlo simulations. It also gives us an opportunity to discuss the selection of geodesic parameters describing the Monte Carlo sample. Section III D introduces our Monte Carlo simulations. In Sec. IV we focus on

spherically symmetric stationary accretion models. A short summary is given in Sec. V.

We will work in standard gravitational units with $c = G = 1$, where c is the speed of light and G denotes the gravitational constant. We assume the signature of the metric in the form $(-, +, +, +)$.

II. ONE-PARTICLE DISTRIBUTION FUNCTION

In the framework of the general-relativistic kinetic theory, the gas is described in terms of a distribution function defined on a one-particle phase space—a subset of the tangent or cotangent bundle of the spacetime manifold (see, e.g., [9] for a recent introduction). In this paper we adhere to the cotangent bundle formulation, but it is, in fact, a matter of convention.

Let (M, g) be a spacetime manifold. The cotangent bundle of M is defined as

$$T^*M = \{(x, p) : x \in M, p \in T_x^*M\}. \quad (1)$$

The one-particle distribution function is a real function defined on suitably chosen subsets $U \subseteq T^*M$, i.e., $\mathcal{F} : U \rightarrow [0, +\infty)$. The precise mathematical definition of $\mathcal{F}$ depends on a convention regarding the choice of U . Let us consider a gas composed of the same rest-mass particles (such a gas is sometimes referred to as “simple” [2]). One possibility is to restrict U to the future mass shell, defined as

$$\Gamma_m^+ = \{(x, p) \in T^*M : g^{\mu\nu} p_\mu p_\nu = -m^2, \quad p \text{ is future-directed}\}, \quad (2)$$

where m denotes the rest mass of a particle. This option has been chosen, e.g., in [9]. Another option is to transfer the mass-shell restriction to the assumed form of $\mathcal{F}$ and to demand that $\mathcal{F} \sim \delta(\sqrt{-p_\mu p^\mu} - m)$. In this paper, we decided to choose the latter option, mainly to be consistent with previous works [10,11].

Let S denote a three-dimensional spacelike hypersurface in M , and let s be a future-directed unit vector normal to S . We define $\mathcal{N}[S]$ as an averaged number of particle trajectories in U , whose projections on M intersect S . In other words, $\mathcal{N}[S]$ denotes the number of particles in S . It can be shown that [cf. [9], Eqs. (94)]

$$\mathcal{N}[S] = - \int_S \left[\int_{P_x^+} \mathcal{F}(x, p) p_\mu s^\mu \text{dvol}_x(p) \right] \eta_S, \quad (3)$$

where

$$P_x^+ = \{p \in T_x^*M : g^{\mu\nu} p_\mu p_\nu < 0, p \text{ is future-directed}\} \quad (4)$$

and η_S denotes the volume element on S . The volume element on P_x^+ is given (in local adapted coordinates) by

$$\text{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)} dp_0 dp_1 dp_2 dp_3. \quad (5)$$

Equation (3) gives rise to the definition of the so-called particle current density

$$\mathcal{J}_\mu(x) = \int_{P_x^+} \mathcal{F}(x, p) p_\mu \text{dvol}_x(p). \quad (6)$$

With this definition one can write $\mathcal{N}[S]$ as

$$\mathcal{N}[S] = - \int_S \mathcal{J}_\mu s^\mu \eta_S. \quad (7)$$

The geodesic motion can be described by Hamilton's equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H(x, p)}{\partial p_\mu}, \quad (8a)$$

$$\frac{dp_\nu}{d\tau} = - \frac{\partial H(x, p)}{\partial x^\nu}, \quad (8b)$$

where $p^\mu = dx^\mu/d\tau$, $H(x, p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu = -\frac{1}{2} m^2$. In the absence of collisions between particles, the distribution function $\mathcal{F}$ satisfies the so-called Vlasov equation, which can be expressed as a requirement that $\mathcal{F}$ should remain constant along a geodesic:

$$\begin{aligned} \frac{d\mathcal{F}}{d\tau} &= \frac{\partial \mathcal{F}}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial \mathcal{F}}{\partial p_\nu} \frac{dp_\nu}{d\tau} = \frac{\partial \mathcal{F}}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial \mathcal{F}}{\partial p_\nu} \frac{\partial H}{\partial x^\nu} \\ &= \{H, \mathcal{F}\} = 0. \end{aligned} \quad (9)$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket. Note, in particular, that a probability function depending on (x, p) via constants of motion $I_i(x, p)$, i.e., $\mathcal{F} = \mathcal{F}(I_1(x, p), \dots, I_s(x, p))$, where $\{H, I_i\} = 0$, $i = 1, \dots, s$, would always satisfy the Vlasov equation. In explicit terms, Eq. (9) can be written as

$$g^{\mu\nu} p_\nu \frac{\partial \mathcal{F}}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial x^\mu} \frac{\partial \mathcal{F}}{\partial p_\mu} = 0. \quad (10)$$

Using Eq. (9), one can show that the particle current density satisfies the conservation law $\nabla_\mu \mathcal{J}^\mu = 0$, which again justifies formula (7). The particle number density can be defined covariantly as $n = \sqrt{-\mathcal{J}_\mu \mathcal{J}^\mu}$. Alternatively, one can work with the components of $\mathcal{J}_\mu$.

III. PARTICLES CONFINED TO A PLANE

We will assume a specific-to-general approach and start with illustrative cases of the gas confined to a plane. More specifically, we will begin with a discussion of a uniform gas of monoenergetic particles restricted to a two-dimensional plane in the flat Minkowski spacetime. In the next step, we turn to a planar accretion problem in the Schwarzschild spacetime.

A. Flat space case

Consider a uniform gas of monoenergetic noncolliding particles confined to a two-dimensional plane in the flat Minkowski spacetime. The gas particles move along straight lines. The only nonvanishing component of the particle current density $\mathcal{J}_t$ can be computed in two equivalent ways. A straightforward way is to work in Cartesian coordinates (t, x, y, z) . Defining

$$\mathcal{F}(x^\alpha, p_\beta) = \delta(z)F(t, x, y; p_\beta), \quad (11)$$

we obtain

$$\mathcal{J}_\mu(x^\alpha) = \delta(z)J_\mu(t, x, y), \quad (12)$$

where

$$J_\mu(t, x, y) = \int F(t, x, y; p_\beta) p_\mu d\text{vol}_x(p). \quad (13)$$

For a uniformly distributed gas of monoenergetic particles of the same mass m_0 , we assume

$$\begin{aligned} F(x, p) &= \alpha m_0 \delta(p_z) \delta(p_t + E_0) \delta(\sqrt{p_t^2 - \mathbf{p}^2} - m_0) \\ &= \alpha \delta(p_z) \delta(\varepsilon - \varepsilon_0) \delta(m - m_0), \end{aligned} \quad (14)$$

where $p_t = -E$, $E = m\varepsilon$, $E_0 = m\varepsilon_0$, and α is a proportionality constant. Hence,

$$\begin{aligned} J_t &= \alpha m_0 \int \delta(p_z) \delta(p_t + E_0) \\ &\quad \times \delta(\sqrt{p_t^2 - \mathbf{p}^2} - m_0) p_t dp_t dp_x dp_y dp_z. \end{aligned} \quad (15)$$

Integrating over p_t and p_z is straightforward, and it gives

$$J_t = -\alpha m_0 E_0 \int \delta(\sqrt{E_0^2 - p_x^2 - p_y^2} - m_0) dp_x dp_y. \quad (16)$$

Introducing polar momentum coordinates $p_x = \zeta \cos \vartheta$, $p_y = \zeta \sin \vartheta$, we obtain $dp_x dp_y = \zeta d\vartheta d\zeta$ and

$$J_t = -2\pi \alpha m_0 E_0 \int_0^\infty \delta(\sqrt{E_0^2 - \zeta^2} - m_0) \zeta d\zeta. \quad (17)$$

On the other hand,

$$\delta(\sqrt{E_0^2 - \zeta^2} - m_0) = \frac{m_0}{\sqrt{E_0^2 - m_0^2}} \delta(\zeta - \sqrt{E_0^2 - m_0^2}), \quad (18)$$

and hence

$$J_t = -2\pi \alpha m_0^2 E_0 = -2\pi \alpha m_0^3 \varepsilon_0, \quad (19)$$

where $E_0 = m_0 \varepsilon_0$.

The second calculation is based on spherical coordinates. While these may seem to be a bad choice for the problem with an explicit translational symmetry, this calculation provides an illustration of some aspects related to the selection of samples of geodesics in the Monte Carlo simulations discussed in this paper. We define

$$m^2 = p_t^2 - p_r^2 - \frac{1}{r^2} p_\theta^2 - \frac{1}{r^2 \sin^2 \theta} p_\varphi^2, \quad (20a)$$

$$E = -p_t, \quad (20b)$$

$$l^2 = p_\theta^2 + \frac{1}{\sin^2 \theta} p_\varphi^2, \quad (20c)$$

$$l_z = p_\varphi \quad (20d)$$

and assume a convention in which $l \geq 0$. Quantities m , E , l , and l_z are constants of motion. A straightforward calculation gives

$$\frac{\partial(m^2, E, l^2, l_z)}{\partial(p_t, p_r, p_\theta, p_\varphi)} = \pm 4 \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}. \quad (21)$$

Treating m , E , l , and l_z as new momentum coordinates, one obtains

$$\begin{aligned} m l d m d E d l d l_z &= \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} \\ &\quad \times \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}} dp_t dp_r dp_\theta dp_\varphi. \end{aligned} \quad (22)$$

Keeping the pair of coordinates (l, l_z) is troublesome in making the restriction to the equatorial plane. A convenient solution is to change $(l, l_z) \mapsto (l, \sigma)$, where

$$\sin \sigma = \frac{l_z}{l}. \quad (23)$$

Thus, $p_\theta = 0$ corresponds to $\sigma = \pm \pi/2$. We have $dl_z = \sqrt{l^2 - l_z^2} d\sigma$. Moreover, at the equatorial plane $\theta = \pi/2$, we find

$$m l d m d E d l d \sigma = \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} dp_t dp_r dp_\theta dp_\varphi, \quad (24)$$

and hence the volume element $d\text{vol}_x(p)$ can be written as

$$d\text{vol}_x(p) = \frac{m l d m d E d l d \sigma}{r^2 \sqrt{E^2 - m^2 - \frac{l^2}{r^2}}}. \quad (25)$$

Similarly,

$$\delta(p_z) = \frac{r}{l} \left[\delta\left(\sigma - \frac{\pi}{2}\right) + \delta\left(\sigma + \frac{\pi}{2}\right) \right]. \quad (26)$$

This gives

$$F(x, p) = \frac{\alpha m_0 r}{l} \delta(m - m_0) \delta(E - E_0) \times \left[\delta\left(\sigma - \frac{\pi}{2}\right) + \delta\left(\sigma + \frac{\pi}{2}\right) \right] \quad (27)$$

and

$$J_t = -\frac{4\alpha m_0^2 E_0}{r} \int_0^{l_{\max}} \frac{dl}{\sqrt{E_0^2 - m_0^2 - \frac{l^2}{r^2}}}, \quad (28)$$

where $l_{\max} = r\sqrt{E_0^2 - m_0^2}$. The additional factor 2 appearing in the above formula comes from the fact that a given set of values E , m , and l corresponds to two possible values of the radial momentum $p_r = \pm|p_r|$ [cf. Eq. (20a)]. Let us introduce the following dimensionless quantities:

$$l = Mm_0\lambda, \quad r = M\xi, \quad E_0 = m_0\varepsilon_0, \quad (29)$$

where M denotes a reference mass parameter. The expression for J_t can be written as

$$J_t = -\frac{4\alpha m_0^3 \varepsilon_0}{\xi} \int_0^{\lambda_{\max}} \frac{d\lambda}{\sqrt{\varepsilon_0^2 - 1 - \frac{\lambda^2}{\xi^2}}}, \quad (30)$$

where $\lambda_{\max} = \xi\sqrt{\varepsilon_0^2 - 1}$. The integral in Eq. (30) reads

$$\int_0^{\lambda_{\max}} \frac{d\lambda}{\sqrt{\varepsilon_0^2 - 1 - \frac{\lambda^2}{\xi^2}}} = \frac{\pi\xi}{2}. \quad (31)$$

Note that it can also be expressed as

$$\frac{\pi}{2\sqrt{\varepsilon_0^2 - 1}} \int_0^{\lambda_{\max}} d\lambda = \frac{\pi\xi}{2}. \quad (32)$$

Thus, we finally obtain

$$J_t = -2\pi\alpha m_0^3 \varepsilon_0, \quad (33)$$

which coincides with Eq. (19). We will return to Eqs. (30)–(32) in Sec. III D, discussing the selection of geodesic parameters. For future use, note that

$$n_s = 2\pi\alpha m_0^3 \varepsilon_0 \quad (34)$$

can be interpreted as a particle surface density of the gas.

B. Planar accretion onto a Schwarzschild black hole

As one of our test models, we consider planar stationary accretion in the Schwarzschild spacetime. It is an

equivalent of the Bondi (or Michel)-type stationary accretion of the collisionless gas occurring in the Schwarzschild spacetime (see Refs. [10,11,23]) restricted to a plane. Within the plane, the gas extends to infinity, and it is assumed to be asymptotically “at rest.” A model of this type for the Kerr spacetime has been analyzed in [17]. In contrast to the Kerr case, the planar accretion model in the Schwarzschild spacetime is relatively simple and can serve as a pedagogical example in our discussion.

We work in standard Schwarzschild coordinates (t, r, θ, φ) . The Schwarzschild metric has the well-known form

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (35)$$

All our calculations can also be repeated using suitably chosen horizon-penetrating Eddington-Finkelstein-type coordinates, yielding essentially the same results. We use the simplest Schwarzschild form of the metric to avoid unnecessary complications.

The geodesic motion is integrable. There exist four independent constants of motion, which, for simplicity, we denote as

$$m^2 = -g^{\mu\nu} p_\mu p_\nu, \quad (36a)$$

$$E = -p_t, \quad (36b)$$

$$l^2 = p_\theta^2 + \frac{1}{\sin^2 \theta} p_\varphi^2, \quad (36c)$$

$$l_z = p_\varphi, \quad (36d)$$

using the symbols which have already been introduced in the flat Minkowski case. As before, we assume $l \geq 0$ and define dimensionless quantities ξ , ε , λ , and λ_z :

$$r = M\xi, \quad E = m\varepsilon, \quad l_z = Mm\lambda_z, \quad l = Mm\lambda, \quad (37)$$

where, this time, M refers explicitly to the Schwarzschild mass. We will also use the momentum-space coordinate σ (23), defined using dimensionless quantities by $\sin \sigma = \lambda_z/\lambda$.

The phase-space region available for the motion can be controlled with almost the same expressions as those used in the spherically symmetric model [10]. In what follows, we will summarize the most important formulas; details of the derivation can be found in [10]. The contravariant radial component of the momentum vector reads

$$p^r = \pm m \sqrt{\varepsilon^2 - U_\lambda(\xi)}, \quad (38)$$

where

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi^2}\right) \quad (39)$$

denotes the dimensionless effective radial potential.

The region available for the radial motion is characterized by the condition $\varepsilon^2 - U_\lambda(\xi) \geq 0$. Since $U_\lambda(\xi) \rightarrow 1$, as $\xi \rightarrow \infty$, unbound orbits are characterized by the energy $\varepsilon \geq 1$. The orbits with the angular momentum $\lambda < \lambda_c(\varepsilon)$, where

$$\lambda_c(\varepsilon)^2 = \frac{12}{1 - \frac{4}{\left(\frac{3\varepsilon}{\sqrt{9\varepsilon^2 - 8}} + 1\right)^2}}, \quad (40)$$

plunge into the black hole—we refer to these orbits as absorbed ones. Unbound orbits with $\lambda > \lambda_c(\varepsilon)$ are scattered off the centrifugal barrier. The maximum value of the angular momentum for a scattered unbound orbit reaching radius ξ reads

$$\lambda_{\max}(\varepsilon, \xi) = \xi \sqrt{\frac{\varepsilon^2}{1 - \frac{2}{\xi}} - 1}, \quad (41)$$

while the minimum allowed energy can be written as

$$\varepsilon_{\min}(\xi) = \begin{cases} \infty & \xi \leq 3 \\ \sqrt{\left(1 - \frac{2}{\xi}\right) \left(1 + \frac{1}{\xi-3}\right)} & 3 < \xi < 4 \\ 1 & \xi \geq 4. \end{cases} \quad (42)$$

1. Monoenergetic particles

For a planar accretion model with monoenergetic particles, we assume a one-particle distribution function of the form

$$\begin{aligned} F(x, p) &= \alpha m_0 \delta(\sqrt{-p_\mu p^\mu} - m_0) \delta(p_t + E_0) \delta(p_z) \\ &= \alpha \delta(m - m_0) \delta(\varepsilon - \varepsilon_0) \delta(p_z) \\ &= \frac{\alpha \xi}{m_0 \lambda} \delta(m - m_0) \delta(\varepsilon - \varepsilon_0) \\ &\quad \times [\delta(\sigma - \pi/2) + \delta(\sigma + \pi/2)]. \end{aligned} \quad (43)$$

One may readily verify that the above choice satisfies the Vlasov equation (9).

The volume element in momentum space reads, in terms of coordinates m , ε , λ , and σ ,

$$\text{dvol}_x(p) = \frac{m^3 \lambda}{\xi^2 \sqrt{\varepsilon^2 - U_\lambda(\xi)}} dm d\varepsilon d\lambda d\sigma. \quad (44)$$

The components of the particle surface current density J_μ can now be expressed as a sum of two parts: a part corresponding to absorbed orbits and a part corresponding to scattered ones, i.e., $J_\mu = J_\mu^{(\text{abs})} + J_\mu^{(\text{scat})}$. According to the characterization given in the previous subsection, one has

$$J_t^{(\text{abs})}(\xi) = -\frac{2\alpha m_0^3}{\xi} \int_1^\infty d\varepsilon \delta(\varepsilon - \varepsilon_0) \varepsilon \int_0^{\lambda_c(\varepsilon)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} = -\frac{2\alpha m_0^3}{\xi} \varepsilon_0 \Theta(\varepsilon_0 - 1) \int_0^{\lambda_c(\varepsilon_0)} \frac{d\lambda}{\sqrt{\varepsilon_0^2 - U_\lambda(\xi)}}, \quad (45a)$$

$$\begin{aligned} J_t^{(\text{scat})}(\xi) &= -\frac{4\alpha m_0^3}{\xi} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon \delta(\varepsilon - \varepsilon_0) \varepsilon \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\varepsilon, \xi)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} \\ &= -\frac{4\alpha m_0^3}{\xi} \varepsilon_0 \Theta(\varepsilon_0 - \varepsilon_{\min}(\xi)) \int_{\lambda_c(\varepsilon_0)}^{\lambda_{\max}(\varepsilon_0, \xi)} \frac{d\lambda}{\sqrt{\varepsilon_0^2 - U_\lambda(\xi)}}, \end{aligned} \quad (45b)$$

where Θ denotes the Heaviside step function. The additional factor 2 in the expression for $J_t^{(\text{scat})}$ is due to equal contributions of ingoing and outgoing trajectories. The radial component reads

$$J^r(\xi) = J_{(\text{abs})}^r(\xi) = -\frac{2\alpha m_0^3}{\xi} \Theta(\varepsilon_0 - 1) \lambda_c(\varepsilon_0). \quad (46)$$

The radial component J^r is directly related with the rest-mass accretion rate, which we define as (see Ref. [17])

$$\dot{M} = -2\pi M m_0 \xi J^r. \quad (47)$$

For the monoenergetic planar model the rest-mass accretion rate reads

$$\dot{M} = 4\pi\alpha M m_0^4 \Theta(\varepsilon_0 - 1) \lambda_c(\varepsilon_0). \quad (48)$$

As a general remark, applying to all models in this paper, let us note that in physical applications the proportionality constant α appearing in the expression for the distribution function is rather hard to control. For planar models, it can be expressed in terms of the asymptotic particle number surface density n_s , which for the gas of monoenergetic particles is given by Eq. (34). This allows us to write Eq. (48) as

$$\dot{M} = 2M m_0 n_s \Theta(\varepsilon_0 - 1) \frac{\lambda_c(\varepsilon_0)}{\varepsilon_0}. \quad (49)$$

For spherical models one can express the proportionality constant α in terms of the asymptotic particle number density [10,11]. Since in this work we focus mainly on the Monte Carlo simulations, we will keep the proportionality constant α for clarity in the resulting formulas.

2. Maxwell-Jüttner distribution

Assuming, instead of monoenergetic particles, that at infinity the gas obeys a Maxwell-Jüttner distribution restricted to a plane [24,25], we set

$$\begin{aligned} F(x, p) &= \alpha \delta\left(\sqrt{-p_\mu p^\mu} - m_0\right) \exp\left(\frac{\beta}{m_0} p_t\right) \delta(p_z) \\ &= \frac{\alpha \xi}{m_0 \lambda} \delta(m - m_0) \exp(-\beta \varepsilon) \\ &\quad \times [\delta(\sigma - \pi/2) + \delta(\sigma + \pi/2)], \end{aligned} \quad (50)$$

where $\beta = m_0/(k_B T)$, k_B denotes the Boltzmann constant, and T is the asymptotic temperature of the gas.

A calculation similar to that for the planar accretion of monoenergetic particles yields

$$J_t^{(\text{abs})}(\xi) = -\frac{2\alpha m_0^3}{\xi} \int_1^\infty \varepsilon \exp(-\beta \varepsilon) \int_0^{\lambda_c(\varepsilon)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}}, \quad (51a)$$

$$J_t^{(\text{scat})}(\xi) = -\frac{4\alpha m_0^3}{\xi} \int_{\varepsilon_{\min}(\xi)}^\infty \varepsilon \exp(-\beta \varepsilon) \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\varepsilon, \xi)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}}, \quad (51b)$$

$$J'(\xi) = -\frac{2\alpha m_0^3}{\xi} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \lambda_c(\varepsilon). \quad (51c)$$

With a slight abuse of terminology, we will refer to the model described in this subsection as the planar

Maxwell-Jüttner model, keeping in mind that the Maxwell-Jüttner distribution is only assumed asymptotically.

C. Monte Carlo integration

Our first approach is to apply a Monte Carlo integration to Eq. (45) or (51) (for a general introduction to Monte Carlo methods see, for instance, [28]). This procedure can be viewed as an intermediate step between analytic solutions given in previous subsections and our final aim to construct actual Monte Carlo simulations. We will also use it as an illustration of the problems related to the selection of the sample of geodesics. Consider an integral

$$I = \int_{\Omega} f(x) dx. \quad (52)$$

Let $X_i \in \Omega$, $i = 1, \dots, N$ denote N samples chosen from the distribution $p(x)$. The Monte Carlo estimator of I reads

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}. \quad (53)$$

For a uniform distribution, one has $p(x) = 1/V$, where $V = \text{vol}(\Omega)$ is the volume of Ω , and thus

$$\langle I \rangle = \frac{V}{N} \sum_{i=1}^N f(X_i). \quad (54)$$

Consider now an integral of the form

$$I = \int_{\Omega} f(x) g(x) dx, \quad (55)$$

where $g(x)$ is a weight. For $p(x) = c g(x)$, where c is a proportionality constant, one obtains

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i) g(X_i)}{p(X_i)} = \frac{1}{cN} \sum_{i=1}^N f(X_i). \quad (56)$$

Since the probability density function $p(x)$ integrates to unity, we obtain

$$c = \left[\int_{\Omega} g(x) dx \right]^{-1}. \quad (57)$$

Thus,

$$\langle I \rangle = \frac{1}{N} \left[\int_{\Omega} g(x) dx \right] \sum_{i=1}^N f(X_i). \quad (58)$$

These two options yield two different approaches to solving our problem. For monoenergetic accretion, we would rather use a uniform-distribution integration. This can be done as follows. Let ξ_0 be the outer radius of the

region of interest, and let $\lambda_i \in [0, \lambda_{\max}(\varepsilon_0, \xi_0)]$ denote uniformly selected random angular momenta. Note that the configuration is assumed to extend to infinity. Selecting a finite ξ_0 allows us to focus our attention on the region $2 < \xi \leq \xi_0$ and to limit the range of the angular momentum.¹ Angular momenta λ_i can be divided into two classes, with the corresponding sets of indices: $I_{\text{abs}} = \{i: 0 \leq \lambda_i < \lambda_c(\varepsilon_0)\}$, $I_{\text{scat}} = \{i: \lambda_c < \lambda_i \leq \lambda_{\max}(\varepsilon_0, \xi_0)\}$, with $N_{\text{abs}} = \#I_{\text{abs}}$, $N_{\text{scat}} = \#I_{\text{scat}}$, where $\#I$ denotes the number of elements in set I . Monte Carlo estimators of $J_t^{(\text{abs})}$ and $J_t^{(\text{scat})}$ in the monoenergetic planar model can be written as

$$\langle J_t^{(\text{abs})} \rangle = -\frac{2am_0^3 \lambda_c(\varepsilon_0)}{N_{\text{abs}} \xi} \times \sum_{i \in I_{\text{abs}}} \frac{\varepsilon_0}{\sqrt{\varepsilon_0^2 - (1 - \frac{2}{\xi})(1 + \frac{\lambda_i^2}{\xi^2})}}, \quad (59a)$$

$$\langle J_t^{(\text{scat})} \rangle = -\frac{4am_0^3 [\lambda_{\max}(\varepsilon_0, \xi_0) - \lambda_c(\varepsilon_0)]}{N_{\text{scat}} \xi} \times \sum_{i \in I_{\text{scat}}} \frac{\varepsilon_0 \Theta(\xi - \xi_{\min}(\lambda_i, \varepsilon_0))}{\sqrt{\varepsilon_0^2 - (1 - \frac{2}{\xi})(1 + \frac{\lambda_i^2}{\xi^2})}}, \quad (59b)$$

where $\xi_{\min}(\lambda, \varepsilon)$ denotes the pericenter radius—the largest zero of the polynomial $\xi^3[\varepsilon^2 - U_\lambda(\xi)]$. Sample results obtained in this way are shown in Fig. 1, assuming $\varepsilon_0 = 1.3$ and $\xi_0 = 20$. Monte Carlo data are plotted with dots. Solid and dashed lines represent exact solutions. We plot the graphs of $-J_t/(\alpha m_0^3)$ and $-J_t^{(\text{abs})}/(\alpha m_0^3)$. We omit the calculation of the Monte Carlo estimator for J^r since it is almost trivial.

The planar Maxwell-Jüttner accretion model can serve as an illustration for the second scheme. Selecting geodesic parameters from the Maxwell-Jüttner distribution can be done using von Neumann's rejection method [29]. As before, let us focus on the region $2 < \xi \leq \xi_0$. We start by introducing a cutoff $\varepsilon_{\text{cutoff}}$ for the allowed energy ε . Subsequently, we select, assuming uniform distributions, $\varepsilon_i \in [1, \varepsilon_{\text{cutoff}}]$, $\lambda_i \in [0, \lambda_{\max}(\varepsilon_{\text{cutoff}}, \xi_0)]$, and an auxiliary variable $y_i \in [0, 1]$. The values of ε_i and λ_i are added to the set of selected parameters, if $y_i < \exp(-\beta \varepsilon_i) / \exp(-\beta)$ and $\lambda_i < \lambda_{\max}(\varepsilon_i, \xi_0)$, and rejected otherwise. This procedure is iterated until a sufficient number of pairs $(\varepsilon_i, \lambda_i)$ is selected. As before, parameters $(\varepsilon_i, \lambda_i)$ are divided into those corresponding to absorbed trajectories [$\lambda_i < \lambda_c(\varepsilon_i)$] and scattered ones [$\lambda_i > \lambda_c(\varepsilon_i)$]. We denote, also as before, $N_{\text{abs}} = \#I_{\text{abs}}$, $N_{\text{scat}} = \#I_{\text{scat}}$.

Monte Carlo estimators of $J_t^{(\text{abs})}$ and $J_t^{(\text{scat})}$ are computed as

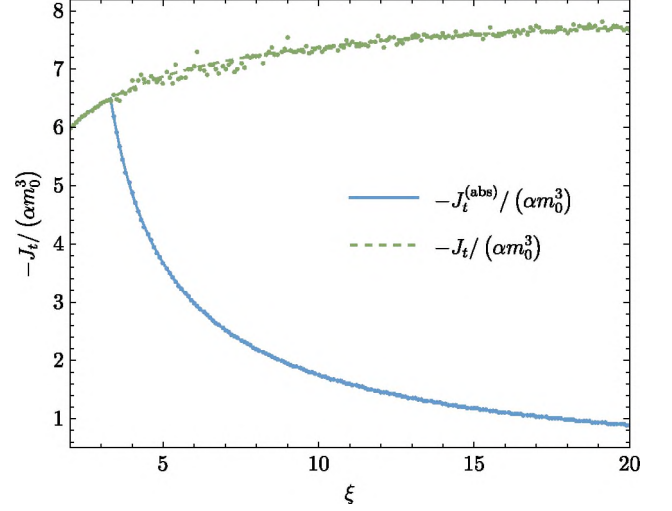


FIG. 1. Components of the particle surface density current J_t for the monoenergetic planar model with $\varepsilon_0 = 1.3$, $\xi_0 = 20$. Exact solutions [Eqs. (45)] are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo integration [Eqs. (59)]. There are 10^5 angular momentum samples: $N_{\text{abs}} = 31814$, $N_{\text{scat}} = 68186$.

$$\langle J_t^{(\text{abs})} \rangle = -\frac{2am_0^3 V_{\text{abs}}}{N_{\text{abs}} \xi} \times \sum_{i \in I_{\text{abs}}} \frac{\varepsilon_i}{\sqrt{\varepsilon_i^2 - (1 - \frac{2}{\xi})(1 + \frac{\lambda_i^2}{\xi^2})}}, \quad (60a)$$

$$\langle J_t^{(\text{scat})} \rangle = -\frac{4am_0^3 V_{\text{scat}}}{N_{\text{scat}} \xi} \times \sum_{i \in I_{\text{scat}}} \frac{\varepsilon_i \Theta(\xi - \xi_{\min}(\lambda_i, \varepsilon_i))}{\sqrt{\varepsilon_i^2 - (1 - \frac{2}{\xi})(1 + \frac{\lambda_i^2}{\xi^2})}}, \quad (60b)$$

where

$$V_{\text{abs}} = \int_1^{\varepsilon_{\text{cutoff}}} \exp(-\beta \varepsilon) \lambda_c(\varepsilon) d\varepsilon, \quad (61a)$$

$$V_{\text{scat}} = \int_1^{\varepsilon_{\text{cutoff}}} \exp(-\beta \varepsilon) [\lambda_{\max}(\varepsilon, \xi_0) - \lambda_c(\varepsilon)] d\varepsilon. \quad (61b)$$

An example of the components $J_t^{(\text{abs})}$ and J_t obtained in this way for $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$ is shown in Fig. 2. In Fig. 2 and in what follows, to assure a fair comparison, both analytic solutions and Monte Carlo estimators are computed assuming the same cutoff value $\varepsilon_{\text{cutoff}}$ for the maximal energy.

In both cases we have introduced small adjustments in the Monte Carlo integration procedure, which make it similar to the actual Monte Carlo simulations discussed in the remainder of this paper. In particular, instead of limiting the selection of geodesic parameters used to compute

¹Accretion from a region of a finite radius is an interesting problem on its own [15].

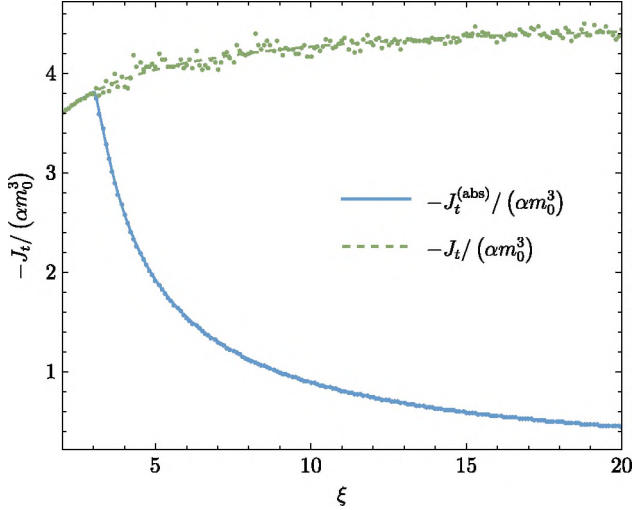


FIG. 2. Same as in Fig. 1 but for the Maxwell-Jüttner planar accretion model with $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$. Exact solutions [Eqs. (51a) and (51b)] are plotted with solid and dashed lines. Blue and green dots show results of the Monte Carlo integration [Eqs. (60)]. The total number of randomly selected triples $(\varepsilon_i, \lambda_i, y_i)$ is 2×10^6 , which gives $N_{\text{abs}} = 10462$, $N_{\text{scat}} = 27268$.

$\langle J_t^{(\text{scat})} \rangle$ with $\lambda_{\text{max}}(\varepsilon, \xi)$, we use an equivalent restriction on ξ_{min} —the pericenter radius.

D. Monte Carlo simulations of stationary planar problems

1. Averages

We construct our Monte Carlo simulations of stationary flows by selecting a set of geodesic trajectories and counting their intersections with suitably chosen surfaces, assuming appropriate weights.

As introductory examples, consider planar problems described in Sec. III. We aim at a Monte Carlo simulation that would allow us to compute the components of, say, J_μ . This poses several problems. The first one is related to the choice of a convenient parametrization of geodesics. The second is to select geodesics assuming the correct probability distribution expressed in terms of geodesic parameters. Finally, one has to design a method of computing suitable averages over momenta. We will start the discussion with this last problem.

The essence of our Monte Carlo simulations is an approximation of a continuous system by a discrete one, given by the distribution function

$$\mathcal{F}(x^\mu, p_\nu) = \int \sum_{i=1}^N \delta^{(4)}(x^\mu - x_{(i)}^\mu(\tau)) \times \delta^{(4)}(p_\nu - p_{(i)}^\nu(\tau)) d\tau, \quad (62)$$

where N denotes the number of particles with trajectories described by coordinates $x_{(i)}^\mu(\tau)$ and momenta $p_{(i)}^\nu(\tau)$ [cf. [5], p. 14, Eq. (A6)]. Our convention regarding the parameter τ is that of Eq. (8), and hence Eq. (62) differs by the factor $1/m$ with respect to the formula in [5]. The particle current density $\mathcal{J}_\mu$ corresponding to this distribution can be computed as

$$\begin{aligned} \mathcal{J}_\mu(x) &= \int \mathcal{F}(x, p) p_\mu \sqrt{-\det g^{\alpha\beta}(x)} dp_0 \dots dp_3 \\ &= \int \sum_{i=1}^N \delta^{(4)}(x^\alpha - x_{(i)}^\alpha(\tau)) \\ &\quad \times p_{(i)}^\mu(\tau) \sqrt{-\det g^{\alpha\beta}(x)} d\tau. \end{aligned} \quad (63)$$

The above expression is compatible with Eq. (7), which can be seen as follows. Let S denote a hypersurface of constant time, and let the spacetime metric g in the vicinity of S be expressed in the standard $3+1$ form

$$g = (-\alpha^2 + \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad (64)$$

where α is the lapse, β^i are the components of the shift vector, and γ denotes the metric induced on S . Then,

$$\begin{aligned} \mathcal{N}[S] &= - \int_S \mathcal{J}_\mu s^\mu \eta_S \\ &= - \int_S \eta_S \int d\tau \sum_{i=1}^N \delta^{(4)}(x^\mu - x_{(i)}^\mu(\tau)) \\ &\quad \times \sqrt{-\det g^{\delta\kappa}(x)} p_{(i)}^\nu(\tau) s_\nu. \end{aligned} \quad (65)$$

Here, $s_\nu = (-\alpha, 0, 0, 0)$, and $\eta_S = \sqrt{\det \gamma_{ij}} d^3x$. Thus,

$$\begin{aligned} \mathcal{N}[S] &= \int_S d^3x \int d\tau \sum_{i=1}^N \delta^{(4)}(x^\mu - x_{(i)}^\mu(\tau)) p_{(i)}^\nu(\tau) \\ &= \sum_{i=1}^N \int d\tau \delta(t - t_{(i)}(\tau)) \frac{dt_{(i)}}{d\tau} = N, \end{aligned} \quad (66)$$

where we have used the fact that $\sqrt{-\det g_{\mu\nu}} = \alpha \sqrt{\det \gamma_{ij}}$.

We will estimate the value of $\mathcal{J}_\mu$ at a point x , by selecting a (small) hypersurface region Σ (a cell), such that $x \in \Sigma$, and computing

$$\langle \mathcal{J}_\mu \rangle = \frac{\int_\Sigma \mathcal{J}_\mu \eta_\Sigma}{\int_\Sigma \eta_\Sigma}. \quad (67)$$

For planar models, we might be interested in the surface density value

$$\langle J_\mu \rangle = \frac{\int_\Sigma J_\mu \eta_\Sigma}{\int_\Sigma \eta_\Sigma} \quad (68)$$

as well.

Perhaps the most natural method would be to select a region within a hypersurface of constant time, discretize this region into cells, and average over trajectories passing through a given cell. The trouble with this approach is that it is not well suited to searching for stationary solutions. Instead, we can select a timelike hypersurface foliated by the orbits of a stationary Killing field and count the intersections of the geodesics with such a surface, assuming additional weights, which we derive below.

For a planar, stationary accretion flow in the Schwarzschild spacetime, we take a segment

$$S = \{(r, \theta, \varphi) : r_1 \leq r \leq r_2, \theta = \pi/2, \varphi = \varphi_0\} \quad (69)$$

and a surface

$$\begin{aligned} \Sigma &= \{(t, r, \theta, \varphi) : t_1 \leq t \leq t_2, r_1 \leq r \leq r_2, \\ &\theta = \pi/2, \varphi = \varphi_0\}. \end{aligned} \quad (70)$$

Let $\Phi_\tau(x_0^i)$ denote the orbit of the timelike Killing vector field $\xi^\mu = (1, 0, 0, 0)$, passing through x_0^i at $\tau = 0$, i.e., $\Phi_0(x_0^i) = x_0^i$. Then, Σ can be expressed as the image

$$\Sigma = \Phi_{[t_1, t_2]}(S). \quad (71)$$

For the planar model we have

$$\begin{aligned} \mathcal{J}_\mu(x) &= \int \sum_{i=1}^N \delta(\theta - \pi/2) \delta^{(3)}(x^\alpha - x_{(i)}^\alpha(\tau)) \\ &\times p_\mu^{(i)}(\tau) \sqrt{-\det g^{\alpha\beta}(x)} d\tau, \end{aligned} \quad (72)$$

where

$$\begin{aligned} \delta^{(3)}(x^\alpha - x_{(i)}^\alpha(\tau)) &= \delta(t - t_{(i)}(\tau)) \delta(r - r_{(i)}(\tau)) \\ &\times \delta(\varphi_0 - \varphi_{(i)}(\tau)). \end{aligned} \quad (73)$$

Since, at the equatorial plane $\delta(\theta - \pi/2) = r\delta(z)$ and $\mathcal{J}_\mu = \delta(z)J_\mu$, we obtain

$$\begin{aligned} J_\mu(x) &= \int \sum_{i=1}^N r \delta^{(3)}(x^\alpha - x_{(i)}^\alpha(\tau)) \\ &\times p_\mu^{(i)}(\tau) \sqrt{-\det g^{\alpha\beta}(x)} d\tau, \end{aligned} \quad (74)$$

where $\sqrt{-\det g^{\alpha\beta}(x)} = (r^2 \sin \theta)^{-1} = 1/r^2$. The volume element induced on Σ reads $\eta_\Sigma = dt dr$. A direct calculation gives

$$\begin{aligned} \int_\Sigma J_\mu \eta_\Sigma &= \int_{t_1}^{t_2} dt \int_{r_1}^{r_2} dr \int d\tau \sum_{i=1}^N \delta^{(3)}(x^\alpha - x_{(i)}^\alpha(\tau)) \frac{p_\mu^{(i)}(\tau)}{r} \\ &= \int d\tau \sum_{i=1}^N \delta(\varphi_0 - \varphi_{(i)}(\tau)) \frac{p_\mu^{(i)}(\tau)}{r_{(i)}(\tau)}, \end{aligned} \quad (75)$$

where in the last sum we only take into account trajectories that intersect Σ . The integral with respect to τ can be computed by writing

$$\delta(\varphi_0 - \varphi_{(i)}(\tau)) = \sum_k \frac{\delta(\tau - \tau_k)}{\left| \frac{d\varphi_{(i)}}{d\tau} \right|_{\tau=\tau_k}}, \quad (76)$$

where $\varphi_{(i)}(\tau_k) = \varphi_0$. The above sum runs over all intersections of the i th trajectory with Σ . Note that

$$\frac{d\varphi_{(i)}}{d\tau} = g^{\varphi\varphi} p_\varphi^{(i)} = \frac{1}{r_{(i)}^2} p_\varphi^{(i)}. \quad (77)$$

Thus, we finally obtain

$$\int_\Sigma J_\mu \eta_\Sigma = \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} r_{(i)}}{l_{(i)}}, \quad (78)$$

where N_{int} is the number of *all intersections* of trajectories with the surface Σ (see Fig. 3). The particle current surface density can now be approximated as

$$\begin{aligned} \langle J_\mu \rangle &= \frac{1}{(t_2 - t_1)(r_2 - r_1)} \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} r_{(i)}}{l_{(i)}} \\ &= \frac{1}{M m (t_2 - t_1) (\xi_2 - \xi_1)} \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} \xi_{(i)}}{\lambda_{(i)}}. \end{aligned} \quad (79)$$

For stationary problems, the result should be independent of the choice of t_1 and t_2 in a sense that the number of trajectories that intersect Σ should be proportional to the length $t_2 - t_1$ if the latter is sufficiently large. In practice, we omit the factor $t_2 - t_1$ and normalize the results by the number of trajectories taken into account. Moreover, instead of considering full orbits in the four-dimensional spacetime, it is sufficient to work with projections of trajectories onto surfaces of constant time t .

2. Selecting trajectories

The setup described above brings us immediately to the problem of selecting the appropriate distribution of geodesics. Let us start the discussion with a homogeneous distribution of a gas within a plane in the Minkowski spacetime, introduced in Sec. III A. In principle, we are interested in an infinite distribution, but for practical reasons, a Monte Carlo simulation has to be restricted to a compact region in space, say, a disk of a dimensionless

radius ξ_0 . The problem we are facing turns out to be a variation of the classic Bertrand problem (or Bertrand paradox), known from the theory of geometrical probability [30]. Bertrand's problem is usually formulated as follows. On a fixed circle, one randomly selects a chord. What is the probability that the length of this chord is larger than the side of an equilateral triangle inscribed in this circle? It is then shown that different methods of selecting the chord "at random" lead to different answers (different probabilities). One of the methods of selecting chords is to demand that they belong to straight lines uniformly distributed in a plane. Such a distribution has been studied, e.g., by Kendall and Moran in [31]. In terms of Cartesian coordinates (x, y) , straight lines can be parametrized by

$$ux + vy + 1 = 0. \quad (80)$$

It can be shown that the probability distribution

$$F(u, v)dudv = (u^2 + v^2)^{-\frac{3}{2}}dudv \quad (81)$$

is invariant under the group of Euclidean symmetries on the plane—rotations and translations [31]. The trouble with parameters u and v is that they are not well suited to a description of chords within a given circle. Instead, a better description can be provided by parametrizing the chords (or, equivalently, straight lines in the plane) by polar coordinates (p, ϑ) of the point on the chord with the smallest distance to the center of the circle. In terms of (p, ϑ) , one has

$$(u^2 + v^2)^{-\frac{3}{2}}dudv = dpd\vartheta, \quad (82)$$

and given that $dudv = dpd\vartheta$, we obtain

$$F(p, \vartheta) = p^{-1}. \quad (83)$$

In the context of Bertrand's paradox, this distribution was already proposed by Poincaré in 1912 [32].

While in the original formulation of Bertrand's paradox choosing a solution may be a matter of preference, Poincaré's solution happens to be well suited to our problem at hand, as the restriction to a given circle is only technical—we are, in fact, interested in a description of a gas of particles with trajectories distributed uniformly in a plane. For monoenergetic particles of energy ε_0 , the parametrization in terms of the coordinate p is equivalent to the one with the total angular momentum λ , as λ and

$$p = M\xi = \frac{M\lambda}{\sqrt{\varepsilon_0^2 - 1}} \quad (84)$$

are linearly related. Let us now return to the angular momentum distribution appearing in Eqs. (30) and (32).

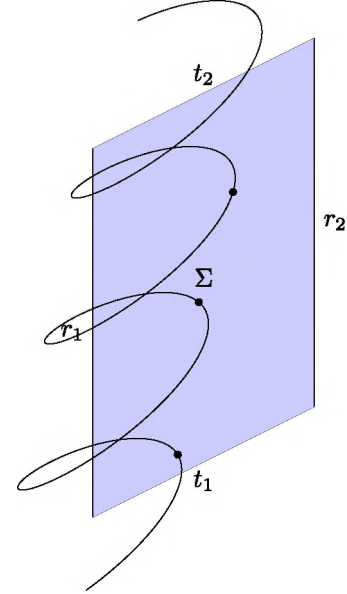


FIG. 3. Illustration of intersections of the i th trajectory with the surface Σ , as defined in Eq. (70). In this example, the number of intersections $N_{\text{int}} = 3$ [see Eq. (78)].

Using Eq. (84), we can express J_t given by Eqs. (30) and (32) as

$$J_t = -\frac{am_0^3\varepsilon_0}{M\xi} \int_0^{M\xi} dp \int_0^{2\pi} d\vartheta. \quad (85)$$

To conclude, in a Monte Carlo simulation of uniformly distributed monoenergetic particles confined to a plane in flat Minkowski spacetime, we would select trajectories distributed uniformly in λ . An example of such a distribution of straight lines in a plane is shown in Fig. 4. Since the total angular momentum λ is a constant of motion and our models of accretion in Schwarzschild spacetime assume a uniform planar asymptotic distribution of particles, a uniform distribution of λ has to be assumed for these models as well.

3. Implementation for the planar accretion problem

Unbound trajectories confined to a plane in the Schwarzschild spacetime can be parametrized by the energy ε , angular momentum λ , the location of the intersection of the orbit with the circle of the given radius ξ_0 (outer boundary of the simulation region), and the sign ϵ_λ distinguishing between the clockwise or anticlockwise direction of motion. If $\lambda = |\lambda_z|$ (the motion occurs in the equatorial plane), we set $\epsilon_\lambda = \text{sgn}\lambda_z$. We compute trajectories using a formalism based on the Biermann-Weierstrass theorem introduced in [33]. Its main ingredient is a universal formula for a radius in terms of the so-called true anomaly angle ψ (the polar angle in the orbital plane). It reads

$$\xi(\psi) = \xi_0 + \frac{-\epsilon_r \sqrt{f(\xi_0)} \wp'(\psi) + \frac{1}{2} f'(\xi_0) [\wp(\psi) - \frac{1}{24} f''(\xi_0)] + \frac{1}{24} f(\xi_0) f'''(\xi_0)}{2[\wp(\psi) - \frac{1}{24} f''(\xi_0)]^2 - \frac{1}{48} f(\xi_0) f^{(4)}(\xi_0)}, \quad (86)$$

where

$$f(\xi) = a_0 \xi^4 + 4a_1 \xi^3 + 6a_2 \xi^2 + 4a_3 \xi, \quad (87)$$

and

$$a_0 = \frac{\epsilon^2 - 1}{\lambda^2}, \quad 4a_1 = \frac{2}{\lambda^2}, \quad 6a_2 = -1, \quad 4a_3 = 2. \quad (88)$$

The Weierstrass $\wp$ function has to be computed with the following Weierstrass invariants:

$$g_2 = \frac{1}{12} - \frac{1}{\lambda^2}, \quad (89a)$$

$$g_3 = \frac{1}{6^3} - \frac{1}{12\lambda^2} - \frac{\epsilon^2 - 1}{4\lambda^2}, \quad (89b)$$

i.e., $\wp(\psi) = \wp(\psi; g_2, g_3)$, and the same applies to the derivative $\wp'(\psi)$. Here, the sign $\epsilon_r = \pm 1$ corresponds to the radial direction of motion at the initial location ξ_0 . Taking ingoing trajectories at ξ_0 , we set $\epsilon_r = -1$.

As another element, we need formulas for the allowed range R_ψ of the true anomaly ψ . It is computed differently for absorbed and scattered orbits. We define the following function:

$$X(\xi) = \int_\xi^\infty \frac{d\xi'}{\sqrt{f(\xi')}}. \quad (90)$$

For generic unbound scattered orbits, $X(\xi)$ can be expressed as

$$X(\xi) = \frac{1}{\sqrt{y_3 - y_1}} \left[\tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi}}{y_2 - y_1}}, k \right) - \tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12}}{y_2 - y_1}}, k \right) \right], \quad (91)$$

where y_1 , y_2 , and y_3 denote the roots of the polynomial $4y^3 - g_2y - g_3$, satisfying $y_1 < y_2 < y_3$,

$$k^2 = \frac{y_2 - y_1}{y_3 - y_1}, \quad (92)$$

and $\tilde{F}(\phi, k)$ is the Legendre elliptic integral defined by

$$\tilde{F}(\phi, k) = \int_0^\phi \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (93)$$

For unbound absorbed orbits, an explicitly real formula for $X(\xi_0)$ is different. Let y_1 denote a real zero of the polynomial $4y^3 - g_2y - g_3 = 4(y - y_1)(y^2 + py + q)$, where $p^2 - 4q < 0$ and thus $y^2 + py + q > 0$. We define $\mu = \sqrt{y_1^2 + py_1 + q}$ and

$$k^2 = \frac{1}{2} \left(1 - \frac{y_1 + p/2}{\mu} \right). \quad (94)$$

The expression for $X(\xi)$ reads

$$X(\xi) = \frac{1}{2\sqrt{\mu}} \left[\tilde{F} \left(2 \arctan \sqrt{\frac{-\frac{1}{12} + \frac{1}{2\xi} - y_1}{\mu}}, k \right) - \tilde{F} \left(2 \arctan \sqrt{\frac{-\frac{1}{12} - y_1}{\mu}}, k \right) \right]. \quad (95)$$

The calculation of the allowed range of ψ is now quite simple. For unbound scattered orbits, one has

$$R_\psi = 2[X(\xi_{\text{peri}}) - X(\xi_0)], \quad (96)$$

where ξ_{peri} denotes the pericenter radius. For unbound absorbed orbits, we set

$$R_\psi = X(2) - X(\xi_0), \quad (97)$$

since $\xi = 2$ corresponds to the black hole horizon.

Note that the range R_ψ of the angle ψ can be larger than 2π , in which case the orbit circles around the black hole. Thus, the last “technical” element in our description is a simple function that gives the number of intersections of a given geodesic trajectory with a radius of constant polar angle φ . To define such a function, one only needs to know the range R_ψ , the polar coordinate φ_0 corresponding to the initial radius ξ_0 , the sign ϵ_λ , and the polar coordinate of the radius φ for which one counts the intersections with the given geodesic. This idea is illustrated in Fig. 5. In this case a sample absorbed geodesic is depicted with a blue line. Radial segments which are intersected twice are marked with solid gray lines. Radial segments with three intersections are marked with dashed lines. A sample Wolfram *Mathematica* [26] code for such a function is as follows (here, `phi` = φ , `phi0` = φ_0 , `range` = R_ψ , `dir` = ϵ_λ):

```

NumberOfIntersections[phi_, phi0_, range_, dir_] := If[dir == 1,
If[phi > phi0,
  If[Mod[range, 2 Pi] > phi-phi0, Quotient[range, 2 Pi] +1, Quotient[range, 2 Pi]],
  If[Mod[range + phi0, 2 Pi] > phi, Quotient[range + phi0, 2 Pi],
  Quotient[range + phi0, 2 Pi]-1]],
If[phi < phi0,
  If[Mod[range, 2 Pi] > phi0-phi, Quotient[range, 2 Pi] +1, Quotient[range, 2 Pi]],
  If[Mod[phi0-range, 2 Pi] < phi, Quotient[range-phi0, 2 Pi] +1,
  Quotient[range-phi0, 2 Pi]]]
]

```

Given all these elements, we select trajectories with a given energy ε_0 and uniformly distributed values of $\varphi_0 \in [0, 2\pi)$, $\lambda \in [0, \lambda_{\max}(\varepsilon_0, \xi_0)]$, and the sign $\varepsilon_\lambda = \pm 1$. For clarity, these parameters can be divided into two classes: those corresponding to absorbed trajectories with $\lambda < \lambda_c(\varepsilon)$ and those corresponding to scattered orbits with $\lambda > \lambda_c(\varepsilon)$. A plot of such trajectories, projected at a hypersurface of constant time, is given in Fig. 6.

According to Eq. (79), the components of J_μ can be computed by counting the number of intersections per length interval at a given radius ($\varphi = \text{const}$), with appropriate weights. Such intersections are illustrated in Fig. 7. Let the set of indices $I_{\text{abs}}(\xi_1, \xi_2)$ number the intersections of absorbed trajectories in the segment $\xi_1 \leq \xi \leq \xi_2$ of a radius of constant φ . Note that a given trajectory can contribute more than once to the set of intersections $I_{\text{abs}}(\xi_1, \xi_2)$ —this can happen for spiraling orbits, similar to the one shown in Fig. 5. An analogous set of indices referring to scattered orbits will be denoted by $I_{\text{scat}}(\xi_1, \xi_2)$. The Monte Carlo

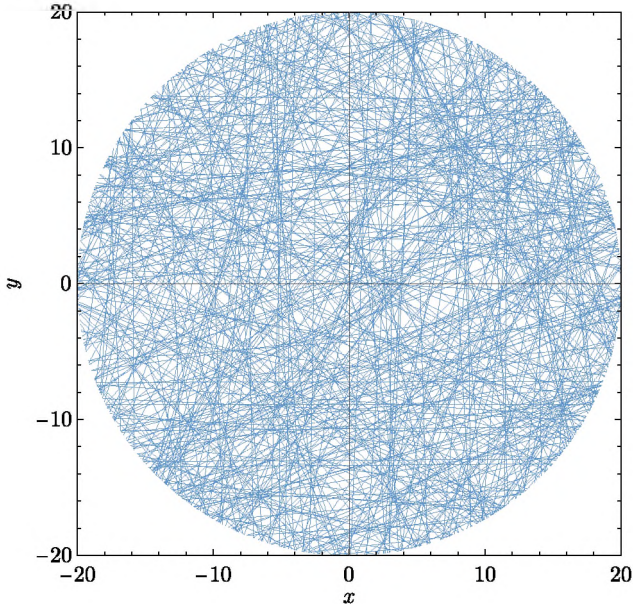


FIG. 4. Randomly distributed straight lines on a plane. There are 500 lines in this picture, selected according to Eq. (82) or (83).

estimators of components $J_t^{(\text{abs})}$ and $J_t^{(\text{scat})}$ at $(\xi = (\xi_1 + \xi_2)/2, \varphi)$ can be computed as

$$\langle J_t^{(\text{abs})} \rangle = -\frac{4\pi\alpha m_0^3 \lambda_c(\varepsilon_0)}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\varepsilon_0 \xi(i)}{\lambda(i)}, \quad (98a)$$

$$\langle J_t^{(\text{scat})} \rangle = -\frac{4\pi\alpha m_0^3 [\lambda_{\max}(\varepsilon_0, \xi_0) - \lambda_c(\varepsilon_0)]}{N_{\text{scat}}(\xi_2 - \xi_1)} \times \sum_{i \in I_{\text{scat}}(\xi_1, \xi_2)} \frac{\varepsilon_0 \xi(i)}{\lambda(i)}, \quad (98b)$$

where N_{abs} and N_{scat} denote the number of absorbed and scattered orbits, respectively. The estimator for J^r can be computed as

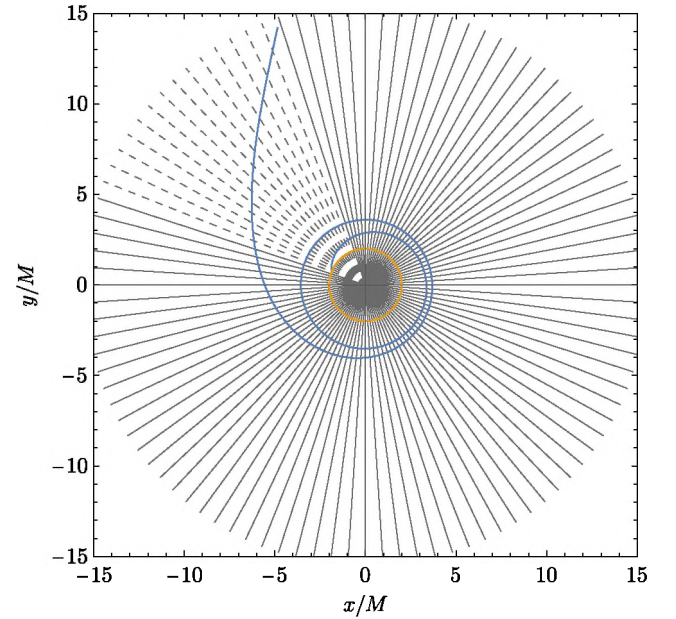


FIG. 5. Illustration of the function controlling the number of intersections of a geodesic and a radial segment of constant φ . A spiraling geodesic is plotted with the blue line. It corresponds to $\varepsilon = 1.1$ and $\lambda = 4.7244$. Radial segments of constant φ which are intersected twice are depicted with solid lines. A dashed line is used for rays intersected three times. The orange circle corresponds to the black hole horizon.

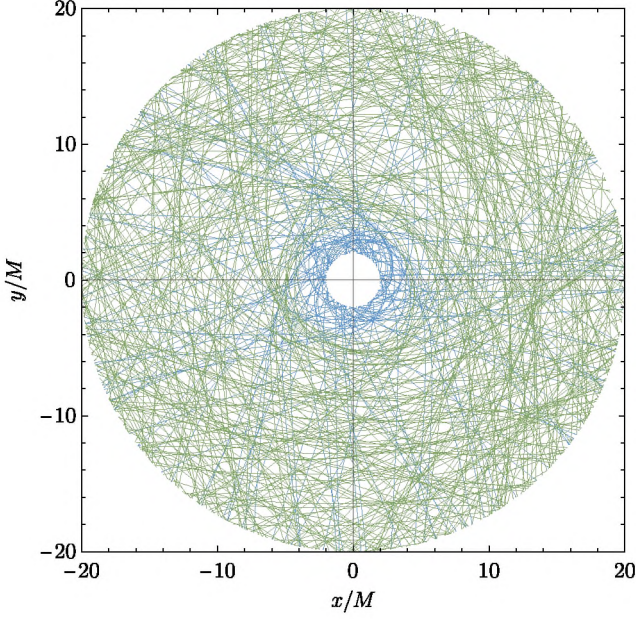


FIG. 6. Randomly distributed monoenergetic ($\varepsilon_0 = 1.3$) geodesics on a plane. Angular momenta are uniformly distributed within $\lambda \in [0, \lambda_{\max}(\varepsilon_0, \xi_0)]$, where $\xi_0 = 20$. Absorbed orbits are depicted in blue. Scattered orbits are plotted in green. There are 500 orbits in this plot.

$$\langle J^r \rangle = -\frac{4\pi\alpha m_0^3 \lambda_c(\varepsilon_0)}{N_{\text{abs}}(\xi_2 - \xi_1)} \times \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\xi_{(i)}}{\lambda_{(i)}} \sqrt{\varepsilon_0^2 - U_{\lambda_{(i)}}(\xi_{(i)})}. \quad (99)$$

We discretize the radial direction according to $\xi_j < \xi < \xi_{j+1}$, $j = 1, \dots, N_\xi - 1$, with $\xi_1 = 2$ and $\xi_{N_\xi} = \xi_0$ (the radius of the outer boundary). Thus, strictly speaking, the values ξ_1 and ξ_2 appearing in all our formulas should be replaced by $\xi_1 \rightarrow \xi_j$, $\xi_2 \rightarrow \xi_{j+1}$. We will, however, keep the notation with ξ_1 and ξ_2 referring to cell boundaries, as it is probably more transparent. A comparison of results obtained according to Eqs. (98) and (99) with exact expressions for J_t and J^r is shown in Figs. 8 and 9.

Apart from the number of trajectories, the resolution of the method is controlled by the number of radii of constant φ and the size $\xi_{j+1} - \xi_j$ of radial segments $[\xi_j, \xi_{j+1}]$. In the example shown in Figs. 8 and 9, there are 120 radii of constant φ , distributed within the full angle, and 100 equidistant radial segments $[\xi_j, \xi_{j+1}]$ in the range $2 < \xi \leq \xi_0 = 20$ at each radius $\varphi = \text{const}$. The values of $\langle J_\mu \rangle$ can be computed separately for each radius of constant φ ; in our examples we additionally average the results over all radii of constant φ .

In a similar fashion one can compute Monte Carlo estimators of the particle current density in the case with the Maxwell-Jüttner asymptotic distribution. It is

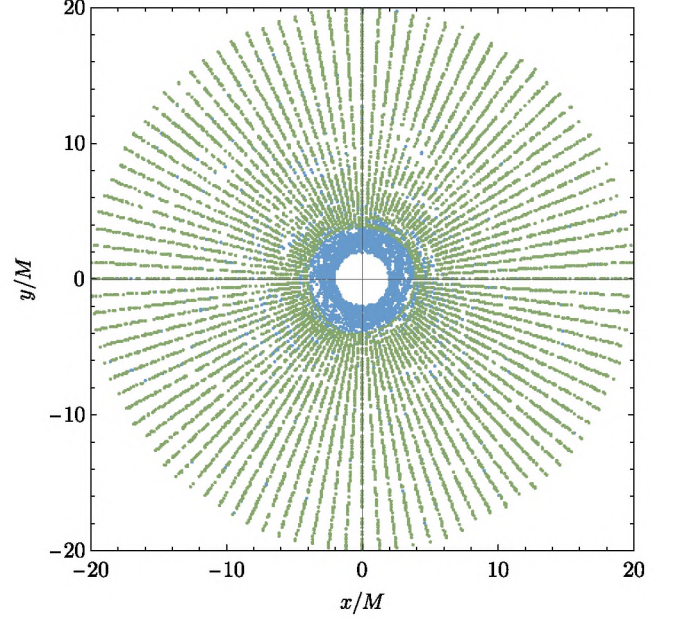


FIG. 7. Intersection points of orbits plotted in Fig. 6 and radii of constant φ .

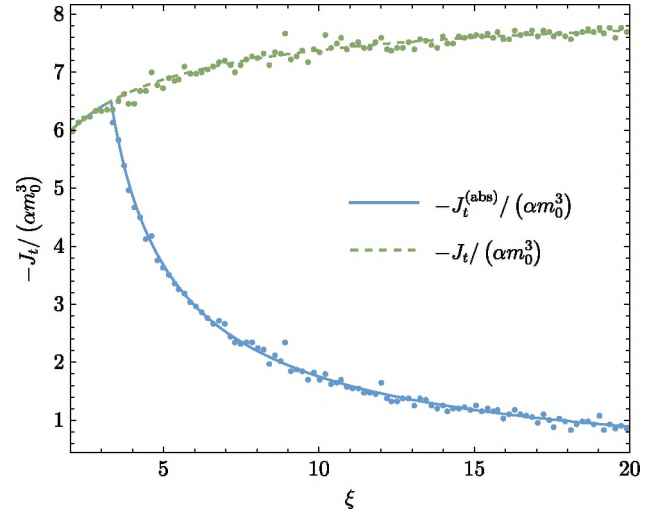


FIG. 8. Time components of the particle surface density current J_t for the planar accretion model with monoenergetic particles. In this case $\varepsilon_0 = 1.3$ and $\xi_0 = 20$. Exact solutions [Eqs. (45)] are plotted with solid and dashed lines. Blue and green points depict results of a Monte Carlo simulation [Eqs. (98)]. There are 5×10^4 orbits: 15,953 absorbed trajectories and 34,047 scattered ones.

convenient to select trajectories directly from the Maxwell-Jüttner distribution. Again, von Neumann's rejection method can be used. We repeat the procedure described in Sec. III C; however, this time four parameters $(\varepsilon_i, \lambda_i, \varphi_{0,i}, \epsilon_{\lambda,i}, y_i)$ are selected in each iteration step, and uniform distributions are assumed for $\varphi_{0,i} \in [0, 2\pi)$, and $\epsilon_{\lambda,i} = \pm 1$.

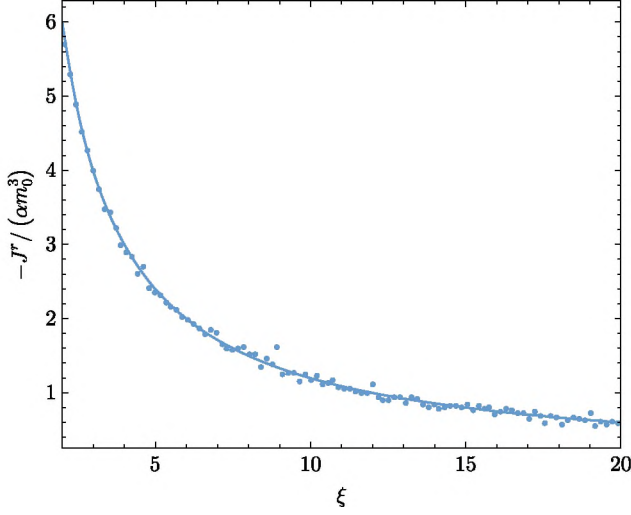


FIG. 9. Radial component of the particle surface density current J^r for the model illustrated in Fig. 8 (planar model with monoenergetic particles; $\varepsilon_0 = 1.3$ and $\xi_0 = 20$). The exact solution [Eq. (46)] is plotted with the solid line. Blue and green dots show results of a Monte Carlo simulation [Eq. (99)]. The sample of geodesic orbits is the same as in Fig. 8. There are 5×10^4 orbits: 15,953 absorbed trajectories and 34,047 scattered ones.

Monte Carlo estimators of the particle current surface density in the asymptotically Maxwell-Jüttner model can be written as

$$\langle J_t^{(\text{abs})} \rangle = -\frac{4\pi\alpha m_0^3 V_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\varepsilon_{(i)} \xi_{(i)}}{\lambda_{(i)}}, \quad (100a)$$

$$\langle J_t^{(\text{scat})} \rangle = -\frac{4\pi\alpha m_0^3 V_{\text{scat}}}{N_{\text{scat}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{scat}}(\xi_1, \xi_2)} \frac{\varepsilon_{(i)} \xi_{(i)}}{\lambda_{(i)}}, \quad (100b)$$

where V_{abs} and V_{scat} are given by Eqs. (61). The estimator for J^r can be computed as

$$\begin{aligned} \langle J^r \rangle &= -\frac{4\pi\alpha m_0^3 V_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)} \\ &\times \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\xi_{(i)}}{\lambda_{(i)}} \sqrt{\varepsilon_{(i)}^2 - U_{\lambda_{(i)}}(\xi_{(i)})}. \end{aligned} \quad (101)$$

A comparison of these estimators and exact solutions given by Eqs. (51) is shown in Figs. 10 and 11.

IV. SPHERICALLY SYMMETRIC SOLUTIONS

A. Spherically symmetric solutions in the Schwarzschild background

We now turn to spherically symmetric solutions. The calculations given in this section are similar to those for planar systems, and thus we will mostly only summarize

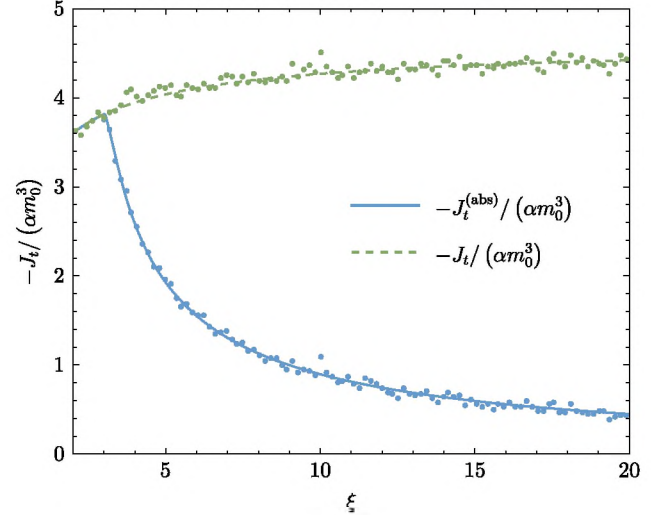


FIG. 10. Time components of the particle surface density current J_t for the planar Maxwell-Jüttner model. In this case $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$. Exact solutions [Eqs. (51a) and (51b)] are plotted with solid and dashed lines. Blue and green dots show results of a Monte Carlo simulation [Eqs. (100)]. There are 2×10^6 sets $(\varepsilon_i, \lambda_i, \varphi_{0,i}, e_{\lambda,i}, y_i)$, giving 10,356 absorbed and 27,056 scattered orbits.

the results. Spherically symmetric models describing stationary accretion of collisionless gas in the Schwarzschild spacetime were derived in [10,11], in particular, assuming the Maxwell-Jüttner distribution at infinity. In the following

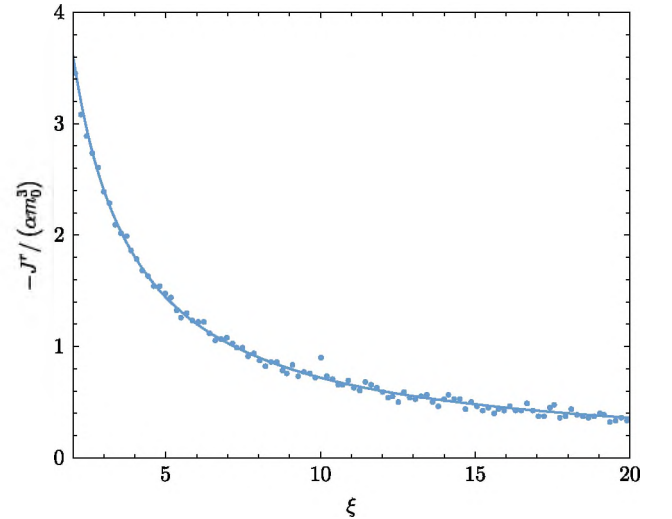


FIG. 11. Radial component of the particle surface density current J_r for the planar accretion model with a Maxwell-Jüttner asymptotic distribution. In this case $\beta = 1$, $\xi_0 = 20$, $\varepsilon_{\text{cutoff}} = 10$. The exact solution [Eq. (51c)] is plotted with a solid line. Blue dots show results of a Monte Carlo simulation [Eq. (101)]. The sample of geodesic orbits is the same as in Fig. 10. There are 2×10^6 sets $(\varepsilon_i, \lambda_i, \varphi_{0,i}, e_{\lambda,i}, y_i)$, giving 10,356 absorbed and 27,056 scattered orbits.

two subsections we give expressions for the particle current density in the spherically symmetric stationary accretion model with monoenergetic particles and recall expressions derived already in [10,11] for the spherical Maxwell-Jüttner model.

1. Monoenergetic particles

For the gas of monoenergetic particles, the one-particle distribution function reads, in our standard setup,

$$\begin{aligned}\mathcal{F}(x, p) &= \alpha m_0 \delta(\sqrt{-p_\mu p^\mu} - m_0) \delta(p_t + E_0) \\ &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \delta(\varepsilon - \varepsilon_0),\end{aligned}\quad (102)$$

where, as before, we only take into account future pointing momenta. The momentum-space volume element can be written as

$$\text{dvol}_x(p) = \frac{m^3 \lambda}{\xi^2 \sqrt{\varepsilon^2 - U_\lambda(\xi)}} d\varepsilon d\mathbf{m} d\lambda d\chi, \quad (103)$$

where the variable χ is defined as

$$p_\theta = M m \lambda \cos \chi, \quad p_\varphi = M m \lambda \sin \theta \sin \chi \quad (104)$$

(see Ref. [10]).

Taking into account the regions in the phase space available for the motion of absorbed and scattered particles, one can write time components of the particle current density as

$$\mathcal{J}_t^{(\text{abs})}(\xi) = -\frac{2\pi\alpha m_0^4 \varepsilon_0}{\xi^2} \Theta(\varepsilon_0 - 1) \int_0^{\lambda_c(\varepsilon_0)} \frac{\lambda d\lambda}{\sqrt{\varepsilon_0^2 - U_\lambda(\xi)}}, \quad (105a)$$

$$\begin{aligned}\mathcal{J}_t^{(\text{scat})}(\xi) &= -\frac{4\pi\alpha m_0^4 \varepsilon_0}{\xi^2} \Theta(\varepsilon_0 - \varepsilon_{\min}(\xi)) \\ &\times \int_{\lambda_c(\varepsilon_0)}^{\lambda_{\max}(\varepsilon_0, \xi)} \frac{\lambda d\lambda}{\sqrt{\varepsilon_0^2 - U_\lambda(\xi)}}.\end{aligned}\quad (105b)$$

The integrals with respect to momenta can be easily computed, yielding

$$\mathcal{J}_t^{(\text{abs})}(\xi) = -\frac{2\pi\alpha m_0^4 \varepsilon_0}{\xi^2} \Theta(\varepsilon_0 - 1) \frac{\lambda_c^2(\varepsilon_0)}{s_{\lambda_c}(\varepsilon_0, \xi) + s_0(\varepsilon_0, \xi)}, \quad (106a)$$

$$\begin{aligned}\mathcal{J}_t^{(\text{scat})}(\xi) &= -\frac{4\pi\alpha m_0^4 \varepsilon_0}{\xi^2} \Theta(\varepsilon_0 - \varepsilon_{\min}(\xi)) \\ &\times \frac{\lambda_{\max}^2(\varepsilon_0, \xi) - \lambda_c^2(\varepsilon_0)^2}{s_{\lambda_c}(\varepsilon_0, \xi) + s_{\lambda_{\max}}(\varepsilon_0, \xi)},\end{aligned}\quad (106b)$$

where

$$s_\lambda(\varepsilon, \xi) = \sqrt{\varepsilon^2 - U_\lambda(\xi)}. \quad (107)$$

Similarly, for $\mathcal{J}^r$, we find

$$\begin{aligned}\mathcal{J}^r(\xi) &= -\frac{2\pi\alpha m_0^4}{\xi^2} \Theta(\varepsilon_0 - 1) \int_0^{\lambda_c(\varepsilon_0)} \lambda d\lambda \\ &= -\frac{\pi\alpha m_0^4}{\xi^2} \Theta(\varepsilon_0 - 1) \lambda_c(\varepsilon_0)^2.\end{aligned}\quad (108)$$

2. Asymptotic Maxwell-Jüttner distribution

Assuming the gas obeying the Maxwell-Jüttner distribution asymptotically, we find the one-particle distribution function in the form

$$\begin{aligned}\mathcal{F}(x, p) &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m} k^\mu p_\mu\right) \\ &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp(-\beta \varepsilon).\end{aligned}\quad (109)$$

This gives the following expressions for $\mathcal{J}_t$ and $\mathcal{J}^r$.

$$\begin{aligned}\mathcal{J}_t^{(\text{abs})}(\xi) &= -\frac{2\pi\alpha m_0^4}{\xi^2} \int_1^\infty d\varepsilon e^{-\beta \varepsilon} \int_0^{\lambda_c(\varepsilon)} d\lambda \frac{\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} \\ &= -\frac{2\pi\alpha m_0^4}{\xi^2} \int_1^\infty d\varepsilon e^{-\beta \varepsilon} \varepsilon \frac{\lambda_c(\varepsilon)^2}{s_{\lambda_c}(\varepsilon, \xi) + s_0(\varepsilon, \xi)},\end{aligned}\quad (110a)$$

$$\begin{aligned}\mathcal{J}_t^{(\text{scat})}(\xi) &= -\frac{4\pi\alpha m_0^4}{\xi^2} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon e^{-\beta \varepsilon} \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\varepsilon, \xi)} d\lambda \frac{\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} \\ &= -\frac{4\pi\alpha m_0^4}{\xi^2} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon e^{-\beta \varepsilon} \varepsilon \frac{\lambda_{\max}^2(\varepsilon, \xi) - \lambda_c^2(\varepsilon)^2}{s_{\lambda_c}(\varepsilon, \xi) + s_{\lambda_{\max}}(\varepsilon, \xi)},\end{aligned}\quad (110b)$$

$$\begin{aligned}\mathcal{J}^r(\xi) &= -\frac{2\pi\alpha m_0^4}{\xi^2} \int_1^\infty d\varepsilon e^{-\beta \varepsilon} \int_0^{\lambda_c(\varepsilon)} d\lambda \lambda \\ &= -\frac{\pi\alpha m_0^4}{\xi^2} \int_1^\infty d\varepsilon e^{-\beta \varepsilon} \lambda_c(\varepsilon)^2.\end{aligned}\quad (110c)$$

B. Monte Carlo simulations of spherically symmetric solutions

Similarly to the planar case, one can perform both Monte Carlo integration and the actual Monte Carlo simulation of stationary spherically symmetric accretion flows. Since Monte Carlo integration is essentially straightforward and does not require any new elements, we will omit this part.

We now take

$$S = \{(r, \theta, \varphi) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2, \varphi = \varphi_0\} \quad (111)$$

and the surface

$$\Sigma = \{(t, r, \theta, \varphi) : t_1 \leq t \leq t_2, r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2, \varphi = \varphi_0\} \quad (112)$$

and compute the estimators

$$\langle \mathcal{J}_\mu \rangle = \frac{\int_\Sigma \mathcal{J}_\mu \eta_\Sigma}{\int_\Sigma \eta_\Sigma}. \quad (113)$$

The calculation is analogous to the previous one, performed for the planar case. The expression for the particle current density reads

$$\mathcal{J}_\mu(x) = \int \sum_{i=1}^N \delta^{(4)}(x^\alpha - x_{(i)}^\alpha(\tau)) \frac{p_\mu^{(i)}(\tau)}{r^2 \sin \theta} d\tau. \quad (114)$$

The volume element on Σ can be written as $\eta_\Sigma = r dt dr d\theta$. Thus,

$$\int_\Sigma \eta_\Sigma = \frac{1}{2} (t_2 - t_1) (r_2 - r_1) (\theta_2 - \theta_1) (r_1 + r_2), \quad (115)$$

and

$$\int_\Sigma \mathcal{J}_\mu \eta_\Sigma = \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} r_{(i)} \sin \theta_{(i)}}{|l_z^{(i)}|}. \quad (116)$$

The values l_z and l are related: $|l_z| = l \cos \iota$, where ι denotes the inclination of the orbit with respect to the equatorial plane of the coordinate system. Thus,

$$\int_\Sigma \mathcal{J}_\mu \eta_\Sigma = \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} r_{(i)} \sin \theta_{(i)}}{l_{(i)} \cos \iota_{(i)}}. \quad (117)$$

This gives the Monte Carlo estimator of $\mathcal{J}_\mu$ in the form

$$\begin{aligned} \langle \mathcal{J}_\mu \rangle &= \frac{1}{(t_2 - t_1)(r_2 - r_1)(\theta_2 - \theta_1)} \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} \sin \theta_{(i)}}{l_{(i)} \cos \iota_{(i)}} \\ &= \frac{1}{M^2 m (t_2 - t_1) (\xi_2 - \xi_1) (\theta_2 - \theta_1)} \\ &\quad \times \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)} \sin \theta_{(i)}}{\lambda_{(i)} \cos \iota_{(i)}} \end{aligned} \quad (118)$$

where we have assumed that $r_2 - r_1 \ll 1$, and, consequently, $\frac{1}{2}(r_1 + r_2) \approx r_{(i)}$.

The easiest way of simulating spherically symmetric flows in the Schwarzschild background uses the fact that each trajectory is in fact a planar one. Thus, one can still select parameters of trajectories in a single plane. In this case, one can ignore the factors depending on θ and the inclination angle, and write

$$\langle \mathcal{J}_\mu \rangle \propto \frac{1}{(t_2 - t_1)(\xi_2 - \xi_1)} \sum_{i=1}^{N_{\text{int}}} \frac{p_\mu^{(i)}}{\lambda_{(i)}}. \quad (119)$$

The key difference with respect to the planar case is related to the selection of geodesics. While in the planar case we select geodesics from the distribution $\propto d\lambda$, the three-dimensional case requires a distribution $\propto \lambda d\lambda$. To see this, one can return to a classic discussion of a uniform random distribution of straight lines in the three-dimensional flat space given by Kendall and Moran in [31]. They parametrize straight lines in $\mathbb{R}^3$ by

$$x = az + p, \quad (120a)$$

$$y = bz + q, \quad (120b)$$

where (x, y, z) denote Cartesian coordinates. The appropriate probability measure, invariant with respect to Euclidean rotations and translations, is given by

$$(1 + a^2 + b^2)^{-2} da db dp dq \quad (121)$$

[cf. Eq. (3.50), p. 74 in [31]]. The plane perpendicular to the line given by Eqs. (120) and passing through the origin of the coordinate system is described by

$$ax + by + z = 0. \quad (122)$$

The unit vector normal to this plane has the coordinates

$$(n_x, n_y, n_z) = \frac{(a, b, 1)}{\sqrt{1 + a^2 + b^2}}. \quad (123)$$

Let us fix the parameters a and b . The intersection of the line (120) with the plane (122) is specified by the parameters p and q . The area element at the plane (122)

defined by such intersection points and obtained by varying the parameters p and q is given by

$$dS = (1 + a^2 + b^2)^{-\frac{1}{2}} dp dq \quad (124)$$

(note an error in [31] in this formula). Let $d\Omega$ denote the solid angle element around the normal vector n^i , obtained by varying a and b . Kendall and Moran show that

$$(1 + a^2 + b^2)^{-2} da db dp dq = dS d\Omega. \quad (125)$$

Let δ be a distance from the center of the coordinate system to the intersection point. It is given by

$$\delta = \sqrt{\frac{(1 + b^2)p^2 - 2abpq + (1 + a^2)q^2}{1 + a^2 + b^2}}. \quad (126)$$

Fix the parameters a and b . Let ϕ denote an angle between the intersection of the plane (122) and the plane $z = 0$ and the line joining the origin of the coordinate system and the intersection point. We have

$$\cos \phi = \frac{aq - bp}{\delta \sqrt{a^2 + b^2}}. \quad (127)$$

One can show that

$$\frac{\partial(\delta, \phi)}{\partial(p, q)} = \pm \frac{1}{\delta \sqrt{1 + a^2 + b^2}}. \quad (128)$$

Thus,

$$dS = \delta d\delta d\phi, \quad (129)$$

as expected $[(\delta, \phi)]$ are standard polar coordinates in the plane (122). Again, the total angular momentum λ associated with a given line is directly proportional to its distance δ from the center of the coordinate system. This justifies our claim that

$$(1 + a^2 + b^2)^{-2} da db dp dq \propto \lambda d\lambda. \quad (130)$$

Assuming the notation from the previous section, we write Monte Carlo estimators for the monoenergetic model in the following form:

$$\langle \mathcal{J}_t^{(\text{abs})} \rangle = -\frac{2\pi^2 \alpha m_0^4 \lambda_c(\varepsilon_0)^2}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\varepsilon_0}{\lambda_{(i)}}, \quad (131a)$$

$$\begin{aligned} \langle \mathcal{J}_t^{(\text{scat})} \rangle &= -\frac{2\pi^2 \alpha m_0^4 [\lambda_{\text{max}}(\varepsilon_0, \xi_0)^2 - \lambda_c(\varepsilon_0)^2]}{N_{\text{scat}}(\xi_2 - \xi_1)} \\ &\times \sum_{i \in I_{\text{scat}}(\xi_1, \xi_2)} \frac{\varepsilon_0}{\lambda_{(i)}}. \end{aligned} \quad (131b)$$

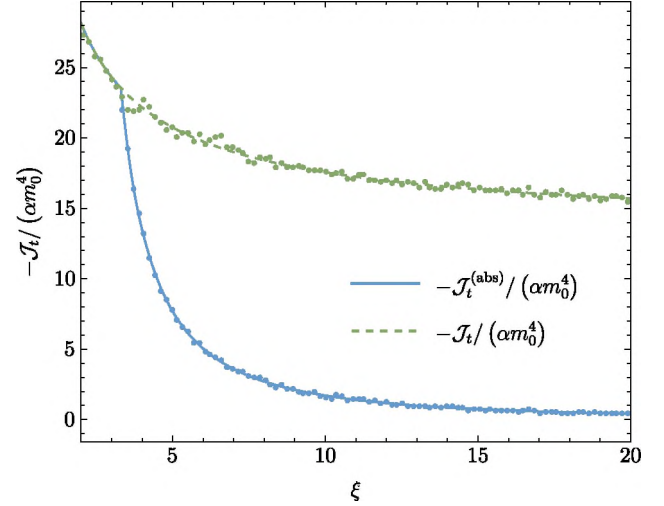


FIG. 12. Time components of the particle current density $\mathcal{J}_t$ for the monoenergetic spherically symmetric model with $\varepsilon_0 = 1.3$ and $\xi_0 = 20$. Exact solutions [Eqs. (106)] are plotted with solid and dashed lines. Blue and green dots show results of a Monte Carlo simulation [Eqs. (131)]. We use von Neumann's rejection method to select appropriate distribution of the total angular momentum. There are 5×10^4 von Neumann iteration steps giving 2477 absorbed and 22,273 scattered trajectories.

The estimator for $\mathcal{J}^r$ can be computed as

$$\langle \mathcal{J}^r \rangle = -\frac{2\pi^2 \alpha m_0^4 \lambda_c(\varepsilon_0)^2}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\sqrt{\varepsilon_0^2 - U_{\lambda_{(i)}}(\xi_{(i)})}}{\lambda_{(i)}}. \quad (132)$$

Here, as in the planar case, we have adapted our normalization to analytic formulas (106) and (108). A comparison of these estimators with the exact solutions is given in Figs. 12 and 13.

In a similar fashion one can compute Monte Carlo estimators of the particle current density in the case with the Maxwell-Jüttner asymptotic distribution. As in the planar case, parameters of the trajectories can be selected from the Maxwell-Jüttner distribution, using von Neumann's rejection method. Once again, we repeat the procedure described in Sec. III C. The only difference is that this time both angular momenta λ and the energies ε are selected from nonuniform distributions. This requires introducing two separate auxiliary parameters (y_i and z_i , say). Monte Carlo estimators of $\mathcal{J}_t^{(\text{abs})}$ and $\mathcal{J}_t^{(\text{scat})}$ can be computed as

$$\langle \mathcal{J}_t^{(\text{abs})} \rangle = -\frac{4\pi^2 \alpha m_0^4 \hat{V}_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\varepsilon_{(i)}}{\lambda_{(i)}}, \quad (133a)$$

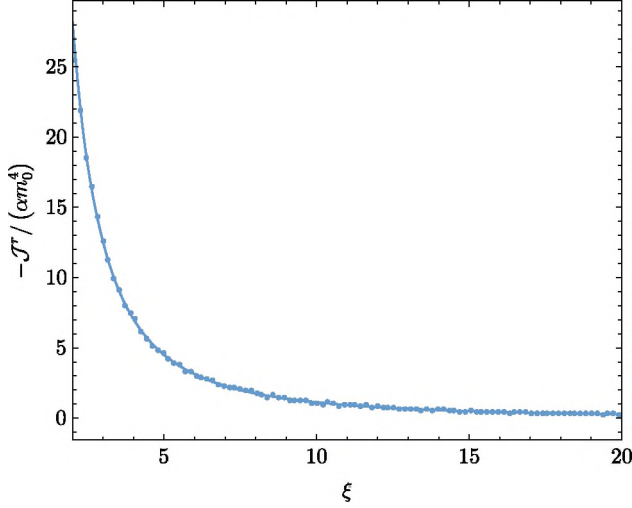


FIG. 13. Radial component of the particle current density $\mathcal{J}^r$ for the monoenergetic spherically symmetric model with $\varepsilon_0 = 1.3$ and $\xi_0 = 20$. The exact solution [Eq. (108)] is plotted with the solid line. Blue dots show results of a Monte Carlo simulation [Eq. (132)]. The sample of geodesic orbits is the same as in Fig. 12. There are 2477 absorbed trajectories.

$$\langle \mathcal{J}_t^{(\text{scat})} \rangle = -\frac{4\pi^2 \alpha m_0^4 \hat{V}_{\text{scat}}}{N_{\text{scat}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{scat}}(\xi_1, \xi_2)} \frac{\varepsilon_{(i)}}{\lambda_{(i)}}. \quad (133b)$$

The estimator for $\mathcal{J}^r$ reads

$$\langle \mathcal{J}^r \rangle = -\frac{4\pi^2 \alpha m_0^4 \hat{V}_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2)} \frac{\sqrt{\varepsilon_{(i)}^2 - U_{\lambda_{(i)}}(\xi_{(i)})}}{\lambda_{(i)}}. \quad (134)$$

Here,

$$\hat{V}_{\text{abs}} = \frac{1}{2} \int_1^{\varepsilon_{\text{cutoff}}} \exp(-\beta \varepsilon) \lambda_c(\varepsilon)^2 d\varepsilon, \quad (135a)$$

$$\hat{V}_{\text{scat}} = \frac{1}{2} \int_1^{\varepsilon_{\text{cutoff}}} \exp(-\beta \varepsilon) [\lambda_{\text{max}}(\varepsilon, \xi_0)^2 - \lambda_c(\varepsilon)^2] d\varepsilon. \quad (135b)$$

A comparison of Monte Carlo estimators defined in this way and exact solutions is given in Figs. 14 and 15.

Of course, one can also adhere to the original prescription given by Eq. (118) and distribute selected trajectories among randomly oriented orbital planes. This requires some technical elements in the calculation but does not change the overall results. The idea is to count (with appropriate weights) intersections of trajectories with meridian half-planes of a fixed spherical coordinate system. As before, we separately count intersections falling in the radial regions $\xi_j < \xi < \xi_{j+1}$, $j = 1, \dots, N_\xi - 1$, with $\xi_1 = 2$ and $\xi_{N_\xi} = \xi_0$, but we also discretize with respect to θ and

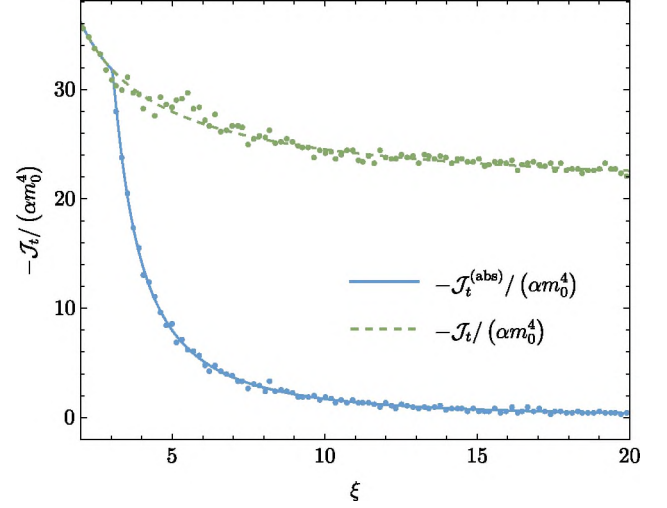


FIG. 14. Time components of the particle current density $\mathcal{J}_t$ for the Maxwell-Jüttner spherically symmetric model with $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$. Exact solutions [Eqs. (110a) and (110b)] are plotted with solid and dashed lines. Blue and green dots show results of a Monte Carlo simulation [Eqs. (133)]. There are 6×10^6 sets $(\varepsilon_i, \lambda_i, \varphi_{0,i}, e_{\lambda,i}, y_i, z_i)$, giving 963 absorbed trajectories and 12,804 scattered trajectories in this simulation.

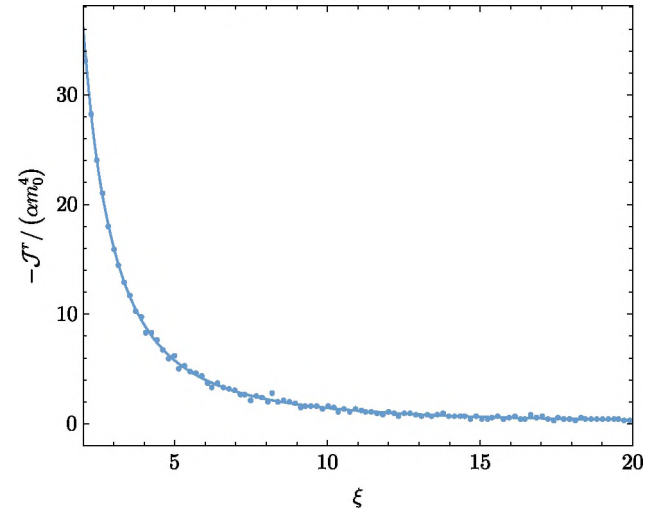


FIG. 15. Radial component of the particle current density $\mathcal{J}^r$ for the Maxwell-Jüttner spherically symmetric model with $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$. The exact solution [Eq. (110c)] is plotted with the solid line. Blue dots show results of a Monte Carlo simulation [Eqs. (134)]. The sample of geodesic orbits is the same as in Fig. 14. There are 6×10^6 sets $(\varepsilon_i, \lambda_i, \varphi_{0,i}, e_{\lambda,i}, y_i, z_i)$, giving 963 absorbed trajectories and 12,804 scattered trajectories in this simulation.

separately count intersections occurring in different regions $\theta_k < \theta < \theta_{k+1}$, $k = 1, \dots, N_\theta - 1$, $\theta_1 = 0$ and $\theta_{N_\theta} = \pi$. As before, for clarity, we keep the notation with ξ_1, ξ_2, θ_1 , and θ_2 referring to cell boundaries.

The Monte Carlo estimators are then computed as

$$\langle \mathcal{J}_t^{(\text{abs})} \rangle = -\frac{8\pi\alpha m_0^4 \hat{V}_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)(\theta_2 - \theta_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2; \theta_1, \theta_2)} \frac{\varepsilon_{(i)} \sin \theta_{(i)}}{\lambda_{(i)} \cos \iota_{(i)}}, \quad (136a)$$

$$\langle \mathcal{J}_t^{(\text{scat})} \rangle = -\frac{8\pi\alpha m_0^4 \hat{V}_{\text{scat}}}{N_{\text{scat}}(\xi_2 - \xi_1)(\theta_2 - \theta_1)} \sum_{i \in I_{\text{scat}}(\xi_1, \xi_2; \theta_1, \theta_2)} \frac{\varepsilon_{(i)} \sin \theta_{(i)}}{\lambda_{(i)} \cos \iota_{(i)}}, \quad (136b)$$

$$\langle \mathcal{J}^r \rangle = -\frac{8\pi\alpha m_0^4 \hat{V}_{\text{abs}}}{N_{\text{abs}}(\xi_2 - \xi_1)(\theta_2 - \theta_1)} \sum_{i \in I_{\text{abs}}(\xi_1, \xi_2; \theta_1, \theta_2)} \frac{\sqrt{\varepsilon_{(i)}^2 - U_{\lambda_{(i)}}(\xi_{(i)})} \sin \theta_{(i)}}{\lambda_{(i)} \cos \iota_{(i)}}, \quad (136c)$$

where $I_{\text{abs}}(\xi_1, \xi_2; \theta_1, \theta_2)$ and $I_{\text{scat}}(\xi_1, \xi_2; \theta_1, \theta_2)$ collect indices corresponding to *intersections* of absorbed and scattered trajectories falling in the regions $\xi_1 \leq \xi_{(i)} \leq \xi_2$ and $\theta_1 \leq \theta_{(i)} \leq \theta_2$.

The orientation of the orbital plane can be controlled by specifying coordinates of the normal vector. Choosing to work with Cartesian coordinates facilitates the selection procedure—again, von Neumann’s rejection method can be used to select normal vectors distributed uniformly in a unit sphere. Taking into account different possible orientations of the orbits would be important for nonspherically symmetric solutions. Examples of such solutions, representing models of a Schwarzschild black hole

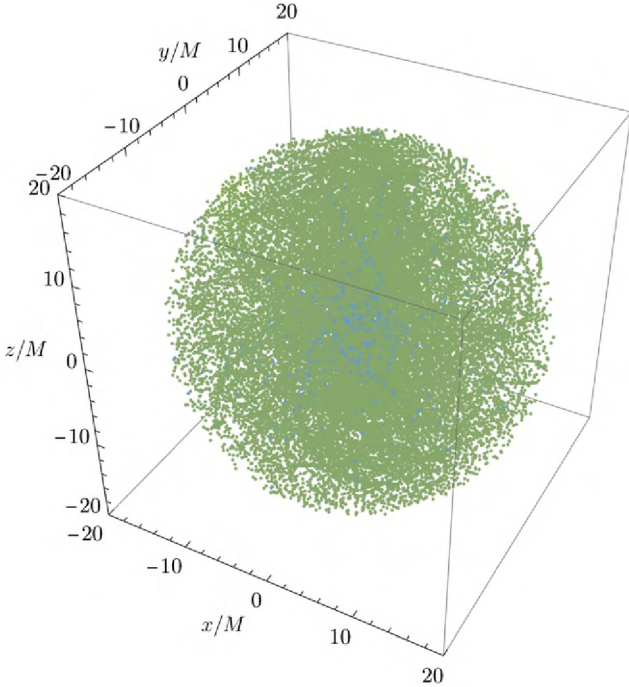


FIG. 16. Intersections of trajectories with a grid of meridional half-planes. Blue and green dots correspond to absorbed and scattered orbits, respectively. There are 79 absorbed orbits and 1077 scattered ones.

moving through the medium, are provided in [13,14]. For spherically symmetric solutions, one can average the results not only over different values of φ_0 , but also over cells $[\theta_1, \theta_2]$.

For completeness, Fig. 16 shows an example of intersections of orbits selected according to the Maxwell-Jüttner distribution with a grid of meridional half-planes. Sample results of a Monte Carlo simulation with a fixed coordinate system and randomly distributed orbital planes of individual trajectories are shown in Fig. 17.

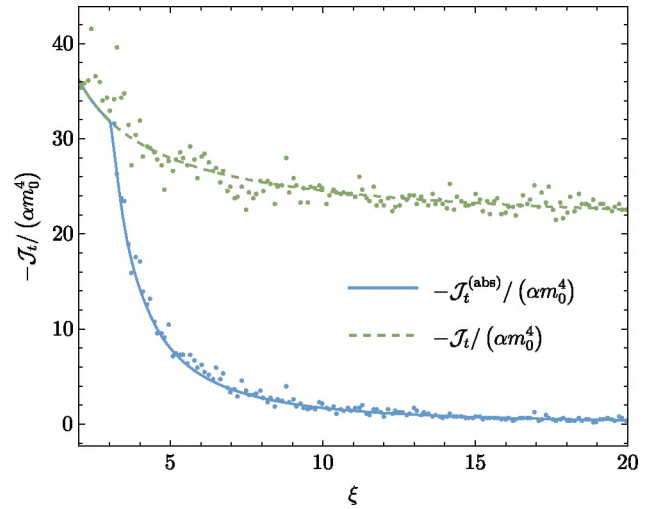


FIG. 17. Time components of the particle current density $\mathcal{J}_t$ for the Maxwell-Jüttner spherically symmetric model with $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 20$. Exact solutions [Eqs. (110a) and (110b)] are plotted with solid and dashed lines. Blue and green dots show results of a Monte Carlo simulation in which orbital planes of individual trajectories are distributed randomly, and we count intersections of orbits with segments of meridional half-planes of a fixed coordinate system (as shown in Fig. 16). We use von Neumann’s rejection method, selecting geodesic parameters and the vector normal to the equatorial plane in a single iteration step. There are 10^7 iteration steps, giving 828 absorbed and 10,979 scattered trajectories.

V. SUMMARY

We have developed Monte Carlo techniques for solving the general-relativistic Vlasov equation. We focused on stationary accretion-type solutions on the Schwarzschild background, but with appropriate adjustments, our methods should also work in other cases, including stationary configurations of the collisionless gas in Kerr spacetime, as well as time-dependent solutions of initial value problems. (A possible future implementation valid for Kerr spacetime should, in particular, recover results of the planar model [17] as a test.)

The emphasis on stationary solutions sets our formulation in contrast to particle-in-cell methods, designed to solve initial value problems and allowing for an efficient coupling of (plasma) particles with electromagnetic fields (see, e.g., [19,34]). Nonspherically symmetric numerical stationary solutions of Einstein-Vlasov equations were obtained already in pioneering works [35,36] and more recently in [37]. Numerical methods used in these papers are focused on Einstein equations; the kinetic part is treated with relatively simple *Ansätze* for the distribution function, and it is restricted to bound orbits. In this paper we were concerned with the kinetic part only, aiming at constructing a possibly versatile framework. Our averaging method for computing observable quantities, such as the particle current density or the energy-momentum tensor, is general [although one might need to adapt the hypersurface Σ in Eqs. (67) and (68) to the problem at

hand]. An appropriate selection of the sample of geodesics can be treated as a separate issue. We have provided examples corresponding to a homogeneous asymptotic distribution of particles, characteristic of Bondi-type accretion problems.

Our discussion emphasized the difference between planar models (in which the motion of particles is confined to a common plane) and nonplanar ones. It is especially tricky (or subtle) for spherically symmetric spacetimes (like the Schwarzschild spacetime), in which each of the geodesics belongs to a single plane.

In all examples, we have concentrated on computing the particle current density, but other observable quantities can be obtained in a similar fashion. A simple Monte Carlo simulation yielding the rest-mass accretion rate in the low-temperature limit of the Bondi-Hoyle-Lyttleton-type models described in [13,14] was reported in [16].

Aside from a purely technical aspect of the proposed Monte Carlo method, we hope that the cases investigated in this paper offer insight into the geometric structure of the general-relativistic kinetic theory and its statistical interpretation.

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- [1] J. L. Synge, *The Relativistic Gas* (North-Holland, Amsterdam, 1957).
 - [2] W. Israel, Relativistic kinetic theory of a simple gas, *J. Math. Phys. (N.Y.)* **4**, 1163 (1963).
 - [3] J. Ehlers, General relativity and kinetic theory, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic Press, New York, 1971).
 - [4] J. Ehlers, General relativistic theory of gases, in *Relativistic Fluid Dynamics*, edited by C. Cattaneo (Cremonese, Rome, 1971).
 - [5] S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland Publishing Company, Amsterdam, 1980).
 - [6] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Basel, 2002).
 - [7] H. Andréasson, The Einstein-Vlasov system/kinetic theory, *Living Rev. Relativity* **14**, 4 (2011).
 - [8] O. Sarbach and Th. Zannias, The geometry of the tangent bundle and the relativistic kinetic theory of gases, *Classical Quantum Gravity* **31**, 085013 (2014).
 - [9] R. O. Acuña-Cárdenas, C. Gabarrete, and O. Sarbach, An introduction to the relativistic kinetic theory on curved spacetimes, *Gen. Relativ. Gravit.* **54**, 23 (2022).
 - [10] P. Rioseco and O. Sarbach, Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole, *Classical Quantum Gravity* **34**, 095007 (2017).
 - [11] P. Rioseco and O. Sarbach, Spherical steady-state accretion of a relativistic collisionless gas into a Schwarzschild black hole, *J. Phys. Conf. Ser.* **831**, 012009 (2017).
 - [12] A. Cieřlik and P. Mach, Accretion of the Vlasov gas on Reissner-Nordström black holes, *Phys. Rev. D* **102**, 024032 (2020).
 - [13] P. Mach and A. Odrzywołek, Accretion of dark matter onto a moving Schwarzschild black hole: An exact solution, *Phys. Rev. Lett.* **126**, 101104 (2021).
 - [14] P. Mach and A. Odrzywołek, Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: Exact solutions, *Phys. Rev. D* **103**, 024044 (2021).
 - [15] A. Gamboa, C. Gabarrete, P. Domínguez-Fernández, D. Núñez, and Olivier Sarbach, Accretion of a Vlasov gas onto a black hole from a sphere of finite radius and the role of angular momentum, *Phys. Rev. D* **104**, 083001 (2021).
 - [16] P. Mach and A. Odrzywołek, Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: Low-temperature limit and numerical aspects, *Acta Phys. Pol. B* **15**, 1 (2022).

- [17] A. Cieřlik, P. Mach, and A. Odrzywołek, Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole, *Phys. Rev. D* **106**, 104056 (2022).
- [18] P. Mach and A. Odrzywołek, Equatorial accretion on the Kerr black hole, *Acta Phys. Pol. B* **16**, 6 (2023).
- [19] K. Parfrey, A. Philippov, and B. Cerutti, First-principles plasma simulations of black-hole jet launching, *Phys. Rev. Lett.* **122**, 035101 (2019).
- [20] A. Bransgrove, B. Ripperda, and A. Philippov, Magnetic hair and reconnection in black hole magnetospheres, *Phys. Rev. Lett.* **127**, 055101 (2021).
- [21] B. Crinquand, B. Cerutti, G. Dubus, K. Parfrey, and A. Philippov, Synthetic images of magnetospheric reconnection-powered radiation around supermassive black holes, *Phys. Rev. Lett.* **129**, 205101 (2022).
- [22] A. Galishnikova, A. Philippov, E. Quataert, F. Bacchini, K. Parfrey, and B. Ripperda, Collisionless accretion onto black holes: Dynamics and flares, *Phys. Rev. Lett.* **130**, 115201 (2023).
- [23] S.L. Shapiro and S.A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, New York, USA, 1983).
- [24] F. Jüttner, Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie, *Ann. Phys. (N.Y.)* **339**, 856 (1911).
- [25] F. Jüttner, Die Dynamik eines bewegten Gases in der Relativtheorie, *Ann. Phys. (N.Y.)* **340**, 145 (1911).
- [26] Wolfram Research, Inc., *Mathematica*, Version 13.1, Champaign, IL (2022).
- [27] Patryk Mach, Wolfram *Mathematica* packages supplementing the article: Patryk Mach, Adam Cieřlik, Andrzej Odrzywołek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes,” Jagiellonian University in Kraków, 2023, [10.57903/UJ/VBSBVD](https://doi.org/10.57903/UJ/VBSBVD).
- [28] M.H. Kalos and P.A. Whitlock, *Monte Carlo Methods* (Wiley-VCH, Weinheim, 2008).
- [29] J. von Neumann, Various techniques used in connection with random digits, in *Monte Carlo Method*, Applied Mathematics Series Vol. 12 (U. S. National Bureau of Standards, Washington D.C., U.S. Government Printing Office, 1951), pp. 36–38.
- [30] J. Bertrand, *Calcul des probabilités* (Gauthier-Villars, Paris, 1889).
- [31] M.G. Kendall and P.A.P. Moran, *Geometrical Probability* (Griffin, London, 1963).
- [32] H. Poincaré, *Calcul des Probabilités* (Gauthier-Villars, Paris, 1912).
- [33] Adam Cieřlik and Patryk Mach, Revisiting timelike and null geodesics in the Schwarzschild spacetime: General expressions in terms of Weierstrass elliptic functions, *Classical Quantum Gravity* **39**, 225003 (2022).
- [34] B. Cerutti and G. Werner, Zeltron: Explicit 3D relativistic electromagnetic Particle-In-Cell code, Astrophysics Source Code Library (record ascl:1911.012) (2019).
- [35] S.L. Shapiro and S.A. Teukolsky, Relativistic stellar systems with spindle singularities, *Astrophys. J.* **419**, 622 (1993).
- [36] S.L. Shapiro and S.A. Teukolsky, Relativistic stellar systems with rotation, *Astrophys. J.* **419**, 636 (1993).
- [37] E. Ames, H. Andreasson, and A. Logg, On axisymmetric and stationary solutions of the self-gravitating Vlasov system, *Classical Quantum Gravity* **33**, 155008 (2016).

2.4. Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole

Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole

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We perform Monte Carlo simulations of stationary planar accretion of a collisionless gas onto a moving Schwarzschild black hole. In this work—a sequel to our previous paper on the Monte Carlo method for stationary general-relativistic Vlasov systems—we demonstrate that our approach can be extended beyond the simplifying assumptions of the spherical symmetry or axial symmetry in the planar case. Our method of computing observable quantities, such as the particle current density, can be regarded as a rigorous coarse graining scheme, adapted to a numerical grid. Main difficulties are related to the appropriate parametrization of particle trajectories and a selection of parameters consistent with assumed requirements on the distribution function.

I. INTRODUCTION

The general-relativistic kinetic model provides a description of gases in strong gravitational fields, complementary to hydrodynamics or magnetohydrodynamics. If collisions between individual gas particles are rare and the mean free path between collisions is large in comparison to the characteristic length scale in the system, the assumption of the local thermodynamical equilibrium may not hold and consequently the hydrodynamical approach may not be valid.

The general-relativistic kinetic description was applied to stellar systems [1, 2], dark matter models [3–7] and matter flows around black holes [4–17]. It also provides a model used in mathematical general relativity [18–26]. A numerical implementation of the general-relativistic kinetic model, the so-called particle-in-cell (PIC) method, is used to simulate the dynamics of magnetized accretion flows onto black holes [27–30].

In a recent paper [31] we proposed a Monte Carlo method of solving the general-relativistic Vlasov (Liouville) equation describing the gas of non-colliding particles moving in a given spacetime. In the kinetic model free particles of the gas move along segments of geodesics, from one individual collision to the next, or simply along geodesics if collisions are absent. To be more precise one should say that the gas is described by a continuous one-particle distribution function which, as long as collisions are neglected, remains constant along geodesics, understood as curves in the phase space. The Monte Carlo method proposed in [31] consists of three elements: *i*) Choosing a finite random set of parameters characterizing geodesic trajectories of individual particles (initial condition and/or constants of motion) selected from a suitable continuous distribution; *ii*) Solving geodesic equations for the selected sample of parameters; *iii*) Computing suitable averages that approximate macroscopic observable quantities—particle current density, the energy-momentum tensor, etc. Out of these three points, only the second—solving geodesic equations—is relatively straightforward. The third point is implemented as follows. We consider observable quantities associated with a Dirac delta type distribution that corresponds to a finite number of point particles moving along given trajectories. Observable quantities associated with a continuous distribution function are then approximated by averaging over suitable hypersurfaces (either spacelike or timelike). This amounts to counting (with appropriate weight factors) the intersections of particle trajectories with given hypersurfaces. One could say that our method provides a particular coarse graining procedure, adapted to a given numerical grid. We emphasize that the novelty of counting over timelike hypersurfaces allows us to deal directly with stationary distributions.

The first point remains notoriously difficult. In principle any selection of geodesics would lead to a valid solution of the Vlasov equation. On the other hand, one would expect that physically sound solutions should correspond to distribution functions characterized by well defined properties (e.g., boundary or initial conditions). In practice, selecting an appropriate sample of geodesic parameters can be problematic. In [31] we solved this task in particular

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examples, providing a collection of results related to the Bondi-type accretion onto a Schwarzschild black hole. We searched for stationary solutions, either spherically symmetric or axially symmetric and planar (all particles moving in single plane).

In this paper we extend our analysis presented in [31] and investigate stationary solutions of a planar accretion problem with asymptotic boundary conditions that describe the gas moving uniformly in one direction. Such models constitute a planar equivalent of the relativistic accretion of the Vlasov gas onto a moving Schwarzschild black hole, derived in [4, 5], and an intermediate step towards a similar analysis in the Kerr spacetime [15]. While a planar model of this type might seem slightly artificial, it provides a perfect opportunity to discuss the details related to the choice of the sample of geodesics as well as the averaging procedure of our Monte Carlo method, outside the safe assumption of spherical symmetry.

The order of this paper is as follows. In Section II we introduce our notation and collect basic definitions regarding the phase space description of gases in the general-relativistic kinetic theory. Section III introduces the planar accretion model in the Schwarzschild spacetime, which we deal with in this paper, and which serves as an illustration and test of our method. Our Monte Carlo method is described in Sec. IV. We start with a discussion of the general coarse graining scheme and then show its application on a polar grid in the Schwarzschild spacetime. A detailed description of our simulations is provided in Sec. V. We discuss, in particular, the selection of particle trajectories corresponding to the assumed asymptotic distribution. Concluding remarks are given in Sec. VI. In Appendix A we show a derivation of the distribution function of our planar accretion model. For the sake of generality, this distribution is derived also for a non-planar flow, in which the motion of the gas is not confined to a single plane. Appendix B provides a derivation of the particle current surface density associated with our planar model.

We used *Wolfram Mathematica* [32] to perform our simulations. Sample packages containing our numerical code will be publicly available in [33].

Throughout this paper we use the geometric system of units with $c = G = 1$, where c is the speed of light, and G denotes the gravitational constant. The signature of the metric is $(-, +, +, +)$.

II. PHASE-SPACE DESCRIPTION

In this section we recall some basic notions of the general-relativistic kinetic theory, mostly to fix the notation. An excellent fresh introduction to the kinetic model in general relativity can be found in [35]. A lot of information can be found in [20, 36]. For the (special) relativistic kinetic theory one can consult [37, 38].

Consider a spacetime manifold $(\mathcal{M}, g)$, where $\mathcal{M}$ is the spacetime and g denotes the metric tensor. Let x denote a point (an event) in $\mathcal{M}$. We will work in a Hamiltonian formulation in which the momenta are expressed as covectors. A particle at $x \in \mathcal{M}$ will be characterized by its four-momentum $p \in T_x^*\mathcal{M}$, the cotangent space at x . The one-particle phase space Γ is a region in the cotangent bundle defined as

$$T^*\mathcal{M} = \{(x, p) : x \in \mathcal{M}, p \in T_x^*\mathcal{M}\}. \quad (1)$$

Ensemble-averaged properties of the gas can be described by a one-particle distribution function $\mathcal{F}$ defined on $\Gamma \subseteq T^*\mathcal{M}$. In this work, we narrow our analysis to the gas of identical particles. Such a model is known as the “simple gas” [39]. In principle, this assumption would allow us to restrict the phase space to the mass shell Γ_m^+ which we define as

$$\Gamma_m^+ = \{(x, p) \in T^*\mathcal{M} : g^{\mu\nu} p_\mu p_\nu = -m^2, p \text{ is future-directed}\}, \quad (2)$$

where $m > 0$ denotes the rest mass of a single gas particle. Another possibility, which we choose in this paper, is to enforce the mass-shell condition by assuming that $\mathcal{F} \sim \delta(\sqrt{-p_\mu p^\mu} - m)$.

The Hamiltonian of a free-particle geodesic motion can be taken in the form

$$H(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu, \quad (3)$$

where (x^μ, p_ν) are regarded as canonical variables. Geodesic equations can be then written in the Hamiltonian form

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}, \quad (4)$$

where the parameter τ is selected as $\tau = s/m$ and s denotes the proper time. Thus $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = -\frac{1}{2} m^2$. For the ensemble of non-colliding particles, the distribution function $\mathcal{F}$ is conserved along geodesics, which can be expressed as

$$\frac{d}{d\tau} \mathcal{F}(x^\mu(\tau), p_\nu(\tau)) = \{H, \mathcal{F}\} = 0, \quad (5)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Equation (5) is known as the Vlasov or Liouville equation. Assuming the Hamiltonian (3), one can rewrite Eq. (5) in the textbook form

$$g^{\mu\nu} p_\nu \frac{\partial \mathcal{F}}{\partial x^\mu} - \frac{1}{2} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial x^\mu} \frac{\partial \mathcal{F}}{\partial p_\mu} = 0. \quad (6)$$

The one-particle distribution function has the following statistical interpretation. Let S be a spacelike hypersurface in $\mathcal{M}$, and let s be a future-directed unit vector normal to S . The averaged number of particle trajectories whose projections on $\mathcal{M}$ intersect S can be expressed as

$$\mathcal{N}[S] = - \int_S \left[\int_{P_x^+} \mathcal{F}(x, p) p_\mu s^\mu \text{dvol}_x(p) \right] \eta_S, \quad (7)$$

where

$$P_x^+ = \{p \in T_x^* \mathcal{M} : g^{\mu\nu} p_\mu p_\nu < 0, p \text{ is future directed}\}, \quad (8)$$

η_S denotes the induced volume element on S , and $\text{dvol}_x(p)$ is the volume element on P_x^+ . In local adapted coordinates $\text{dvol}_x(p)$ is given as

$$\text{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)} dp_0 dp_1 dp_2 dp_3. \quad (9)$$

Equation (7) gives rise to the following definition of the so-called particle current density:

$$\mathcal{J}_\mu(x) = \int_{P_x^+} \mathcal{F}(x, p) p_\mu \text{dvol}_x(p). \quad (10)$$

Thus, the number of particles in S can be re-expressed as

$$\mathcal{N}[S] = - \int_S \mathcal{J}_\mu s^\mu \eta_S. \quad (11)$$

Using Eq. (5), one can show that the particle current density satisfies the conservation law $\nabla_\mu \mathcal{J}^\mu = 0$, which again justifies formula (11). The particle number density can be defined covariantly as

$$n = \sqrt{-\mathcal{J}_\mu \mathcal{J}^\mu}. \quad (12)$$

Such a definition of the particle number density—as a scalar quantity—allows for a connection with general-relativistic hydrodynamics, where the particle number density as well as the energy density and the pressure are usually defined as scalar functions. Alternatively, one can work with the components of $\mathcal{J}_\mu$.

One should also note that in integral formulas of type (7) or (10) one can choose to transfer some information about the distribution function $\mathcal{F}$ from the distribution function itself to the specification of the integration domain (e.g., S, P_x^+), or vice versa—one can consider a fairly general phase space and impose restrictions directly on the distribution function. To some degree we will also make use of this freedom in this paper, e.g., in Eqs. (40), where we will specify a convenient form of the distribution function, but restrict the integrals over momenta to regions in the phase space available to the motion of particles of a given type (particles absorbed and scattered by the black hole).

In this work we deal with stationary systems. In particular, in our prescription of the Monte Carlo estimators for the particle current density $\mathcal{J}_\mu$ we will, at some stage, explicitly assume that both the distribution function $\mathcal{F}$ and the components $\mathcal{J}_\mu$ are stationary. A spacetime $(\mathcal{M}, g)$ is called stationary if there exist a Killing vector k that is asymptotically timelike and generates one parameter groups of isometries. A definition of a stationary distribution function $\mathcal{F}$ defined on $T^* \mathcal{M}$ can be formulated in terms of the lift of the Killing vector k to the cotangent bundle, defined as follows. For a general Killing vector field ξ on $(\mathcal{M}, g)$ such that its value at point $x \in \mathcal{M}$ can be expressed as

$$\xi_x = \xi^\mu(x) \left. \frac{\partial}{\partial x^\mu} \right|_x, \quad (13)$$

one defines an associated vector field on the cotangent bundle $T^*\mathcal{M}$, the so-called lift of ξ , by

$$\hat{\xi}_{(x,p)} = \xi^\mu(x) \left. \frac{\partial}{\partial x^\mu} \right|_{(x,p)} - p_\alpha \frac{\partial \xi^\alpha}{\partial x^\mu}(x) \left. \frac{\partial}{\partial p_\mu} \right|_{(x,p)}. \quad (14)$$

The lifts defined in this way satisfy Killing equations with respect to the so-called Sasaki metric—a natural metric on $T^*\mathcal{M}$ [35]. We say that a distribution function $\mathcal{F}$ is stationary if

$$\mathcal{L}_{\hat{k}} \mathcal{F} = k^\mu \frac{\partial \mathcal{F}}{\partial x^\mu} - p_\alpha \frac{\partial k^\alpha}{\partial x^\mu} \frac{\partial \mathcal{F}}{\partial p_\mu} = 0, \quad (15)$$

where $\mathcal{L}_{\hat{k}}$ denotes the Lie derivative with respect to the lift $\hat{k}$.

III. PLANAR STATIONARY ACCRETION IN THE SCHWARZCHILD SPACETIME

A. Geodesic motion in the Schwarzschild spacetime

For simplicity, we work in standard Schwarzschild coordinates (t, r, θ, φ) , and use the metric in the form

$$g = -N dt^2 + \frac{1}{N} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (16)$$

where

$$N = 1 - \frac{2M}{r}. \quad (17)$$

We have $\sqrt{-\det g^{\mu\nu}} = 1/(r^2 \sin \theta)$.

The Hamiltonian associated with a geodesic motion in the spacetime with the metric (16), is given by

$$H = \frac{1}{2} \left[-\frac{1}{N} p_t^2 + N p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right) \right]. \quad (18)$$

It is well-known that the energy $E = -p_t$, the azimuthal component of the angular momentum $l_z = p_\varphi$, and the total angular momentum

$$l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}} \quad (19)$$

are conserved. The fourth constant of motion is the Hamiltonian itself, or equivalently the particle rest mass $m = \sqrt{-2H}$.

For geodesics with fixed values of E , l , l_z , and m , the radial momentum is determined by

$$p_r = \frac{\epsilon_r}{N} \sqrt{E^2 - U(r)}, \quad (20)$$

where $\epsilon_r = \mp 1$ indicate an inward $(-)$ /outward $(+)$ motion and,

$$U(r) = N \left(m^2 + \frac{l^2}{r^2} \right) \quad (21)$$

represents an effective radial potential. The component p_θ can be expressed as

$$p_\theta = \epsilon_\theta \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}. \quad (22)$$

For the motion confined to the equatorial plane we have $p_\theta = 0$ and consequently $l^2 = l_z^2$. In our planar models we will generally use the constant $l \geq 0$, and define $l_z = \epsilon_\varphi l$, where $\epsilon_\varphi = \pm 1$ corresponds to the motion with a growing $(+)$ /decreasing $(-)$ angle φ .

In this paper we will frequently use dimensionless quantities ξ , ε , λ_z , and λ defined as (see, e.g., [8])

$$r = M\xi, \quad E = m\varepsilon, \quad l_z = Mm\lambda_z, \quad l = Mm\lambda. \quad (23)$$

The radial potential (21) can be expressed in the form $U(r) = m^2 U_\lambda(\xi)$, where

$$U_\lambda(\xi) = \left(1 - \frac{2}{\xi}\right) \left(1 + \frac{\lambda^2}{\xi^2}\right). \quad (24)$$

The expression for p_r reads

$$p_r = \epsilon_r \frac{m}{N} \sqrt{\varepsilon^2 - U_\lambda(\xi)}. \quad (25)$$

In this work we only take into account unbound orbits that can reach $\xi \rightarrow \infty$. The region in which the motion is permitted is defined by the condition $\varepsilon^2 - U_\lambda(\xi) \geq 0$. Since $U_\lambda(\xi)$ approaches 1 when $\xi \rightarrow \infty$, unbound orbits are characterized by the energy $\varepsilon \geq 1$. The properties of the radial potential $U_\lambda(\xi)$ are well known (see, for instance, the Appendix A in [8] for a detailed discussion), and they allow for the following further classification of unbound trajectories. Orbits with the angular momentum $\lambda < \lambda_c(\varepsilon)$, where

$$\lambda_c(\varepsilon)^2 = \frac{12}{1 - \frac{4}{\left(\frac{3\varepsilon}{\sqrt{9\varepsilon^2 - 8}} + 1\right)^2}}, \quad (26)$$

plunge into the black hole. We will refer to such orbits as absorbed ones. Unbound orbits with $\lambda > \lambda_c(\varepsilon)$ are scattered by the centrifugal barrier. We will refer to these trajectories as scattered ones. The maximum angular momentum for a scattered unbound orbit reaching radius ξ is given by

$$\lambda_{\max}(\varepsilon, \xi) = \xi \sqrt{\frac{\varepsilon^2}{1 - \frac{2}{\xi}} - 1}, \quad (27)$$

while the minimum permitted energy is expressed as

$$\varepsilon_{\min}(\xi) = \begin{cases} \infty, & \xi \leq 3, \\ \sqrt{\left(1 - \frac{2}{\xi}\right) \left(1 + \frac{1}{\xi - 3}\right)}, & 3 < \xi < 4, \\ 1, & \xi \geq 4. \end{cases} \quad (28)$$

The infinity symbol in Eq. (28) means that no unbound scattered trajectory can reach the region with $\xi < 3$. The radius $\xi = 3$ corresponds to the so-called circular photon orbit (or the photon sphere). A circular orbit at $\xi = 4$ is known as the marginally bound one. The Innermost Stable Circular Orbit (ISCO) occurs at $\xi = 6$. These orbits are, to some degree, relevant to the interpretation of our results presented in Sec. VD.

B. Planar distribution function

We will consider a planar accretion model, in which the gas particles are only moving within the equatorial plane. The restriction to the equatorial plane can be imposed on the form of the distribution function as follows. We set

$$\mathcal{F}(t, r, \theta, \varphi; p_\alpha) = \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) F(t, r, \varphi; p_\alpha), \quad \mathcal{J}_\mu(t, r, \theta, \varphi) = \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) J_\mu(t, r, \varphi). \quad (29)$$

Note that the factor $\delta(\theta - \pi/2)/r$ can be written as $\delta(\theta - \pi/2)/r = \delta(z)$, where $z = r \cos \theta$. The vector ∂_z , associated with the coordinate z , is normal to the equatorial plane $\theta = \pi/2$. A substitution of Eq. (29) in Eq. (10) implies that the particle current surface density J_μ can be written as

$$J_\mu(t, r, \varphi) = \int_{P_x^+} F(t, r, \varphi; p_\alpha) p_\mu d\text{vol}_x(p). \quad (30)$$

The restriction of the coordinates (t, r, θ, φ) to the equatorial plane $\theta = \pi/2$ implies a corresponding restriction in momenta ($p_\theta = 0$) as well. We set

$$F(t, r, \varphi, p_t, p_r, p_\theta, p_\varphi) = r\delta(p_\theta)f(t, r, \varphi, p_t, p_r, p_\varphi), \quad (31)$$

where again the factor $r\delta(p_\theta)$ can be written as $r\delta(p_\theta) = \delta(p_z)$ (at the equatorial plane). This follows immediately from the relation $p_z = -p_\theta/r$, holding at the equatorial plane. Note that distribution functions $\mathcal{F}$ and f are conveniently related as

$$\mathcal{F}(t, r, \theta, \varphi, p_t, p_r, p_\theta, p_\varphi) = \delta\left(\theta - \frac{\pi}{2}\right)\delta(p_\theta)f(t, r, \varphi, p_t, p_r, p_\varphi). \quad (32)$$

The particle current surface density can be now expressed in the form

$$J_\mu(t, r, \varphi) = \int_{P_x^+} r\delta(p_\theta)f(t, r, \varphi, p_t, p_r, p_\varphi)p_\mu \frac{dp_t dp_r dp_\theta dp_\varphi}{r^2} = \int_{P_x^+} f(t, r, \varphi, p_t, p_r, p_\varphi)p_\mu \frac{dp_t dp_r dp_\varphi}{r}, \quad (33)$$

where

$$\bar{P}_x^+ = \{(p_t, p_r, p_\varphi) : \bar{g}^{\mu\nu} p_\mu p_\nu < 0, p \text{ is future directed}\}, \quad (34)$$

and where we have used the fact that at the equatorial plane the volume element $\text{dvol}_x(p)$ reads

$$\text{dvol}_x(p) = \frac{1}{r^2 \sin \theta} dp_t dp_r dp_\theta dp_\varphi = \frac{1}{r^2} dp_t dp_r dp_\theta dp_\varphi. \quad (35)$$

Here

$$\bar{g} = -Ndt^2 + Ndr^2 + r^2 d\varphi^2 \quad (36)$$

denotes the metric induced at the equatorial plane.

In the remainder of this paper we will also use the following covariant particle number surface density

$$n_s = \sqrt{-\bar{g}^{\mu\nu} J_\mu J_\nu}. \quad (37)$$

C. Planar accretion model

We consider a stationary gas of particles moving within the equatorial plane. Asymptotically, the gas is assumed to be homogeneous and described by the two dimensional Maxwell-Jüttner distribution [40, 41], boosted with a constant velocity v along the x axis. In other words, we think of a planar distribution of gas that is in equilibrium, and moves with a uniform velocity v in a given direction. This is mathematically equivalent to a model with a black hole moving with a constant velocity with respect to a gas remaining asymptotically at rest, and a majority of works dealing with the accretion onto a moving black hole assume this point of view. We show in Appendix A that the distribution function corresponding to these asymptotic conditions can be written as

$$f(\xi, \varphi, m, \varepsilon, \lambda) = \alpha\delta(m - m_0) \exp \left\{ -\beta\gamma \left[\varepsilon - \epsilon_r v \sqrt{\varepsilon^2 - 1} \cos [\varphi + \epsilon_\varphi \epsilon_r X(\xi, \varepsilon, \lambda)] \right] \right\}, \quad (38)$$

where the elliptic function $X(\xi, \varepsilon, \lambda)$ is defined as

$$X(\xi, \varepsilon, \lambda) = \lambda \int_{\xi}^{\infty} \frac{d\xi'}{\xi'^2 \sqrt{\varepsilon^2 - U_\lambda(\xi')}} = \lambda \int_{\xi}^{\infty} \frac{d\xi'}{\xi'^2 \sqrt{\varepsilon^2 - \left(1 - \frac{2}{\xi'}\right) \left(1 + \frac{\lambda^2}{\xi'^2}\right)}}. \quad (39)$$

Here $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor associated with the velocity v , m_0 denotes the rest mass of a single particle, α and β are constants. The constant β is related to the asymptotic temperature of the gas by $\beta = m_0/(k_B T)$, where k_B denotes the Boltzmann constant. The distribution function specified in Eq. (38) is a planar equivalent of the distribution function derived in [4, 5].

In our models we will only take into account unbound orbits—trajectories that can reach (or originate at) infinity. The particle current surface density J_μ can be expressed as a sum $J_\mu = J_\mu^{(\text{abs})} + J_\mu^{(\text{scat})}$, where the part $J_\mu^{(\text{abs})}$ refers to absorbed orbits and the part $J_\mu^{(\text{scat})}$ refers to scattered trajectories.

The components $J_\mu^{(\text{abs})}$ and $J_\mu^{(\text{scat})}$ can be computed using the following formulas (details of the derivation are given in Appendix B)

$$J_t^{(\text{abs})}(\xi, \varphi) = -\frac{2\alpha m_0^3}{\xi} \int_1^\infty d\varepsilon \varepsilon \int_0^{\lambda_c(\varepsilon)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} e^{-\beta\gamma[\varepsilon + v\sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)]]} \quad (40a)$$

$$\times \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right],$$

$$J_t^{(\text{scat})}(\xi, \varphi) = -\frac{4\alpha m_0^3}{\xi} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon e^{-\beta\gamma\varepsilon} \varepsilon \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\xi, \varepsilon)} \frac{d\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)] \right] \quad (40b)$$

$$\times \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right],$$

$$J_r^{(\text{abs})}(\xi, \varphi) = -\frac{2\alpha m_0^3}{\xi - 2} \int_1^\infty d\varepsilon \int_0^{\lambda_c(\varepsilon)} d\lambda e^{-\beta\gamma[\varepsilon + v\sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)]]} \quad (40c)$$

$$\times \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right],$$

$$J_r^{(\text{scat})}(\xi, \varphi) = \frac{4\alpha m_0^3}{\xi - 2} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon e^{-\beta\gamma\varepsilon} \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\xi, \varepsilon)} d\lambda \sinh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)] \right] \quad (40d)$$

$$\times \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right],$$

$$J_\varphi^{(\text{abs})}(\xi, \varphi) = -\frac{2\alpha m_0^3 M}{\xi} \int_1^\infty d\varepsilon \int_0^{\lambda_c(\varepsilon)} d\lambda \frac{\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} e^{-\beta\gamma[\varepsilon + v\sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)]]} \quad (40e)$$

$$\times \sinh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right],$$

$$J_\varphi^{(\text{scat})}(\xi, \varphi) = -\frac{4\alpha m_0^3 M}{\xi} \int_{\varepsilon_{\min}(\xi)}^\infty d\varepsilon e^{-\beta\gamma\varepsilon} \int_{\lambda_c(\varepsilon)}^{\lambda_{\max}(\xi, \varepsilon)} d\lambda \frac{\lambda}{\sqrt{\varepsilon^2 - U_\lambda(\xi)}} \sinh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \sin(\varphi) \sin[X(\xi, \varepsilon, \lambda)] \right] \quad (40f)$$

$$\times \cosh \left[\beta\gamma v \sqrt{\varepsilon^2 - 1} \cos(\varphi) \cos[X(\xi, \varepsilon, \lambda)] \right].$$

The parameter α can be expressed in terms of the asymptotic surface number density, which we define as $n_{s,\infty} = \lim_{r \rightarrow \infty} n_s = \lim_{r \rightarrow \infty} \sqrt{-J_\mu J^\mu}$. Since, asymptotically, the gas is described by a two dimensional Maxwell-Jüttner distribution, one can show that

$$n_{s,\infty} = 2\pi\alpha m_0^3 \frac{1+\beta}{\beta^2} e^{-\beta}. \quad (41)$$

A calculation leading to Eq. (41) can be performed assuming the flat Minkowski metric, and it can be found in [6]. Note that a boosted Maxwell-Jüttner distribution must satisfy

$$\lim_{r \rightarrow \infty} J_t = -n_{s,\infty} \gamma, \quad (42)$$

which can also be checked by a direct calculation.

IV. MONTE CARLO APPROACH

A. Coarse graining method

We will now discuss the main ideas behind our Monte Carlo method, starting with a general averaging scheme. The choice of trajectories of individual particles will be discussed in the subsequent section in a particular example of the planar accretion model.

Consider a sample of N particle trajectories in the phase space $T^*\mathcal{M}$ understood as mappings (curves) $\gamma_{(i)}: \tau \mapsto (x_{(i)}^\mu(\tau), p_{(i)}^\nu(\tau))$, $i = 1, \dots, N$. The distribution function associated with this sample can be written in the form

$$\mathcal{F}^{(N)}(x^\mu, p_\nu) = \sum_{i=1}^N \int \delta^{(4)}(x^\mu - x_{(i)}^\mu(\tau)) \delta^{(4)}(p_\nu - p_{(i)}^\nu(\tau)) d\tau, \quad (43)$$

which can be understood as a general-relativistic version of an analogous expression known from the (special) relativistic kinetic theory ([37], p. 14, Eq. (A6)). In the special relativistic setting this function was also introduced in [42], although with a slightly different normalization, i.e., divided by the number of trajectories N . The distribution function $\mathcal{F}^{(N)}$ represents a collection of point-like test particles, and therefore it provides a fine-grained description of matter. The essence of the Monte Carlo method introduced in [31] and developed in this paper is a suitable averaging method, allowing for an approximation of the coarse-grained smooth distribution function $\mathcal{F}$. Since ultimately one is interested in observable quantities, such as the particle current density $\mathcal{J}_\mu$, we apply this coarse graining procedure explicitly to the components of $\mathcal{J}_\mu$. A general discussion of coarse graining in the context of the special-relativistic kinetic theory can be found in [42].

The particle current density associated with the distribution function (43) reads

$$\mathcal{J}_\mu^{(N)}(x) = \int_{P_x^+} \mathcal{F}^{(N)}(x, p) p_\mu \sqrt{-\det g^{\delta\kappa}(x)} dp_0 \dots dp_3 = \sum_{i=1}^N \int \delta^{(4)}(x^\alpha - x_{(i)}^\alpha(\tau)) p_\mu^{(i)}(\tau) \sqrt{-\det g^{\delta\kappa}(x)} d\tau, \quad (44)$$

where we have assumed that all momenta $p_\mu^{(i)}(\tau)$ belong to $P_{x(\tau)}^+$. It was shown in [31] that this expressions is compatible with Eq. (7), i.e., if S is a spacelike hypersurface in $\mathcal{M}$ such that the projections of all trajectories $\gamma_{(i)}$, $i = 1, \dots, N$, on $\mathcal{M}$ intersect S , then $\mathcal{N}[S] = N$, where $\mathcal{N}[S]$ is computed according to Eq. (7), assuming the distribution function $\mathcal{F}^{(N)}$.

Let us view $\mathcal{F}^{(N)}$ as an approximation to a smooth distribution function $\mathcal{F}$. Select a hypersurface Σ in $\mathcal{M}$ (not necessarily a spacelike one) and a small neighbourhood $\sigma \subset \Sigma$ (a numerical cell) such that $x \in \sigma$. We estimate $\mathcal{J}_\mu(x)$ corresponding to $\mathcal{F}$ by the average

$$\langle \mathcal{J}_\mu(x) \rangle = \frac{\int_\sigma \mathcal{J}_\mu^{(N)} \eta_\Sigma}{\int_\sigma \eta_\Sigma}, \quad (45)$$

where η_Σ denotes the volume element on Σ . For stationary systems, Σ can be chosen as a timelike hypersurface, adapted to isometry groups generated by the Killing vector field k . We will see in the next section that the coarse graining (or smoothing) procedure given by Eq. (45) amounts to counting (with suitable weights) the intersections of particle trajectories with selected surfaces.

Equation (45) can be justified as follows. Suppose that $\mathcal{F}^{(N)}$ tends to a smooth distribution function $\mathcal{F}$ in the sense that

$$\begin{aligned} & \int dx^0 \dots dx^3 dp_0 \dots dp_3 \frac{\mathcal{F}^{(N)}(x, p) - \mathcal{F}(x, p)}{N} \phi(x, p) \\ &= \int \sqrt{-\det g_{\alpha\beta}(x)} dx^0 \dots dx^3 \sqrt{-\det g^{\delta\kappa}(x)} dp_0 \dots dp_3 \frac{\mathcal{F}^{(N)}(x, p) - \mathcal{F}(x, p)}{N} \phi(x, p) \rightarrow 0, \end{aligned} \quad (46)$$

as $N \rightarrow \infty$, where $\phi(x, p)$ is a test function on $T^*\mathcal{M}$. Choosing $\phi(x, p) = \chi(p \in P_x^+) p_\mu \psi(x)$, we get

$$\int \sqrt{-\det g_{\alpha\beta}(x)} dx^0 \dots dx^3 \frac{\mathcal{J}_\mu^{(N)}(x)}{N} \psi(x) \rightarrow \int \sqrt{-\det g_{\alpha\beta}(x)} dx^0 \dots dx^3 \frac{\mathcal{J}_\mu(x)}{N} \psi(x). \quad (47)$$

Here $\chi(p \in P_x^+)$ denotes the characteristic function of the set P_x^+ . Suppose that Σ is a hypersurface in $\mathcal{M}$ chosen as in Eq. (45). Let γ be a three dimensional metric induced on Σ . Suppose further that $\tilde{\alpha}$ is a function such that $\sqrt{-\det g_{\alpha\beta}(x)} = \tilde{\alpha}(x) \sqrt{|\det \gamma_{\alpha\beta}(x)|}$. Let $(x^0, \dots, x^3)$ denote a coordinate system adapted to the hypersurface Σ , so that Σ is characterised by $x^0 = \bar{x}^0$ (say) and $\eta_\Sigma = \sqrt{|\det \gamma_{\alpha\beta}(x)|} dx^1 dx^2 dx^3$. Choose $\psi(x) = \chi(x \in \sigma) \delta(x^0 - \bar{x}^0) / \tilde{\alpha}(x)$. This gives

$$\int_\sigma \frac{\mathcal{J}_\mu^{(N)}(x)}{N} \eta_\Sigma \rightarrow \int_\sigma \frac{\mathcal{J}_\mu(x)}{N} \eta_\Sigma = \frac{\mathcal{J}_\mu(x_0)}{N} \int_\sigma \eta_\Sigma, \quad (48)$$

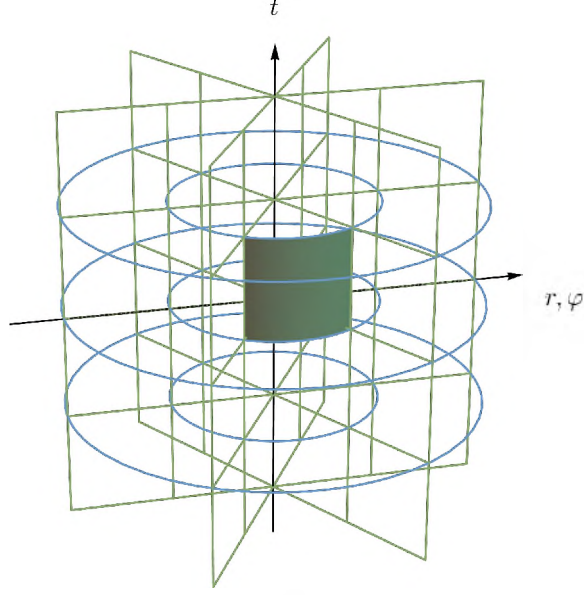


FIG. 1: A visual representation of a cell $\tilde{\sigma}$ (green slab) defined by Eqs. (51) or (53).

where $x_0 \in \sigma$. The last equality follows from the mean value theorem, provided that σ is closed, bounded, and connected, and the components of $\mathcal{J}_\mu(x)$ are continuous on σ . Note that this equality holds for each component separately, i.e., x_0 might be different for different $\mu = 0, \dots, 3$ (the mean value theorem does not hold for vector-valued functions). Equation (45) follows directly from the limit (48).

In the analysis of planar models investigated in this paper, we will rather use the estimate of the particle current surface density

$$\langle J_\mu(x) \rangle = \frac{\int_{\tilde{\sigma}} J_\mu^{(N)} \eta_{\tilde{\Sigma}}}{\int_{\tilde{\sigma}} \eta_{\tilde{\Sigma}}}, \quad (49)$$

defined in the analogous way, but with $\tilde{\sigma}$ and $\tilde{\Sigma}$ being two dimensional surfaces.

B. Intersections of trajectories with arcs of constant radius

As an example adapted to a planar stationary accretion flow in the Schwarzschild spacetime we select surfaces of constant $r = \bar{r}$ defined by

$$\tilde{\Sigma} = \{(t, r, \theta, \varphi) : t \in \mathbb{R}, r = \bar{r}, \theta = \pi/2, \varphi \in [0, 2\pi)\} \quad (50)$$

and cells (cf. Fig. 1)

$$\tilde{\sigma} = \{(t, r, \theta, \varphi) : t_1 \leq t \leq t_2, r = \bar{r}, \theta = \pi/2, \varphi_1 \leq \varphi \leq \varphi_2\}. \quad (51)$$

It is also convenient to denote

$$S = \{(r, \theta, \varphi) : r = \bar{r}, \theta = \pi/2, \varphi_1 \leq \varphi \leq \varphi_2\}. \quad (52)$$

The choice defined by Eq. (52) is different than the one made in [31]. In our previous work, we chose S as segments of constant angle φ . Both choices are adapted to stationary flows in the sense that $\tilde{\sigma}$ can be foliated by orbits of the timelike Killing vector field $k^\mu = (1, 0, 0, 0)$. More precisely Let $\Phi_\tau(x_0^i)$ denote the orbit of k , passing through x_0^i at $\tau = 0$, i.e., $\Phi_0(x_0^i) = x_0^i$. Then $\tilde{\sigma}$ can be expressed as the image

$$\tilde{\sigma} = \Phi_{[t_1, t_2]}(S). \quad (53)$$

Both choices are also compatible with a standard grid of polar coordinates (Fig. 1).

For the planar model (particles moving in the equatorial plane) the fine-grained particle current density can be expressed as

$$\mathcal{J}_\mu^{(N)}(t, r, \theta, \varphi) = \sum_{i=1}^N \int \delta\left(\theta - \frac{\pi}{2}\right) \delta^{(3)}\left(x^\alpha - x_{(i)}^\alpha(\tau)\right) p_\mu^{(i)}(\tau) \sqrt{-\det g^{\delta\kappa}(x)} d\tau, \quad (54)$$

where

$$\delta^{(3)}\left(x^\alpha - x_{(i)}^\alpha(\tau)\right) = \delta\left(t - t_{(i)}(\tau)\right) \delta\left(r - r_{(i)}(\tau)\right) \delta\left(\varphi - \varphi_{(i)}(\tau)\right). \quad (55)$$

A corresponding particle current surface density $J_\mu^{(N)}$ is defined as in Eq. (29), i.e.,

$$\mathcal{J}_\mu^{(N)}(t, r, \theta, \varphi) = \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) J_\mu^{(N)}(t, r, \varphi). \quad (56)$$

Thus,

$$J_\mu^{(N)}(t, r, \varphi) = \sum_{i=1}^N \int r \delta^{(3)}\left(x^\alpha - x_{(i)}^\alpha(\tau)\right) p_\mu^{(i)}(\tau) \sqrt{-\det g^{\delta\kappa}(x)} d\tau, \quad (57)$$

where, from (16) at the equatorial plane, $\sqrt{-\det g^{\delta\kappa}(x)} = (r^2 \sin \theta)^{-1} = 1/r^2$. The volume element on $\tilde{\Sigma}$ reads

$$\eta_{\tilde{\Sigma}} = \sqrt{-\det \tilde{g}_{\mu\nu}(x)} dt d\varphi = \sqrt{1 - \frac{2M}{r}} r dt d\varphi, \quad (58)$$

where $\tilde{g}$ denotes the metric induced on the surface $\tilde{\Sigma}$:

$$\tilde{g} = -\left(1 - \frac{2M}{r}\right) dt^2 + r^2 d\varphi^2. \quad (59)$$

A direct calculation gives

$$\begin{aligned} \int_{\tilde{\sigma}} J_\mu^{(N)} \eta_{\tilde{\Sigma}} &= \int_{t_1}^{t_2} dt \int_{\varphi_1}^{\varphi_2} d\varphi \sum_{i=1}^N \int d\tau \sqrt{1 - \frac{2M}{r}} \delta^{(3)}\left(x^\alpha - x_{(i)}^\alpha(\tau)\right) p_\mu^{(i)}(\tau) \\ &= \sum_{i \in I(\tilde{\sigma})} \int d\tau \sqrt{1 - \frac{2M}{r}} \delta(\bar{r} - r_{(i)}(\tau)) p_\mu^{(i)}(\tau), \end{aligned} \quad (60)$$

where $I(\tilde{\sigma}) \subseteq \{1, \dots, N\}$ denotes the set of indices corresponding to trajectories that intersect $\tilde{\sigma}$. The integral with respect to τ can be computed by writing

$$\delta(\bar{r} - r_{(i)}(\tau)) = \sum_k \frac{\delta(\tau - \tau_k)}{\left| dr_{(i)}/d\tau \right|_{\tau=\tau_k}}, \quad (61)$$

where $r_{(i)}(\tau_k) = \bar{r}$. The sum in Eq. (61) runs over all intersections of the i -th trajectory with $\tilde{\sigma}$. Note that

$$\frac{dr_{(i)}}{d\tau} = g^{rr} p_r^{(i)} = \left(1 - \frac{2M}{\bar{r}}\right) p_r^{(i)} = \epsilon_r \sqrt{E_{(i)}^2 - \left(1 - \frac{2M}{\bar{r}}\right) \left(m^2 + \frac{l_{(i)}^2}{\bar{r}^2}\right)}. \quad (62)$$

Returning to Eq. (60), we get

$$\begin{aligned} \int_{\tilde{\sigma}} J_\mu^{(N)} \eta_{\tilde{\Sigma}} &= \sum_{i \in I(\tilde{\sigma})} \sum_k \int d\tau \sqrt{1 - \frac{2M}{\bar{r}}} \frac{\delta(\tau - \tau_k)}{\left| dr_{(i)}/d\tau \right|_{\tau=\tau_k}} p_\mu^{(i)}(\tau) \\ &= \sqrt{1 - \frac{2M}{\bar{r}}} \sum_{j=1}^{N_{\text{int}}} \frac{p_\mu^{(j)}}{\sqrt{E_{(j)}^2 - (1 - 2M/\bar{r}) (m^2 + l_{(j)}^2/\bar{r}^2)}}, \end{aligned} \quad (63)$$

where the index j in the last sum runs over all N_{int} intersections of trajectories with the cell $\tilde{\sigma}$. The area (cf. Fig. 1) of $\tilde{\sigma}$ reads

$$\int_{\tilde{\sigma}} \eta_{\Sigma} = \int_{t_1}^{t_2} dt \int_{\varphi_1}^{\varphi_2} d\varphi \bar{r} \sqrt{1 - \frac{2M}{\bar{r}}} = \bar{r} \sqrt{1 - \frac{2M}{\bar{r}}} (t_2 - t_1) (\varphi_2 - \varphi_1). \quad (64)$$

This gives the Monte Carlo estimator of J_{μ} in the form

$$\begin{aligned} \langle J_{\mu} \rangle &= \frac{\int_{\tilde{\sigma}} J_{\mu} \eta_{\Sigma}}{\int_{\tilde{\sigma}} \eta_{\Sigma}} = \frac{1}{\bar{r}(t_2 - t_1)(\varphi_2 - \varphi_1)} \sum_{j=1}^{N_{\text{int}}} \frac{p_{\mu}^{(j)}}{\sqrt{E_{(j)}^2 - (1 - 2M/\bar{r}) (m^2 + l_{(j)}^2/\bar{r}^2)}} \\ &= \frac{1}{M\bar{\xi}(t_2 - t_1)(\varphi_2 - \varphi_1)} \sum_{j=1}^{N_{\text{int}}} \frac{p_{\mu}^{(j)}}{\sqrt{\varepsilon_{(j)}^2 - (1 - 2/\bar{\xi}) (1 + \lambda_{(j)}^2/\bar{\xi}^2)}}, \end{aligned} \quad (65)$$

where we changed to dimensionless variables (23).

For a stationary distribution in the Schwarzschild spacetime Eq. (15) reduces to the condition $\partial_t \mathcal{F} = 0$, which implies a translational invariance of the distribution function with respect to time t . Consequently, instead of computing all spacetime components of trajectories $x_{(i)}^{\mu}(\tau) = (t_{(i)}(\tau), r_{(i)}(\tau), \theta_{(i)}(\tau), \varphi_{(i)}(\tau))$, $i = 1, \dots, N$, including the time coordinate $t_{(i)}(\tau)$, it suffices to consider their projections $(r_{(i)}(\tau), \theta_{(i)}(\tau), \varphi_{(i)}(\tau))$ on hypersurfaces of constant time. The condition that a trajectory intersects $\tilde{\sigma}$ present in Eq. (65) can be then replaced with a simpler condition that its projection onto a hypersurface of constant time intersects a segment S . Also, instead of the normalization by $(t_2 - t_1)$ in Eq. (65), one can normalize by a factor proportional to the total number of trajectories N . The details of our implementation are given in Eqs. (73) below.

V. MONTE CARLO METHOD FOR THE PLANAR ACCRETION PROBLEM

A. Selection of geodesic parameters

In the first step of our simulations we select a set of particle trajectories, corresponding to the assumed distribution function. In our simulations of the planar accretion in the Schwarzschild spacetime we use the parameters $\{\xi_0, \varphi_i^{(\text{init})}, \varepsilon_i, \lambda_i\}$, representing the radial and the azimuthal coordinates of the initial position, the energy, and the total angular momentum of i -th particle, respectively. The first coordinate is the same for all trajectories—all particles start at a fixed radius $r_0 = M\xi_0$. It is important to ensure that this value is sufficiently large. The coordinate values $\varphi_i^{(\text{init})}$ and ε_i are sampled from the planar asymptotic ($\xi \rightarrow \infty$) distribution function (A32) (derived in Appendix A):

$$f(x, p) = \alpha \delta(\sqrt{-p_{\mu} p^{\mu}} - m_0) \exp \left[-\beta \gamma \left(\varepsilon - \epsilon_r v \sqrt{\varepsilon^2 - 1} \cos \varphi \right) \right], \quad (66)$$

where we assumed that all particles have the same mass m_0 .

For a given speed v , distribution (66) depends on three parameters: $\varphi_i^{(\text{init})}$, ε_i and ϵ_r . The parameter ϵ_r determines the direction of the radial motion: $\epsilon_r = -1$ means that a particle moves towards the center of the coordinate system, $\epsilon_r = +1$ means that a particle moves in the opposite direction. See also definition (20). Absorbed particles start at $\xi = \xi_0$, and are characterized by $\epsilon_r = -1$. Our treatment of scattered particles is different. In principle we could say that all scattered particles (moving along unbound orbits) originate also at $\xi = \xi_0$ with $\epsilon_r = -1$, they reach a pericenter and then move to $\xi = \xi_0$ again, this time with $\epsilon_r = +1$. That means that a scattered trajectory of a particle originating at $(\xi_0, \varphi^{(\text{init})})$ with $\epsilon_r = -1$ contributes also to the distribution function at $(\xi_0, \varphi^{(\text{end})})$ with $\epsilon_r = +1$. While for an individual trajectory $\varphi^{(\text{end})}$ can be easily related to $\varphi^{(\text{init})}$, adjusting the overall selection procedure to the distribution function (66) is difficult. In practice we circumvent this difficulty by dividing each of the scattered trajectories in half. We consider segments running from ξ_0 to the pericenter with $\epsilon_r = -1$ and those running from the pericenter to ξ_0 with $\epsilon_r = +1$. Both segments are selected *separately* from the distribution function (66).

To randomize the parameters φ_i, ε_i according to the distribution function (66), we use the Markov Chain Monte Carlo (MCMC) method, implemented in the *Wolfram Mathematica* [32]. A sample of the *Wolfram Mathematica* code used to select the parameters of trajectories (with $\epsilon_r = -1$) looks as follows:

```

num = 10^6;
normM = NIntegrate[
  Exp[-β/Sqrt[1 - v^2]*(ε + v*Sqrt[ε^2 - 1]*Cos[φ])],
  {ε, 1, energycutoff}, {φ, 0, 2*Pi}];
distM = ProbabilityDistribution[
  Exp[-β/Sqrt[1 - v^2]*(ε + v*Sqrt[ε^2 - 1]*Cos[φ])/normM,
  {ε, 1, energycutoff}, {φ, 0, 2*Pi}];
dataM0 = RandomVariate[distM, num, Method -> {"MCMC", "Thinning" -> 2, "InitialVariance" -> 1}];

```

Here num denotes the number of generated sets $\{\varphi_i^{(\text{init})}, \varepsilon_i\}$, and normM is a normalization factor of the truncated distribution function specified as distM . In the last line we generate the set of parameters $\varphi_i^{(\text{init})}$ and ε_i , using the *Wolfram Mathematica* function `RandomVariate`. `RandomVariate` allows one to select among multiple methods of generating data, which can be specified in the `Method` parameter. The *Wolfram Mathematica* implementation of the MCMC method can be controlled by a number of parameters. In our example we set the so-called “thinning” and the “initial variance.”

The method of selecting the remaining parameter λ_i is described in detail in [31]. In our scenario, the values of λ_i are distributed uniformly. Both the energy ε_i and the angular momentum λ_i are randomized within specific limits. Their choice is arbitrary, but the numbers should be large enough to provide a wide range of different trajectories. All selected energies ε_i are smaller than a cutoff value $\varepsilon_{\text{cutoff}}$, represented in our sample code as `energycutoff`. All angular momenta are smaller than $\lambda_{\text{max}}(\varepsilon_{\text{cutoff}}, \xi_0)$. Once the parameters of individual orbits are selected, they are subject to a test. Specifically, they should satisfy the inequality

$$\lambda_i \leq \lambda_{\text{max}}(\varepsilon_i, \xi_0), \quad (67)$$

where λ_{max} is given by the Eq. (27). Parameters satisfying Eq. (67) are divided into those corresponding to absorbed trajectories—if the angular momentum λ_i is less or equal to $\lambda_c(\varepsilon_i)$ given by Eq. (26)—and those corresponding to scattered trajectories (if $\lambda_i > \lambda_c(\varepsilon_i)$).

B. Geodesics

Given initial parameters $\{\xi_0, \varphi_i^{(\text{init})}, \varepsilon_i, \lambda_i\}$, the geodesic motion of a test particle can be determined in a couple of reasonable ways. One can either express the radial distance $r = M\xi$ as a function of the azimuthal angle φ or use the inverted relation $\varphi(\xi)$. Both possibilities are described in [43]. Alternatively, it is possible to use the parametrization by the proper time [44]. All these methods are equivalent, and the choice of the most convenient one depends on the problem at hand. In Section IV B we described our method of estimating $\langle J_\mu \rangle$ on surfaces $\tilde{\Sigma}$ with a fixed radial distance $\bar{r} = M\tilde{\xi}$. Consequently, our description of geodesic trajectories should provide the values of the angle φ corresponding to a given radius ξ . This can be done with the help of the function (closely related to (39))

$$\varphi(\xi) = \epsilon_r \epsilon_\varphi \lambda \int_{\xi_0}^{\xi} \frac{d\xi'}{\xi'^2 \sqrt{\varepsilon^2 - U_\lambda(\xi')}} = \epsilon_r \epsilon_\varphi [X(\xi_0, \varepsilon, \lambda) - X(\xi, \varepsilon, \lambda)], \quad (68)$$

which was described in detail in [43] (p. 26, Eq. (C.2)). In Equation (68) we assume that ϵ_r is constant along the segment of the geodesic between ξ_0 and ξ , and that $\varphi(\xi_0) = 0$. In our previous study [31] the surfaces $\tilde{\Sigma}$ were characterized by $\varphi = \text{const}$. Consequently, we used formulas for $\xi(\varphi)$, based on the Biermann-Weierstrass theorem.

It can be shown that for unbound scattered and absorbed geodesic trajectories the integral in Eq. (68) can be expressed as

$$\varphi(\xi)^{(\text{scat})} = -\frac{\epsilon_r \epsilon_\varphi}{\sqrt{y_3 - y_1}} \left[\tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi}}{y_2 - y_1}}, k \right) - \tilde{F} \left(\arccos \sqrt{\frac{y_2 + \frac{1}{12} - \frac{1}{2\xi_0}}{y_2 - y_1}}, k \right) \right], \quad (69a)$$

$$\varphi(\xi)^{(\text{abs})} = \frac{\epsilon_\varphi}{2\sqrt{\mu}} \left[\tilde{F} \left(2 \arctan \sqrt{\frac{\frac{1}{2\xi} - \frac{1}{12} - \tilde{y}_1}{\mu}}, \tilde{k} \right) - \tilde{F} \left(2 \arctan \sqrt{\frac{\frac{1}{2\xi_0} - \frac{1}{12} - \tilde{y}_1}{\mu}}, \tilde{k} \right) \right], \quad (69b)$$

where in Eq. (69a) y_1 , y_2 , and y_3 denote the roots of the polynomial $4y^3 - g_2y - g_3$, satisfying $y_1 < y_2 < y_3$. In Equation (69b) $\tilde{y}_1$ denotes a real zero of the polynomial $4y^3 - g_2y - g_3 = 4(y - y_1)(y^2 + py + q)$, where $p^2 - 4q < 0$

and thus $y^2 + py + q > 0$. The constants g_2 and g_3 —Weierstrass invariants—are given by

$$g_2 = \frac{1}{12} - \frac{1}{\lambda^2}, \quad (70a)$$

$$g_3 = \frac{1}{6^3} - \frac{1}{12\lambda^2} - \frac{\varepsilon^2 - 1}{4\lambda^2}. \quad (70b)$$

The remaining parameters are defined as follows:

$$k^2 = \frac{y_2 - y_1}{y_3 - y_1}, \quad (71a)$$

$$\tilde{k}^2 = \frac{1}{2} \left(1 - \frac{\tilde{y}_1 + p/2}{\mu} \right), \quad (71b)$$

where $\mu = \sqrt{\tilde{y}_1^2 + p\tilde{y}_1 + q}$. Here $\tilde{F}(\phi, k)$ is the Legendre elliptic integral [34] defined by

$$\tilde{F}(\phi, k) = \int_0^\phi \frac{d\chi}{\sqrt{1 - k^2 \sin^2 \chi}}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}. \quad (72)$$

C. Monte Carlo estimators

We apply Eqs. (65) on a standard equidistant polar grid, with the radial coordinate in the range $2 < \xi \leq \tilde{\xi}_0$. The parameter $\tilde{\xi}_0$ is, in principle, arbitrary. The angle φ is discretized as $\varphi_i = i\Delta\varphi$, $i = 1, \dots, N_\varphi$, where $\Delta\varphi = 2\pi/N_\varphi$, and N_φ denotes the number of segments in the angular direction. Similarly, the radial coordinate is discretized according to $\xi_j = 2 + j\Delta\xi$, $j = 1, \dots, N_\xi$, where $\Delta\xi = (\tilde{\xi}_0 - 2)/N_\xi$. Thus for $j = N_\xi$, we have $\xi_j = \tilde{\xi}_0$, which corresponds to the outer boundary of the grid. Note that $\tilde{\xi}_0$ may be smaller than ξ_0 , which is related to the selection of the sample of trajectories. For the results presented in Sec. VD, we assumed $N_\varphi = 360$, $N_\xi = 100$, $\tilde{\xi}_0 = 20$.

Formula (65) is then applied to all segments $\tilde{\sigma}$ (Fig. 1) specified in Eq. (51), with $\varphi_1 = \varphi_{i-1}$, $\varphi_2 = \varphi_i$, and $\bar{r} = M\xi_j$, where $i = 1, \dots, N_\varphi$, $j = 1, \dots, N_\xi$, and we allow ourselves for a slight abuse of notation. For simplicity, we will keep the notation with φ_1 , φ_2 , and ξ in the remainder of this section, when referring to a single segment of the numerical grid.

Our choice of $\tilde{\sigma}$ (or S) has some advantages. An unbound absorbed trajectory crosses the circle of constant radius $\bar{r} = M\xi$ only once. A generic unbound scattered trajectory crosses a circle of constant $\bar{r} = M\xi$ twice, or it does not cross it at all. The precise angular coordinates of these crossing points can be computed from Eqs. (69a) and (69b). This stays in contrast to our previous choice made in [31], where a trajectory wrapping around the black hole could, in principle, cross a segment of a constant φ an arbitrary number of times.

The important difference with respect to our previous work [31] is related to the treatment of scattered trajectories. Ingoing and outgoing segments of scattered trajectories are selected separately, and in practice no scattered trajectory appears in our calculation as a whole one. As described in Sec. VA, this fact is not related to the discretization of the numerical grid, but rather to the asymptotic distribution function, which takes into account both ingoing and outgoing trajectories in the overall budget. All absorbed trajectories are characterized by $\epsilon_r = -1$. Ingoing segments of scattered trajectories should have $\epsilon_r = -1$; outgoing segments correspond to $\epsilon_r = +1$. There should be roughly the same number of ingoing and outgoing segments in a given simulation.

Let $I_{\text{abs}}(\varphi_1, \varphi_2)$ denote the set of indices numbering absorbed trajectories crossing a segment of a circle with $\xi = \bar{\xi}$ and $\varphi_1 \leq \varphi < \varphi_2$. In a similar way we define the sets of indices corresponding to ingoing and outgoing segments of scattered trajectories. They are denoted by $I_{\text{scat}}^-(\varphi_1, \varphi_2)$ and $I_{\text{scat}}^+(\varphi_1, \varphi_2)$, respectively. The total number of absorbed trajectories is denoted by N_{abs} . Symbols N_{scat}^- and N_{scat}^+ denote, respectively, the total numbers of ingoing and outgoing segments of scattered trajectories in the sample. Using this notation, the Monte Carlo estimators of the

particle current surface density components can be expressed as

$$\langle J_t^{(\text{abs})} \rangle = -\frac{2\alpha m_0^3 V_{\text{abs}}}{N_{\text{abs}}(\varphi_2 - \varphi_1)\bar{\xi}} \sum_{i \in I_{\text{abs}}(\varphi_1, \varphi_2)} \frac{\varepsilon_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}}, \quad (73a)$$

$$\begin{aligned} \langle J_t^{(\text{scat})} \rangle = & -\frac{2\alpha m_0^3}{(\varphi_2 - \varphi_1)\bar{\xi}} \left[\frac{V_{\text{scat}}^+}{N_{\text{scat}}^+} \sum_{i \in I_{\text{scat}}^+(\varphi_1, \varphi_2)} \frac{\varepsilon_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}} \right. \\ & \left. + \frac{V_{\text{scat}}^-}{N_{\text{scat}}^-} \sum_{i \in I_{\text{scat}}^-(\varphi_1, \varphi_2)} \frac{\varepsilon_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}} \right], \end{aligned} \quad (73b)$$

$$\langle J_r^{(\text{abs})} \rangle = -\frac{2\alpha m_0^3 V_{\text{abs}}}{N_{\text{abs}}(\varphi_2 - \varphi_1)\bar{\xi}} \sum_{i \in I_{\text{abs}}(\varphi_1, \varphi_2)} \frac{1}{(1 - 2/\bar{\xi})}, \quad (73c)$$

$$\langle J_r^{(\text{scat})} \rangle = \frac{2\alpha m_0^3}{(\varphi_2 - \varphi_1)\bar{\xi}} \left[\frac{V_{\text{scat}}^+}{N_{\text{scat}}^+} \sum_{i \in I_{\text{scat}}^+(\varphi_1, \varphi_2)} \frac{1}{(1 - 2/\bar{\xi})} - \frac{V_{\text{scat}}^-}{N_{\text{scat}}^-} \sum_{i \in I_{\text{scat}}^-(\varphi_1, \varphi_2)} \frac{1}{(1 - 2/\bar{\xi})} \right], \quad (73d)$$

$$\langle J_\varphi^{(\text{abs})} \rangle = \frac{2\alpha m_0^3 M V_{\text{abs}}}{N_{\text{abs}}(\varphi_2 - \varphi_1)\bar{\xi}} \sum_{i \in I_{\text{abs}}(\varphi_1, \varphi_2)} \frac{\epsilon_{\varphi, i} \lambda_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}}, \quad (73e)$$

$$\begin{aligned} \langle J_\varphi^{(\text{scat})} \rangle = & -\frac{2\alpha m_0^3 M}{(\varphi_2 - \varphi_1)\bar{\xi}} \left[\frac{V_{\text{scat}}^+}{N_{\text{scat}}^+} \sum_{i \in I_{\text{scat}}^+(\varphi_1, \varphi_2)} \frac{\epsilon_{\varphi, i} \lambda_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}} \right. \\ & \left. - \frac{V_{\text{scat}}^-}{N_{\text{scat}}^-} \sum_{i \in I_{\text{scat}}^-(\varphi_1, \varphi_2)} \frac{\epsilon_{\varphi, i} \lambda_i}{\sqrt{\varepsilon_i^2 - (1 - 2/\bar{\xi})(1 + \lambda_i^2/\bar{\xi}^2)}} \right], \end{aligned} \quad (73f)$$

where

$$V_{\text{abs}} = \int_0^{2\pi} \int_1^{\varepsilon_{\text{cutoff}}} \exp \left[-\beta \gamma \left(\varepsilon + v \sqrt{\varepsilon^2 - 1} \cos \varphi \right) \right] \lambda_c(\varepsilon) d\varepsilon d\varphi, \quad (74a)$$

$$V_{\text{scat}}^\pm = \int_0^{2\pi} \int_1^{\varepsilon_{\text{cutoff}}} \exp \left[-\beta \gamma \left(\varepsilon \mp v \sqrt{\varepsilon^2 - 1} \cos \varphi \right) \right] [\lambda_{\text{max}}(\varepsilon, \xi_0) - \lambda_c(\varepsilon)] d\varepsilon d\varphi, \quad (74b)$$

and the sign $\epsilon_{\varphi, i} = \pm 1$ is selected randomly from a set $\{-1, +1\}$ using *Mathematica* [32] `RandomChoice` command for all trajectories.

In the plots shown in Sec. VD, the values the estimators $\langle J_\mu(x) \rangle$ corresponding to each cell are assigned to coordinates $(\xi_j, (\varphi_i + \varphi_{i-1})/2)$. Consequently the angular coordinates change in the range $0.5^\circ \leq \varphi \leq 359.5^\circ$. To calculate the angular position of the particle, we use one of Eqs. (69a) and (69b), depending on the type of trajectory. For a given radial coordinate ξ_j , we determine the corresponding i -th cell by using the relation $\varphi_i = \lfloor \varphi(\xi_j)/\Delta\varphi \rfloor + 1$.

D. Results

Results of our Monte Carlo simulations are shown in Figs. 2–13. We compare the components of the particle current surface density J_μ and the surface density n_s computed according to exact expressions (40) with the Monte Carlo estimators (73). Figures 2–13 show the results obtained for 3 models with the following parameters: $v = 0.95$, $\beta = 1$ (Figs. 2–5), $v = 0.5$, $\beta = 1$ (Figs. 6–9), and $v = 0.5$, $\beta = 8$ (Figs. 10–13). In all cases the maximum energy `energycutoff` was set to $\varepsilon_{\text{cutoff}} = 10$. For a fair comparison, the same maximal energy is also used in the computation of integrals in Eqs. (40)—instead of improper integrals taken over energies running up to infinity, we compute the same integrals up to $\varepsilon_{\text{cutoff}}$. In all 3 models we assumed $\xi_0 = 1000$, as described in Sec. VA.

The graphs illustrating all 3 models share the same layout. In Figures (2–4), (6–8), and (10–12) we compare the components of the particle current surface density at different angles φ . Exact solutions are shown with solid lines.

Results of our Monte Carlo simulations are plotted with points. As described in Sec. V C, they are only computed in a region outside the black hole horizon. Blue and green colors are used for the components corresponding to absorbed trajectories ($J_\mu^{(\text{abs})}$) and the total current ($J_\mu = J_\mu^{(\text{abs})} + J_\mu^{(\text{scat})}$), respectively. Figures 5, 9, and 13 depict the surface particle density. To illustrate the directions of the flow, we also plot directions of the vector field (J^x, J^y) defined as

$$J^x = J^r \cos \varphi - J^\varphi r \sin \varphi, \quad (75a)$$

$$J^y = J^r \sin \varphi + J^\varphi r \cos \varphi. \quad (75b)$$

In each graph, we have marked the location of the horizon ($\xi = 2$), the photon orbit ($\xi = 3$), and the ISCO ($\xi = 6$). Additionally, the location of the marginally bound orbit ($\xi = 4$) is indicated on the figures showing radial density profiles.

Numerical results shown in our examples require generating approximately 10^8 initial geodesic parameters (this number is denoted as *num* in Sec. V A), from which we subsequently select those that satisfy Eq. (67). In the numerical examples presented in this paper we set the number of those initial parameter sets to 2×10^8 . This results in a different number of geodesics, depending on the assumed values of v and β .

All our examples exhibit a remarkable agreement between exact expressions and results of our Monte Carlo simulations. The planar model investigated in this paper shares several common features with the three dimensional model of [4, 5]. There is a bow wave in front of the black hole, roughly between photon and marginally bound circles (cf. Figs. 5, 9, 13) and a wake behind it at ISCO distance. Around the stagnation point, the particle number surface density drops to small values, see also Fig. 1 in [15]. Since in this paper we mainly concentrate on the construction of our Monte Carlo scheme, we postpone the detailed discussion of this planar accretion model to some other occasion.

VI. CONCLUDING REMARKS

The Monte Carlo method introduced in [31] and developed in this work should provide stationary solutions of the general-relativistic Vlasov equation in cases in which controlling the underlying phase-space structure is hard, and consequently exact solutions may be difficult to obtain. We have demonstrated our method by computing the particle current density, but an adaptation to other quantities, such as the components of the energy momentum tensor is straightforward.

Precise numerical simulations may, of course, be time consuming. On the other hand, two-dimensional solutions analyzed in this paper, involving the numbers of individual trajectories of the order of 10^8 , can be obtained using *Wolfram Mathematica* [32] within days on a modest computer equipped with less than 100 computing cores. We supplement this paper with a sample of *Wolfram Mathematica* packages used to produce our numerical results. They will be available at [33]. We believe that these packages can serve as an illustration of many details of our method, which are otherwise difficult to describe. A clear evidence for the robustness of our Monte Carlo technique for stationary accretion problems is presented in Fig. 5 where, except for the statistical noise, both the surface particle density and currents are nearly identical to those obtained in a direct numerical integration. This paves a road towards problems which are unsolvable analytically at present, in particular the stationary accretion onto the Kerr black hole moving in an arbitrary direction with respect to its spin.

We have applied our method specifically to collisionless non-magnetized systems, but its main elements are purely geometrical. We believe that it should allow for a generalization to general-relativistic Vlasov systems coupled with the electromagnetic field (see, e.g., [45]). In this case geodesic orbits should be replaced by trajectories of charged particles in a given spacetime and a given electromagnetic field (external or dynamically coupled to the gas).

Acknowledgments

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Appendix A: Boosted planar Maxwell-Jüttner distribution

In this appendix we derive Eq. (38), which provides the distribution function corresponding to a planar stationary accretion of a collisionless gas onto a moving Schwarzschild black hole, investigated in this paper. In the asymptotic limit the distribution function given by Eq. (38) tends to the Maxwell-Jüttner distribution boosted with a constant velocity along the x axis. Equation (38) is derived using the action-angle formalism, developed in [8], and used also in [4–6, 11, 15]. In contrast to our previous works [6, 15] dealing with planar models, we consider a three dimensional

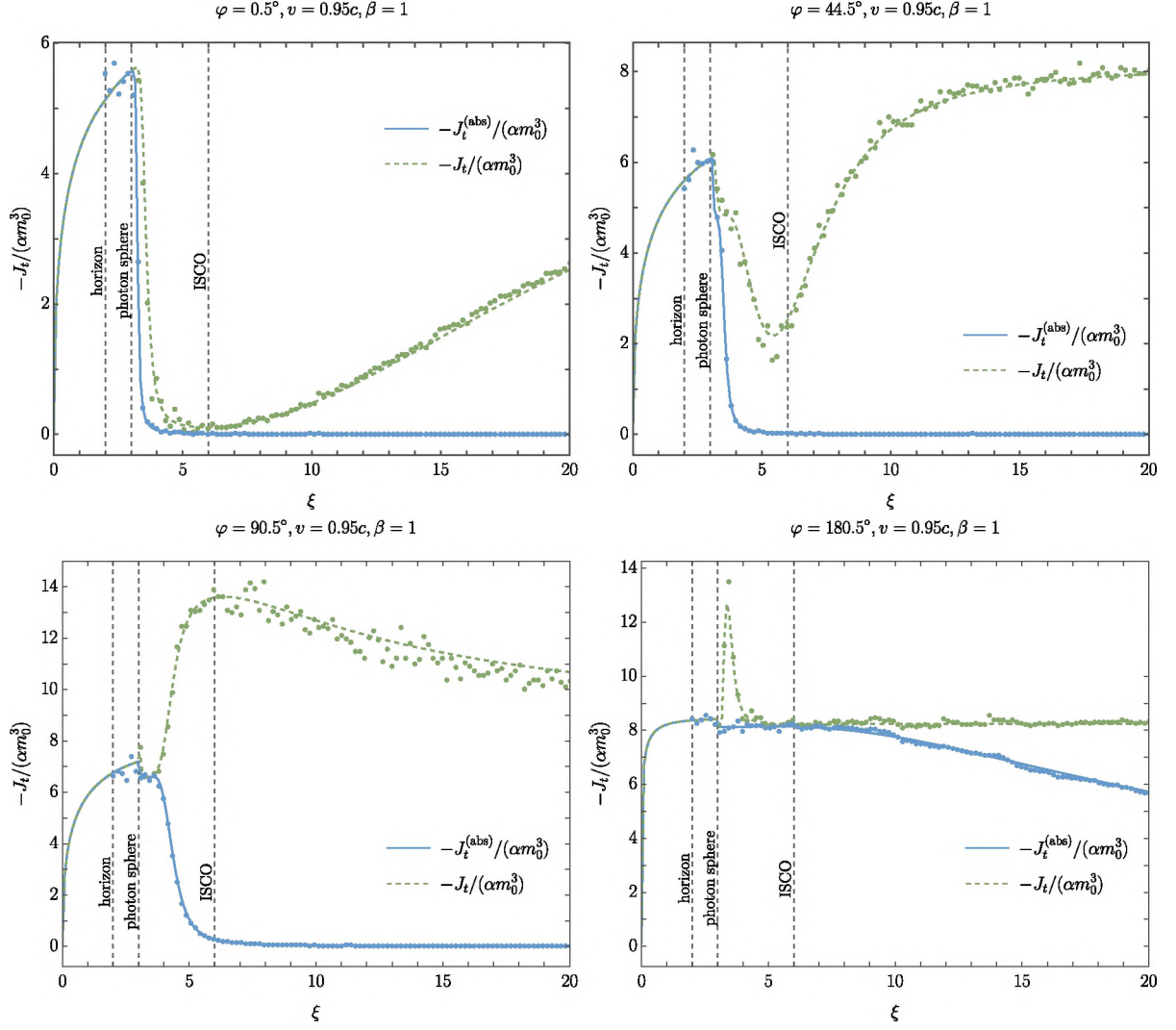


FIG. 2: Time components of the particle current surface density J_t in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

distribution corresponding to gas particles that are not confined to the equatorial plane, and impose the restriction to a single plane at the end. This calculation follows closely the footsteps outlined in [4], where the boost was applied along the z axis of standard spherical coordinates (A5). Boosting the distribution function along the x axis instead allows for the subsequent adaptation to the planar problem.

1. Boosted Maxwell-Jüttner distribution in the Minkowski spacetime

The Maxwell-Jüttner distribution describing a gas of same rest mass particles in the Minkowski spacetime endowed with the metric

$$g = -dt^2 + dx^2 + dy^2 + dz^2 \quad (\text{A1})$$

is given by

$$\mathcal{F}(x, p) = \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m_0} k^\mu p_\mu\right) = \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m_0} p_t\right), \quad (\text{A2})$$

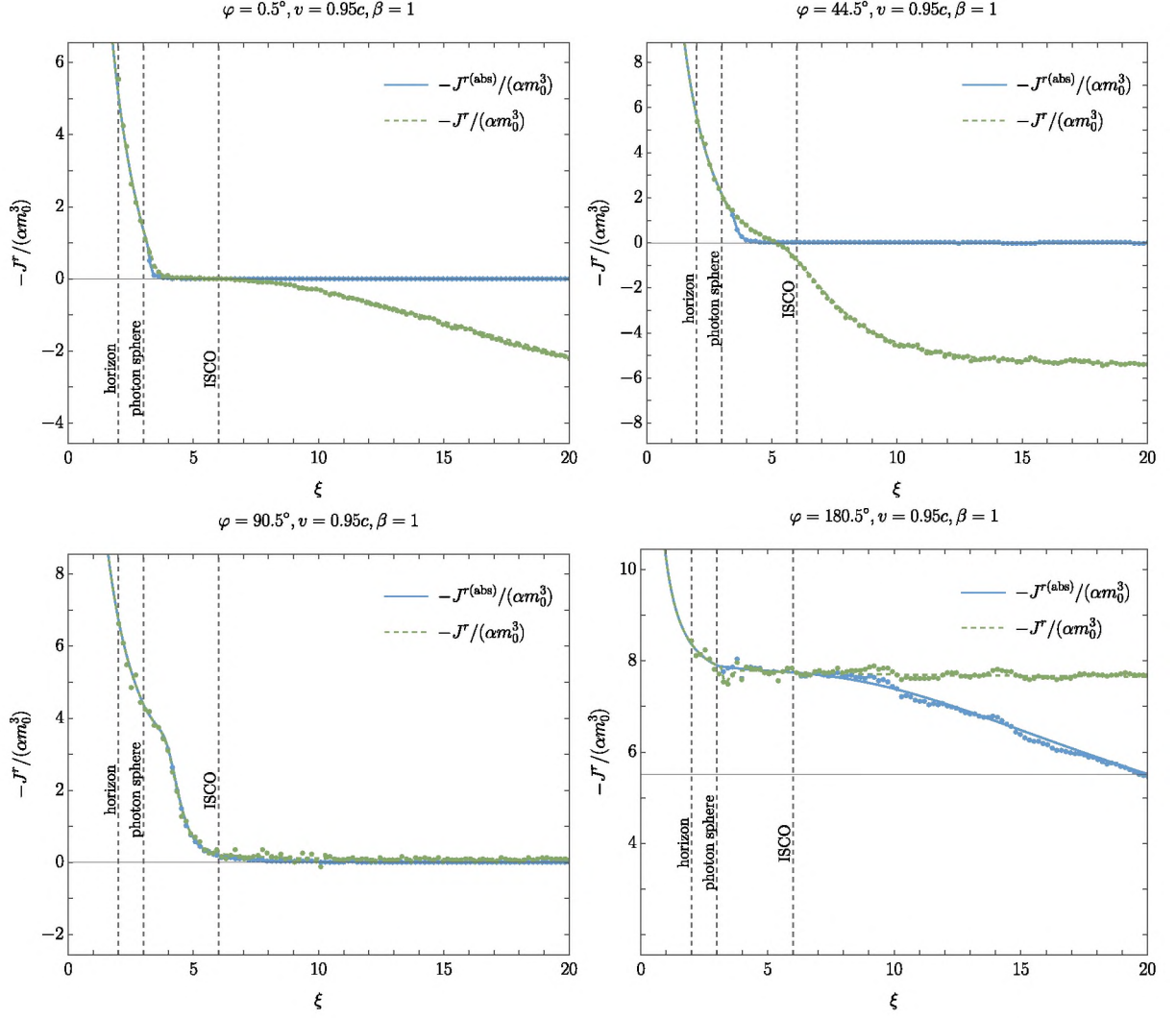


FIG. 3: Radial components of the particle current surface density J^r in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

where α and β are constants, $k^\mu = (1, 0, 0, 0)$ is a timelike Killing vector, and m_0 denotes the rest mass of a single particle.

Applying a Lorentz boost $B^\mu{}_\nu$ with the velocity v along the x -axis to this distribution is straightforward. Vector k^μ transforms as

$$k'^\mu = B^\mu{}_\nu k^\nu = (\gamma, \gamma v, 0, 0), \quad \gamma = \frac{1}{\sqrt{1 - v^2}} \quad (\text{A3})$$

in Cartesian coordinates (t, x, y, z) . Consequently, the boosted Maxwell-Jüttner distribution becomes

$$\mathcal{F}'(x, p) = \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left[\frac{\beta}{m_0} \gamma (p_t + v p_x) \right]. \quad (\text{A4})$$

Note that the gas is boosted in the positive direction of the x axis. This is equivalent to the motion of the black hole

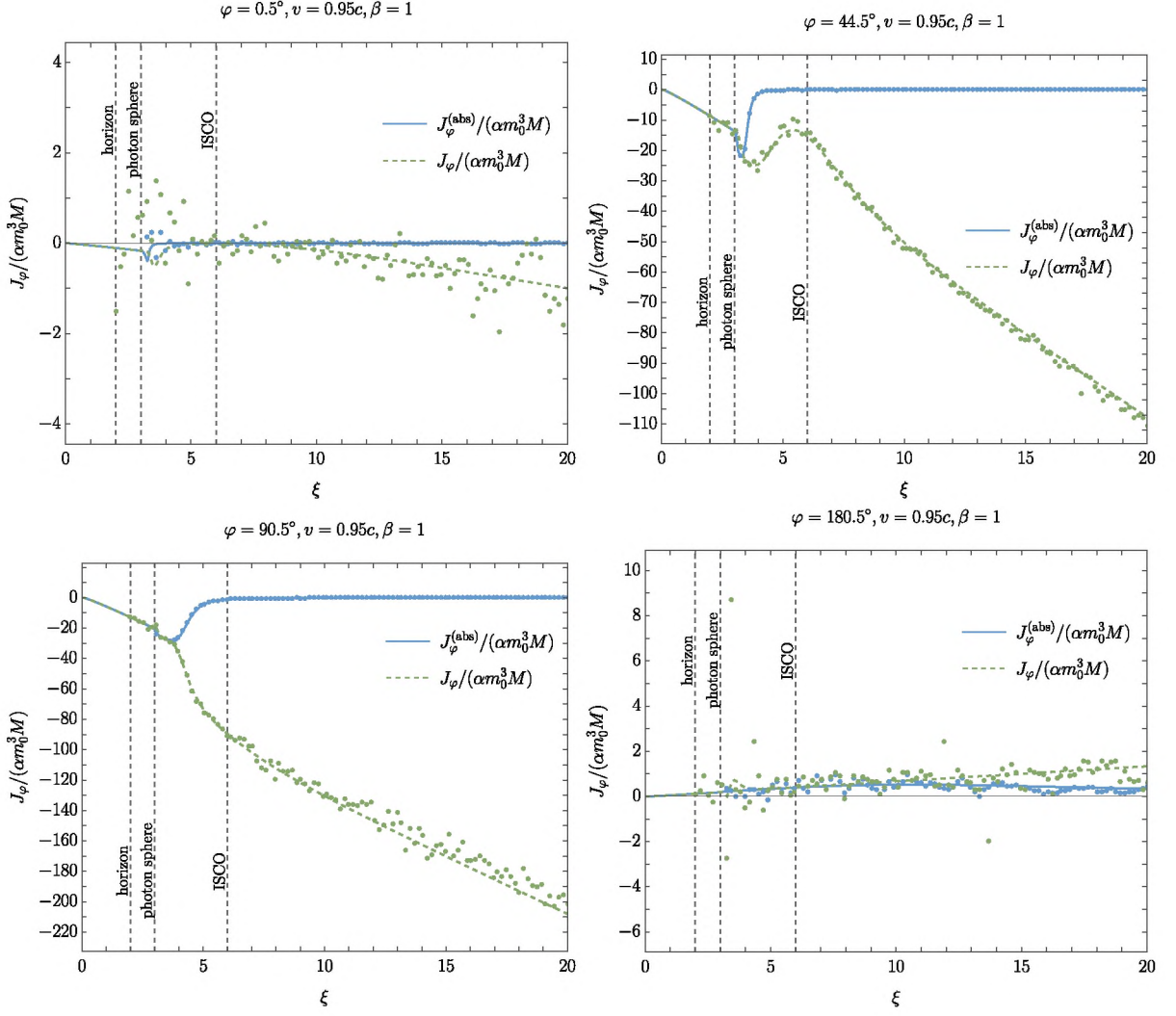


FIG. 4: Angular components of the particle current surface density J_φ in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 168 486 945 trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

in the negative direction ($\varphi = 180^\circ$) in our accretion model. By converting to spherical coordinates (t, r, θ, φ) ,

$$x = r \cos \varphi \sin \theta, \quad (\text{A5a})$$

$$y = r \sin \varphi \sin \theta, \quad (\text{A5b})$$

$$z = r \cos \theta, \quad (\text{A5c})$$

we obtain the distribution if the form

$$\mathcal{F}'(x, p) = \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ \frac{\beta}{m_0} \gamma \left[p_t + v \left(\sin \theta \cos \varphi p_r + \frac{\cos \varphi \cos \theta}{r} p_\theta - \frac{\sin \varphi}{r \sin \theta} p_\varphi \right) \right] \right\}. \quad (\text{A6})$$

For simplicity, we will henceforth omit the prime in $\mathcal{F}$.

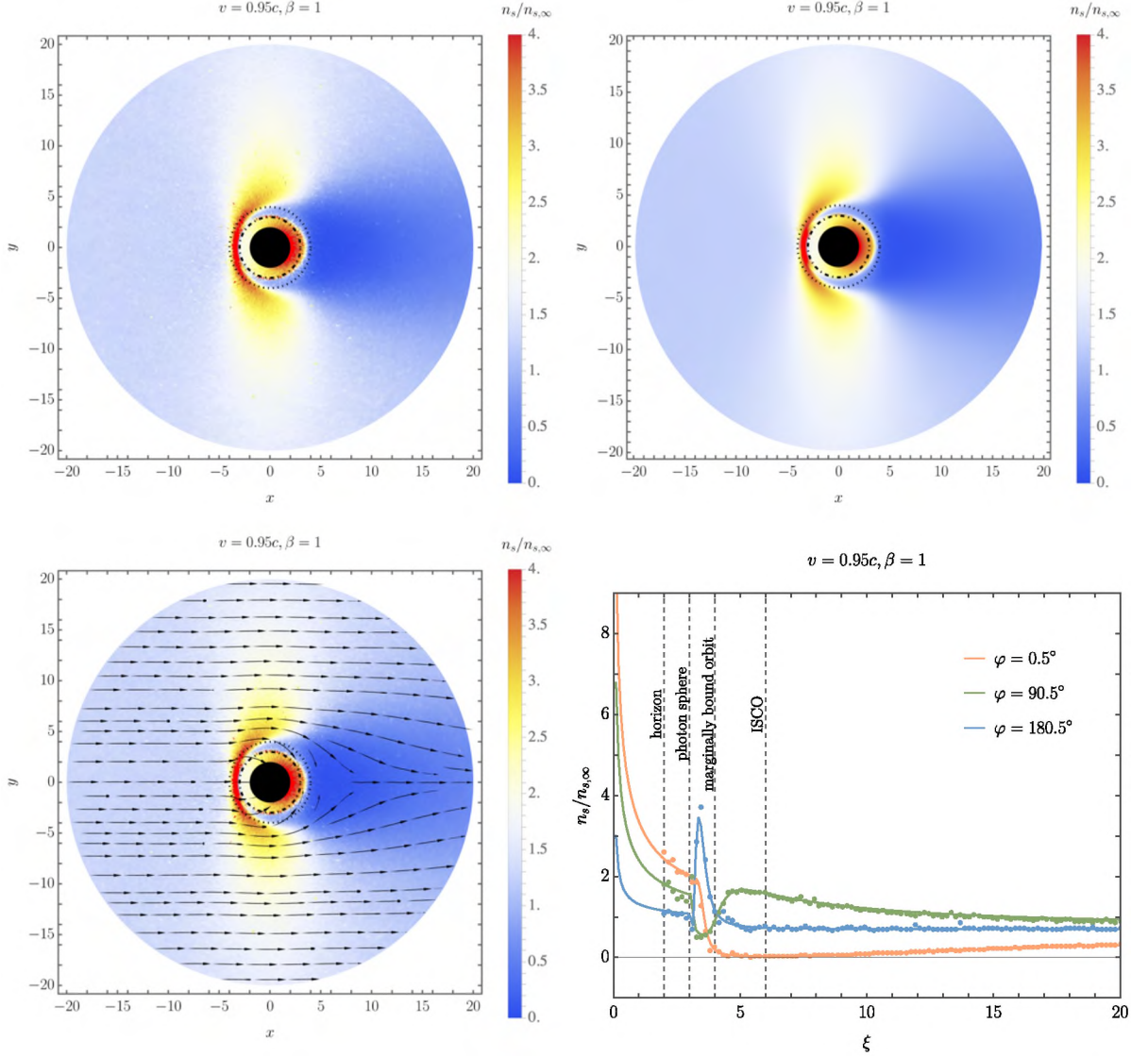


FIG. 5: The particle surface density ratio $n_s/n_{s,\infty}$ in the model with $v = 0.95$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. The black hole moves in the negative direction of the x axis, i.e., $\varphi = 180^\circ$. The first three plots show the particle number surface density in the equatorial plane. Plots on the left show the results obtained in the Monte Carlo simulation. The upper right plot shows the exact result based on Eqs. (40). The remaining graph depicts the radial density profiles for three selected values of the angle φ . The simulation was prepared for 168 486 945 particle trajectories: $N_{\text{abs}} = 450\,198$, $N_{\text{scat}}^- = 84\,010\,986$, $N_{\text{scat}}^+ = 84\,025\,761$.

2. Boosted Maxwell-Jüttner distribution and action-angle variables

The Hamiltonian describing a geodesic motion of a free particle in Minkowski spacetime in spherical coordinates is given by

$$H = \frac{1}{2} \left[-p_t^2 + p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right) \right]. \quad (\text{A7})$$

The corresponding solutions of the Hamilton equations—geodesics in the Minkowski spacetime—are simply straight lines. This formulation implies that the energy $E = -p_t$, the azimuthal component of the angular momentum $l_z = p_\varphi$,

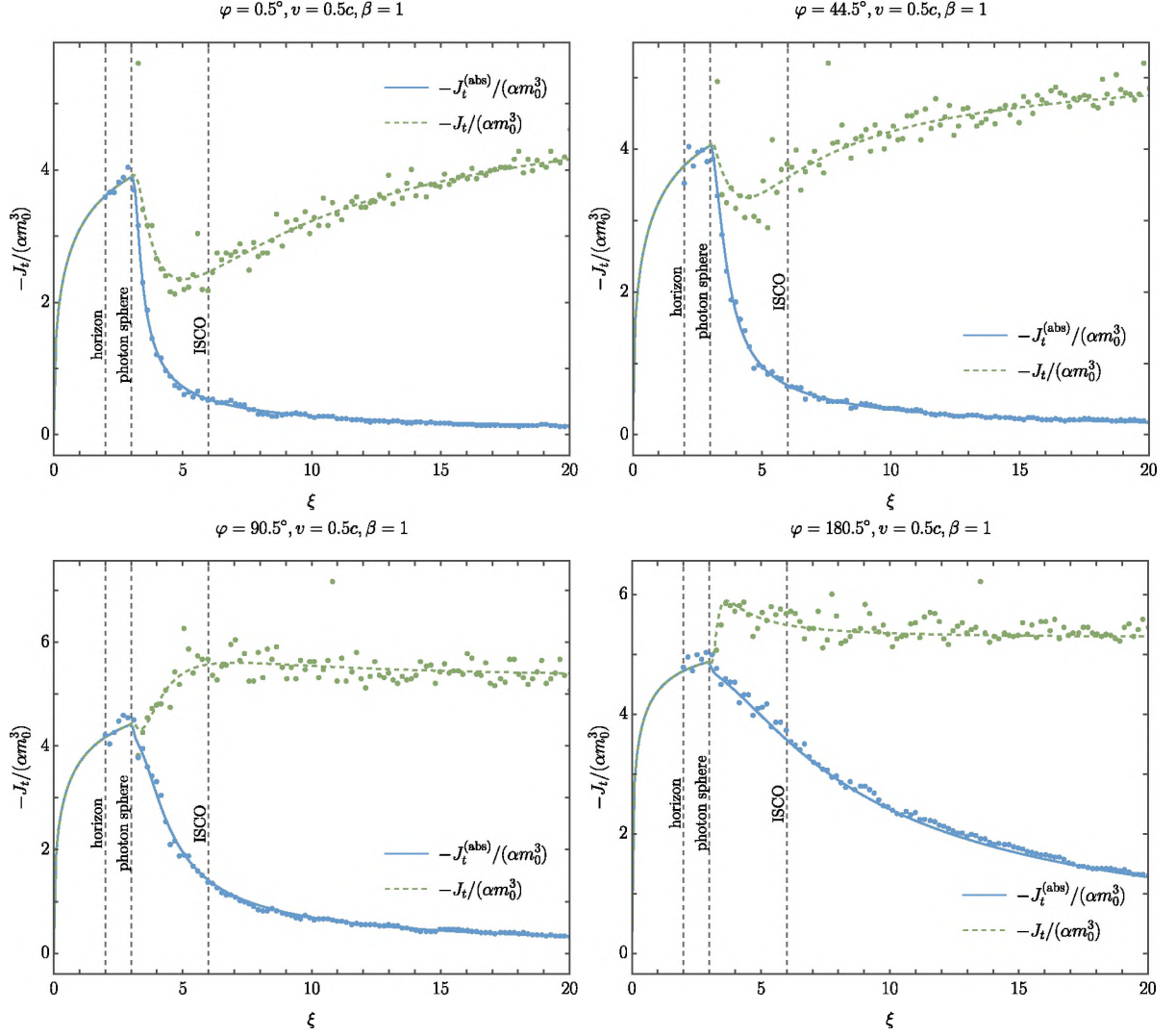


FIG. 6: Time components of the particle current surface density J_t in the model with $v = 0.5$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 78 361 180 trajectories: $N_{\text{abs}} = 227\,889$, $N_{\text{scat}}^- = 39\,064\,112$, $N_{\text{scat}}^+ = 39\,069\,179$.

and the particle rest mass $m = \sqrt{-2H}$, are all conserved. Additionally, the total angular momentum

$$l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}} \quad (\text{A8})$$

is also conserved. For geodesics with fixed values of E , l , l_z , and m , the radial momentum is determined by

$$p_r = \epsilon_r \sqrt{E^2 - m^2 - \frac{l^2}{r^2}}, \quad (\text{A9})$$

where $\epsilon_r = \pm 1$ differentiates between ingoing and outgoing particles. The component p_θ is expressed as

$$p_\theta = \epsilon_\theta \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}, \quad (\text{A10})$$

with $\epsilon_\theta = \pm 1$ similarly defined.

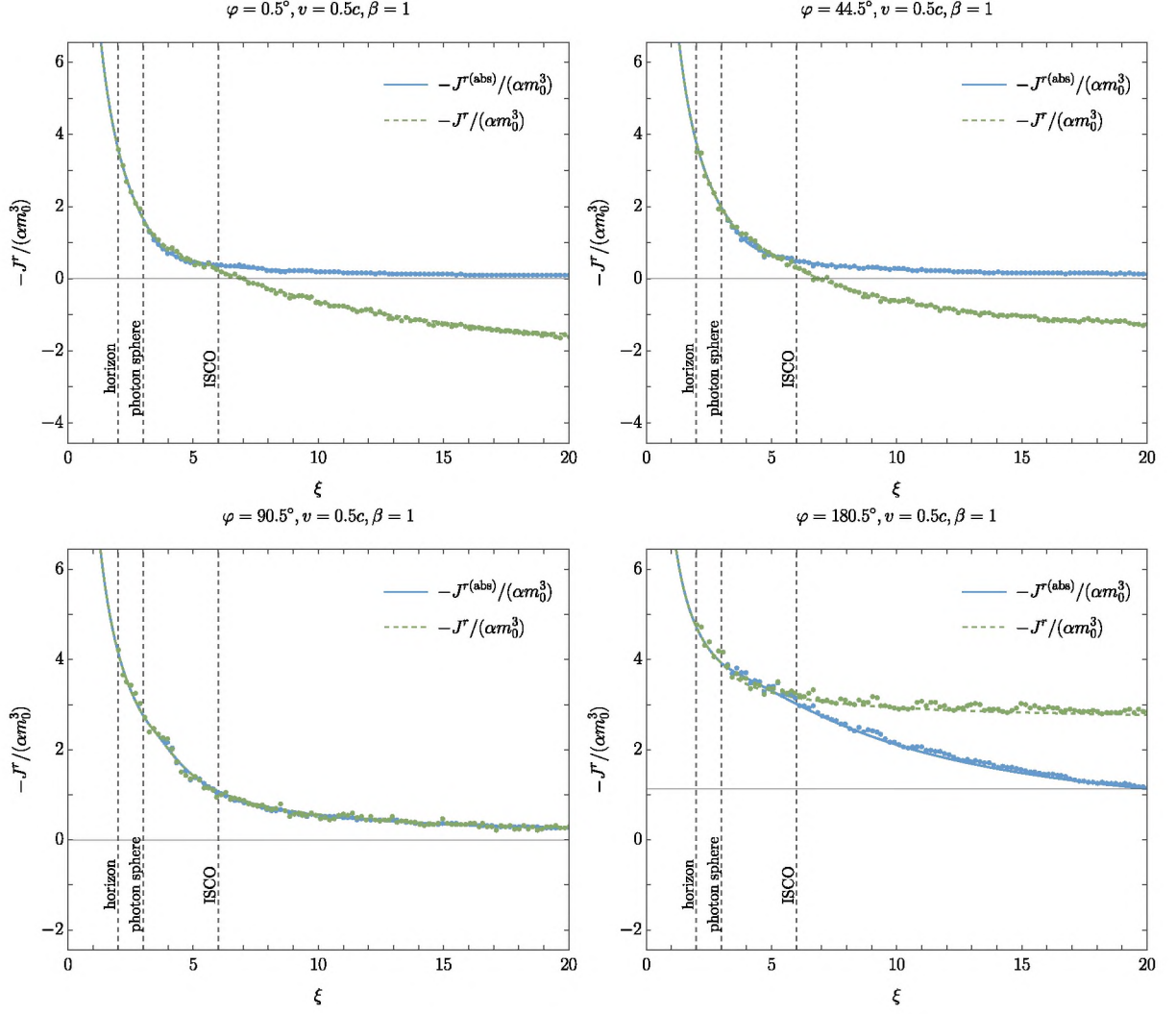


FIG. 7: Radial components of the particle current surface density J^r in the model with $v = 0.5$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Components of the particle current surface density J^r for $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 78 361 180 trajectories: $N_{\text{abs}} = 227\,889$, $N_{\text{scat}}^- = 39\,064\,112$, $N_{\text{scat}}^+ = 39\,069\,179$.

For a motion along a segment of a geodesic $\hat{\gamma}$ with constant E , l , l_z , m , and constant signs ϵ_r and ϵ_θ , we define the abbreviated action as

$$\begin{aligned}
 S &= \int_{\hat{\gamma}} p_\mu dx^\mu = -Et + l_z \varphi + \int_{\hat{\gamma}} p_r dr + \int_{\hat{\gamma}} p_\theta d\theta \\
 &= -Et + l_z \varphi + \epsilon_r \int \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} dr + \epsilon_\theta \int \sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}} d\theta \\
 &= -Et + l_z \varphi + \epsilon_r \left[r \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} - l \arctan \left(\frac{r}{l} \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} \right) \right] \\
 &\quad + \epsilon_\theta \left[-l \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) + l_z \arctan \left(\frac{l_z \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) \right] + \text{const.}
 \end{aligned} \tag{A11}$$

Here, p_r and p_θ are defined in Eqs. (A9) and (A10). We introduce the following canonical transformation:

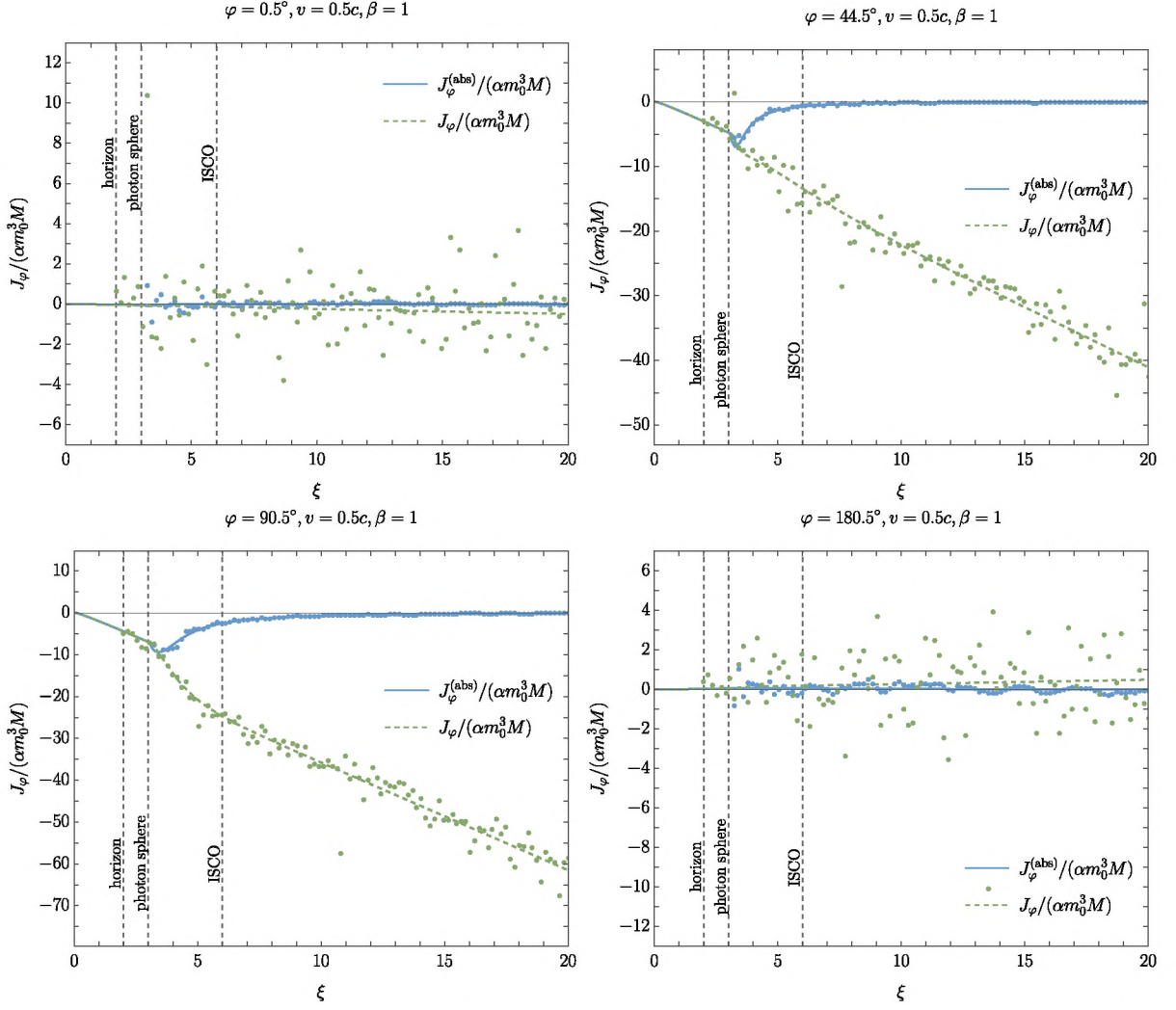


FIG. 8: Angular components of the particle current surface density J_φ in the model with $v = 0.5$, $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 78 361 180 trajectories: $N_{\text{abs}} = 227\,889$, $N_{\text{scat}}^- = 39\,064\,112$, $N_{\text{scat}}^+ = 39\,069\,179$.

$(t, r, \theta, \varphi, p_t, p_r, p_\theta, p_\varphi) \longrightarrow (Q^\mu, P_\nu)$, where the new momenta are constants of motion:

$$P_0 = m = \sqrt{p_t^2 - p_r^2 - \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right)}, \quad (\text{A12a})$$

$$P_1 = E = -p_t, \quad (\text{A12b})$$

$$P_2 = l_z = p_\varphi, \quad (\text{A12c})$$

$$P_3 = l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}, \quad (\text{A12d})$$

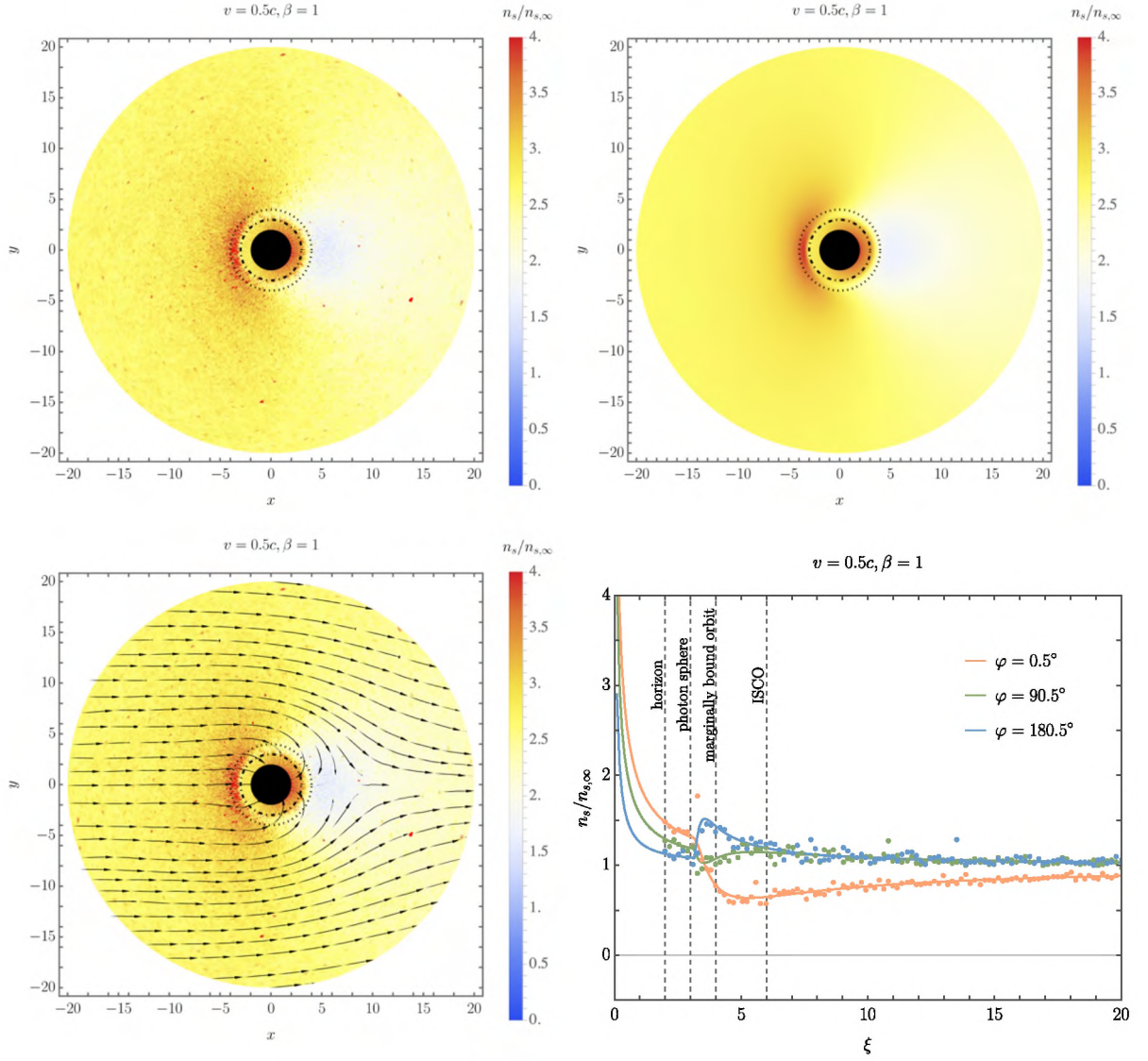


FIG. 9: Same as in Fig. 5 for the model with smaller velocity $v = 0.5$. Remaining parameters $\beta = 1$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. The simulation was prepared for 78 361 180 particle trajectories: $N_{\text{abs}} = 227\,889$, $N_{\text{scat}}^- = 39\,064\,112$, $N_{\text{scat}}^+ = 39\,069\,179$.

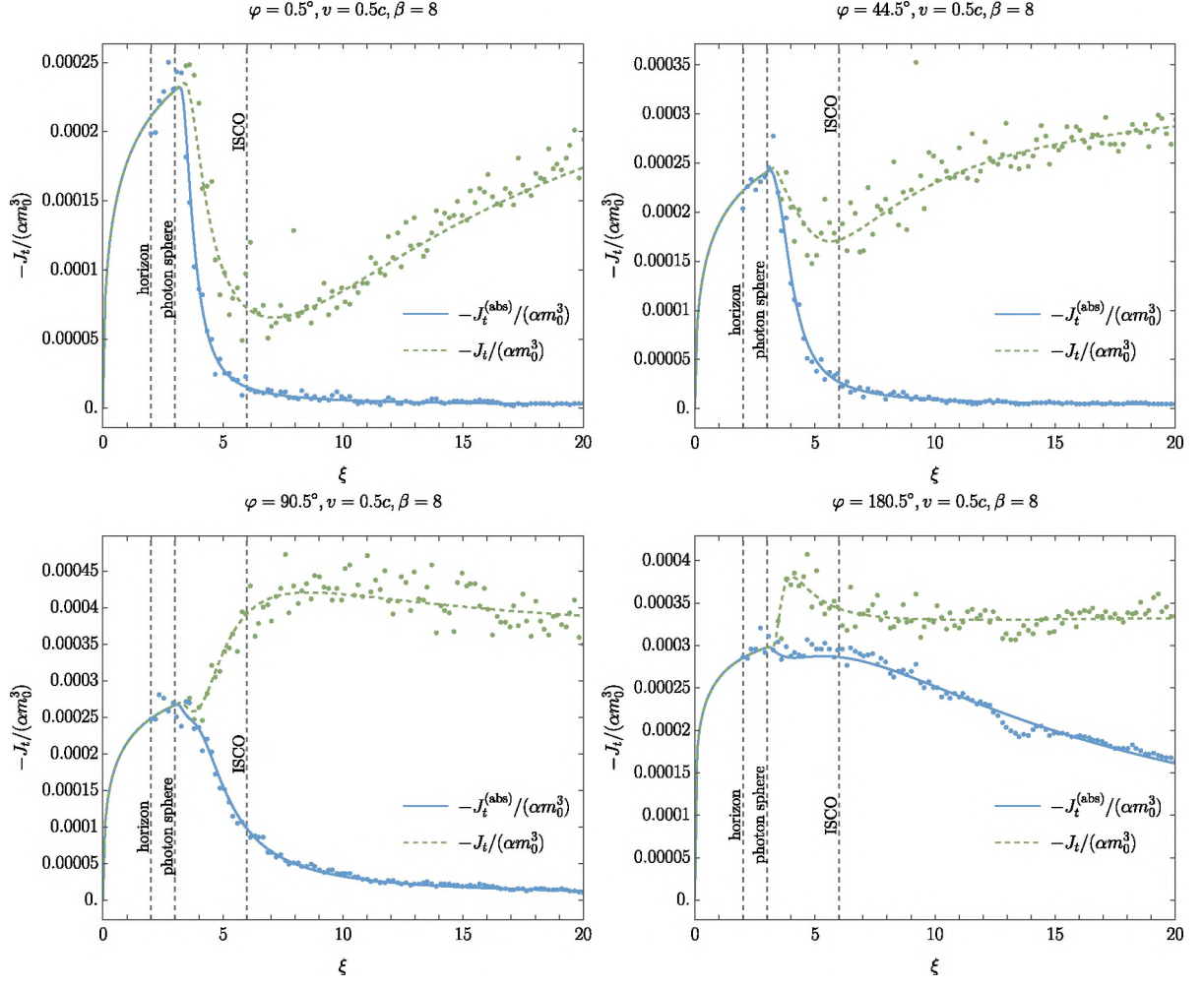


FIG. 10: Time components of the particle current surface density J_t in the model with $v = 0.5$, $\beta = 8$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 30 900 617 trajectories: $N_{\text{abs}} = 118\,561$, $N_{\text{scat}}^- = 15\,391\,121$, $N_{\text{scat}}^+ = 15\,390\,935$.

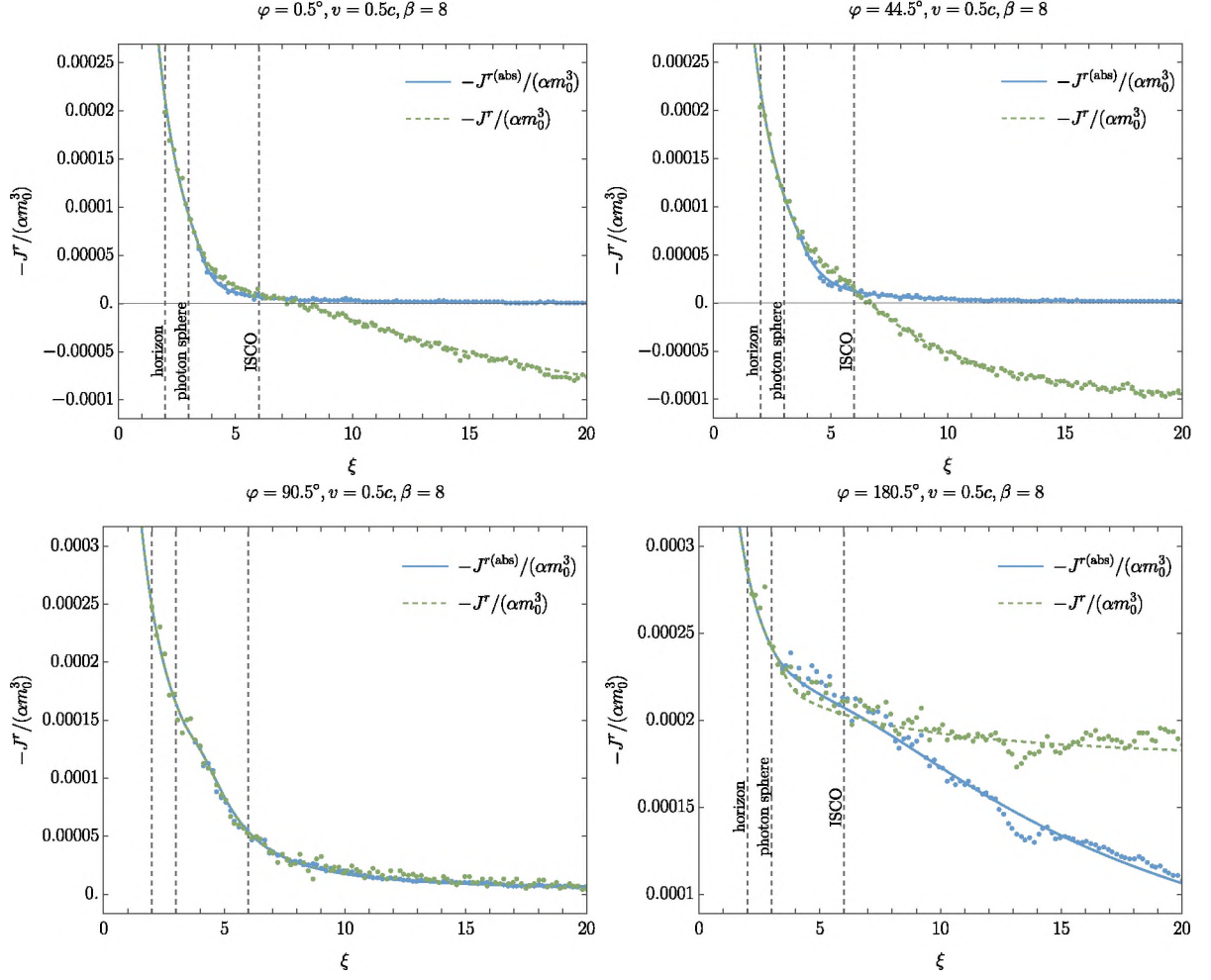


FIG. 11: Radial components of the particle current surface density J^r in the model with $v = 0.5$, $\beta = 8$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 30 900 617 particles: $N_{\text{abs}} = 118\,561$, $N_{\text{scat}}^- = 15\,391\,121$, $N_{\text{scat}}^+ = 15\,390\,935$.

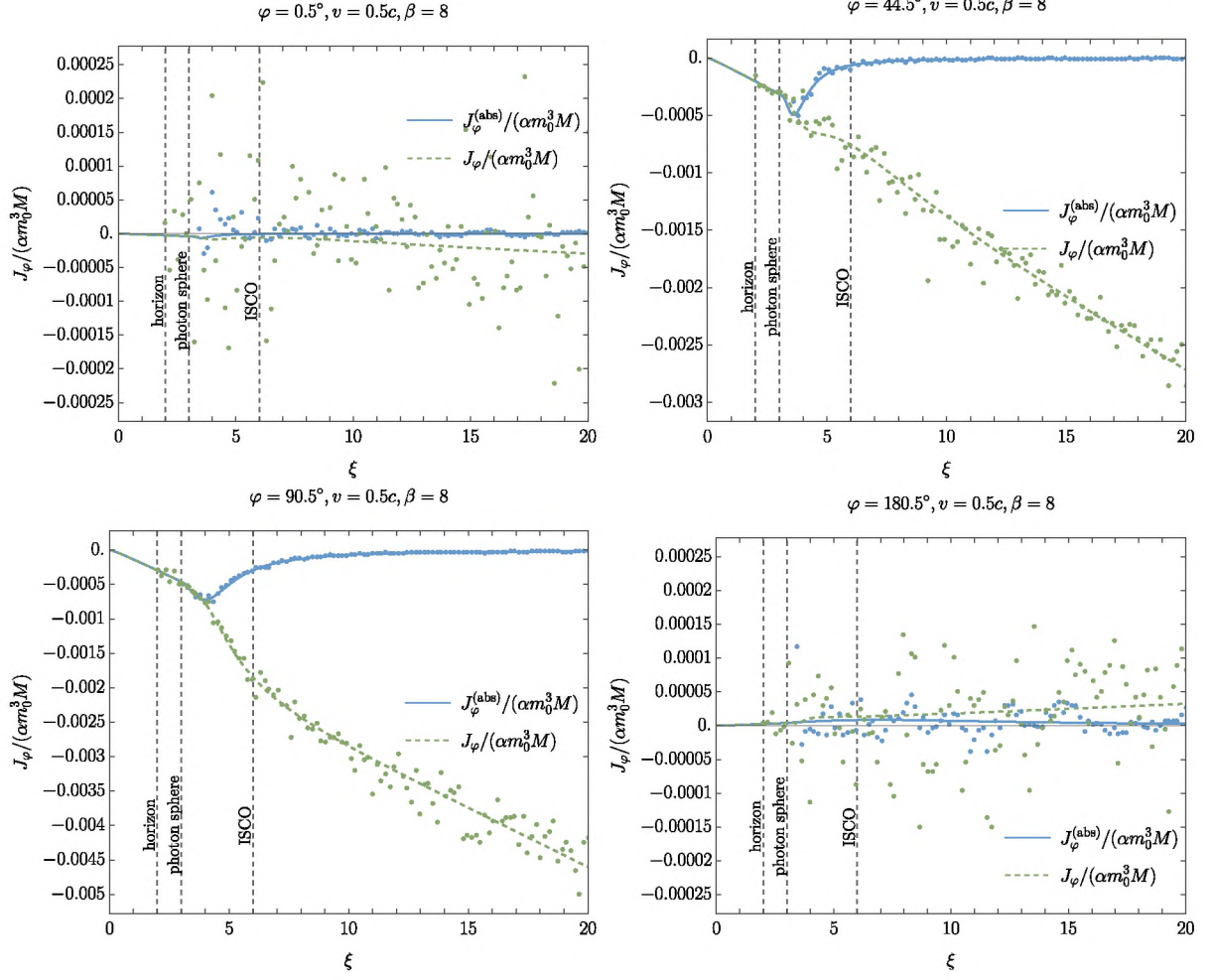


FIG. 12: Angular components of the particle current surface density J_φ in the model with $v = 0.5$, $\beta = 8$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. Exact solutions (Eqs. (40)) are plotted with solid and dashed lines. Dots (blue and green) represent sample results obtained by the Monte Carlo simulation (Eqs. (73)). There are 30 900 617 trajectories: $N_{\text{abs}} = 118\,561$, $N_{\text{scat}}^- = 15\,391\,121$, $N_{\text{scat}}^+ = 15\,390\,935$.

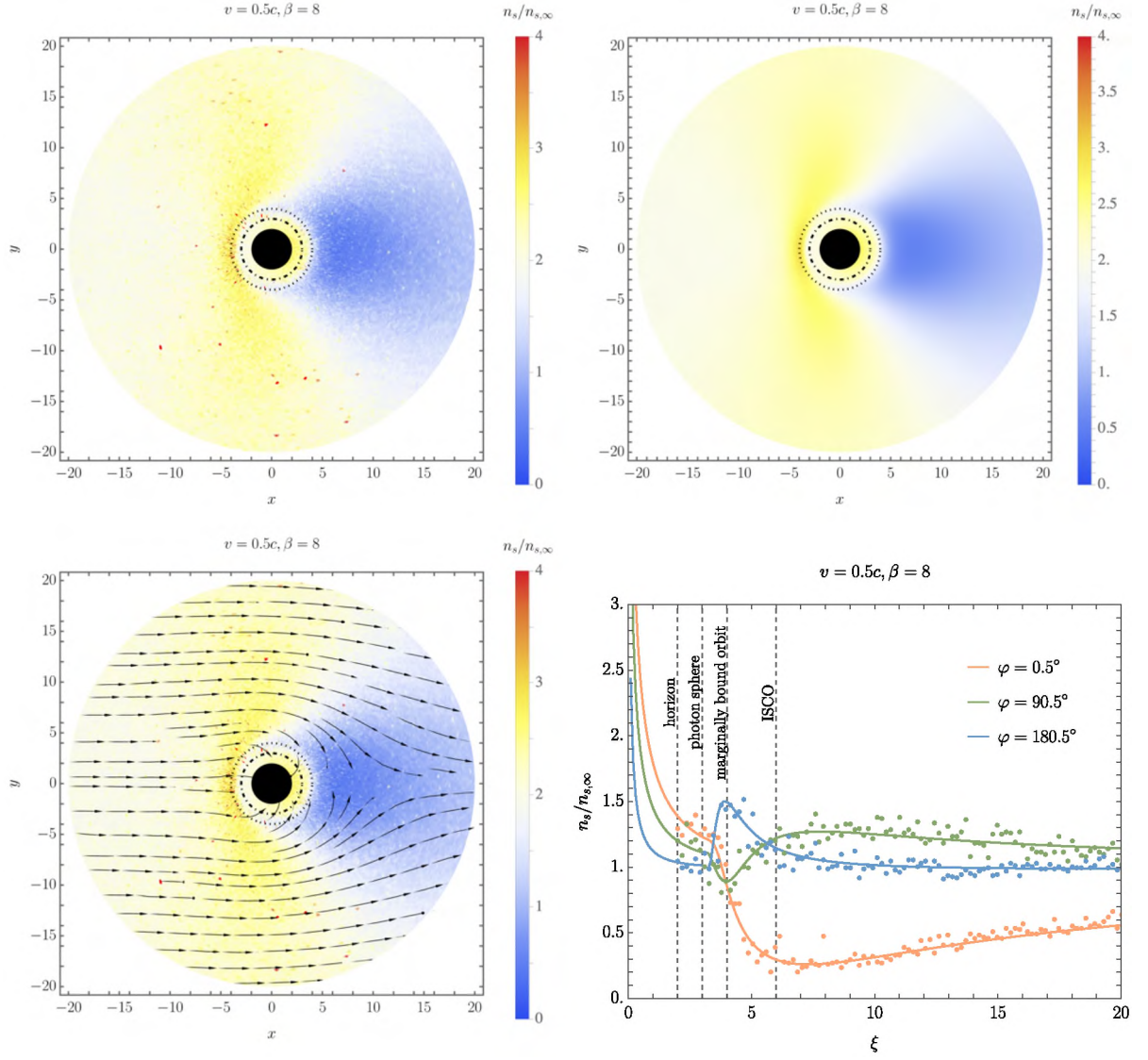


FIG. 13: Same as in Fig. 9 but with lower temperature. Parameters of the model are $v = 0.5$, $\beta = 8$, $\varepsilon_{\text{cutoff}} = 10$, and $\xi_0 = 1000$. The simulation was prepared for 30 900 617 particle trajectories: $N_{\text{abs}} = 118\,561$, $N_{\text{scat}}^- = 15\,391\,121$, $N_{\text{scat}}^+ = 15\,390\,935$.

and

$$Q^0 = \frac{\partial S}{\partial m} = -\frac{\epsilon_r m r \sqrt{E^2 - m^2 - \frac{l^2}{r^2}}}{E^2 - m^2} = -\frac{r p_r}{p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right)} \sqrt{p_t^2 - p_r^2 - \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right)}, \quad (\text{A13a})$$

$$Q^1 = \frac{\partial S}{\partial E} = -t + \frac{\epsilon_r E r \sqrt{E^2 - m^2 - \frac{l^2}{r^2}}}{E^2 - m^2} = -t - \frac{r p_t p_r}{p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right)}, \quad (\text{A13b})$$

$$Q^2 = \frac{\partial S}{\partial l_z} = \varphi + \epsilon_\theta \arctan \left(\frac{l_z \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) = \varphi + \arctan \left(\frac{p_\varphi \cot \theta}{p_\theta} \right), \quad (\text{A13c})$$

$$\begin{aligned} Q^3 = \frac{\partial S}{\partial l} &= -\epsilon_r \arctan \left(\frac{r}{l} \sqrt{E^2 - m^2 - \frac{l^2}{r^2}} \right) - \epsilon_\theta \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) \\ &= -\arctan \left(\frac{r p_r}{\sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}} \right) - \arctan \left(\frac{\sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}} \cot \theta}{p_\theta} \right). \end{aligned} \quad (\text{A13d})$$

(see [4]). In the above formulas, the last form is given in terms of the momenta $(p_t, p_r, p_\theta, p_\varphi)$. To derive expressions (A13), we assume that the constant in Eq. (A11) is independent of m , E , l_z , and l .

Basing on [4], one can note several relationships between the variables (Q^μ, P_ν) and (x^μ, p_ν) . These relationships will be useful in the following sections:

$$\frac{\cot \theta}{p_\theta} = \frac{1}{P_3} \tan \left[-Q^3 - \arctan \left(\frac{r p_r}{\sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}} \right) \right], \quad (\text{A14})$$

$$\varphi = Q^2 - \arctan \left(\frac{p_\varphi \cot \theta}{p_\theta} \right) = Q^2 - \arctan \left\{ \frac{P_2}{P_3} V \right\}, \quad (\text{A15})$$

where

$$V = \tan \left[-Q^3 - \arctan \left(\frac{r p_r}{\sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}} \right) \right]. \quad (\text{A16})$$

In a similar fashion we get

$$\sin^2 \theta = \frac{P_3^2 + P_2^2 V^2}{P_3^2 (1 + V^2)}. \quad (\text{A17})$$

3. Asymptotic relations

In the asymptotic limit of $r \rightarrow \infty$, Eq. (A6) becomes

$$\mathcal{F}(x, p) = \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ \frac{\beta}{m_0} \gamma [p_t + v \sin \theta \cos \varphi p_r] \right\}. \quad (\text{A18})$$

Following the footsteps outlined in [4], we write the following asymptotic expressions. The radial component of the four-momentum tends to

$$p_r = \epsilon_r \sqrt{E^2 - m^2} = \epsilon_r \sqrt{P_1^2 - P_0^2}. \quad (\text{A19})$$

In the asymptotic limit, (A16) takes the following form

$$V = -\tan \left(Q^3 + \epsilon_r \frac{\pi}{2} \right) = \cot Q^3, \quad (\text{A20})$$

and Eq. (A17) tends to

$$\sin \theta = \sqrt{\sin^2 Q^3 + \frac{P_2^2}{P_3^2} \cos^2 Q^3}, \quad (\text{A21})$$

where we used the fact that $0 < \theta < \pi$. Therefore $\cos \varphi$ can be now expressed as

$$\cos \varphi = \cos \left\{ Q^2 - \arctan \left[\frac{P_2}{P_3} \cot Q^3 \right] \right\} = \frac{\cos Q^2 |\sin Q^3| + \frac{P_2}{P_3} \frac{|\sin Q^3|}{\sin Q^3} \sin Q^2 \cos Q^3}{\sqrt{\sin^2 Q^3 + \frac{P_2^2}{P_3^2} \cos^2 Q^3}}. \quad (\text{A22})$$

Hence,

$$\cos \varphi \sin \theta = \cos Q^2 |\sin Q^3| + \frac{P_2}{P_3} \frac{|\sin Q^3|}{\sin Q^3} \sin Q^2 \cos Q^3, \quad (\text{A23})$$

and

$$\sin Q^3 = -\sin \left[\arctan(\omega) + \epsilon_r \frac{\pi}{2} \right] = -\epsilon_r \cos [\arctan(\omega)], \quad (\text{A24})$$

where $\omega = \frac{l \cot \theta}{p_\theta}$. Since $-\frac{\pi}{2} \leq \arctan \omega \leq \frac{\pi}{2}$, it follows that $\cos [\arctan(\omega)] \geq 0$. Consequently, one obtains

$$\sin Q^3 = -\epsilon_r |\sin Q^3|. \quad (\text{A25})$$

Finally substituting the above expression in Eq. (A18), we get

$$\mathcal{F}(x, p) = \alpha \delta(P_0 - m_0) \exp \left\{ -\frac{\beta}{m_0} \gamma \left[P_1 + v \sqrt{P_1^2 - P_0^2} \left(\cos Q^2 \sin Q^3 + \frac{P_2}{P_3} \sin Q^2 \cos Q^3 \right) \right] \right\}. \quad (\text{A26})$$

It is important to note that this expression is valid for the three-dimensional motion of the gas. In the following section, we deal with asymptotic expressions corresponding to the planar motion of the gas.

4. Planar motion in the asymptotic limit

To analyze the motion confined to the equatorial plane, we consider the limit $\theta \rightarrow \pi/2$ and $p_\theta \rightarrow 0$, leading to $l_z \rightarrow \epsilon_\varphi l$, where $\epsilon_\varphi = \text{sgn}(l_z)$. An application of those limits to (A26) is straightforward. Equations (A12) yield

$$P_2 \rightarrow \epsilon_\varphi P_3. \quad (\text{A27})$$

Consequently, using the notation (32), we have

$$\mathcal{F}(x, p) = \delta \left(\theta - \frac{\pi}{2} \right) \delta(p_\theta) f(x, p), \quad (\text{A28})$$

where

$$f(x, p) = \alpha \delta(P_0 - m_0) \exp \left\{ -\frac{\beta}{m_0} \gamma \left[P_1 + v \sqrt{P_1^2 - P_0^2} \sin(Q^3 + \epsilon_\varphi Q^2) \right] \right\}. \quad (\text{A29})$$

Additionally, from Eqs. (A13), we obtain

$$Q^3 + \epsilon_\varphi Q^2 \rightarrow \epsilon_\varphi \varphi - \epsilon_r \frac{\pi}{2}. \quad (\text{A30})$$

Therefore,

$$f(x, p) = \alpha \delta(P_0 - m_0) \exp \left\{ -\frac{\beta}{m_0} \gamma \left[P_1 + v \sqrt{P_1^2 - P_0^2} \sin \left(\epsilon_\varphi \varphi - \epsilon_r \frac{\pi}{2} \right) \right] \right\}. \quad (\text{A31})$$

Substituting (A12), we get

$$\begin{aligned} f(x, p) &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ -\beta \gamma \left[\varepsilon + \epsilon_\varphi v \sqrt{\varepsilon^2 - 1} \sin \left(\varphi - \epsilon_\varphi \epsilon_r \frac{\pi}{2} \right) \right] \right\} \\ &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ -\beta \gamma \left[\varepsilon - \epsilon_r v \sqrt{\varepsilon^2 - 1} \cos \varphi \right] \right\}. \end{aligned} \quad (\text{A32})$$

5. Action-angle variables in the Schwarzschild spacetime

As previously, we consider the motion along a segment of a geodesic $\hat{\gamma}$ with constant E , l , l_z , m , and constant signs ϵ_r and ϵ_θ , and we define the abbreviated action as

$$S = \int_{\hat{\gamma}} p_\mu dx^\mu = -Et + l_z \varphi + \int_{\hat{\gamma}} p_r dr + \int_{\hat{\gamma}} p_\theta d\theta. \quad (\text{A33})$$

In this case, p_r and p_θ are defined by Eqs. (20) and (22). We now introduce the following canonical transformation:

$$P_0 = m = \sqrt{\frac{1}{N}E^2 - Np_r^2 - \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \right)}, \quad (\text{A34a})$$

$$P_1 = E = -p_t, \quad (\text{A34b})$$

$$P_2 = l_z = p_\varphi, \quad (\text{A34c})$$

$$P_3 = l = \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}, \quad (\text{A34d})$$

and

$$Q^0 = \frac{\partial S}{\partial m} = -m\epsilon_r \int_{\hat{\gamma}} \frac{dr}{\sqrt{E^2 - U(r)}} = -m \int_{\hat{\gamma}} \frac{dr}{Np_r} = -m \int_{\hat{\gamma}} \frac{dr}{g^{rr}p_r}, \quad (\text{A35a})$$

$$Q^1 = \frac{\partial S}{\partial E} = -t + \epsilon_r E \int_{\hat{\gamma}} \frac{dr}{N\sqrt{E^2 - U(r)}} = -t + \int_{\hat{\gamma}} \frac{\frac{1}{N}E}{Np_r} dr = -t - \int_{\hat{\gamma}} \frac{g^{tt}E}{g^{rr}p_r} dr, \quad (\text{A35b})$$

$$Q^2 = \frac{\partial S}{\partial l_z} = \varphi + \epsilon_\theta \arctan \left(\frac{l_z \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) = \varphi + \arctan \left(\frac{p_\varphi \cot \theta}{p_\theta} \right), \quad (\text{A35c})$$

$$Q^3 = \frac{\partial S}{\partial l} = -\epsilon_\theta \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) - l\epsilon_r \int_{\hat{\gamma}} \frac{dr}{r^2 \sqrt{E^2 - U(r)}} \\ - \arctan \left(\frac{\sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}} \cot \theta}{p_\theta} \right) - l \int_{\hat{\gamma}} \frac{dr}{r^2 N p_r}. \quad (\text{A35d})$$

Using (39) we could reformulate Eq. (A35d) as

$$Q^3 = -\epsilon_\theta \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) + \epsilon_r X(\xi, \varepsilon, \lambda). \quad (\text{A36})$$

However, $\lim_{\xi \rightarrow \infty} X(\xi, \varepsilon, \lambda) = 0$, which would disagree with the asymptotic value of Q^3 calculated from Eq. (A13d), i.e.,

$$\lim_{r \rightarrow \infty} Q^3 = -\epsilon_r \frac{\pi}{2} - \epsilon_\theta \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right). \quad (\text{A37})$$

Therefore, to ensure consistency with the asymptotic behavior, we choose

$$Q^3 = -\epsilon_\theta \arctan \left(\frac{l \cot \theta}{\sqrt{l^2 - \frac{l_z^2}{\sin^2 \theta}}} \right) + \epsilon_r X(\xi, \varepsilon, \lambda) - \epsilon_r \frac{\pi}{2}. \quad (\text{A38})$$

6. Boosted Maxwell-Jüttner distribution on the equatorial plane

Equation (A26) yields a general form the distribution function expressed in the action-angle coordinates

$$\mathcal{F}(x, p) = \alpha \delta(P_0 - m_0) \exp \left\{ -\frac{\beta}{m_0} \gamma \left[P_1 + v \sqrt{P_1^2 - P_0^2} \left(\cos Q^2 \sin Q^3 + \frac{P_2}{P_3} \sin Q^2 \cos Q^3 \right) \right] \right\}. \quad (\text{A39})$$

As in Appendix A 4, we take a limit $\theta \rightarrow \pi/2$ and $p_\theta \rightarrow 0$, so that $l_z \rightarrow \epsilon_\varphi l$. Similarly

$$P_2 \rightarrow \epsilon_\varphi P_3, \quad (\text{A40a})$$

$$Q^3 + \epsilon_\varphi Q^2 \rightarrow \epsilon_\varphi \varphi + \epsilon_r X(\xi, \varepsilon, \lambda) - \epsilon_r \frac{\pi}{2}, \quad (\text{A40b})$$

and therefore

$$f(x, p) = \alpha \delta(P_0 - m_0) \exp \left\{ -\frac{\beta}{m_0} \gamma \left[P_1 + v \sqrt{P_1^2 - P_0^2} \sin \left[\epsilon_\varphi \varphi + \epsilon_r X(\xi, \varepsilon, \lambda) - \epsilon_r \frac{\pi}{2} \right] \right] \right\}. \quad (\text{A41})$$

Substituting Eq. (A34), one obtains the final expression

$$\begin{aligned} f(x, p) &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ -\beta \gamma \left[\varepsilon + \epsilon_\varphi v \sqrt{\varepsilon^2 - 1} \sin \left[\varphi + \epsilon_\varphi \epsilon_r \left(X(\xi, \varepsilon, \lambda) - \frac{\pi}{2} \right) \right] \right] \right\} \\ &= \alpha \delta(\sqrt{-p_\mu p^\mu} - m_0) \exp \left\{ -\beta \gamma \left[\varepsilon - \epsilon_r v \sqrt{\varepsilon^2 - 1} \cos [\varphi + \epsilon_\varphi \epsilon_r X(\xi, \varepsilon, \lambda)] \right] \right\}. \end{aligned} \quad (\text{A42})$$

Appendix B: Particle current surface density

To evaluate momentum integrals in Eq. (33), we change the variables according to $(p_t, p_r, p_\varphi) \rightarrow (m^2, E, l_z)$, where

$$m^2 = \frac{1}{N} p_t^2 - N p_r^2 - \frac{1}{r^2} p_\varphi^2, \quad (\text{B1a})$$

$$E = -p_t, \quad (\text{B1b})$$

$$l_z = p_\varphi. \quad (\text{B1c})$$

The Jacobian of this transformation is simply

$$\frac{\partial(m^2, E, l_z)}{\partial(p_t, p_r, p_\varphi)} = -2N p_r = -2\epsilon_r \sqrt{E^2 - U(r)} = -2\epsilon_r m \sqrt{\varepsilon^2 - U_\lambda}. \quad (\text{B2})$$

Consequently,

$$\frac{dp_t dp_r dp_\varphi}{r} = \frac{dm dE dl_z}{r \sqrt{\varepsilon^2 - U_\lambda}} = \frac{m^2 dm d\varepsilon d\lambda_z}{\xi \sqrt{\varepsilon^2 - U_\lambda}}, \quad (\text{B3})$$

where we introduced the dimensionless variables ε and λ_z . Thus Eq. (33) reads

$$\begin{aligned} J_\mu(t, r, \varphi) &= \int f(t, r, \varphi, p_t, p_r, p_\varphi) p_\mu \frac{dp_t dp_r dp_\varphi}{r} = \frac{1}{\xi} \int f(t, \xi, \varphi, m, \varepsilon, \lambda_z) p_\mu \frac{m^2 dm d\varepsilon d\lambda_z}{\sqrt{\varepsilon^2 - U_\lambda}} \\ &= \sum_{\epsilon_\varphi = \pm 1} \frac{1}{\xi} \int f(t, \xi, \varphi, m, \varepsilon, \epsilon_\varphi, \lambda) p_\mu \frac{m^2 dm d\varepsilon d\lambda}{\sqrt{\varepsilon^2 - U_\lambda}}, \end{aligned} \quad (\text{B4})$$

where in the last step we used the relation $\lambda_z = \epsilon_\varphi \lambda$, valid at the equatorial plane, and where we also abuse the notation by allowing ourselves to change freely the arguments in f .

Assuming f in the form of Eq. (A42) and

$$(p_t, p_r, p_\varphi) = \left(-m\varepsilon, \frac{\epsilon_r m}{N} \sqrt{\varepsilon^2 - U_\lambda}, M m \lambda_z \right) = m \left(-\varepsilon, \frac{\epsilon_r}{N} \sqrt{\varepsilon^2 - U_\lambda}, \epsilon_\varphi M \lambda \right) \quad (\text{B5})$$

we get Eqs. (40).

[1] S. L. Shapiro and S. A. Teukolsky, Relativistic stellar systems with spindle singularities, *Astrophys. J.* 419, 622 (1993).

- [2] S. L. Shapiro and S. A. Teukosky, Relativistic stellar systems with rotation, *Astrophys. J.* 419, 636 (1993).
- [3] P. Domínguez-Fernández, E. Jiménez-Vázquez, M. Alcubierre, et al., Description of the evolution of inhomogeneities on a dark matter halo with the Vlasov equation, *Gen. Relativ. Gravit.* 49, 123 (2017).
- [4] P. Mach and A. Odrzywolek, Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: Exact solutions, *Phys. Rev. D* 103, 024044 (2021).
- [5] P. Mach, A. Odrzywolek, Accretion of Dark Matter onto a Moving Schwarzschild Black Hole: An Exact Solution, *Phys. Rev. Lett.* 126, 101104 (2021).
- [6] A. Cieřlik, P. Mach and A. Odrzywolek, Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole, *Phys. Rev. D* 106, 104056 (2022).
- [7] S. L. Shapiro, Spikes and accretion of unbound, collisionless matter around black holes, *Phys. Rev. D* 108, 083037 (2023).
- [8] P. Rioseco and O. Sarbach, Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole, *Class. Quantum Grav.* 34, 095007 (2017).
- [9] P. Rioseco and O. Sarbach, Spherical steady-state accretion of a relativistic collisionless gas into a Schwarzschild black hole, *J. Phys. Conf. Ser.*, 831, 012009 (2017).
- [10] P. Rioseco and O. Sarbach, Phase space mixing in the equatorial plane of a Kerr black hole, *Phys. Rev. D* 98, 124024 (2018).
- [11] A. Cieřlik and P. Mach, Accretion of the Vlasov gas on Reissner-Nordström black holes, *Phys. Rev. D* 102, 024032 (2020).
- [12] A. Gamboa, C. Gabarrete, P. Domínguez-Fernández, D. Núñez, and Olivier Sarbach, Accretion of a Vlasov gas onto a black hole from a sphere of finite radius and the role of angular momentum, *Phys. Rev. D* 104, 083001 (2021).
- [13] C. Gabarrete and O. Sarbach, Kinetic Gas Disks Surrounding Schwarzschild Black Holes, *Acta Phys. Pol. B Proc. Suppl.* 15, 1-A10 (2022).
- [14] P. Mach, A. Odrzywolek, Accretion of the Relativistic Vlasov Gas onto a Moving Schwarzschild Black Hole: Low-temperature Limit and Numerical Aspects, *Acta Phys. Pol. B Proc. Suppl.* 15, 1-A7 (2022).
- [15] P. Mach and A. Odrzywolek, Equatorial Accretion on the Kerr Black Hole, *Acta Phys. Pol. B Proc. Suppl.* 16, 6-A13 (2023).
- [16] C. Gabarrete and O. Sarbach, Axisymmetric, stationary collisionless gas configurations surrounding Schwarzschild black holes, *Class. Quantum Grav.* 40, 055012 (2023).
- [17] P. Rioseco, O. Sarbach, Phase Space Mixing of a Vlasov Gas in the Exterior of a Kerr Black Hole, *Commun. Math. Phys.* 405, 105 (2024).
- [18] H. Andréasson and G. Rein, On the steady states of the spherically symmetric Einstein-Vlasov system, *Class. Quantum Grav.* 24, 1809 (2007).
- [19] H. Andréasson, M. Kunze, and G. Rein, Existence of axially symmetric static solutions of the Einstein-Vlasov system, *Commun. Math. Phys.* 308, 23 (2011).
- [20] H. Andréasson, The Einstein-Vlasov System/Kinetic Theory, *Living Rev. Relativ.* 14, 4 (2011).
- [21] H. Andréasson, M. Kunze, and G. Rein, Rotating, Stationary, Axially Symmetric Spacetimes with Collisionless Matter, *Commun. Math. Phys.* 329, 787 (2014).
- [22] E. Ames, H. Andréasson, and A. Logg, On axisymmetric and stationary solutions of the self-gravitating Vlasov system, *Class. Quantum Grav.* 33, 155008 (2016).
- [23] E. Ames, H. Andréasson, O. and Rinne, Dynamics of gravitational collapse in the axisymmetric Einstein-Vlasov system, *Class. Quantum Grav.* 38 105003 (2021).
- [24] H. Andréasson H, Existence of steady states of the massless Einstein-Vlasov system surrounding a Schwarzschild black hole, *Ann. Henri Poincaré* 22, 4271, (2021).
- [25] M. Hadzic, Z. Lin, and G. Rein, Stability and Instability of Self-Gravitating Relativistic Matter Distributions, *Archive for Rational Mechanics and Analysis*, 241, 1 (2021).
- [26] E. Ames, H. Andréasson, Stationary solutions of the axially symmetric Einstein-Vlasov system: present status and open problems, *Class. Quantum Grav.* 41, 073001 (2024).
- [27] K. Parfrey, A. Philippov, and B. Cerutti, First-Principles Plasma Simulations of Black-Hole Jet Launching, *Phys. Rev. Lett.* 122, 035101 (2019).
- [28] A. Bransgrove, B. Ripperda, and A. Philippov, Magnetic Hair and Reconnection in Black Hole Magnetospheres, *Phys. Rev. Lett.* 127, 055101 (2021).
- [29] B. Crinquand, B. Cerutti, G. Dubus, K. Parfrey, and A. Philippov, Synthetic Images of Magnetospheric Reconnection-Powered Radiation around Supermassive Black Holes, *Phys. Rev. Lett.* 129, 205101 (2022).
- [30] A. Galishnikova, A. Philippov, E. Quataert, F. Bacchini, K. Parfrey, and B. Ripperda, Collisionless Accretion onto Black Holes: Dynamics and Flares, *Phys. Rev. Lett.* 130, 115201 (2023).
- [31] P. Mach, A. Cieřlik, and A. Odrzywolek, Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Collisionless accretion onto black holes, *Phys. Rev. D* 108, 124057 (2023).
- [32] Wolfram Research, Inc., *Mathematica*, Version 13.2.1, Champaign, IL (2023).
- [33] A. Cieřlik, Wolfram *Mathematica* packages supplementing the article: Adam Cieřlik, Patryk Mach, Andrzej Odrzywolek, “Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems: Planar accretion onto a moving Schwarzschild black hole,” Jagiellonian University in Kraków, 2024.
- [34] NIST Digital Library of Mathematical Functions. <https://dlmf.nist.gov/19.2#E4>, Release 1.2.0 of 2024-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [35] R. O. Acuña-Cárdenas, C. Gabarrete, and O. Sarbach, An introduction to the relativistic kinetic theory on curved space-

- times, *General Relativity and Gravitation* 54, 23 (2022).
- [36] O. Sarbach and Th. Zannias, The geometry of the tangent bundle and the relativistic kinetic theory of gases, *Class. Quantum Grav.* 31, 085013 (2014).
 - [37] S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory*, North-Holland Publishing Company, Amsterdam 1980.
 - [38] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications*, Birkhäuser, Basel 2002.
 - [39] W. Israel, Relativistic Kinetic Theory of a Simple Gas, *J. Math. Phys.* 4, 1163 (1963).
 - [40] F. Jüttner, Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie, *Annal. Phys.* 339, 856 (1911).
 - [41] F. Jüttner, Die Dynamik eines bewegten Gases in der Relativtheorie, *Annal. Phys.* 340, 145 (1911).
 - [42] N. G. van Kampen, Lorentz-invariance of the distribution in phase space, *Physica* 43, 244 (1969).
 - [43] A. Cieřlik and P. Mach, Revisiting timelike and null geodesics in the Schwarzschild spacetime: general expressions in terms of Weierstrass elliptic functions, *Class. Quantum Grav.* 39 225003 (2022).
 - [44] A. Cieřlik and P. Mach, Timelike and Null Geodesics in the Schwarzschild Space-time: Analytical Solutions, *Acta Phys. Pol. B Proc. Suppl.* 16, 6-A10 (2023).
 - [45] M. Thaller, Rotating clouds of charged Vlasov matter in general relativity, *Class. Quantum Grav.* 37, 035008 (2020).

2.5. Kerr geodesics in terms of Weierstrass elliptic functions

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Kerr geodesics in terms of Weierstrass elliptic functions

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We derive novel analytical solutions describing timelike and null geodesics in the Kerr spacetime. The solutions are parametrized explicitly by constants of motion—the energy, the angular momentum, and the Carter constant—and initial coordinates. A single set of formulas is valid for all null and timelike geodesics, irrespectively of their radial and polar type. This uniformity has been achieved by applying a little-known result due to Biermann and Weierstrass, regarding solutions of a certain class of ordinary differential equations. Different from other expressions in terms of Weierstrass functions, our solution is explicitly real for all types of geodesics. In particular, for the first time the so-called transit orbits are now expressed by explicitly real Weierstrass functions.

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I. INTRODUCTION

Few results in general relativity have more astrophysical implications than the theory of timelike and null Kerr geodesics. Properties of geodesic motion in the Kerr spacetime are key in modeling accretion disks [1,2] or the propagation of light in the Kerr geometry [3,4]. Geodesic motion is used as a zero-order approximation in the self-force approach [5] and serves as a basis for our understanding of the Penrose process [6].

The theory of Kerr geodesics developed quickly after a fundamental discovery made by Carter that the Hamilton-Jacobi equation is completely separable in the Kerr spacetime [7,8]. The geometrical nature of this fact was understood by Walker and Penrose in terms of Killing tensors [9], and equatorial orbits were soon analyzed, see e.g. [10–12]. An analysis of nonequatorial geodesics is, of course, much more involved. Vortical orbits were first discussed by De Felice and Calvani in [13] and later in [14]; in [15] Wilkins described bound orbits. Another account of timelike and null geodesics can be found in Bardeen's lectures [16]. An early review of the theory of geodesics in black hole spacetimes was given by Sharp

in [17], and an extensive text-book discussion of various generic types of Kerr geodesics was provided in Chandrasekhar's book [18].

Much has been achieved in subsequent years. Another detailed text-book introduction to the theory of Kerr geodesics was published by O'Neill in [19]. Negative energy geodesics within the ergosphere were studied in [20] and later in [21,22]. Spherical orbits were investigated in [23–25]. A particularly convenient geodesic parametrization, allowing for partial decoupling of geodesic equations, was introduced by Mino in [5]. In [26] Schmidt derived action-angle variables for Kerr geodesics and computed corresponding fundamental frequencies. A combination of Schmidt's results and the Mino parametrization was given in [27]. Analytic solutions for bound timelike orbits (and the associated characteristic frequencies) were obtained in [28,29], using Jacobian elliptic functions and Legendre's integrals. Various solutions expressed in terms of Jacobian elliptic functions were also given in [30,31]. General methods of obtaining analytical solutions in terms of Weierstrass elliptic functions were described by Hackmann and Lämmerzahl in [32–34].

The phase-space picture of Kerr geodesics was investigated by Levin, Perez-Giz, Stein, and Warburton in [35–37]. In particular, Ref. [35] discusses equatorial homoclinic orbits and the separatrix between stable orbits

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and orbits plunging into the black hole. In [37] Stein and Warburton considered nonequatorial orbits as well. Special classes of orbits characterized by constants of motion that also describe circular orbits were recently analyzed by Mummery and Balbus in [38,39] (see also [40]). The separatrix problem has also been discussed earlier in the context of gravitational radiation by Glampedakis, Kennefick and O’Shaughnessy [41,42], and it appears naturally in the study of accretion of the Vlasov gas onto the Kerr black hole [43]. Periodic orbits in the Kerr spacetime were studied in [44]. Reference [45] provides a relation between scattering and bound geodesics in the Kerr spacetime using the Hamilton-Jacobi action.

In [46] Gralla and Lupsasca revisited null geodesics in the Kerr exterior, providing a convenient classification and analytic solutions. A very detailed classification of radial motion for timelike and null geodesics has recently been provided in [47].

In addition, many authors investigated geodesic motion in the near-horizon approximation for high-spin black holes (see, e.g., [48–52]). References [51,52] contain solutions for the generic polar motion in the Kerr spacetime as well. Last but not least, an important progress was recently made in the analysis of intricate properties of null Kerr geodesics, related to strictly observational problems—gravitational lensing, observable properties of photon rings, etc. [3,4,53–56].

In this paper we give exact solutions of the geodesic equations in the Kerr spacetime. Using an old, but relatively little known result by Biermann and Weierstrass [57] (see also [58] and [59–61] for the proof), we were able to find a single set of formulas, valid for all generic timelike and null geodesics, that yield an explicitly real solution for arbitrary admissible initial data. Our analysis is a sequel to Ref. [61], which is also based on the Biermann-Weierstrass theorem and provides solutions for generic timelike and null geodesics in the Schwarzschild spacetime. Benefits of this new approach are at least twofold. All solutions can be effectively written in terms of Weierstrass elliptic functions $\wp$, σ , and ζ , and a single set of formulas remains valid for all generic geodesics. Solutions are explicitly specified by prescribing constants of motion—the energy, the angular momentum, the Carter constant—and the initial position. This means, in practice, that no *a priori* knowledge about the type of the orbit is required in order to select an appropriate formula. Secondly, radial motions of the so-called transit type [19], for which the radial potential has no real zeros, can be described in a straightforward way, by explicitly real formulas.

Many geodesic types, which are usually treated as special, nongeneric cases in other approaches, are incorporated naturally in our formalism. Interesting examples of such types include “whirling” near-separatrix orbits.

All our formulas have been coded in *Wolfram Mathematica* [62], and a collection of sample solutions is

plotted in Sec. V of this paper. A *Wolfram Mathematica* package implementing our formulas is available at PANGAEA and at the GitHub [63].

The remainder of this paper is organized as follows. Section II contains preliminary material: we introduce our notation conventions, discuss a simple classification of orbital types, and recall the Biermann-Weierstrass theorem. Section III provides general solutions to the geodesic equations. We discuss the general method of deriving our solutions, but many technical details are given in Appendix A. Sections IV and V give a short description of our implementation in *Wolfram Mathematica* and a collection of examples, respectively. Section VI provides a short discussion of our results. Appendix B is devoted to Schwarzschild and extreme-Kerr limits of our solutions. In Appendix C we provide a table of integrals to which we refer in our derivation.

II. PRELIMINARIES

A. Metric conventions

We use standard geometric units with $c = G = 1$, where c denotes the speed of light, and G is the gravitational constant. The signature of the metric is $(-, +, +, +)$.

We will work in Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \varphi)$. The Kerr metric can be written as

$$g = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + 2g_{t\varphi} dt d\varphi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \quad (1)$$

where

$$g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad (2a)$$

$$g_{t\varphi} = -\frac{2Mar \sin^2 \theta}{\rho^2}, \quad (2b)$$

$$g_{rr} = \frac{\rho^2}{\Delta}, \quad (2c)$$

$$g_{\theta\theta} = \rho^2, \quad (2d)$$

$$g_{\varphi\varphi} = \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2} \right) \sin^2 \theta, \quad (2e)$$

and we denote

$$\Delta = r^2 - 2Mr + a^2, \quad (3a)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta. \quad (3b)$$

The Kerr spacetime is characterized by the mass M and the angular momentum $J = Ma$. The two solutions of the

equation $\Delta = 0$, $r_H^\pm = M \pm \sqrt{M^2 - a^2}$, correspond to the inner and outer Kerr horizons.

B. Equations of motion

Timelike and null geodesic equations can be expressed in the Hamiltonian form

$$\frac{dx^\mu}{d\tilde{\tau}} = \frac{\partial H}{\partial p_\mu}, \quad (4a)$$

$$\frac{dp_\nu}{d\tilde{\tau}} = -\frac{\partial H}{\partial x^\nu}, \quad (4b)$$

where $p^\mu = dx^\mu/d\tilde{\tau}$, $H(x^\alpha, p_\beta) = \frac{1}{2}g^{\mu\nu}(x^\alpha)p_\mu p_\nu = -\frac{1}{2}m^2$, and m denotes the particle rest mass. The above normalization of the four-momentum p^μ is a useful convention. The four-velocity u^μ is normalized as $g_{\mu\nu}u^\mu u^\nu = -\delta_1$, where

$$\delta_1 = \begin{cases} 1 & \text{for timelike geodesics,} \\ 0 & \text{for null geodesics.} \end{cases} \quad (5)$$

Thus, for timelike geodesics, $p^\mu = mu^\mu$, and the proper time can be expressed as $\tau = m\tilde{\tau}$.

By standard arguments, H , $p_t \equiv -E$ and $p_\phi \equiv l_z$ are constants of motion. The fourth constant—the so-called Carter constant $\mathcal{K}$ —follows from the separation of variables in the Hamilton-Jacobi equation [7,18]. In explicit terms, geodesic equations can be written as

$$\rho^2 \frac{dr}{d\tilde{\tau}} = \epsilon_r \sqrt{R(r)},$$

$$\rho^2 \frac{d\theta}{d\tilde{\tau}} = \epsilon_\theta \sqrt{\Theta(\theta)},$$

$$\rho^2 \frac{d\varphi}{d\tilde{\tau}} = \frac{a[(r^2 + a^2)E - al_z]}{\Delta} + \frac{1}{\sin^2\theta}(l_z - aE \sin^2\theta),$$

$$\rho^2 \frac{dt}{d\tilde{\tau}} = \frac{(r^2 + a^2)[(r^2 + a^2)E - al_z]}{\Delta} + a(l_z - aE \sin^2\theta).$$

Here

$$R(r) \equiv [(r^2 + a^2)E - al_z]^2 - \Delta(m^2 r^2 + \mathcal{K}) \quad (6)$$

and

$$\Theta(\theta) \equiv \mathcal{K} - m^2 a^2 \cos^2\theta - \left(\frac{l_z}{\sin\theta} - a \sin\theta E \right)^2. \quad (7)$$

With a slight abuse of terminology, we will refer to $R(r)$ and $\Theta(\theta)$ as radial and polar potentials, respectively. The signs ϵ_r and ϵ_θ correspond to directions of motion in r and θ , respectively. More precisely,

and

$$p_r = \epsilon_r \frac{\sqrt{R(r)}}{\Delta}, \quad (8)$$

$$p_\theta = \epsilon_\theta \sqrt{\Theta(\theta)}. \quad (9)$$

There is a convenient way to partially decouple the above equations, introduced by Mino in [5]. It consists in reparametrizing geodesics so that

$$\rho^2 \frac{dx^\mu}{d\tilde{s}} = \frac{dx^\mu}{d\tilde{s}} \quad (10)$$

or

$$\tilde{\tau} = \int_0^{\tilde{s}} \rho^2 ds. \quad (11)$$

This yields

$$\frac{dr}{d\tilde{s}} = \epsilon_r \sqrt{R(r)}, \quad (12a)$$

$$\frac{d\theta}{d\tilde{s}} = \epsilon_\theta \sqrt{\Theta(\theta)}, \quad (12b)$$

$$\frac{d\varphi}{d\tilde{s}} = \frac{a[(r^2 + a^2)E - al_z]}{\Delta} + \frac{1}{\sin^2\theta}(l_z - aE \sin^2\theta), \quad (12c)$$

$$\frac{dt}{d\tilde{s}} = \frac{(r^2 + a^2)[(r^2 + a^2)E - al_z]}{\Delta} + a(l_z - aE \sin^2\theta). \quad (12d)$$

It is convenient to work in dimensionless rescaled variables. For timelike geodesics, we define dimensionless variables as in [64,65], i.e., by

$$a = M\alpha, \quad t = MT, \quad r = M\xi, \quad E = m\epsilon, \quad \mathcal{K} = M^2 m^2 \kappa, \quad l_z = Mm\lambda_z. \quad (13)$$

In addition, the rescaled Mino time s is defined by

$$\tilde{s} = \frac{1}{Mm} s. \quad (14)$$

For null geodesics $m = 0$, and the parameter m in Eqs. (13) should be replaced with any positive mass parameter $\tilde{m} > 0$.

In terms of dimensionless variables, geodesic equations (12) can be written as

$$\frac{d\xi}{ds} = \epsilon_r \sqrt{\tilde{R}}, \quad (15a)$$

$$\frac{d\theta}{ds} = \epsilon_\theta \sqrt{\tilde{\Theta}}, \quad (15b)$$

$$\frac{d\varphi}{ds} = \frac{\alpha[(\xi^2 + \alpha^2)\varepsilon - \alpha\lambda_z]}{\xi^2 - 2\xi + \alpha^2} + \frac{1}{\sin^2\theta}(\lambda_z - \alpha\varepsilon\sin^2\theta), \quad (15c)$$

$$\frac{dT}{ds} = \frac{(\xi^2 + \alpha^2)[(\xi^2 + \alpha^2)\varepsilon - \alpha\lambda_z]}{\xi^2 - 2\xi + \alpha^2} + \alpha(\lambda_z - \alpha\varepsilon\sin^2\theta), \quad (15d)$$

where

$$\tilde{R} \equiv [(\xi^2 + \alpha^2)\varepsilon - \alpha\lambda_z]^2 - (\xi^2 - 2\xi + \alpha^2)(\delta_1\xi^2 + \kappa), \quad (16)$$

$$\tilde{\Theta} \equiv \kappa - \delta_1\alpha^2\cos^2\theta - \frac{1}{\sin^2\theta}(\lambda_z - \alpha\varepsilon\sin^2\theta)^2. \quad (17)$$

In Sec. III we give explicit solutions of Eqs. (15) for the functions $\xi(s)$, $\theta(s)$, $\varphi(s)$, and $T(s)$. They are specified by prescribing δ_1 and the constants of motion ε , λ_z , and κ , which in turn can be computed from initial velocities. Defining, for timelike geodesics, components of the velocity $v^i = u^i/u^t = p^i/p^t$, $i = r, \theta, \varphi$, we get

$$\left(\frac{p^t}{m}\right)^2 = -\frac{1}{g_{tt} + 2g_{t\varphi}v^\varphi + g_{rr}(v^r)^2 + g_{\theta\theta}(v^\theta)^2 + g_{\varphi\varphi}(v^\varphi)^2}. \quad (18)$$

Constants ε , λ_z , and κ can be expressed as

$$\varepsilon = -\frac{p^t}{m}(g_{tt} + g_{t\varphi}v^\varphi), \quad (19)$$

$$\lambda_z = \frac{p^t}{Mm}(g_{t\varphi} + g_{\varphi\varphi}v^\varphi), \quad (20)$$

$$\kappa = \frac{(p^t)^2}{M^2m^2} \left\{ \rho^4(v^\theta)^2 + \left[\frac{g_{\varphi\varphi}v^\varphi + g_{t\varphi}}{\sin\theta} + a\sin\theta(g_{tt} + g_{t\varphi}v^\varphi) \right]^2 \right\} + \alpha^2\cos^2\theta. \quad (21)$$

It is well known [18,46] that for null geodesics solutions of Eqs. (12) depend on E , l_z , and $\mathcal{K}$ via the ratios l_z/E and $\mathcal{K}/E^2$. This corresponds to the choice $\tilde{m} = E$ in Eqs. (13) and (15), which, assuming $E > 0$, yields $\varepsilon = 1$. Constants E , λ_z and κ can be then expressed as

$$E = -g_{tt}p^t - g_{t\varphi}p^\varphi, \quad (22)$$

$$\lambda_z = \frac{1}{ME}(g_{t\varphi}p^t + g_{\varphi\varphi}p^\varphi), \quad (23)$$

$$\kappa = \frac{\rho^4}{M^2E^2}(p^\theta)^2 + \frac{1}{\sin^2\theta}(\lambda_z - \alpha\sin^2\theta)^2, \quad (24)$$

where p^μ is a null vector.

C. Types of orbits

Although the main idea of this paper is to provide a single set of formulas describing solutions of all generic timelike and null geodesic trajectories, it is useful to introduce a general classification of different types of possible orbits and some terminology, which can facilitate the discussion. A detailed discussion of various orbital types can be found, e.g., in [19].

Generic orbits can be classified according to the type of radial motion. Allowing, in the discussion, for negative values of ξ , we adopt the following terminology from [19] (p. 209, Def. 4.6.3): A geodesic is said to be of transit type, if $\xi(s)$ goes from $\pm\infty$ to $\mp\infty$, as the parameter s goes from $-\infty$ to $+\infty$. A geodesics, for which $\xi(s)$ goes from $\pm\infty$ to $\pm\infty$ is called a flyby. Both types represent unbound orbits. A geodesic is said to be interval-bound, if $\xi_1 \leq \xi(s) \leq \xi_2$, for some $-\infty < \xi_1 < \xi_2 < +\infty$. All other geodesics are exceptional in some sense, and this includes orbits terminating in the $\rho^2 = 0$ singularity or geodesics related with an occurrence of multiple zeros of $\tilde{R}$. The character of generic orbits is mainly governed by the number of real zeros of the radial potential $\tilde{R}$. The following cases are generically possible:

- (I) The potential $\tilde{R}$ has no real zeros, and $\tilde{R} > 0$ for all ξ . This allows for transit orbits.
- (II) The potential $\tilde{R}$ has two real zeros $\xi_1 < \xi_2$, and $\tilde{R} > 0$ for $\xi < \xi_1$ and $\xi_2 < \xi$. This case allows for flyby orbits.
- (III) The potential $\tilde{R}$ has four real zeros $\xi_1 < \xi_2 < \xi_3 < \xi_4$, and $\tilde{R} > 0$ for $\xi_1 < \xi < \xi_2$ and $\xi_3 < \xi < \xi_4$. Interval-bound orbits are possible.
- (IV) The potential $\tilde{R}$ has four real zeros $\xi_1 < \xi_2 < \xi_3 < \xi_4$, and $\tilde{R} > 0$ for $\xi < \xi_1$, $\xi_2 < \xi < \xi_3$, and $\xi_4 < \xi$. Both flyby and interval-bound orbits are possible.
- (V) The potential $\tilde{R}$ has two real zeros $\xi_1 < \xi_2$, and $\tilde{R} > 0$ for $\xi_1 < \xi < \xi_2$. Only interval-bound orbits are possible.

Since Boyer-Lindquist coordinates used in this paper are singular at Kerr horizons ($\xi = \xi_H^\pm$), solutions of the whole system of equations (15) cannot be continued through $\xi = \xi_H^\pm$. On the other hand, the radial and polar equations (15a) and (15b) remain unaffected by horizon singularities of Boyer-Lindquist coordinates, and this fact allows for the general classification given above, without invoking explicitly regular coordinate systems.

A very detailed classification of the radial geodesic motion has recently been given in [47] for timelike trajectories. In contrast to the terminology summarized above, the authors of [47] concentrate on possible configurations of roots of $\tilde{R}$ in the region outside the outer Kerr horizon, which is of course of main physical relevance. A similar classification for null geodesics was given in [46] and subsequently slightly expanded in [47].

A classification with respect to the range of the θ coordinate is simpler [18,19]. Defining $\mu = \cos \theta$, one can express $\tilde{\Theta}$ as $\tilde{\Theta} = g(\mu)/\sin^2 \theta$, where $g(\mu)$ is a biquadratic polynomial defined in Eq. (37) below. It allows for one or two zeros in the range $0 \leq \mu^2 \leq 1$. Moreover, $g(\mu = 0) = \kappa - (\alpha\epsilon - \lambda_z)^2$ and $g(\mu = 1) = -\lambda_z^2 \leq 0$. The condition $\tilde{\Theta} > 0$, or equivalently $g(\mu) > 0$, leads to two generic types of motion. For $\kappa - (\alpha\epsilon - \lambda_z)^2 > 0$, the trajectory oscillates around the equator. In this case $\mu^2 < \mu_+^2 \leq 1$, where μ_+ is a zero of the polynomial $g(\mu)$. For $\kappa - (\alpha\epsilon - \lambda_z)^2 < 0$, if the motion is at all possible, it is restricted to one hemisphere, and the allowed range of μ^2 is $0 < \mu_-^2 \leq \mu^2 \leq \mu_+^2 \leq 1$, where again $\mu_{\pm}$ denote zeros of $g(\mu)$.

D. Biermann–Weierstrass theorem

Solutions derived in this work are largely based on the following result due to Biermann and Weierstrass. The original formulation of this theorem was published in [57]. More detailed proofs can be found in [59–61].

Let

$$f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \quad (25)$$

be a quartic polynomial, and let g_2 and g_3 denote Weierstrass invariants of f , i.e.,

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad (26a)$$

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \quad (26b)$$

Let

$$z(x) = \int_{x_0}^x \frac{dx'}{\sqrt{f(x')}}, \quad (27)$$

where x_0 is any constant, not necessarily a zero of $f(x)$. Then,

$$x = x_0 + \frac{-\sqrt{f(x_0)}\wp'(z) + \frac{1}{2}f'(x_0)[\wp(z) - \frac{1}{24}f''(x_0)] + \frac{1}{24}f(x_0)f'''(x_0)}{2[\wp(z) - \frac{1}{24}f''(x_0)]^2 - \frac{1}{48}f(x_0)f^{(4)}(x_0)}, \quad (28)$$

where $\wp(z) = \wp(z; g_2, g_3)$ is the Weierstrass function corresponding to invariants (26), i.e., the function $z(x)$ can be inverted. In addition

$$\wp(z) = \frac{\sqrt{f(x)f(x_0)} + f(x_0)}{2(x - x_0)^2} + \frac{f'(x_0)}{4(x - x_0)} + \frac{f''(x_0)}{24}, \quad (29a)$$

$$\begin{aligned} \wp'(z) = & -\left[\frac{f(x)}{(x - x_0)^3} - \frac{f'(x)}{4(x - x_0)^2} \right] \sqrt{f(x_0)} \\ & - \left[\frac{f(x_0)}{(x - x_0)^3} + \frac{f'(x_0)}{4(x - x_0)^2} \right] \sqrt{f(x)}. \end{aligned} \quad (29b)$$

III. SOLUTIONS OF THE GEODESIC EQUATIONS

A. Radial motion

A solution for $\xi(s)$ can be obtained by a direct application of the Biermann–Weierstrass theorem to Eq. (15a). The polynomial $\tilde{R}(\xi)$ can be written as

$$\tilde{R}(\xi) = a_0\xi^4 + 4a_1\xi^3 + 6a_2\xi^2 + 4a_3\xi + a_4, \quad (30)$$

where

$$a_0 = \epsilon^2 - \delta_1, \quad (31a)$$

$$a_1 = \frac{1}{2}\delta_1, \quad (31b)$$

$$a_2 = -\frac{1}{6}(\delta_1\alpha^2 + \kappa - 2\alpha^2\epsilon^2 + 2\alpha\epsilon\lambda_z), \quad (31c)$$

$$a_3 = \frac{1}{2}\kappa, \quad (31d)$$

$$a_4 = \alpha^4\epsilon^2 - \alpha^2\kappa - 2\alpha^3\epsilon\lambda_z + \alpha^2\lambda_z^2 = -\alpha^2[\kappa - (\alpha\epsilon - \lambda_z)^2]. \quad (31e)$$

For a segment of a trajectory for which ϵ_r is constant, we get

$$s = \epsilon_r \int_{\xi_0}^{\xi} \frac{d\bar{\xi}}{\sqrt{\tilde{R}(\bar{\xi})}}, \quad (32)$$

where ξ_0 is an arbitrarily chosen (initial) radius corresponding $s = 0$. Weierstrass invariants of the polynomial $\tilde{R}$ read

$$g_{\tilde{R}2} = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad (33a)$$

$$g_{\tilde{R}3} = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4. \quad (33b)$$

Using the Biermann–Weierstrass theorem, we can write the formula for $\xi = \xi(s)$ as

$$\xi(s) = \xi_0 + \frac{-\epsilon_r \sqrt{\tilde{R}(\xi_0)} \wp'_R(s) + \frac{1}{2} \tilde{R}'(\xi_0) \left[\wp_R(s) - \frac{1}{24} \tilde{R}''(\xi_0) \right] + \frac{1}{24} \tilde{R}(\xi_0) \tilde{R}'''(\xi_0)}{2 \left[\wp_R(s) - \frac{1}{24} \tilde{R}''(\xi_0) \right]^2 - \frac{1}{48} \tilde{R}(\xi_0) \tilde{R}^{(4)}(\xi_0)}, \quad (34)$$

where $\wp_R(s) = \wp(s; g_{\tilde{R},2}, g_{\tilde{R},3})$, and the polynomial $\tilde{R}$ is defined in Eqs. (30) and (31).

It is important to stress that the above solution remains valid also when the sign ϵ_r changes along a trajectory. In this case ϵ_r in Eq. (34) should be understood as an initial sign value, corresponding to $s = 0$. A longer discussion of this fact in the context of Schwarzschild geodesics can be found in [61].

Note that if ξ_0 is chosen as a root of the polynomial $\tilde{R}$, i.e., $\tilde{R}(\xi_0) = 0$, expression (34) can be reduced to

$$\xi(s) = \xi_0 + \frac{\tilde{R}'(\xi_0)}{4\wp_R(s) - \frac{1}{6} \tilde{R}''(\xi_0)}. \quad (35)$$

This can be a useful parametrization for trajectories with radial turning points. On the other hand, if no real zeros of $\tilde{R}$ exist, Eq. (34) still provides a valid, explicitly real solution. This is the case of transit orbits of radial type I.

B. Polar motion

Equation (15b) can be transformed to the Biermann-Weierstrass form by a substitution $\nu = \cos^2 \theta$ or $\mu = \cos \theta$. The first option yields a convenient form of a third order polynomial under the square root in Eq. (15b), but it does not constitute a one to one function in the region $0 \leq \theta \leq \pi$. In this work we choose the second possibility and define $\mu = \cos \theta$, which yields a one to one mapping for $0 \leq \theta \leq \pi$. We get, in this case,

$$\frac{d\mu}{ds} = -\epsilon_\theta \sin \theta \sqrt{\tilde{\Theta}} = -\epsilon_\theta \sqrt{\sin^2 \theta \tilde{\Theta}} = -\epsilon_\theta \sqrt{g(\mu)}, \quad (36)$$

where

$$g(\mu) = b_0 \mu^4 + 6b_2 \mu^2 + b_4 \quad (37)$$

and

$$b_0 = -\alpha^2(\epsilon^2 - \delta_1) = -\alpha^2 a_0, \quad (38a)$$

$$b_2 = \frac{1}{6}(-\alpha^2 \delta_1 + 2\alpha^2 \epsilon^2 - \kappa - 2\alpha \epsilon \lambda_z) = a_2, \quad (38b)$$

$$b_4 = -\alpha^2 \epsilon^2 + \kappa + 2\alpha \epsilon \lambda_z - \lambda_z^2 = \kappa - (\alpha \epsilon - \lambda_z)^2 = -\frac{a_4}{\alpha^2}. \quad (38c)$$

The fact that the coefficients b_0 , b_2 , and b_4 are directly related to the coefficients a_0 , a_2 , and a_4 is somewhat surprising, since Eqs. (15a) and (15b) are decoupled. The appropriate Weierstrass invariants read

$$g_{g,2} = b_0 b_4 + 3b_2^2, \quad (39a)$$

$$g_{g,3} = b_0 b_2 b_4 - b_2^3, \quad (39b)$$

and the solution for $\mu = \mu(s)$ can be written as

$$\mu(s) = \mu_0 + \frac{\epsilon_\theta \sqrt{g(\mu_0)} \wp'_g(s) + \frac{1}{2} g'(\mu_0) \left[\wp_g(s) - \frac{1}{24} g''(\mu_0) \right] + \frac{1}{24} g(\mu_0) g'''(\mu_0)}{2 \left[\wp_g(s) - \frac{1}{24} g''(\mu_0) \right]^2 - \frac{1}{48} g(\mu_0) g^{(4)}(\mu_0)}, \quad (40)$$

where $\wp_g(s) = \wp(s; g_{g,2}, g_{g,3})$ and $\mu_0 = \mu(0) = \cos \theta_0 = \cos \theta(0)$ is the initial value corresponding to $s = 0$. Here, as in the radial case, the sign ϵ_θ is to be understood as referring to the polar momentum component at the initial location θ_0 .

Note that due to the relation between b_0 , b_2 , b_4 and a_0 , a_2 , a_4 , we have

$$g_{\tilde{R},2} = g_{g,2} - \delta_1 \kappa, \quad (41a)$$

$$g_{\tilde{R},3} = g_{g,3} + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4. \quad (41b)$$

C. Azimuthal motion

In order to find solutions for $\varphi(s)$, we split the right-hand side of Eq. (15c) into two parts depending on ξ and θ , respectively. Integrating the result with respect to the Mino time s , we get

$$\begin{aligned} \varphi(s) - \varphi(0) &= \int_0^s \frac{\alpha[(\xi(\bar{s})^2 + \alpha^2)\epsilon - \alpha\lambda_z]}{\xi(\bar{s})^2 - 2\xi(\bar{s}) + \alpha^2} d\bar{s} \\ &\quad + \int_0^s \frac{1}{\sin^2 \theta(\bar{s})} [\lambda_z - \alpha \epsilon \sin^2 \theta(\bar{s})] d\bar{s} \\ &= I_\xi(s) + I_\theta(s), \end{aligned} \quad (42)$$

where

$$I_\xi(s) = \alpha \int_0^s \frac{2\xi(\bar{s})\varepsilon - \alpha\lambda_z}{\xi(\bar{s})^2 - 2\xi(\bar{s}) + \alpha^2} d\bar{s}, \quad (43)$$

and

$$I_\theta(s) = \lambda_z \int_0^s \frac{d\bar{s}}{\sin^2\theta(\bar{s})}. \quad (44)$$

The first step in computing the integral $I_\xi(s)$ consists in applying the partial fraction decomposition with respect to $\xi(s)$. Note that $\xi^2 - 2\xi + \alpha^2 = (\xi - \xi_H^+)(\xi - \xi_H^-)$, where $\xi_H^\pm = 1 \pm \sqrt{1 - \alpha^2}$ denote the dimensionless radii of the inner and outer Kerr horizons. This yields, assuming $-1 < \alpha < 1$, the following partial fraction decomposition

$$\begin{aligned} I_\xi(s) &= \frac{\alpha\varepsilon}{\xi_H^+ - \xi_H^-} \int_0^s \left\{ \left(\xi_H^{+2} + \alpha^2 - \frac{\alpha\lambda_z}{\varepsilon} \right) \frac{1}{\xi(\bar{s}) - \xi_H^+} \right. \\ &\quad \left. - \left(\xi_H^{-2} + \alpha^2 - \frac{\alpha\lambda_z}{\varepsilon} \right) \frac{1}{\xi(\bar{s}) - \xi_H^-} \right\} d\bar{s} \\ &= \frac{\alpha\varepsilon}{\xi_H^+ - \xi_H^-} \left[\left(\xi_H^{+2} + \alpha^2 - \frac{\alpha\lambda_z}{\varepsilon} \right) N_H^+(s) \right. \\ &\quad \left. - \left(\xi_H^{-2} + \alpha^2 - \frac{\alpha\lambda_z}{\varepsilon} \right) N_H^-(s) \right], \end{aligned} \quad (45)$$

where

$$N_H^\pm(s) := \int_0^s \frac{1}{\xi(\bar{s}) - \xi_H^\pm} d\bar{s}. \quad (46)$$

One way of computing integrals $N_H^\pm(s)$ would be to express the integrand in terms of derivatives of the Weierstrass ζ function, and to adhere to a general integration scheme described in [32]. Here, we decided to follow a more straightforward approach, which once again makes use of the Biermann-Weierstrass result. We start with the following substitution

$$u_\pm(s) = \frac{1}{\xi(s) - \xi_H^\pm}, \quad (47)$$

so that

$$N_H^\pm = \int_0^s u_\pm(\bar{s}) d\bar{s}. \quad (48)$$

Differentiating $u_\pm$ with respect to s , we get, using Eq. (15a),

$$\frac{du_\pm}{ds} = -\varepsilon_r u_\pm^2 \sqrt{\tilde{R}\left(\frac{1}{u_\pm} + \xi_H^\pm\right)} = -\varepsilon_r \sqrt{h(u_\pm)}, \quad (49)$$

where

$$h(u_\pm) \equiv c_0 u_\pm^4 + 4c_1 u_\pm^3 + 6c_2 u_\pm^2 + 4c_3 u_\pm + c_4, \quad (50)$$

and

$$c_0 = (2\xi_H^\pm \varepsilon - \alpha\lambda_z)^2, \quad (51a)$$

$$c_1 = \frac{1}{2} \{ \xi_H^\pm [-\alpha^2 \delta_1 + 2\alpha^2 \varepsilon^2 - 2\alpha\lambda_z \varepsilon + \xi_H^\pm (-2\delta_1 \xi_H^\pm + 3\delta_1 + 2\xi_H^\pm \varepsilon^2)] - \kappa(\xi_H^\pm - 1) \}, \quad (51b)$$

$$c_2 = \frac{1}{6} [-\alpha^2 (\delta_1 - 2\varepsilon^2) - \kappa - 2\alpha\varepsilon\lambda_z + 6(\delta_1 - \delta_1 \xi_H^\pm + \varepsilon^2 \xi_H^\pm) \xi_H^\pm], \quad (51c)$$

$$c_3 = \delta_1 \left(\frac{1}{2} - \xi_H^\pm \right) + \varepsilon^2 \xi_H^\pm, \quad (51d)$$

$$c_4 = \varepsilon^2 - \delta_1. \quad (51e)$$

A simpler expression for c_1 can be obtained by replacing the terms $\xi_H^\pm$ with $\xi_H^\pm = 1 \pm \sqrt{1 - \alpha^2}$. This yields

$$\begin{aligned} c_1 &= -2\alpha^2 \varepsilon^2 + \frac{1}{2} \alpha^2 \delta_1 (\pm \sqrt{1 - \alpha^2} + 2) \\ &\quad - \delta_1 (\pm \sqrt{1 - \alpha^2} + 1) \pm \frac{1}{2} \sqrt{1 - \alpha^2} (8\varepsilon^2 - \kappa) \\ &\quad - \alpha\lambda_z \varepsilon (\pm \sqrt{1 - \alpha^2} + 1) + 4\varepsilon^2. \end{aligned} \quad (52)$$

One can check that invariants of the polynomial h are the same as for the polynomial $\tilde{R}$, i.e., they are given by Eq. (33). This is not surprising. It can be easily proved [59] that a substitution (we keep the original notation of [59] at this point)

$$x = \frac{lx' + m}{l'x' + m'} \quad (53)$$

transforms a differential form $dx/\sqrt{X(x)}$, where $X(x)$ denotes a 4th order polynomial in x , into

$$\frac{(lm' - l'm)dx'}{\sqrt{X'(x')}}, \quad (54)$$

where $X'(x')$ is a 4th order polynomial of x' . The Weierstrass invariants g_2 and g_3 of the polynomial $X(x)$ and the invariants g'_2, g'_3 of $X'(x')$ are related:

$$g'_2 = (lm' - l'm)^4 g_2, \quad (55a) \quad \text{and thus}$$

$$g'_3 = (lm' - l'm)^6 g_3. \quad (55b)$$

In our case, $\xi(s) = [\xi_H^\pm u_\pm(s) + 1]/u_\pm(s)$, i.e., $m = 1$, $l = \xi_H^\pm$, $l' = 1$, and $m' = 0$ in substitution (53). Consequently $l'm - lm' = 1$, and $g'_2 = g_2$, $g'_3 = g_3$.

Before proceeding further, let us note that Eqs. (47) and (49) give

$$\frac{1}{u_\pm^2} = \frac{\sqrt{\bar{R}(\xi)}}{\sqrt{h(u_\pm)}}, \quad \xi = \frac{1}{u_\pm} + \xi_H^\pm, \quad (56)$$

$$h(u_\pm) = u_\pm^4 \bar{R}(\xi). \quad (57)$$

It follows that $h(u_\pm)$ is positive, as long as $\bar{R}(\xi)$ remains positive. Note, however, that $\xi \rightarrow \xi_H^\pm$, corresponds to $u_\pm \rightarrow \infty$, i.e., the change of variables given by Eq. (47) is singular at one of horizons.

Keeping the above fact in mind, we can now apply the Biermann-Weierstrass theorem to Eq. (49) and express $u_\pm$ as a function of s :

$$u_\pm(s) = u_{0\pm} + \frac{\epsilon_r \sqrt{h(u_{0\pm})} \wp'_R(s) + \frac{1}{2} h'(u_{0\pm}) [\wp_R(s) - \frac{1}{24} h''(u_{0\pm})] + \frac{1}{24} h(u_{0\pm}) h'''(u_{0\pm})}{2[\wp_R(s) - \frac{1}{24} h''(u_{0\pm})]^2 - \frac{1}{48} h(u_{0\pm}) h^{(4)}(u_{0\pm})}, \quad (58)$$

where $u_{0\pm} = u_\pm(0) = 1/(\xi_0 - \xi_H^\pm)$. Formula (58) is written in terms of the Weierstrass function $\wp_R = \wp(s; g_{R,2}, g_{R,3})$, since both polynomials $\bar{R}$ and h are characterized by the same Weierstrass invariants.

Expression (58) can be integrated. To simplify subsequent calculations, we introduce the following new symbols:

$$\mathcal{A}_{1\pm} = \frac{1}{48} h(u_{0\pm}) h'''(u_{0\pm}) - \frac{1}{96} h'(u_{0\pm}) h''(u_{0\pm}), \quad (59a)$$

$$\mathcal{A}_{2\pm} = \frac{1}{4} h'(u_{0\pm}), \quad (59b)$$

$$\mathcal{A}_{3\pm} = \frac{1}{2} \epsilon_r \sqrt{h(u_{0\pm})}, \quad (59c)$$

$$\mathcal{A}_{4\pm} = \frac{1}{24} h''(u_{0\pm}), \quad (59d)$$

$$\mathcal{A}_{5\pm} = \frac{1}{48} h(u_{0\pm}) h^{(4)}(u_{0\pm}), \quad (59e)$$

$$\wp_{1\pm} = \mathcal{A}_{4\pm} + \sqrt{\frac{\mathcal{A}_{5\pm}}{2}}, \quad (59f)$$

$$\wp_{2\pm} = \mathcal{A}_{4\pm} - \sqrt{\frac{\mathcal{A}_{5\pm}}{2}}. \quad (59g)$$

Hence Eq. (58) takes the form

$$u_\pm(s) = u_{0\pm} + \frac{2\mathcal{A}_{3\pm} \wp'_R(s) + 2\mathcal{A}_{2\pm} \wp_R(s) + 2\mathcal{A}_{1\pm}}{2[\wp_R(s) - \wp_{1\pm}][\wp_R(s) - \wp_{2\pm}]}, \quad (60)$$

and integral (46) can be written as

$$\begin{aligned} N_H^\pm &= \int_0^s \left\{ u_{0\pm} + \frac{\mathcal{A}_{1\pm} + \mathcal{A}_{2\pm} \wp_R(\bar{s}) + \mathcal{A}_{3\pm} \wp'_R(\bar{s})}{[\wp_R(\bar{s}) - \wp_{1\pm}][\wp_R(\bar{s}) - \wp_{2\pm}]} \right\} d\bar{s} \\ &= u_{0\pm} s + \mathcal{A}_{1\pm} K_1(s; p_{1\pm}, p_{2\pm}) + \mathcal{A}_{2\pm} K_2(s; p_{1\pm}, p_{2\pm}) \\ &\quad + \mathcal{A}_{3\pm} K_3(s; p_{1\pm}, p_{2\pm}), \end{aligned} \quad (61)$$

where

$$K_1(s; p_{1\pm}, p_{2\pm}) := \int_0^s \frac{1}{[\wp(\bar{s}) - \wp_{1\pm}][\wp(\bar{s}) - \wp_{2\pm}]} d\bar{s}, \quad (62a)$$

$$K_2(s; p_{1\pm}, p_{2\pm}) := \int_0^s \frac{\wp(\bar{s})}{[\wp(\bar{s}) - \wp_{1\pm}][\wp(\bar{s}) - \wp_{2\pm}]} d\bar{s}, \quad (62b)$$

$$K_3(s; p_{1\pm}, p_{2\pm}) := \int_0^s \frac{\wp'(\bar{s})}{[\wp(\bar{s}) - \wp_{1\pm}][\wp(\bar{s}) - \wp_{2\pm}]} d\bar{s}, \quad (62c)$$

and $p_{1\pm}$ and $p_{2\pm}$ satisfy $\wp(p_{1\pm}; g_{R,2}, g_{R,3}) = \mathcal{A}_{4\pm} + \sqrt{\frac{\mathcal{A}_{5\pm}}{2}}$

and $\wp(p_{2\pm}; g_{R,2}, g_{R,3}) = \mathcal{A}_{4\pm} - \sqrt{\frac{\mathcal{A}_{5\pm}}{2}}$. Weierstrass functions appearing in the expressions for K_1 , K_2 , and K_3 in Eq. (61) should be computed assuming invariants $g_{R,2}$ and $g_{R,3}$. Integrals (62) are calculated in Appendix C [Eqs. (C1), (C2), (C7), and (C27)–(C29)].

Note that $h^{(4)}(u_{0\pm}) = 24c_0 = 24(2\xi_H^\pm \epsilon - \alpha\lambda_\tau)^2$ is non-negative, and hence $\mathcal{A}_{5\pm}$ must also be non-negative. Thus both $\wp_{1\pm}$ and $\wp_{2\pm}$ are real. Nevertheless $p_{1\pm}$ or $p_{2\pm}$ may be complex, even though both Weierstrass invariants $g_{R,2}$, $g_{R,3}$ are real. This happens, for instance, when $\wp(p; g_{R,2}, g_{R,3})$ is strictly positive on the real line, and $\wp_{1\pm}$ or $\wp_{2\pm}$ becomes

negative. However, even in this case, the integral K_1 given by Eqs. (C1) and (C27) is explicitly real. A potentially complex term in the expression for K_1 [Eq. (C27)] has the form

$$I_1(s; p) = \frac{1}{\wp'(p)} \left[2\zeta(p)s + \ln \frac{\sigma(s-p)}{\sigma(s+p)} \right],$$

where s is real. In specific examples $I_1(s; p)$ can indeed become complex, but the imaginary part of $I_1(s; p)$ is independent of s , and hence it cancels out in the definite integral K_1 . This fact follows immediately from Eq. (C1), which we write as

$$\frac{d}{ds} I_1(s; p) = \frac{1}{\wp(s) - \wp(p)}.$$

Since both $\wp(p)$ and $\wp(s)$ are real, one gets

$$\frac{d}{ds} \text{Im} I_1(s; p) = 0.$$

In a similar way, one can demonstrate that K_2 must be real. The integral K_3 is given by an explicitly real formula.

The integral I_θ defined in Eq. (44) can be computed using the substitution $\mu = \cos \theta$. We have

$$\frac{1}{\sin^2 \theta(s)} = \frac{1}{1 - \mu^2(s)} \equiv q(s). \quad (63)$$

The derivative of $q(s)$ reads

$$\frac{dq}{ds} = 2q^2 \mu \frac{d\mu}{ds} = -2\epsilon_\theta \epsilon_\mu q^2 \sqrt{g(\mu)} = -\epsilon_\theta \epsilon_\mu \sqrt{4q^4 \mu^2 g(\mu)}, \quad (64)$$

where $\epsilon_\mu = \text{sgn} \mu$. On the other hand

$$4q^4 \mu^2 g(\mu) = 4q(q-1)[b_0(q-1)^2 + 6b_2(q^2 - q) + b_4 q^2] \equiv w(q). \quad (65)$$

Thus q satisfies the equation

$$\frac{dq(s)}{ds} = -\epsilon_\theta \epsilon_\mu \sqrt{w(q)}, \quad (66)$$

where

$$w(q) \equiv d_0 q^4 + 4d_1 q^3 + 6d_2 q^2 + d_3 q, \quad (67)$$

and

$$d_0 = -4\lambda_z^2, \quad (68a)$$

$$d_1 = -\alpha^2 \delta_1 + \kappa + 2\alpha \epsilon \lambda_z + \lambda_z^2, \quad (68b)$$

$$d_2 = \frac{2}{3} [\alpha^2 (2\delta_1 - \epsilon^2) - \kappa - 2\alpha \epsilon \lambda_z], \quad (68c)$$

$$d_3 = \alpha^2 (-\delta_1 + \epsilon^2). \quad (68d)$$

Weierstrass invariants of the polynomial $w(q)$ read

$$g_{w,2} = -4d_1 d_3 + 3d_2^2, \quad (69a)$$

$$g_{w,3} = 2d_1 d_2 d_3 - d_2^3 - d_0 d_3^2. \quad (69b)$$

Thus,

$$q(s) = q_0 + \frac{\epsilon_\theta \epsilon_\mu \sqrt{w(q_0)} \wp'_w(s) + \frac{1}{2} w'(q_0) [\wp_w(s) - \frac{1}{24} w''(q_0)] + \frac{1}{24} w(q_0) w'''(q_0)}{2[\wp_w(s) - \frac{1}{24} w''(q_0)]^2 - \frac{1}{48} w(q_0) w^{(4)}(q_0)}, \quad (70)$$

where $\wp_w(s) = \wp(s; g_{w,2}, g_{w,3})$. Since $w(q) = 4q^2 \mu^2 g(\mu)$, $w(q)$ is non-negative, whenever $g(\mu)$ remains non-negative. Note also that $q \rightarrow +\infty$ at the axes, while $\mu = 0$ at the equatorial plane. We define

$$\mathcal{B}_1 = \frac{1}{48} w(q_0) w'''(q_0) - \frac{1}{96} w'(q_0) w''(q_0), \quad (71a)$$

$$\mathcal{B}_2 = \frac{1}{4} w'(q_0), \quad (71b)$$

$$\mathcal{B}_3 = \frac{1}{2} \epsilon_\theta \epsilon_\mu \sqrt{w(q_0)}, \quad (71c)$$

$$\mathcal{B}_4 = \frac{1}{24} w''(q_0), \quad (71d)$$

$$\mathcal{B}_5 = \frac{1}{48} w(q_0) w^{(4)}(q_0), \quad (71e)$$

$$\wp_{w,1} = \mathcal{B}_4 + \sqrt{\frac{\mathcal{B}_5}{2}}, \quad (71f)$$

$$\wp_{w,2} = \mathcal{B}_4 - \sqrt{\frac{\mathcal{B}_5}{2}}. \quad (71g)$$

Hence

$$\begin{aligned}
I_\theta(s) &= \lambda_z \int_0^s q(\bar{s}) d\bar{s} \\
&= \lambda_z \int_0^s \left\{ q_0 + \frac{\mathcal{B}_1 + \mathcal{B}_2 \wp_w(\bar{s}) + \mathcal{B}_3 \wp'_w(\bar{s})}{[\wp_w(\bar{s}) - \wp_{w,1}][\wp_w(\bar{s}) - \wp_{w,2}]} \right\} d\bar{s} \\
&= \lambda_z [q_0 s + \mathcal{B}_1 K_1(s; p_{w,1}, p_{w,2}) \\
&\quad + \mathcal{B}_2 K_2(s; p_{w,1}, p_{w,2}) + \mathcal{B}_3 K_3(s; p_{w,1}, p_{w,2})], \quad (72)
\end{aligned}$$

where $p_{w,1}$ and $p_{w,2}$ satisfy $\wp(p_{w,1}; g_{w,2}, g_{w,3}) = \mathcal{B}_4 + \sqrt{\frac{\mathcal{B}_5}{2}}$ and $\wp(p_{w,2}; g_{w,2}, g_{w,3}) = \mathcal{B}_4 - \sqrt{\frac{\mathcal{B}_5}{2}}$. Moreover, Weierstrass functions appearing in the expressions for K_1, K_2, K_3 in the above formula should be computed with the invariants $g_{w,2}$ and $g_{w,3}$.

Integrals K_1, K_2 , and K_3 appearing in Eq. (72) are again real, although the constants $p_{w,1}, p_{w,2}$ are in general complex. The situation encountered here is, however, different than the one described for integrals $N_H^\pm(s)$. We

have $\mathcal{B}_5 \leq 0$, and thus $\wp_{w,1}$ and $\wp_{w,2}$ are, in general, complex. The fact that the integral K_1 is real can be demonstrated as follows. Observe that

$$\wp(p_1) = \mathcal{B}_4 + \sqrt{\frac{\mathcal{B}_5}{2}} = \overline{\wp(p_2)} = \wp(\bar{p}_2), \quad (73)$$

where the bar denotes complex conjugation, and where, for simplicity, we omit the Weierstrass invariants $g_{w,2}$ and $g_{w,3}$, as well as the subscript w in the symbols $p_{w,1}$ and $p_{w,2}$. Therefore, p_1 and $\bar{p}_2$ are the two solutions, in the period parallelogram, of the equation $\wp(x) = \mathcal{B}_4 + \sqrt{\frac{\mathcal{B}_5}{2}}$, and hence they are related by

$$\bar{p}_2 = z_{NM} - p_1, \quad (74)$$

where $z_{NM} = 2N\omega + 2M\omega'$ is a period of $\wp$, and N, M are integers. Keeping in mind that $\bar{z}_{NM} = z_{nm}$ is a period of $\wp$ as well (this happens for real Weierstrass invariants), we find the following relations:

$$\wp'(p_2) = \wp'(z_{nm} - \bar{p}_1) = -\wp'(\bar{p}_1) = -\overline{\wp'(p_1)}, \quad (75)$$

$$\zeta(p_2) = \zeta(z_{nm} - \bar{p}_1) = -\zeta(\bar{p}_1) + 2n\eta + 2m\eta' = -\overline{\zeta(p_1)} + 2n\eta + 2m\eta', \quad (76)$$

$$\begin{aligned}
\sigma(x + p_2) &= \sigma(x - \bar{p}_1 + z_{nm}) \\
&= (-1)^{n+m+nm} \sigma(x - \bar{p}_1) \exp [2(n\eta + m\eta')(n\omega + m\omega' + x - \bar{p}_1)], \quad (77)
\end{aligned}$$

$$\begin{aligned}
\sigma(x - p_2) &= \sigma(x + \bar{p}_1 - z_{nm}) \\
&= (-1)^{-n-m+nm} \sigma(x + \bar{p}_1) \exp [2(-n\eta - m\eta')(-n\omega - m\omega' + x + \bar{p}_1)], \quad (78)
\end{aligned}$$

$$\frac{\sigma(x - p_2)}{\sigma(x + p_2)} = \frac{\sigma(x + \bar{p}_1)}{\sigma(x - \bar{p}_1)} \exp [-4(n\eta + m\eta')x], \quad (79)$$

where η and η' are the periods of the second kind. Inserting the above formulas in the expressions for integrals K_1 and K_2 [see Eqs. (C1), (C7), (C27), and (C28)], we find, for instance,

$$\begin{aligned}
K_1 &= \frac{1}{\wp(p_1) - \wp(\bar{p}_1)} \left[2s \left(\frac{\zeta(p_1)}{\wp'(p_1)} - \frac{\zeta(\bar{p}_1)}{\wp'(\bar{p}_1)} \right) + \frac{4s(n\eta + m\eta')}{\wp'(\bar{p}_1)} + \frac{1}{\wp'(p_1)} \ln \frac{\sigma(s - p_1)}{\sigma(s + p_1)} \right. \\
&\quad \left. + \frac{1}{\wp'(\bar{p}_1)} \ln \left(\frac{\sigma(s + \bar{p}_1)}{\sigma(s - \bar{p}_1)} e^{-4s(n\eta + m\eta')} \right) \right] \\
&= \frac{1}{\wp(p_1) - \wp(\bar{p}_1)} \left[2s \left(\frac{\zeta(p_1)}{\wp'(p_1)} - \frac{\zeta(\bar{p}_1)}{\wp'(\bar{p}_1)} \right) + \frac{1}{\wp'(p_1)} \ln \frac{\sigma(s - p_1)}{\sigma(s + p_1)} - \frac{1}{\wp'(\bar{p}_1)} \ln \frac{\sigma(s - \bar{p}_1)}{\sigma(s + \bar{p}_1)} \right]. \quad (80)
\end{aligned}$$

We see that pairs of the type $x - \bar{x}$ appear in Eq. (80) both in the nominator and the denominator and, therefore, K_1 is real. Similarly, the integral K_2 is basically of the same form. Integral K_3 does not involve any sigma functions and reads

$$K_3 = \frac{\ln[\wp(s) - \wp(p_1)] - \ln[\wp(s) - \wp(\bar{p}_1)]}{\wp(p_1) - \wp(\bar{p}_1)}. \quad (81)$$

Summarizing all results from this section, ultimately, we can write Eq. (42) as

$$\varphi(s) - \varphi(0) = \frac{\alpha\epsilon}{\xi_H^+ - \xi_H^-} \left[\left(\xi_H^{+2} + \alpha^2 - \frac{\alpha\lambda_z}{\epsilon} \right) N_H^+(s) - \left(\xi_H^{-2} + \alpha^2 - \frac{\alpha\lambda_z}{\epsilon} \right) N_H^-(s) \right] + I_\theta(s), \quad (82)$$

with integrals $N_H^\pm$ given in Eq. (61) and integral I_θ given in Eq. (72).

The case with $\alpha = \pm 1$ (the extreme Kerr limit) requires a separate treatment, as the partial fraction decomposition given by Eq. (45) has a different form. Appropriate formulas are given in Appendix B.

D. Time coordinate

The solution for $T(s)$ can be obtained in a similar way. A direct integration of Eq. (15d) yields

$$\begin{aligned} T(s) - T(0) &= \int_0^s \frac{[\xi^2(\bar{s}) + \alpha^2] \{ [\xi^2(\bar{s}) + \alpha^2] \epsilon - \alpha\lambda_z \}}{\xi^2(\bar{s}) - 2\xi(\bar{s}) + \alpha^2} d\bar{s} \\ &\quad + \int_0^s \alpha[\lambda_z - \alpha\epsilon \sin^2 \theta(\bar{s})] d\bar{s} \\ &= J_\xi(s) - \alpha^2 \epsilon J_\theta(s), \end{aligned} \quad (83)$$

where

$$J_\xi(s) = \int_0^s \frac{[\xi^2(\bar{s}) + \alpha^2]^2 \epsilon - 2\alpha\lambda_z \xi(\bar{s})}{\xi(\bar{s})^2 - 2\xi(\bar{s}) + \alpha^2} d\bar{s} \quad (84)$$

$$J_\theta(s) = \int_0^s \sin^2 \theta(\bar{s}) d\bar{s}. \quad (85)$$

For $-1 < \alpha < 1$, the expression

$$\frac{(\xi^2 + \alpha^2)^2 \epsilon - 2\alpha\lambda_z \xi}{\xi^2 - 2\xi + \alpha^2} = \frac{(\xi^2 + \alpha^2)^2 \epsilon - 2\alpha\lambda_z \xi}{(\xi - \xi_H^+)(\xi - \xi_H^-)} \quad (86)$$

has the following partial fraction decomposition

$$\begin{aligned} \frac{(\xi^2 + \alpha^2)^2 \epsilon - 2\alpha\lambda_z \xi}{(\xi - \xi_H^+)(\xi - \xi_H^-)} &= (4 + \alpha^2)\epsilon + 2\epsilon\xi + \epsilon\xi^2 \\ &\quad + \frac{A_-}{\xi - \xi_H^-} + \frac{A_+}{\xi - \xi_H^+}, \end{aligned} \quad (87)$$

where

$$\begin{aligned} A_\pm &= \frac{2(\mp \alpha^2 + 2\sqrt{1 - \alpha^2} \pm 2)\epsilon - \alpha(\sqrt{1 - \alpha^2} \pm 1)\lambda_z}{\sqrt{1 - \alpha^2}} \\ &= \frac{2\epsilon(\xi_H^\pm)^2 - \alpha\lambda_z \xi_H^\pm}{\xi_H^\pm - 1}. \end{aligned} \quad (88)$$

Consequently,

$$\begin{aligned} J_\xi(s) &= (4 + \alpha^2)\epsilon s + 2\epsilon N_1(s) + \epsilon N_2(s) + A_- N_H^-(s) \\ &\quad + A_+ N_H^+(s), \end{aligned} \quad (89)$$

where, apart from the integrals $N_H^\pm$ computed in the previous section, we have denoted

$$N_1(s) = \int_0^s \xi(\bar{s}) d\bar{s} \quad (90)$$

and

$$N_2(s) = \int_0^s \xi(\bar{s})^2 d\bar{s}. \quad (91)$$

Detailed expressions for $N_1(s)$, $N_2(s)$, and $J_\theta(s)$ are given in Appendix A.

Finally, we can write Eq. (83) as

$$\begin{aligned} T(s) - T(0) &= (4 + \alpha^2)\epsilon s + 2\epsilon N_1(s) + \epsilon N_2(s) \\ &\quad + A_- N_H^-(s) + A_+ N_H^+(s) - \alpha^2 \epsilon J_\theta(s). \end{aligned} \quad (92)$$

As before, the extreme Kerr case with $\alpha = \pm 1$ has to be considered separately. The appropriate partial fraction expansion of the radial part of Eq. (83) can be found in Appendix B.

E. Proper time

The relation between the proper time $\tilde{\tau}$ and the rescaled Mino time s can also be integrated. Equations (10) or (11) give

$$\frac{d\tilde{\tau}}{ds} = \frac{M}{m} (\xi^2 + \alpha^2 \cos^2 \theta)$$

and, upon integration,

$$\tilde{\tau}(s) - \tilde{\tau}(0) = \frac{M}{m} \int_0^s [\xi(\bar{s})^2 + \alpha^2 \cos^2 \theta(\bar{s})] d\bar{s},$$

which can be expressed in terms of $N_2(s)$ and $J_\theta(s)$ as

$$\tilde{\tau}(s) - \tilde{\tau}(0) = \frac{M}{m} \{ N_2(s) + \alpha^2 [s - J_\theta(s)] \}. \quad (93)$$

IV. IMPLEMENTATION IN WOLFRAM MATHEMATICA

All formulas expressing the solutions obtained in Sec. III have been encoded in a *Wolfram Mathematica* [62] package, which is available at PANGAEA and at the GitHub [63]. The implementation of our formulas is essentially straightforward, but it is tedious, due to the number of different integrals appearing in our calculation. Perhaps the only place which requires special attention is related with computing the terms of the form

$$\ln \frac{\sigma(s-y; g_2, g_3)}{\sigma(s+y; g_2, g_3)}, \quad (94)$$

where σ denotes the Weierstrass function sigma, s is the Mino time, and y is, in general, a complex parameter. Expressions of this type appear in integrals (C1), (C3), and (C7). A proper implementation of our formulas, yielding continuous solutions for $\varphi(s)$ and $T(s)$, requires careful selection of the complex logarithm branch in Eq. (94). A “brute force” approach to this problem is to start at $s = 0$ and to follow the appropriate logarithm branch up to a given nonzero value of s , assuring that both real and imaginary parts of Eq. (94) remain continuous. The logarithm in Eq. (94) can be thus computed as

$$\ln \frac{\sigma(s-y; g_2, g_3)}{\sigma(s+y; g_2, g_3)} = \text{Log} \frac{\sigma(s-y; g_2, g_3)}{\sigma(s+y; g_2, g_3)} + 2\pi ki, \quad (95)$$

where Log denotes the principal value of the complex logarithm, and $k \in \mathbb{Z}$, but one has to select an appropriate integer k , which can change as s increases. Another straightforward (but numerically expensive) approach is to express (94) as

$$\ln \frac{\sigma(s-y; g_2, g_3)}{\sigma(s+y; g_2, g_3)} = \int_0^s [\zeta(\bar{s}-y; g_2, g_3) - \zeta(\bar{s}+y; g_2, g_3)] d\bar{s}, \quad (96)$$

and evaluate the above integral numerically.

There is a partial work-around to the above problem, based on the properties of the Weierstrass function sigma. Let ω_1 and ω_3 denote half-periods of $\wp(z)$, and let ω_1 be real. The function sigma can be expressed as

$$\begin{aligned} \sigma(z) &= \frac{2\omega_1}{\pi} \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right) \sin\left(\frac{\pi z}{2\omega_1}\right) \\ &\quad \times \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos\left(\frac{\pi z}{\omega_1}\right) + q^{4n}}{(1 - q^{2n})^2} \\ &= \frac{2\omega_1}{\pi} \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right) \frac{\theta_1(\pi z/(2\omega_1), q)}{\theta_1'(0, q)}, \end{aligned} \quad (97)$$

where $q = e^{i\pi\omega_3/\omega_1}$, $\eta_1 = \zeta(\omega_1)$, $\theta_1(z, q)$ denotes the Jacobi theta function, and where we omit temporarily the Weierstrass invariants g_2 and g_3 in the argument of $\sigma(z)$ [66]. For arguments z of the form $z = s \pm y$, where y is a (possibly) complex parameter, and s is real, we get a product of an exponential factor $\exp[\eta_1 z^2/(2\omega_1)]$ and the remaining part, periodic in s . To disentangle these two different behaviors, we define

$$\tilde{\sigma}(z) = \sigma(z) \exp[-\eta_1 z^2/(2\omega_1)]. \quad (98)$$

Thus

$$\ln \frac{\sigma(s-y)}{\sigma(s+y)} = \ln \frac{\tilde{\sigma}(s-y)}{\tilde{\sigma}(s+y)} - \frac{2\eta_1 sy}{\omega_1}. \quad (99)$$

Here the key task is to control the phase of $\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)$. The principal value of the argument of this ratio, $\text{Arg}[\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)]$, is discontinuous at $s = 0$ and $s = 2\omega_1$. This can be seen by expanding $\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)$ with respect to s around $s = 0$, which gives, up to terms linear in s ,

$$\frac{\tilde{\sigma}(s-y)}{\tilde{\sigma}(s+y)} = -1 + 2\left[\zeta(y) - \frac{\eta_1 y}{\omega_1}\right]s + O(s^2). \quad (100)$$

Hence, $\text{Arg}[\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)]$ changes at $s = 0$ from $-\pi$ to π , if $\text{Im}[\zeta(y) - \eta_1 y/\omega_1] > 0$, and from π to $-\pi$, if $\text{Im}[\zeta(y) - \eta_1 y/\omega_1] < 0$. By periodicity, the same happens also at $s = 2\omega_1$. In the simplest case $\text{Arg}[\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)]$ would exhibit no additional discontinuities within the range $0 < s < 2\omega_1$. If, however, the argument $\text{Arg}[\tilde{\sigma}(s-y)/\tilde{\sigma}(s+y)]$ changes sufficiently fast, additional jumps may also occur. To correct for this general behavior, we write

$$\begin{aligned} \ln \frac{\tilde{\sigma}(s-y)}{\tilde{\sigma}(s+y)} &= \ln \left\{ \frac{\tilde{\sigma}(s-y)}{\tilde{\sigma}(s+y)} \exp \left[k\pi i \left(\frac{s}{\omega_1} - 1 \right) \right] \right\} \\ &\quad - k\pi i \left(\frac{s}{\omega_1} - 1 \right), \end{aligned} \quad (101)$$

where k is a fixed integer.

A final prescription for a continuous choice of the logarithm branch can be written as

$$\begin{aligned} \ln \frac{\sigma(s-y)}{\sigma(s+y)} &= \text{Log} \left\{ \frac{\tilde{\sigma}(s-y)}{\tilde{\sigma}(s+y)} \exp \left[k\pi i \left(\frac{s}{\omega_1} - 1 \right) \right] \right\} \\ &\quad - k\pi i \left(\frac{s}{\omega_1} - 1 \right) - \frac{2\eta_1 sy}{\omega_1}. \end{aligned} \quad (102)$$

In practical examples, the above prescription works well, saving time with respect to the “brute force” algorithm described above. Controlling the changes of the phase of

TABLE I. Parameters of geodesics shown in Figs. 1–10. Black hole horizons are located at $\xi_H^- = 1 - \sqrt{1 - \alpha^2} = 0.4$ and $\xi_H^+ = 1 + \sqrt{1 - \alpha^2} = 1.6$.

Figure Number	δ_1	ε^2	κ	λ_z	α	Radial type	Real zeros of $\tilde{R}(\xi)$	ξ_0
1	1	1.1	12	−1	0.8	II, flyby	−24.3351, 0.254136	8
2	1	0.95	12	3	0.8	III, interval-bound	0.22019, 1.63896, 8.44487, 29.696	10
3	1	0.95	12	3	0.8	III, interval-bound	0.22019, 1.63896, 8.44487, 29.696	1.55
4	1	1.1	12	3	0.8	IV, flyby	−26.4861, 0.230431, 1.67987, 4.57578	10
5	1	0.5	12	−1	0.8	V, interval-bound	0.291099, 2.3974	2.3
6	1	30	12	−0.05	0.8	I, transit	—	10
7	0	1	60	4.47214	0.8	IV, flyby	−8.927, 0.296172, 1.60191, 7.02891	10
8	0	1	60	4.47214	0.8	IV, interval-bound	−8.927, 0.296172, 1.60191, 7.02891	1.5
9	0	1	0.6	−0.111803	0.8	II, flyby	−0.721257, −0.137167	10
10	0	1	0.4	−0.00912871	0.8	I, transit	—	10

$\sigma(s - y)/\sigma(s + y)$ within one period $2\omega_1$ suffices to compute the suitable value of k . Let

$$c = \text{Im} \left[\ln \frac{\sigma(s + 2\omega_1 - y)}{\sigma(s + 2\omega_1 + y)} \right] - \text{Im} \left[\ln \frac{\sigma(s - y)}{\sigma(s + y)} \right], \quad (103)$$

which can be computed, e.g., from Eq. (96). Assuming Eq. (102) and the equality

$$\begin{aligned} & \text{Arg} \left\{ \frac{\tilde{\sigma}(s + 2\omega_1 - y)}{\tilde{\sigma}(s + 2\omega_1 + y)} \exp \left[k\pi i \left(\frac{s + 2\omega_1}{\omega_1} - 1 \right) \right] \right\} \\ &= \text{Arg} \left\{ \frac{\tilde{\sigma}(s - y)}{\tilde{\sigma}(s + y)} \exp \left[k\pi i \left(\frac{s}{\omega_1} - 1 \right) \right] \right\}, \end{aligned} \quad (104)$$

one gets immediately

$$c = -2k\pi - \text{Im}(4\eta_1 y), \quad (105)$$

and thus

$$k = -\frac{1}{2\pi} [\text{Im}(4\eta_1 y) + c]. \quad (106)$$

V. EXAMPLES

Examples of solutions obtained in Sec. III are plotted in Figs. 1–11. In Figs. 1–10 we plot projections of the orbits on the xy and xz planes together with projections on the three-dimensional spaces of constant time. Here the Cartesian coordinates (x, y, z) are defined as

TABLE II. Parameters of special equatorial geodesics shown in Fig. 11.

Figure Number	δ_1	ε^2	κ	λ_z	α	ξ_0
11	1	0.77064	2.81619	2.38044	0.8	2.89664
11	0	1	5.94042	3.2373	0.8	1.80109

$$x = \xi \cos \varphi \sin \theta, \quad (107a)$$

$$y = \xi \sin \varphi \sin \theta, \quad (107b)$$

$$z = \xi \cos \theta. \quad (107c)$$

The last plot (lower right panel) in Figs. 1 to 10 shows the coordinate time T versus the proper time $\bar{\tau}$ for timelike geodesics and the affine parameter for null ones. Our analytic solutions are depicted with an orange line. For comparison, we also plot numerical solutions corresponding to the same initial data. They are depicted with black dotted lines. In all plots we mark initial positions $(\xi_0, \theta_0, \varphi_0)$, but in many cases the solutions are evolved both forward and backward in time. Blue lines or gray cones in the plots show the limiting values of the angle θ , marking the region available for the motion. Black hole horizons (inner and outer) are drawn as black circles or gray spheres.

The coordinate singularities present in Boyer-Lindquist coordinates prevent us from continuing the solutions through horizons. The equations for $\xi(s)$ and $\theta(s)$ are unaffected, but the ones for $\varphi(s)$ and $T(s)$ are. Consequently, we only plot our solutions up to horizons.

For convenience, main parameters of solutions shown in Figs. 1–10 have been collected in Table I, together with information about their radial types and real zeros of the radial potential $\tilde{R}$. In all cases $\alpha = 0.8$, and consequently the horizons are located at $\xi_H^- = 1 - \sqrt{1 - \alpha^2} = 0.4$ and $\xi_H^+ = 1 + \sqrt{1 - \alpha^2} = 1.6$. Figures 1–6 show examples of timelike orbits ($\delta_1 = 1$). Null orbits ($\delta_1 = 0$) are shown in Figs. 7–10.

Figure 1 shows a generic timelike flyby orbit of radial type II, plunging into the black hole. The orbit crosses the equatorial plane. Figure 2 depicts a generic timelike type III interval-bound orbit, restricted to a region outside the black hole horizon, oscillating around the equatorial plane. In Fig. 3 we assume the same constants of motion as in Fig. 2, but the choice of the initial radius $\xi_0 = 1.55$ yields an inner bound orbit. We follow the motion both forward and

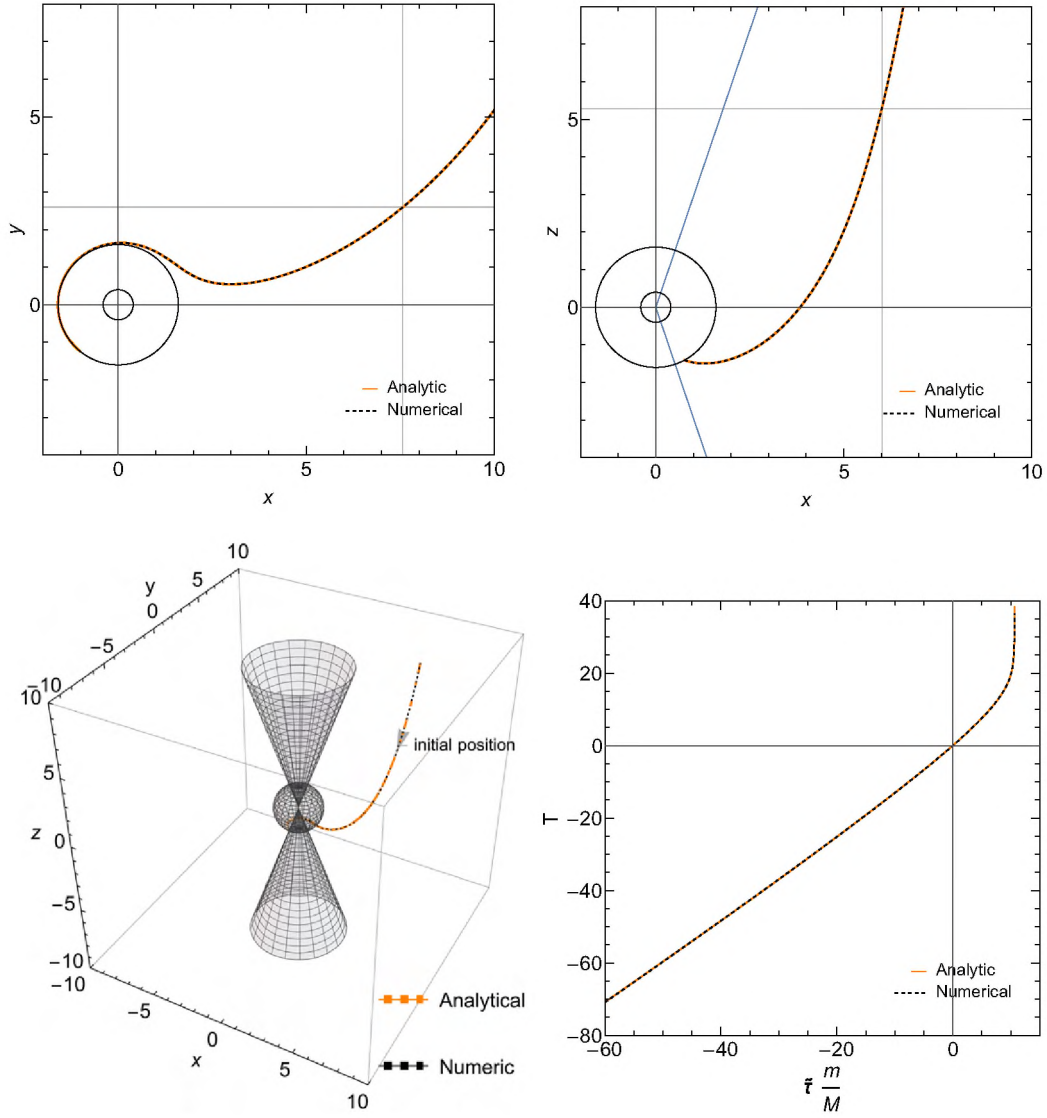


FIG. 1. An example of a timelike unbound absorbed orbit with $\varepsilon^2 = 1.1$, $\lambda_z = -1$, $\alpha = 0.8$, $\kappa = 12$ and initial position at $\xi_0 = 8$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$. Solid color lines correspond to solutions obtained with Eqs. (34), (40), (82). Dotted lines depict corresponding numerical solutions. The motion is depicted in two planes: xy on the right and xz on the left. The intersection of thin black lines marks the initial position. Blue lines in the xz plane and cones in the three-dimensional plot correspond to extremal values of θ . Gray spheres and dark circles depict black hole horizons. The figure in the lower right corner shows to the coordinate time T versus the proper time $\bar{\tau}$.

backward in time, up to horizons at $\xi = \xi_H^\pm$. Figure 4 gives a generic example of a type IV flyby orbit. In this case the turning point is located at $\xi \approx 4.58 > \xi_H^+$. Consequently, the particle incoming from $\xi = +\infty$ is scattered by the black hole, and moves to $\xi = +\infty$. Figure 5 shows an example of an interval-bound orbit of radial type V, with $\xi_0 = 2.3 > \xi_H^+$. The motion is continued forward and backward in time up to $\xi = \xi_H^\pm$. Perhaps the most interesting case is shown in Fig. 6. The trajectory is of type I and represents a transit orbit. The motion is restricted to the “northern” hemisphere. We emphasize once more that the

radial motion is described by a manifestly real expression (34), even though the radial potential $\tilde{R}$ has no real zeros in this case.

In Fig. 7 we plot a generic null flyby trajectory of radial type IV. This trajectory crosses the equatorial plane. For the null trajectory depicted in Fig. 8 we assume the same constants of motion as in Fig. 8, but we take $\xi_0 = 1.5$. Consequently, the trajectory is of interval-bound type, and the part of the orbit shown in Fig. 8 is restricted to the region enclosed by the two horizons $\xi = \xi_H^-$ and $\xi = \xi_H^+$. Figures 9 and 10 show two examples of null orbits

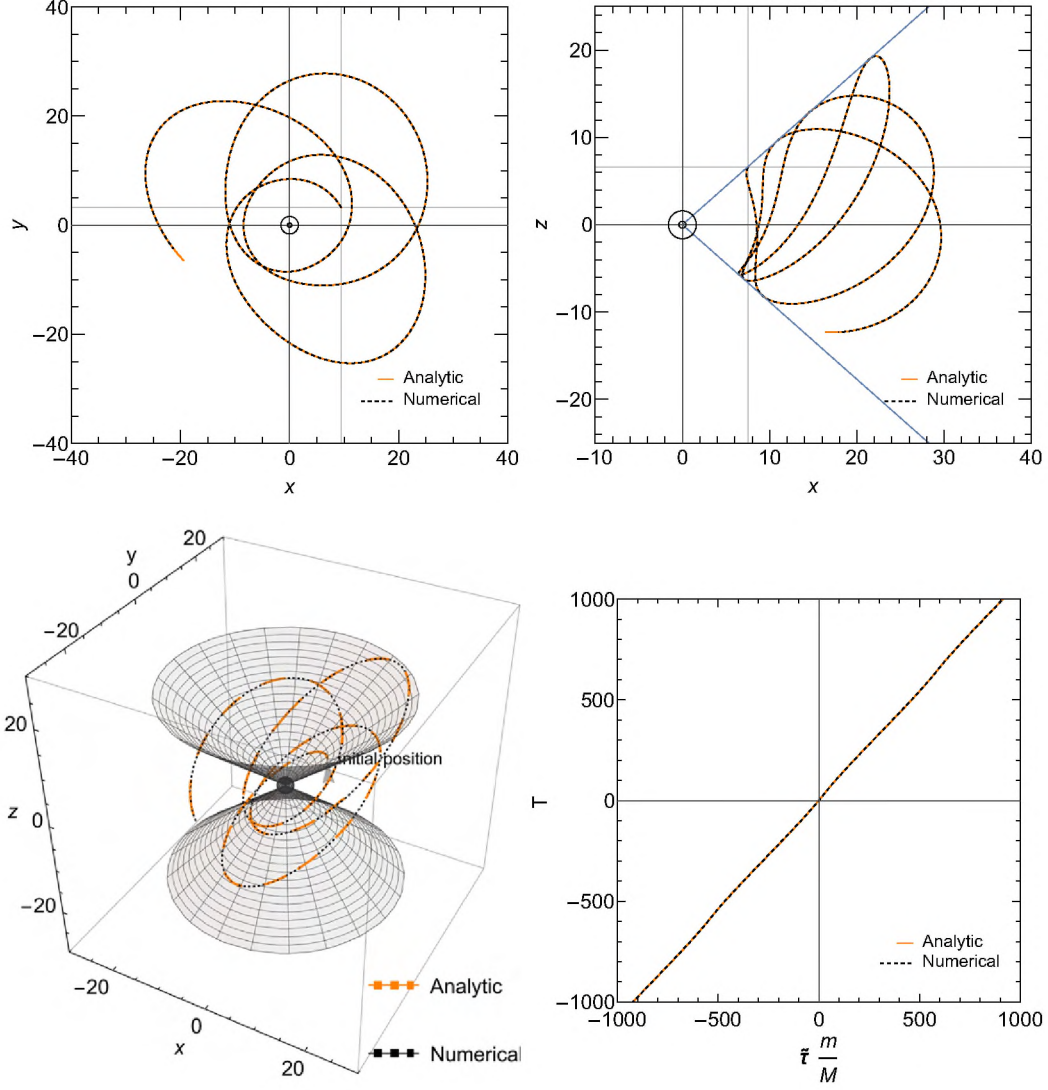


FIG. 2. Same as in Fig. 1, but for a timelike bound orbit with $\epsilon^2 = 0.95$, $\lambda_z = 3$, $\alpha = 0.8$, $\kappa = 12$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $e_r = -1$, $e_\theta = 1$.

plunging into the black hole. Both trajectories originate at $\xi_0 = 10$. The trajectory shown in Fig. 9 is of radial type II (flyby). The one depicted in Fig. 10 belongs to radial type I (transit). Both trajectories are restricted to the “northern” hemisphere.

Figure 11 shows examples of two special equatorial trajectories with constants of motion corresponding to circular orbits [38,39]. Parameters of these solutions are collected in Table II. Upper plots in Fig. 11 depict a timelike spiraling orbit with constants of motion ϵ , λ_z , and κ characteristic for a marginally stable corotating circular orbit with the radius ξ_{ms} . In this case

$$\tilde{R} = (\epsilon^2 - 1)\xi(\xi - \xi_{\text{ms}})^3, \quad (108)$$

i.e., ξ_{ms} is a triple root of $\tilde{R}$. The radius ξ_{ms} can be computed as

$$\xi_{\text{ms}} = 3 + Z_2 - \epsilon_\lambda \alpha \sqrt{\frac{(3 - Z_1)(3 + Z_1 + 2Z_2)}{\alpha^2}}, \quad (109)$$

where $Z_1 = 1 + \sqrt[3]{1 - \alpha^2}(\sqrt[3]{1 + \epsilon_\lambda \alpha} + \sqrt[3]{1 - \epsilon_\lambda \alpha})$, $Z_2 = \sqrt{3\alpha^2 + Z_1^2}$ [12]. Here the sign $\epsilon_\lambda = \text{sgn}\alpha$ corresponds to a corotating orbit, and $\epsilon_\lambda = -\text{sgn}\alpha$ to a counterrotating one. For $\alpha = 0.8$ and $\epsilon_\lambda = +1$, we get $\xi_{\text{ms}} \approx 2.90664$. In this case a corotating marginally stable orbit is characterized by $\epsilon^2 = 0.77064$, $\kappa = 2.81619$, and $\lambda_z = 2.38044$. For the timelike orbit plotted in Fig. 11 we assume the same values of ϵ , λ_z , and κ , but the initial radius is set to be $\xi_0 = \xi_{\text{ms}} - 0.01$.

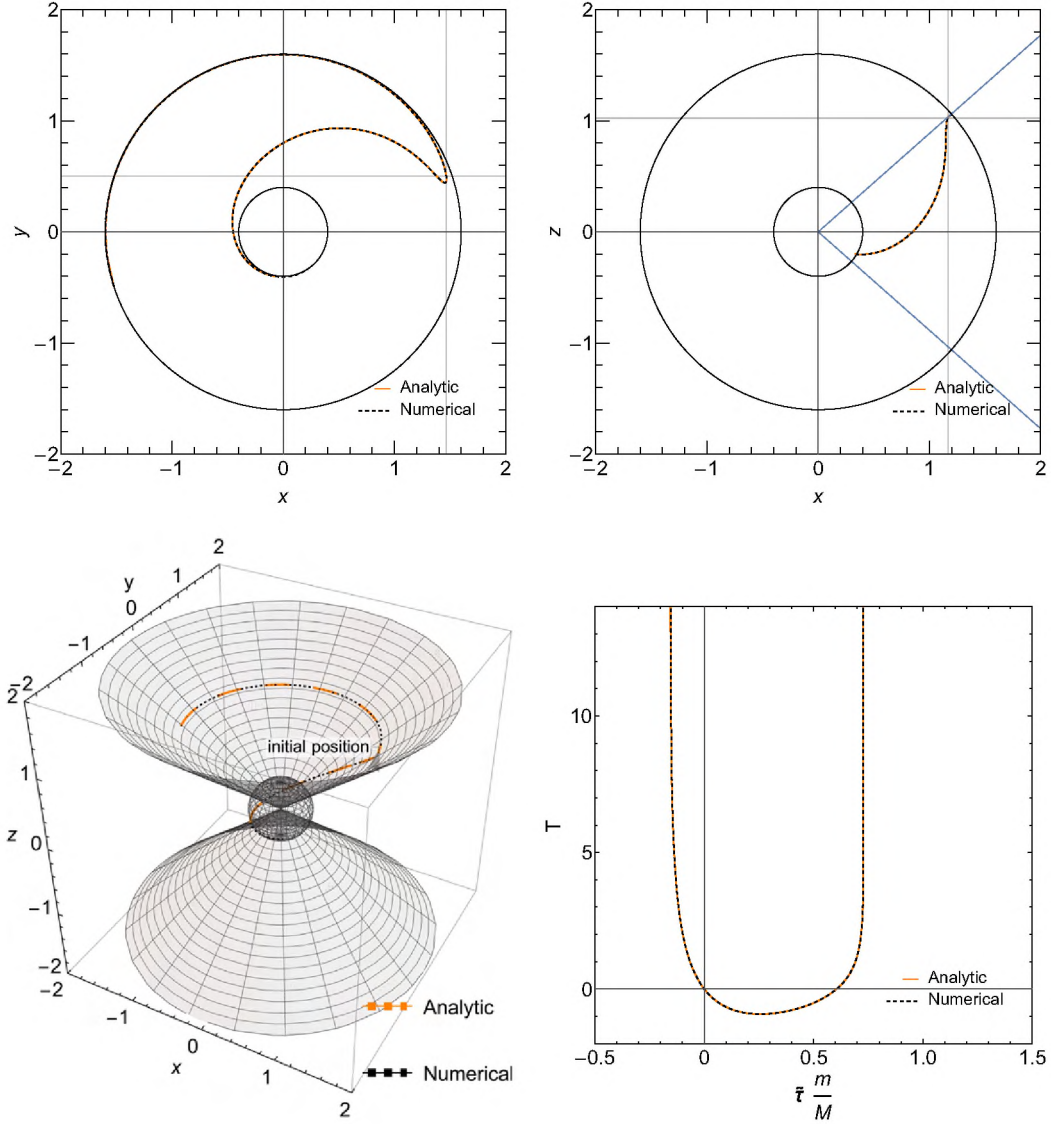


FIG. 3. Same as in Fig. 1, but for an inner bound orbit with $\epsilon^2 = 0.95$, $\lambda_z = 3$, $\alpha = 0.8$, $\kappa = 12$, and an initial position at $\xi_0 = 1.55$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $e_r = -1$, $e_\theta = 1$.

Lower panels in Fig. 11 show a similar picture for an equatorial null orbit. In this case we assume the constants ϵ , λ_z , and κ corresponding to a circular null orbit of radius

$$\xi_{\text{ph}} = 2 + 2 \cos \left[\frac{2}{3} \arccos(-\epsilon_\lambda \alpha) \right]. \quad (110)$$

This gives, for $\alpha = 0.8$ and $\epsilon_\lambda = +1$, $\xi_{\text{ph}} = 1.81109$, and the constants of motion $\lambda_z/\epsilon = 3.2373$, and $\kappa/\epsilon^2 = 5.94042$. The potential $\tilde{R}$ has a double zero at $\xi = \xi_{\text{ph}}$. For the null orbit plotted in Fig. 11 we assume the initial radius at $\xi_0 = \xi_{\text{ph}} - 0.01$.

VI. DISCUSSION

We have derived a single set of general, closed form analytic solutions describing all types of timelike and null Kerr geodesics in terms of Weierstrass elliptic functions. Our derivation follows largely the footsteps of Ref. [32], but a new ingredient is an application of the Biermann-Weierstrass theorem, which allows us to parametrize solutions with the constants of motion and arbitrary admissible initial coordinates. In particular, there is no need in our formalism to calculate any turning points of the motion, which is generally necessary in other methods.

We tried to be coherent in our approach, and to express our solutions in terms of Weierstrass elliptic functions only, but one is free to mix our formulas with the existing

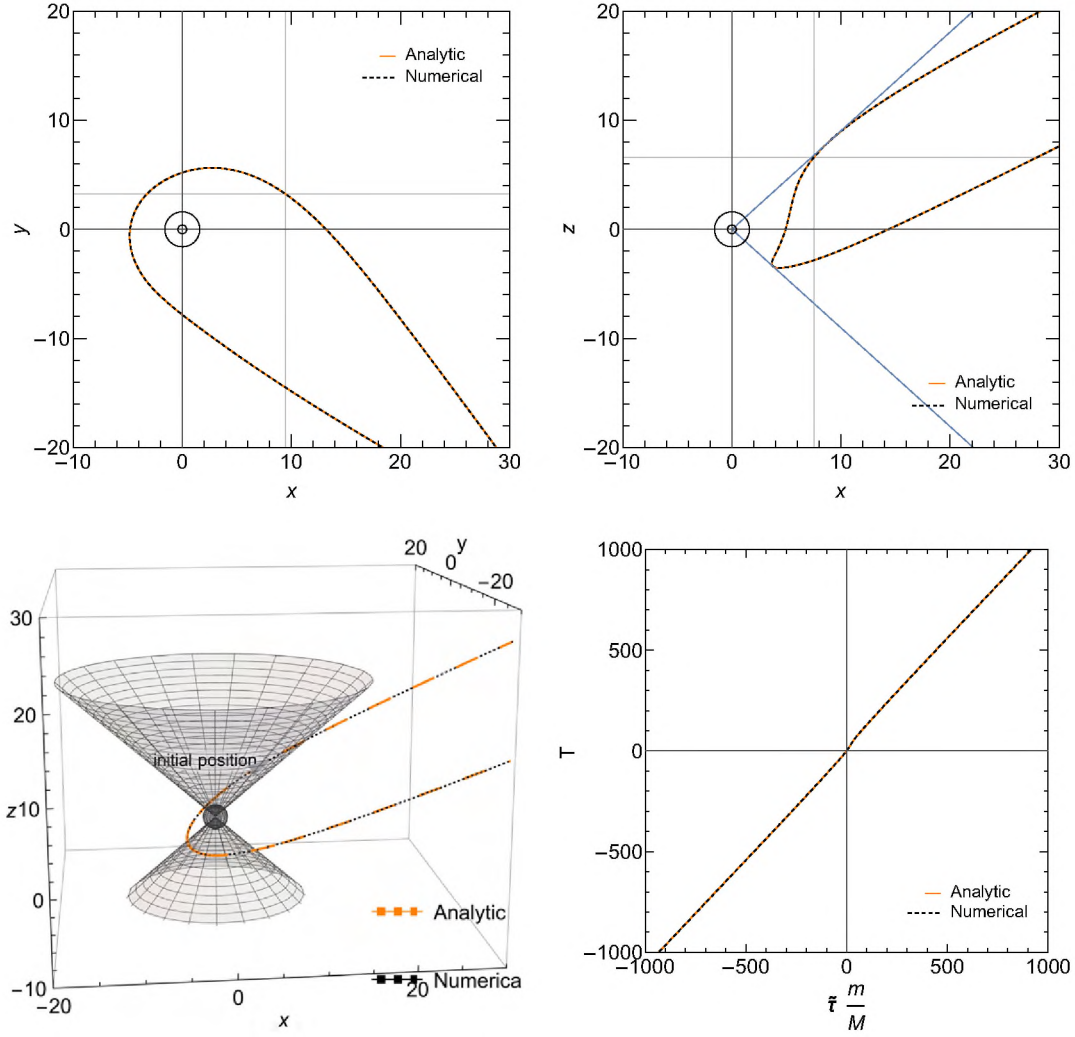


FIG. 4. Same as in Fig. 1, but for a timelike unbound scattered orbit with $\epsilon^2 = 1.1$, $\lambda_z = 3$, $\alpha = 0.8$, $\kappa = 12$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$.

solutions, perhaps more convenient in some cases. Our method provides very general and clear expressions for radial and polar motions. As usual, expressions for $\varphi(s)$ and the time coordinate $T(s)$ are much more involved, as several elliptic integrals appear that are solved using logarithms of the Weierstrass sigma function. However, progress has been made in the required careful selection of appropriate branches of complex logarithms, as described in Sec. IV. This has been achieved by extracting the linear parts and reducing the problem to the determination of a single constant for the complete trajectory. On the other hand, the big advantage of our derivation is that it does not differentiate between many possible algebraic types of solutions, perhaps except for (zero-measure) special cases with $\epsilon = \delta_1$, described in Appendix B 3, and a necessity to handle some aspects of the derivation in the extreme-Kerr case separately (Appendix B 2).

ACKNOWLEDGMENTS

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APPENDIX A: EXPRESSIONS FOR J_θ , N_1 , AND N_2

In this appendix we collect expressions for $J_\theta(s)$, $N_1(s)$, and $N_2(s)$, appearing in our formulas for $T(s)$. The derivations are similar to the ones for $N_H^\pm$ and $I_\theta(s)$, given in Sec. III C.

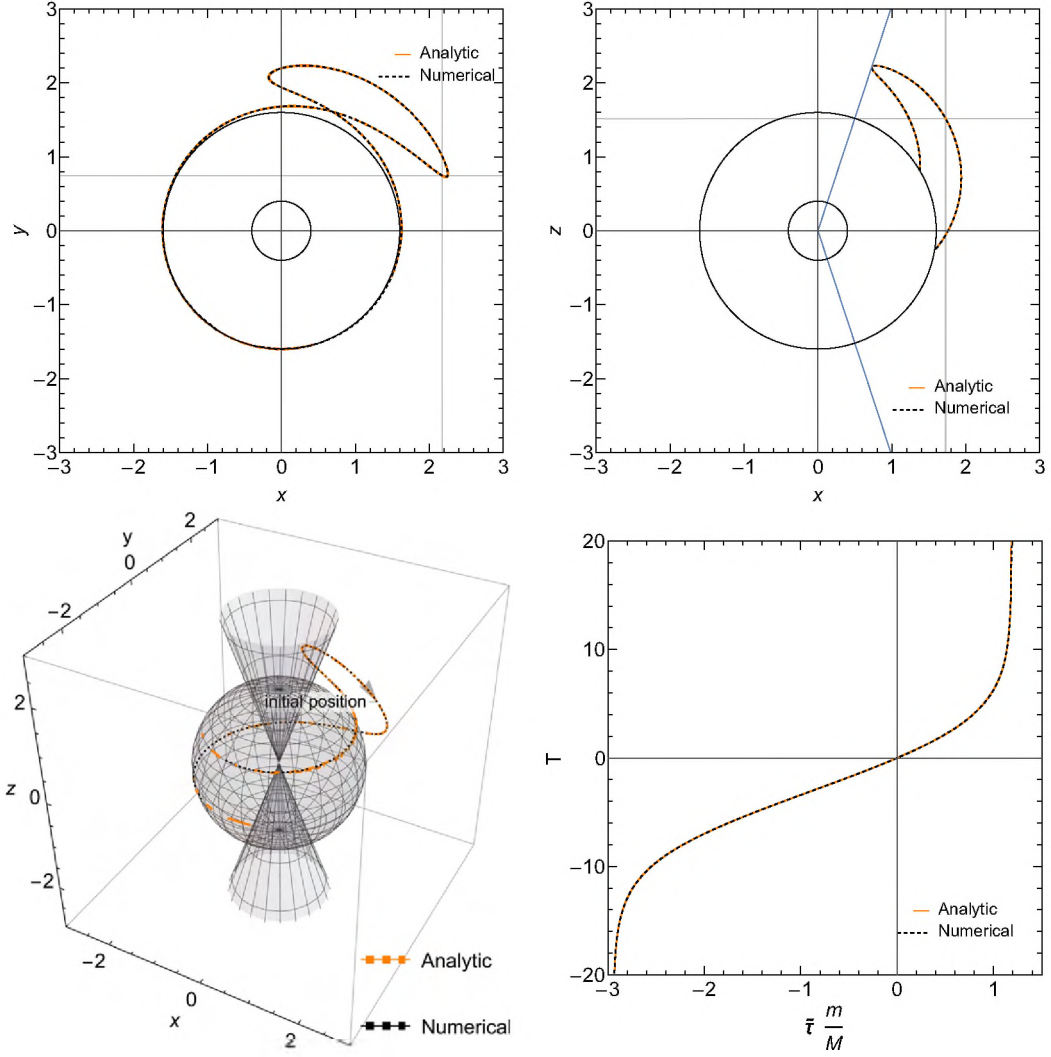


FIG. 5. Same as in Fig. 1, but for a timelike bound orbits with $\epsilon^2 = 0.5$, $\lambda_z = -1$, $\alpha = 0.8$, $\kappa = 12$, and an initial position at $\xi_0 = 2.3$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $e_r = -1$, $e_\theta = 1$.

1. Expressions for N_1 and N_2

Integrals N_1 and N_2 are defined as

$$N_1(s) = \int_0^s \xi(\bar{s}) d\bar{s} \quad (\text{A1})$$

and

$$N_2(s) = \int_0^s \xi(\bar{s})^2 d\bar{s}. \quad (\text{A2})$$

As before, we define

$$C_1 = \frac{1}{48} \tilde{R}(\xi_0) \tilde{R}'''(\xi_0) - \frac{1}{96} \tilde{R}'(\xi_0) \tilde{R}''(\xi_0), \quad (\text{A3a})$$

$$C_2 = \frac{1}{4} \tilde{R}'(\xi_0), \quad (\text{A3b})$$

$$C_3 = -\frac{1}{2} \epsilon_r \sqrt{\tilde{R}(\xi_0)}, \quad (\text{A3c})$$

$$C_4 = \frac{1}{24} \tilde{R}''(\xi_0), \quad (\text{A3d})$$

$$C_5 = \frac{1}{48} \tilde{R}(\xi_0) \tilde{R}^{(4)}(\xi_0), \quad (\text{A3e})$$

$$\wp_{\tilde{R},1} = C_4 + \sqrt{\frac{C_5}{2}}, \quad (\text{A3f})$$

$$\wp_{\tilde{R},2} = C_4 - \sqrt{\frac{C_5}{2}}. \quad (\text{A3g})$$

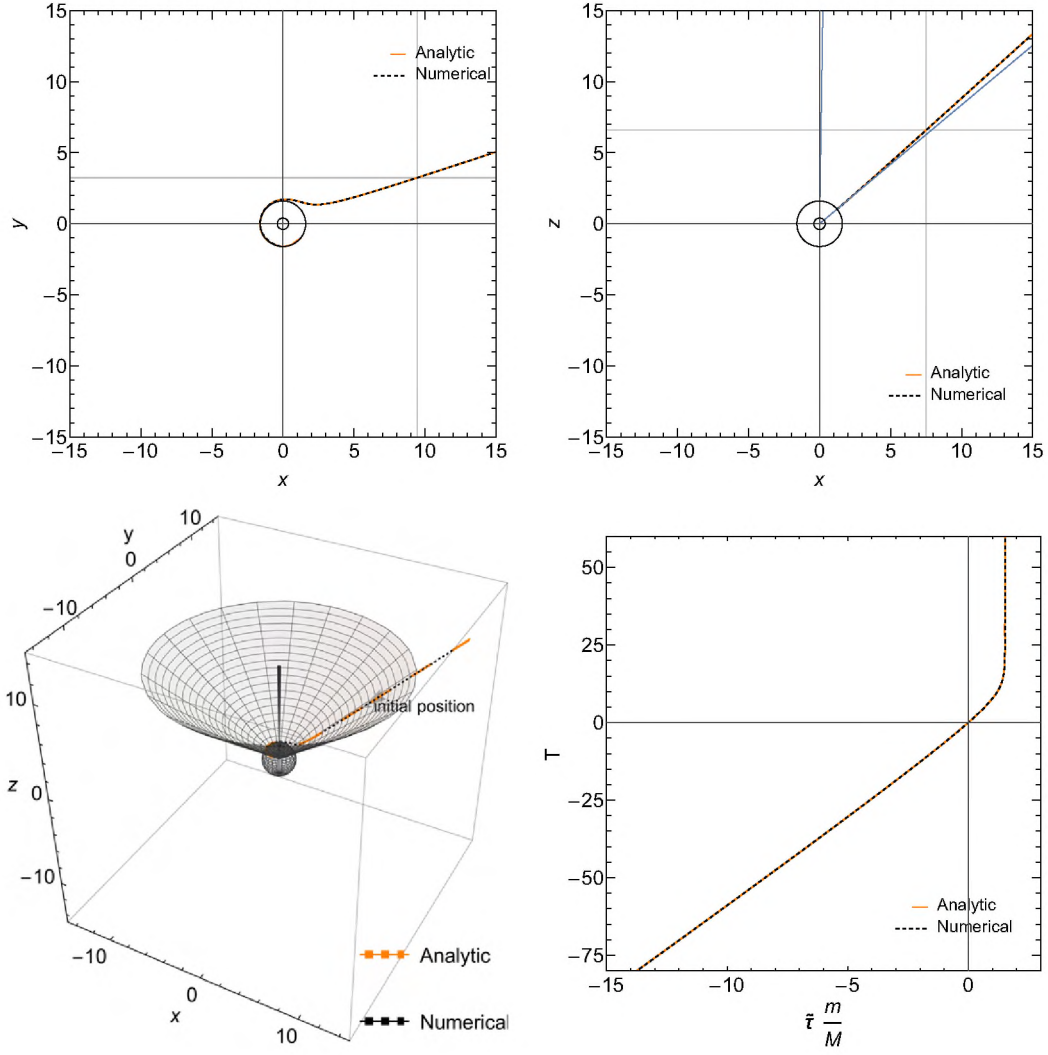


FIG. 6. Same as in Fig. 1, but for a timelike unbound absorbed orbit with $\varepsilon^2 = 30$, $\lambda_z = -0.05$, $\alpha = 0.8$, $\kappa = 12$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $e_r = -1$, $e_\theta = 1$.

Hence Eq. (34) takes the form

$$\xi(s) = \xi_0 + \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\tilde{R}}(s) + \mathcal{C}_3 \wp'_{\tilde{R}}(s)}{(\wp_{\tilde{R}}(s) - \wp_{\tilde{R},1})(\wp_{\tilde{R}}(s) - \wp_{\tilde{R},2})}, \quad (\text{A4})$$

and therefore

$$\begin{aligned} N_1(s) &= \int_0^s \xi(\tilde{s}) d\tilde{s} = \int_0^s \left\{ \xi_0 + \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\tilde{R}}(s) + \mathcal{C}_3 \wp'_{\tilde{R}}(s)}{[\wp_{\tilde{R}}(\tilde{s}) - \wp_{\tilde{R},1}][\wp_{\tilde{R}}(\tilde{s}) - \wp_{\tilde{R},2}]} \right\} d\tilde{s} \\ &= \xi_0 s + \mathcal{C}_1 K_1(s; p_{\tilde{R},1}, p_{\tilde{R},2}) + \mathcal{C}_2 K_2(s; p_{\tilde{R},1}, p_{\tilde{R},2}) + \mathcal{C}_3 K_3(s; p_{\tilde{R},1}, p_{\tilde{R},2}), \end{aligned} \quad (\text{A5})$$

where $p_{\tilde{R},1}$ and $p_{\tilde{R},2}$ satisfy $\wp(p_{\tilde{R},1}; g_{\tilde{R},2}, g_{\tilde{R},3}) = \mathcal{C}_4 + \sqrt{\frac{\mathcal{C}_3}{2}}$ and $\wp(p_{\tilde{R},2}; g_{\tilde{R},2}, g_{\tilde{R},3}) = \mathcal{C}_4 - \sqrt{\frac{\mathcal{C}_3}{2}}$. Weierstrass functions appearing in expressions for K_1 , K_2 , K_3 in the above formula should be computed with the invariants $g_{\tilde{R},2}$ and $g_{\tilde{R},3}$.

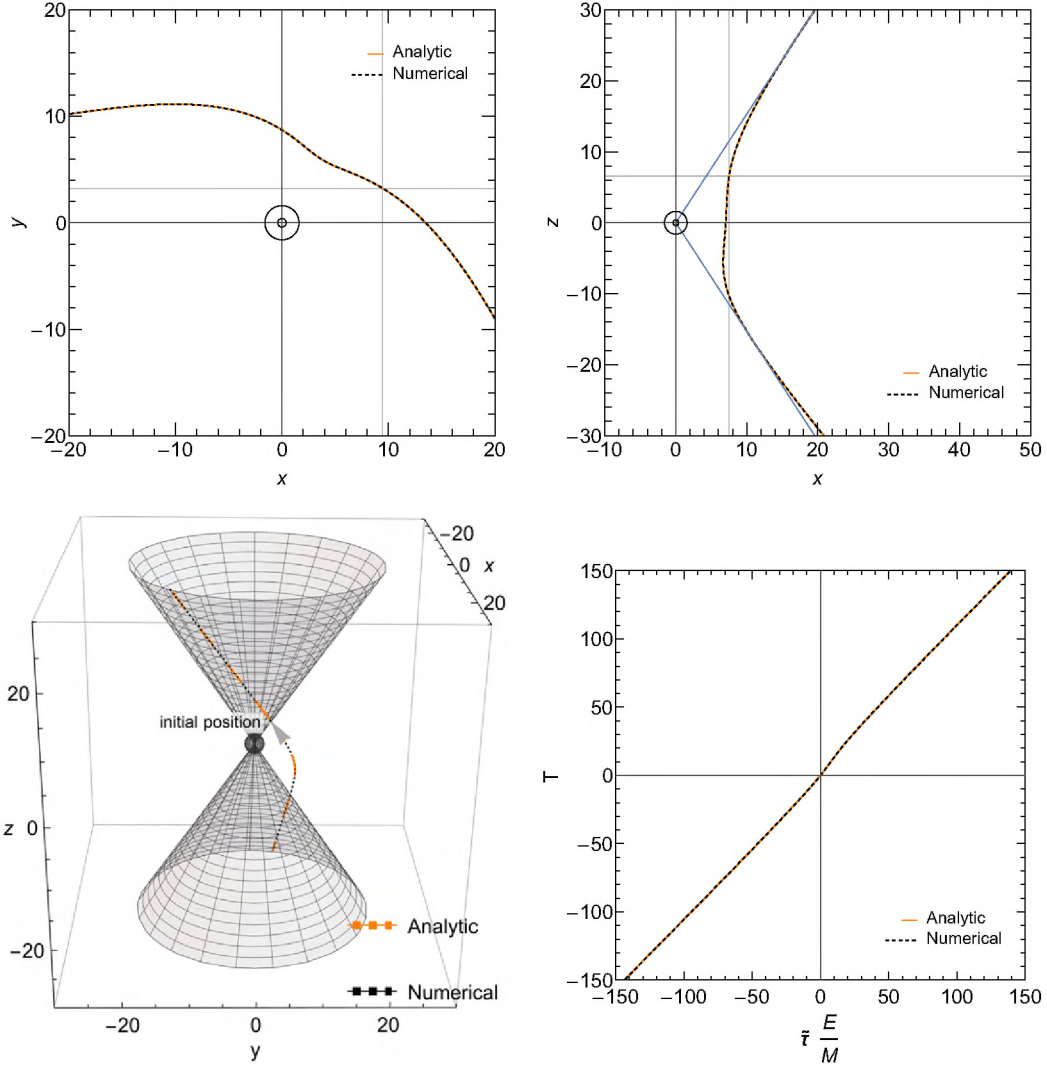


FIG. 7. Same as in Fig. 1, but for a null unbound orbit with $\varepsilon^2 = 1$, $\lambda_z \approx 4.47214$, $\alpha = 0.8$, $\kappa = 60$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $e_r = -1$, $e_\theta = 1$.

The integral $N_2(s)$ can be calculated as follows. We have

$$\begin{aligned}
 N_2(s) &= \int_0^s \xi(\bar{s})^2 d\bar{s} = \int_0^s \left\{ \xi_0 + \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},1}][\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},2}]} \right\}^2 d\bar{s} \\
 &= \int_0^s \xi_0^2 + 2\xi_0 \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},1}][\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},2}]} + \left\{ \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},1}][\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},2}]} \right\}^2 d\bar{s} \\
 &= \xi_0^2 s + 2\xi_0 [N_1(s) - \xi_0 s] + N_3(s) = 2\xi_0 N_1(s) - \xi_0^2 s + N_3(s),
 \end{aligned} \tag{A6}$$

where

$$N_3(s) = \int_0^s \left\{ \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},1}][\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},2}]} \right\}^2 d\bar{s}. \tag{A7}$$

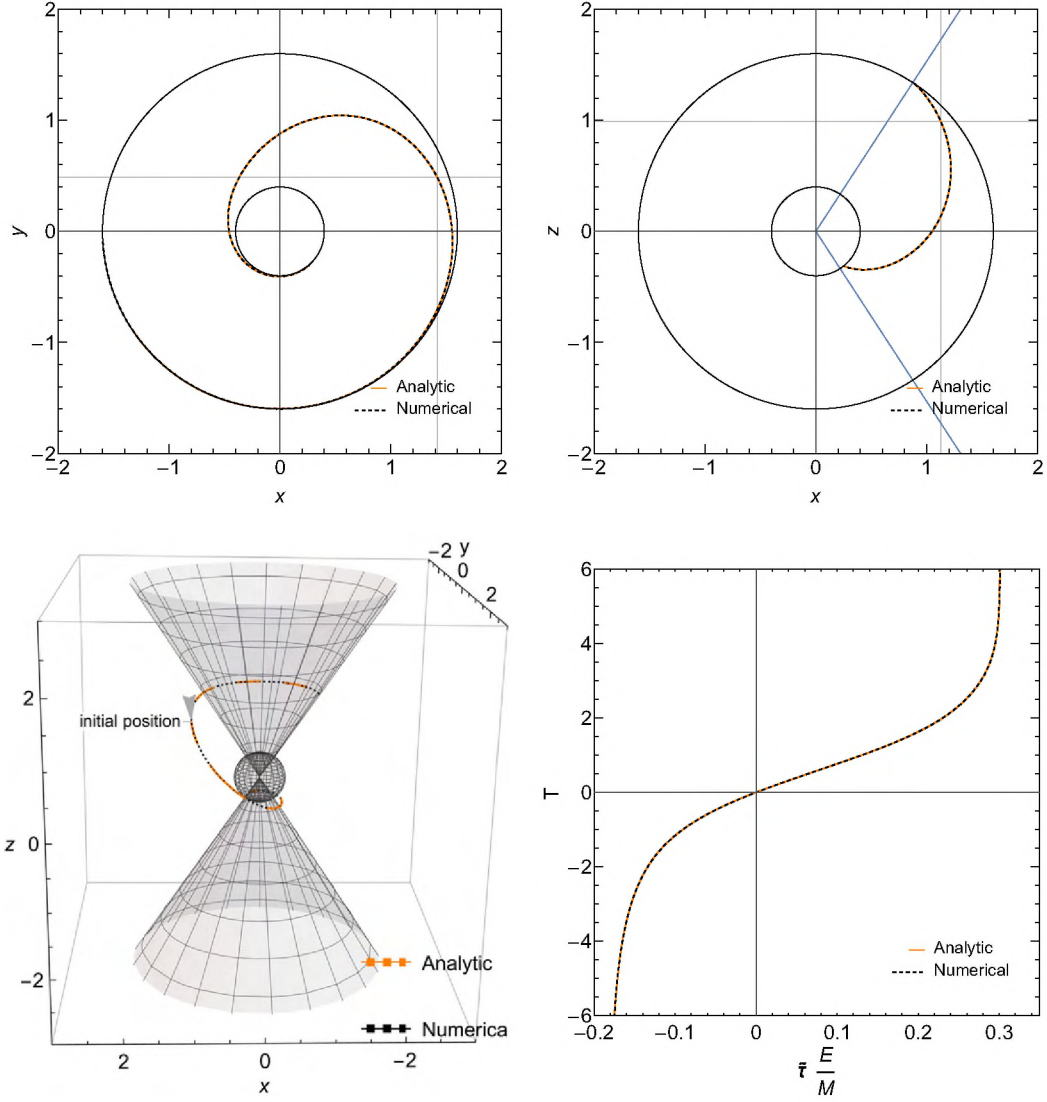


FIG. 8. Same as in Fig. 1, but for a null bound orbit with $\varepsilon^2 = 1$, $\lambda_z \approx 4.47214$, $\alpha = 0.6$, $\kappa = 60$, and an initial position at $\xi_0 = 1.5$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$.

The integral N_3 can be written in the form

$$N_3(s) = \mathcal{C}_1^2 L_1(s; p_{\bar{R},1}, p_{\bar{R},2}) + 2\mathcal{C}_1 \mathcal{C}_2 L_2(s; p_{\bar{R},1}, p_{\bar{R},2}) \\ + 2\mathcal{C}_1 \mathcal{C}_3 L_3(s; p_{\bar{R},1}, p_{\bar{R},2}) + 2\mathcal{C}_2 \mathcal{C}_3 L_4(s; p_{\bar{R},1}, p_{\bar{R},2}) \\ + \mathcal{C}_2^2 L_5(s; p_{\bar{R},1}, p_{\bar{R},2}) + \mathcal{C}_3^2 L_6(s; p_{\bar{R},1}, p_{\bar{R},2}), \quad (\text{A8})$$

where

$$L_1(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}, \quad (\text{A9a})$$

$$L_2(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{\wp(\bar{s}) d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}, \quad (\text{A9b})$$

$$L_3(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{\wp'(\bar{s}) d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}, \quad (\text{A9c})$$

$$L_4(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{\wp'(\bar{s}) \wp(\bar{s}) d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}, \quad (\text{A9d})$$

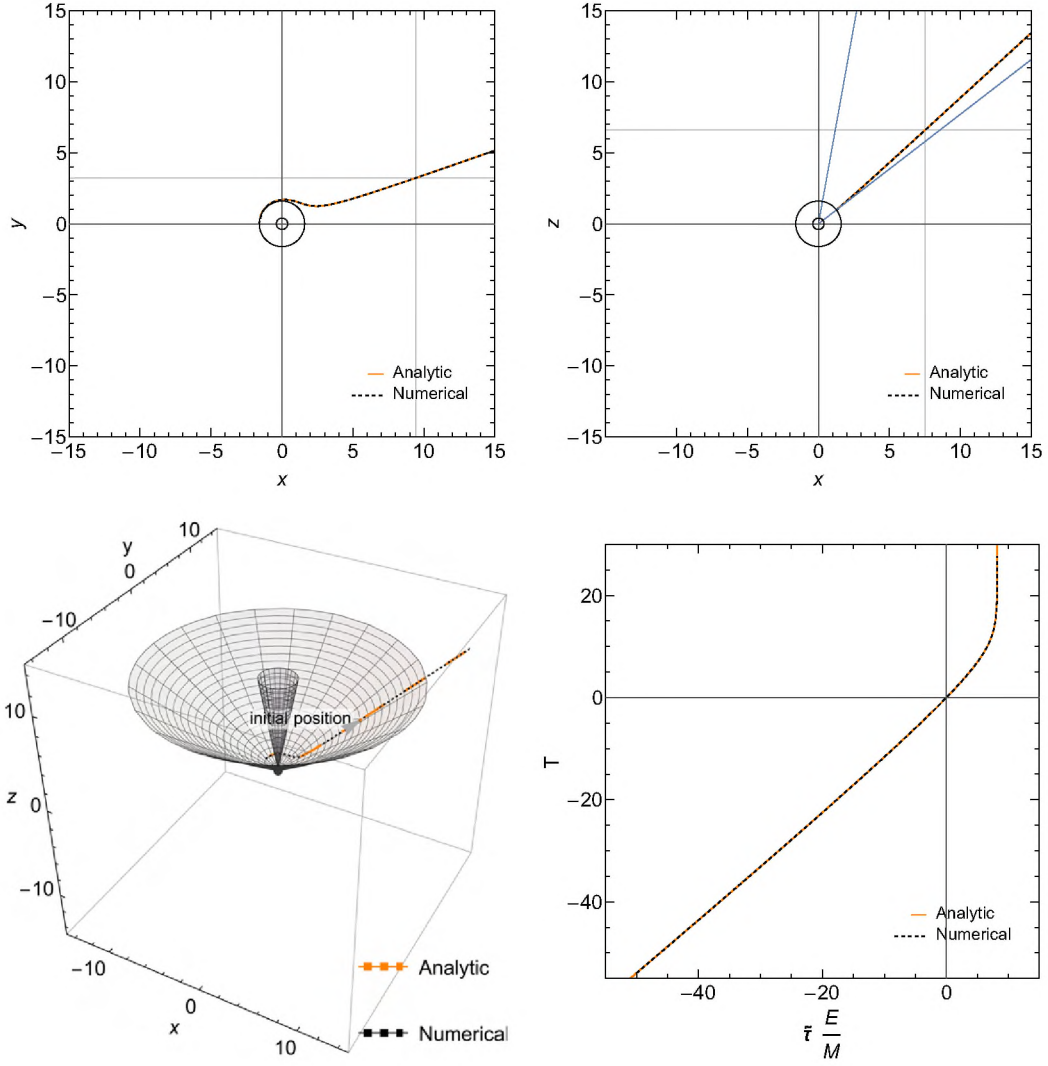


FIG. 9. Same as in Fig. 1, but for a null unbound orbit with $\varepsilon^2 = 1$, $\lambda_z \approx -0.111803$, $\alpha = 0.8$, $\kappa = 0.6$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$.

$$L_5(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{\wp(\bar{s})^2 d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}, \quad (\text{A9e})$$

$$L_6(s; p_{\bar{R},1}, p_{\bar{R},2}) := \int_0^s \frac{\wp'(\bar{s})^2 d\bar{s}}{\left[\left(\wp(\bar{s}) - \wp_{\bar{R},1} \right) \left(\wp(\bar{s}) - \wp_{\bar{R},2} \right) \right]^2}. \quad (\text{A9f})$$

Integrals (A9) are calculated in Appendix C. For $\varepsilon = \delta_1$ integrals (A5) and (A7) have a different form. They are listed in Appendix B 3.

2. Expression for J_θ

The integral

$$J_\theta(s) = \int_0^s \sin^2 \theta(\bar{s}) d\bar{s} \quad (\text{A10})$$

can be computed in a manner similar to I_θ . Let

$$z = \sin^2 \theta = 1 - \mu^2, \quad (\text{A11})$$

where $\mu = \cos \theta$. This gives

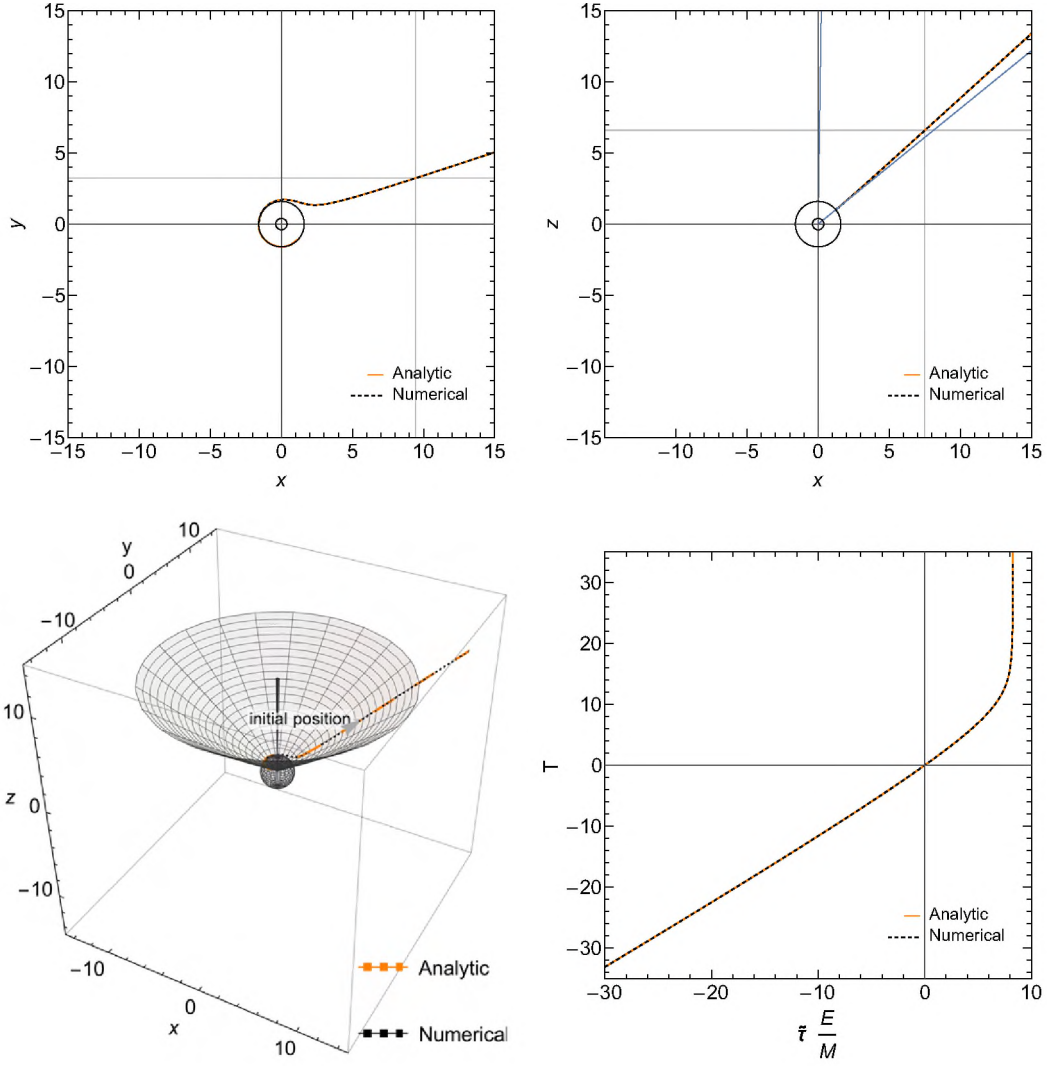


FIG. 10. Same as in Fig. 1, but for a null unbound orbit with $\varepsilon^2 = 1$, $\lambda_z \approx -0.00912871$, $\alpha = 0.8$, $\kappa = 0.4$, and an initial position at $\xi_0 = 10$, $\theta_0 = 0.85$, $\varphi_0 = 0.33$, $\epsilon_r = -1$, $\epsilon_\theta = 1$.

$$\begin{aligned} \frac{dz}{ds} &= -2\mu \frac{d\mu}{ds} = \epsilon_\theta \sqrt{4\mu^2 g(\mu)} \\ &= \epsilon_\theta \sqrt{4(1-z)[b_0(1-z)^2 + 6b_2(1-z) + b_4]} \\ &\equiv \epsilon_\theta \epsilon_\mu \sqrt{v(z)}. \end{aligned} \quad (\text{A12})$$

Here

$$v(z) \equiv 4e_1 z^3 + 6e_2 z^2 + 4e_3 z + e_4, \quad (\text{A13})$$

and

$$e_1 = \alpha^2(\varepsilon^2 - \delta_1) = d_3, \quad (\text{A14a})$$

$$e_2 = \frac{2}{3}[\alpha^2(2\delta_1 - \varepsilon^2) - \kappa - 2\alpha\lambda_z \varepsilon] = d_2, \quad (\text{A14b})$$

$$e_3 = \kappa - \alpha^2 \delta_1 + 2\alpha\lambda_z \varepsilon + \lambda_z^2 = d_1, \quad (\text{A14c})$$

$$e_4 = -4\lambda_z^2 = d_0, \quad (\text{A14d})$$

where coefficients $d_0, \dots, d_3$ are given by Eq. (68). Weierstrass invariants of the polynomial $v(z)$ read

$$g_{v,2} = -4e_1 e_3 + 3e_2^2, \quad (\text{A15a})$$

$$g_{v,3} = 2e_1 e_2 e_3 - e_2^3 - e_1^2 e_4, \quad (\text{A15b})$$

and not surprisingly, $g_{v,2} = g_{w,2}$ and $g_{v,3} = g_{w,3}$. Consequently,

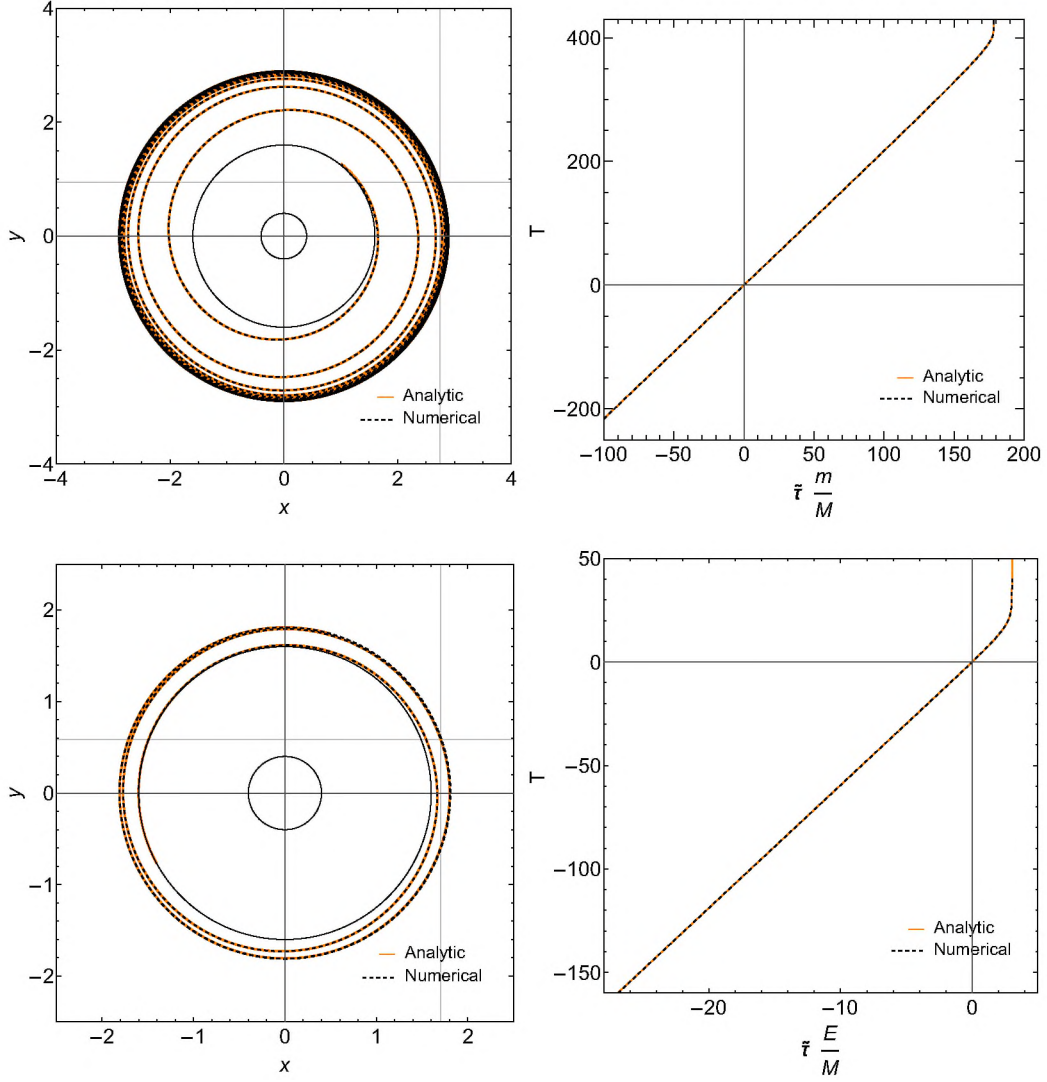


FIG. 11. Examples of spiraling orbits in the equatorial plane. Upper figures correspond to a timelike orbit with $\varepsilon^2 \approx 0.77064$, $\lambda_z \approx 2.38044$, $\alpha = 0.8$, $\kappa = (\lambda_z - \alpha\varepsilon)^2 \approx 2.81619$, and an initial position at $\xi_0 = \xi_{\text{ms}} - 0.01 \approx 2.89664$, $\theta_0 = \pi/2$, $\varphi_0 = 0.33$, $e_r = -1$. Lower figures correspond to a null orbit with $\varepsilon^2 = 1$, $\lambda_z \approx 3.2373$, $\alpha = 0.8$, $\kappa = (\lambda_z - \alpha\varepsilon)^2 \approx 5.94042$ and an initial position at $\xi_0 = \xi_{\text{ph}} - 0.01 \approx 1.80109$, $\theta_0 = \pi/2$, $\varphi_0 = 0.33$, $e_r = -1$.

$$z(s) = z_0 + \frac{-\epsilon_\theta \epsilon_\mu \sqrt{v(z_0)} \wp'_w(s) + \frac{1}{2} v'(z_0) [\wp_w(s) - \frac{1}{24} v''(z_0)] + \frac{1}{24} v(z_0) v'''(z_0)}{2[\wp_w(s) - \frac{1}{24} v''(z_0)]^2}, \quad (\text{A16})$$

where $z_0 = \sin^2 \theta_0$, $\wp_w(s) = \wp(s; g_{w,2}, g_{w,3})$. As before, we define

$$\mathcal{D}_1 = \frac{1}{48} v(z_0) v'''(z_0) - \frac{1}{96} v'(z_0) v''(z_0), \quad (\text{A17a})$$

$$\mathcal{D}_2 = \frac{1}{4} v'(z_0), \quad (\text{A17b})$$

Hence

$$\mathcal{D}_3 = -\frac{1}{2} \epsilon_\theta \epsilon_\mu \sqrt{v(z_0)}, \quad (\text{A17c})$$

$$\mathcal{D}_4 = \frac{1}{24} v''(z_0), \quad (\text{A17d})$$

$$\wp_{v,1} = \mathcal{D}_4 = \frac{1}{24} v''(z_0). \quad (\text{A17e})$$

$$J_\theta(s) = \int_0^s z(\bar{s}) d\bar{s} = \int_0^s \left\{ z_0 + \frac{\mathcal{D}_1 + \mathcal{D}_2 \wp_w(\bar{s}) + \mathcal{D}_3 \wp'_w(\bar{s})}{[\wp_w(\bar{s}) - \wp_{v,1}]^2} \right\} d\bar{s} \quad (B1d)$$

$$= z_0 s + \mathcal{D}_1 I_3(s; p_{v,1}) + \mathcal{D}_2 I_8(s; p_{v,1}) + \mathcal{D}_3 I_4(s; p_{v,1}), \quad (A18)$$

$$a_3 = \frac{1}{2} \kappa, \quad (B1e)$$

$$a_4 = 0, \quad (B1e)$$

where

$$I_3(s; p_{v,1}) := \int_0^s \frac{1}{[\wp(\bar{s}) - \wp_{v,1}]^2} d\bar{s}, \quad (A19a)$$

$$I_8(s; p_{v,1}) := \int_0^s \frac{\wp(\bar{s})}{[\wp(\bar{s}) - \wp_{v,1}]^2} d\bar{s}, \quad (A19b)$$

$$I_4(s; p_{v,1}) := \int_0^s \frac{\wp'(\bar{s})}{[\wp(\bar{s}) - \wp_{v,1}]^2} d\bar{s}, \quad (A19c)$$

and $p_{v,1}$ satisfies $\wp(p_{v,1}; g_{w,2}, g_{w,3}) = \mathcal{D}_4$. Integrals (A19) are calculated in Appendix C.

APPENDIX B: LIMITING CASES

In this appendix we discuss two special limiting cases: the Schwarzschild limit with $\alpha \rightarrow 0$, and the extreme Kerr limit with $\alpha \rightarrow \pm 1$.

1. Schwarzschild limit

The Biermann-Weierstrass formula has been used to provide an explicit description of Schwarzschild geodesics in [61,67]. The description given in [61] exploits the fact that the motion occurs in a single plane and uses the true anomaly as a geodesic parameter. The solution discussed briefly in [67] is adjusted to the setup of this paper—geodesics are parametrized by an equivalent of the Mino time. In this subsection we discuss $\alpha = 0$ limits of our solutions for $\xi(s)$, $\theta(s)$, $\varphi(s)$, and $T(s)$ and show that they coincide with the expressions given in [67].

a. Solution for $\xi(s)$

The form of the solution for $\xi(s)$ —Eq. (34)—does not change for $\alpha = 0$, but the coefficients (31) and Weierstrass invariants (33) of the polynomial $\tilde{R}$ acquire the following simpler form:

$$a_0 = \varepsilon^2 - \delta_1, \quad (B1a)$$

$$a_1 = \frac{1}{2} \delta_1, \quad (B1b)$$

$$a_2 = -\frac{1}{6} \kappa, \quad (B1c)$$

$$g_{\tilde{R},2} = -4a_1 a_3 + 3a_2^2 = \frac{1}{12} \kappa^2 - \delta_1 \kappa, \quad (B2a)$$

$$g_{\tilde{R},3} = 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 = \frac{1}{6} \kappa^3 - \frac{\delta_1}{12} \kappa^2 - \frac{\varepsilon^2 - \delta_1}{4} \kappa^2. \quad (B2b)$$

b. Solution for $\theta(s)$

For $\alpha = 0$ the solution for $\theta(s)$ —Eq. (40)—can be expressed in a simple form, involving elementary functions. This can be seen as follows. The coefficients of the polynomial $g(\mu)$ read, for $\alpha = 0$,

$$b_0 = 0, \quad (B3a)$$

$$b_2 = -\frac{1}{6} \kappa, \quad (B3b)$$

$$b_4 = \kappa - \lambda_z^2. \quad (B3c)$$

The appropriate Weierstrass invariants can be written as

$$g_{g,2} = 3b_2^2 = \frac{1}{12} \kappa^2, \quad (B4)$$

$$g_{g,3} = -b_2^3 = \frac{1}{6^3} \kappa^3. \quad (B5)$$

The value of $\wp(s; g_{g,2}, g_{g,3})$ can be computed by noting the following two identities ([68], p. 652, Eq. 18.12.27 and [66], Eq. 23.10.17):

$$\wp\left(z; \frac{1}{12}, \frac{1}{6^3}\right) = \frac{1}{2(1 - \cos z)} - \frac{1}{12}, \quad (B6)$$

$$\wp(z; c^4 g_2, c^6 g_3) = c^2 \wp(cz; g_2, g_3), \quad (B7)$$

where c is any nonzero real or complex constant. This gives

$$\begin{aligned} \wp(s; g_{g,2}, g_{g,3}) &= \wp\left(s; \frac{1}{12} \kappa^2, \frac{1}{6^3} \kappa^3\right) = \kappa \wp\left(\sqrt{\kappa} s; \frac{1}{12}, \frac{1}{6^3}\right) \\ &= \frac{\kappa}{2[1 - \cos(\sqrt{\kappa} s)]} - \frac{\kappa}{12}. \end{aligned} \quad (B8)$$

Differentiating the above expression with respect to s , we get

$$\wp'(s; g_{g,2}, g_{g,3}) = -\frac{\kappa^{3/2}}{2} \frac{1 + \cos(\sqrt{\kappa}s)}{[1 - \cos(\sqrt{\kappa}s)] \sin(\sqrt{\kappa}s)}. \quad (\text{B9})$$

Using Eqs. (B8) and (B9) in Eq. (40), it is easy to obtain the following expression for $\mu(s)$:

$$\mu(s) = \mu_0 \cos(\sqrt{\kappa}s) - \epsilon_\theta \sqrt{1 - \mu_0^2 - \frac{\lambda_z^2}{\kappa}} \sin(\sqrt{\kappa}s). \quad (\text{B10})$$

Denoting $\beta_0 = \frac{\epsilon_\theta}{\sqrt{\kappa}} \arcsin \frac{\mu_0}{\sqrt{1 - \frac{\lambda_z^2}{\kappa}}}$, we finally get

$$\mu(s) = -\epsilon_\theta \sqrt{1 - \frac{\lambda_z^2}{\kappa}} \sin[\sqrt{\kappa}(s - \beta_0)]. \quad (\text{B11})$$

This form has already been obtained for the Schwarzschild metric in [67].

c. Solution for $\varphi(s)$

The only term remaining in the expression for $\varphi(s) - \varphi(0)$ in the Schwarzschild limit is the integral I_θ . We have

$$\varphi(s) - \varphi(0) = I_\theta = \lambda_z \int_0^s q(\bar{s}) d\bar{s}. \quad (\text{B12})$$

The expression for $q(s) = 1/\sin^2 \theta(s)$ —Eq. (70)—can also be simplified for $\alpha = 0$. The coefficients and Weierstrass invariants of the polynomial $w(q)$ have the form

$$d_0 = -4\lambda_z^2, \quad (\text{B13a})$$

$$d_1 = \kappa + \lambda_z^2, \quad (\text{B13b})$$

$$d_2 = -\frac{2}{3}\kappa, \quad (\text{B13c})$$

$$d_3 = 0, \quad (\text{B13d})$$

and

$$g_{w,2} = \frac{4}{3}\kappa^2, \quad (\text{B14a})$$

$$g_{w,3} = \left(\frac{2}{3}\right)^3 \kappa^3. \quad (\text{B14b})$$

We find that

$$\wp(s; g_{w,2}, g_{w,3}) = \wp\left(s; \frac{4}{3}\kappa^2, \frac{2^3}{3}\kappa^3\right) = \kappa \left[\frac{1}{\sin^2(\sqrt{\kappa}s)} - \frac{1}{3} \right] \quad (\text{B15})$$

and

$$\wp'(s; g_{w,2}, g_{w,3}) = -2\kappa^{\frac{3}{2}} \frac{\cos(\sqrt{\kappa}s)}{\sin^3(\sqrt{\kappa}s)}. \quad (\text{B16})$$

An explicit expression for $q(s)$ is lengthy, but it can be shown that it simplifies to $q(s) = 1/[1 - \mu(s)^2]$, where $\mu(s)$ is given by Eq. (B10) or, equivalently, Eq. (B11). This gives

$$\begin{aligned} \varphi(s) - \varphi(0) &= I_\theta = \lambda_z \int_0^s \frac{d\bar{s}}{1 - \mu^2(\bar{s})} \\ &= \lambda_z \int_0^s \frac{d\bar{s}}{1 - \left(1 - \frac{\lambda_z^2}{\kappa}\right) \sin^2[\sqrt{\kappa}(s - \beta_0)]}. \end{aligned} \quad (\text{B17})$$

Evaluating the last integral, we obtain

$$\begin{aligned} \varphi(s) - \varphi(0) &= \arctan \left[\frac{\lambda_z}{\sqrt{\kappa}} \tan[\sqrt{\kappa}(s - \beta_0)] \right] \\ &\quad + \arctan \left[\frac{\lambda_z}{\sqrt{\kappa}} \tan(\sqrt{\kappa}\beta_0) \right] + n\pi, \end{aligned} \quad (\text{B18})$$

where $n \in \mathbb{Z}$, in agreement with a formula given in [67].

d. Solution for $T(s)$

Assuming $\alpha = 0$, we get, from Eq. (83),

$$T(s) - T(0) = \varepsilon \int_0^s \frac{\xi^3(\bar{s})}{\xi(\bar{s}) - 2} d\bar{s}. \quad (\text{B19})$$

Noting that

$$\frac{\xi^3}{\xi - 2} = \xi^2 + 2\xi + 4 + \frac{8}{\xi - 2}, \quad (\text{B20})$$

one can write

$$\begin{aligned} T(s) - T(0) &= \varepsilon \int_0^s \xi^2(\bar{s}) d\bar{s} + 2\varepsilon \int_0^s \xi(\bar{s}) d\bar{s} + 4\varepsilon s \\ &\quad + 8\varepsilon \int_0^s \frac{d\bar{s}}{\xi(\bar{s}) - 2} \\ &= \varepsilon N_2(s) + 2\varepsilon N_1(s) + 4\varepsilon s + 8\varepsilon N_H^+(s). \end{aligned} \quad (\text{B21})$$

Integrals N_1 and N_2 are given by Eqs. (90) and (91). The integral N_H^+ can still be computed as in Sec. III, i.e.,

$$N_H^+(s) = \int_0^s \frac{d\bar{s}}{\xi(\bar{s}) - 2} = \int_0^s u_+(\bar{s}) d\bar{s}. \quad (\text{B22})$$

Here u_+ satisfies Eq. (49) with the polynomial $h(u_+)$ given by Eq. (50). On the other hand, coefficients $c_0, \dots, c_4$ have, for $\alpha = 0$, the following simple form

$$c_0 = 16\varepsilon^2, \quad (\text{B23})$$

$$c_1 = 8\varepsilon^2 - \frac{1}{2}\kappa - 2\delta_1, \quad (\text{B24})$$

$$c_2 = 4\varepsilon^2 - \frac{1}{6}\kappa - 2\delta_1, \quad (\text{B25})$$

$$c_3 = 2\varepsilon^2 - \frac{3}{2}\delta_1, \quad (\text{B26})$$

$$c_4 = \varepsilon^2 - \delta_1. \quad (\text{B27})$$

As in the general case, the Weierstrass invariants $g_{h,2}$ and $g_{h,3}$ of the polynomial $h(u_+)$ coincide with $g_{\bar{R},2}$ and $g_{\bar{R},3}$, respectively.

2. Extreme Kerr limit

A few equations have to be modified in the limit of $\alpha \rightarrow \pm 1$ (extreme Kerr spacetime). In this case $\Delta = r^2 - 2Mr + a^2 = (r - M)^2$ and $\xi^2 - 2\xi + \alpha^2 = (\xi - 1)^2$.

Equation (43) reads

$$I_\xi(s) = \alpha \int_0^s \frac{2\xi(\bar{s})\varepsilon - \alpha\lambda_z}{[\xi(\bar{s}) - 1]^2} d\bar{s}. \quad (\text{B28})$$

A partial fraction expansion now gives

$$I_\xi(s) = 2\alpha\varepsilon N_4(s) + (2\alpha\varepsilon - \lambda_z)N_5(s), \quad (\text{B29})$$

where

$$N_4(s) = \int_0^s \frac{d\bar{s}}{\xi(\bar{s}) - 1}, \quad N_5(s) = \int_0^s \frac{d\bar{s}}{[\xi(\bar{s}) - 1]^2}. \quad (\text{B30})$$

Integral N_4 can be computed as in Eqs. (46)–(62). Defining $u(s) = 1/[\xi(s) - 1]$, we get

$$\frac{du}{ds} = -\varepsilon_r \sqrt{h(u)}, \quad (\text{B31})$$

where

$$h(u) = c_0 u^4 + 4c_1 u^3 + 6c_2 u^2 + 4c_3 u + c_4, \quad (\text{B32})$$

and

$$c_0 = 4\varepsilon^2 - 4\alpha\varepsilon\lambda_z + \lambda_z^2, \quad (\text{B33a})$$

$$c_1 = 2\varepsilon^2 - \alpha\varepsilon\lambda_z, \quad (\text{B33b})$$

$$c_2 = \frac{1}{6}(-\delta_1 + 8\varepsilon^2 - \kappa - 2\alpha\varepsilon\lambda_z), \quad (\text{B33c})$$

$$c_3 = \varepsilon^2 - \frac{1}{2}\delta_1, \quad (\text{B33d})$$

$$c_4 = \varepsilon^2 - \delta_1. \quad (\text{B33e})$$

Solution for u is then given, as in Eq. (58), and integrals N_4 and N_5 can be computed as

$$N_4(s) = \int_0^s u(\bar{s}) d\bar{s}, \quad N_5(s) = \int_0^s u^2(\bar{s}) d\bar{s}. \quad (\text{B34})$$

Similarly, the integral $J_\xi(s)$ appearing in Eq. (83) reads, for $\alpha = \pm 1$,

$$\begin{aligned} J_\xi(s) &= \int_0^s \frac{[\xi^2(\bar{s}) + 1]^2 \varepsilon - 2\alpha\lambda_z \xi(\bar{s})}{[\xi(\bar{s}) - 1]^2} d\bar{s} \\ &= 5\varepsilon s + 2\varepsilon \int_0^s \xi(\bar{s}) d\bar{s} + \varepsilon \int_0^s \xi^2(\bar{s}) d\bar{s} \\ &\quad + 2(4\varepsilon - \alpha\lambda_z)N_4(s) + 2(2\varepsilon - \alpha\lambda_z)N_5(s). \end{aligned} \quad (\text{B35})$$

3. The case with $\varepsilon = \delta_1$

In this subsection we compute integrals (A5) and (A7) in the special case with $\varepsilon = \delta_1$. In this case $\mathcal{C}_5 = 0$ [Eq. (A3e)], and Eq. (A4) has the following form

$$\xi(s) = \xi_0 + \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(s) - \wp_{\bar{R},3}]^2}, \quad (\text{B36})$$

where $\wp_{\bar{R},3} = \wp(p_{\bar{R},3}; g_{\bar{R},2}, g_{\bar{R},3}) = \mathcal{C}_4$. Thus

$$\begin{aligned} N_1(s) &= \int_0^s \xi(\bar{s}) d\bar{s} = \int_0^s \left\{ \xi_0 + \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(s) - \wp_{\bar{R},3}]^2} \right\} d\bar{s} \\ &= \xi_0 s + \mathcal{C}_1 I_3(s; p_{\bar{R},3}) + \mathcal{C}_2 I_8(s; p_{\bar{R},3}) + \mathcal{C}_3 I_4(s; p_{\bar{R},3}), \end{aligned} \quad (\text{B37})$$

and

$$\begin{aligned}
N_3(s) = & \int_0^s \left\{ \frac{\mathcal{C}_1 + \mathcal{C}_2 \wp_{\bar{R}}(s) + \mathcal{C}_3 \wp'_{\bar{R}}(s)}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^2} \right\}^2 d\bar{s} = \mathcal{C}_1^2 I_{19}(s; p_{\bar{R},3}) \\
& + 2\mathcal{C}_1 \mathcal{C}_2 I_{20}(s; p_{\bar{R},3}) + 2\mathcal{C}_1 \mathcal{C}_3 I_{21}(s; p_{\bar{R},3}) \\
& + 2\mathcal{C}_2 \mathcal{C}_3 I_{22}(s; p_{\bar{R},3}) + \mathcal{C}_2^2 I_{23}(s; p_{\bar{R},3}) + \mathcal{C}_3^2 I_{25}(s; p_{\bar{R},3}),
\end{aligned} \tag{B38}$$

where

$$I_3(s; p_{\bar{R},3}) := \int_0^s \frac{1}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^2} d\bar{s}, \tag{B39a}$$

$$I_4(s; p_{\bar{R},3}) := \int_0^s \frac{\wp'_{\bar{R}}(\bar{s})}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^2} d\bar{s}, \tag{B39b}$$

$$I_8(s; p_{\bar{R},3}) := \int_0^s \frac{\wp_{\bar{R}}(\bar{s})}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^2} d\bar{s}, \tag{B39c}$$

$$I_{19}(s; p_{\bar{R},3}) := \int_0^s \frac{1}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}, \tag{B39d}$$

$$I_{20}(s; p_{\bar{R},3}) := \int_0^s \frac{\wp_{\bar{R}}(\bar{s})}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}, \tag{B39e}$$

$$I_{21}(s; p_{\bar{R},3}) := \int_0^s \frac{\wp'_{\bar{R}}(\bar{s})}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}, \tag{B39f}$$

$$I_{22}(s; p_{\bar{R},3}) := \int_0^s \frac{\wp'_{\bar{R}}(\bar{s}) \wp_{\bar{R}}(\bar{s})}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}, \tag{B39g}$$

$$I_{23}(s; p_{\bar{R},3}) := \int_0^s \frac{\wp_{\bar{R}}(\bar{s})^2}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}, \tag{B39h}$$

$$I_{25}(s; p_{\bar{R},3}) := \int_0^s \frac{\wp'_{\bar{R}}(\bar{s})^2}{[\wp_{\bar{R}}(\bar{s}) - \wp_{\bar{R},3}]^4} d\bar{s}. \tag{B39i}$$

A analogous analysis should be carried out for initial parameters yielding $\mathcal{A}_{5\pm} = 0$ or $\mathcal{B}_5 = 0$.

APPENDIX C: INTEGRALS

In this appendix, we collect the formulas for all elliptic integrals appearing in this paper and used to solve Kerr geodesic equations. We adopt a convention according to which in all following formulas x denotes the variable, while y and z are treated as parameters. This is emphasized by separating the variable x from the parameters y and z with the semicolon. The integrals are based on the results presented in [69] (pp. 626) [70] (pp. 311), [71] (vol.4, pp. 109–110)

$$I_1(x; y) = \int \frac{dx}{\wp(x) - \wp(y)} = \frac{1}{\wp'(y)} \left(2\zeta(y)x + \ln \frac{\sigma(x-y)}{\sigma(x+y)} \right), \tag{C1}$$

$$I_2(x; y) = \int \frac{\wp'(x) dx}{\wp(x) - \wp(y)} = \ln(\wp(x) - \wp(y)), \tag{C2}$$

$$\begin{aligned}
I_3(x; y) = & \int \frac{dx}{(\wp(x) - \wp(y))^2} = -\frac{\wp''(y)}{\wp'(y)^3} \ln \frac{\sigma(x-y)}{\sigma(x+y)} \\
& - \frac{1}{\wp'(y)^2} \left(\zeta(x+y) + \zeta(x-y) + \left(2\wp(y) + \frac{2\wp''(y)\zeta(y)}{\wp'(y)} \right) x \right),
\end{aligned} \tag{C3}$$

$$I_4(x; y) = \int \frac{\wp'(x) dx}{(\wp(x) - \wp(y))^2} = -\frac{1}{\wp(x) - \wp(y)}, \tag{C4}$$

$$I_5(x; y) = \int \frac{\wp'(x) \wp(x) dx}{\wp(x) - \wp(y)} = \wp(y) \ln(\wp(x) - \wp(y)) + \wp(x), \tag{C5}$$

$$I_6(x; y) = \int \frac{\wp'(x) \wp(x) dx}{(\wp(x) - \wp(y))^2} = \ln(\wp(x) - \wp(y)) - \frac{\wp(y)}{\wp(x) - \wp(y)}, \tag{C6}$$

$$I_7(x; y) = \int \frac{\wp(x)dx}{\wp(x) - \wp(y)} = x + \frac{\wp(y)}{\wp'(y)} \left(2\zeta(y)x + \ln \frac{\sigma(x-y)}{\sigma(x+y)} \right), \quad (C7)$$

$$I_8(x; y) = \int \frac{\wp(x)dx}{(\wp(x) - \wp(y))^2} = I_1(x; y) + \wp(y)I_3(x; y), \quad (C8)$$

$$I_9(x; y) = \int \frac{\wp^2(x)dx}{\wp(x) - \wp(y)} = \wp(y)I_7(x; y) - \zeta(x), \quad (C9)$$

$$I_{10}(x; y) = \int \frac{\wp^3(x)dx}{\wp(x) - \wp(y)} = \wp(y)I_9(x; y) + \frac{1}{6}\wp'(x) + \frac{g_2}{12}x, \quad (C10)$$

$$I_{11}(x; y) = \int \frac{\wp^2(x)dx}{\wp(x) - \wp(y)} = 4I_{10}(x; y) - g_2I_7(x; y) - g_3I_1(x; y), \quad (C11)$$

$$I_{12}(x; y) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))^2} = I_7(x; y) + \wp(y)I_8(x; y), \quad (C12)$$

$$I_{13}(x; y) = \int \frac{\wp^3(x)dx}{(\wp(x) - \wp(y))^2} = I_9(x; y) + \wp(y)I_{12}(x; y), \quad (C13)$$

$$I_{14}(x; y) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))^2} = 4I_{13}(x; y) - g_2I_8(x; y) - g_3I_3(x; y), \quad (C14)$$

$$I_{15}(x; y) = \int \frac{dx}{(\wp(x) - \wp(y))^3} = \frac{1}{\wp'(y)^3} (\wp(x+y) - \wp(x-y) - 2\wp'(y)x - 12\wp(y)\wp'(y)I_1(x; y) - 3\wp'(y)\wp''(y)I_3(x; y)), \quad (C15)$$

$$I_{16}(x; y) = \int \frac{\wp'(x)dx}{(\wp(x) - \wp(y))^3} = -\frac{1}{2(\wp(x) - \wp(y))^2}, \quad (C16)$$

$$I_{17}(x; y) = \int \frac{\wp(x)dx}{(\wp(x) - \wp(y))^3} = I_3(x; y) + \wp(y)I_{15}(x; y), \quad (C17)$$

$$I_{18}(x; y) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))^3} = I_8(x; y) + \wp(y)I_{17}(x; y), \quad (C18)$$

$$I_{19}(x; y) = \int \frac{dx}{(\wp(x) - \wp(y))^4} = \frac{1}{6\wp'(y)^4} \left[\wp'(x+y) + \wp'(x-y) - 2\wp''(y)x - \wp^{(4)}(y)I_1(x; y) - (3\wp''(y)^2 + 48\wp(y)\wp'(y))I_3(x; y) - 12\wp''(y)\wp'(y)^2I_{19}(x; y) \right], \quad (C19)$$

$$I_{20}(x; y) = \int \frac{\wp(x)dx}{(\wp(x) - \wp(y))^4} = I_{15}(x; y) + \wp(y)I_{19}(x; y), \quad (C20)$$

$$I_{21}(x; y) = \int \frac{\wp'(x)dx}{(\wp(x) - \wp(y))^4} = -\frac{1}{3(\wp(x) - \wp(y))^3}, \quad (C21)$$

$$I_{22}(x; y) = \int \frac{\wp(x)\wp'(x)dx}{(\wp(x) - \wp(y))^4} = -\frac{3\wp(x) - \wp(y)}{6(\wp(x) - \wp(y))^3}, \quad (C22)$$

$$I_{23}(x; y) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))^4} = I_{17}(x; y) + \wp(y)I_{20}(x; y), \quad (C23)$$

$$I_{24}(x; y) = \int \frac{\wp^3(x)dx}{(\wp(x) - \wp(y))^4} = I_{18}(x; y) + \wp(y)I_{23}(x; y), \quad (C24)$$

$$I_{25}(x; y) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))^4} = 4I_{24}(x; y) - g_2I_{20}(x; y) - g_3I_{19}(x; y), \quad (C25)$$

$$K_1(x; y, z) = \int \frac{dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_1(x; y) - I_1(x; z)}{\wp(y) - \wp(z)}, \quad (C26)$$

$$K_2(x; y, z) = \int \frac{\wp(x)dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_7(x; y) - I_7(x; z)}{\wp(y) - \wp(z)}, \quad (C27)$$

$$K_3(x; y, z) = \int \frac{\wp'(x)dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_2(x; y) - I_2(x; z)}{\wp(y) - \wp(z)}, \quad (C28)$$

$$K_4(x; y, z) = \int \frac{\wp(x)\wp'(x)dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_5(x; y) - I_5(x; z)}{\wp(y) - \wp(z)}, \quad (C29)$$

$$K_5(x; y, z) = \int \frac{\wp^2(x)dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_9(x; y) - I_9(x; z)}{\wp(y) - \wp(z)}, \quad (C30)$$

$$K_6(x; y, z) = \int \frac{\wp'^2(x)dx}{(\wp(x) - \wp(y))(\wp(x) - \wp(z))} = \frac{I_{11}(x; y) - I_{11}(x; z)}{\wp(y) - \wp(z)}, \quad (C31)$$

$$L_1(x; y, z) = \int \frac{dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_3(x; y) - 2K_1(x; y, z) + I_3(x; z)}{(\wp(y) - \wp(z))^2}, \quad (C32)$$

$$L_2(x; y, z) = \int \frac{\wp(x)dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_8(x; y) - 2K_2(x; y, z) + I_8(x; z)}{(\wp(y) - \wp(z))^2}, \quad (C33)$$

$$L_3(x; y, z) = \int \frac{\wp'(x)dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_4(x; y) - 2K_3(x; y, z) + I_4(x; z)}{(\wp(y) - \wp(z))^2}, \quad (C34)$$

$$L_4(x; y, z) = \int \frac{\wp'(x)\wp(x)dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_6(x; y) - 2K_4(x; y, z) + I_6(x; z)}{(\wp(y) - \wp(z))^2}, \quad (C35)$$

$$L_5(x; y, z) = \int \frac{\wp(x)^2dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_{12}(x; y) - 2K_5(x; y, z) + I_{12}(x; z)}{(\wp(y) - \wp(z))^2}, \quad (C36)$$

$$L_6(x; y, z) = \int \frac{\wp'(x)^2dx}{[(\wp(x) - \wp(y))(\wp(x) - \wp(z))]^2} = \frac{I_{14}(x; y) - 2K_6(x; y, z) + I_{14}(x; z)}{(\wp(y) - \wp(z))^2}. \quad (C37)$$

- [1] I. D. Novikov and K. S. Thorne, in *Les Houches Summer School of Theoretical Physics: Black Holes*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1973).
- [2] D. N. Page and K. S. Thorne, Disk-accretion onto a black hole. Time-averaged structure of accretion disk, *Astrophys. J.* **191**, 499 (1974).
- [3] S. E. Gralla and A. Lupsasca, Lensing by Kerr black holes, *Phys. Rev. D* **101**, 044031 (2020).
- [4] S. E. Gralla and A. Lupsasca, Observable shape of black hole photon rings, *Phys. Rev. D* **102**, 124003 (2020).
- [5] Y. Mino, Perturbative approach to an orbital evolution around a supermassive black hole, *Phys. Rev. D* **67**, 084027 (2003).
- [6] T. Piran, J. Shaham, and J. Katz, High efficiency of the Penrose mechanism for particle collisions, *Astrophys. J.* **196**, L107 (1975).
- [7] B. Carter, Global structure of the Kerr family of gravitational fields, *Phys. Rev.* **174**, 1559 (1968).
- [8] B. Carter, Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations, *Commun. Math. Phys.* **10**, 280 (1968).
- [9] M. Walker and R. Penrose, On quadratic first integrals of the geodesic equations for type {22} spacetimes, *Commun. Math. Phys.* **18**, 265 (1970).
- [10] F. De Felice, Equatorial geodesic motion in the gravitational field of a rotating source, *Il Nuovo Cimento Soc. Ital. Fis.* **57B**, 351 (1968).
- [11] J. M. Bardeen, Stability of circular orbits in stationary, axisymmetric space-times, *Astrophys. J.* **161**, 103 (1970).
- [12] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation, *Astrophys. J.* **178**, 347 (1972).
- [13] F. De Felice and M. Calvani, Orbital and vortical motion in the Kerr metric, *Il Nuovo Cimento Soc. Ital. Fis.* **10B**, 447 (1972).
- [14] M. Calvani and F. De Felice, Vortical null orbits, repulsive barriers, energy confinement in Kerr metric, *Gen. Relativ. Gravit.* **9**, 889 (1978).
- [15] D. C. Wilkins, Bound geodesics in the Kerr metric, *Phys. Rev. D* **5**, 814 (1972).
- [16] J. M. Bardeen, Timelike and null geodesics in the Kerr metric, in *Les Houches Summer School of Theoretical Physics: Black Holes*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1973).
- [17] N. A. Sharp, Geodesics in black hole space-times, *Gen. Relativ. Gravit.* **10**, 659 (1979).
- [18] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford 1983).
- [19] B. O'Neill, *The Geometry of Kerr Black Holes* (A. K. Peters, Ltd., Wellesley, Massachusetts, 1995).
- [20] G. Contopoulos, Orbits through the ergosphere of a Kerr black hole, *Gen. Relativ. Gravit.* **16**, 43 (1984).
- [21] A. A. Grib, Yu. V. Pavlov, and V. D. Vertogradov, Geodesics with negative energy in the ergosphere of rotating black holes, *Mod. Phys. Lett. A* **29**, 1450110 (2014).
- [22] V. D. Vertogradov, Geodesics for particles with negative energy in Kerr's metric, *Gravitation Cosmol.* **21**, 171 (2015).
- [23] E. Teo, Spherical photon orbits around a Kerr black hole, *Gen. Relativ. Gravit.* **35**, 1909 (2003).
- [24] E. Teo, Spherical orbits around a Kerr black hole, *Gen. Relativ. Gravit.* **53**, 10 (2021).
- [25] A. Tavlayan and B. Tekin, Exact formulas for spherical photon orbits around Kerr black holes, *Phys. Rev. D* **102**, 104036 (2020).
- [26] W. Schmidt, Celestial mechanics in Kerr spacetime, *Classical Quantum Gravity* **19**, 2743 (2002).
- [27] S. Drasco and S. A. Hughes, Rotating black hole orbit functionals in the frequency domain, *Phys. Rev. D* **69**, 044015 (2004).
- [28] R. Fujita and W. Hikida, Analytical solutions of bound timelike geodesic orbits in Kerr spacetime, *Classical Quantum Gravity* **26**, 135002 (2009).
- [29] P. Rana and A. Mangalam, Astrophysically relevant bound trajectories around a Kerr black hole, *Classical Quantum Gravity* **36**, 045009 (2019).
- [30] G. Slezáková, Geodesic geometry of black holes, Ph.D. thesis, University of Waikato, 2006.
- [31] M. van de Meent, Analytic solutions for parallel transport along generic bound geodesics in Kerr spacetime, *Classical Quantum Gravity* **37**, 145007 (2020).
- [32] E. Hackmann, Geodesic equations in black hole space-times with cosmological constant, Ph.D. thesis, Bremen, 2010.
- [33] E. Hackmann and C. Lämmerzahl, Analytical solution methods for geodesic motion, *AIP Conf. Proc.* **1577**, 78 (2014).
- [34] C. Lämmerzahl and E. Hackmann, Analytical solutions for geodesic equation in black hole spacetimes, *Springer Proc. Phys.* **170**, 43 (2016).
- [35] J. Levin and G. Perez-Giz, Homoclinic orbits around spinning black holes, I. Exact solution for the Kerr separatrix, *Phys. Rev. D* **79**, 124013 (2009).
- [36] G. Perez-Giz and J. Levin, Homoclinic orbits around spinning black holes II: The phase space portrait, *Phys. Rev. D* **79**, 124014 (2009).
- [37] L. C. Stein and N. Warburton, Location of the last stable orbit in Kerr spacetime, *Phys. Rev. D* **101**, 064007 (2020).
- [38] A. Mummery and S. Balbus, Inspirals from the Innermost Stable Circular Orbit of the Kerr Black Holes: Exact Solutions and Universal Radial Flow, *Phys. Rev. Lett.* **129**, 161101 (2022).
- [39] A. Mummery and S. Balbus, Complete characterization of the orbital shapes of the noncircular Kerr geodesic solutions with circular orbit constants of motion, *Phys. Rev. D* **107**, 124058 (2023).
- [40] C. Dyson and M. van de Meent, Kerr-fully diving into the abyss: Analytic solutions to plunging geodesics in Kerr, [arXiv:2302.03704](https://arxiv.org/abs/2302.03704).
- [41] K. Glampedakis and D. Kennefick, Zoom and whirl: Eccentric equatorial orbits around spinning black holes and their evolution under gravitational radiation reaction, *Phys. Rev. D* **66**, 044002 (2002).
- [42] R. W. O'Shaughnessy, Transition from inspiral to plunge for eccentric equatorial Kerr orbits, *Phys. Rev. D* **67**, 044004 (2003).
- [43] A. Cieřlik, P. Mach, and A. Odrzywoć, Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole, *Phys. Rev. D* **106**, 104056 (2022).

- [44] J. Levin and G. Perez-Giz, A periodic table for black hole orbits, *Phys. Rev. D* **77**, 103005 (2008).
- [45] R. Gonzo and C. Shi, Boundary to bound dictionary for generic Kerr orbits, [arXiv:2304.06066](https://arxiv.org/abs/2304.06066).
- [46] S. E. Gralla and A. Lupsasca, Null geodesics of the Kerr exterior, *Phys. Rev. D* **101**, 044032 (2020).
- [47] G. Compère, Y. Liu, and J. Long, Classification of the radial Kerr geodesic motion, *Phys. Rev. D* **105**, 024075 (2022).
- [48] S. Hod, Marginally bound (critical) geodesics of rapidly rotating black holes, *Phys. Rev. D* **88**, 087502 (2013).
- [49] S. Hadar, A. P. Porfyriadis, and A. Strominger, Gravity waves from extreme-mass-ratio plunges into Kerr black holes, *Phys. Rev. D* **90**, 064045 (2014).
- [50] S. E. Gralla, A. P. Porfyriadis, and N. Warburton, Particle on the innermost stable circular orbit of a rapidly spinning black hole, *Phys. Rev. D* **92**, 064029 (2015).
- [51] D. Kapec and A. Lupsasca, Particle motion near high-spin black holes, *Classical Quantum Gravity* **37**, 015006 (2020).
- [52] G. Compère and A. Druart, Near-horizon geodesics of high spin black holes, *Phys. Rev. D* **101**, 084042 (2020); **102**, 029901(E) (2020).
- [53] K. P. Rauch and R. D. Blandford, Optical caustics in a Kerr spacetime and the origin of rapid x-ray variability in active galactic nuclei, *Astrophys. J.* **421**, 46 (1994).
- [54] S. E. Gralla, A. Lupsasca, and D. P. Marrone, The shape of the black hole photon ring: A precise test of strong-field general relativity, *Phys. Rev. D* **102**, 124004 (2020).
- [55] D. E. A. Gates, S. Hadar, and A. Lupsasca, Maximum observable blueshift from circular equatorial Kerr orbiters, *Phys. Rev. D* **102**, 104041 (2020).
- [56] E. Himwich, M. D. Johnson, A. Lupsasca, and A. Strominger, Universal polarimetric signatures of the black hole photon ring, *Phys. Rev. D* **101**, 084020 (2020).
- [57] W. Biermann, *Problemata quaedam mechanica functionum ellipticarum ope soluta*, Ph.D. thesis, Schultze, Berlin, 1865.
- [58] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).
- [59] A. G. Greenhill, *The Applications of Elliptic Functions* (Macmillan, London, 1892).
- [60] M. J. Reynolds, An exact solution in non-linear oscillations, *J. Phys. A* **22**, L723 (1989).
- [61] A. Cieřlik and P. Mach, Revisiting timelike and null geodesics in the Schwarzschild spacetime: General expressions in terms of Weierstrass elliptic functions, *Classical Quantum Gravity* **39**, 225003 (2022).
- [62] Wolfram Research, Inc., *Mathematica*, Version 13.1, Champaign, IL (2022).
- [63] The package created for this paper is available at <https://doi.pangaea.de/10.1594/PANGAEA.960855>, and possible updates at <https://github.com/AdamCieslik/Kerr-Geodesics-in-Terms-of-Weierstrass-Elliptic-Functions>.
- [64] P. Rioseco and O. Sarbach, Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole, *Classical Quantum Gravity* **34**, 095007 (2017).
- [65] P. Rioseco and O. Sarbach, Phase space mixing in the equatorial plane of the Kerr black hole, *Phys. Rev. D* **98**, 124024 (2018).
- [66] F. W. J. Olver, A. B. Olde Daalhuis *et al.*, NIST Digital Library of Mathematical Functions, Release 1.1.8 of 2022-12-15, <http://dlmf.nist.gov/>.
- [67] A. Cieřlik and P. Mach, Timelike and null geodesics in the Schwarzschild space-time: Analytical solutions, *Acta Phys. Pol. B Proc. Suppl.* **16**, 6-A10 (2023).
- [68] M. Abramowitz, *Handbook of Mathematical Functions* (United States Department of Commerce, National Bureau of Standards, 1964).
- [69] I. S. Gradshteyn, I. M. Ryzhik, and A. Jeffrey, *Table of Integrals, Series, and Products* (Academic Press, Amsterdam, Boston, 2007).
- [70] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer, Berlin, Heidelberg, 1971).
- [71] J. Tannery and J. Molk, *Éléments de la théorie des Fonctions Elliptiques* (Gauthier-Villars, Paris, 1893–1902), Vol. 4.

2.6. Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole

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Accretion of the relativistic Vlasov gas in the equatorial plane of the Kerr black hole

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We investigate stationary accretion of the collisionless Vlasov gas onto the Kerr black hole, occurring in the equatorial plane. The solution is specified by imposing asymptotic boundary conditions: at infinity the gas obeys the Maxwell-Jüttner distribution, restricted to the equatorial plane (both in positions and momenta). In the vicinity of the black hole, the motion of the gas is governed by the spacetime geometry. We compute accretion rates of the rest-mass, the energy, and the angular momentum, as well as the particle number surface density, focusing on the dependence of these quantities on the asymptotic temperature of the gas and the black hole spin. The rest-mass and energy-accretion rates, normalized by the black hole mass and appropriate asymptotic surface densities of the gas, increase with increasing asymptotic temperature. The accretion slows down the rotation of the black hole.

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I. INTRODUCTION

General-relativistic kinetic theory provides a description of gases in strong gravitational fields, alternative to the more popular hydrodynamic approach. When collisions between gas particles are rare and the mean free path is large, fluid dynamics does not apply. In this case, a more reasonable approximation is provided by the kinetic theory of noncolliding gas particles. A standard kinetic description uses a distribution function defined on a suitable one-particle phase space. In general-relativistic literature this approximation is sometimes referred to as the Vlasov gas. Vlasov models are often used in stellar dynamics to describe galaxies and clusters of galaxies (see a mathematical analysis in [1,2] or [3] for the Newtonian account). Another physical realization of Vlasov systems would be in the form of noninteracting dark matter massive particles [4]. Last but not least, the gas surrounding supermassive black holes, such as M87* and Sgr A*, observed recently by the Event Horizon Telescope Collaboration, is also believed to be nearly collisionless and magnetized [5–12]. Classic reviews of the general-relativistic kinetic theory can be found e.g., in [13–18]; see also [19–21] for more recent accounts.

Rigorous relativistic models of accretion flows onto black holes derived within the framework of the kinetic theory are relatively recent. Spherically-symmetric steady accretion of the collisionless Vlasov gas onto a Schwarzschild black hole (a kinetic counterpart of the

Michel accretion model [22,23]) was investigated in 2017 by Rioseco and Sarbach [24,25]. An extension of this model for Reissner-Nordström black holes is given in [26]. Accretion of the collisionless Vlasov gas onto a moving Schwarzschild black hole (a general-relativistic counterpart of the Bondi-Hoyle-Lyttleton accretion problem) was investigated in 2021 [27–29]. In the same year Gamboa *et al.* described accretion of the Vlasov gas onto a Schwarzschild black hole occurring from a sphere of a finite radius [30]. A kinetic description of the phase-space mixing at the equatorial plane of the Kerr spacetime is given in [31]. Some aspects of the dynamics of the massless Vlasov gas in a slowly-rotating Kerr spacetime are analyzed in [32]. We would also like to mention recent studies of self-gravitating static Vlasov configurations around black holes [33,34]. At the same time, there is an ongoing work on kinetic simulations of magnetized plasma in black hole spacetimes [35,36]. These, highly sophisticated numerical simulations complement existing works based on the magnetohydrodynamical approximation, and seem to be crucial in understanding the physics of black hole magnetospheres, as well as interpreting future horizon-resolving observations.

In this paper we derive stationary axially-symmetric solutions of the collisionless Vlasov equation corresponding to a gas accreting onto a Kerr black hole at the equatorial plane. They constitute a model of an infinite, geometrically thin accretion disk. At infinity, the gas remains at rest, and it is assumed to be described by the Maxwell-Jüttner distribution confined to the equatorial plane. The flow of the gas in the vicinity of the black hole is still restricted to the equatorial plane, but otherwise it is governed by the geometry of the spacetime. In a sense,

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our model is an analogue of the spherically symmetric one obtained for Schwarzschild black holes in [24]. The restriction to the equatorial plane of the Kerr black hole is justified by the algebraic simplicity of equations governing the geodesic motion at the equatorial plane. On the other hand, this model seems to be of interest on its own, as a model of a thin accretion disk around the Kerr black hole. It allows us to provide explicit expressions for the particle current surface density, as well as mass and angular momentum accretion rates. We find that the accretion process described by this model slows down the black hole rotation, i.e., the sign of the angular momentum accretion rate is opposite to the sign of the black hole spin parameter.

In many respects our two-dimensional model of a thin disk of the Vlasov gas differs from spherically-symmetric, three-dimensional models known from [24–26]. We find, for instance, that the rest-mass accretion rate, normalized by the black hole mass and the asymptotic rest-mass surface density, increases with the asymptotic temperature. This stays in contrast to the spherically-symmetric case, known for the Schwarzschild or Reissner-Nordström metric, in which the rest-mass accretion rate, normalized by the square of the black hole mass and the asymptotic rest-mass density, decreases with the asymptotic temperature. We should state that this behavior depends on the parametrization of solutions and is related to the properties of the two dimensional Maxwell-Jüttner distribution, assumed at infinity.

Thin accretion disks appear naturally in numerical simulations of accreting plasma in the vicinity of Kerr black holes, both kinetic and based on general-relativistic magnetohydrodynamics [35–38]. Since our model does not involve magnetic fields, we would rather think of its potential applications in modeling of dark matter accretion, as in [27].

In fact, a better understanding of astrophysical black holes in the environment dominated by the omnipresent dark matter is one of main motivations behind our work. For example, changes in the Kerr spin parameter [39,40] could explain the ON/OFF behavior of active galactic nuclei powered by supermassive black holes [41,42]. The specific physical mechanism responsible for this behavior remains elusive, and major black hole mergers are required to significantly alter the black hole spin [43]. Accretion of noninteracting matter could, in principle, provide a new mechanism of this kind.

Equations governing the Vlasov gas at the equatorial plane of the Kerr spacetime can be derived in two frameworks. We will work in a $(2 + 1)$ -dimensional setup, based on the 3 dimensional metric induced at the equatorial plane. Alternatively, one can derive the same equations, working in the full $(3 + 1)$ -dimensional framework and imposing conditions which effectively restrict the motion to the equatorial plane. For clarity and completeness, this alternative formulation will be summarized in Appendix A.

The order of this paper is as follows. Section II contains mathematical preliminaries and conventions concerning the Kerr metric and the geodesic motion in the Kerr spacetime. In Sec. III we introduce our model of the disk of the collisionless Vlasov gas confined to the equatorial plane of the Kerr metric. We specify the distribution function and derive the expressions for the particle current surface density. Section IV discusses the phase space of unbound orbits. In Sec. V we compute the particle current surface density and the rest-mass surface density of the disk. Section VI is devoted to the rest-mass, energy, and angular momentum accretion rates; in particular, we derive analytic limits of these accretion rates for zero and infinite asymptotic temperatures of the gas. Section VII contains a short discussion of the results.

II. PRELIMINARIES

A. Metric conventions

We use standard geometric units with $c = G = 1$, where c denotes the speed of light, and G is the gravitational constant. The signature of the metric is $(-, +, +, +)$.

We will work in Boyer-Lindquist coordinates (t, r, θ, φ) . They are divergent at the black hole horizon, but, in comparison to more refined coordinate choices, they result in relatively simple formulas. In Boyer-Lindquist coordinates the Kerr metric can be written as

$$g = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2, \quad (1)$$

where

$$g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad (2a)$$

$$g_{t\varphi} = -\frac{2Mar \sin^2 \theta}{\rho^2}, \quad (2b)$$

$$g_{rr} = \frac{\rho^2}{\Delta}, \quad (2c)$$

$$g_{\theta\theta} = \rho^2, \quad (2d)$$

$$g_{\varphi\varphi} = \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\rho^2} \right) \sin^2 \theta, \quad (2e)$$

and we denote

$$\Delta = r^2 - 2Mr + a^2, \quad (3a)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta. \quad (3b)$$

Note that many authors use the symbol Σ in place of ρ^2 (cf. [44]). The contravariant metric component read

$$g^{rr} = -\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Delta \rho^2} = -\frac{1}{\Delta} \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right), \quad (4a)$$

$$g^{t\varphi} = -\frac{2Mra}{\Delta \rho^2}, \quad (4b)$$

$$g^{rr} = \frac{\Delta}{\rho^2}, \quad (4c)$$

$$g^{\theta\theta} = \frac{1}{\rho^2}, \quad (4d)$$

$$g^{\varphi\varphi} = \frac{1}{\Delta \rho^2} \left(-a^2 + \frac{\Delta}{\sin^2 \theta} \right). \quad (4e)$$

The Kerr spacetime is characterized by the mass M and the angular momentum $J = Ma$. In general, we will restrict ourselves to the region outside the black hole horizon, with $\Delta > 0$.

The metric induced at the equatorial plane $\theta = \pi/2$ has the form

$$\gamma = \gamma_{tt} dt^2 + 2\gamma_{t\varphi} dt d\varphi + \gamma_{rr} dr^2 + \gamma_{\varphi\varphi} d\varphi^2, \quad (5)$$

where

$$\gamma_{tt} = -1 + \frac{2M}{r}, \quad (6a)$$

$$\gamma_{t\varphi} = -\frac{2Ma}{r}, \quad (6b)$$

$$\gamma_{rr} = \frac{r^2}{\Delta}, \quad (6c)$$

$$\gamma_{\varphi\varphi} = r^2 + a^2 \left(1 + \frac{2M}{r} \right). \quad (6d)$$

Contravariant components of γ read

$$\gamma^{tt} = -\frac{1}{\Delta} \left(r^2 + a^2 + \frac{2Ma^2}{r} \right), \quad (7a)$$

$$\gamma^{t\varphi} = -\frac{2Ma}{\Delta r}, \quad (7b)$$

$$\gamma^{rr} = \frac{\Delta}{r^2}, \quad (7c)$$

$$\gamma^{\varphi\varphi} = \frac{1}{\Delta} \left(1 - \frac{2M}{r} \right). \quad (7d)$$

An easy calculation yields $\det g^{\mu\nu} = -1/(\rho^4 \sin^2 \theta)$ and $\det \gamma^{\mu\nu} = -1/r^2$. Note that γ is not a solution of the vacuum

Einstein equations in $2 + 1$ dimensions. Such an equivalent of the Kerr solution does not exist, except for the so-called Bañados-Teitelboim-Zanelli (BTZ) solution [45], which assumes a negative cosmological constant.

B. Timelike geodesics

Timelike geodesic equations can be expressed in the Hamiltonian form

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad (8a)$$

$$\frac{dp_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}, \quad (8b)$$

where $p^\mu = dx^\mu/d\tau$, $H(x^\alpha, p_\beta) = \frac{1}{2} g^{\mu\nu}(x^\alpha) p_\mu p_\nu = -\frac{1}{2} m^2$, and m denotes the particle rest mass. By standard arguments, H , $p_t \equiv -E$ and $p_\varphi \equiv l_z$ are constants of motion. The fourth constant—the so-called Carter constant l —follows from the separation of variables in the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + H(x^\alpha, \partial_\beta S) = 0. \quad (9)$$

The ansatz

$$S = -H\tau + W = \frac{1}{2} m^2 \tau + W \quad (10)$$

with

$$W = \int p_t dt + \int p_\varphi d\varphi + \int p_r dr + \int p_\theta d\theta = -Et + l_z \varphi + W_r(r) + W_\theta(\theta)$$

yields the “time-independent” Hamilton-Jacobi equation

$$g^{\mu\nu} \partial_\mu W \partial_\nu W = -m^2 \quad (11)$$

or, in explicit terms,

$$-\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Delta} E^2 + \frac{4Mra}{\Delta} E l_z + \Delta (W'_r)^2 + (W'_\theta)^2 + \frac{1}{\Delta} \left(-a^2 + \frac{\Delta}{\sin^2 \theta} \right) l_z^2 = -m^2 \rho^2. \quad (12)$$

Rearranging terms, we get

$$(W'_\theta)^2 + m^2 a^2 \cos^2 \theta + \left(\frac{l_z}{\sin \theta} - a \sin \theta E \right)^2 = -m^2 r^2 - \Delta (W'_r)^2 + \frac{1}{\Delta} [(r^2 + a^2)E - a l_z]^2. \quad (13)$$

Since the left-hand side depends only on θ , while the right-hand side depends only on r , both sides have to be constant. We denote this constant value by l^2 [note that the left-hand side of Eq. (13) is explicitly non-negative]. Thus

$$(W'_\theta)^2 \equiv \Theta(\theta) = l^2 - m^2 a^2 \cos^2 \theta - \left(\frac{l_z}{\sin \theta} - a \sin \theta E \right)^2, \quad (14)$$

and

$$\Delta^2 (W'_r)^2 \equiv R(r) = [(r^2 + a^2)E - al_z]^2 - \Delta(m^2 r^2 + l^2). \quad (15)$$

The equations of motion can be integrated following the standard procedure, i.e., by computing the derivatives $\partial S/\partial m$, $\partial S/\partial E$, $\partial S/\partial l_z$, $\partial S/\partial l$, and equating the results to a new set of constants $\beta_0, \beta_1, \beta_2, \beta_3$ (say) [46,47].

C. Timelike geodesics at the equatorial plane

The description of time-like geodesics restricted to the equatorial plane ($\theta = \pi/2$) is much simpler. We have $p_\theta = 0$, $W_\theta = 0$. Equation (14) reduces to

$$l^2 = (l_z - aE)^2. \quad (16)$$

We will assume a convention with $l \geq 0$. Thus,

$$\epsilon_\sigma l = l_z - aE, \quad (17)$$

where $\epsilon_\sigma = \pm 1$. Consequently,

$$\begin{aligned} R(r) &= (r^2 E - \epsilon_\sigma a l)^2 - \Delta(m^2 r^2 + l^2) \\ &= r^4 \left[E^2 - \left(1 - \frac{2M}{r} \right) \left(m^2 + \frac{l^2}{r^2} \right) - \frac{2\epsilon_\sigma a E l + a^2 m^2}{r^2} \right]. \end{aligned} \quad (18)$$

Momentum components associated with the geodesic motion confined to the equatorial plane can be written as

$$p_t = -E, \quad (19a)$$

$$p_r = \epsilon_r \frac{\sqrt{R}}{\Delta}, \quad (19b)$$

$$p_\theta = 0, \quad (19c)$$

$$p_\phi = l_z = \epsilon_\sigma l + aE = \epsilon_\sigma (l + \epsilon_\sigma aE), \quad (19d)$$

where $\epsilon_r = \pm 1$ corresponds to the radial direction of motion.

Alternatively, one can obtain equations governing the motion confined to the equatorial plane directly from the Hamiltonian

$$H = \frac{1}{2} (\gamma^t p_t^2 + 2\gamma^{t\phi} p_t p_\phi + \gamma^{rr} p_r^2 + \gamma^{\phi\phi} p_\phi^2). \quad (20)$$

Here again, $p_t = -E$, $p_\phi = l_z$, and $H = -\frac{1}{2}m^2$ have to be constant. Solving the equation $H = -\frac{1}{2}m^2$ for p_r yields Eq. (19b).

Following [24,31], we use dimensionless variables defined as

$$\alpha = \frac{a}{M}, \quad \varepsilon = \frac{E}{m}, \quad \lambda = \frac{l}{Mm}, \quad \xi = \frac{r}{M}. \quad (21)$$

Thus,

$$R(r) = M^4 m^2 \bar{R}(\xi), \quad (22)$$

where

$$\begin{aligned} \bar{R}(\xi) &= (\xi^2 \varepsilon - \epsilon_\sigma \alpha \lambda)^2 - (\xi^2 - 2\xi + \alpha^2)(\xi^2 + \lambda^2) \\ &= \xi^4 \left[\varepsilon^2 - \left(1 - \frac{2}{\xi} \right) \left(1 + \frac{\lambda^2}{\xi^2} \right) - \frac{2\epsilon_\sigma \alpha \varepsilon \lambda + \alpha^2}{\xi^2} \right], \end{aligned} \quad (23)$$

and

$$p_t = -E = -m\varepsilon, \quad (24a)$$

$$p_r = \epsilon_r \frac{\sqrt{R}}{\Delta} = \frac{\epsilon_r m \sqrt{R}}{\xi^2 - 2\xi + \alpha^2}, \quad (24b)$$

$$p_\theta = 0, \quad (24c)$$

$$\begin{aligned} p_\phi = l_z = \epsilon_\sigma l + aE &= Mm(\epsilon_\sigma \lambda + \alpha \varepsilon) \\ &= Mm\epsilon_\sigma (\lambda + \epsilon_\sigma \alpha \varepsilon). \end{aligned} \quad (24d)$$

In this work we will generally parametrize geodesics using the Carter constant l (or λ) instead of l_z . This is, of course, a matter of preference. Our choice was partially motivated by a similar substitution in Eq. (3) of [31]; it also simplifies a few algebraic relations used in this paper. We should emphasize that the sign ϵ_σ is associated with the quantity $l_z - aE$ and not with l_z . In particular $l_z = 0$ implies $l = -\epsilon_\sigma aE$. On the other hand in many important cases the signs of l_z and ϵ_σ coincide. Thus, we will generally refer to trajectories with $\epsilon_\sigma \alpha > 0$ as prograde ones, and to those satisfying $\epsilon_\sigma \alpha < 0$ as retrograde ones.

There are three well-known types of circular orbits at the equatorial plane, two of which will play an important role in the subsequent analysis [44,48]. There is the circular photon orbit with the radius

$$\xi_{\text{ph}} = 2 + 2 \cos \left[\frac{2}{3} \arccos(-\epsilon_\sigma \alpha) \right], \quad (25)$$

a solution of $(\xi - 3)\sqrt{\xi} + 2\epsilon_\sigma \alpha = 0$ or $(\xi - 3)^2 \xi - 4\alpha^2 = 0$. The circular photon orbit corresponds to a null geodesic, but it appears naturally as a limit of time-like orbits. There is the so-called marginally bound orbit with the dimensionless radius

$$\xi_{\text{mb}} = 2 - \epsilon_\sigma \alpha + 2\sqrt{1 - \epsilon_\sigma \alpha}, \quad (26)$$

which is a solution to $\xi(\xi - 4) + 4\epsilon_\sigma \alpha \sqrt{\xi} - \alpha^2 = 0$ [see also Eq. (60)]. For unbound orbits ξ_{mb} is related to the so-called fly-by radius [49]. The dimensionless radius of the innermost stable circular orbit (marginally stable orbit, ISCO) is given by

$$\xi_{\text{ms}} = 3 + Z_2 - \epsilon_\sigma \alpha \sqrt{\frac{(3 - Z_1)(3 + Z_1 + 2Z_2)}{\alpha^2}}, \quad (27)$$

where $Z_1 = 1 + \sqrt[3]{1 - \alpha^2}(\sqrt[3]{1 + \epsilon_\sigma \alpha} + \sqrt[3]{1 - \epsilon_\sigma \alpha})$, $Z_2 = \sqrt{3\alpha^2 + Z_1^2}$ [44]. The radius ξ_{ms} is a solution of the equation $(\xi - 6)\xi + 8\alpha\sqrt{\xi} - 3\alpha^2 = 0$. The location of the innermost stable circular orbit does not seem to play a crucial role in our analysis. However, the related polynomial in $\sqrt{\xi}$, as well as those defining ξ_{ph} and ξ_{mb} , do appear in the integrands of our formulas for accretion rates.

III. VLASOV EQUATION AT THE EQUATORIAL PLANE

The Vlasov gas considered in this paper is a collection of free particles moving along timelike future-directed geodesic lines. It is described in terms of the distribution function $f = f(x^\mu, p_\nu)$, defined on the one-particle phase space. For collisionless particles, the Vlasov equation can be expressed as a requirement that the probability function f is constant along a geodesic,

$$\begin{aligned} \frac{df}{d\tau} &= \frac{\partial f}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial f}{\partial p_\nu} \frac{dp_\nu}{d\tau} = \frac{\partial f}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial f}{\partial p_\nu} \frac{\partial H}{\partial x^\nu} \\ &= \{H, f\} = 0. \end{aligned} \quad (28)$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket.

A formal solution of the above equation has been obtained in the context of the Kerr metric in [20]. We will repeat this reasoning here, assuming that the motion is restricted to the equatorial plane. In this case, $(x^\mu, p_\nu) = (t, r, \varphi, p_t, p_r, p_\varphi)$, and the Hamiltonian H is given by Eq. (20). The idea is based on a suitable canonical (symplectic) transformation of the phase-space variables $(t, r, \varphi, p_t, p_r, p_\varphi) \mapsto (Q^\mu, P_\nu)$. It is given in terms of a generating function (the so-called abbreviated action)

$$\begin{aligned} W &= W(t, r, \varphi, m, E, l_z) = \int p_t dt + \int p_r dr + \int p_\varphi d\varphi \\ &= -Et + l_z \varphi + W_r. \end{aligned} \quad (29)$$

We choose new momenta in the form

$$P_0 = m, \quad P_1 = E, \quad P_2 = l_z. \quad (30)$$

The corresponding conjugate variables Q^μ are defined as $Q^\mu = \partial W / \partial P_\mu$, i.e.,

$$Q^0 = \frac{\partial W}{\partial m}, \quad Q^1 = \frac{\partial W}{\partial E}, \quad Q^2 = \frac{\partial W}{\partial l_z}. \quad (31)$$

The Vlasov equation (28), written in terms of the Poisson bracket, is covariant with respect to canonical transformations. Since $H = -\frac{1}{2}P_0^2$, it reads, in the new variables (Q^μ, P_ν) ,

$$-P_0 \frac{\partial f}{\partial Q^0} = 0. \quad (32)$$

Consequently, any distribution function of the form

$$f = f(Q^1, Q^2, P_0, P_1, P_2) \quad (33)$$

satisfies the Vlasov equation. Further restrictions on the form of the distribution function can be obtained by imposing symmetry conditions.

The distribution function f is not an observable quantity on its own. We define the components of the particle current surface density as

$$J_\mu(x) = \int f(x, p) p_\mu \sqrt{-\det \gamma^{\mu\nu}} dp_t dp_r dp_\varphi, \quad (34)$$

where $\mu = t, r, \varphi$, and $f = f(t, r, \varphi, p_t, p_r, p_\varphi)$. Similarly, the surface energy-momentum tensor components are

$$T_{\mu\nu}(x) = \int f(x, p) p_\mu p_\nu \sqrt{-\det \gamma^{\mu\nu}} dp_t dp_r dp_\varphi. \quad (35)$$

A covariant particle number surface density n_s can be defined as

$$n_s = \sqrt{-\gamma_{\mu\nu}} J^\mu J^\nu. \quad (36)$$

We consider a model of a stationary accretion disk consisting of same-mass particles, confined to the equatorial plane and extending to infinity. At infinity the gas is assumed to be essentially “at rest” and to be characterized by a constant particle number surface density $n_{s,\infty}$. More precisely, we require that at infinity the distribution function should be given by the two dimensional Maxwell-Jüttner distribution [50,51]. On the other hand,

the motion of the gas near the black hole should be governed by its gravitational field. A distribution function fulfilling these assumptions can be chosen as

$$f = A\delta(\sqrt{-p_\mu p^\mu} - m_0) \exp\left(\frac{\beta}{m_0} p_t\right) \\ = A\delta(P_0 - m_0) \exp\left(-\frac{\beta}{m_0} P_1\right), \quad (37)$$

where m_0 (the particle mass) and A are constants. Here $\beta = m_0/(k_B T)$, T is the asymptotic temperature of the gas, and k_B denotes the Boltzmann constant. The factor $\delta(\sqrt{-p_\mu p^\mu} - m_0)$ ensures that the momenta satisfy the mass shell condition. Here, and in what follows, we only take into account future-pointing momenta. Clearly, the distribution function given by Eq. (37) is of the form (33); it satisfies the collisionless Vlasov equation (32) and coincides with the standard Maxwell-Jüttner distribution at the infinity.

In what follows, we will also use the rest-mass surface density, defined as

$$\rho_s = m_0 n_s. \quad (38)$$

The components of the particle current surface density J_t , J_r , and J_φ can be computed according to Eqs. (34). To evaluate the integrals over momenta appearing in Eqs. (34), one needs to control the domain in the phase space occupied by relevant particle trajectories—this is probably the most difficult part of the calculation. The coordinate transformations introduced in the remainder of this section and the analysis of Sec. IV are devoted to this problem.

We will only take into account unbound trajectories, i.e., trajectories that reach infinity. Bound trajectories could also play a role, but since collisions between gas particles are neglected and we are working on a fixed spacetime, contributions corresponding to bound and unbound trajectories can be treated separately. Thus, for simplicity, we assume that no bound trajectories are present. The reader interested in the phase-space mixing occurring for bound trajectories in the equatorial plane of the Kerr spacetime is referred to [31].

Unbound trajectories are naturally divided into two classes. There are orbits characterized by relatively small values of angular momentum, which end in the black hole. We refer to such orbits as absorbed ones. The second class consists of orbits with sufficiently high angular momentum, which are scattered off the centrifugal barrier. They originate and end at infinity. We will refer to those orbits as scattered trajectories. Even though we are mainly interested in the accretion process, both types of unbound trajectories have to be taken into account, as they all contribute to the total “budget” of trajectories at infinity, where the Maxwell-Jüttner distribution is assumed.

The analysis of unbound trajectories is simpler in terms of conserved variables such as m , E , l_z (or l), instead of momenta p_t , p_r , p_φ . Thus, we change momentum variables p_t , p_r , p_φ to

$$m^2 = -(\gamma^{tt} p_t^2 + 2\gamma^{t\varphi} p_t p_\varphi + \gamma^{rr} p_r^2 + \gamma^{\varphi\varphi} p_\varphi^2), \quad (39a)$$

$$E = -p_t, \quad (39b)$$

$$l = e_\sigma(p_\varphi + a p_t). \quad (39c)$$

The Jacobian of the above transformation reads

$$\frac{\partial(m^2, E, l)}{\partial(p_t, p_r, p_\varphi)} = -\frac{2e_\sigma p_r \Delta}{r^2}. \quad (40)$$

Consequently,

$$dp_t dp_r dp_\varphi = \frac{r^2}{2|p_r|\Delta} dm^2 dE dl = \frac{r^2 m}{|p_r|\Delta} dm dE dl, \quad (41)$$

where $|p_r|\Delta = \sqrt{R(r)}$.

In terms of dimensionless quantities (21) the momentum integration element can be written as

$$\sqrt{-\det\gamma^{\mu\nu}} dp_t dp_r dp_\varphi = \frac{\xi m^2 dm dE d\lambda}{\sqrt{R}}. \quad (42)$$

This yields formal expressions for the components J_μ in the form

$$J_t = -Am_0^3 \xi \sum_{\epsilon_\sigma=\pm 1} \int \exp(-\beta E) \epsilon \frac{dE d\lambda}{\sqrt{R}}, \quad (43a)$$

$$J_r = \frac{AM^2 m_0^3 \xi}{\Delta} \sum_{\epsilon_\sigma=\pm 1} \int \epsilon_r \exp(-\beta E) dE d\lambda, \quad (43b)$$

$$J_\varphi = AMm_0^3 \xi \sum_{\epsilon_\sigma=\pm 1} \int \exp(-\beta E) (\epsilon_\sigma \lambda + a\epsilon) \frac{dE d\lambda}{\sqrt{R}}. \quad (43c)$$

In the following we will also need explicit expressions for

$$J^r = \frac{Am_0^3}{\xi} \sum_{\epsilon_\sigma=\pm 1} \int \epsilon_r \exp(-\beta E) dE d\lambda \quad (44)$$

and

$$T^r_t = -\frac{Am_0^4}{\xi} \sum_{\epsilon_\sigma=\pm 1} \int \epsilon_r \exp(-\beta E) \epsilon dE d\lambda, \quad (45)$$

$$T^r_\varphi = \frac{Am_0^4 M}{\xi} \sum_{\epsilon_\sigma=\pm 1} \int \epsilon_r \exp(-\beta E) (\epsilon_\sigma \lambda + a\epsilon) dE d\lambda. \quad (46)$$

There is a simple test of correctness that can be performed at this point—one can compute the component J_t in the flat Minkowski space limit $M \rightarrow 0$, $a \rightarrow 0$. This has to be done with some caution. In particular the mass M

in Eqs. (21) defining the dimensionless quantities should be replaced with any positive constant with the mass dimension. The expression for $\tilde{R}$ yields, in the Minkowski limit,

$$\tilde{R} \rightarrow \xi^4 \left(\varepsilon^2 - 1 - \frac{\lambda^2}{\xi^2} \right),$$

and hence

$$J_t = -\frac{Am_0^3}{\xi} \sum_{\epsilon_\sigma = \pm 1} \int \exp(-\beta \varepsilon) \varepsilon \frac{d\varepsilon d\lambda}{\sqrt{\varepsilon^2 - 1 - \frac{\lambda^2}{\xi^2}}}.$$

The key point is to control the range in the phase space over which the above integral is performed. The trajectories are straight lines with $\varepsilon > 1$ and $\lambda < \xi \sqrt{\varepsilon^2 - 1}$. In addition, a set of values ε , m , and λ , correspond to two possible values of the radial momentum $p_r = \pm |p_r|$. This gives J_t of the form

$$J_t = -\frac{4Am_0^3}{\xi} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \varepsilon \int_0^{\xi \sqrt{\varepsilon^2 - 1}} \frac{d\lambda}{\sqrt{\varepsilon^2 - 1 - \frac{\lambda^2}{\xi^2}}}, \quad (47)$$

The last integral, over λ , is simply $\pi \xi / 2$. Thus,

$$\begin{aligned} J_t &= -2\pi Am_0^3 \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \varepsilon \\ &= -2\pi Am_0^3 \frac{e^{-\beta}(1+\beta)}{\beta^2}. \end{aligned} \quad (48)$$

This result agrees with the standard computation of J_t based on Cartesian coordinates. For completeness, we report this computation in Appendix B. Notably formula (48) agrees with Eq. (B8) and formula (117) of [21], which uses the Bessel function

$$K_{3/2}(\beta) = \sqrt{\frac{\pi}{2\beta}} \frac{e^{-\beta}(1+\beta)}{\beta}. \quad (49)$$

The above result allows us also to write the expression for the asymptotic rest-mass surface density of matter at the equatorial plane

$$\rho_{s,\infty} \equiv m_0 n_{s,\infty} = -m_0 J_t|_\infty = 2\pi Am_0^4 \frac{1+\beta}{\beta^2} \exp(-\beta). \quad (50)$$

Hence, the factor A in the integral expressions (43) for J_μ can be replaced by

$$A = \frac{\rho_{s,\infty}}{2\pi m_0^4} \frac{\beta^2}{1+\beta} \exp(\beta). \quad (51)$$

IV. CLASSIFICATION OF ORBITS

We are interested in both absorbed and scattered unbound trajectories. By inspecting the expression (23) for $\tilde{R}$ at $\xi \rightarrow \infty$, we see that the unbound trajectories must satisfy $\varepsilon \geq 1$.

As in the Schwarzschild case (see, e.g., [24]), a particle traveling on an absorbed trajectory cannot encounter the centrifugal barrier, when traveling from infinity towards the black hole. The centrifugal barrier is effectively controlled by the value of λ . An absorbed trajectory is characterized by the condition $\lambda \leq \lambda_c(\varepsilon, \epsilon_\sigma)$, where $\lambda_c(\varepsilon, \epsilon_\sigma)$ is a solution (with respect to λ) of the set of equations

$$\tilde{R}(\xi) = 0, \quad (52a)$$

$$\frac{d\tilde{R}(\xi)}{d\xi} = 0, \quad (52b)$$

and $\tilde{R}(\xi)$ is given by (23). The function $\lambda_c(\varepsilon, \epsilon_\sigma)$ can be given in a parametric (therefore the subscript p) form as

$$\lambda_c = \lambda_p(\xi, \epsilon_\sigma) \equiv \frac{\xi^{5/4} - \alpha \epsilon_\sigma \xi^{3/4}}{\sqrt{2\alpha \epsilon_\sigma + (\xi - 3)\sqrt{\xi}}}, \quad (53a)$$

$$\varepsilon = \varepsilon_p(\xi, \epsilon_\sigma) \equiv \frac{\alpha \epsilon_\sigma + (\xi - 2)\sqrt{\xi}}{\xi^{3/4} \sqrt{2\alpha \epsilon_\sigma + (\xi - 3)\sqrt{\xi}}}. \quad (53b)$$

It can be readily verified that the above expressions solve Eqs. (52). Zeros of the expression under the square root in the denominator of (53) correspond to circular photon orbits (25). We have $\varepsilon_p(\xi_{\text{mb}}, \epsilon_\sigma) = 1$, where ξ_{mb} is the dimensionless radius of the marginally bound orbit, given by Eq. (26). On the other hand $\varepsilon_p(\xi, \epsilon_\sigma) \rightarrow \infty$ for $\xi \rightarrow \xi_{\text{ph}}$, where ξ_{ph} is the dimensionless radius of the circular photon orbit at the equatorial plane, given by Eq. (25). Consequently, the range $1 \leq \varepsilon_p(\xi, \epsilon_\sigma) < \infty$ corresponds to $\xi_{\text{mb}} \leq \xi < \xi_{\text{ph}}$. Note that

$$\frac{d\lambda_c(\varepsilon)}{d\varepsilon} = \frac{\frac{\partial \lambda_p(\xi, \epsilon_\sigma)}{\partial \xi}}{\frac{\partial \varepsilon_p(\xi, \epsilon_\sigma)}{\partial \xi}} = \frac{3}{\xi^2}. \quad (54)$$

The function $\lambda_c(\varepsilon, \epsilon_\sigma)$ cannot be expressed in a closed form, however it satisfies a relatively simple equation, polynomial in both λ and ε , which can be found as follows. The expressions on the left hand-side of Eqs. (52) are polynomials in ξ . It is well known that the substitution $u = 1/\xi$ yields polynomials of a lower order. We define $U = \tilde{R}/\xi^4$, and change the variables to $u = 1/\xi$. This gives the conditions

$$\begin{aligned} U(u) &= -1 + 2u - u^2 \alpha^2 + \varepsilon^2 - 2\epsilon_\sigma u^2 \alpha \varepsilon \lambda \\ &\quad - u^2 \lambda^2 + 2u^3 \lambda^2 = 0, \end{aligned} \quad (55a)$$

$$\begin{aligned} \frac{dU(u)}{du} &= 2 - 2u\alpha^2 - 4\epsilon_\sigma u\alpha\epsilon\lambda - 2u\lambda^2 \\ &\quad + 6u^2\lambda^2 = 0. \end{aligned} \quad (55b)$$

The term u^3 in Eq. (55a) can be easily removed by multiplying Eq. (55b) by u , solving for u^3 , and substituting the result in Eq. (55a). This yields the set of equations

$$3(\epsilon^2 - 1) + 4u - u^2(\alpha^2 + 2\epsilon_\sigma\alpha\epsilon\lambda + \lambda^2) = 0, \quad (56a)$$

$$1 - u(\alpha^2 + 2\epsilon_\sigma\alpha\epsilon\lambda + \lambda^2) + 3u^2\lambda^2 = 0. \quad (56b)$$

instead of Eqs. (55). Substituting temporarily

$$X = \alpha^2 + 2\epsilon_\sigma\alpha\epsilon\lambda + \lambda^2, \quad (57)$$

we get

$$3(\epsilon^2 - 1) + 4u - u^2X = 0, \quad (58a)$$

$$1 - uX + 3u^2\lambda^2 = 0. \quad (58b)$$

One could continue this way in order to finally eliminate u , but the most straightforward method to achieve this aim is to compute the so-called Gröbner basis for the above system. The result reads

$$\begin{aligned} Y &\equiv (\epsilon^2 - 1)[27(\epsilon^2 - 1)\lambda^4 - X^3 + 18\lambda^2X] \\ &\quad + 16\lambda^2 - X^2 = 0. \end{aligned} \quad (59)$$

Gröbner's basis is an invariant of the polynomial elimination, and it can be computed directly from Eqs. (55) or (52). On the other hand, to get a relatively simple form of Eq. (59) one has to guess substitution (57), suggested by Eqs. (58).

In practice, we compute $\lambda_c(\epsilon, \epsilon_\sigma)$ numerically, as a solution of Eq. (59), or we use the parametric form (53). In the former case, one has to be careful to select the physical branch of solutions. Sample solution branches of Eq. (59) are shown in Fig. 1. The physical branch, parametrized by Eqs. (53) is depicted with a red line, but other branches are also present. We show, for instance, the existence of a branch (blue) with vertical asymptote $\epsilon = 1$ (thin line). The function Y is a 6th degree polynomial in λ and a 5th degree polynomial in ϵ . The term in Y with the highest degree in λ is $-(\epsilon^2 - 1)\lambda^6$. Thus, for sufficiently

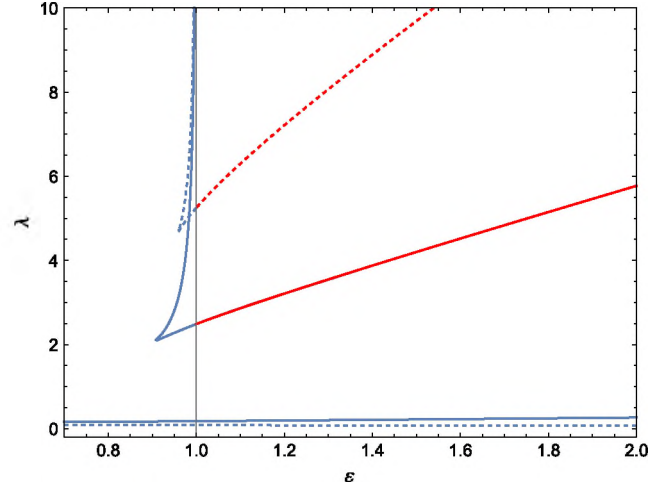


FIG. 1. Sample solutions of Eq. (59) for $\alpha = 2/3$, $\epsilon_\sigma = +1$ (continuous line), $\epsilon_\sigma = -1$ (dashed line). Physically relevant branches, given by the parametric formula (53), are depicted with red lines. Note the existence of mirrored branches in the remaining three quadrants of the full $\epsilon - \lambda$ plane (not shown).

large λ , Y becomes positive for $\epsilon^2 < 1$. On the other hand, Eq. (59) has a simple solution for $\epsilon = 1$. In this case $X = (\epsilon_\sigma\alpha + \lambda)^2$, and Eq. (59) can be reduced to

$$Y = 16\lambda^2 - (\epsilon_\sigma\alpha + \lambda)^4 = 0. \quad (60)$$

Non-negative solutions of this equation read $\lambda = 2 - \epsilon_\sigma\alpha \pm 2\sqrt{1 - \epsilon_\sigma\alpha}$, and the larger of the two solutions is

$$\lambda_* = 2 - \epsilon_\sigma\alpha + 2\sqrt{1 - \epsilon_\sigma\alpha} \quad (61)$$

(note an algebraic coincidence $\lambda_* = \xi_{\text{mb}}$). The remaining two nonpositive solutions are $\lambda = -2 - \epsilon_\sigma\alpha \pm 2\sqrt{1 + \epsilon_\sigma\alpha}$. It follows that for $\epsilon = 1$ and $\lambda > \lambda_*$, we have $Y < 0$. Consequently, Eq. (59) has an asymptotic solution with $\lambda \rightarrow +\infty$ for $\epsilon \rightarrow 1$.

Equation (59) has a simple solution for $\alpha = \pm 1$ and $\epsilon_\sigma\alpha = +1$, i.e., for prograde orbits in the extremal Kerr metric. In this case $\lambda_c(\epsilon, \epsilon_\sigma) = \epsilon$. This solution is shown in Fig. 2.

For scattered orbits, $\lambda_c(\epsilon, \epsilon_\sigma) < \lambda \leq \lambda_{\text{max}}(\xi, \epsilon, \epsilon_\sigma)$. The formula for the upper limit $\lambda_{\text{max}}(\xi, \epsilon, \epsilon_\sigma)$ follows from the requirement that $\tilde{R} \geq 0$. It reads

$$\lambda_{\text{max}}(\xi, \epsilon, \epsilon_\sigma) = \frac{\xi}{\xi - 2} \left\{ \sqrt{\alpha^2 \left(\epsilon^2 + \frac{2}{\xi} - 1 \right) + (\xi - 2)[\xi(\epsilon^2 - 1) + 2] - \epsilon_\sigma\alpha\epsilon} \right\}. \quad (62)$$

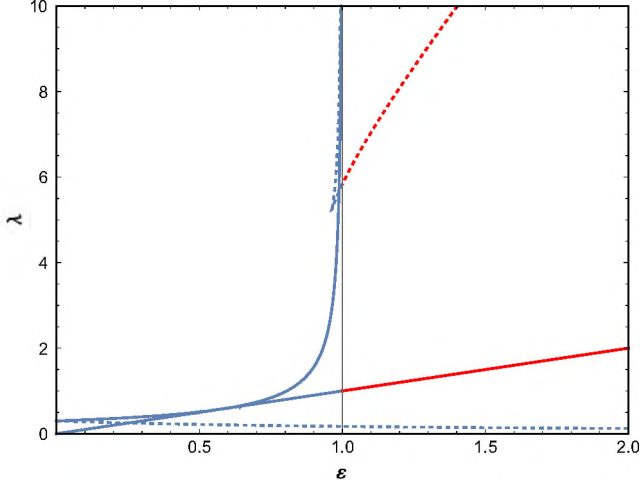


FIG. 2. Sample solutions of Eq. (59) for $\alpha = 1$, $\epsilon_\sigma = +1$ (continuous line), $\epsilon_\sigma = -1$ (dashed line). Physically relevant branches are depicted with red lines. For $\alpha = 1$ and $\epsilon_\sigma = +1$, $\lambda_c(\epsilon, \epsilon_\sigma) = +1$.

Note that for $|\alpha| \geq \sqrt{2}/2$ we have $\xi_{\text{ph}} \leq 2$ and λ_{max} must be evaluated at the removable discontinuity $\xi = 2$ for prograde trajectories. The relevant limit for $\epsilon_\sigma \alpha > 0$ is

$$\lim_{\xi \rightarrow 2} \lambda_{\text{max}}(\xi, \epsilon, \epsilon_\sigma) = \frac{4\epsilon^2 - \alpha^2}{2\epsilon_\sigma \alpha \epsilon}. \quad (63)$$

Scattered unbound orbits exist only for sufficiently high energies ϵ . The main bound, $\epsilon > 1$, applies to all trajectories reaching infinity. On the other hand, a condition defining a scattered orbit is the existence of a centrifugal barrier—a particle on a scattered orbit travels from infinity, reaches a turning point, at which $\tilde{R}(\xi) = 0$, and moves again to infinity. In other words, a scattered orbit

parametrized by ϵ , λ , and ϵ_σ can only pass through a point of radius ξ , if there is a turning point at $\xi_0 \leq \xi$ (i.e., $\tilde{R}(\xi_0) = 0$) and the region with radii $\xi' \geq \xi$ is available for motion [i.e., $\tilde{R}(\xi') \geq 0$]. In order to obtain the value of the minimal energy $\epsilon_{\text{min}}(\xi, \epsilon_\sigma)$ allowing for a scattered orbit at some radius ξ , we solve Eqs. (58) with respect to ϵ . More precisely, we start by solving Eq. (58b) with respect to λ . There are two possible solutions, which we can substitute in Eq. (58a). Solving the resulting equations with respect to ϵ yields four solutions, two of which can be positive. These are

$$\epsilon = \sqrt{\frac{\pm 2 \sqrt{\frac{\alpha^3 + \alpha(\xi-2)\xi^2}{\xi}} + \alpha^2(5-3\xi) + (\xi-3)(\xi-2)^2\xi}{\xi[(\xi-3)^2\xi - 4\alpha^2]}}; \quad (64)$$

they do not depend explicitly on ϵ_σ , but in fact one of them corresponds to prograde, and the other to retrograde orbits. One can distinguish between the two cases by inspecting the asymptotics of the two expressions. The two solutions tend to infinity for

$$\xi \rightarrow 2 + 2 \cos \left[\frac{2}{3} \arccos(\pm \alpha) \right], \quad (65)$$

i.e., solutions to $(\xi-3)^2\xi - 4\alpha^2 = 0$. These solutions can be easily identified as the locations of circular photon orbits at the equatorial plane. The latter are given by $\xi = \xi_{\text{ph}}$, where ξ_{ph} is defined by Eq. (25).

Matching the asymptotics of Eq. (64) with the locations of circular photon orbits, we can select the signs in (64), leading to the formula

$$\epsilon = \sqrt{\frac{-2\epsilon_\sigma \alpha [\alpha^2 + (\xi-2)\xi] \xi^{-1/2} + \alpha^2(5-3\xi) + (\xi-3)(\xi-2)^2\xi}{\xi[(\xi-3)^2\xi - 4\alpha^2]}}. \quad (66)$$

For both solutions (with $\epsilon_\sigma = \pm 1$) the limiting energy given by the above formula decreases from $+\infty$ for $\xi > \xi_{\text{ph}}$. The value $\epsilon = 1$ is reached at $\xi = \xi_{\text{mb}}$, where ξ_{mb} is given by Eq. (26). Thus, the minimal allowed energy for scattered orbits can be written as

$$\epsilon_{\text{min}}(\xi, \epsilon_\sigma) = \begin{cases} \infty & \text{for } \xi < \xi_{\text{ph}}, \\ \sqrt{\frac{-2\epsilon_\sigma \alpha [\alpha^2 + (\xi-2)\xi] \xi^{-1/2} + \alpha^2(5-3\xi) + (\xi-3)(\xi-2)^2\xi}{\xi[(\xi-3)^2\xi - 4\alpha^2]}} & \text{for } \xi_{\text{ph}} < \xi < \xi_{\text{mb}}, \\ 1 & \text{for } \xi \geq \xi_{\text{mb}}. \end{cases} \quad (67)$$

Note that no scattered trajectory can extend to $\xi < \xi_{\text{ph}}$. Sample graphs of $\epsilon_{\text{min}}(\xi, \epsilon_\sigma)$ are shown in Fig. 3.

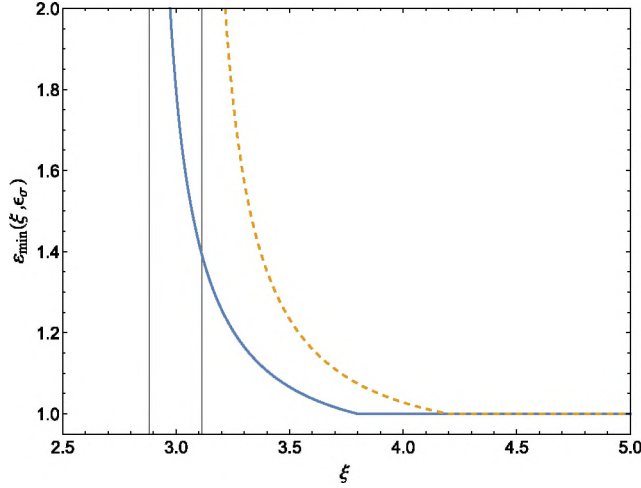


FIG. 3. Sample graphs of $\varepsilon_{\min}(\xi, \varepsilon_\sigma)$ for $\alpha = 1/10$ and $\varepsilon_\sigma = +1$ (blue continuous line) and $\varepsilon_\sigma = -1$ (orange dashed line). Vertical lines denote locations of circular photon orbits ξ_{ph} for $\varepsilon_\sigma = \pm 1$. Note that $\varepsilon_{\min}(\xi, +1)$ is regular at ξ_{ph} corresponding to $\varepsilon_\sigma = -1$.

V. PARTICLE CURRENT SURFACE DENSITY

We divide the expressions for the particle current surface density J_μ into two parts; a part corresponding to absorbed trajectories and a part corresponding to scattered ones. The quantities referring to those two parts will be denoted with the superscripts (abs) and (scat), respectively. Consequently, $J_\mu = J_\mu^{(\text{abs})} + J_\mu^{(\text{scat})}$. As discussed in the previous section, the phase-space region corresponding to absorbed trajectories is characterized by $\varepsilon \geq 1$ and $\lambda \leq \lambda_c(\varepsilon, \varepsilon_\sigma)$. Scattered orbits passing through a point at radius ξ are characterized by $\varepsilon_{\min}(\xi, \varepsilon_\sigma) < \varepsilon < \infty$ and $\lambda_c(\varepsilon, \varepsilon_\sigma) < \lambda \leq \lambda_{\max}(\xi, \varepsilon, \varepsilon_\sigma)$. Explicit expressions for $J_\mu^{(\text{abs})}$ and $J_\mu^{(\text{scat})}$ read

$$J_t^{(\text{abs})}(\xi) = -Am_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \varepsilon \int_0^{\lambda_c(\varepsilon, \varepsilon_\sigma)} \frac{d\lambda}{\sqrt{R}}, \quad (68a)$$

$$J_r^{(\text{abs})}(\xi) = -\frac{AM^2 m_0^3 \xi}{\Delta} \sum_{\varepsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \lambda_c(\varepsilon, \varepsilon_\sigma), \quad (68b)$$

$$J_\varphi^{(\text{abs})}(\xi) = AMm_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta\varepsilon) \times \int_0^{\lambda_c(\varepsilon, \varepsilon_\sigma)} d\lambda \frac{\varepsilon_\sigma \lambda + \alpha \varepsilon}{\sqrt{R}}, \quad (68c)$$

and

$$J_t^{(\text{scat})}(\xi) = -2Am_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_{\varepsilon_{\min}(\xi, \varepsilon_\sigma)}^\infty d\varepsilon \exp(-\beta\varepsilon) \times \varepsilon \int_{\lambda_c(\varepsilon, \varepsilon_\sigma)}^{\lambda_{\max}(\xi, \varepsilon, \varepsilon_\sigma)} \frac{d\lambda}{\sqrt{R}}, \quad (69a)$$

$$J_r^{(\text{scat})}(\xi) = 0, \quad (69b)$$

$$J_\varphi^{(\text{scat})}(\xi) = 2AMm_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_{\varepsilon_{\min}(\xi, \varepsilon_\sigma)}^\infty d\varepsilon \exp(-\beta\varepsilon) \times \int_{\lambda_c(\varepsilon, \varepsilon_\sigma)}^{\lambda_{\max}(\xi, \varepsilon, \varepsilon_\sigma)} d\lambda \frac{\varepsilon_\sigma \lambda + \alpha \varepsilon}{\sqrt{R}}. \quad (69c)$$

The factor 2 in Eqs. (69a) and (69c) is due to two possible radial directions of motion along a scattered trajectory, $\varepsilon_r = \pm 1$. Note that in dimensionless variables $\Delta = M^2(\xi^2 - 2\xi + \alpha^2)$.

Using the parametric solution for $\lambda_c(\varepsilon, \varepsilon_\sigma)$ given by Eq. (53), we get

$$J_t^{(\text{abs})}(\xi) = -Am_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_{\xi_{\text{mb}}}^{\xi_{\text{ph}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \varepsilon_\sigma) \varepsilon_p(\bar{\xi}, \varepsilon_\sigma) \times \exp[-\beta\varepsilon_p(\bar{\xi}, \varepsilon_\sigma)] \times \int_0^{\lambda_p(\bar{\xi}, \varepsilon_\sigma)} \frac{d\lambda}{\sqrt{R}[\varepsilon_p(\bar{\xi}, \varepsilon_\sigma), \lambda, \xi, \varepsilon_\sigma]}, \quad (70a)$$

$$J_r^{(\text{abs})}(\xi) = -\frac{AM^2 m_0^3 \xi}{\Delta} \sum_{\varepsilon_\sigma = \pm 1} \int_{\xi_{\text{mb}}}^{\xi_{\text{ph}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \varepsilon_\sigma) \times \exp[-\beta\varepsilon_p(\bar{\xi}, \varepsilon_\sigma)] \lambda_p(\bar{\xi}, \varepsilon_\sigma), \quad (70b)$$

$$J_\varphi^{(\text{abs})}(\xi) = AMm_0^3 \xi \sum_{\varepsilon_\sigma = \pm 1} \int_{\xi_{\text{mb}}}^{\xi_{\text{ph}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \varepsilon_\sigma) \times \exp[-\beta\varepsilon_p(\bar{\xi}, \varepsilon_\sigma)] \times \int_0^{\lambda_p(\bar{\xi}, \varepsilon_\sigma)} d\lambda \frac{\varepsilon_\sigma \lambda + \alpha \varepsilon_p(\bar{\xi}, \varepsilon_\sigma)}{\sqrt{R}[\varepsilon_p(\bar{\xi}, \varepsilon_\sigma), \lambda, \xi, \varepsilon_\sigma]}, \quad (70c)$$

where $\varepsilon'_p(\xi, \varepsilon_\sigma) = \partial \varepsilon_p(\xi, \varepsilon_p)/\partial \xi$, and where we have explicitly listed the arguments in $\bar{R}$, i.e., we set

$$\bar{R}(\varepsilon, \lambda, \xi, \varepsilon_\sigma) = \xi^4 \left[\varepsilon^2 - \left(1 - \frac{2}{\xi} \right) \left(1 + \frac{\lambda^2}{\xi^2} \right) - \frac{2\varepsilon_\sigma \alpha \varepsilon \lambda + \alpha^2}{\xi^2} \right], \quad (71)$$

to avoid confusion. In principle, a similar parametrization can be used to express the integrals corresponding to scattered trajectories, however it is difficult to find the integration limit corresponding to $\varepsilon_{\min}(\xi, \varepsilon_\sigma)$, i.e., $\bar{\xi}$ such that $\varepsilon_{\min}(\xi, \varepsilon_\sigma) = \varepsilon_p(\bar{\xi}, \varepsilon_\sigma)$.

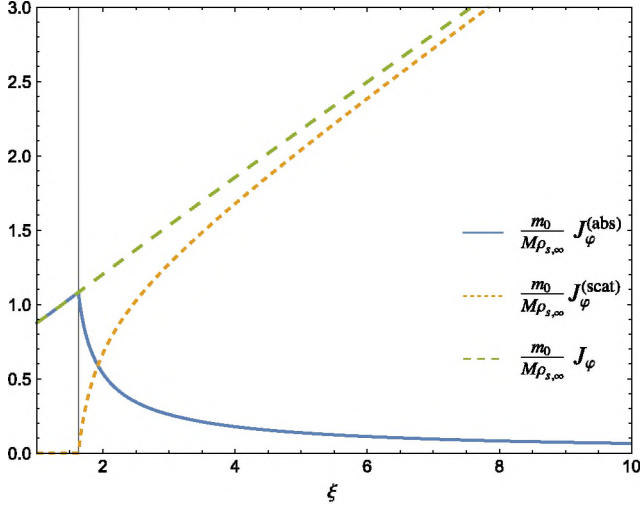


FIG. 4. Components of the particle current surface density J_φ , $J_\varphi^{(\text{abs})}$, and $J_\varphi^{(\text{scat})}$ corresponding to $\epsilon_\sigma = +1$. The spin parameter $\alpha = 7/8$; $\beta = 0.1$. The vertical line marks the location of the circular photon orbit ξ_{ph} .

Sample graphs of various components of the particle current surface density are shown in Figs. 4–10. The main feature visible in these plots is the cutoff for the scattered components $J_t^{(\text{scat})}$, $J_r^{(\text{scat})}$, and $J_\varphi^{(\text{scat})}$ at the location of the circular photon orbit ξ_{ph} , which in turn depends on ϵ_σ : $\epsilon_\sigma = +1$ (prograde motion, Fig. 4); $\epsilon_\sigma = -1$ (retrograde motion, Fig. 5). The difference is best visible in Fig. 9. Note that the graphs of the components $J_t^{(\text{abs})}$, $J_t^{(\text{scat})}$, $J_\varphi^{(\text{abs})}$, and $J_\varphi^{(\text{scat})}$ are not smooth, due to the cutoff behavior at $\xi = \xi_{\text{ph}}$.

Figure 11 shows the directions of the flow at the equatorial plane for $\alpha = 1/2$ and $\beta = 1$. We use standard Cartesian coordinates defined by $t' = t$, $x = r \cos \varphi$, $y = r \sin \varphi$. This yields the following contravariant components of the particle current surface density,

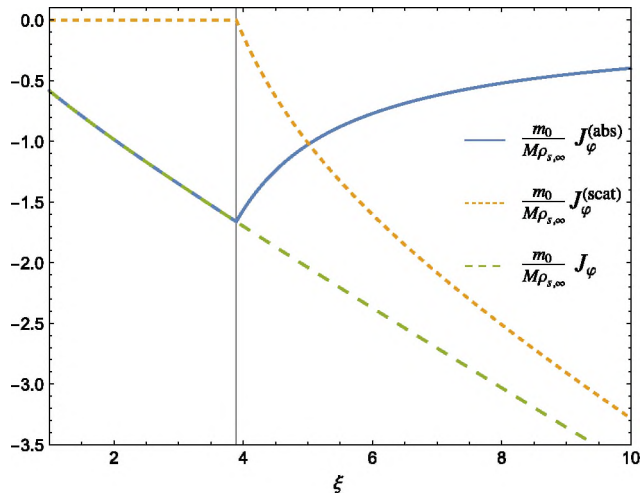


FIG. 5. Same as in Fig. 4, but for $\epsilon_\sigma = -1$.

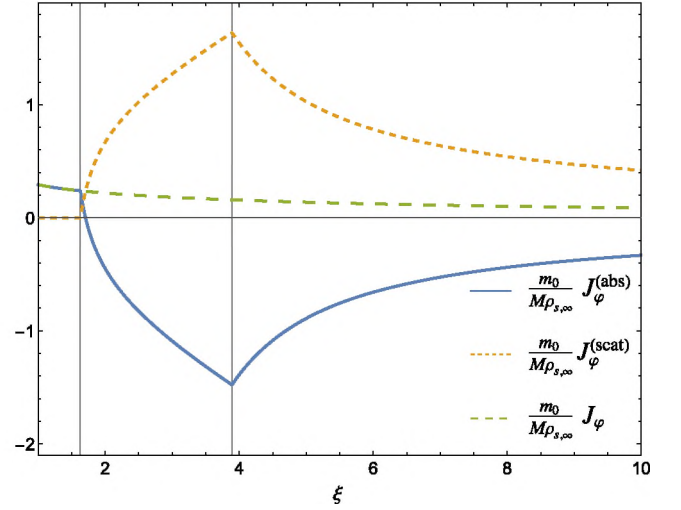


FIG. 6. Components of the particle current surface density J_φ , $J_\varphi^{(\text{abs})}$, and $J_\varphi^{(\text{scat})}$ obtained by summing the parts corresponding to $\epsilon_\sigma = \pm 1$. The spin parameter $\alpha = 7/8$; $\beta = 0.1$.

$$J^t = -\frac{J_t[\xi^3 + (\xi + 2)\alpha^2] + 2\alpha J_\varphi/M}{(\xi^2 - 2\xi + \alpha^2)\xi}, \quad (72a)$$

$$J^x = J_r \frac{\xi^2 - 2\xi + \alpha^2}{\xi^2} \cos \varphi - \frac{J_\varphi/M \times (\xi - 2) - 2\alpha J_t}{\xi^2 - 2\xi + \alpha^2} \sin \varphi, \quad (72b)$$

$$J^y = J_r \frac{\xi^2 - 2\xi + \alpha^2}{\xi^2} \sin \varphi + \frac{J_\varphi/M \times (\xi - 2) - 2\alpha J_t}{\xi^2 - 2\xi + \alpha^2} \cos \varphi. \quad (72c)$$

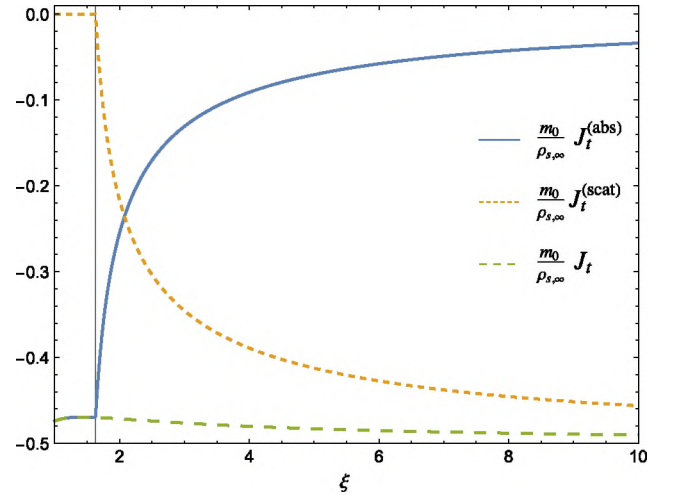
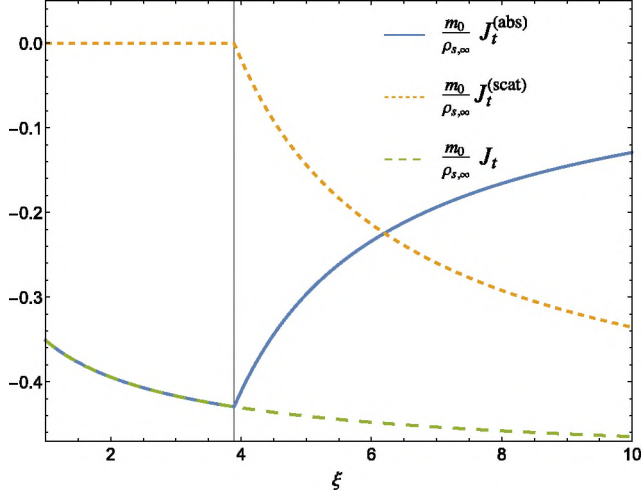
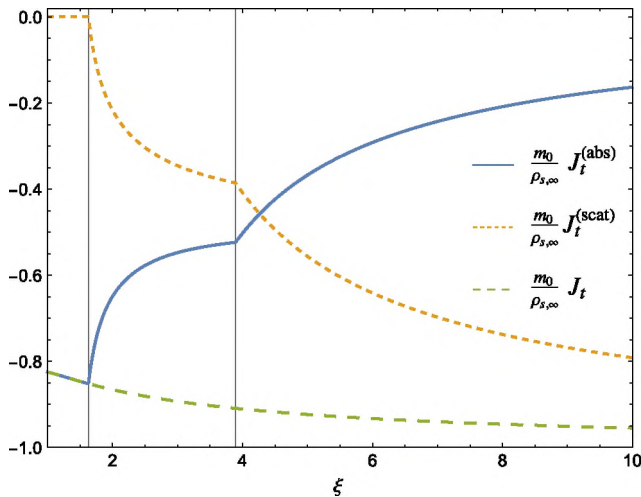
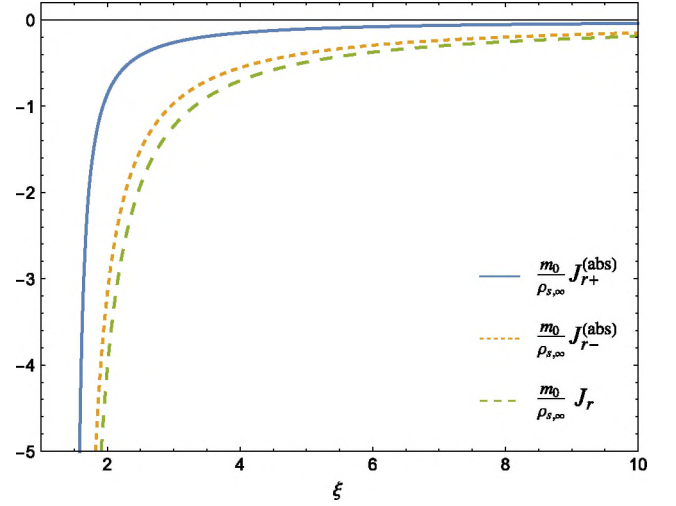


FIG. 7. Components of the particle current surface density J_t , $J_t^{(\text{abs})}$, and $J_t^{(\text{scat})}$ corresponding to $\epsilon_\sigma = +1$. The spin parameter $\alpha = 7/8$; $\beta = 0.1$. The vertical line marks the location of the circular photon orbit ξ_{ph} .

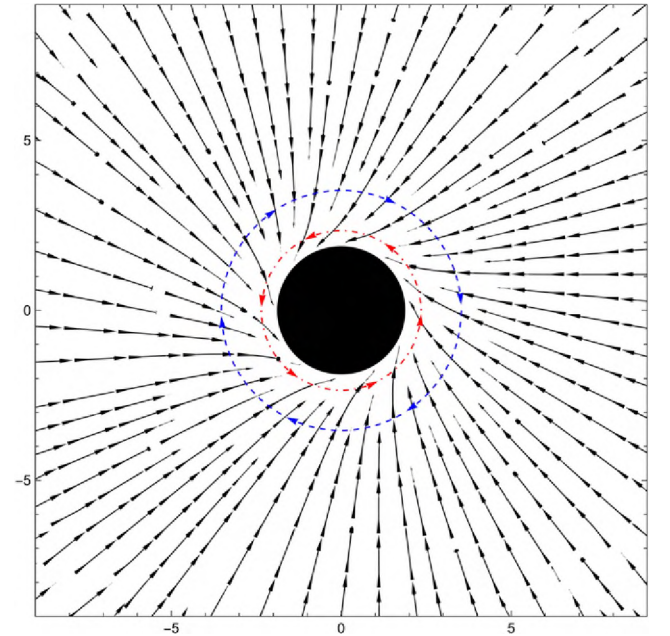

 FIG. 8. Same as in Fig. 7, but for $\epsilon_\sigma = -1$.

The arrows in Fig. 11 correspond to the components (J^x, J^y) . The accretion flow is nearly radial outside the outer (retrograde) circular photon orbit; it twists towards the prograde circular photon orbit, where the dragging effect becomes much more pronounced. Finally, the matter sinks below the black hole horizon.

We should emphasize that the vector components J_μ depend on the assumed metric gauge, in particular on the chosen time foliation. This is best visible for the component J_r (cf. Fig. 10), which in the Boyer-Lindquist coordinates is divergent at the black hole horizon. On the other hand, the particle surface density n_s defined by Eq. (36) and plotted in Fig. 12 is a gauge invariant quantity. The particle surface density n_s is continuous across the outer horizon, but it diverges at the inner Kerr horizon (Fig. 12, right panel). This latter behavior resembles the so-called mass inflation


 FIG. 9. Components of the particle current surface density J_t , $J_t^{(abs)}$, and $J_t^{(scat)}$ obtained by summing the parts corresponding to $\epsilon_\sigma = \pm 1$. The spin parameter $\alpha = 7/8$; $\beta = 0.1$.

 FIG. 10. Components of the particle current surface density $J_{r+}^{(abs)}$ (for $\epsilon_\sigma = 1$), $J_{r-}^{(abs)}$ (for $\epsilon_\sigma = -1$) and J_r obtained by summing the parts corresponding to $\epsilon_\sigma = \pm 1$. The spin parameter $\alpha = 7/8$; $\beta = 0.1$.

discovered by Poisson and Israel [52]. A peculiar feature illustrated in Fig. 12 is a local minimum of the particle surface density n_s near the retrograde marginally stable orbit, occurring for high values of the parameter β . This behavior seems to be related with the dimensionality of the model—it has not been observed for spherically symmetric models investigated in [24–26].


 FIG. 11. Streamlines of the vector field (J^x, J^y) . Circular photon orbits are shown in red and blue colors. The region inside the black hole horizon is marked in black. The black hole rotates counterclockwise. The parameters used to create this plot are $\alpha = 0.5$ and $\beta = 1$. The units on the axes are mass units M .

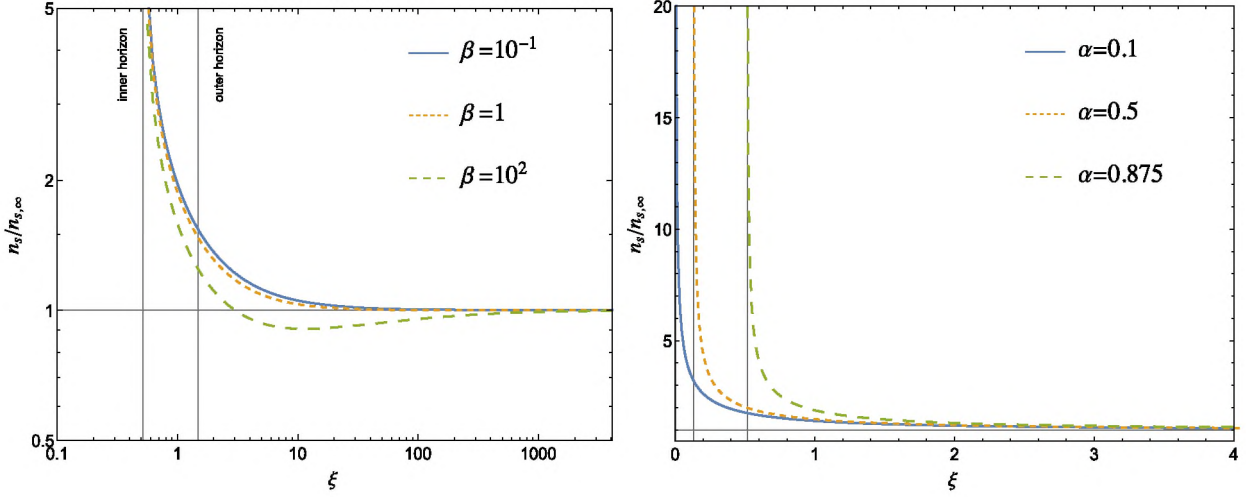


FIG. 12. The invariant particle surface density n_s vs the dimensionless radius ξ . The left panel shows the dependence of n_s on the value of the parameter β for $\alpha = 7/8$. In the right panel we set $\beta = 1$ and plot the graphs of n_s for three different values of α . Vertical lines in the right plot correspond to inner Kerr horizons.

VI. ACCRETION RATES

The two Killing vectors admitted by the metric γ induced at the equatorial plane (5), i.e., $\xi^\mu = (\xi^t, \xi^r, \xi^\varphi) = (1, 0, 0)$ and $\chi^\mu = (\chi^t, \chi^r, \chi^\varphi) = (0, 0, 1)$, give rise to two independent conserved currents $J_{(t)}^\mu = T^\mu{}_\nu \xi^\nu$ and $J_{(\varphi)}^\mu = T^\mu{}_\nu \chi^\nu$, satisfying $\nabla_\mu J_{(t)}^\mu = 0$ and $\nabla_\mu J_{(\varphi)}^\mu = 0$. Together with the conserved particle current surface density J^μ , they yield three accretion rates, which can be defined based on Stokes theorem.

Let $\mathcal{M}$ be a region in a d dimensional spacetime equipped with metric $^{(d)}g$. Let $\partial\mathcal{M}$ denote the $d-1$ dimensional boundary of $\mathcal{M}$, and let $^{(d-1)}g$ be the metric induced on $\partial\mathcal{M}$. A general-relativistic version of Stokes theorem can be written as

$$\int_{\mathcal{M}} d^d x \sqrt{|\det(^{(d)}g)|} \nabla_\mu V^\mu = \int_{\partial\mathcal{M}} d^{d-1} y \sqrt{|\det(^{(d-1)}g)|} n_\mu V^\mu. \quad (73)$$

Here the covariant derivative ∇_μ is defined with respect to the metric $^{(d)}g$, V^μ is a vector field on $\mathcal{M}$, and n^μ is the vector field normal to the boundary $\partial\mathcal{M}$. If $\nabla_\mu V^\mu = 0$, we get

$$\int_{\partial\mathcal{M}} d^{d-1} y \sqrt{|\det(^{(d-1)}g)|} n_\mu V^\mu = 0. \quad (74)$$

Working in the $(2+1)$ -dimensional case restricted to the equatorial plane only, we take $d = 3$ and $^{(d)}g = ^{(3)}g = \gamma$. Assume $\mathcal{M}$ to be a region enclosed by two surfaces of constant time Σ_1, Σ_2 and restricted by $r_1 < r < r_2$. The metric induced on the surface $r = \text{const}$, $\theta = \pi/2$ reads

$$\begin{aligned} {}^{(2)}g = \tilde{\gamma} = & \left(-1 + \frac{2M}{r} \right) dt^2 - \frac{4Ma}{r} dt d\varphi \\ & + \left[r^2 + a^2 \left(1 + \frac{2M}{r} \right) \right] d\varphi^2. \end{aligned} \quad (75)$$

We have $\det \tilde{\gamma}_{\mu\nu} = -\Delta$. The vector normal to the surface $r = \text{const}$ has the components $\tilde{n}_r = \pm r/\sqrt{\Delta}$, $\tilde{n}_t = \tilde{n}_\varphi = 0$. The flux through the boundary at $r = r_1$ can be written as

$$\int_{r=r_1} d^2 y \sqrt{|\det \tilde{\gamma}|} \tilde{n}_\mu V^\mu = - \int_{r=r_1} dt d\varphi r V^r, \quad (76)$$

giving rise to the accretion rate

$$\dot{V} = - \int_{r=r_1} r V^r d\varphi. \quad (77)$$

The minus sign in the above definition is a matter of convention, but it has a meaning. Here, and in what follows, we choose the signs so that the accretion rates correspond to the fluxes into the black hole.

Assuming $V^\mu = m_0 J^\mu$, $-J_{(t)}^\mu$, or $J_{(\varphi)}^\mu$, we get three accretion rates, respectively; the rest-mass accretion rate given by $\dot{M} = -2\pi M m_0 \xi J^r$, the energy accretion rate $\dot{\mathcal{E}} = 2\pi M \xi T^r_t$, and the angular momentum accretion rate given by $\dot{\mathcal{L}} = -2\pi M \xi T^r_\varphi$. Sign conventions in the definitions of $\dot{\mathcal{E}}$ and $\dot{\mathcal{L}}$ can be checked by noticing that for perfect fluids $T^r_t \propto u^r u_t$ and $T^r_\varphi \propto u^r u_\varphi$, where u^μ denote the components of the four-velocity of the fluid.

The expression for J^r given by Eq. (44) yields the rest-mass accretion rate

$$\begin{aligned}\dot{M} &= -2\pi M m_0 \xi J^r \\ &= -2\pi A M m_0^4 \int e_r \exp(-\beta \varepsilon) d\varepsilon d\lambda.\end{aligned}\quad (78)$$

Since $J_r^{(\text{scat})} = 0$, we only get a contribution from absorbed trajectories with $1 \leq \varepsilon$ and $\lambda \leq \lambda_c(\varepsilon, \epsilon_\sigma)$, i.e.,

$$\dot{M} = 2\pi A M m_0^4 \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \lambda_c(\varepsilon, \epsilon_\sigma). \quad (79)$$

For the energy and the angular momentum accretion rates we get

$$\dot{\mathcal{E}} = 2\pi M \xi T^r_t = 2\pi A m_0^4 M \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma), \quad (80)$$

$$\begin{aligned}\dot{\mathcal{L}} &= -2\pi M \xi T^r_\phi = 2\pi A m_0^4 M^2 \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \\ &\times \left[\frac{\epsilon_\sigma}{2} \lambda_c^2(\varepsilon, \epsilon_\sigma) + \alpha \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma) \right],\end{aligned}\quad (81)$$

respectively. Note that $\dot{\mathcal{L}} = 0$ for a nonrotating black hole, i.e., for $\alpha = 0$, since in this case $\lambda_c(\varepsilon, +1) = \lambda_c(\varepsilon, -1)$.

Finally, expressing the normalization constant A in terms of the asymptotic surface rest-mass density $\rho_{s,\infty}$ (51) or the asymptotic surface energy density $\varepsilon_{s,\infty} = -\lim_{\xi \rightarrow \infty} T^r_t$, we get

$$\dot{M} = M \rho_{s,\infty} \frac{\beta^2 \exp(\beta)}{1 + \beta} \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \lambda_c(\varepsilon, \epsilon_\sigma), \quad (82a)$$

$$\dot{\mathcal{E}} = M \varepsilon_{s,\infty} \frac{\beta^3 \exp(\beta)}{2 + 2\beta + \beta^2} \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma), \quad (82b)$$

$$\begin{aligned}\dot{\mathcal{L}} &= M^2 \varepsilon_{s,\infty} \frac{\beta^3 \exp(\beta)}{2 + 2\beta + \beta^2} \sum_{\epsilon_\sigma = \pm 1} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \\ &\times \left[\frac{\epsilon_\sigma}{2} \lambda_c^2(\varepsilon, \epsilon_\sigma) + \alpha \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma) \right].\end{aligned}\quad (82c)$$

The results of the remainder of this section suggest that the parametrization in terms of the asymptotic surface energy density $\varepsilon_{s,\infty}$ seems to be more relevant to accretion rates $\dot{\mathcal{E}}$ and $\dot{\mathcal{L}}$ than an alternative one which uses $\rho_{s,\infty}$. The two quantities are related by

$$\rho_{s,\infty} = \frac{\beta(1 + \beta)}{2 + 2\beta + \beta^2} \varepsilon_{s,\infty}. \quad (83)$$

We derive this relation in Appendix B.

The expressions for $\dot{M}$, $\dot{\mathcal{E}}$, $\dot{\mathcal{L}}$, understood as functions of β , are given by Laplace transforms of certain functions. For instance, $\dot{M}$ turns out to be a Laplace transform of λ_c . This Laplace form implies certain standard relations, such as

$$-\frac{\partial}{\partial \beta} \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \lambda_c(\varepsilon, \epsilon_\sigma) = \int_1^\infty d\varepsilon \exp(-\beta \varepsilon) \varepsilon \lambda_c(\varepsilon, \epsilon_\sigma). \quad (84)$$

Thus $\dot{M}$ and $\dot{\mathcal{E}}$ are related.

Using the parametric representation of $\lambda_c(\varepsilon)$ given by Eqs. (53), one can express the accretion rates as

$$\begin{aligned}\dot{M} &= -M \rho_{s,\infty} \frac{\beta^2 \exp(\beta)}{1 + \beta} \sum_{\epsilon_\sigma = \pm 1} \int_{\xi_{\text{ph}}}^{\xi_{\text{mb}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \epsilon_\sigma) \\ &\times \exp[-\beta \varepsilon_p(\bar{\xi}, \epsilon_\sigma)] \lambda_p(\bar{\xi}, \epsilon_\sigma),\end{aligned}\quad (85a)$$

$$\begin{aligned}\dot{\mathcal{E}} &= -M \varepsilon_{s,\infty} \frac{\beta^3 \exp(\beta)}{2 + 2\beta + \beta^2} \sum_{\epsilon_\sigma = \pm 1} \int_{\xi_{\text{ph}}}^{\xi_{\text{mb}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \epsilon_\sigma) \\ &\times \exp[-\beta \varepsilon_p(\bar{\xi}, \epsilon_\sigma)] \varepsilon_p(\bar{\xi}, \epsilon_\sigma) \lambda_p(\bar{\xi}, \epsilon_\sigma),\end{aligned}\quad (85b)$$

$$\begin{aligned}\dot{\mathcal{L}} &= -M^2 \varepsilon_{s,\infty} \frac{\beta^3 \exp(\beta)}{2 + 2\beta + \beta^2} \sum_{\epsilon_\sigma = \pm 1} \int_{\xi_{\text{ph}}}^{\xi_{\text{mb}}} d\bar{\xi} \varepsilon'_p(\bar{\xi}, \epsilon_\sigma) \\ &\times \exp[-\beta \varepsilon_p(\bar{\xi}, \epsilon_\sigma)] \left[\frac{\epsilon_\sigma}{2} \lambda_p^2(\bar{\xi}, \epsilon_\sigma) + \alpha \varepsilon_p(\bar{\xi}, \epsilon_\sigma) \lambda_p(\bar{\xi}, \epsilon_\sigma) \right],\end{aligned}\quad (85c)$$

where $\varepsilon'_p(\xi, \epsilon_\sigma) = \partial \varepsilon_p(\xi, \epsilon_\sigma) / \partial \xi$.

The above parametric representation fails for $\alpha = \pm 1$, i.e., for the extreme Kerr metric. For $\epsilon_\sigma \alpha = +1$ (prograde motion), we have $\xi_{\text{ph}} = \xi_{\text{mb}} = 1$, and the lower and upper integration limits in the integrals corresponding to $\epsilon_\sigma = \alpha$ coincide. On the other hand, correct expressions given by Eqs. (82) yield nonzero results for those integrals. This behavior is not a fault of Boyer-Lindquist coordinates, as it is discussed in a beautiful article by Jacobson [53], but it is a failure of parametrization (53) for the extreme Kerr metric. Since for $\alpha = \pm 1$ and $\epsilon_\sigma \alpha = +1$ we have $\lambda_c(\varepsilon, \epsilon_\sigma) = \varepsilon$ (cf. Fig. 2), integrals appearing in Eqs. (82) can be easily evaluated. We get in this case (assuming for simplicity positive signs $\alpha = +1$, $\epsilon_\sigma = +1$)

$$\dot{M}|_{\alpha=+1, \epsilon_\sigma=+1} = M \rho_{s,\infty}, \quad (86a)$$

$$\dot{\mathcal{E}}|_{\alpha=+1, \epsilon_\sigma=+1} = M \varepsilon_{s,\infty}, \quad (86b)$$

$$\dot{\mathcal{L}}|_{\alpha=+1, \epsilon_\sigma=+1} = \frac{3}{2} M^2 \varepsilon_{s,\infty}. \quad (86c)$$

Although, $\dot{M}$ and $\dot{\mathcal{E}}$ appear on a similar footing, the quantity responsible for the growth of the black hole mass M is the energy accretion rate $\dot{\mathcal{E}}$ rather than $\dot{M}$. This is shown in spherical symmetry and in the context of perfect fluids in [54] (see also [55]).

While the integrals in Eqs. (85) can be computed numerically, relatively simple analytic expressions can be obtained in the limits of $\beta \rightarrow \infty$ (zero asymptotic temperature) and $\beta \rightarrow 0$ (ultrarelativistic particles).

The asymptotic behavior of $\dot{M}$ for $\beta \rightarrow \infty$ (zero asymptotic temperature) follows from Watson's lemma on asymptotic expansions of Laplace-type integrals [56]. It gives

$$\int_1^\infty e^{-\beta\epsilon} \lambda_c(\epsilon, \epsilon_\sigma) d\epsilon \sim \frac{e^{-\beta}}{\beta} \sum_{n=0}^\infty \frac{1}{\beta^n} \left. \frac{\partial^n \lambda_c(\epsilon, \epsilon_\sigma)}{\partial \epsilon^n} \right|_{\epsilon=1}. \quad (87)$$

Note that the lower integration limit is 1 instead of zero, which leads to an additional factor $e^{-\beta}$, as compared to the standard textbook version of Watson's lemma. Since $\lambda_c(1, \epsilon_\sigma) = \lambda_*(\epsilon_\sigma) \equiv 2 - \epsilon_\sigma \alpha + 2\sqrt{1 - \epsilon_\sigma \alpha}$, and

$$\left. \frac{\partial \lambda_c(\epsilon, \epsilon_\sigma)}{\partial \epsilon} \right|_{\epsilon=1} = \lambda_*(\epsilon_\sigma)^{\frac{3}{2}}, \quad (88)$$

we get, up to the first order,

$$\int_1^\infty e^{-\beta\epsilon} \lambda_c(\epsilon, \epsilon_\sigma) d\epsilon \sim \frac{e^{-\beta}}{\beta} \left[\lambda_*(\epsilon_\sigma) + \frac{1}{\beta} \lambda_*(\epsilon_\sigma)^{\frac{3}{2}} \right]. \quad (89)$$

Since

$$\sum_{\epsilon_\sigma=\pm 1} \lambda_*(\epsilon_\sigma) = 4 + 2\sqrt{1 - \alpha} + 2\sqrt{1 + \alpha}, \quad (90)$$

we obtain the asymptotic expansion of $\dot{M}$ in the form

$$\begin{aligned} \dot{M} \sim M \rho_{s,\infty} \frac{\beta}{1 + \beta} & \left\{ 4 + 2\sqrt{1 - \alpha} + 2\sqrt{1 + \alpha} \right. \\ & + \frac{1}{\beta} [(2 - \alpha + 2\sqrt{1 - \alpha})^{\frac{3}{2}} \\ & \left. + (2 + \alpha + 2\sqrt{1 + \alpha})^{\frac{3}{2}}] \right\}. \end{aligned} \quad (91)$$

The limit for $\beta \rightarrow \infty$ reads

$$\lim_{\beta \rightarrow \infty} \dot{M} = 2M \rho_{s,\infty} (2 + \sqrt{1 - \alpha} + \sqrt{1 + \alpha}). \quad (92)$$

The above expression can also be written in the following trigonometric form

$$\lim_{\beta \rightarrow \infty} \dot{M} = 8M \rho_{s,\infty} \cos^2 \left(\frac{\arcsin \alpha}{4} \right). \quad (93)$$

In the case of $\dot{\mathcal{E}}$, we get, for $\beta \rightarrow \infty$, a similar expansion

$$\begin{aligned} & \int_1^\infty \exp(-\beta\epsilon) \epsilon \lambda_c(\epsilon, \epsilon_\sigma) d\epsilon \\ & \sim \frac{e^{-\beta}}{\beta} \sum_{n=0}^\infty \frac{1}{\beta^n} \frac{\partial^n}{\partial s^n} [(s+1) \lambda_c(s+1, \epsilon_\sigma)] \Big|_{s=0}. \end{aligned} \quad (94)$$

This yields the limit

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \dot{\mathcal{E}} &= 2M \rho_{s,\infty} (2 + \sqrt{1 - \alpha} + \sqrt{1 + \alpha}) \\ &= 2M \epsilon_{s,\infty} (2 + \sqrt{1 - \alpha} + \sqrt{1 + \alpha}). \end{aligned} \quad (95)$$

Clearly, the expressions for $\dot{M}$ and $\dot{\mathcal{E}}$ coincide for $\beta \rightarrow \infty$, as expected.

A similar calculation for $\dot{\mathcal{L}}$ leads to an asymptotic expansion for large values of β of the form

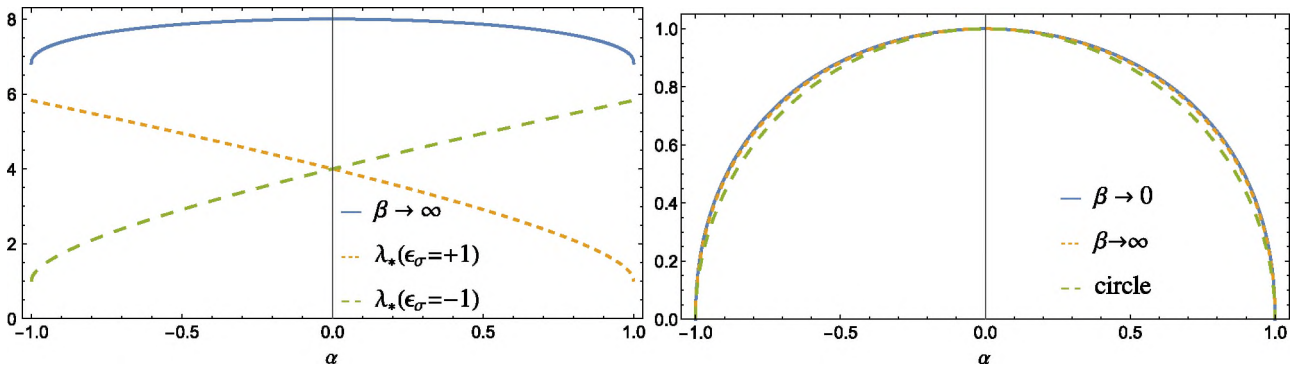


FIG. 13. Accretion rate $\dot{M}$ vs black hole spin parameter α . The abscissa in the left panel shows $\lim_{\beta \rightarrow \infty} \dot{M} / (M \rho_{s,\infty}) = \sum_{\epsilon_\sigma=\pm 1} \lambda_*(\epsilon_\sigma)$ (solid blue line), $\lambda_*(\epsilon_\sigma = +1)$ (dotted orange line), and $\lambda_*(\epsilon_\sigma = -1)$ (dashed green line). The right panel depicts accretion rates $\dot{M}$, rescaled according to Eq. (130). Blue and orange graphs correspond to the limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, respectively. A graph of the half circle $\sqrt{1 - \alpha^2}$ (dashed green line) is shown for comparison.

$$\sum_{\epsilon_\sigma=\pm 1} \int_1^\infty d\epsilon \exp(-\beta\epsilon) \left[\frac{\epsilon_\sigma}{2} \lambda_c^2(\epsilon, \epsilon_\sigma) + \alpha \epsilon \lambda_c(\epsilon, \epsilon_\sigma) \right] \sim \frac{4e^{-\beta}}{\beta} (\sqrt{1-\alpha} - \alpha - \sqrt{1+\alpha}). \quad (96)$$

Thus,

$$\lim_{\beta \rightarrow \infty} \dot{\mathcal{L}} = 4M^2 \rho_{s,\infty} (\sqrt{1-\alpha} - \alpha - \sqrt{1+\alpha}) \quad (97)$$

or, using trigonometric functions,

$$\lim_{\beta \rightarrow \infty} \dot{\mathcal{L}} = -16M^2 \rho_{s,\infty} \cos^2\left(\frac{\arcsin \alpha}{4}\right) \sin\left(\frac{\arcsin \alpha}{2}\right). \quad (98)$$

To find the limits for $\beta \rightarrow 0$, we first observe that the integrand of, e.g., (85a) has a sharp maximum near the beginning of the integration interval, i.e., near ξ_{ph} . Our method is based on identifying the leading terms in the integrand function and estimating the remainder. The latter can be done using the general Lebesgue dominated convergence theorem [57] (page 89, Theorem 19). It is a version of the standard Lebesgue dominated convergence theorem in which the dominating function is replaced by a suitable series of functions. We will illustrate this method on two particular examples, before proceeding with the general result.

Let us start with the $\alpha = 0$ case. The goal is to estimate the integral

$$\begin{aligned} I_1 &= -\frac{e^\beta \beta^2}{1+\beta} \int_{\xi_{\text{ph}}}^{\xi_{\text{mb}}} d\xi e'_p(\xi) \exp[-\beta e_p(\xi)] \lambda_p(\xi) \\ &= -\frac{e^\beta \beta^2}{1+\beta} \int_3^4 \frac{\xi-6}{2(\xi-3)^2 \sqrt{\xi}} \exp\left[-\frac{\beta(\xi-2)}{\sqrt{(\xi-3)\xi}}\right] d\xi. \end{aligned} \quad (99)$$

It cannot be evaluated analytically. However, since the maximum of the integrand is, in the limit, very close to $\xi = 3$, one can try to substitute $\xi = 3$ everywhere except for terms $\xi - 3$, i.e., divergent ones. To be more precise, we split the integral I_1 into two parts $I_1 = I_2 + I_3$, where

$$I_2 = \frac{e^\beta \beta^2}{\beta+1} \int_3^4 \frac{\sqrt{3}}{2(\xi-3)^2} \exp\left(-\frac{\beta}{\sqrt{3}\sqrt{\xi-3}}\right) d\xi \quad (100)$$

and

$$I_3 = -\frac{e^\beta \beta^2}{1+\beta} \int_3^4 G_1(\xi, \beta) d\xi, \quad (101)$$

with

$$\begin{aligned} G_1(\xi, \beta) &= \frac{\sqrt{3}}{2(\xi-3)^2} \exp\left(-\frac{\beta}{\sqrt{3}\sqrt{\xi-3}}\right) \\ &\quad - \frac{6-\xi}{2(\xi-3)^2 \sqrt{\xi}} \exp\left[-\frac{\beta(\xi-2)}{\sqrt{(\xi-3)\xi}}\right]. \end{aligned} \quad (102)$$

The integral I_2 can be evaluated. We have

$$I_2 = \frac{\sqrt{3}(\sqrt{3}\beta+3)}{\beta+1} \exp\left(\beta - \frac{\beta}{\sqrt{3}}\right). \quad (103)$$

Moreover,

$$\lim_{\beta \rightarrow 0} I_2 = 3\sqrt{3}. \quad (104)$$

We now show that $\lim_{\beta \rightarrow 0} I_3 = 0$. Noticing elementary inequalities

$$\sqrt{3} \geq \frac{6-\xi}{\sqrt{\xi}} \quad (105)$$

and

$$\frac{\xi-2}{\sqrt{\xi}} \geq \frac{1}{\sqrt{3}}, \quad (106)$$

valid for $3 \leq \xi \leq 4$, we see that the integrand $G_1(\xi, \beta)$ is non-negative. It is also easy to check that it tends to zero, as $\xi \rightarrow 3$ from above. Clearly,

$$G_1(\xi, \beta) < \frac{\sqrt{3}}{2(\xi-3)^2} \exp\left(-\frac{\beta}{\sqrt{3}\sqrt{\xi-3}}\right) \quad (107)$$

in the range $3 < \xi < 4$. The right-hand side of inequality (107) is integrable, as follows from Eqs. (100) and (103). We may now apply the general Lebesgue dominated convergence theorem [57] to the integral I_3 . Since

$$\lim_{\beta \rightarrow 0} \frac{e^\beta \beta^2}{1+\beta} G_1(\xi, \beta) = 0, \quad (108)$$

we also have $\lim_{\beta \rightarrow 0} I_3 = 0$. Thus $\lim_{\beta \rightarrow 0} I_1 = \lim_{\beta \rightarrow 0} I_2 = 3\sqrt{3}$, and finally

$$\lim_{\beta \rightarrow 0} (\dot{M}|_{\alpha=0}) = 6\sqrt{3} M \rho_{s,\infty}. \quad (109)$$

The additional overall factor 2 in the result is due to the fact that for $\alpha = 0$ the integrals corresponding to $\epsilon_\sigma = \pm 1$ coincide.

As the second example we take $\alpha = 1$, $\epsilon_\sigma = -1$, i.e., the extreme Kerr metric and retrograde orbits. The integral to estimate is now

$$I_4 = \int_{\xi_{\text{ph}}}^{\xi_{\text{mb}}} d\xi e'_p(\xi, -1) \exp[-\beta \varepsilon_p(\xi, -1)] \lambda_p(\xi, -1) \\ = \int_4^{3+2\sqrt{2}} d\xi \frac{\sqrt{\xi} - 3}{2\xi(\sqrt{\xi} - 2)^2} e^{-\frac{\beta(\xi - \sqrt{\xi} - 1)}{\xi^{3/4}\sqrt{\sqrt{\xi} - 2}}}, \quad (110)$$

where we deliberately skip the term $-e^{\beta\beta^2/(1+\beta)}$, to shorten the notation. This integral can be expressed as a sum $I_4 = I_5 + I_6$, where

$$I_5 = - \int_4^{3+2\sqrt{2}} \frac{d\xi}{8(\sqrt{\xi} - 2)^2} \exp\left(-\frac{\beta}{2\sqrt{2}\sqrt{\sqrt{\xi} - 2}}\right) \quad (111)$$

and

$$I_6 = \int_4^{3+2\sqrt{2}} G_2(\xi, \beta) d\xi, \quad (112)$$

with

$$G_2(\xi, \beta) = \frac{1}{8(\sqrt{\xi} - 2)^2} \exp\left(-\frac{\beta}{2\sqrt{2}\sqrt{\sqrt{\xi} - 2}}\right) \\ - \frac{3 - \sqrt{\xi}}{2\xi(\sqrt{\xi} - 2)^2} \exp\left[-\frac{\beta(\xi - \sqrt{\xi} - 1)}{\xi^{3/4}\sqrt{\sqrt{\xi} - 2}}\right]. \quad (113)$$

We have

$$0 \leq G_2(\xi, \beta) \leq \frac{1}{8(\sqrt{\xi} - 2)^2} \exp\left(-\frac{\beta}{2\sqrt{2}\sqrt{\sqrt{\xi} - 2}}\right). \quad (114)$$

The first inequality follows again from two elementary inequalities

$$\frac{1}{4} \geq \frac{3 - \sqrt{\xi}}{\xi} \quad (115)$$

and

$$\frac{1}{2\sqrt{2}} \leq \frac{\xi - \sqrt{\xi} - 1}{\xi^{3/4}}, \quad (116)$$

valid in the range $4 \leq \xi \leq 3 + 2\sqrt{2}$. The second inequality in Eq. (114) is obvious, as $3 - \sqrt{\xi} > 0$ for $4 \leq \xi \leq 3 + 2\sqrt{2}$. The integral I_5 can be evaluated with the result

$$I_5 = -\frac{2\sqrt{2(1+\sqrt{2})}\beta + 8}{\beta^2} \exp\left(-\frac{\beta}{2}\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}\right) \\ + \frac{1}{2} \text{Ei}\left(-\frac{\beta}{2}\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}\right). \quad (117)$$

Thus, $G_2(\xi, \beta)$ is bounded by an integrable function. We have

$$\lim_{\beta \rightarrow 0} \left[\frac{\beta^2 e^\beta}{1 + \beta} I_5 \right] = -8 \quad (118)$$

and

$$\lim_{\beta \rightarrow 0} \left[\frac{\beta^2 e^\beta}{1 + \beta} G_2(\xi, \beta) \right] = 0. \quad (119)$$

By the generalized Lebesgue theorem

$$\lim_{\beta \rightarrow 0} \left[\frac{\beta^2 e^\beta}{1 + \beta} I_6 \right] = 0, \quad (120)$$

and consequently

$$\lim_{\beta \rightarrow 0} \left[\frac{\beta^2 e^\beta}{1 + \beta} I_4 \right] = -8. \quad (121)$$

Thus,

$$\lim_{\beta \rightarrow 0} (\dot{M}|_{\alpha=1, \epsilon_\sigma=-1}) = 8M\rho_{s,\infty}. \quad (122)$$

The corresponding result for $\alpha = 1$ and $\epsilon_\sigma = +1$ is given by Eq. (86a), which is independent of β .

Repeating the above procedure for any $-1 < \alpha < 1$ gives, with the help of a computer algebra system (we use Wolfram Mathematica [58]), the following formula

$$\lim_{\beta \rightarrow 0} \dot{M} = M\rho_{s,\infty} \left[\left(2 + \sqrt[3]{-\alpha + i\sqrt{1-\alpha^2}} \left(-i\sqrt{1-\alpha^2} + \sqrt[3]{-\alpha + i\sqrt{1-\alpha^2} - \alpha} \right) \right)^{3/2} \right. \\ \left. + \left(2 + \sqrt[3]{\alpha + i\sqrt{1-\alpha^2}} \left(-i\sqrt{1-\alpha^2} + \sqrt[3]{\alpha + i\sqrt{1-\alpha^2} + \alpha} \right) \right)^{3/2} \right]. \quad (123)$$

An alternative, explicitly real form of this expression can be written as

$$\lim_{\beta \rightarrow 0} \dot{M} = 6\sqrt{3}M\rho_{s,\infty} \cos\left(\frac{1}{3} \arcsin \alpha\right). \quad (124)$$

In a similar way, one can compute the limit of $\dot{\mathcal{E}}$ for $\beta \rightarrow 0$. The result reads

$$\lim_{\beta \rightarrow 0} \dot{\mathcal{E}} = 6\sqrt{3}M\epsilon_{s,\infty} \cos\left(\frac{1}{3}\arcsin\alpha\right). \quad (125)$$

This is quite surprising. In addition to the equality

$$\lim_{\beta \rightarrow \infty} \frac{\dot{M}}{M\rho_{s,\infty}} = \lim_{\beta \rightarrow \infty} \frac{\dot{\mathcal{E}}}{M\epsilon_{s,\infty}}, \quad (126)$$

established so far, we now also have

$$\lim_{\beta \rightarrow 0} \frac{\dot{M}}{M\rho_{s,\infty}} = \lim_{\beta \rightarrow 0} \frac{\dot{\mathcal{E}}}{M\epsilon_{s,\infty}}, \quad (127)$$

Finally, for $\dot{\mathcal{L}}$ we get

$$\begin{aligned} \lim_{\beta \rightarrow 0} \dot{\mathcal{L}} &= -6\sqrt{3}M^2\epsilon_{s,\infty} \sin\left(\frac{2\arcsin\alpha}{3}\right) \\ &\times \left[2 + \cos\left(\frac{2\arcsin\alpha}{3}\right)\right]. \end{aligned} \quad (128)$$

In a good approximation, the dependence of $\dot{\mathcal{L}}/(M^2\epsilon_{s,\infty})$ on α is just linear (cf. Fig. 17). The coefficient

$$\delta \equiv \frac{d}{d\alpha} \left(\frac{\dot{\mathcal{L}}}{M^2\epsilon_{s,\infty}} \right) \Big|_{\alpha=0} \quad (129)$$

varies from $\delta = -8$ in the limit $\beta \rightarrow \infty$ to $\delta = -12\sqrt{3}$ for $\beta \rightarrow 0$.

Figures 14–17 show numerical values of the accretion rates $\dot{M}$, $\dot{\mathcal{E}}$, and $\dot{\mathcal{L}}$ together with the corresponding limits, obtained for $\beta \rightarrow \infty$ and $\beta \rightarrow 0$. Figure 14 depicts the dependence of the rest-mass accretion rate on the black hole spin parameter α . Different colors represent solutions corresponding to different values of the parameter β . The first observation is that $\dot{M}/(M\rho_{s,\infty})$ decreases with $|\alpha|$. The dependence of $\dot{M}/(M\rho_{s,\infty})$ on α is, in fact, similar for different values of β and can be illustrated by the limiting cases $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, given by Eqs. (124) and (92). After a suitable rescaling, the graphs of $\dot{M}/(M\rho_{s,\infty})$ vs α have an almost circular shape, as shown in Fig. 13. The quantity plotted in this figure (the rescaled $\dot{M}(\beta, \alpha)/(M\rho_{s,\infty})$) is

$$\frac{\dot{M}(\beta, \alpha) - \dot{M}(\beta, 1)}{\dot{M}(\beta, 0) - \dot{M}(\beta, 1)} \quad (130)$$

for fixed values of M and $\rho_{s,\infty}$. Upon such a rescaling, the difference between the values corresponding to $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ is less than 1%, and it is barely visible on the plot. The nearly circular shape can be explained by the zero temperature limit, $\beta \rightarrow \infty$, in which $\dot{M}/(M\rho_{s,\infty})$ is just the sum $\lambda_*(\epsilon_\sigma = +1) + \lambda_*(\epsilon_\sigma = -1) = 4 + 2\sqrt{1-\alpha} + 2\sqrt{1+\alpha}$.

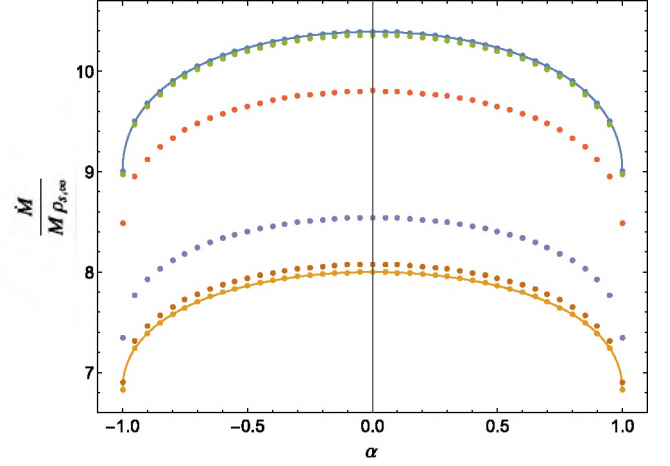


FIG. 14. $\dot{M}/(M\rho_{s,\infty})$ versus α . The dots represent numerically computed values. Solid lines depict the limits $\beta \rightarrow 0$ (blue line) and $\beta \rightarrow \infty$ (orange line), given by Eqs. (124) and (92), respectively. Numerical data are computed for $\beta = 1/4000, 1/10, 1, 10, 100$, and 10000 .

The obtained dependence of $\dot{M}/(M\rho_{s,\infty})$ on α confirms our previous results reported in [26], where two of us investigated a spherical model of the steady accretion of the collisionless Vlasov gas on the Reissner-Nordström black hole. In this work the charged black hole was treated as a toy model of a rotating one, allowing for a simple analysis in spherical symmetry. We have found that the rest-mass accretion rate decreases with the charge parameter (see Fig. 7 in [26]).

Another fact is related to the dependence of $\dot{M}/(M\rho_{s,\infty})$ on β . We see both from Figs. 14 and 15 that the ratio $\dot{M}/(M\rho_{s,\infty})$ is actually decreasing with β , i.e., it increases

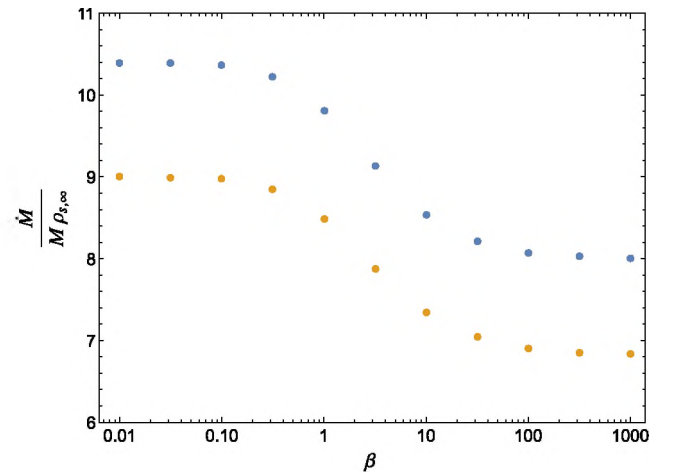


FIG. 15. $\dot{M}/(M\rho_{s,\infty})$ versus β . Blue dots correspond to $\alpha = 0$. Orange dots correspond to $\alpha = \pm 1$. For $\alpha = 0$ the data interpolate between $6\sqrt{3}$ and 8. For $\alpha = \pm 1$, between 9 and $2(2 + \sqrt{2})$. The inflection point is located around $\beta \simeq 2.8$.

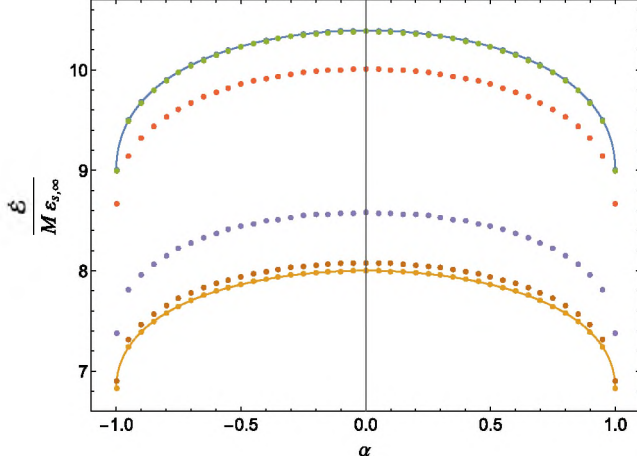


FIG. 16. $\dot{\mathcal{E}}/(M\epsilon_{s,\infty})$ versus α . The dots represent numerically computed values. Solid lines depict the limits $\beta \rightarrow 0$ (blue line) and $\beta \rightarrow \infty$ (orange line), given by Eqs. (125) and (95), respectively. Numerical data are computed for $\beta = 1/1000, 1/10, 1, 10, 100$, and 10000 .

with the asymptotic temperature of the gas. This is surprising, as the situation known for the spherically symmetric accretion of the Vlasov gas onto the Schwarzschild black hole [24,25] or the Reissner-Nordström black hole [26] is exactly opposite. In the latter case, the mass accretion rate $\dot{M}$, normalized by M^2 and the asymptotic rest-mass density, decreases with the asymptotic temperature of the gas. Moreover, it is divergent for $\beta \rightarrow \infty$. It turns out that the factor responsible for the reversed behavior in our case is the parametrization of $\dot{M}$ by the asymptotic surface rest-mass density, and its dependence on the parameter β . We will return to the discussion of this behavior in the subsequent section.

The energy accretion rate $\dot{\mathcal{E}}$ exhibits a very similar dependence on α and β , provided that it is normalized by the asymptotic surface energy density $\epsilon_{s,\infty}$, and not by $\rho_{s,\infty}$. This behavior is illustrated in Fig. 16.

The most interesting outcome of our study concerns, in our view, the angular momentum accretion rate. The dependence of $\dot{\mathcal{L}}/(M^2\epsilon_{s,\infty})$ is shown in Fig. 17. The sign of $\dot{\mathcal{L}}$ is always opposite to the sign of the black hole spin parameter α , meaning that the accretion process slows down the black hole rotation. This seems to be reasonable—in our model the black hole is placed in a large (infinite) reservoir of the gas with essentially no net angular momentum. A steady accretion of the gas can, therefore, decrease the black hole angular momentum. Moreover, the dependence of $\dot{\mathcal{L}}/(M^2\epsilon_{s,\infty})$ on α is to a good approximation linear, except for very high values of $|\alpha|$. The dependence of $\dot{\mathcal{L}}/(M^2\epsilon_{s,\infty})$ on the parameter β is qualitatively similar to the dependence of the rest-mass and energy accretion rates on β . The values interpolate between two limits obtained for $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, given by Eqs. (128) and (97).

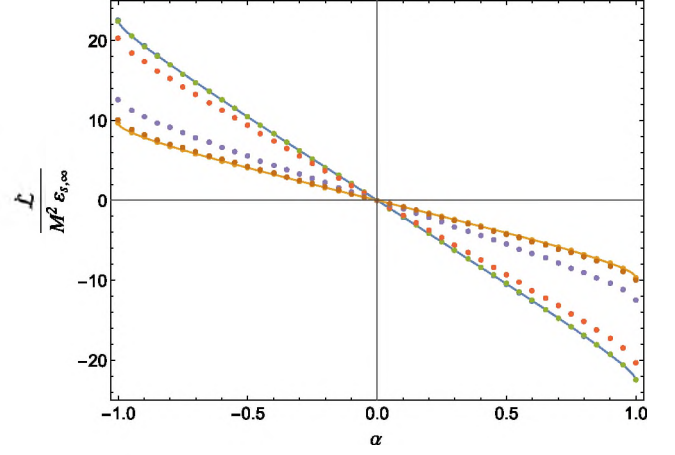


FIG. 17. $\dot{\mathcal{L}}/(M^2\epsilon_{s,\infty})$ versus α . The dots represent numerically computed values. Solid lines depict the limits $\beta \rightarrow 0$ (blue line) and $\beta \rightarrow \infty$ (orange line), given by Eqs. (128) and (97), respectively. Numerical data are computed for $\beta = 1/100, 1/10, 1, 10, 100$, and 1000 . Note a nearly linear behavior for moderate values of α ; the deviation from a first order series expansion around $\alpha = 0$ is of the order of 3.5% at $\alpha = \pm 0.5$.

VII. DISCUSSION

We have investigated a model of a geometrically thin, stationary accretion disk composed of the collisionless Vlasov gas moving in the equatorial plane of the Kerr spacetime. This model serves as a two dimensional kinetic analogue of the Bondi (or Michel) type accretion—at infinity the gas is assumed to be homogeneous and at rest, while the motion in the vicinity of the black hole is induced by the gravitational attraction.

One of subtle points in our analysis is related to the parametrization of solutions. The normalization constant A appearing in the distribution function (37) is not an observable quantity. To obtain a consistent formalism, we express this constant by the asymptotic rest-mass surface density $\rho_{s,\infty}$ or the asymptotic energy surface density $\epsilon_{s,\infty}$. The relation of these quantities with the asymptotic temperature T or the parameter β , characteristic for the two dimensional Maxwell-Jüttner distribution, leads to an unexpected dependence of the rest-mass and energy accretion rates on β —both $\dot{M}/(M\rho_{s,\infty})$ and $\dot{\mathcal{E}}/(M\epsilon_{s,\infty})$ decrease with increasing β . The situation known for spherically symmetric models of steady accretion of the collisionless Vlasov gas on Schwarzschild or Reissner-Nordström black holes is the opposite—the mass accretion rate $\dot{M}$, normalized by M^2 and the asymptotic rest-mass density ρ_{∞} , grows with β [24–26]. We have checked that replacing the constant A in our formula for $\dot{M}$ with a relation characteristic for a three dimensional Maxwell-Jüttner distribution, i.e.,

$$A = \frac{\beta \rho_{\infty}}{4\pi m_0^5 K_2(\beta)}, \quad (131)$$

[cf. Eq. (B20)], would restore (qualitatively) the same standard behavior. This implies a natural question about the physically relevant parametrization. Note that even a tiny momentum component p_z , normal to the equatorial plane, would cause a deviation from the equatorial plane sufficiently far from the black hole and imply a three-dimensional asymptotic distribution.

Another aspect in which the dimensionality of our model is clearly visible is related to the changes of the black hole spin implied by the angular momentum accretion rate. Let us estimate these changes assuming a quasistationary approximation. Assume that $\dot{\mathcal{L}}$ is a time derivative of the black hole angular momentum $J = M^2\alpha$ and that the energy accretion rate $\dot{\mathcal{E}}$ gives the time derivative of the black hole mass M . Thus,

$$\frac{d\alpha}{dt} = \frac{d}{dt} \frac{J}{M^2} = \frac{J}{M^2} \left(\frac{\dot{\mathcal{L}}}{J} - 2 \frac{\dot{\mathcal{E}}}{M} \right) = \alpha \left(\frac{\dot{\mathcal{L}}}{J} - 2 \frac{\dot{\mathcal{E}}}{M} \right). \quad (132)$$

In the limit of $\beta \rightarrow \infty$ (cold matter) we get, for small values of α ,

$$\frac{d\alpha}{dt} \simeq -24\rho_{s,\infty}\alpha. \quad (133)$$

Therefore, in this limit the rotation of the black hole slows down on the time scale $\tau = c/(24G\rho_{s,\infty})$ (in SI units). Note that in our two-dimensional model the above result does not depend on the black hole mass M . As an illustration, we could estimate the time scale τ for the dark matter component present in the Milky Way. Projecting the local cold dark matter density $\rho_{\text{DM}} \simeq 5 \times 10^{-22} \text{ kg/m}^3$ [59] on the Galactic plane, we get $\rho_{s,\infty} = 2r_0\rho_{\text{DM}}$, where $r_0 = 50 \text{ kpc}$ is the dark matter halo scale [60]. This gives the time scale τ of the order of 4×10^9 years. We emphasize that the above result clearly depends on the dimensionality of the model. Improving of the above estimate would require a fully three-dimensional accretion model.

Another aspect not discussed in this paper is related to the stability of the presented model. We would expect various kinetic instabilities known from the theory of accretion disks to be relevant in the present context, especially if magnetic fields are taken into account (see e.g., [61] and references therein).

ACKNOWLEDGMENTS

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APPENDIX A: FOUR-DIMENSIONAL CALCULATION

In this appendix we recover main formulas of this paper, performing a fully four-dimensional calculation in the phase-space and imposing a restriction to the equatorial plane afterwards.

In the four-dimensional setting, the distribution function has the form $\mathcal{F} = \mathcal{F}(t, r, \theta, \varphi, p_t, p_r, p_\theta, p_\varphi)$. The particle current density and the energy momentum tensor are defined as

$$\mathcal{J}_\mu(x) = \int \mathcal{F}(x, p) p_\mu \text{dvol}_x(p), \quad (A1)$$

and

$$\mathcal{T}_{\mu\nu}(x) = \int \mathcal{F}(x, p) p_\mu p_\nu \text{dvol}_x(p), \quad (A2)$$

where

$$\text{dvol}_x(p) = \sqrt{-\det(g^{\mu\nu})} dp_t dp_r dp_\theta dp_\varphi. \quad (A3)$$

For the motion restricted to the equatorial plane we take

$$\mathcal{F}(x, p) = \delta(z) F(x, p), \quad (A4)$$

where z is a coordinate such that the vector ∂_z is normal to the equatorial plane at $z = 0$. In terms of spherical coordinates used in this paper we can assume the standard formula $z = r \cos \theta$. This gives $\delta(z) = \delta(\theta - \pi/2)/r$. Thus,

$$\mathcal{J}_\mu(x) = \delta(z) J_\mu(x), \quad (A5a)$$

$$\mathcal{T}_{\mu\nu}(x) = \delta(z) T_{\mu\nu}(x), \quad (A5b)$$

where

$$J_\mu(x) = \int F(x, p) p_\mu \text{dvol}_x(p), \quad (A6)$$

and

$$T_{\mu\nu}(x) = \int F(x, p) p_\mu p_\nu \text{dvol}_x(p). \quad (A7)$$

The distribution function $\mathcal{F}$ satisfies the Vlasov equation

$$\begin{aligned} \frac{d\mathcal{F}}{d\tau} &= \frac{\partial \mathcal{F}}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial \mathcal{F}}{\partial p_\nu} \frac{dp_\nu}{d\tau} = \frac{\partial \mathcal{F}}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial \mathcal{F}}{\partial p_\nu} \frac{\partial H}{\partial x^\nu} \\ &= \{H, \mathcal{F}\} = 0. \end{aligned} \quad (A8)$$

To compute momentum integrals in Eqs. (A6) and (A7), we continue in a way similar to our three-dimensional calculation and consider a transformation of momentum variables $(p_t, p_r, p_\theta, p_\varphi) \mapsto (m^2, E, l^2, l_z)$, given by

$$m^2 = -[g^{tt}(p_t)^2 + 2g^{t\varphi}p_t p_\varphi + g^{rr}(p_r)^2 + g^{\theta\theta}(p_\theta)^2 + g^{\varphi\varphi}(p_\varphi)^2], \quad (\text{A9a})$$

$$E = -p_t, \quad (\text{A9b})$$

$$l^2 = p_\theta^2 + m^2 a^2 \cos^2 \theta + \left(\frac{p_\varphi}{\sin \theta} + a \sin \theta p_t \right)^2, \quad (\text{A9c})$$

$$l_z = p_\varphi, \quad (\text{A9d})$$

where m^2 in Eq. (A9c) has to be replaced by the right-hand side of Eq. (A9c). By a straightforward computation, we get the Jacobian determinant

$$\frac{\partial(m^2, E, l^2, l_z)}{\partial(p_t, p_r, p_\theta, p_\varphi)} = -\frac{4\Delta p_r p_\theta}{\rho^2} = \pm \frac{4\sqrt{R}\sqrt{\Theta}}{\rho^2}. \quad (\text{A10})$$

Consequently,

$$\begin{aligned} dm^2 dE dl^2 dl_z &= 4(mdm) dE (ldl) dl_z \\ &= \frac{4\sqrt{R}\sqrt{\Theta}}{\rho^2} dp_t dp_r dp_\theta dp_\varphi. \end{aligned} \quad (\text{A11})$$

For the motion confined to the equatorial plane $p_\theta = \Theta = 0$, and hence $l^2 - (l_z - aE)^2 = 0$. This motivates the following definition,

$$\sin \sigma = \frac{l_z - aE}{l}, \quad (\text{A12})$$

and thus $dl_z = \sqrt{l^2 - (l_z - aE)^2} d\sigma$. The momentum integration element reads

$$dp_t dp_r dp_\theta dp_\varphi = \frac{\rho^2 \sqrt{l^2 - (l_z - aE)^2}}{\sqrt{R}\sqrt{\Theta}} (mdm) dE (ldl) d\sigma. \quad (\text{A13})$$

At the equatorial plane we get simply

$$\frac{\rho^2 \sqrt{l^2 - (l_z - aE)^2}}{\sqrt{R}\sqrt{\Theta}} = \frac{r^2}{\sqrt{R}}. \quad (\text{A14})$$

Finally,

$$\text{dvol}_x(p) = \frac{1}{\sqrt{R(r)}} (mdm) dE (ldl) d\sigma, \quad (\text{A15})$$

where $\sigma = \pm\pi/2$ and we have used the fact that $\sqrt{-\det(g^{\mu\nu})} = 1/r^2$ and $\sqrt{\Theta} = \sqrt{l^2 - (l_z - aE)^2}$ at the equatorial plane. In order to control the sign of σ at the equatorial plane we assume a convention with $l \geq 0$ and $l = \epsilon_\sigma(l_z - aE)$ or, equivalently, $l_z = \epsilon_\sigma l + aE$. Thus $\sigma = \epsilon_\sigma \pi/2$. This yields, at the equatorial plane,

$$\begin{aligned} R(r) &= (r^2 E - \epsilon_\sigma a l)^2 - \Delta(m^2 r^2 + l^2) \\ &= M^4 m^2 \tilde{R}(\xi), \end{aligned} \quad (\text{A16})$$

where $\tilde{R}(\xi)$ is given by Eq. (23).

Using dimensionless variables (21) we get, at the equatorial plane,

$$\text{dvol}_x(p) = \frac{m^3 \lambda}{\sqrt{\tilde{R}(\xi)}} dm d\epsilon d\lambda d\sigma. \quad (\text{A17})$$

In order to compute integrals (A6) and (A7), one needs to invert relations (A9). For the motion confined to the equatorial plane outside of the black hole horizon, one gets

$$p_t = -E = -m\epsilon, \quad (\text{A18})$$

$$p_r = \epsilon_r \frac{\sqrt{R}}{\Delta} = \frac{\epsilon_r m \sqrt{\tilde{R}}}{\xi^2 - 2\xi + \alpha^2}, \quad (\text{A19})$$

$$p_\theta = 0, \quad (\text{A20})$$

$$p_\varphi = l_z = \epsilon_\sigma l + aE = Mm(\epsilon_\sigma \lambda + \alpha\epsilon), \quad (\text{A21})$$

where we have expressed the results in terms of m , ϵ , and λ . It is also convenient to have an explicit expression for p^r , which reads

$$p^r = \frac{\epsilon_r m \sqrt{\tilde{R}}}{\xi^2}. \quad (\text{A22})$$

We now take

$$\begin{aligned} F &= A \delta(m - m_0) \frac{\xi}{m\lambda} [\delta(\sigma - \pi/2) + \delta(\sigma + \pi/2)] \\ &\times \exp(-\beta\epsilon) \end{aligned} \quad (\text{A23})$$

and

$$\begin{aligned} \mathcal{F} &= \frac{A}{m_0 M \lambda} \delta(m - m_0) [\delta(\sigma - \pi/2) + \delta(\sigma + \pi/2)] \\ &\times \delta\left(\theta - \frac{\pi}{2}\right) \exp(-\beta\epsilon). \end{aligned} \quad (\text{A24})$$

Note that at the equatorial plane

$$\frac{\xi}{m\lambda} [\delta(\sigma - \pi/2) + \delta(\sigma + \pi/2)] = \delta(p_z). \quad (\text{A25})$$

This relation follows from equations $p_\theta = -rp_z$ and $l^2 = p_\theta^2 + (p_\varphi + ap_t)^2$, which are satisfied at the equatorial plane. Thus

$$p_z = -\epsilon_\theta \frac{l \cos \sigma}{r}, \quad (\text{A26})$$

where ϵ_θ is the sign of p_θ . One can check that $\mathcal{F}$ given by Eq. (A24) satisfies Eq. (A8).

This gives

$$J_t(\xi) = - \sum_{\epsilon_\theta = \pm 1} A m_0^3 \xi \int \exp(-\beta \epsilon) \frac{\epsilon}{\sqrt{\tilde{R}(\xi)}} d\epsilon d\lambda, \quad (\text{A27})$$

$$J_r(\xi) = \sum_{\epsilon_\theta = \pm 1} \frac{A m_0^3 \xi}{\xi^2 - 2\xi + \alpha^2} \int \epsilon_r \exp(-\beta \epsilon) d\epsilon d\lambda, \quad (\text{A28})$$

$$J_\theta(\xi) = 0, \quad (\text{A29})$$

$$J_\varphi(\xi) = \sum_{\epsilon_\theta = \pm 1} A M m_0^3 \xi \int \exp(-\beta \epsilon) \frac{\epsilon_\sigma \lambda + \alpha \epsilon}{\sqrt{\tilde{R}(\xi)}} d\epsilon d\lambda, \quad (\text{A30})$$

in agreement with Eqs. (43).

In the same four-dimensional setup we can also recover the formulas for the accretion rates. A unit vector normal to the boundary $r = r_1$ and tangent to the hypersurface of constant time has the components $\hat{n}_\mu = (0, \pm Q/\sqrt{\Delta}, 0, 0)$. The determinant of the metric induced at $r = r_1$ reads $\det \hat{\gamma} = \Delta Q^2 \sin^2 \theta$.

The flux of the vector field $\hat{V}^\mu$ through the boundary at $r = r_1$ can be written as

$$\int_{r=r_1} d^3 y \sqrt{|\det \hat{\gamma}|} \hat{n}_\mu \hat{V}^\mu = \pm \int_{r=r_1} dt d\theta d\varphi Q^2 \sin \theta \hat{V}^r, \quad (\text{A31})$$

where, as in our previous calculation, we choose the minus sign. This gives rise to the accretion rate

$$\dot{V} = - \int_{r=r_1} d\theta d\varphi Q^2 \sin \theta \hat{V}^r. \quad (\text{A32})$$

For the current $\hat{V}^\mu$ confined to the equatorial plane with $\hat{V}^r = V^r \delta(z) = V^r \delta(\theta - \pi/2)/r$, where V^r does not depend on θ , we get

$$\dot{V} = -r_1 \int_{r=r_1} d\varphi V^r. \quad (\text{A33})$$

The remaining part of the derivation follows the footsteps of our three-dimensional calculation. The only difference is that in the four-dimensional case, the conserved vector currents are given by $\mathcal{J}^\mu$, $\mathcal{T}^\mu_{\nu} \xi^\nu$, and $\mathcal{T}^\mu_{\nu} \chi^\nu$, where the Killing vectors associated with the four dimensional metric g have the components $\xi^\mu = (\xi^t, \xi^r, \xi^\theta, \xi^\varphi) = (1, 0, 0, 0)$ and $\chi^\mu = (\chi^t, \chi^r, \chi^\theta, \chi^\varphi) = (0, 0, 0, 1)$. This again leads to accretion rates defined as in Eqs. (79)–(81).

APPENDIX B: MAXWELL-JÜTTNER DISTRIBUTION FOR THE GAS CONFINED TO A SINGLE PLANE IN THE MINKOWSKI SPACETIME

Thermodynamic relations characterizing the Maxwell-Jüttner distribution in arbitrary dimensions are given, e.g., in [21], Sec. 3.3. In this appendix we derive some of these relations for $2+1$ dimensions, and we do so mainly to keep the same conventions as in the remainder of this paper. In particular, we compute the rest-mass surface density of a gas in thermal equilibrium, confined to a single plane in the Minkowski spacetime, as well as the corresponding energy surface density.

We work in Cartesian coordinates (t, x, y, z) . The Minkowski metric has the form

$$g = -dt^2 + dx^2 + dy^2 + dz^2. \quad (\text{B1})$$

We assume the same notation as in Appendix A. The particle current density is denoted by $\mathcal{J}^\mu$, and the energy-momentum tensor by $\mathcal{T}^{\mu\nu}$. They are defined by Eqs. (A1) and (A2) with $\text{dvol}_x(p) = dp_t dp_x dp_y dp_z$. Surface quantities J^μ and $T^{\mu\nu}$ are defined as in Eqs. (A5). Distribution functions $\mathcal{F}$ and F are related by Eq. (A4).

The Maxwell-Jüttner distribution of the gas confined to a single plane $z = 0$ is defined by the distribution function

$$\begin{aligned} F &= A \delta(\sqrt{-p_\mu p^\mu} - m_0) \delta(p_z) \exp\left(\frac{\beta}{m_0} p_t\right) \\ &= A \delta(\sqrt{p_t^2 - \mathbf{p}^2} - m_0) \delta(p_z) \exp\left(\frac{\beta}{m_0} p_t\right). \end{aligned} \quad (\text{B2})$$

The rest-mass density is equal to $-m_0 \mathcal{J}_t$. The rest-mass surface density, used in this paper, reads $\rho_s = -m_0 J_t$. Similarly, the energy density equals $-\mathcal{T}^t_t$, and the energy surface density is given by $\epsilon_s = -T^t_t$.

The component J_t can be computed as

$$\begin{aligned} J_t &= A \int \delta(\sqrt{p_t^2 - \mathbf{p}^2} - m_0) \delta(p_z) \\ &\quad \times \exp\left(\frac{\beta}{m_0} p_t\right) p_t dp_t dp_x dp_y dp_z, \end{aligned} \quad (\text{B3})$$

where $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$, and where we only take into account future-pointing momenta. In the first step we use the identity

$$\delta(\sqrt{p_t^2 - \mathbf{p}^2} - m_0) = \frac{m_0}{\sqrt{m_0^2 + \mathbf{p}^2}} \delta\left(p_t + \sqrt{m_0^2 + \mathbf{p}^2}\right). \quad (\text{B4})$$

Integrating over p_t , we obtain

$$J_t = -Am_0 \int \delta(p_z) \exp\left(-\frac{\beta}{m_0} \sqrt{m_0^2 + \mathbf{p}^2}\right) dp_x dp_y dp_z. \quad (\text{B5})$$

Next, a straightforward integration over p_z gives

$$J_t = -Am_0 \int \exp\left(-\frac{\beta}{m_0} \sqrt{m_0^2 + p_x^2 + p_y^2}\right) dp_x dp_y. \quad (\text{B6})$$

The next step is standard—we change momentum coordinates (p_x, p_y) to polar ones (ζ, ϑ) , defined by $p_x = \zeta \cos \vartheta$, $p_y = \zeta \sin \vartheta$. This gives $dp_x dp_y = \zeta d\vartheta d\zeta$, where $p_x^2 + p_y^2 = \zeta^2$. Hence,

$$J_t = -2\pi Am_0 \int_0^\infty \exp\left(-\frac{\beta}{m_0} \sqrt{m_0^2 + \zeta^2}\right) \zeta d\zeta. \quad (\text{B7})$$

We can now substitute $\chi = \frac{\beta}{m_0} \sqrt{m_0^2 + \zeta^2}$. Thus, $\zeta d\zeta = (m_0/\beta)^2 \chi d\chi$, and

$$\begin{aligned} J_t &= -\frac{2\pi Am_0^3}{\beta^2} \int_\beta^\infty \exp(-\chi) \chi d\chi \\ &= -2\pi Am_0^3 \frac{1+\beta}{\beta^2} \exp(-\beta). \end{aligned} \quad (\text{B8})$$

The particle number surface density reads

$$n_s = 2\pi Am_0^3 \frac{1+\beta}{\beta^2} \exp(-\beta), \quad (\text{B9})$$

and the rest-mass surface density can be expressed as $\rho_s = m_0 n_s$.

In a similar fashion, one can compute the expression for the energy surface density

$$\varepsilon_s = -T_t = - \int F(p_\nu) p^t p_t dp_x dp_y dp_z, \quad (\text{B10})$$

where $F(p_\nu)$ is given by Eq. (B2). The result reads

$$\varepsilon_s = 2\pi Am_0^4 \frac{2+2\beta+\beta^2}{\beta^3} \exp(-\beta). \quad (\text{B11})$$

It is also worth noticing that the equivalent of the pressure, i.e., $P_s = T^x_x = T^y_y$ reads

$$P_s = \int F(p_\nu) p^x p_x dp_t dp_x dp_y dp_z \quad (\text{B12})$$

$$= 2\pi Am_0^4 \frac{1+\beta}{\beta^3} \exp(-\beta). \quad (\text{B13})$$

Thus,

$$\frac{P_s}{n_s} = \frac{m_0}{\beta} = k_B T. \quad (\text{B14})$$

We see that the two-dimensional Maxwell-Jüttner distribution gives rise to the standard ideal gas equation.

The specific enthalpy associated with the two dimensional Maxwell-Jüttner distribution is

$$\frac{\varepsilon_s + P_s}{\rho_s} = \frac{3+3\beta+\beta^2}{\beta(1+\beta)}. \quad (\text{B15})$$

These results should be contrasted with the standard formula obtained for the gas not restricted to one plane, but filling the whole space uniformly. In the latter case, we assume

$$\mathcal{F} = A' \delta\left(\sqrt{p_t^2 - \mathbf{p}^2} - m_0\right) \exp\left(\frac{\beta}{m_0} p_t\right). \quad (\text{B16})$$

This gives, as before

$$\mathcal{J}_t = -A' m_0 \int \exp\left(-\frac{\beta}{m_0} \sqrt{m_0^2 + \mathbf{p}^2}\right) dp_x dp_y dp_z. \quad (\text{B17})$$

Changing to spherical-momentum coordinates we get

$$\mathcal{J}_t = -4\pi A' m_0 \int_0^\infty \exp\left(-\frac{\beta}{m_0} \sqrt{m_0^2 + \zeta^2}\right) \zeta^2 d\zeta. \quad (\text{B18})$$

The substitution $\chi = \frac{\beta}{m_0} \sqrt{m_0^2 + \zeta^2}$ now gives

$$\begin{aligned} \mathcal{J}_t &= -\frac{4\pi A' m_0^4}{\beta^2} \int_\beta^\infty \exp(-\chi) \chi \sqrt{\left(\frac{\chi}{\beta}\right)^2 - 1} d\chi \\ &= -4\pi A' m_0^4 \frac{K_2(\beta)}{\beta}. \end{aligned} \quad (\text{B19})$$

Thus, the particle number density for the three-dimensional Maxwell-Jüttner distribution can be written as

$$n = 4\pi A' m_0^4 \frac{K_2(\beta)}{\beta}. \quad (\text{B20})$$

- [1] M. Hadzic, Z. Lin, and G. Rein, Stability and instability of self-gravitating relativistic matter distributions, *Arch. Ration. Mech. Anal.* **241**, 1 (2021).
- [2] H. Andréasson, M. Kunze, and G. Rein, Rotating, Stationary, axially symmetric spacetimes with collisionless matter, *Commun. Math. Phys.* **329**, 787 (2014).
- [3] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, New Jersey, 2008).
- [4] P. Domínguez-Fernández, E. Jiménez-Vázquez, M. Alcubierre, E. Montoya, and D. Núñez, Description of the evolution of inhomogeneities on a dark matter halo with the Vlasov equation, *Gen. Relativ. Gravit.* **49**, 123 (2017).
- [5] S. L. Shapiro, A. P. Lightman, and D. M. Eardley, A two-temperature accretion disk model for Cygnus X-1: Structure and spectrum, *Astrophys. J.* **204**, 187 (1976).
- [6] R. Narayan, I. Yi, and R. Mahadevan, Explaining the spectrum of Sagittarius A* with a model of an accreting black hole, *Nature (London)* **374**, 623 (1995).
- [7] R. Mahadevan and E. Quataert, Are particles in advection-dominated accretion flows thermal?, *Astrophys. J.* **490**, 605 (1997).
- [8] B. Ryan, S. Ressler, J. Dolence, Ch. Gammie, and E. Quataert, Two-temperature GRRMHD simulations of M87, *Astrophys. J.* **864**, 126 (2018).
- [9] The Event Horizon Telescope Collaboration, First M87 event horizon telescope results. I. The shadow of the supermassive black hole, *Astrophys. J. Lett.* **875**, L1 (2019).
- [10] The Event Horizon Telescope Collaboration, First M87 event horizon telescope results. V. Physical origin of the asymmetric ring, *Astrophys. J. Lett.* **875**, L5 (2019).
- [11] The Event Horizon Telescope Collaboration, First Sagittarius A* event horizon telescope results. I. The shadow of the supermassive black hole in the center of the Milky Way, *Astrophys. J. Lett.* **930**, L12 (2022).
- [12] The Event Horizon Telescope Collaboration, First Sagittarius A* event horizon telescope results. V. Testing astrophysical models of the galactic center black hole, *Astrophys. J. Lett.* **930**, L16 (2022).
- [13] J. L. Synge, *The Relativistic Gas* (North-Holland, Amsterdam, 1957).
- [14] W. Israel, Relativistic kinetic theory of a simple gas, *J. Math. Phys. (N.Y.)* **4**, 1163 (1963).
- [15] J. Ehlers, General relativity and kinetic theory, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic Press, New York, 1971).
- [16] J. Ehlers, General relativistic theory of gases, in *Relativistic Fluid Dynamics*, edited by C. Cattaneo (Cremonese, Rome, 1971).
- [17] S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland Publishing Company, Amsterdam, 1980).
- [18] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Basel, 2002).
- [19] H. Andréasson, The Einstein-Vlasov system/kinetic theory, *Living Rev. Relativity* **14**, 4 (2011).
- [20] O. Sarbach and Th. Zannias, The geometry of the tangent bundle and the relativistic kinetic theory of gases, *Classical Quantum Gravity* **31**, 085013 (2014).
- [21] R. O. Acuña-Cárdenas, C. Gabarrete, and O. Sarbach, An introduction to the relativistic kinetic theory on curved spacetimes, *Gen. Relativ. Gravit.* **54**, 23 (2022).
- [22] F. C. Michel, Accretion of matter by condensed objects, *Astrophys. Space Sci.* **15**, 153 (1972).
- [23] E. Chaverra, P. Mach, and O. Sarbach, Michel accretion of a polytropic fluid with adiabatic index $\gamma > 5/3$: Global flows versus homoclinic orbits, *Classical Quantum Gravity* **33**, 105016 (2016).
- [24] P. Rioseco and O. Sarbach, Accretion of a relativistic, collisionless kinetic gas into a Schwarzschild black hole, *Classical Quantum Gravity* **34**, 095007 (2017).
- [25] P. Rioseco and O. Sarbach, Spherical steady-state accretion of a relativistic collisionless gas into a Schwarzschild black hole, *J. Phys. Conf. Ser.* **831**, 012009 (2017).
- [26] A. Cieřlik and P. Mach, Accretion of the Vlasov gas on Reissner-Nordström black holes, *Phys. Rev. D* **102**, 024032 (2020).
- [27] P. Mach and A. Odrzywołek, Accretion of Dark Matter onto a Moving Schwarzschild Black Hole: An Exact Solution, *Phys. Rev. Lett.* **126**, 101104 (2021).
- [28] P. Mach and A. Odrzywołek, Accretion of the relativistic Vlasov gas onto a moving Schwarzschild black hole: Exact solutions, *Phys. Rev. D* **103**, 024044 (2021).
- [29] P. Mach and A. Odrzywołek, Accretion of the relativistic vlasov gas onto a moving schwarzschild black hole: Low-temperature limit and numerical aspects, *Acta Phys. Pol. B Proc. Suppl.* **15**, 1-A7 (2022).
- [30] A. Gamboa, C. Gabarrete, P. Domínguez-Fernández, D. Núñez, and Olivier Sarbach, Accretion of a Vlasov gas onto a black hole from a sphere of finite radius and the role of angular momentum, *Phys. Rev. D* **104**, 083001 (2021).
- [31] P. Rioseco and O. Sarbach, Phase space mixing in the equatorial plane of the Kerr black hole, *Phys. Rev. D* **98**, 124024 (2018).
- [32] L. Andersson, P. Blue, and J. Joudioux, Hidden symmetries and decay for the Vlasov equation on the Kerr spacetime, *Commun. Partial Differ. Equations* **43**, 47 (2018).
- [33] H. Andréasson, Existence of steady states of the massless Einstein-Vlasov system surrounding a schwarzschild black hole, *Ann. Inst. Henri Poincaré* **22**, 4271 (2021).
- [34] S. Günther, G. Rein, and Ch. Straub, A birman-schwinger principle in general relativity: Linearly stable shells of collisionless matter surrounding a black hole, [arXiv:2204.10620](https://arxiv.org/abs/2204.10620).
- [35] A. Bransgrove, B. Ripperda, and A. Philippov, Magnetic Hair and Reconnection in Black Hole Magnetospheres, *Phys. Rev. Lett.* **127**, 055101 (2021).
- [36] K. Parfrey, A. Philippov, and B. Cerutti, First-Principles Plasma Simulations of Black-Hole Jet Launching, *Phys. Rev. Lett.* **122**, 035101 (2019).
- [37] B. Ripperda, M. Liska, K. Chatterjee, G. Musoke, A. A. Philippov, S. B. Markoff, A. Tchekhovskoy, and Z. Younsi, Black hole flares: Ejection of accreted magnetic flux through 3D plasmoid-mediated reconnection, *Astrophys. J. Lett.* **924**, L32 (2022).
- [38] S. Ressler, Ch. White, E. Quataert, and J. Stone, Ab Initio horizon-scale simulations of magnetically arrested accretion in Sagittarius A* fed by stellar winds, *Astrophys. J. Lett.* **896**, L6 (2020).

- [39] M. Volonteri, M. Sikora, J.-P. Lasota, and A. Merloni, The evolution of active galactic nuclei and their spins, *Astrophys. J.* **775**, 94 (2013).
- [40] D. Fiacconi, D. Sijacki, and J.E. Pringle, Galactic nuclei evolution with spinning black holes: method and implementation, *Mon. Not. R. Astron. Soc.* **477**, 3807 (2018).
- [41] A. Wójtowicz, Ł. Stawarz, C.C. Cheung, N. Werner, and D. Rudka, Radio emission of nearby early-type galaxies at low and very-low radio luminosity range, [arXiv: 2209.14638](https://arxiv.org/abs/2209.14638).
- [42] A. Wilson and E. Colbert, The difference between radio-loud and radio-quiet active galaxies, *Astrophys. J.* **438**, 62 (1995).
- [43] Y. Dubois, M. Volonteri, and J. Silk, Black hole evolution III. Statistical properties of mass growth and spin evolution using large-scale hydrodynamical cosmological simulations, *Mon. Not. R. Astron. Soc.* **440**, 1590 (2014).
- [44] J.M. Bardeen, W.H. Press, and S.A. Teukolsky, Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation, *Astrophys. J.* **178**, 347 (1972).
- [45] M. Bañados, C. Teitelboim, and J. Zanelli, Black Hole in Three-Dimensional Spacetime, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [46] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1992).
- [47] L.D. Landau and E.M. Lifshitz, *Mechanics. Course of Theoretical Physics*, 2nd ed. (Pergamon Press, Oxford, 1969).
- [48] J.M. Bardeen, Timelike and null geodesics in the Kerr metric, in *Black Holes*, edited by C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1973).
- [49] S.J. Szybka, Black hole flyby, *Am. J. Phys.* **89**, 783 (2021).
- [50] F. Jüttner, Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie, *Ann. Phys. (Berlin)* **339**, 856 (1911).
- [51] F. Jüttner, Die Dynamik eines bewegten Gases in der Relativtheorie, *Ann. Phys. (Berlin)* **340**, 145 (1911).
- [52] E. Poisson and E. Israel, Inner-Horizon Instability and Mass Inflation in Black Holes, *Phys. Rev. Lett.* **63**, 1663 (1989).
- [53] T. Jacobson, Where is the extremal Kerr ISCO?, *Classical Quantum Gravity* **28**, 187001 (2011).
- [54] P. Mach and E. Malec, Steady critical accretion onto black holes: Self-gravity and sonic point characteristics, *Phys. Rev. D* **105**, 104012 (2022).
- [55] J. Karkowski, B. Kinasiewicz, P. Mach, E. Malec, and Z. Świerczyński, Universality and backreaction in a general-relativistic accretion of steady fluids, *Phys. Rev. D* **73**, 021503(R) (2006).
- [56] G.N. Watson, The harmonic functions associated with the parabolic cylinder, *Proc. London Math. Soc.* **s2-17**, 116 (1918).
- [57] H.L. Royden and P.M. Fitzpatrick, *Real Analysis*, 4th ed. (Pearson, Hoboken, New Jersey, 2010).
- [58] Wolfram Research, Inc., *Mathematica*, Version 13.1, Champaign, IL (2022).
- [59] P. Salucci, F. Nesti, G. Gentile, and C. Frigerio Martins, The dark matter density at the Sun's location, *Astron. Astrophys.* **523**, A83 (2010).
- [60] F. Nesti and P. Salucci, The dark matter halo of the Milky Way, AD 2013, *J. Cosmol. Astropart. Phys.* **07** (2013) 016.
- [61] F. Foucart, M. Chandra, C.F. Gammie, E. Quataert, and A. Tchekhovskoy, How important is non-ideal physics in simulations of sub-Eddington accretion on to spinning black holes?, *Mon. Not. R. Astron. Soc.* **470**, 2240 (2017).