

Isotopic effects in nuclear multifragmentation: Studies with the Lattice Gas Model.

Ph.D. Dissertation

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Oświadczenie

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podpis doktoranta

I dedicate my dissertation work to my father,
who is waiting for the scientific news the most,
and to my wife, who is waiting for... me the most.

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Streszczenie

Multifragmentacja jądrowa - rozpad silnie wzbudzonych jąder atomowych na wiele lekkich cząstek i fragmentów o różnych masach - była przedmiotem intensywnych badań w ostatnich dziesięcioleciach. Spodziewano się, że badania te dostarczą informacji o diagramie fazowym materii jądrowej, a w szczególności o przewidywanym przejściu fazowym typu ciecz-gaz. W wielu pracach wnioskowano o obecności przejścia fazowego, jednak jego rodzaj nie został jednoznacznie ustalony.

Niniejszej praca jest poświęcona badaniu procesów multifragmentacji z zastosowaniem dwuskładnikowego Modelu Gazu Sieciowego, w którym rozróżniane są protony i neutrony (Isospin dependent Lattice Gas Model). Umożliwia on badanie efektów izotopowych w multifragmentacji związanych z asymetrią neutronowo-protonową systemu oraz składem izotopowym fragmentów. Obliczenia przeprowadzone zostały w zespole kanonicznym metodą Monte Carlo. Szczególne znaczenie miało badanie charakterystyk rozmiaru największego fragmentu, który odgrywa rolę parametru porządku w układach skończonych, zapewniając lepszy wgląd w zachowanie fazowe systemu. Specyficzne własności kumulant charakteryzujących rozkład prawdopodobieństwa tej wielkości stanowiły wartościową sygnaturę zjawiska krytycznego (punktu pseudokrytycznego) w systemach skończonych.

Przewidywania Modelu Gazu Sieciowego zostały szczegółowo porównane z danymi eksperymentalnymi ALADIN z pomiarów fragmentacji spektatorów pocisku ^{197}Au w zderzeniach ciężkich jonów przy energiach relatywistycznych. Eksperymentalne rozkłady ładunku fragmentów oraz charakterystyka obserwowanego punktu pseudokrytycznego zostały bardzo dobrze opisane przez model, w którym gęstości systemów są znacznie większe niż gęstość oczekiwana dla przejścia fazowego ciecz-gaz, która stanowi około jednej trzeciej normalnej gęstości jądrowej, ρ_0 .

W dalszej analizie zbadano zależność fragmentacji od izospinu systemu wykorzystując dane z pomiarów ALADIN wykonanych dla reakcji z wiązkami neutronowo-nadmiarowymi ^{124}Sn oraz radioaktywnymi wiązkami neutronowo-niedomiarowymi ^{107}Sn i ^{124}La o energiach 600 AMeV. Wyznaczone temperatury systemu w punktach pseudokrytycznych są prawie niezależne od stosunku A/Z pocisków i wynoszą około 5,5 MeV. Analogiczne temperatury wyznaczone w modelu zależą w istotny sposób od gęstości systemu, wykazując zgodność z wartością eksperymentalną dla gęstości około 0,7 ρ_0 . Uzyskane wyniki wskazują, że proces klasteryzacji następuje w gęstym ośrodku jądrowym zgodnie ze scenariuszem multifragmentacji określanego mianem „Małego Wielkiego Wybuchu”.

Abstract

Nuclear multifragmentation - the prompt disintegration of highly excited nuclei into many light particles and complex fragments - have been extensively studied in past decades. It was expected that these investigations would provide information on the phase diagram of nuclear matter, in particular on the predicted liquid-gas phase transition. The presence of a phase transition was often deduced, however, its kind is not unambiguously identified.

In this thesis the nuclear multifragmentation process has been studied with the isospin-dependent Lattice Gas Model with the distinction between protons and neutrons, allowing for investigation of the isotopic effects as represented by the system neutron-proton asymmetry and the fragment isotopic compositions. The calculations were performed in the canonical ensemble using a Monte Carlo method. A special emphasis has been paid to the distributions of the largest fragment size, which plays the role of the order parameter in finite systems, providing a better insight into the phase behavior of the fragmenting system. Specific features of cumulants characterizing the distribution allowed for a robust indication of a critical behavior (the pseudocritical point) in finite systems.

The Lattice Gas Model predictions have been extensively compared with the ALADIN experimental data on fragmentation of ^{197}Au spectator projectiles in heavy-ion collisions at relativistic energies. The experimental fragment charge distributions and characteristics of the observed pseudocritical point are very well reproduced by the model with the system densities significantly larger than the density expected for the liquid-gas phase transition, which is about one third of the normal nuclear density, ρ_0 .

Further analysis investigated the isotopic dependence of fragmentation using the ALADIN data for neutron-rich ^{124}Sn and radioactive neutron-poor ^{107}Sn and ^{124}La projectiles of energy 600 AMeV. The temperatures associated with the observed transition (pseudocritical) points are nearly independent of the projectile A/Z ratio, of about 5.5 MeV. The transition temperature in the model depend significantly on the system density, showing an agreement with the experimental value for the density about $0.7\rho_0$. The obtained results indicate that clusters are formed in the dense medium, in accordance with the “Little big bang” scenario of multifragmentation.

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1 Introduction

Nuclear multifragmentation - the prompt disintegration of highly excited nuclei into many light particles and complex fragments - have been extensively studied both theoretically and experimentally in past decades. It was expected that these investigations would help to determine the phase diagram of nuclear matter. Many works deduced the presence of a phase transition in nuclear multifragmentation that was interpreted as a first-order liquid-gas phase transition as predicted by theoretical models [1–3]. However, the type of phase transition is not unambiguously identified, since signals characteristic of a second-order (continuous) phase transition or critical behavior were also observed. In particular, multifragmentation induced by high energy heavy-ion collisions shows a strong similarity to percolation processes which contain a continuous phase transition [4–8].

The difficulties in recognizing the transition type arises from the fact that phase transitions are well defined in the thermodynamic limit and are significantly modified in small nuclear systems due to the surface and isotopic effects, and Coulomb interactions. These effects are studied within the isospin dependent Lattice Gas Model in this thesis. The model is expected to contain a liquid-gas phase transition and percolation-like critical behavior, which is examined for different system densities and neutron-proton asymmetries. The analysis is focused on a phase transition signature associated with fluctuations of the largest fragment size (charge or mass). This new signature was proposed as a robust indication of a phase transition in small systems and explored in analysis of the ALADIN experimental data [8, 9].

This work is organized as follows. Sec. 2 provides a brief description of the predicted phase diagram of nuclear matter and its relation to multifragmentation processes. The signatures of phase transitions applied to nuclear multifragmentation studies are discussed in Sec. 3. This is followed by a description of the Lattice Gas Model with a brief overview of its applications to nuclear multifragmentation (Sec. 4). Details of the model Monte Carlo algorithm are given in the Appendix. Subsequently, the model predictions are confronted with the ALADIN experimental data on fragmentation of projectile spectators in heavy-ion collisions at relativistic energies [10, 11]. Section 5 examines the fragment charge partitions observed in fragmentation of Au projectiles to determine the source density and temperature at the breakup transition. In Sec. 6 the isotopic dependence of the transition temperature and the fragment isotopic composition are studied using data from the ALADIN S254 experiment with neutron-rich ^{124}Sn and radioactive neutron-poor ^{107}Sn and ^{124}La beams of 600 MeV/nucleon incident energy. Summary and conclusions are presented in Sec. 7.

2 Phase diagram of nuclear matter and nuclear multifragmentation

Nuclear matter is an idealized infinite system of uniformly distributed nucleons (protons and neutrons) interacting only by nuclear forces. Due to the similarity of the radial part of the nuclear force to the van der Waals interaction one can expect that the phase diagram of nuclear matter contains a liquid-gas phase transition [1]. Many theoretical calculations confirm this prediction, showing the presence a liquid-gas phase coexistence zone at densities below the normal nuclear density $\rho_0 \approx 0.17$ nucleons per fm^3 , with a critical point located within a density range $0.3 \div 0.5\rho_0$ and a temperature between 14 and 18 MeV for the symmetric nuclear matter (with equal numbers of protons and neutrons) (see e.g. [12–15]). For example, Fig. 1 shows recent results obtained from a lattice effective field theory [14]. The isotherms plotted in the temperature vs density plane exhibit similar behavior as for the van der Waals fluid. The black line shows the upper boundary of the coexistence region, derived from the Maxwell construction. The predicted critical temperature and density are about 15.8 MeV and $0.43\rho_0$, respectively. The cyan rectangle marks the critical point location deduced from an analysis of nuclear multifragmentation data [14]. The negative slopes of the isotherms (negative isothermal compressibility) indicate the isothermal spinodal region where spontaneous phase separation takes place due to mechanical instabilities.

The phase diagram of asymmetric nuclear matter was extensively studied by Müller and Serot within a relativistic mean-field model [15]. Figure 2 compares the results obtained for symmetric nuclear matter (left panel) and for asymmetric nuclear matter with a proton fraction of 0.3 (right panel). In the case of asymmetric matter, the coexistence region is narrower with somewhat lower critical temperature. In addition to mechanical instabilities in the isothermal and adiabatic spinodal regions, a chemical instability related to fluctuations in the neutron-proton concentration is predicted to occur in the diffusive spinodal zone. The coexisting liquid and gas phases have different neutron-proton concentrations as a result of the increasing symmetry energy with density. The gas is enriched in the more abundant species (neutrons).

The strict phase transitions occur in infinite systems, i.e. in the thermodynamic limit. The predicted phase behavior of nuclear matter is significantly modified in finite systems. In particular, the critical temperature can be lowered by 5 to 6 MeV due to the surface effects, while the Coulomb interactions lead to an additional reduction of approximately 2 to 3 MeV [12]. The estimated change of the critical temperature depends on the mass and charge of the system, and also on

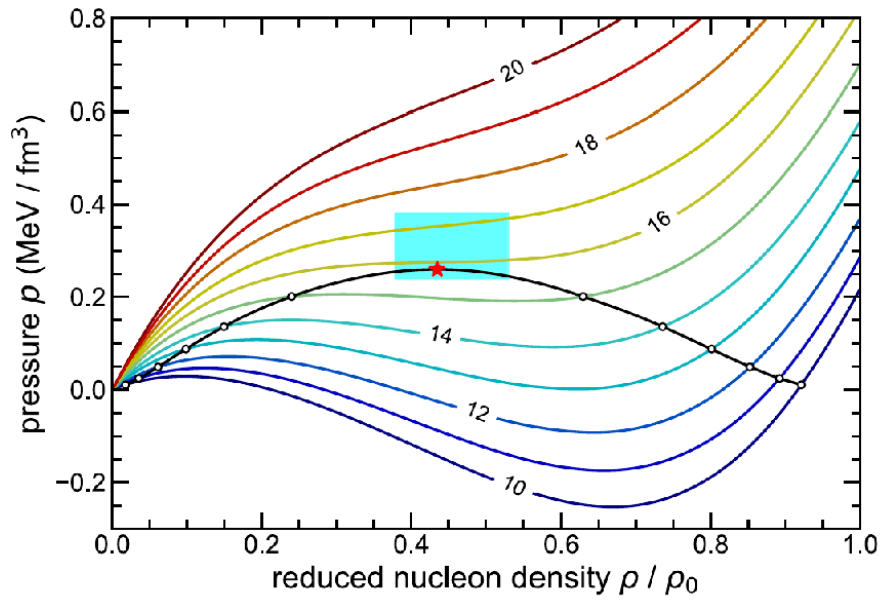


Figure 1: *Ab initio* calculations of isotherms in the pressure-density plane for symmetric nuclear matter. The numbers on the lines are temperatures in MeV. The black line indicates the liquid-vapor coexistence line derived from Maxwell construction, and the red star marks the calculated critical point. The cyan rectangle marks the empirical critical point deduced from nuclear multifragmentation [16]. Figure from [14].

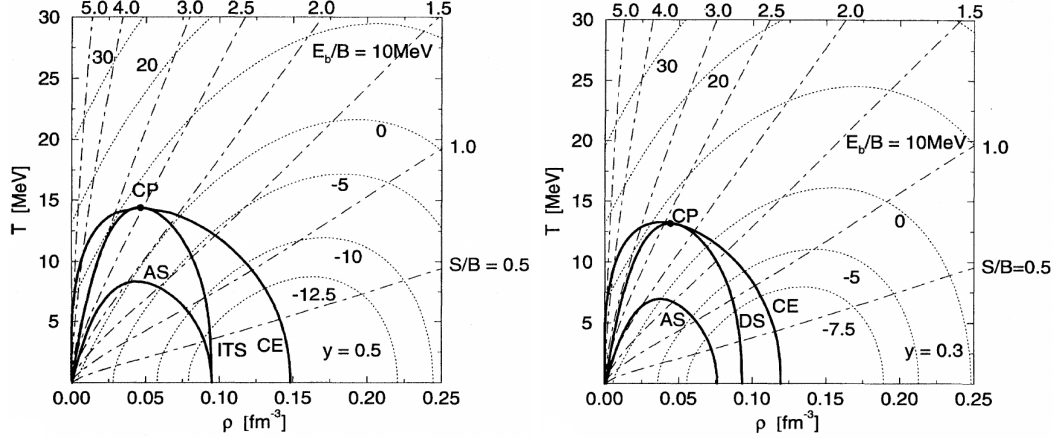


Figure 2: Phase diagram of symmetric nuclear matter (a) and asymmetric nuclear matter with proton fraction of $y = 0.3$ (b) in the temperature-density plane. The solid lines show the upper boundaries of the coexistence (CE), isothermal spinodal (ITS), adiabatic spinodal (AS), and diffusive spinodal (DS) regions. The critical point is denoted by CP. The dotted lines are contours of equal energy per baryon (in MeV), and the dot-dashed lines are adiabats. The drawing source: [15].

the applied theoretical approach.

In the commonly accepted scenario of multifragmentation, schematically sketched in Fig. 3, a highly excited nuclear system created in a heavy-ion collision expands due to the internal thermal pressure. Once it reaches the liquid-gas coexistence region, it breaks into fragments through either the spinodal decomposition or nucleation process. Single nucleons and light charged particles are considered as gas, while larger fragments as liquid droplets. The hot primary fragments undergo secondary decays and propagate in the mutual Coulomb field. Such a scenario assumes fast equilibration of the system during its expansion.

Different models have been developed to describe nuclear multifragmentation. They can be categorized into two main types: dynamical models, which study the time evolution of the system created in a nuclear reaction (e.g., Boltzmann-Uehling-Uhlenbeck (BUU) model [18], Quantum Molecular Dynamics (QMD) model [19] or Antisymmetrized Molecular Dynamics (AMD) model [20]), and statistical models assuming a statistical equilibrium during fragment formation (freeze-out stage) (e.g. Statistical Multifragmentation Model (SMM) [21], Canonical Thermodynamical Model (CTM) [22], Microcanonical Statistical Model [23]).

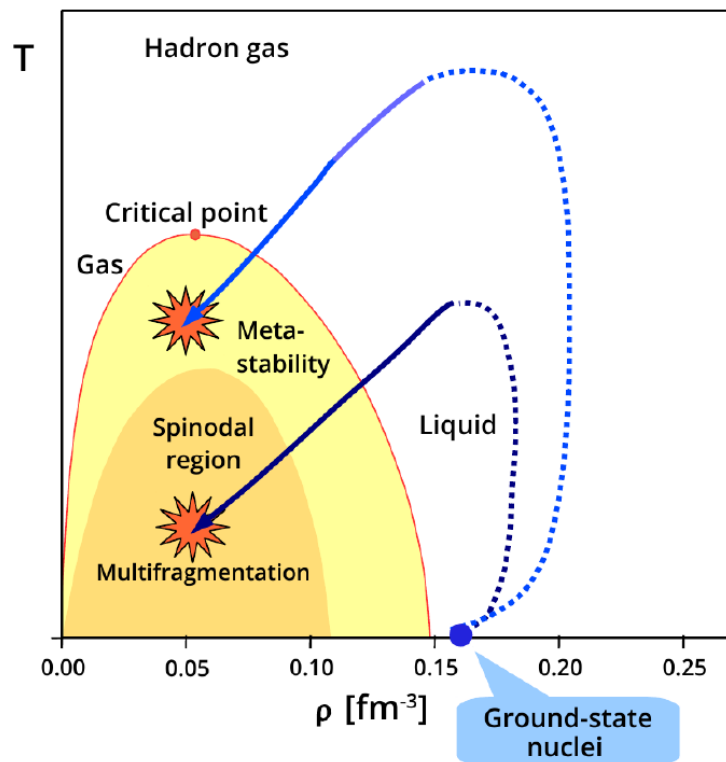


Figure 3: Sketch of temperature-density evolution of highly excited systems created in nuclear collisions that undergo fragmentation in the liquid-gas coexistence zone. Figure from [17].

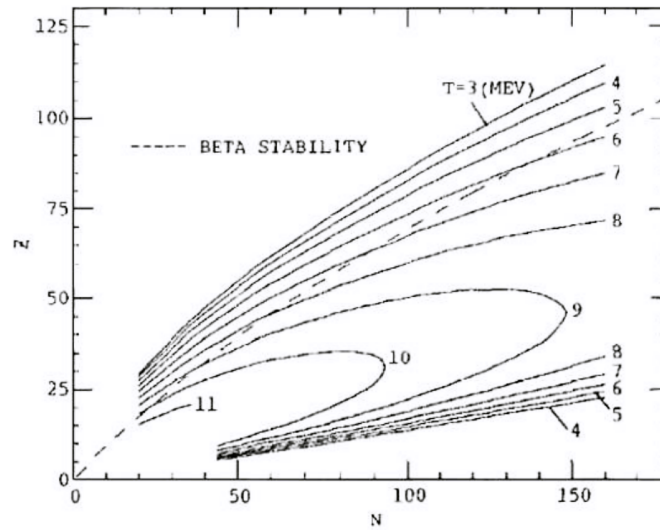


Figure 4: The limiting temperatures of nuclei in the plane of atomic number Z versus neutron number N [25]. The contour lines indicate constant temperatures.

Calculations with a temperature-dependent Hartree-Fock model suggest another scenario of multifragmentation. They showed that a hot charged nuclear liquid drop being in equilibrium with surrounding vapor becomes unstable and break into pieces when temperature exceeds a certain value called the limiting temperature [24]. The limiting temperatures are much lower than the critical temperature of nuclear matter and their values significantly differ depending on the mass and neutron-proton asymmetry of the nucleus as shown in Fig. 4. This Coulomb instability scenario may be however questionable because the system evolves before fragmentation without the presence of the outer vapor.

Particularly successful in describing many features of the nuclear fragmentation are statistical multifragmentation models, which assume that a statistical equilibrium is reached at low density in the freeze-out region, near $1/3$ of the normal nuclear density [21,22]. These models show signatures characteristic of a first-order liquid-gas phase transition. On the other hand, fragment properties observed in multifragmentation induced by high energy collisions are very well reproduced by percolation models with a second-order phase transition (see [6,8] and references therein). This strong resemblance to percolation suggests that clusters are formed in the dense medium, in accordance with the "Little big bang" scenario of multifragmentation proposed in [6]. The present work aims to resolve this ambiguity by examining predictions of the isospin dependent Lattice Gas Model, which contains a liquid-gas phase transition and percolation-like critical behavior at higher densities.

3 Signatures of phase transitions in multifragmentation

In the early 1980s, experiments performed by the Purdue University group revealed a power law dependence of the fragment mass yields $Y(A) = Y_0 A^{-\tau}$ in reactions of high energy protons with Kr and Xe targets [26,27]. This observation was interpreted as a criticality signal according to the Fisher's model prediction for droplet formation at the critical point [28]. The investigations showed that the power law exponent τ was about 2.6, consistently with the range from 2.0 to 3.0 given by the theory of critical phenomena [29]. These results initiated intensive theoretical and experimental studies of phase transitions in nuclear multifragmentation.

Since the simple inclusive mass measurements did not provide sufficient insight into the fragmentation process, the construction of detection systems allowing for simultaneous measurement of multiple fragments was stimulated [30]. In particular 4π detectors, such as, e.g., the MSU Miniball at the Michigan State University [31], ISiS at the Indiana University [32], INDRA at the GANIL facility in Caen [33], and CHIMERA at LNS Catania [34], were capable of measuring nearly all charged reaction products on the event-by-event basis, providing data on the fragment partitions, energy and angular distributions. The measurements allowed to study fragment size (charge or mass) distributions and correlations on the event-by-event basis with sorting events according to quantities correlated with the system excitation energy, and thus serving as a control parameter.

Campi proposed to analyze the conditional moments of the fragment charge distributions [35,36]. As based on percolation model predictions, the critical region can be revealed by maxima in the second moment and the reduced variance distributions, as well as by the event-by-event scatter plot of the logarithm of the largest fragment charge versus the logarithm of the normalized second moment. In such a plot the points are concentrated along two branches corresponding to low and high excitation energies. The transition between them indicates the region of maximal fluctuations associated with the critical behavior. This method applied to experimental data has shown the transition at excitation energies between 4.3 and 6 MeV/nucleon [37–39].

The fragment hierarchy distribution in nuclear disassembly may also carry information about a phase transition or critical behavior [39–42]. This distribution is defined by the so-called Zipf plot, i.e. the mean size (charge) of the largest (rank = 1), second largest (rank = 2), etc. fragments plotted as a function of their rank. At a phase transition (critical) point the distribution is expected to obey the Zipf's

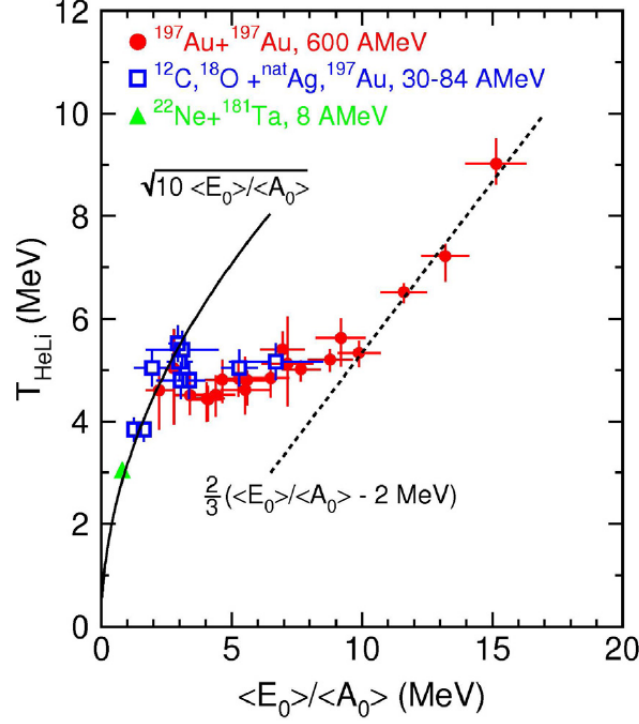


Figure 5: Dependence of the isotope temperature T_{HeLi} on the excitation energy per nucleon (caloric curve) reported by the ALADIN collaboration. The meaning of lines is explained in the text. Figure from [43].

power-law

$$\langle Z_{rank} \rangle \propto rank^{-\lambda} \quad (1)$$

with the exponent $\lambda \approx 1$. Although, as it was shown in [42], the Zipf's law is a mathematical consequence of the Fisher's fragment size power-law with $\tau \approx 2$ for infinite systems, it may serve as a useful additional signature of criticality in nuclear multifragmentation [8, 40].

Many works have been devoted to study the thermodynamic properties of fragmenting nuclear systems such as the relation between the temperature and the excitation energy (the caloric curve). The excitation energy can be obtained with the calorimetry method from the kinetic energies of the detected products and the mass balance [44], while the temperature can be estimated from the slope of fragment kinetic energy spectra, from the double ratios of isotopic yields, and from the population of excited states in light clusters [45]. The first experimental result of the nuclear caloric curve was derived by the ALADIN collaboration from

fragment distributions measured in Au+Au collisions at an incident energy of 600 AMeV [43] (see Fig. 5). The temperatures were deduced from the yield ratios of He and Li isotopes using the Albergo method [46]. As can be seen in the figure, the caloric curve exhibits three distinct parts. In the evaporation regime, characterized by excitation energies below 2 MeV/nucleon, the temperature increase aligns with the low-temperature approximation typical of a fermionic system, as indicated by the solid line. At high excitation energies above 10 MeV/nucleon, the caloric curve exhibits an approximately linear rise, similar to that of a classical Boltzmann gas, signalling the onset of the vaporization regime. Within the excitation energy range of 3 to 10 MeV/nucleon, where multifragment breakup prevails, a nearly constant value of T_{HeLi} of approximately 4.5 to 5 MeV is observed. This plateau in the caloric curve was interpreted as a signature of a first-order liquid-gas phase transition, as it was predicted by statistical models of multifragmentation [21, 47], and also by molecular dynamics calculations (e.g. [48]).

The plateau or flattening in the caloric curve was often observed in further experimental studies. Natowitz et al. performed a systematics of caloric curves for different regions of nuclear mass as shown in Fig. 6 [49]. The observed decrease of the plateau temperature with increasing the system mass from about 9 to 6 MeV was interpreted as a result of Coulomb instabilities in heated and expanded nuclei. Based on these results, the authors estimated a critical temperature of about 16.6 MeV for symmetric infinite nuclear matter [50].

The expansion of hot nuclear systems may introduce them into the spinodal region within the liquid-gas coexistence zone where they disintegrate due to mechanical instabilities, leading to the observation of abnormal kinetic energy fluctuations associated with the negative micro-canonical heat capacity [37, 51, 52]. However, such a feature was deduced with several controversial assumptions, as it was discussed in Ref. [53]. Another proposed signature of spinodal decomposition was the abnormal production of equally sized fragments [54].

In searching for phase transition signals, a special emphasis has been paid to distributions of the largest fragment size (charge or mass), which plays the role of the order parameter in finite systems (see, e.g., Refs. [8, 9, 55] and references given therein). The investigations were focused on a rapid decrease of the largest fragment size associated with a transition from a “liquid” to a “gaseous” phase, on the appearance of large event-to-event fluctuations expected in the critical region [36, 39, 56], on the so-called Δ -scaling [39, 57, 58], and on bimodal characteristics representative of a two-phase coexistence [59–61].

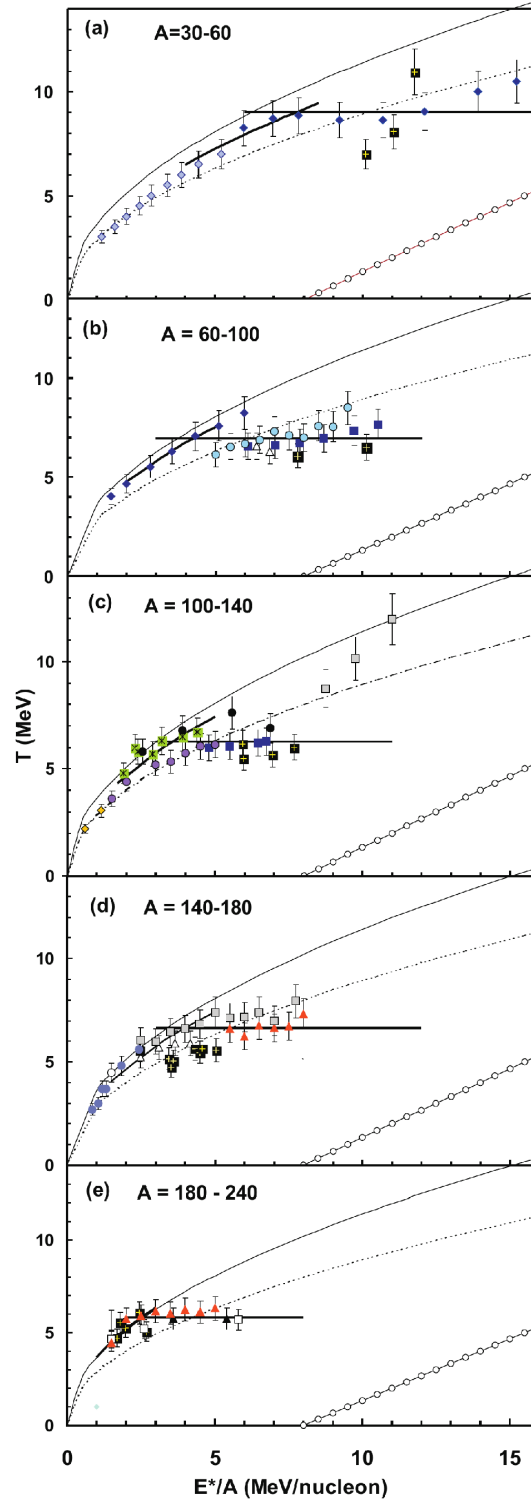


Figure 6: Caloric curves for five selected mass regions. Figure from [49].

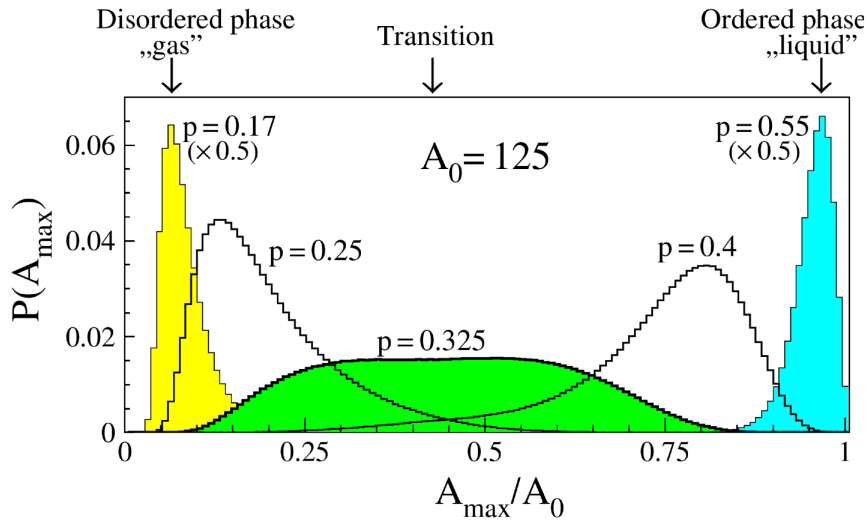


Figure 7: Bond percolation calculations on a cubic lattice of size $A_0 = 5 \times 5 \times 5$: Probability distributions of the largest cluster size $P(A_{max})$ for several values of the bond probability p . The drawing source: [17].

Recently, the largest fragment distributions were extensively studied with a three-dimensional bond percolation model on simple cubic lattices, which contains a second order (or continuous) phase transition at the critical point in infinite systems [62]. Events were generated with a Monte Carlo procedure, in which the bonds are created randomly with a probability p . Free boundary conditions were applied to mimic the presence of surface in real finite systems. As an example representative for small systems (lattices), Fig. 7 shows the dependence of the largest fragment size distribution on the bond probability p being the control parameter. A trace of the phase transition is observed as the broadest, flattened, and nearly symmetrical distribution at $p = 0.325$. Such a transition signal corresponding to the true critical point located in infinite systems at $p_c \simeq 0.25$, appears in finite systems at the pseudocritical point at different values of p depending on the system size.

The probability distributions of the largest cluster size $P(A_{max})$ were characterized by their statistical measures as the mean, the variance, and the following dimensionless cumulant ratios

$$\begin{aligned}
 K_2 &\equiv \mu_2 / \langle A_{max} \rangle^2 = \kappa_2 / \kappa_1^2 \\
 K_3 &\equiv \mu_3 / \mu_2^{3/2} = \kappa_3 / \kappa_2^{3/2} \\
 K_4 &\equiv \mu_4 / \mu_2^2 - 3 = \kappa_4 / \kappa_2^2
 \end{aligned} \tag{2}$$

where $\langle A_{max} \rangle$ is the mean value, $\mu_i = \langle (A_{max} - \langle A_{max} \rangle)^i \rangle$ is the i -th central moment, and κ_i is the i -th cumulant of the $P(A_{max})$ distribution. K_2 is the variance normalized to the squared mean, K_3 is the skewness measuring the distribution asymmetry, and K_4 is the kurtosis excess measuring the degree of peakedness.

It was shown that the transitional $P(A_{max})$ distribution at the percolation pseudocritical point is characterized by $K_3 = 0$ and the minimum value of K_4 of about -1 independently of the system size. The left part of Fig. 8 shows how the cumulant ratios indicate the locations of the pseudocritical and critical points for three different system sizes. Further studies have shown that zero crossing of K_3 coinciding with a minimum of K_4 indicates not only the second-order percolation transition but also the transition observed in the canonical thermodynamic model that contains a first-order phase transition in the thermodynamic limit [9] as illustrated in Fig. 8 (b). This new signature has been applied in analysis of the ALADIN data providing a robust indication of a phase transition or critical behavior [8,9]. It will be also examined with the Lattice Gas Model in this dissertation.

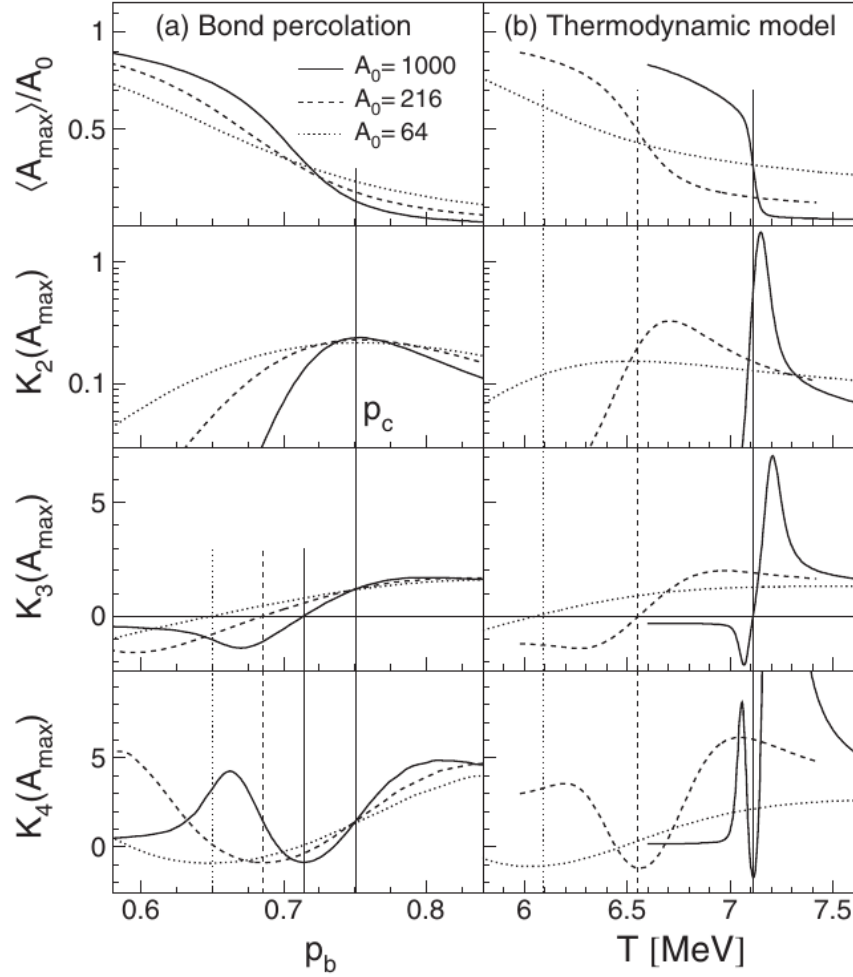


Figure 8: The normalized mean and the cumulant ratios of the largest cluster size distribution for the three indicated system sizes. (a) Percolation results plotted as a function of the bond breaking probability $p_b \equiv 1 - p$ in the transition region. The long vertical line indicates the critical point p_c in the continuous limit, while the short lines indicate the pseudocritical points ($K_3 = 0$ and minimum of K_4) for the finite systems. (b) Results of the canonical thermodynamic model as a function of the temperature. The vertical lines mark the transition temperatures corresponding to the maximum specific heats of the three systems. The drawing source: [8].

4 Lattice Gas Model

4.1 Model description

In a one-component Lattice Gas Model [63–65] sites of a lattice are occupied or not by one-kind particles. The Hamiltonian of this model is expressed as follows:

$$H = \sum_{i=1}^{N_s} n_i \frac{p_i^2}{2m} - \epsilon \sum_{\langle i,j \rangle} n_i n_j, \quad (3)$$

where $n_i(n_j)$ is the occupation number of a particle $i(j)$, which is 1 for occupied site and 0 for vacancy, ϵ is the interaction energy between particles, and N_s is the number of lattice sites. The second sum runs only over neighboring particles.

The calculations are performed on a 3-dimensional cubic lattice with the most compact shape. The length of each side of the elementary cubic lattice is 1.8 fm. A density of a lattice with all sites occupied by particles corresponds to the normal nuclear density, which is about 0.17 fm^{-3} . Particles momenta are obtained from a Maxwell-Boltzmann distribution.

In the two-component Lattice Gas Model [66,67] sites can be occupied by two-kind particles (protons and neutrons). The Hamiltonian of the two-component LGM is expressed as follows:

$$H = \sum_{i=1}^{N_s} \frac{p_i^2}{2m} + \sum_{\langle i,j \rangle} \epsilon_{s_i s_j} s_i s_j + \sum_{s_i=s_j=1, i \neq j}^{N_s} \frac{I_C}{r_{ij}}, \quad (4)$$

where $s_i(s_j)$ is the occupation number of nucleon $i(j)$, taking values of 1 (-1) for proton (neutron) and 0 for vacancy, $\epsilon_{s_i s_j}$ represents the short-range interaction (nuclear potential), $I_C = 1.44 \text{ [MeV} \cdot \text{fm]}$ is the Coulomb energy for two protons at a distance of 1 fm, r_{ij} is the distance between protons, and N_s is a number of lattice sites.

The first term in the Hamiltonian describes the kinetic energy of nucleons, which follows the Maxwell-Boltzmann distribution at an arbitrarily given temperature. The second term represents the potential energy caused by nuclear interaction, which only applies to the nearest neighboring particles. The third term represents the Coulomb energy. The parameter s_i may serve as the isospin of a particle and facilitates the calculations of nuclear and Coulomb interactions. The nuclear potential was set to reproduce the saturation energy (-16 MeV) in the thermodynamic limit. The exact values were adopted from previous studies [68–71]:

$$\epsilon_{nn} = \epsilon_{pp} = 0 \text{ [MeV]} \quad \text{and} \quad \epsilon_{pn} = \epsilon_{np} = -5.33 \text{ [MeV]} \quad (5)$$

Model is evaluated numerically by Monte Carlo procedure with the Metropolis sampling technique for event generation. When a configuration is generated, each of nucleons can be given a momentum. Then, geometrical clusters are identified. This is based on a criterion that neighboring particles belong to the same cluster. However, they will form a bound cluster if the kinetic energy of relative motion is insufficient to overcome the attraction between them. This can be done by introducing a percolation-like procedure in which bonds are created in geometric clusters and breaks with a given probability. The final clusters consists of particles that are connected by active bonds. Bond breaking probability is defined as: $p(T) = \exp^{-|\epsilon|/2T}$. This prescription is known as the Coniglio-Klein's (C-K) [72]. In a case of normal nuclear density (when all nodes are occupied) and only one type of particles is considered, the C-K procedure is equivalent to bond-percolation one. Alternative procedure was proposed by Pan-Das (P-D) [64], denoted also as E_{REL} , which refers to a “relative energy” between particles. In this procedure nucleons belong to the same cluster if a sum of relative kinetic energy and nuclear and coulomb potential is lower or equal zero.

The samples are generated in canonical ensemble for a constant temperature and constant volume. To account for the presence of a surface, free boundary conditions are applied. Details of the algorithm can be found in Appendix A.

4.2 Application to multifragmentation studies

Several authors applied the Lattice Gas Model to study the fragment size distributions and phase transitions in nuclear multifragmentation [40, 41, 64–67, 73–83]. Das Gupta and Pan [64, 65] examined the fragment size distributions with a one-component LGM. They found that a power-law distribution $A^{-\tau}$ with a minimal critical exponent of about 2.2 is observed at the density and temperature corresponding to the thermal critical point. As highlighted by Campi and Krivine [73] the mere observation of a power-law in fragment size distributions is not sufficient to prove that the thermal critical point has been reached. They observed, that such a distribution appears also at supercritical densities and temperatures located near so called the Kertész line [84]. It may explain why this criticality signature is commonly observed in nuclear fragmentation (see e.g.: [85, 86]), which may occurs for different system densities and temperatures. In a later work of Das Gupta and Pan [74] the LGM calculations were compared with experimental data for Ar+Sc and Au+Au central collisions [87, 88]. The authors concluded that the observed discrepancies for the larger system are due to the lack of Coulomb interactions in the model.

Gulminelli and Chomaz [75] used the one-component LGM to investigate the phase diagram of finite systems. They found that the critical line (the Kertész line in the thermodynamic limit) determined from the Fisher scaling in fragment size distributions extends to subcritical densities and temperatures inside the liquid-gas coexistence zone. This feature was corroborated by examining fluctuations of the largest fragment size [76]. Therefore, it was concluded that the first-order liquid-gas phase transition may mimic a critical behavior as a result of finite size effects.

In Ref. [67], Das Gupta and Pan performed calculations using the two-component Lattice Gas Model to explore the effect of Coulomb interactions. The results have shown that the model retains the essential features of the standard LGM, while offering more realistic description of nuclear multifragmentation. The impact of isospin on the fragment distributions and associated signatures of critical behavior was explored with this model for medium-size Xe isotopes in Ref. [79]. It was observed that criticality signals and corresponding temperatures are only weakly dependent on the system neutron-proton asymmetry. The model was also applied to study the caloric curve. The backbending in the caloric curve (the negative specific heat) does not appear in the canonical ensemble at constant volume, but was observed in microcanonical simulations with fluctuating system volume at liquid-gas phase transition [89, 90].

Several studies were devoted to the isoscaling phenomenon [82, 91–96] observed in nuclear multifragmentation experiments, which is expected to provide information about the symmetry energy in the nuclear equation of state. This scaling describes the relationship between isotope yield ratios obtained from two reactions with different isospin asymmetry in the entrance channel. Overall, the LGM predicts the light fragment isotopic compositions obey the isoscaling relation. In Ref. [97] the examination of isoscaling is performed and confronted with experimental data [10]. It was reported that the isoscaling is obeyed very well but for the LGM calculations the values of isoscaling parameters are reproduced only qualitatively and underestimated by a factor of 2.

In Ref. [77] preliminary results from Lattice Gas Model have shown that the fragment charge distributions can be very well reproduced for Au fragmentation data from ALADIN experiment [10] in a wide range of system densities. This work will present a more detailed analysis.

4.3 Phase transition signals

4.3.1 Standard one-component model

The standard (one-component) Lattice Gas Model serves as an useful reference model, as its phase diagram is known in the thermodynamic limit [73]. This phase diagram is depicted in Fig. 9. It contains a liquid-gas coexistence region, which is characteristic for a first-order phase transition. The coexistence region extends from density 0 to 1 of the normal nuclear density. The thermal critical point on the top of this region is located at the density $0.5\rho_0$. The critical temperature equals 1.1275ϵ [73]. For lower densities, above the liquid-gas coexistence zone, the dashed-dotted line indicates the equivalence of the model to the site percolation with Ising clusters. The dashed line represents the Kertész line [84]. It extends from the thermal critical point to the percolation critical point as the model is equivalent to the bond percolation model at the limit of $\rho/\rho_0 = 1$ with the C-K clusterization method. Along the Kertész line a critical behaviour is observed, which manifests as a power-law behavior of the fragment size distribution. The phase behavior in finite small systems was studied in Ref. [75]. As shown in Fig. 10 the critical line (the Kertész line for infinite system) is strongly modified by finite size effects. It extends to subcritical densities, emerging in the coexistence region, which blurs the distinction between different types of phase transitions. The first-order phase transition may resemble a critical behavior.

In this study the critical behavior is identified by analyzing the cumulant ratios K_3 and K_4 of the largest fragment size distribution. In Fig. 11 they are plotted together with the mean largest fragment size and the normalized variance K_2 as a function of the temperature for a system with 1000 particles at various densities (0.3, 0.5, 0.7, and 0.9 ρ_0). In addition, the top panel shows the specific heat, C_V , distributions. The maximum of C_V can be interpreted as a trace of the first-order liquid-gas phase transition according to statistical model predictions [9]. The pseudocritical points are indicated by the zero crossing of K_3 and minimum of K_4 . The derived from such an analysis locations of the pseudocritical points and the maxima of C_V for different system sizes are presented in Fig. 12. As can be seen at low densities, the signatures $K_3 = 0$ and maximum of C_V , nearly overlap each other resulting in inability to distinguish the transition type.

The upper panels of Fig. 13 shows the largest fragment size distributions at the pseudocritical points. Their characteristics are similar to those in percolation model. The associated fragment size distributions shown in the lower panels exhibit power-law dependencies with exponents around 2.2, confirming that the $K_3 = 0$ min. K_4 signature indicates in the Lattice Gas Model a percolation-

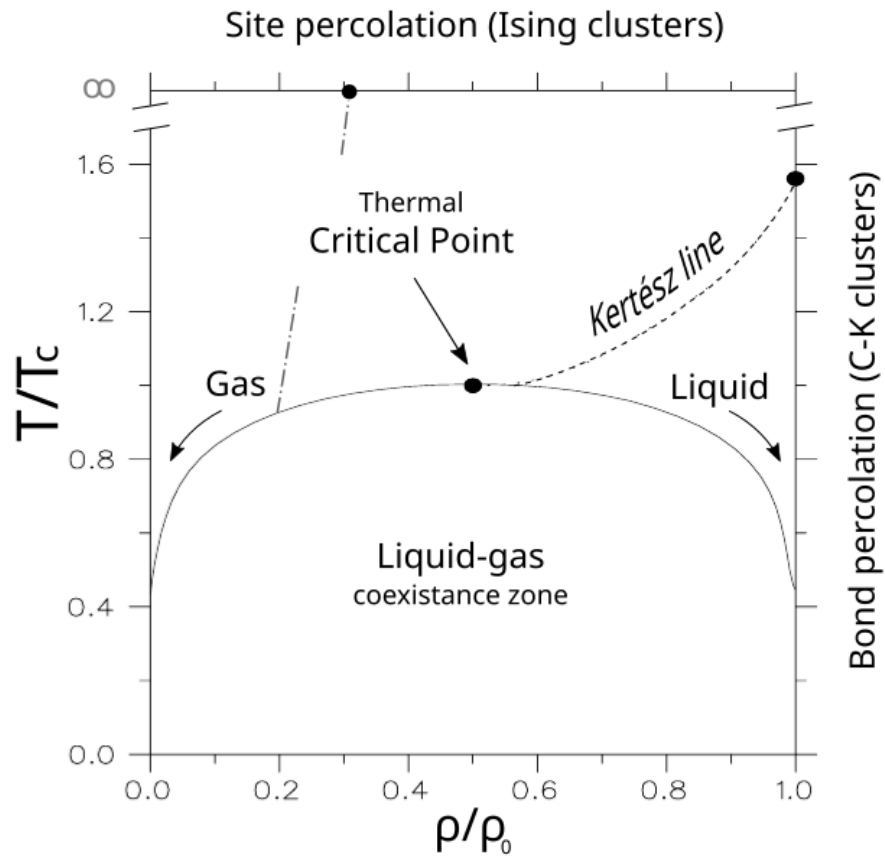


Figure 9: Phase diagram of the standard Lattice Gas Model in the temperature-density plane. See text for the description. Figure adopted from [73].

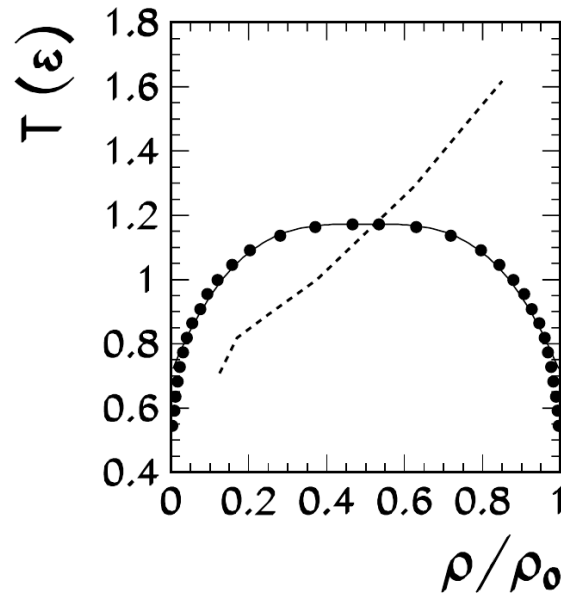


Figure 10: Location of the critical line deduced from fluctuations of the largest fragment size within the LGM phase diagram for a small system (the dashed line) as predicted in Ref. [76].

like critical behavior. This is also corroborated by characteristic for percolation the evolution of the largest fragment size distribution with the control parameter (temperature in LGM), as can be seen in Fig. 14.

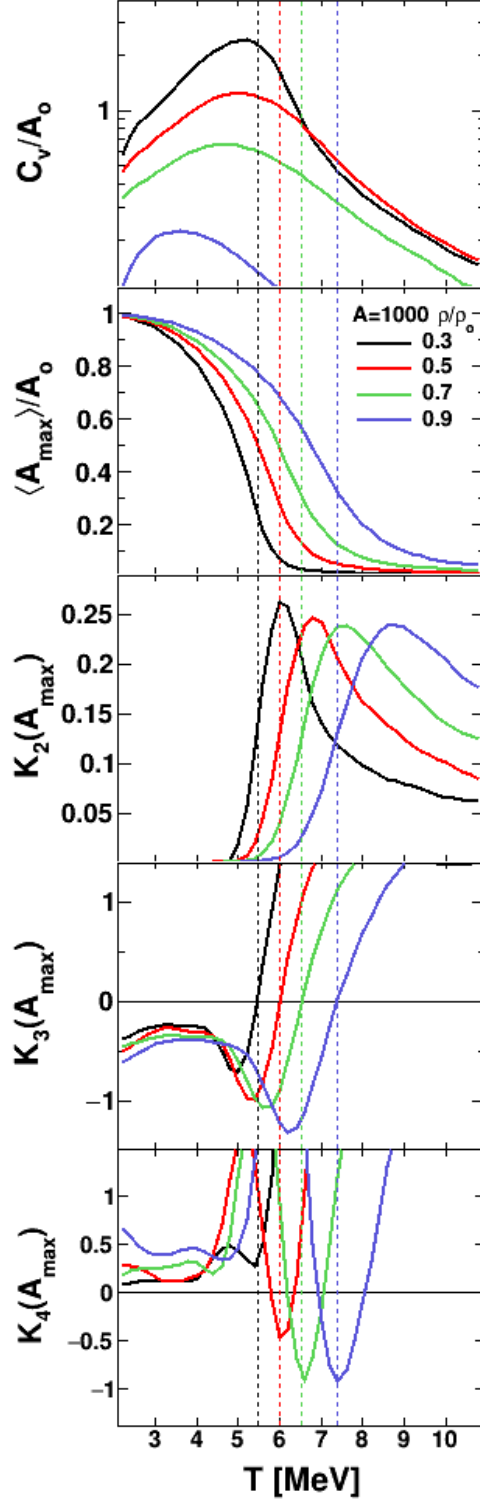


Figure 11: Results of one-component Lattice Gas Model (LGM) simulations for system with 1000 particles for various densities (see the legend for color coding). From top to bottom: normalized to the system size the specific heat C_V , the normalized mean largest fragment size $A_{\max}$, and cumulants K_2 , K_3 , K_4 of the largest fragment size distribution. The vertical dashed lines indicate the locations of the zero crossing of K_3 and minimum of K_4 .

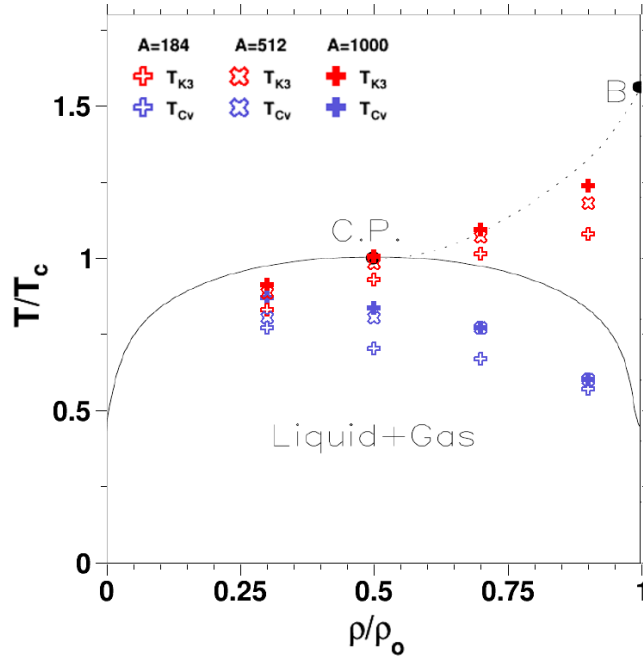


Figure 12: Locations of phase transition signals in the temperature vs density plane for three different system sizes. The red symbols indicate the pseudocritical points ($K_3 = 0$). The blue symbols correspond to the maxima of C_V .

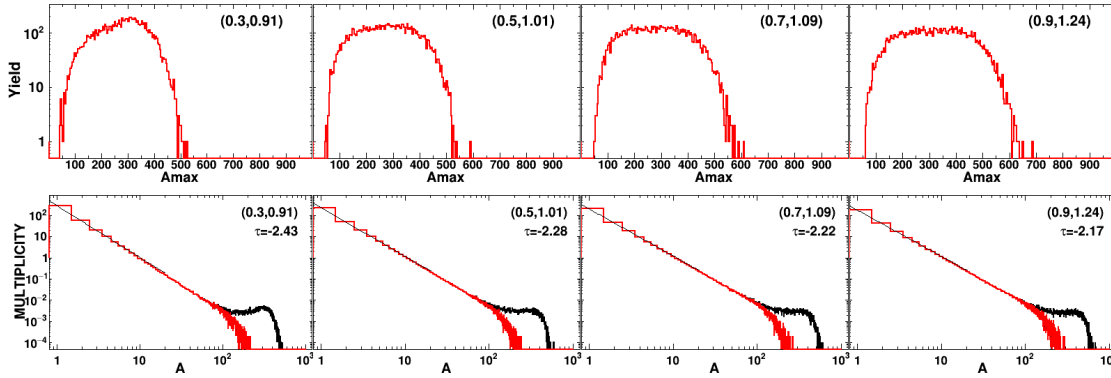


Figure 13: Upper panel: the largest fragment size distributions for the system size $A = 1000$ at the pseudocritical points. Lower panel: the corresponding fragment size distributions with the largest fragment included (black histograms) and excluded in each event (red histograms). Power-law fits to the red distributions are shown by the black thin straight lines. The obtained from these fits the values of the critical exponent τ are displayed. The meaning of the numbers in brackets: $(\rho/\rho_0, T/T_C)$.

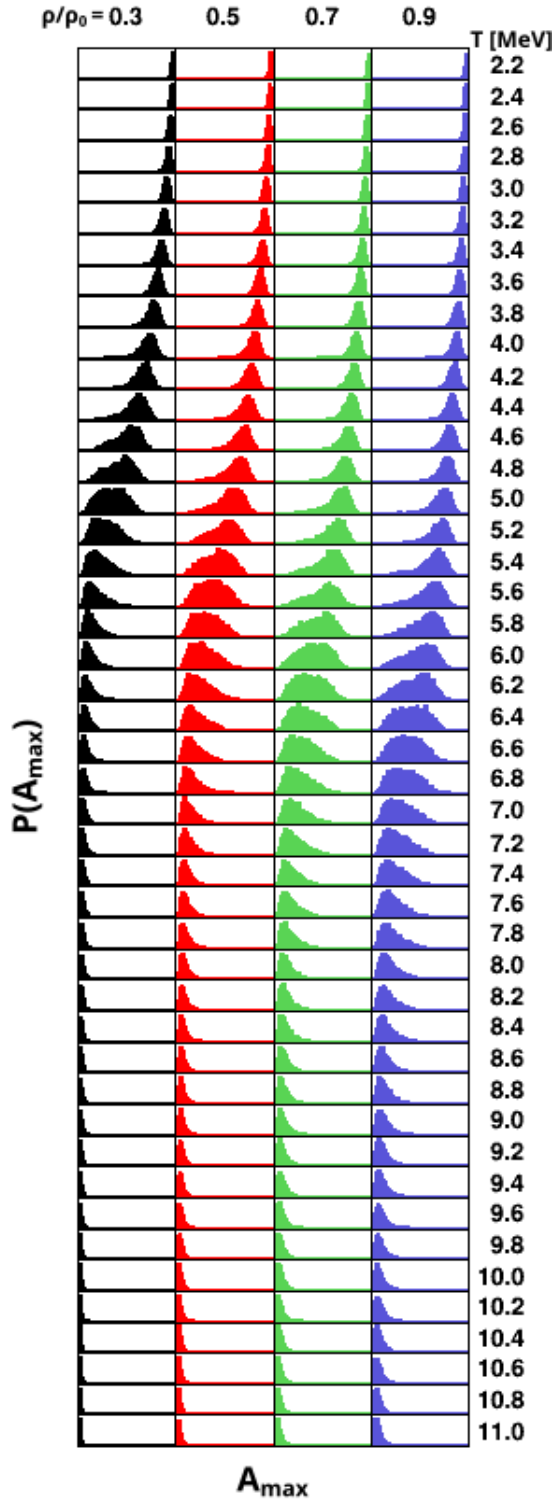


Figure 14: Evolution of the largest fragment size probability distribution with the temperature for several system densities. The system size $A = 1000$.

4.3.2 Isospin dependent model

The phase diagram of the two-component LGM is not well established. Similar analysis as in the previous section allowed to determine the critical lines in the density-temperature plane for various systems. Exemplary results of these calculations are presented in the Fig. 15. They are shown with the one-component LGM phase diagram in the background as a reference. The calculations were performed for a system mass number of 184 and different N/Z ratios, without (left panel) and with Coulomb interactions (right panel). The red symbols are the results from the one-component LGM model for comparison. As can be seen, the transition temperatures are weakly dependent on the system isospin, but they are significantly lowered by the Coulomb interactions, especially at lower densities.

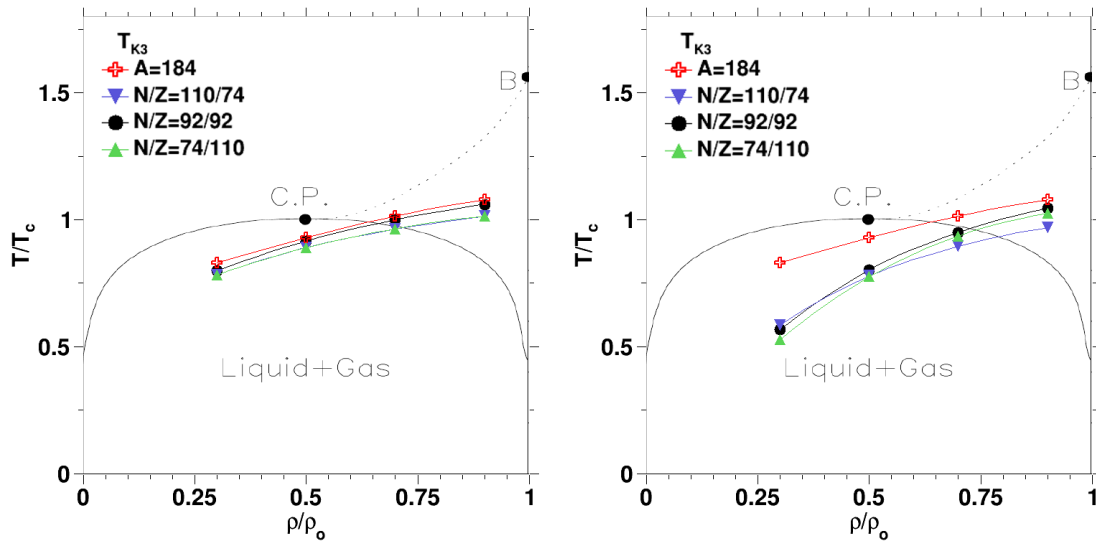


Figure 15: Phase diagrams of the two-component LGM without (left diagram) and with Coulomb interactions (right diagram). for three systems of the same mass $A = 184$ but different A/Z ratio. The red symbols show results for the one-component model.

Figure 16 shows the determined transition (breakup) temperatures as a function of the system size for different system densities for the two system A/Z ratios with comparisons to the predictions of the Statistical Multifragmentation Model (SMM), Hartree-Fock calculations, and Canonical Thermodynamic Model (CTM). The observed weak dependence on the system size and isospin is in agreement with the SMM and CTM predictions but in contrast to the Hartree-Fock model predictions that represent the limiting temperatures associated with the Coulomb

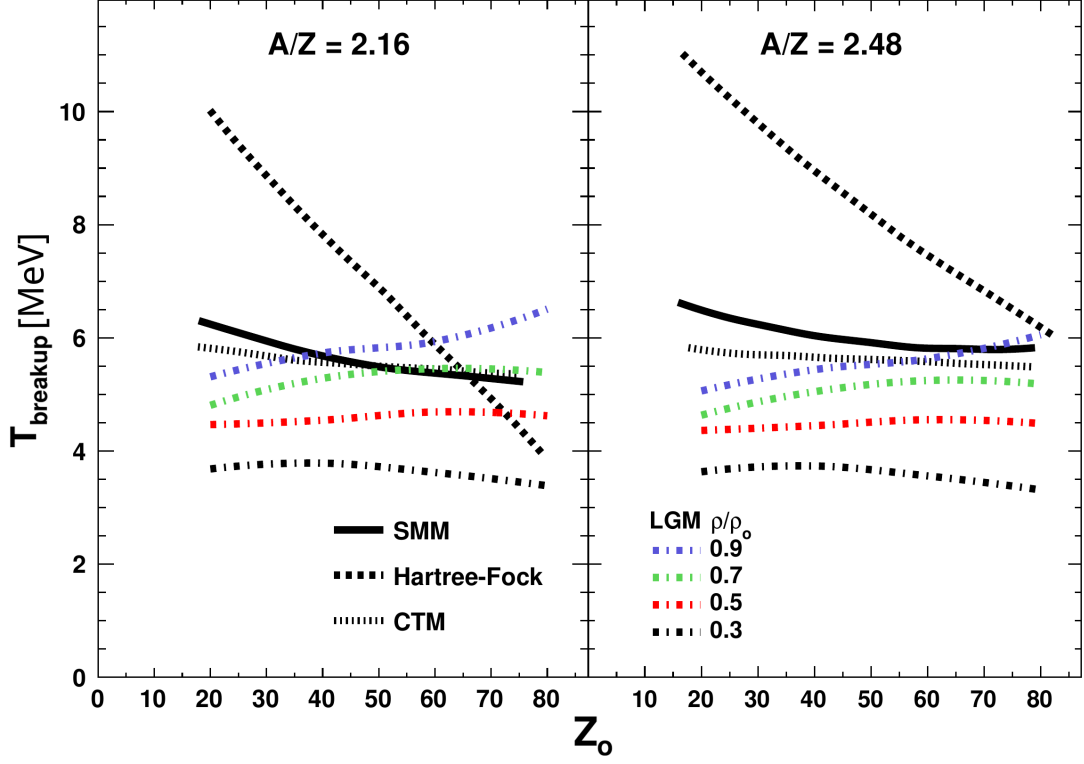


Figure 16: Breakup temperatures as a function of the system atomic number for the four indicated system densities and two different A/Z ratios. The results are compared with the SMM, Hartree-Fock and CTM model predictions showed in Ref. [9].

instabilities. The LGM results for the higher densities consistently align with the statistical models predictions, even though the freeze-out volume assumed in these models is one third of ρ_0 . The LGM breaking temperature strongly decreases with decreasing the system density, from about 6 MeV for $0.9\rho_0$ to about 3.5 MeV for $0.3\rho_0$. This results suggest that the determining the breakup temperatures can provide information on the system properties at the freeze-out stage.

5 ALADIN data on Au spectator fragmentation at relativistic energies and the Lattice Gas Model predictions

5.1 ALADIN S114 experiment

The experiment ALADIN S114 was performed at GSI in Darmstadt [10]. The experimental setup was specifically designed to investigate projectile spectator fragmentation in heavy-ion reactions at relativistic energies. It is characterized by high detection efficiency and charge identification of all fragments with $Z \geq 2$. The experimental data employed in this study comprises collisions of ^{197}Au projectiles with Cu, In, and Au targets at the incident energies of 600 (Cu, In, Au targets), 800 (Au), and 1000 AMeV (Cu, Au).

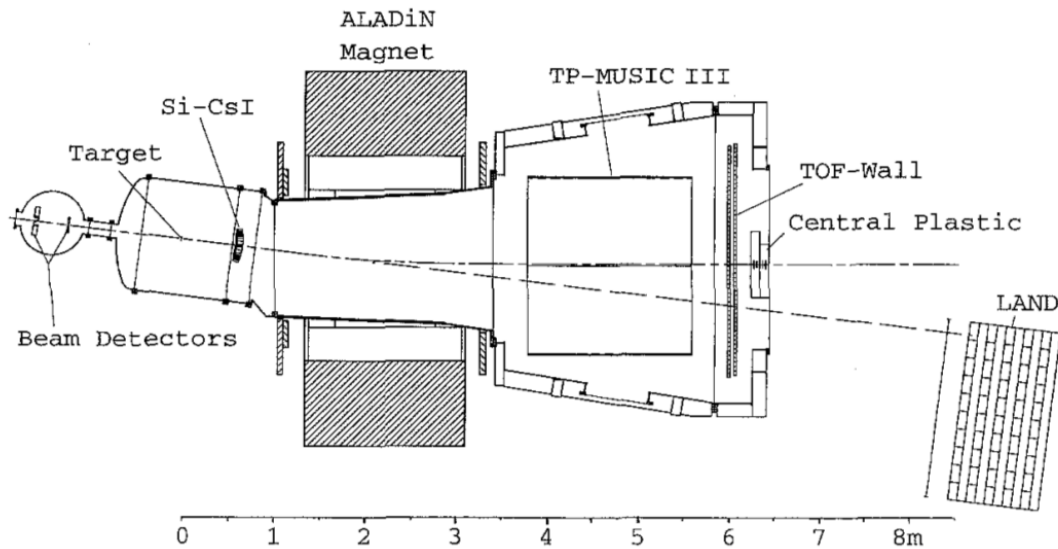


Figure 17: Schematic view of the ALADIN setup [10].

In the setup shown in Fig. 17, the beam enters from the left and interacts with the target placed in the vacuum chamber. The forward emitted reaction products, mostly from projectile fragmentation, pass through the magnetic field of the ALADIN magnet. The charged fragments are deflected while neutrons pass undisturbed to the LAND neutron detector. Projectile fragments are tracked and identified in the time-projection chamber (TP-MUSIC III) and in the time-of-flight (TOF) wall. Reconstruction of tracks in TP-MUSIC provides particle rigidities, p/Z (particle momentum divided by its charge). The time of flight and energy loss measurements

allow to determine the fragment velocity and charge (atomic number Z), and in case of light fragments also to identify isotopes. Comprehensive details regarding the experimental setup and essential characteristics of the data can be found in Refs: [10, 98–100].

The data exhibits a prominent characteristic known as the Z_{bound} universality, which is manifested in the fragment multiplicity distributions and correlations. The quantity Z_{bound} is defined as the sum of the atomic numbers Z of all projectile fragments with $Z \geq 2$. It plays the role of the control parameter being correlated with the impact parameter and the excitation energy of the fragmenting system. Observables such as the mean fragment multiplicities and the mean largest fragment charge plotted as a function of Z_{bound} remain invariant irrespective of the target and the incident energy [10]. Figure 18 illustrates this feature for the mean multiplicity of intermediate mass fragments. In the left panel no significant dependence on the bombarding energy is observed, while the right panel demonstrates the persistence of the distribution regardless of the chosen target. These invariant properties of spectator nuclei indicate that a high degree of equilibrium is reached before the fragmentation stage, which is required for the study of the thermodynamic behavior of highly excited nuclear matter. They also justify using in analysis the summed data sets (all targets and energies) to achieve better statistics.

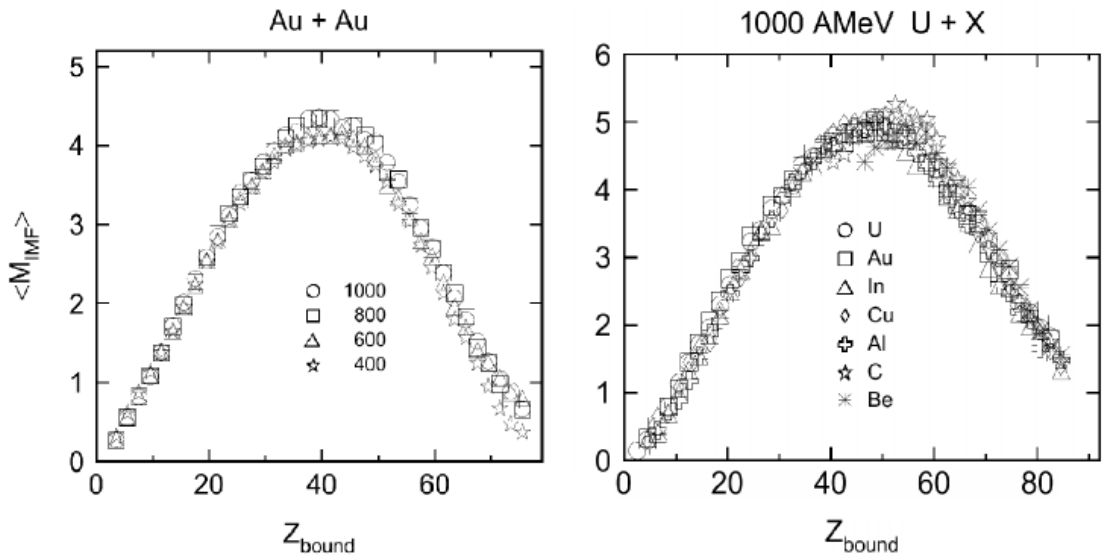


Figure 18: Mean multiplicity of intermediate mass fragments as a function of Z_{bound} for the reaction $^{197}\text{Au} + ^{197}\text{Au}$ at the incident energies of 400, 600, 800 and 1000 MeV/nucleon (left panel), and for the reactions of ^{238}U projectiles at the energy of 1000 MeV/nucleon with various indicated targets (right panel). Figures from [10].

5.2 Fragment size distributions and percolation description

Recently, the data on fragment charge partitions from the ALADIN S114 experiment were analyzed with focus on the cumulants of the largest fragment charge distributions [8]. The results were compared with predictions of a bond percolation model, in which the sites were arranged on the simple cubic lattice in the most compact configuration, and the bonds were created randomly with a given probability p using a Monte Carlo procedure. Clusters were recognized with applying free boundary conditions. The number of sites in a cluster was considered as the fragment atomic number, Z , which serves as a measure of the fragment size in the experimental data. Percolation events were generated for the bond probabilities uniformly distributed in the interval between 0 and 1, and then sorted according to Z_{bound} .

Figure 19 presents the experimental cumulant ratios K_2 , K_3 and K_4 of the largest fragment charge distribution compared with predictions of the bond percolation model. The percolation results are plotted in the left diagrams as a function of Z_{bound} normalized to the system size Z_0 . Three different system sizes are considered, covering a range expected in the transition region. In this representation, the distributions are weakly dependent on the system size with the same location of the pseudocritical points ($K_3 = 0$ and minimum of K_4) at $Z_{\text{bound}}/Z_0 \approx 0.84$ as indicated by the squares and the vertical dotted line. The results corresponding to the percolation true critical point (filled circles) are located near the maximum of K_2 but their positions are somewhat dependent on the system size.

The right diagrams display the experimental results for the $^{197}\text{Au} + ^{197}\text{Au}$ systems at 600 and 1000 AMeV, and for the summed data sets (all targets and energies). The cumulants are plotted here as a function of Z_{bound} . Overall, the cumulant patterns in percolation and the experimental data exhibit remarkable similarities. The signature of the percolation pseudocritical point is observed in the data at $Z_{\text{bound}} \approx 54$. This correspondence allows to estimate the mean system size at $Z_{\text{bound}} = 54$ as $Z_0 \approx Z_{\text{bound}}/0.84 \approx 64$. With the correspondence relying on K_3 and K_4 equalities, the true point can be approximately localized at $Z_{\text{bound}} = 36$ with a system size $Z_0 \approx 36/0.69 \approx 52$.

Figure 20 compares the experimental fragment charge distributions with percolation calculations at the identified critical and pseudocritical points. The distributions are plotted in log-log scale, in which a power-law dependence is seen as a straight line. The model well reproduces the data over four orders of magnitude. At $Z_{\text{bound}} = 54$, for fragments of size much smaller than the system size, the distri-

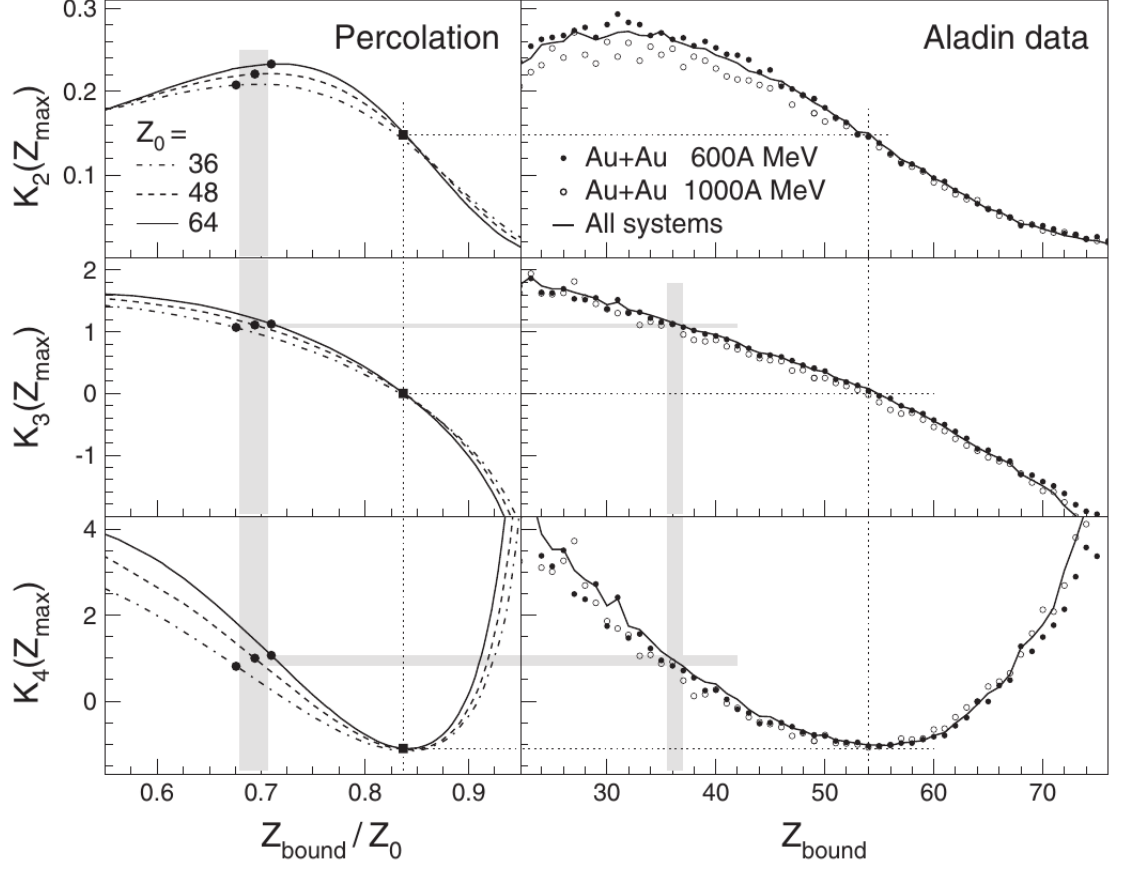


Figure 19: Cumulant ratios of the largest fragment size (charge) probability distribution $P(Z_{\max})$ plotted as a function of Z_{bound} . The left panels display the results of bond percolation calculations for the three indicated system sizes Z_0 . In these plots Z_{bound} is normalized to Z_0 . The right panels show the ALADIN experimental data for ^{197}Au fragmentation resulting from collisions with ^{197}Au targets at two energies (symbols), along with the results for the summed data sets (solid lines). See text for the meaning of lines and symbols in the left panel. Figure from [8].

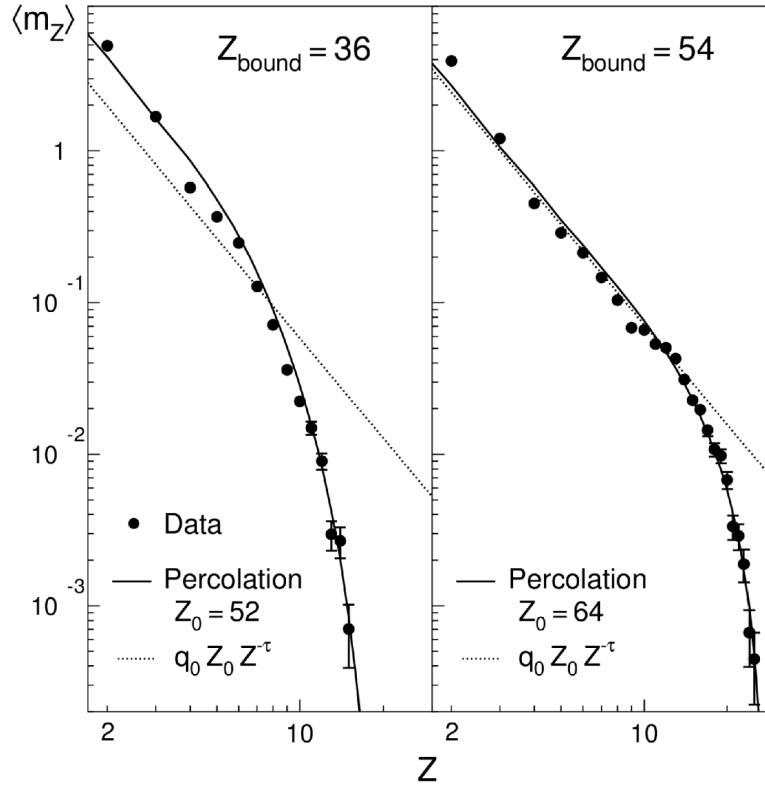


Figure 20: The mean fragment multiplicity as a function of the fragment charge Z (excluding the largest fragment in each event) at $Z_{\text{bound}} = 36$ and 54 . Comparison of the experimental data for all systems (filled circles) with percolation calculations performed for the estimated system sizes $Z_0 = 52$ and 64 , respectively (solid lines). The asymptotic power-law dependencies are shown by the dotted straight lines. The statistical error bars are shown for errors larger than the symbol size. Figure from [17].

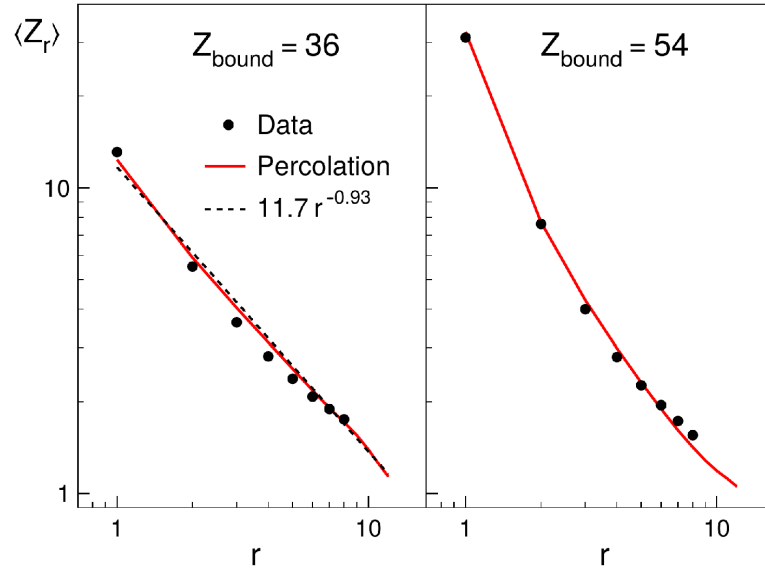


Figure 21: The mean fragment charge as a function of the fragment rank r . The dashed line in the left diagram shows a power-law fit to the percolation results. Figure from [17].

butions follow the asymptotic power-law dependence with the exponent $\tau = 2.189$ and the normalization constant $q_0 = 0.173$, as expected for the percolation pseudocritical point [62]. Figure 21 shows the Zipf-type plots, i.e. the mean size (charge) of the largest, second largest ... r -largest fragments as a function of their rank r . As can be seen, the Zipf's power-law appears at the “critical” point while the power-law behavior in the fragment size distribution is observed at the pseudocritical point. This difference in locations of the two criticality signatures is a consequence of finite-size effects.

More detailed confrontation of the data with percolation predictions is presented in Fig. 22. Various fragment charge characteristics and correlations are plotted as a function of Z_{max} for four Z_{bound} selections. As it was concluded in Ref. [8], the bond percolation model, containing a second-order phase transition, remarkably well describes the whole fragmentation pattern observed in the experiment as represented by the fragment sizes and their event-to-event fluctuations. Therefore, the percolation results may serve as a reference for further analysis with the Lattice Gas Model and for sensitive tests of its performance.

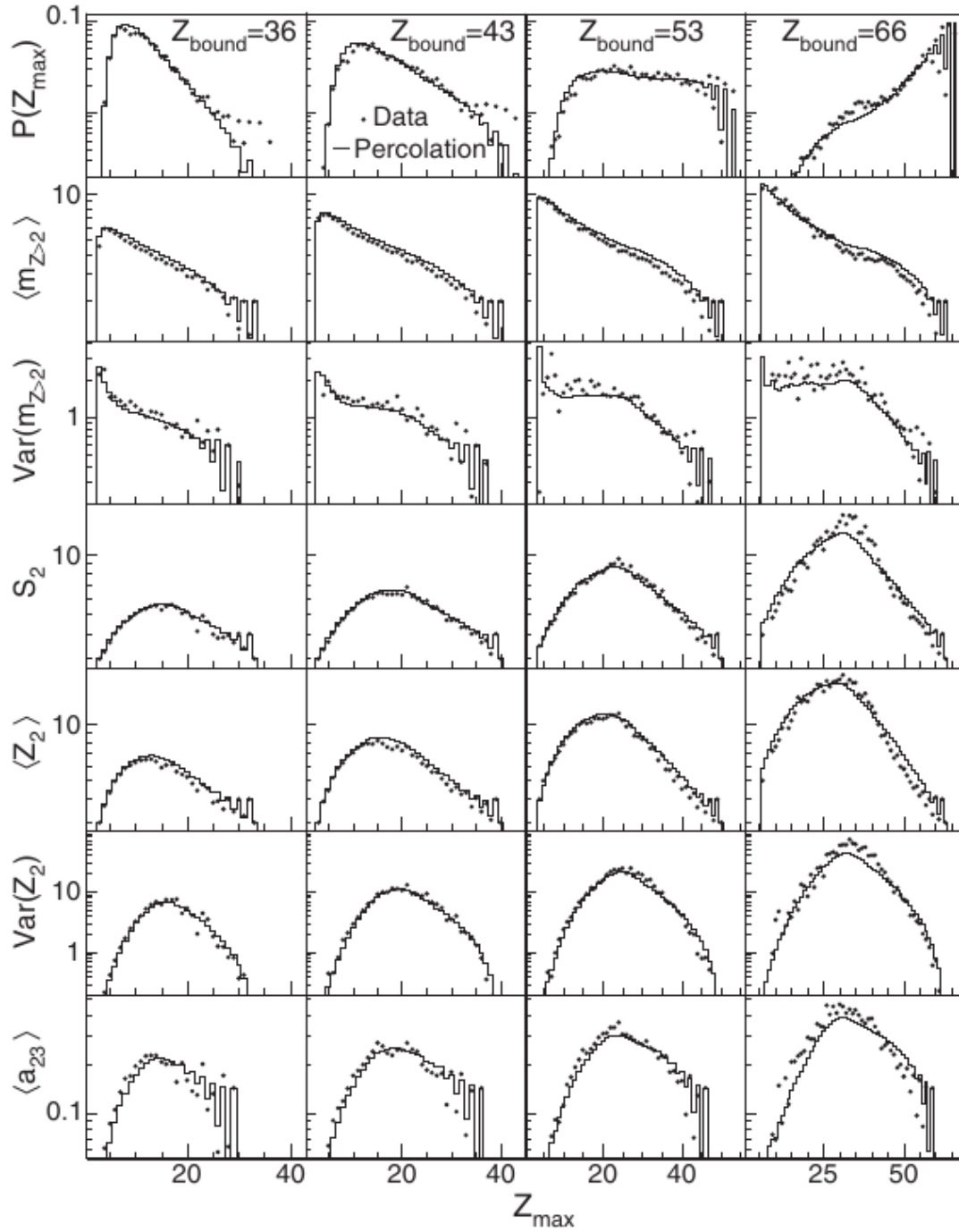


Figure 22: Percolation (solid histograms) vs the experimental data (dots; all systems). Characteristics of fragments with $Z > 2$ are plotted as a function of $Z_{\max}$ for four different Z_{bound} selections. From top to bottom: the $Z_{\max}$ probability distribution, the mean and variance of the fragment multiplicity, the mean fragment size, the mean and variance of the second largest fragment charge, the mean size asymmetry between the second and the third largest fragments. The statistical errors are of the order of the scatter of data points. Figure from [8].

5.3 Determination of the systems sizes in the Lattice Gas Model simulations

To ensure a proper comparison between the Lattice Gas Model results and experimental data, it is essential to establish appropriate system sizes in simulations. As it was shown in the previous section, the system size Z_0 can be determined from comparisons between the experimental and simulated cumulants K_3 and K_4 of the largest fragment charge distribution. Maximal experimental system size can be 79 (^{197}Au projectile's charge). Proton to neutron ratio is assumed to be the same as for the projectile, thus the model calculations will be performed for system's $A/Z = 2.49$. In particular, to determine Z_0 for the experimental pseudocritical point located at $Z_{\text{bound}} = 54$ (see Fig. 19) the following procedure is performed.

Let us choose some exemplary system sizes: around the experimental maximum, which is 80, and 10% and 20% smaller ones, i. e. 69 and 60. The density for this example is set to be a half of the normal nuclear density. In Fig. 23 the cumulants K_3 and K_4 of the largest fragment distribution are plotted against Z_{bound} normalized to the system charge Z_0 - left panel of the figure. In such a representation the critical behavior signal locations are nearly independent of the system size. It can be estimated as $Z_{\text{bound}}/Z_0 \approx 0.65$. Since the experimental critical Z_{bound} is about 54 (see Fig. 19), we can deduce that $Z_0 \approx 83$.

As mentioned, the dependence of K_3 and K_4 on the normalized Z_{bound} for the three system sizes shows small differences despite substantial disparities in system sizes. When plotted as a function of Z_{bound} as in the right panel of the Fig. 23, the curves are widely spaced, allowing for fine-tuning of the Z_0 value. The previous step yielded $Z_0 \approx 83$. By examining the data plotted as a function of Z_{bound} , we can verify if $K_3 = 0$ and $\min(K_4)$ occur at the desired critical Z_{bound} . If not, a small adjustment in the Z_0 value should be satisfactory. In this case, after “fine-tuning”, Z_0 equals 80 or 81.

Above procedure was applied to determine system size at the pseudocritical point, which was relied on the finding $K_3 = 0$ coinciding with $\min(K_4)$. In order to estimate system sizes for other Z_{bound} values we can search for other K_3 and K_4 equalities in the Z_{bound} domain. Such a procedure was described in Ref. [8]. In Fig. 24 cumulants K_3 and K_4 of three different system sizes (Z_0): 77, 81, and 85 are plotted. The case with $Z_0 = 81$ approximately coincide with the experimental data at discussed earlier pseudocritical point. For the case with $Z_0 = 77$ (blue curve) and $Z_0 = 85$ (green curve), the crossing of the simulated and experimental lines, determines the corresponding Z_{bound} . In this cases, the crossings occurs around $Z_{\text{bound}} = 44$ for $Z_0 = 77$ and $Z_{\text{bound}} = 67$ for the $Z_0 = 85$.

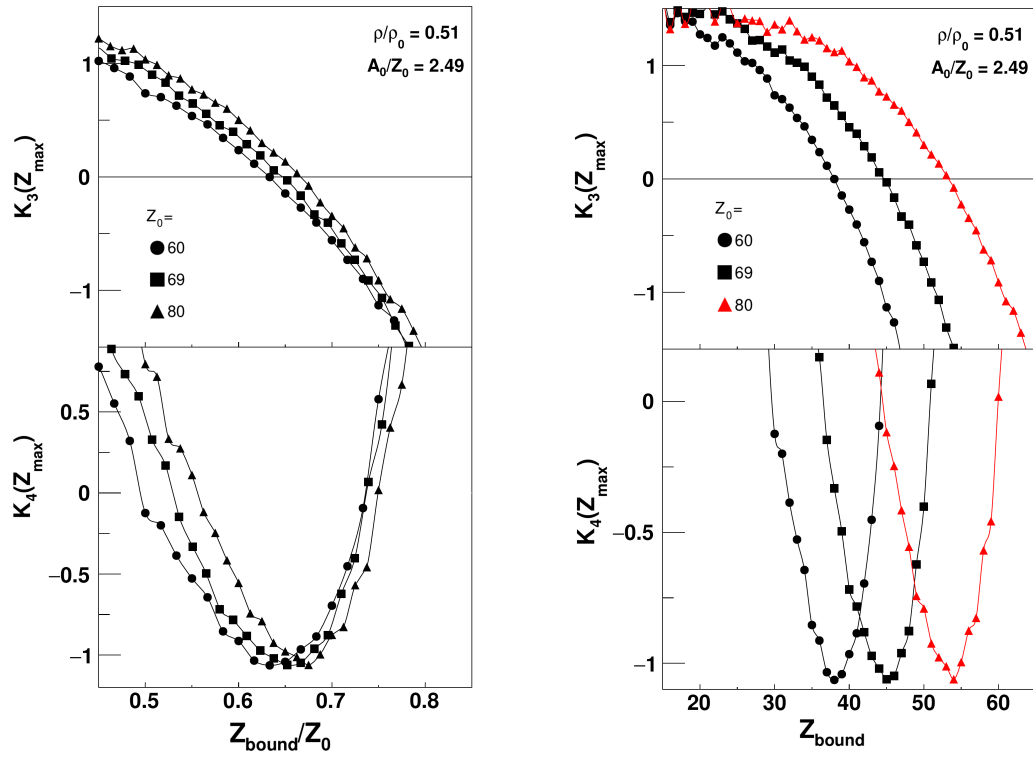


Figure 23: Simulated cumulant ratios K_3 (skewness) and K_4 (kurtosis) of the largest fragment charge distribution as a function of Z_{bound} for three different system sizes Z_0 . In the left plots Z_{bound} is normalized to Z_0 .

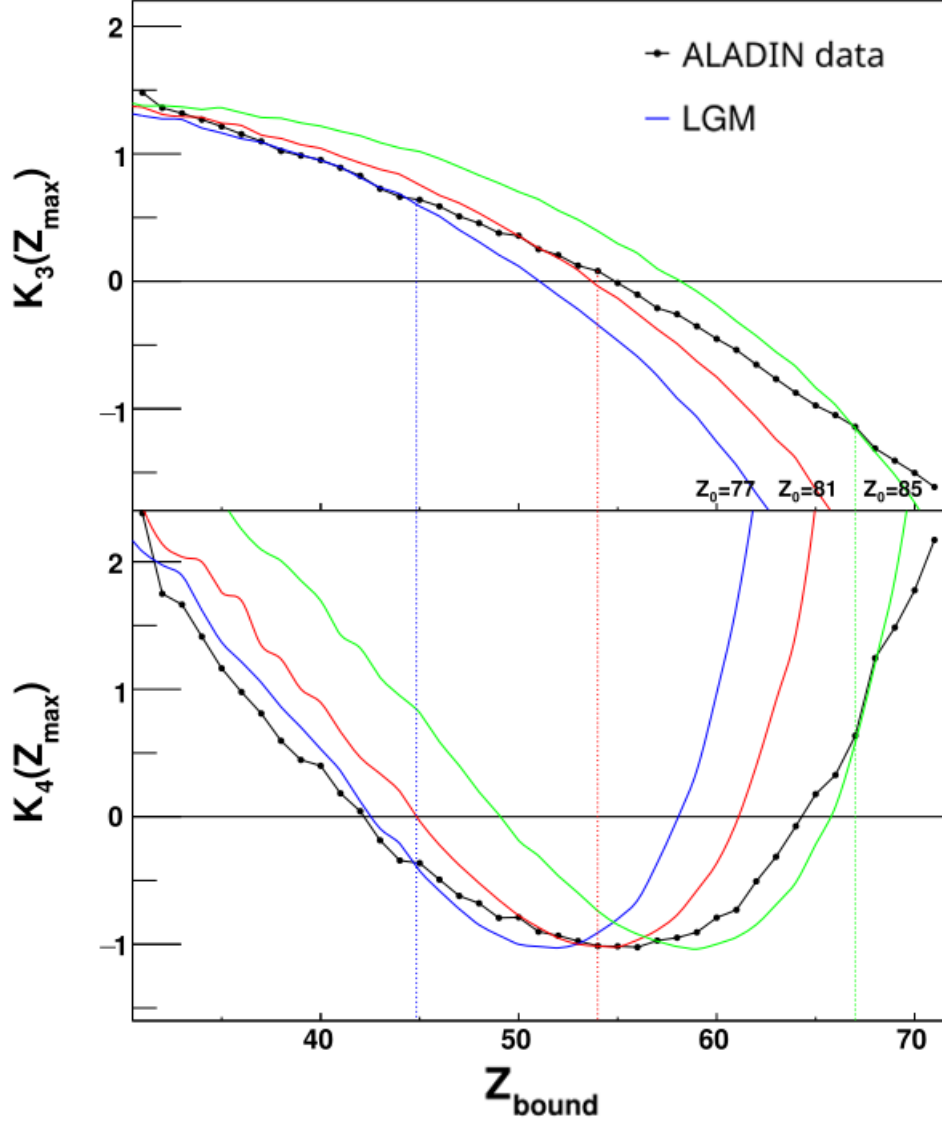


Figure 24: Skewness and kurtosis of the $Z_{\max}$ distribution against Z_{bound} . The black dots connected with black line represent the experimental data. The blue, red and green lines show LGM results for system sizes $Z_0 = 77, 81, 85$, respectively, with a system density of $\approx 0.5\rho_0$. The vertical dashed lines indicate the crossings of model and data lines.

5.4 Comparisons of the experimental data with the Lattice Gas Model predictions

5.4.1 Cumulant analysis of the largest fragment charge distributions

The following figures depict the skewness (K_3) and kurtosis (K_4) for both the experimental data (ALADIN S114 [10]) and a series of simulated data obtained for four densities $\approx 0.3, 0.5, 0.7, 0.9$ of the normal nuclear density ρ_0 . System sizes used for simulations were match to three Z_{bound} cases: 54, 36, 66, as the system size changes with the control parameter, which is Z_{bound} . The system A/Z ratio of 2.49 is assumed as for the projectile (Au). The Figs 25, 26 and 27 show the K_3 and K_4 of the Z_{max} distributions plotted as a function of Z_{bound} . They are intended to determined system sizes for $Z_{\text{bound}} = 54, 36$ and 66.

Simulations were performed using the two previously described clusterization methods, E_{REL} and C-K. The left panels shows results for E_{REL} , while the right panels show results the C-K clustering. The experimental data are depicted by the black full circles, while the colored symbols indicate the simulation results for various densities. Figure 25 shows cumulant analysis for simulated data with system sizes corresponding to the $Z_{\text{bound}} = 54$.

For simulations with system sizes corresponding to other than 54 Z_{bound} values, results are shown in Fig. 26 and 27. These plots serve as confirmation of the appropriate selection of system sizes Z_0 based on the experimental data. Both clusterization methods provide comparable accuracy in the data description, however the sizes obtained with the E_{REL} method are found to be systematically larger than those obtained with the C-K method.

It is noteworthy that in the case of low density $0.3\rho_0$ and simultaneously large $Z_{\text{bound}} = 66$, it is necessary to apply a data cut to remove the contributions from the fission. Due to the influence of long-range Coulomb interactions, the model produces a portion of fission-like channels that involve two substantial fragments [101]. However, these fission-like occurrences are generally excluded from the investigation of fragment size distributions in studies concerning phase transitions (criticality). This exclusion is essential to prevent their influence from compromising the accuracy and integrity of the derived outcomes. Figure 28 illustrates the contribution of the fission events in the model (top) and in the experimental data (bottom).

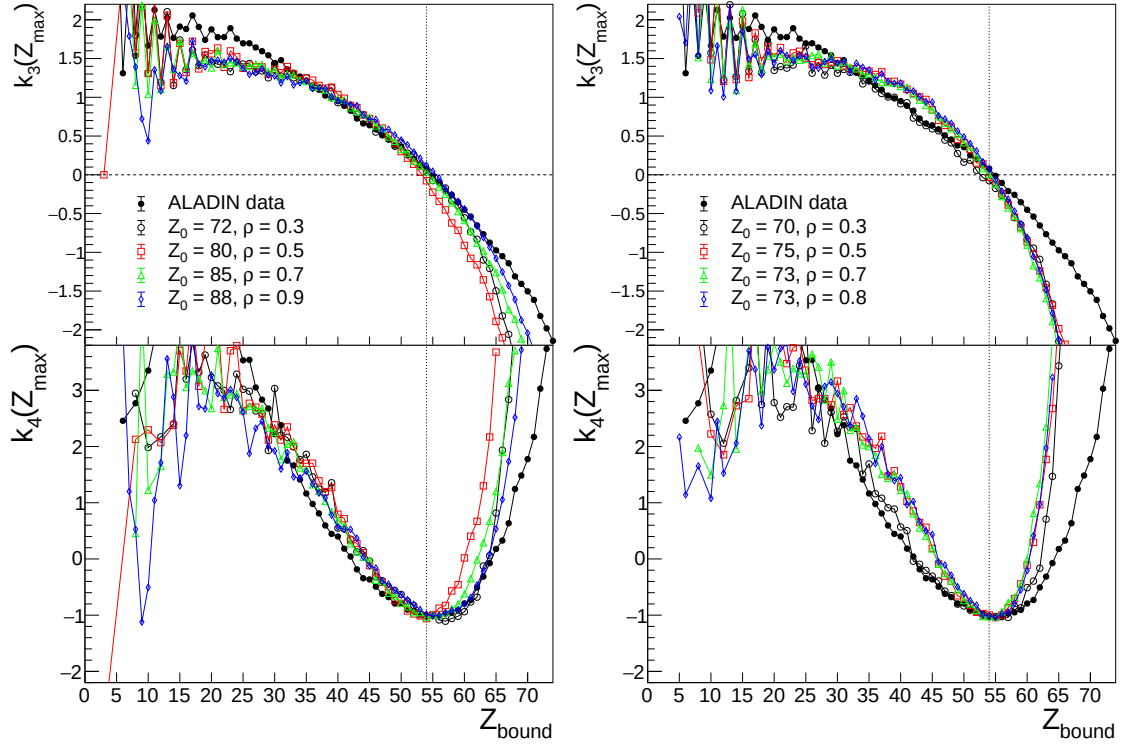


Figure 25: Cumulants K_3 and K_4 against Z_{bound} . The experimental data are shown by the black full circles. The LGM simulation results for indicated system sizes and densities are shown by the open symbols. The left (right) panels show calculations using the E_{REL} (C-K) clusterization method. The system sizes have been determined to match the experimental data at $Z_{\text{bound}} = 54$ indicated by the vertical dotted line.

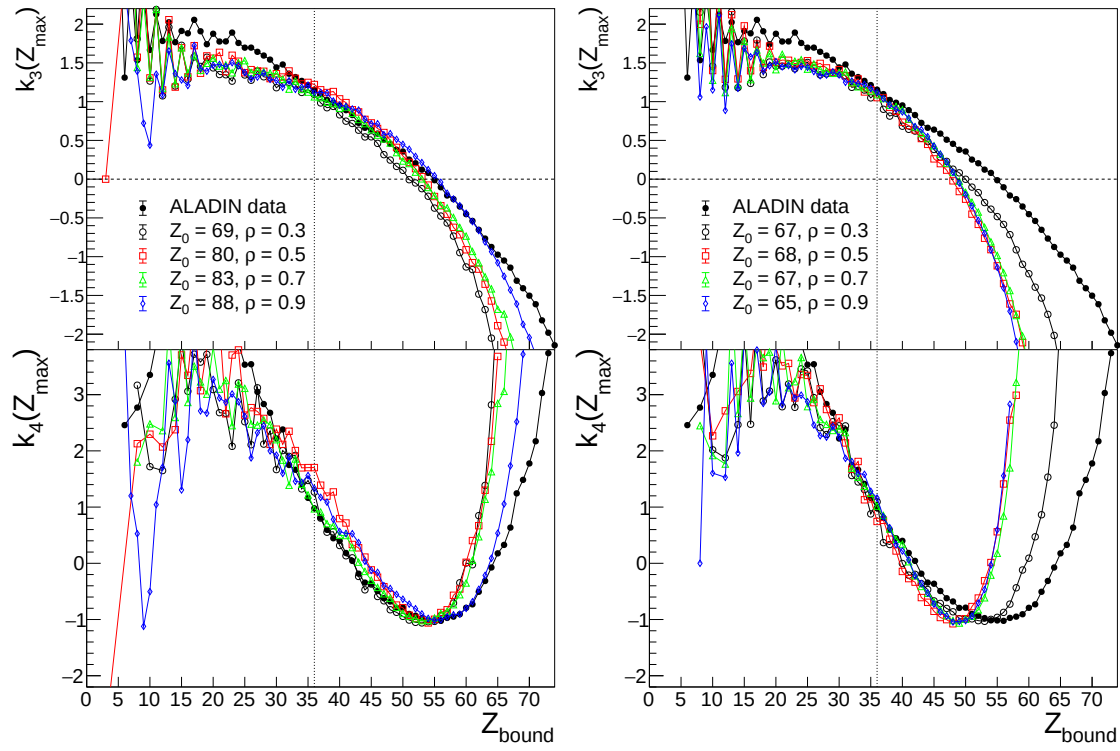


Figure 26: The same as in Fig. 25, but for the matching at $Z_{\text{bound}} = 36$.

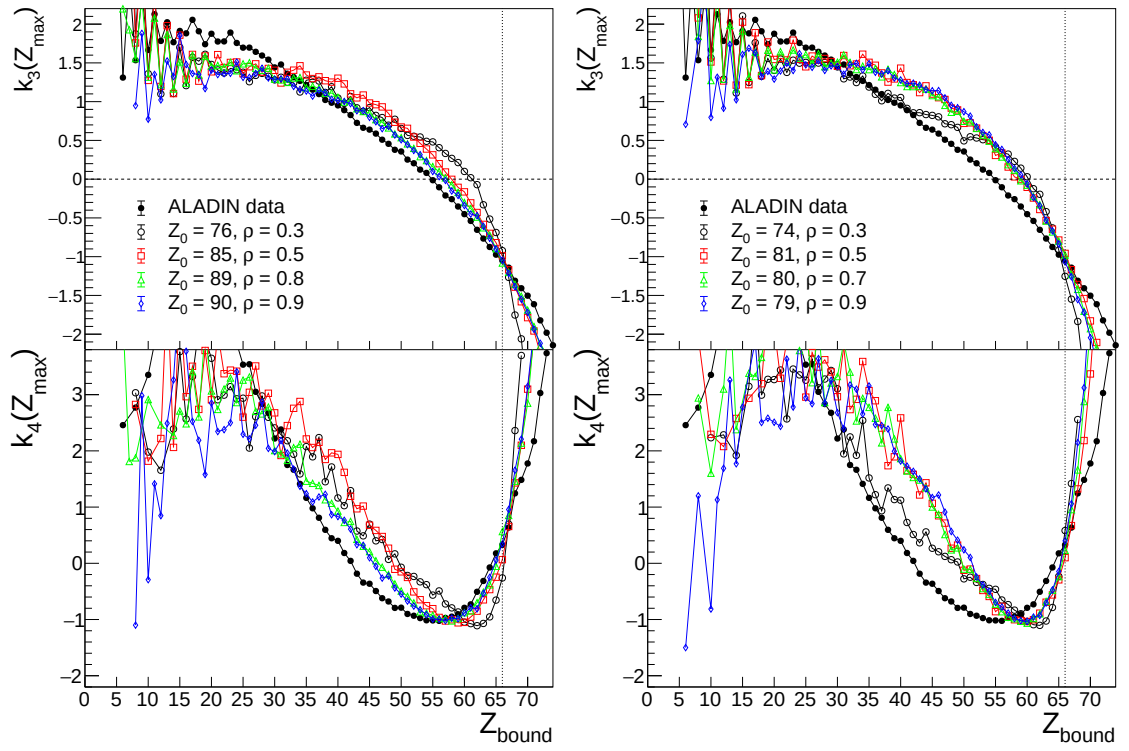


Figure 27: The same as in Fig. 25, but for the matching at $Z_{\text{bound}} = 66$.

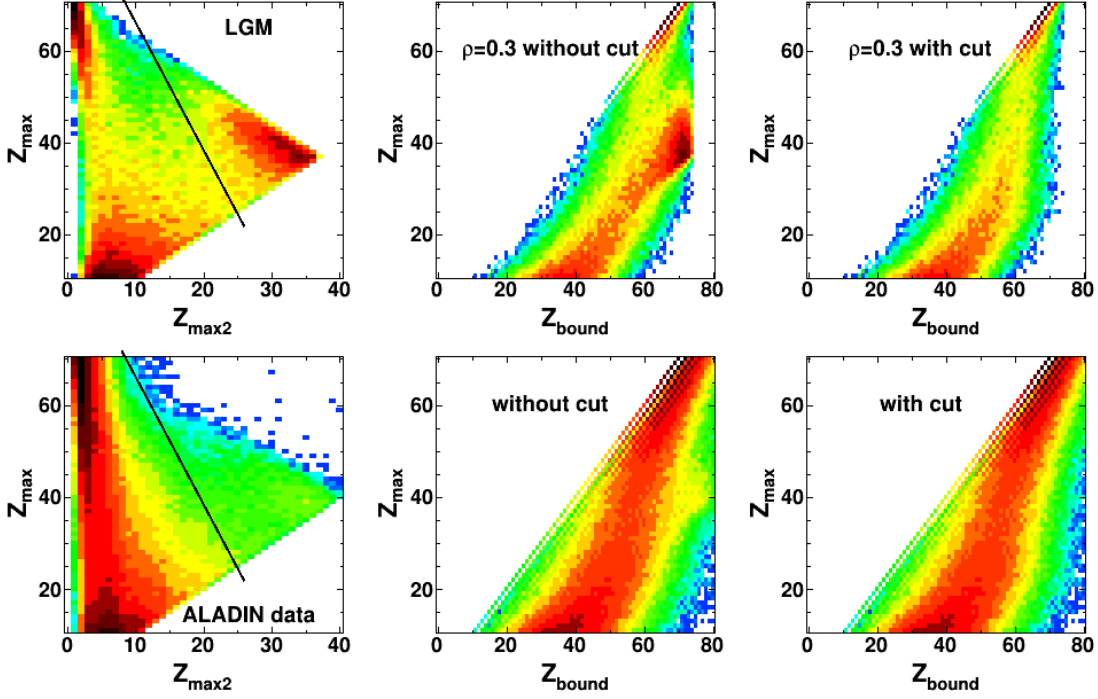


Figure 28: Comparison of LGM simulations performed for a low density of (upper panels) with the ALADIN data (lower panels) revealing contributions from fission events. $Z_{\max}$ vs $Z_{\max2}$ distributions showing the contribution of the fission events located on the right side of the black line: LGM - upper left panel, ALADIN data - lower left panel. Plots in the middle - distributions before cut, plots on the right - distributions after cut (see text). Simulations performed for density $\rho/\rho_0 = 0.3$ and $Z_0 = 76$. The probabilities are indicated by a color scale going from blue (minimum) to red (maximum).

Left top panel of Fig. 28 shows a contribution of the fission events to the partitions generated by LGM for the source at density $0.3\rho_0$ and at $Z_{\text{bound}} = 66$. The fission contribution is best visible in the largest, $Z_{\max}$ vs second largest, $Z_{\max2}$, representation, where it is located around the $Z_{\max} \simeq Z_{\max2} \simeq Z_{\text{projectile}}/2$. In the experimental data the contribution of fission is less pronounced, but still visible in the bottom left panel of the Figure. Lines (identical) in the left panels define the graphical cut used to reduce the unwanted contribution of fission in the model and experimental ensembles. The middle and right panels of the Figure show the contribution of fission in the $Z_{\max}$ vs Z_{bound} representation before and after applying the cut condition from the left panels, respectively. As can be seen the cut is quite effective in removing the fission events. Without removing the fission events it was not possible to construct reasonable K_3 and K_4 analysis.

Further investigations of the size of the disassembling system can be conducted using the acquired datasets. Figure 29 shows cases the system size Z_0 corresponding to specific Z_{bound} values, alongside a comparison to the experimental estimation obtained by the ALADIN collaboration [98]. Both clusterization methods are depicted, revealing a divergence as the density of the simulated system increases.

Alternatively, the system size Z_0 can be presented as a function of a system density. Figure 30 exhibits the system size Z_0 for $Z_{\text{bound}} = 54$. The red band represents the experimental estimates for the size of the disassembling system Z_0 for this Z_{bound} value. Both clusterization methods are depicted. It is evident that the E_{REL} method predicts a system size larger than the size of the projectile Z_{proj} (except for the lowest density of $0.3\rho_0$). The CK clusterization method also overestimates the system size, but the predicted values remain within physical limits (sizes lower than the size of the projectile).

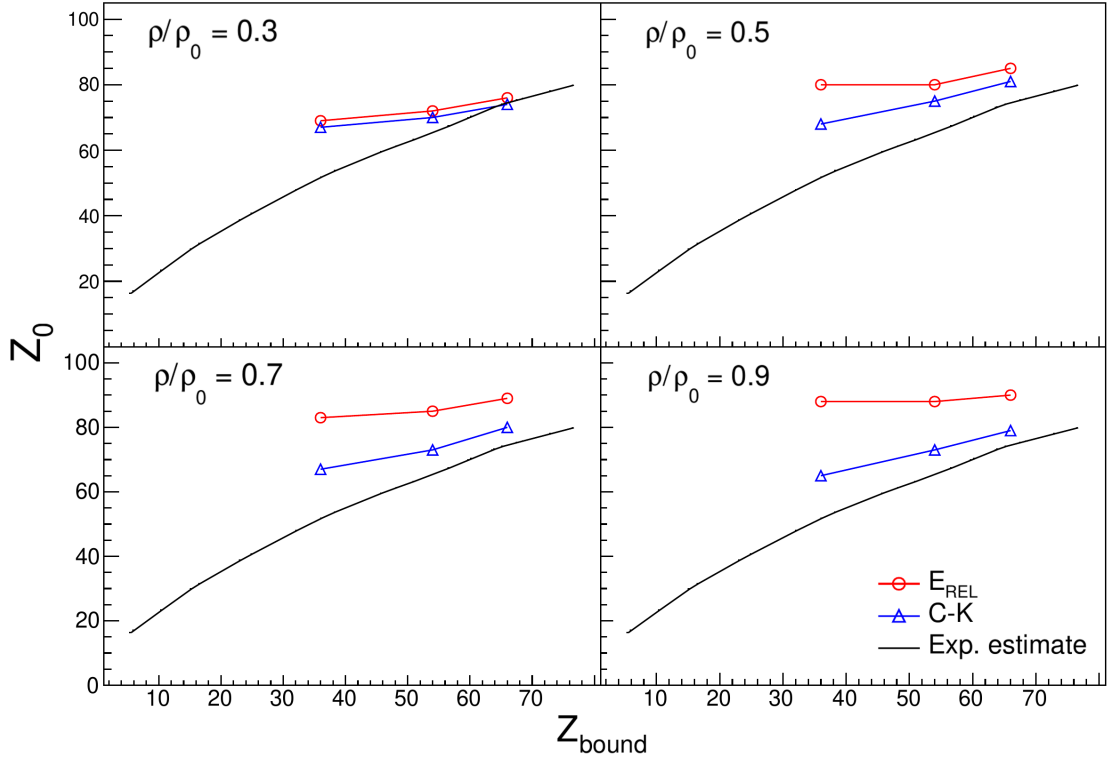


Figure 29: Estimated system size Z_0 vs Z_{bound} . The red and blue lines present the LGM results obtained with the two clusterization methods. The black solid line shows the experimental estimate [98]. The panels are for different system densities as indicated in each panel.

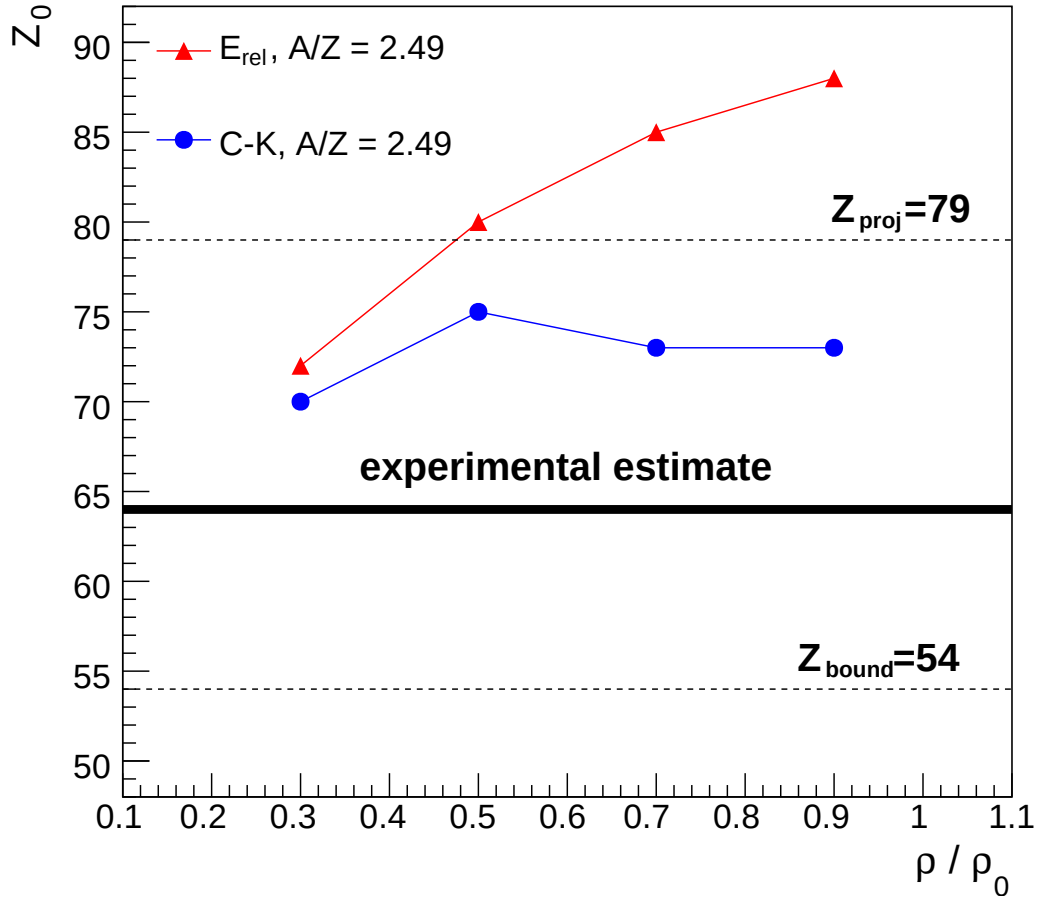


Figure 30: LGM estimations of the system size vs the system density at $Z_{\text{bound}} = 54$ obtained with the two clusterization methods. The experimental estimate and the projectile (Au) charge (size) Z_{proj} are indicated.

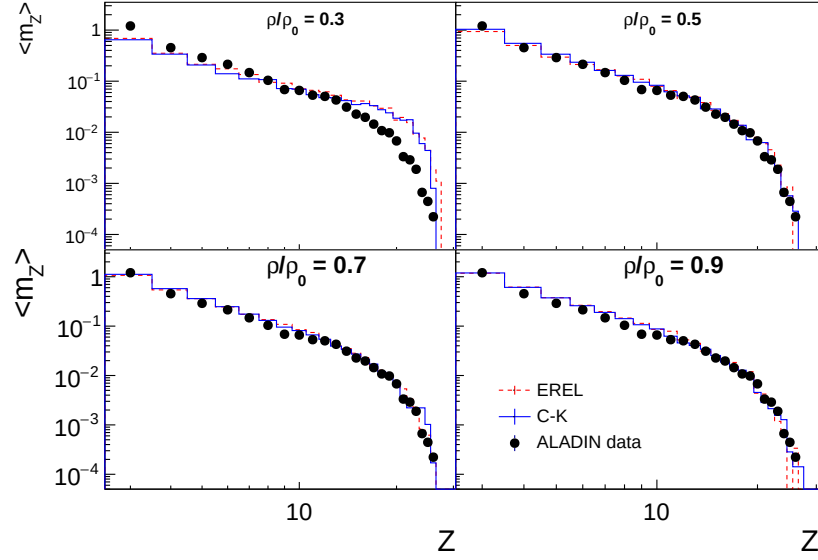


Figure 31: Mean fragment multiplicity as a function of the fragment size Z (the largest fragment excluded) for events with $Z_{\text{bound}} = 54$. The model predictions (histograms) are compared with the ALADIN data (filled circles). The panels are for different system densities as indicated in each panel.

5.4.2 Examination of the charge partitions

The following plots present four characteristics of charge partitions:

1. The mean fragment multiplicity as a function of the fragment size Z (the largest fragment excluded)
2. The mean fragment size as a function of the fragment rank r
3. The probability distribution of the second largest fragment size Z_2
4. The multiplicity distribution of fragments with $Z > 2$, $m_{Z>2}$

These plots depict simulated data obtained using the two clusterization methods, E_{rel} and C-K, along with experimental data. The four panels represent the results for $\rho/\rho_0 = 0.3, 0.5, 0.7$ and 0.9 . The plots are organized into three sets corresponding to three Z_{bound} values: below, at and above the pseudocritical point.

The following set of the Figs (31 - 34) refers to $Z_{\text{bound}} = 54$, which corresponds to the pseudocritical point.

Figure 31 depicts the mean fragment multiplicity vs fragment charge. Both clusterization methods, E_{rel} and C-K, exhibit a consistent agreement with each other as well as with the experimental data, except for the density $\rho = 0.3\rho_0$. For this particular density, the mean multiplicity of small fragments is underestimated,

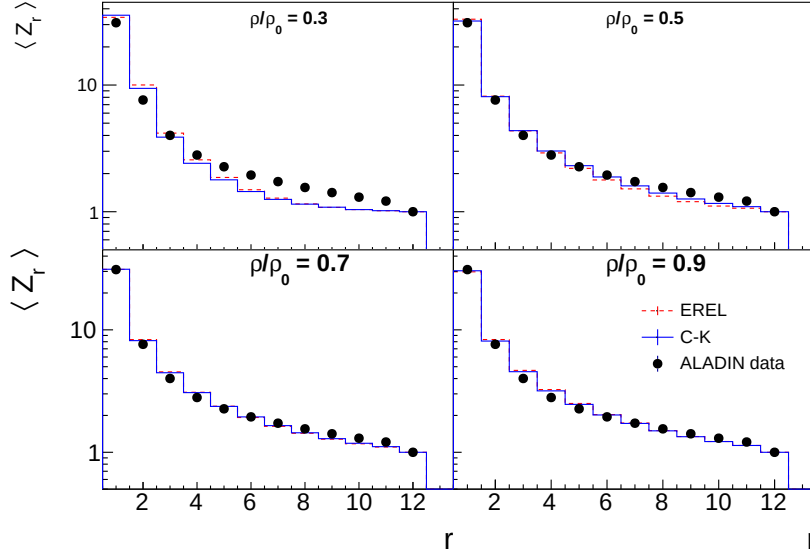


Figure 32: Mean fragment size as a function of the fragment rank r for events with $Z_{\text{bound}} = 54$.

while for large fragments, it is overestimated.

Figure 32 shows the mean fragment charge as a function of the fragment rank, i.e. the mean charge of the largest ($r = 1$), second largest ($r = 2$), third largest ($r = 3$), and so on, fragments. The simulation results exhibit nearly perfect agreement for both clusterization methods and are in very good alignment with the experimental results, except for the case of the lowest ($0.3\rho_0$) density.

Figure 33 presents the probability distribution of the second largest fragment size, Z_2 . The agreement between the simulation and experimental data remains quite good, except the lowest density.

Figure 34 illustrates the multiplicity distribution of fragments with $Z > 2$, $m_{Z>2}$. Results exhibit a very good agreement with the experimental data, however, for the lowest density case ($0.3\rho_0$), discrepancies are observed the most. What is more, the E_{REL} clusterization method underestimate higher multiplicities.

The subsequent Figs (35 - 42) present the same distributions for the $Z_{\text{bound}}=36$ and 66.

The comparisons show that the Coniglio-Klein clusterization method demonstrates slightly better agreement with experimental data compared to E_{REL} . The advantage of the C-K method is its ability to better reproduce the system sizes. These

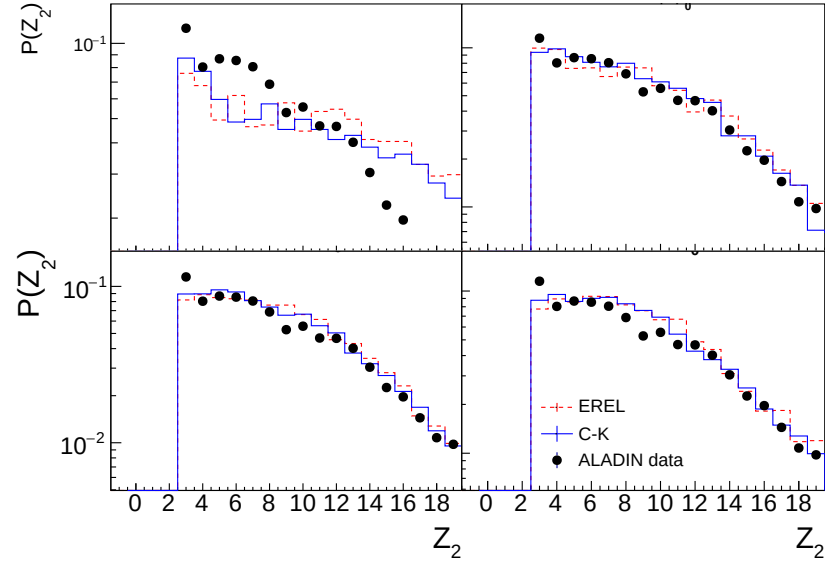


Figure 33: Probability distribution of the second largest fragment size Z_2 for events with $Z_{\text{bound}} = 54$.

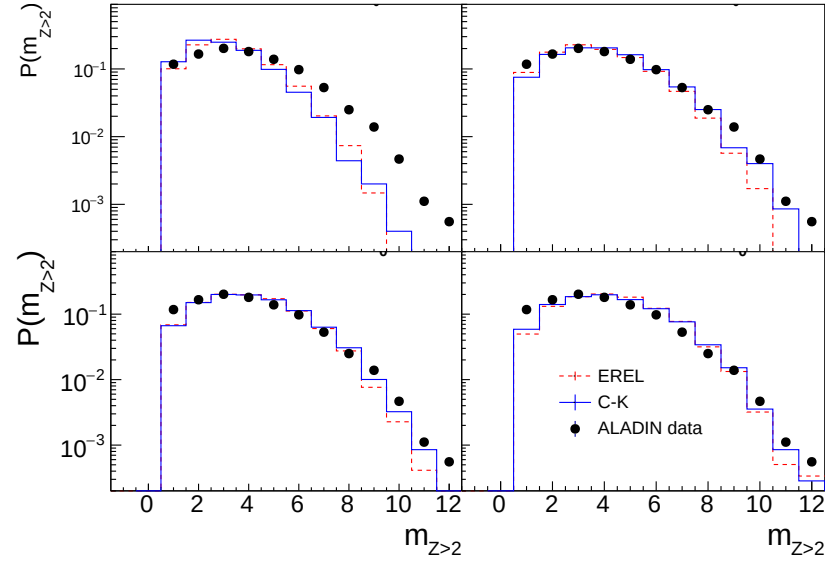


Figure 34: Multiplicity distribution of fragments with $Z > 2$ for events with $Z_{\text{bound}} = 54$.

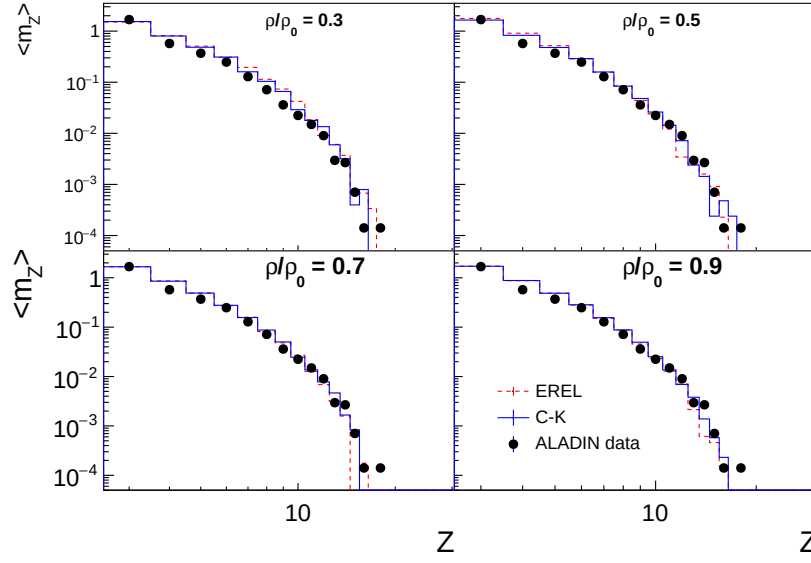


Figure 35: Mean fragment multiplicity as a function of the fragment size Z (the largest fragment excluded) for events with $Z_{\text{bound}} = 36$.

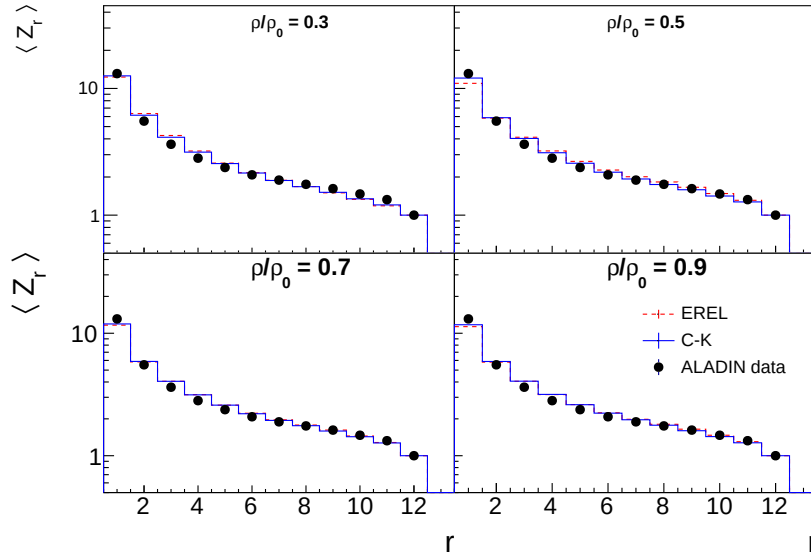


Figure 36: Mean fragment size as a function of the fragment rank r for events with $Z_{\text{bound}} = 36$.

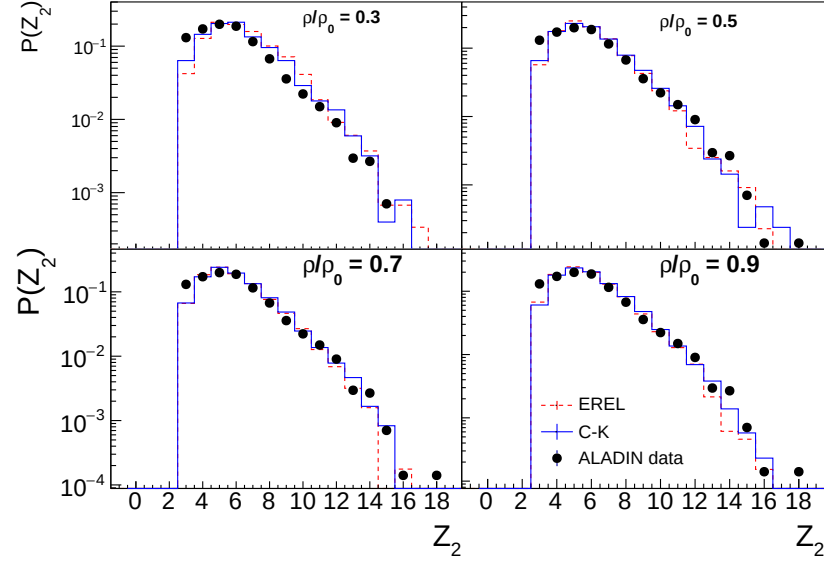


Figure 37: Probability distribution of the second largest fragment size Z_2 for events with $Z_{\text{bound}} = 36$.

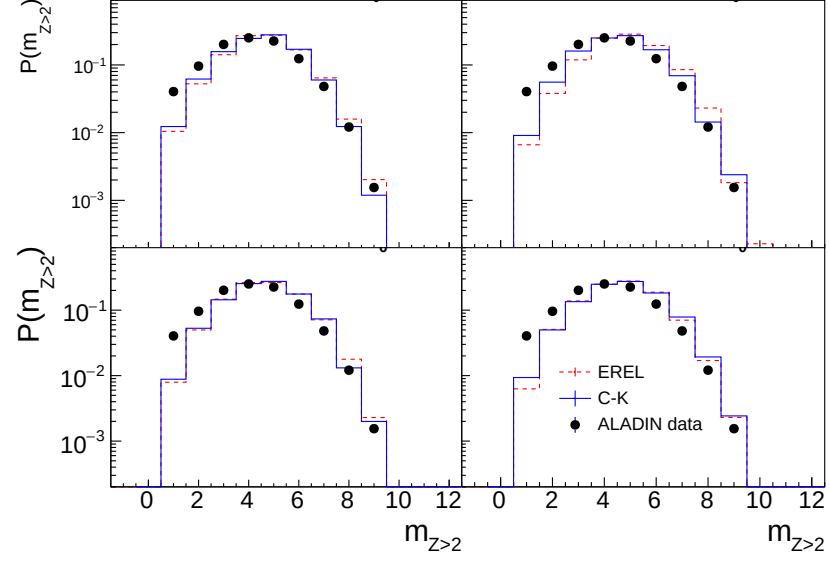


Figure 38: Multiplicity distribution of fragments with $Z > 2$ for events with $Z_{\text{bound}} = 36$.

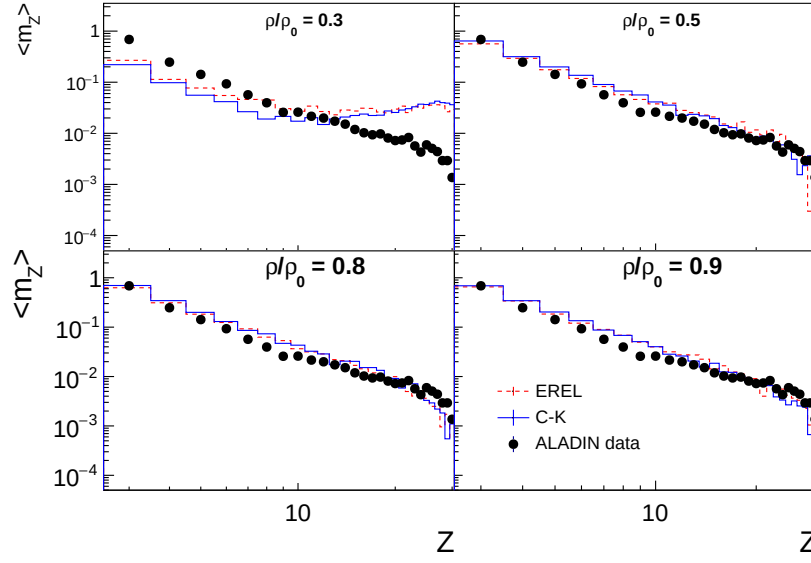


Figure 39: Mean fragment multiplicity as a function of the fragment size Z (the largest fragment excluded) for events with $Z_{\text{bound}} = 66$.

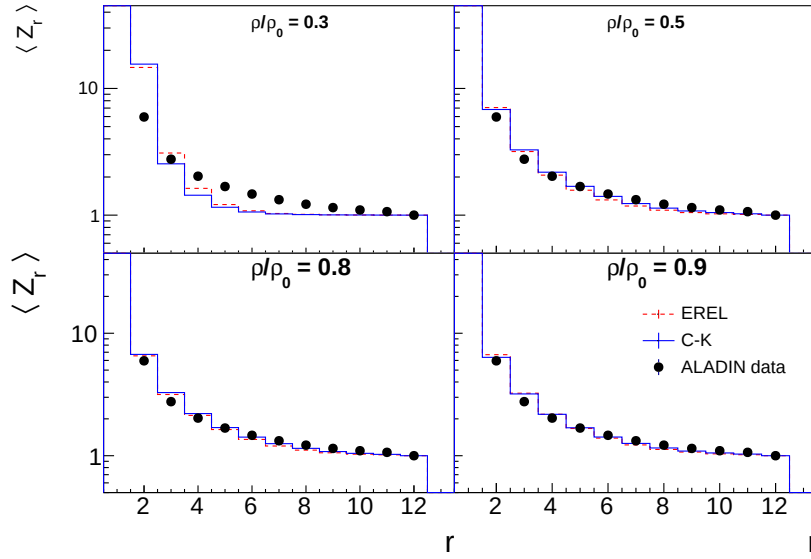


Figure 40: Mean fragment size as a function of the fragment rank r for events with $Z_{\text{bound}} = 66$.

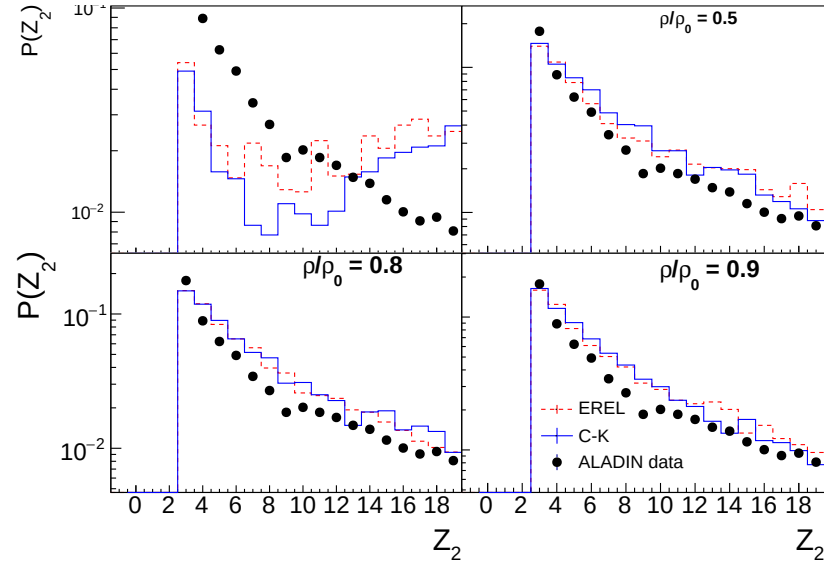


Figure 41: Probability distribution of the second largest fragment size Z_2 for events with $Z_{\text{bound}} = 66$.

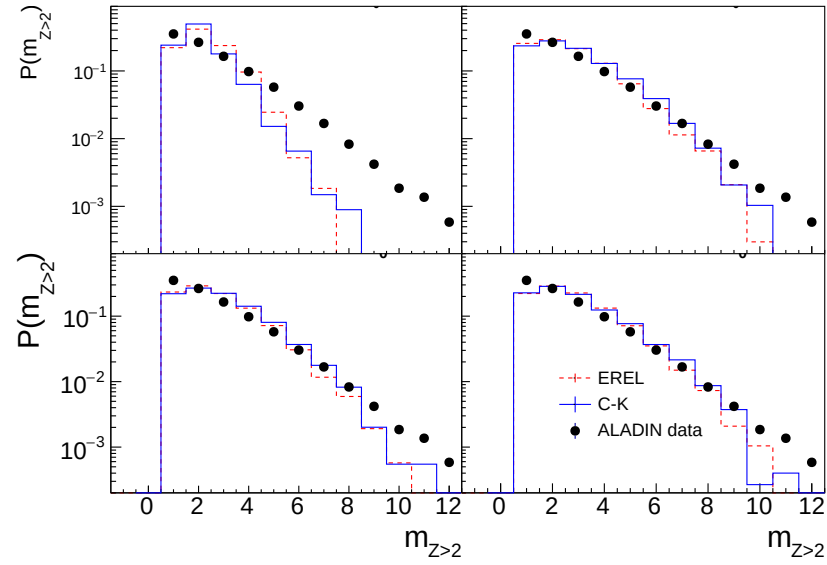


Figure 42: Multiplicity distribution of fragments with $Z > 2$ for events with $Z_{\text{bound}} = 66$.

arguments leads to prioritize the C-K clusterization method, and as a result, all further investigations will be exclusively conducted using this approach.

Figures 43 and 44 show the same characteristics of fragments as presented in Sec.5.2 in Fig. 22. There, the bond percolation model was compared to the experimental data. The following figures presents similar comparison but for the LGM results. The top diagrams shows the $P(Z_{\max})$ distributions. The next panels examine: the mean and variance of the fragment multiplicity, the mean fragment size S_2 defined as the second moment of the fragment size distribution normalized to the first moment (the largest fragment excluded), mean and variance of the second largest fragment charge, mean charge asymmetry between the second and the third largest fragments. All characteristics in the Figure, except the top panel are for fragments with $Z > 2$. The characteristics focus on the largest fragments domain because this allows for a better insights into the fragmentation patterns, which is explained in detail in Ref. [8]. For clarity the investigated densities are split into two separate figures: Fig. 43 refers to lower densities ($\rho = 0.3$ and 0.5) while the Fig. 44 illustrates higher densities ($\rho = 0.7$ and 0.9). In general, a good agreement between the experimental data and simulation results is evident across different Z_{bound} values. Overall agreement is similar to that of percolation model [8], however, for lower densities LGM results noticeably differ from the others, and from experimental data.

In the Fig. 45 other quantities characterizing fragments charge distributions are presented, which were used in multifragmentation studies. They maxima were considered as a signature of a critical behavior [40, 73]. Those observables are: second moment of the fragment charge with $Z_{\max}$ excluded (M_2), mean value of the normalized width of the fragment-charge distribution γ_2 , mean charge asymmetry between the largest and the second largest fragments and mean asymmetry between three largest fragments a_3 .

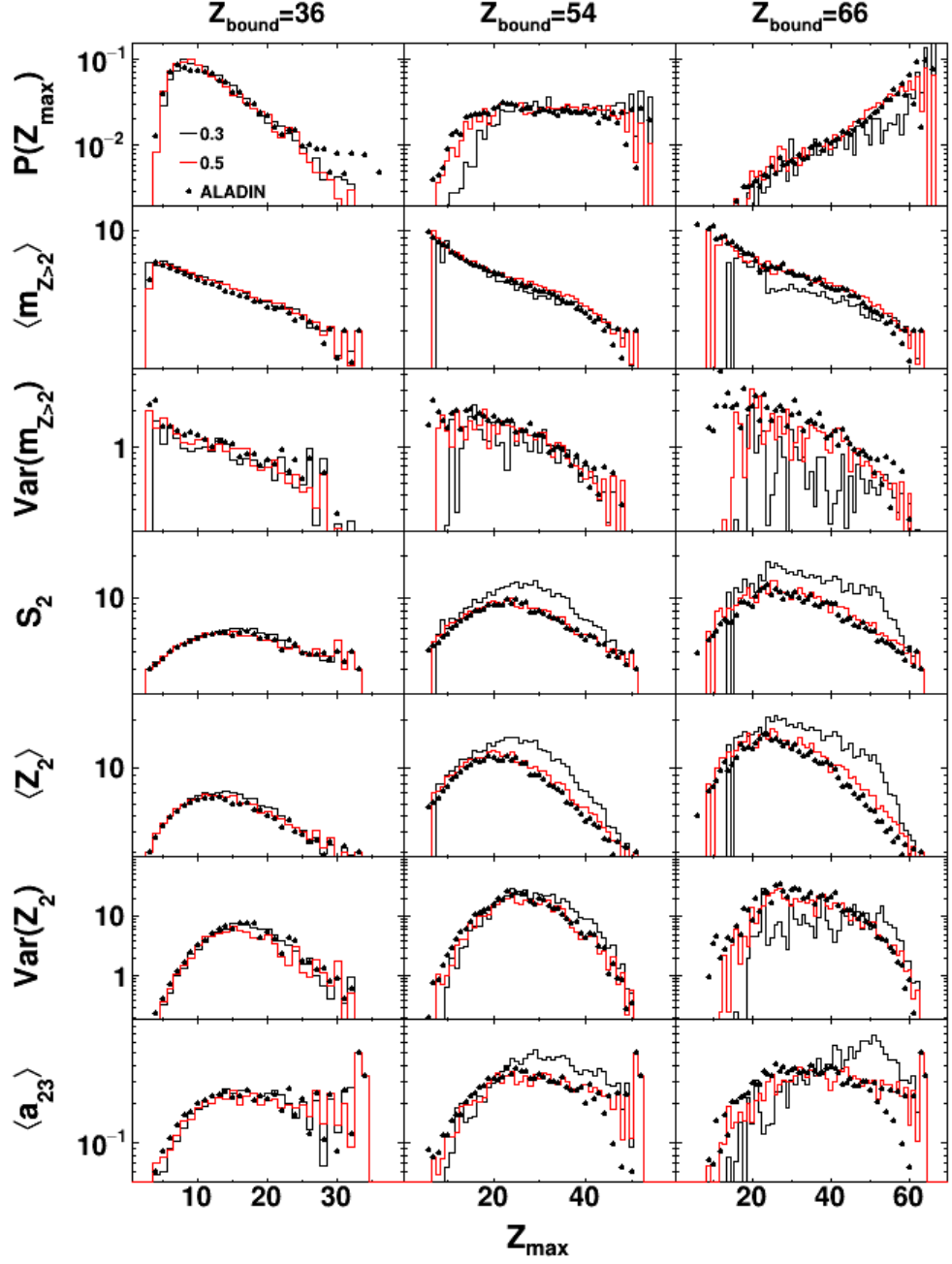


Figure 43: LGM results for $\rho/\rho_o = 0.3$ and 0.5 , (black and red histograms) vs experimental data (filled circles). Various characteristics of fragments with $Z > 2$ are plotted as a function of Z_{max} for $Z_{\text{bound}} = 36, 54$ and 66 . From top to bottom: the Z_{max} distribution, the mean and variance of the fragment multiplicity, the mean fragment size, the mean and variance of the second largest fragment charge, the mean charge asymmetry between the second and the third largest fragments. The statistical errors are of the order of the scatter of data points.

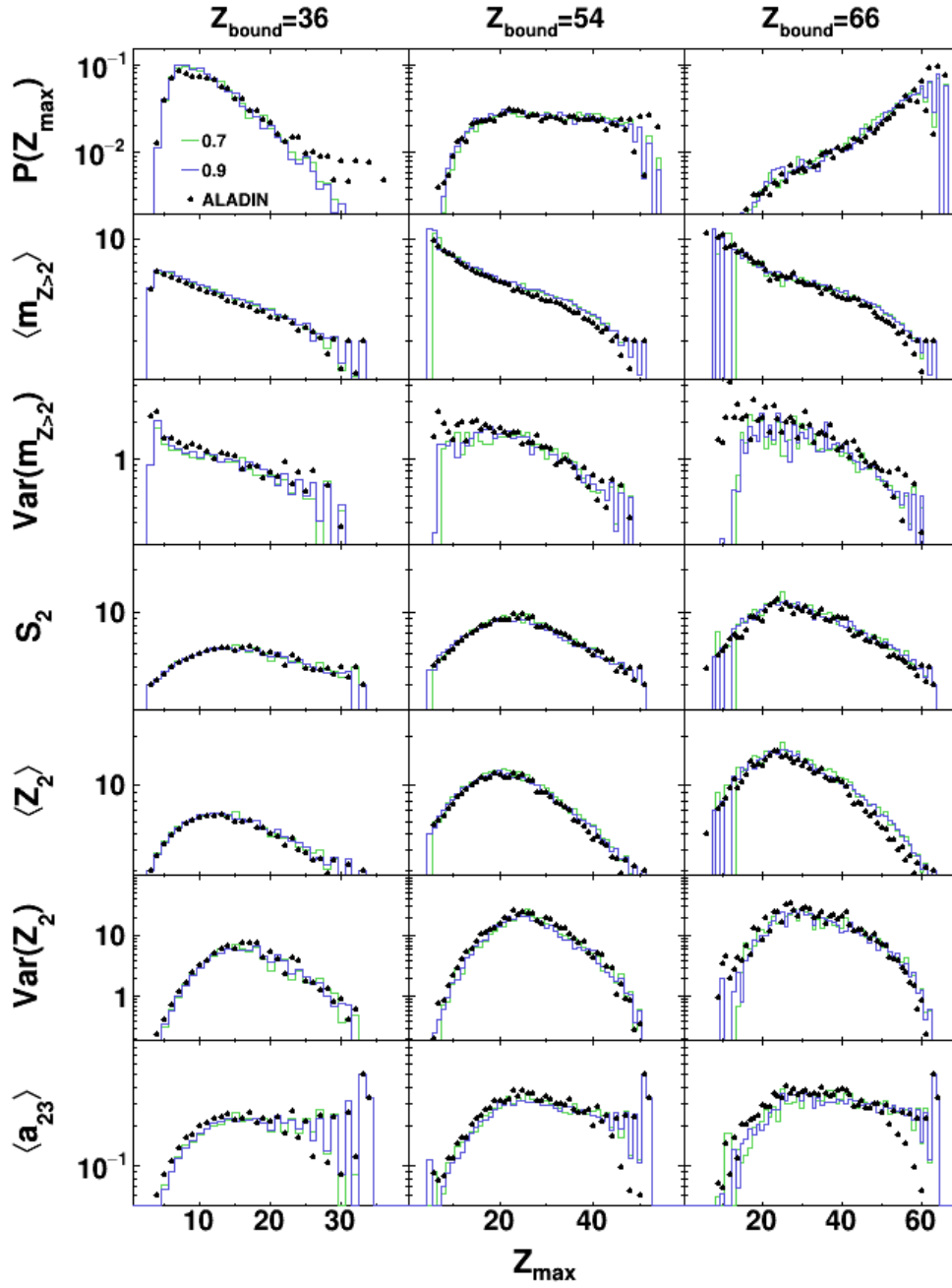


Figure 44: Same as Fig. 43 but for LGM system densities $\rho/\rho_o = 0.7$ and 0.9 (green and blue histograms).

The k 'th moment of the fragment charge distribution is defined by:

$$M_k = \sum_{Z \neq Z_{max}} Z^k n_Z \quad (6)$$

where Z is the fragment charge and n_Z is its multiplicity in an event.

The mean value of the normalized width of the fragment charge distribution γ_2 is defined as in eq. (7).

$$\gamma_2 = \frac{M_2 M_0}{M_1^2} \quad (7)$$

The definitions of charge asymmetries can be found in [10]. The charge asymmetry between the two largest fragments a_{12} , the charge asymmetry between the second and third largest fragment a_{23} , and the three-fragment charge asymmetry a_3 are defined as follows:

$$a_{12} = \frac{Z_{max} - Z_2}{Z_{max} + Z_2} \quad (8)$$

$$a_3 = \frac{\sqrt{(Z_{max} - \langle Z \rangle)^2 + (Z_2 - \langle Z \rangle)^2 + (Z_3 - \langle Z \rangle)^2}}{\sqrt{6} \langle Z \rangle} \quad (9)$$

where:

$$\langle Z \rangle = \frac{1}{3}(Z_{max} + Z_2 + Z_3) \quad (10)$$

denotes the mean value of the three largest atomic numbers in an event.

The parameter a_3 takes values close to one when the charge partition is dominated by a single large fragment, while it approaches zero when the three fragments are of equal size. For the calculation of a_{12} , only events with at least two fragments having $Z \geq 2$ were considered, whereas for $\langle a_{23} \rangle$ and $\langle a_3 \rangle$, only events with at least three fragments having $Z \geq 2$ were selected.

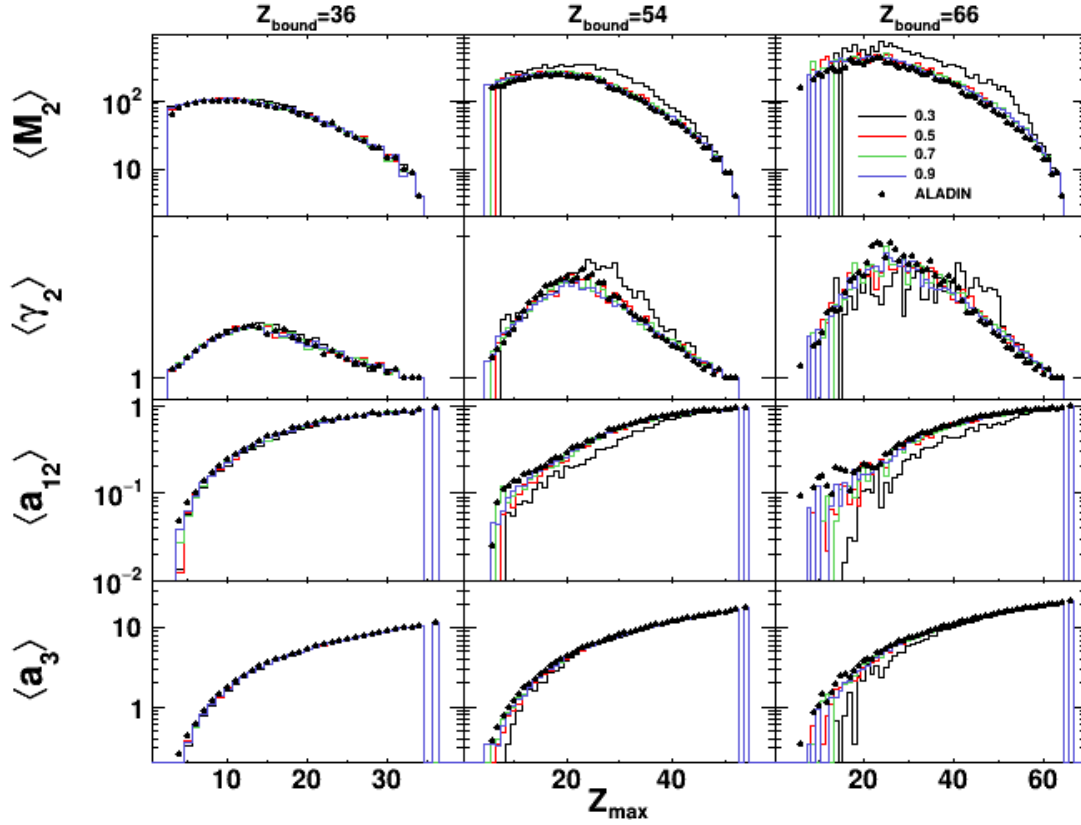


Figure 45: Experimental data vs LGM predictions for the four indicated system densities. From top to bottom: mean of the distribution of the second moment of the fragment charge with Z_{max} excluded, the mean normalized width of the fragment charge distribution, the mean charge asymmetry between the largest and second largest fragments, and the mean charge asymmetry between the three largest fragments. See text for definitions of the quantities.

This extended comparison confirm previous conclusions that the predictions for the largest densities are in good agreement and the lowest density shows discrepancies for largest Z_{bound} values.

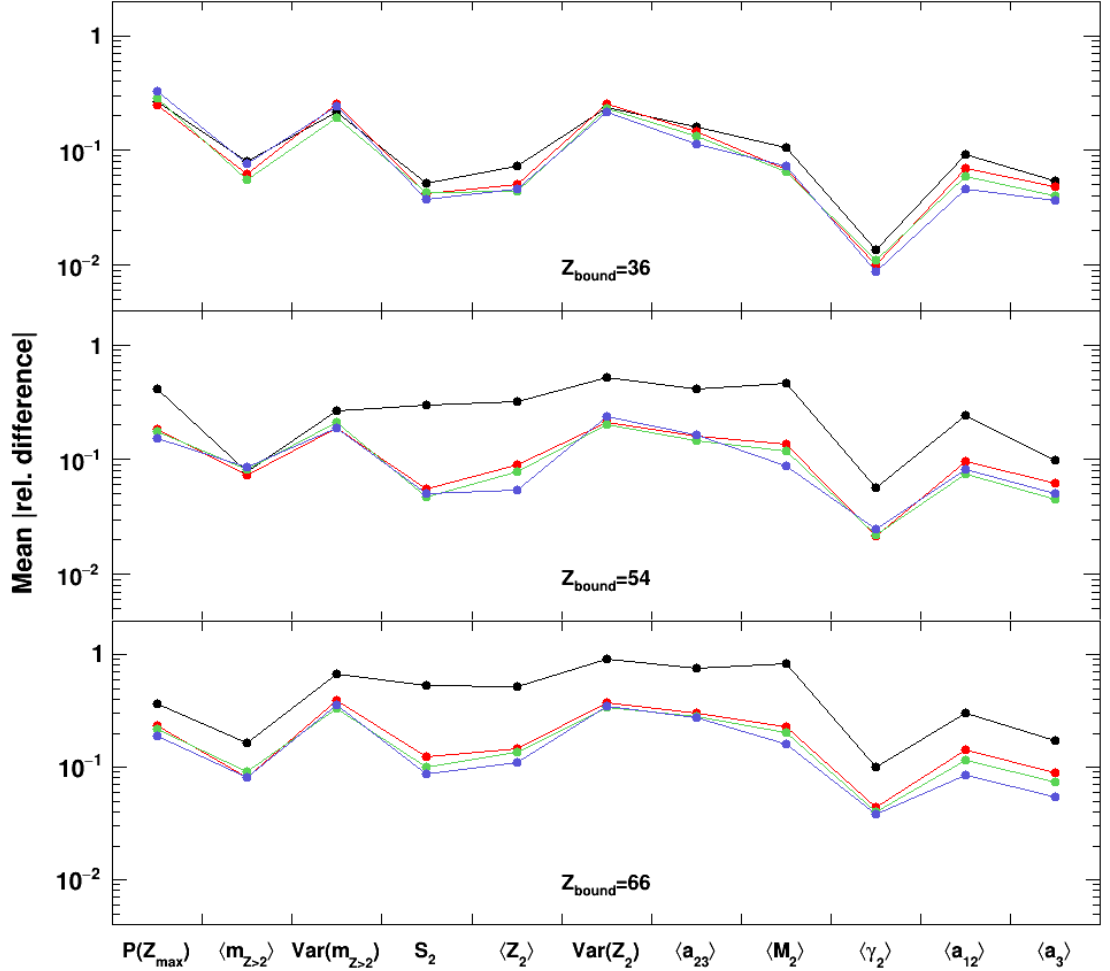


Figure 46: Mean values of the relative differences between experimental and model results obtained for observables studied in Figs. 43, 44 and 45. Color coding is the same as in those figures.

Figure 46 depicts the average relative differences between the experimental data and the model results obtained from Figs 43, 44 and 45. The color scheme employed is consistent with previous figures. Notably, the highest density case ($\rho = 0.9$, represented by blue markers) consistently exhibits the smallest deviations between the simulated and experimental data while the low density results ($\rho = 0.3$) show

a clearly worse agreement with the experimental data. The disagreement increases with the specific Z_{bound} value, i.e. for more peripheral collisions. The largest discrepancy is for $Z_{\text{bound}} = 66$ for $Var(Z_2)$ and $\rho = 0.3\rho_0$, which amounts to 90%. It can be also observed that the dependence on density is quite weak for these observables, particularly for densities over 0.3. Additionally, this dependence is even less pronounced for Z_{bound} below the pseudocritical value.

Basing on the comparisons of various characteristics of the simulated partitions with the experimental ones, one can conclude that the LGM results suggest that it is rather unlikely that fragmentation in peripheral collisions occurs at densities lower than $\sim 0.5\rho_0$. This is important observation, since other statistical models, such as SMM, MMMC, CTM, etc [9, 11], assume and require rather low breakup densities (large freeze out volumes) in order to describe the experimental results.

5.4.3 Breakup temperatures

The distribution of the cumulant ratios K_3 and K_4 as it was shown, can indicate a critical-like behavior or a phase transition. It is of particular interest to estimate the temperature associated with this transition, which will be called the breakup temperature. Figure 47 shows the temperature distribution for this transition point at $Z_{\text{bound}} = 54$ for different investigated system densities.

It has to be clarified that the temperature distributions are plotted in two versions: assuming uniform temperature distribution in the simulated ensemble and assuming a triangular one with weights proportional to $(12 - T)[\text{MeV}]$, where T is a temperature set in simulation. In the case of simulation the same number of events was generated for all (discrete) values of the temperature. In the case of experimental data a probability of event at a certain temperature is connected to an impact parameter. Explanation and formulas can be found in Ref. [102]. Here, the simplest solution was adapted, which is a triangular temperature distribution in the simulated ensemble where events get weights of $(12 - T)[\text{MeV}]$. These weights follow from the assumption of a linear correlation between the temperature and an impact parameter [102].

From Fig. 47 one can conclude, that both assumptions of the temperature distribution in the simulated ensemble result in very similar temperature distributions for a specific Z_{bound} and thus, very similar values of mean breakup temperatures. This can be explained by the fact that the Z_{bound} is correlated with temperature and fixing this at some value selects from all events a subgroup of events with a narrow spread of temperatures.

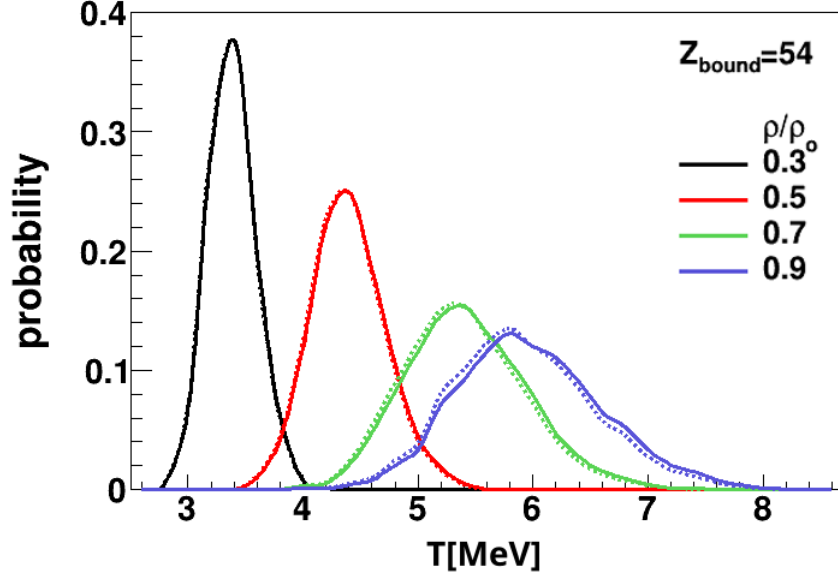


Figure 47: Temperature distributions at $Z_{\text{bound}} = 54$ for the system densities specified in the legend. The dashed lines (visible at higher densities) show the results of simulations using a triangular temperature distribution in the whole simulated ensemble (see text).

As a further confirmation of this effect the largest fragment distribution for those two cases is presented in Fig. 48 for uniform temperature distribution and in Fig. 49 for triangular temperature distribution. The figures are organized as follows: each column represent one of the densities in the order 0.3, 0.5, 0.7 and 0.9, respectively; rows presents the distributions in the following planes: top: $Z_{\text{max}} - T$, middle: $Z_{\text{bound}} - T$, bottom: $Z_{\text{max}} - Z_{\text{bound}}$. Horizontal lines in the middle row and vertical lines in the bottom row indicate $Z_{\text{bound}} = 54$. The comparison between those figures shows a very similar results, except for large temperatures or small Z_{bound} values.

In the top row ($Z_{\text{max}} - T$) of both Figs 48 and 49, it can be observed that the temperature derived from the temperature distribution selects the widest and most symmetric slice of the Z_{max} distribution. In the middle row ($Z_{\text{bound}} - T$), the horizontal line represents $Z_{\text{bound}} = 54$. In the bottom row, it can be seen that the pseudocritical Z_{bound} also selects the widest and most symmetric (K_4 and $K_3(Z_{\text{max}})$) slice of the Z_{max} distribution. Unlike the two previous rows of the plot, these distributions are experimentally measurable. The model introduces an unknown parameter T and allows for their separation into two (data presented in

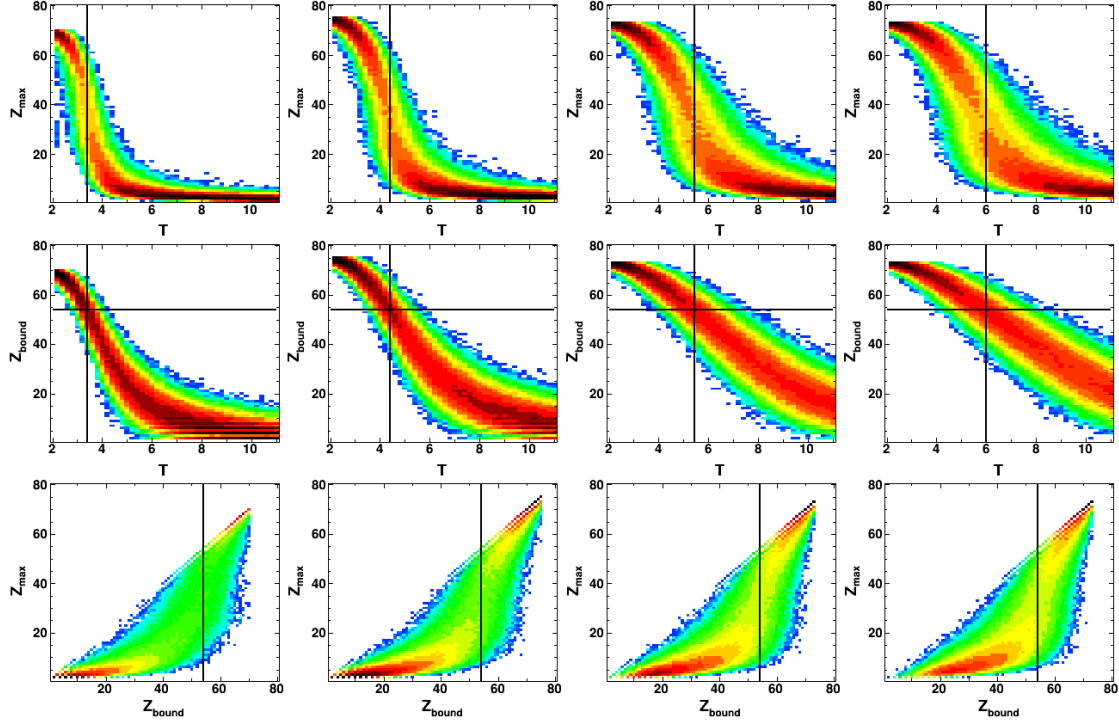


Figure 48: LGM calculations with the uniform temperature distribution in the simulated ensemble for $Z_{\text{bound}} = 54$. The top and middle rows show the temperature dependence of the $Z_{\max}$ and Z_{bound} distributions. The bottom row presents the correlation between $Z_{\max}$ and Z_{bound} . Columns from left to right show results for system densities of 0.3 , 0.5 , 0.7 , and $0.9 \rho_0$. The color scale is logarithmic. The vertical lines in the top and middle row indicate the breakup temperatures. The horizontal lines in the middle row and vertical lines in the bottom row indicate $Z_{\text{bound}} = 54$.

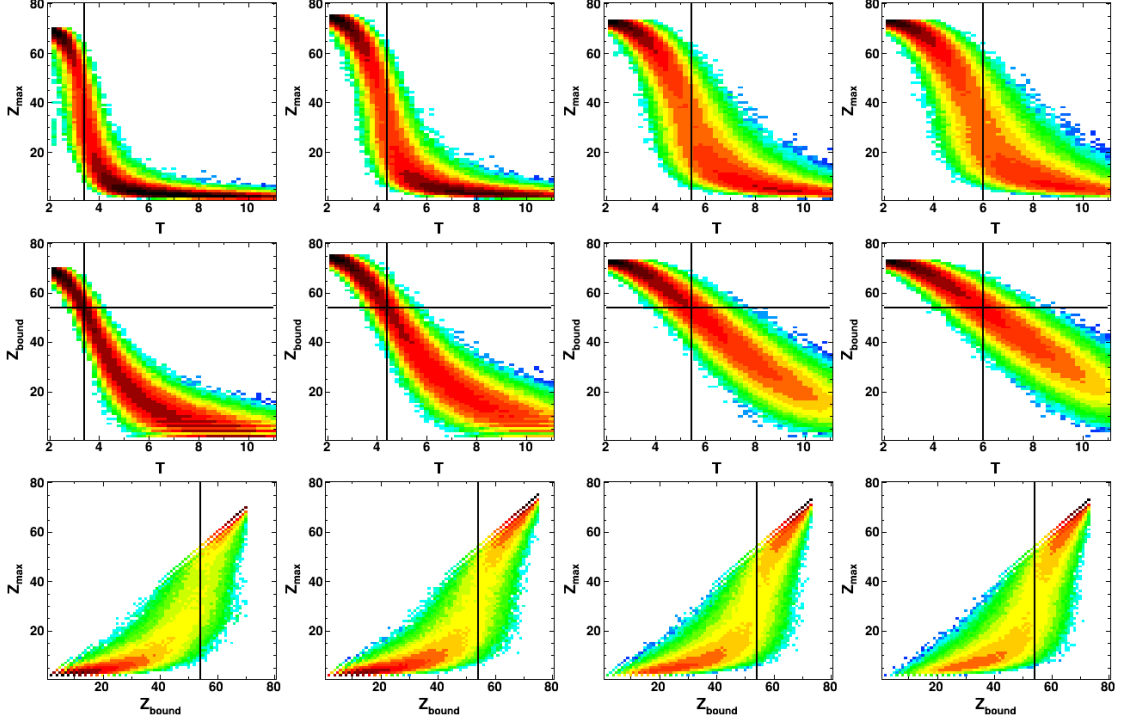


Figure 49: The same as in Fig. 48 but for the triangular temperature distribution in the simulated ensemble.

the two upper rows of the plot). It can be observed that the line passes through the inflection point and separates the regions where large $Z_{\max}$ (liquid) and small $Z_{\max}$ (more gas, less liquid) dominate.

Finally, the breakup temperatures for all investigated Z_{bound} values can be summarized in Fig. 50. The data points represent densities for sources with $A/Z = 2.49$, with error bars derived from the spread of the respective distributions (see Fig. 47 for the case of $Z_{\text{bound}} = 54$). Additionally, the breakup temperatures of Au-like spectators obtained from experimental apparent temperatures T_{HeLi} from Au+Au reactions at 600 and 1000 AMeV measured in the Z_{bound} domain are included for comparison. The experimental data has been corrected for the effect of secondary decays using the predictions of the quantum statistical model (see Figures 6 and 7 of Ref. [103]) for densities $\rho_0 = 0.1, 0.3$, and 0.5 .

A more detailed analysis of the plot reveals that for $Z_{\text{bound}} = 66$, the experimental range of temperatures is best described by a density around $0.9 \rho_0$. Similarly, for $Z_{\text{bound}} = 54$, the experimental range of temperatures is also best described by a density around $0.7\text{--}0.9 \rho_0$. For $Z_{\text{bound}} = 36$, densities between $0.5\rho_0$ and $0.7\rho_0$ are

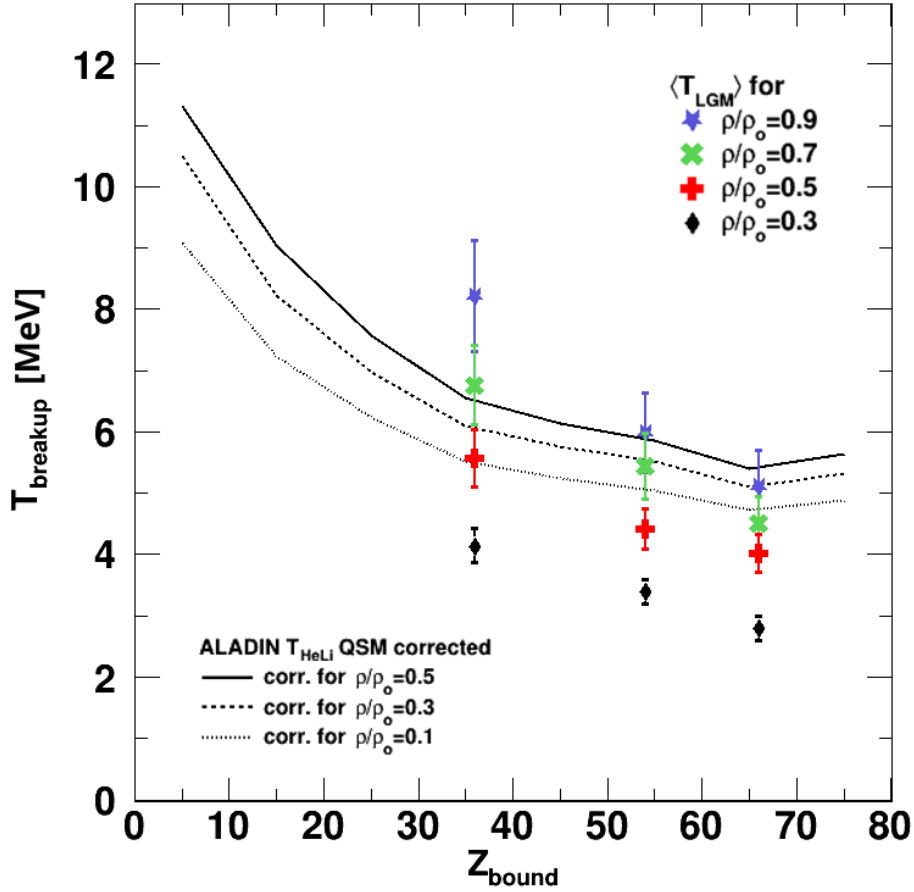


Figure 50: Breakup temperatures of the Au-like spectators as a function of Z_{bound} obtained from the experimental apparent temperatures T_{HeLi} (Fig. 7 of Ref. [103]) corrected for the effect of secondary decays by using the quantum statistical model calculations (Fig. 6 of Ref. [103]) for system densities 0.1, 0.3 and 0.5 ρ_o (lines). The symbols show LGM results for the four indicated system densities.

found to be the most appropriate. These results indicate that the commonly adopted scenario of fragmentation at low densities (e.g., around $0.3 \rho_0$ in the SMM model) is not confirmed by the LGM model, which predicts a gradual increase in breakup densities for more peripheral events. In particular, it appears that the density at the critical point ($Z_{\text{bound}} = 54$) lies between $0.7\rho_0$ and $0.9\rho_0$.

It should be noted that the isotopic effects in the LGM model are not sufficiently pronounced/correct to be used to reconstruct the breakup temperatures from the isotopic double-ratios (see. [46]), as in the experimental analysis. These Albergo temperatures are driven by fine quantum effects (binding energies, Fermi statistics, spin degeneracies, densities of states) that are missing in this simple model. Instead it has been assumed that the experimentally derived breakup temperatures correspond to the Maxwell-Boltzman temperatures used in the model.

6 Isotopic effects in spectator fragmentation

6.1 ALADIN S254 experiment

Isotopic effects in spectator fragmentation at relativistic energies have been a subject of experimental studies at GSI Darmstadt with the use of the upgraded ALADIN setup. The data collected in 2003 on fragmentation of stable and radioactive Sn and La isotopes will serve here as a reference for checking the abilities of LGM in reproducing the measured isotopic effects and its predictive power in terms of extracting the break-up conditions (temperatures and densities).

Experiment S254 was performed by the ALADIN collaboration to investigate the isotopic effects in projectile-spectator fragmentation at relativistic energies. The study involved the use of a stable beam of neutron-rich ^{124}Sn and radioactive neutron-poor ^{107}Sn and ^{124}La secondary beams, allowing for exploration across a wide range of isotopic compositions [11, 104–106].

It is worth highlighting that the secondary beams of ^{107}Sn and ^{124}La directed towards the reaction target actually encompassed a distribution of neighboring isotopes along with the designated projectiles. The mean $\langle A/Z \rangle$ values for these two secondary beams are 2.19 for the nominal “ ^{124}La ” beam and 2.16 for the “ ^{107}Sn ” beam, corresponding to ^{108}Sn (refer to Fig. 6 in Ref. [107]). Due to the prevalent representation of ^{108}Sn within the radioactive beam, the current study opted for this isotope as the primary choice.

The experimental setup of ALADIN, described in Section 5.1 (Fig. 17), was upgraded for the S254 experiment. This upgrade involved the construction of new eight proportional readout chambers for the TP-MUSIC IV detector and redesign of the electronic readout chain [105]. The setup enabled the detection and individual identification of fragments based on their atomic number Z , with a detection threshold below $Z = 2$ and a resolution that extends up to the atomic number Z_{proj} of the projectile [11]. A key feature of the setup was its high detection efficiency, close to 100%, for the detection of fragments with Z up to that of the projectile and ability to resolve the masses of fragments with $2 \leq Z \leq 10$ thanks to application of the proportional counters.

6.2 Determination of the systems sizes in the Lattice Gas Model

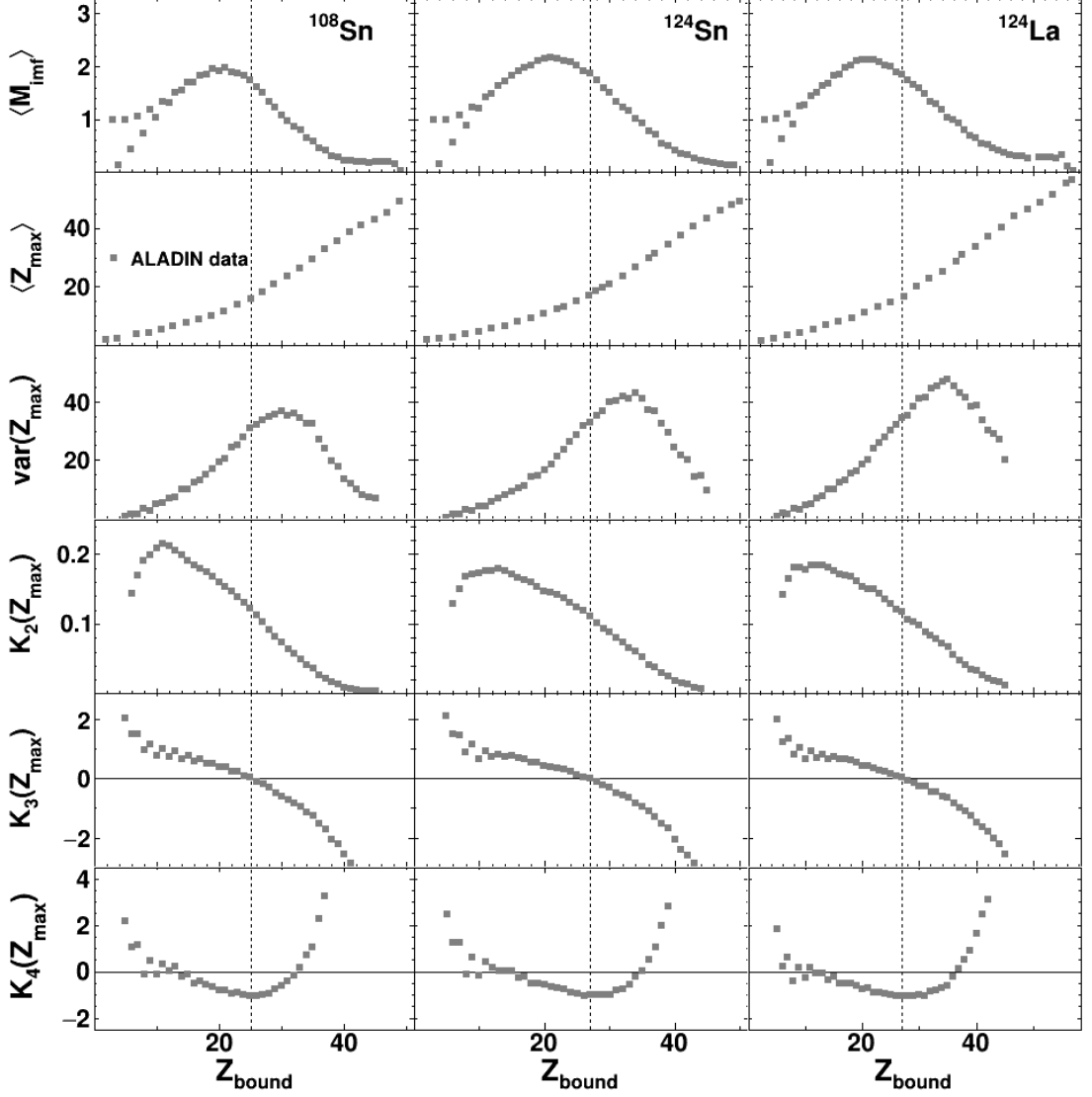


Figure 51: Experimental ALADIN S254 results for ^{108}Sn , ^{124}Sn and ^{124}La projectiles (derived from Refs [9, 11]). From top to bottom: the mean multiplicity $\langle M_{\text{imf}} \rangle$ of intermediate mass fragments, the mean $\langle Z_{\text{max}} \rangle$, variance $\text{var}(Z_{\text{max}})$, and the cumulant ratios K_2 , K_3 , K_4 of the largest fragment charge distribution, as a function of Z_{bound} . The dotted vertical lines indicate the zero transitions of K_3 and minima of K_4 .

Some of the experimental results are presented in Fig. 51. Those results were adopted from Ref. [9]. They are for ^{108}Sn , ^{124}Sn and ^{124}La fragmentation. The pseudocritical points are indicated by vertical lines indicated by $K_3 = 0$ and minimum of K_4 . The top row shows the mean multiplicity of intermediate mass fragments (IMF: $3 \leq Z \leq 20$). The next rows present the mean $\langle Z_{\text{max}} \rangle$, variance $\text{var}(Z_{\text{max}})$ and cumulant ratios K_2 , K_3 and K_4 of the largest fragment charge.

Fig. 52 shows the same distributions for the system sizes determined at the pseudocritical points obtained from the LGM simulation. Therefore the experimental data can be compared with those points and they are indicated by the boxes. Model predictions provide overall good description of the data for all investigated system densities.

The isotopic composition (A/Z) of the fragmenting sources was assumed to be that of the initial projectiles. The transition points indicated in Fig. 51 by vertical lines ($Z_{\text{bound}} = 25, 27$ and 27) allowed to evaluate the system sizes. The system sizes were determined for each density. The results are presented in Tab. 1.

Projectile	Z_{bound}	A_0/Z_0	Z_0			
			$\rho/\rho_0 = 0.3$	$\rho/\rho_0 = 0.5$	$\rho/\rho_0 = 0.7$	$\rho/\rho_0 = 0.9$
^{108}Sn	25	2.16	37	36	34	34
^{124}Sn	27	2.48	40	38	37	37
^{124}La	27	2.18	40	39	38	35

Table 1: LGM estimations of the system sizes Z_0 at the observed transition points $Z_{\text{bound}} = 25, 27$, and 27 for the ^{108}Sn , ^{124}Sn and ^{124}La projectiles, respectively. Calculations performed for the four system densities.

6.3 Temperature distributions at the transition points

Figure 54 depicts the dependence of breakup temperature on isotopic composition and system size. As a comparison, predictions from the SMM, CTM, and Hartree-Fock calculations are included. The experimental apparent temperatures T_{HeLi} at the transition points (extracted from Fig. 3 of Ref. [104]) are included for comparison. The two panels present results separately for the neutron-poor and neutron-rich cases. Additionally, the value of 5.4 ± 0.4 MeV for the fragmentation of ^{197}Au [103] is included in the right panel, it corresponds to $Z_0 = 64$.

The experimental result have shown a weak dependence of the breakup temperature on the system A/Z and the system size which is in agreement with predictions

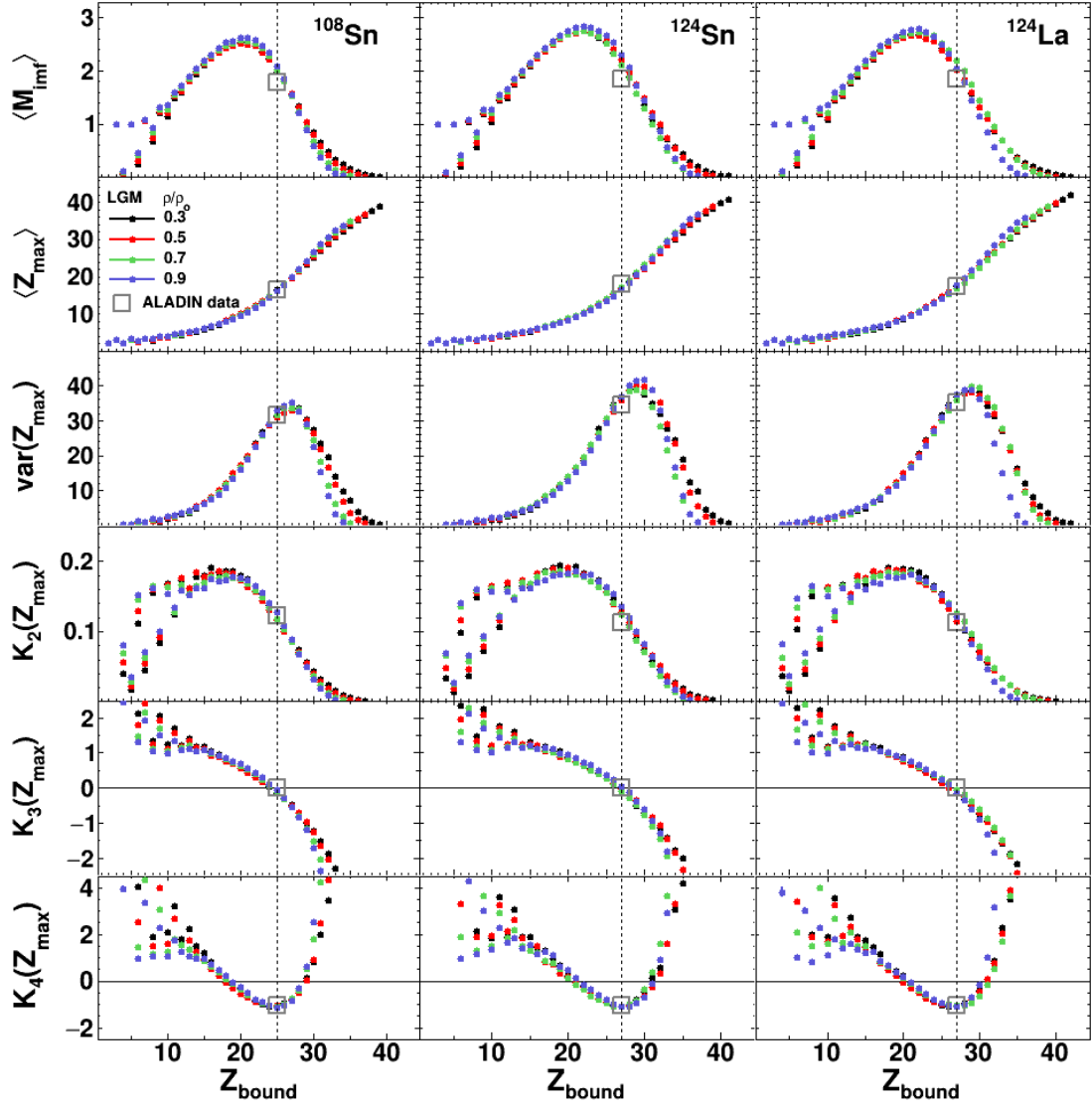


Figure 52: As in Fig. 51 for LGM simulations with systems specified in Tab. 1. The boxes show the experimental results at the transition points (zero crossings of K_3).

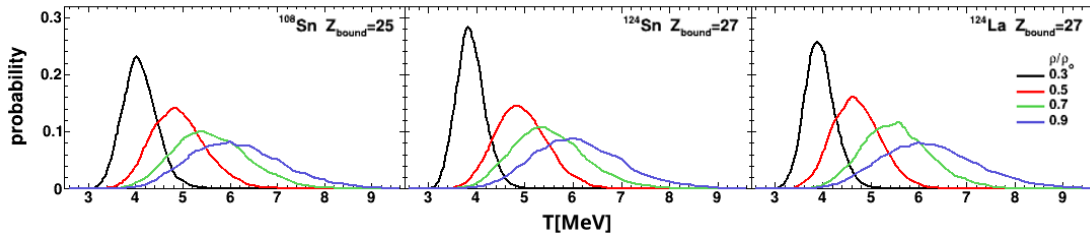


Figure 53: LGM predictions for the temperature distributions at the transition points depending on the system density.

Projectile	Z_{bound}	T_{HeLi}	T_{mean} (MeV)			
			$\rho/\rho_0 = 0.3$	$\rho/\rho_0 = 0.5$	$\rho/\rho_0 = 0.7$	$\rho/\rho_0 = 0.9$
^{108}Sn	25	5.4	4.1 ± 0.4	4.9 ± 0.6	5.6 ± 0.8	6.2 ± 1.0
^{124}Sn	27	5.8	3.9 ± 0.3	4.9 ± 0.6	5.5 ± 0.8	6.1 ± 0.9
^{124}La	27	5.5	3.9 ± 0.3	4.7 ± 0.5	5.5 ± 0.7	6.3 ± 1.0

Table 2: Mean and standard deviation (error) of the temperature distributions at the transition points shown in Fig. 53). The experimentally estimated apparent temperatures T_{HeLi} at the transition points are taken from Ref. [104].

of the statistical models (SMM and CTM). The results presented in Fig. 54, together with the consistent results of Fig. 50 and with pure model predictions of Fig. 16, indicate that, according to LGM, nuclei that undergo fragmentation most likely at densities around 0.7-0.9 ρ/ρ_0 . This result is somewhat contrary to the results of SMM or CTM models which assume low breakup densities, around 0.3 ρ_0 . LGM breakup temperatures obtained for these densities are closest to the experimental ones. This observation of a relatively high density at the breakup supports the scenario proposed by Campi et al. [108] within their classical molecular dynamics model. The obtained here the breakup temperature exhibit a significant dependence on the system density, indicating the agreement with the experimental results at larger densities 0.7-0.9 ρ_0 .

Another conclusion from Fig. 54, or more precisely from the mean temperatures specified in Table 2, is that the breakup temperatures are nearly independent on the system A/Z .

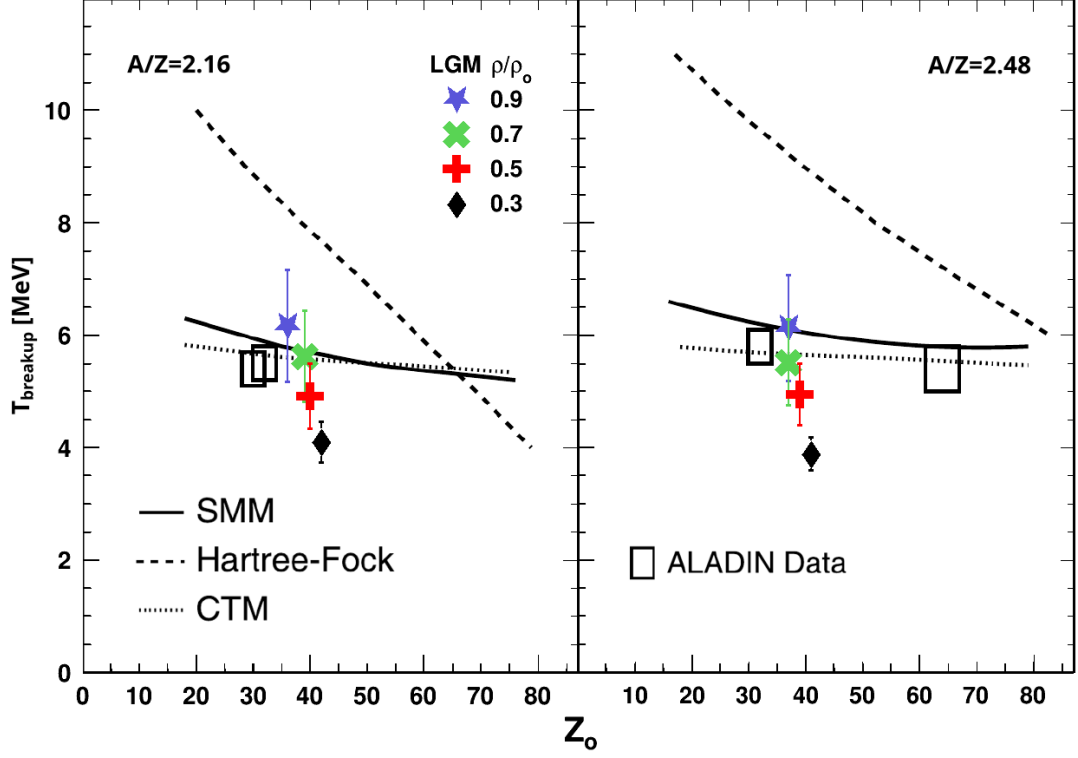


Figure 54: Breakup temperatures as a function of the reconstructed atomic number Z_0 of the disintegrating system at the pseudocritical transition point for neutron-poor (left) and neutron-rich (right) systems. The boxes represent the apparent chemical breakup temperatures T_{HeLi} while the lines give the Statistical Multifragmentation Model (SMM), Hartree-Fock, and Canonical Thermodynamic Model (CTM) predictions as published in Ref. [9]. The LGM results, dependent on the assumed system density, are shown by the color symbols with error bars. They are the mean values with standard deviations of the temperature distributions at the pseudocritical transition points, as given in Tab. 2.

6.4 Isotopic composition of fragments

Figures 55 - 57 examine the isotopic composition of fragments.

Figures 55, 56 show the mass distributions of lithium and neon elements for ^{108}Sn and ^{124}Sn , integrated over the range $0.2 \leq Z_{\text{bound}}/Z_0 \leq 0.8$. The experimental data are indicated by the black points (Ref. [11]). The crosses are the LGM predictions different densities as indicated in the figure.

It is evident that the LGM overestimates the yield of low-mass isotopes, such as ^4Li and ^5Li (Fig. 55), and ^{18}Ne and ^{19}Ne (Fig. 56). This discrepancy is most likely due to the limitations of the model. In particular ^5Li , which is unbound and does not appear in nature, appears in the model because of lack of fine quantum and structure effects. However, the predicted yield of stable isotopes is in very good agreement with the experimental data. Notably, no clear density dependency is observed, as the predicted yields remain close for all investigated densities.

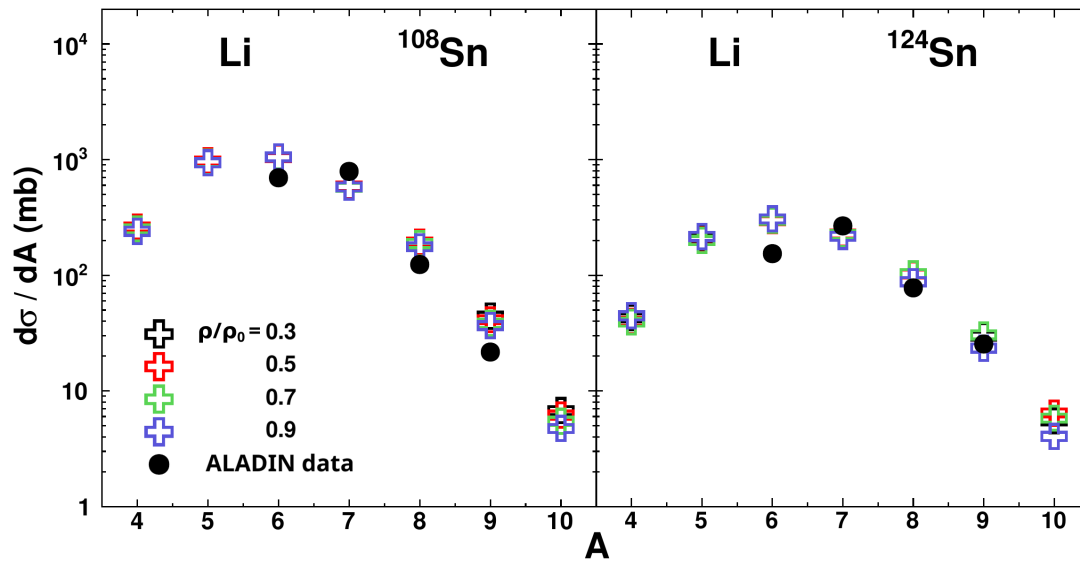


Figure 55: Mass distributions of Li isotopes for ^{108}Sn (left panel) and ^{124}Sn (right panel) projectiles. The filled circles represent the experimental data integrated over $0.2 \leq Z_{\text{bound}}/Z_0 \leq 0.8$ (from Fig. 18 of Ref. [11]). The crosses show the normalized LGM results for $Z_{\text{bound}} = 25 \pm 5$ and 27 ± 5 for ^{108}Sn and ^{124}Sn , respectively. The color code for LGM system densities is specified in the legend.

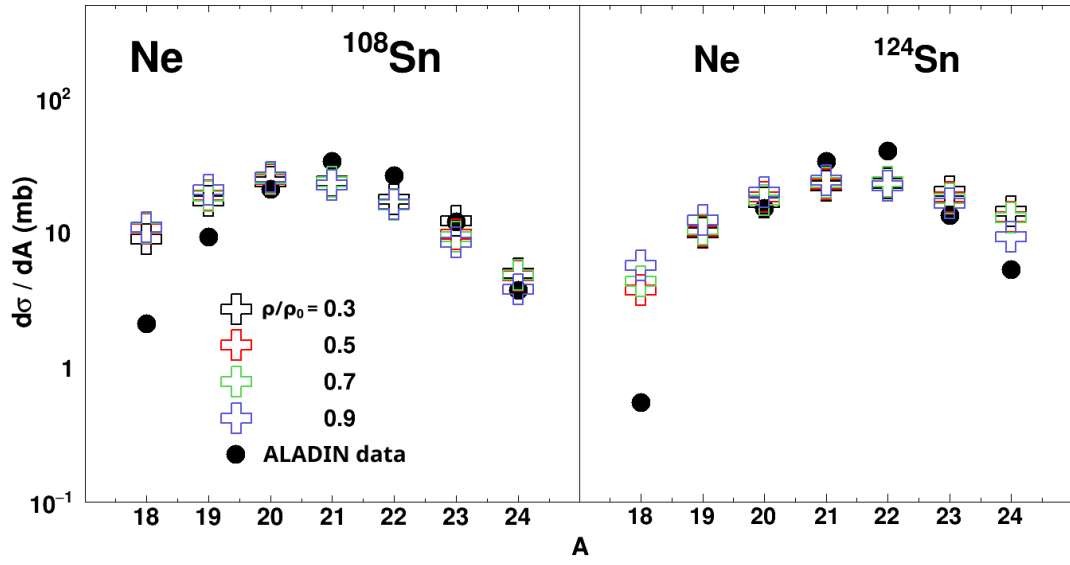


Figure 56: Same as Fig. 55 but for Ne isotopes (see Fig. 19 of Ref. [11] for experimental data).

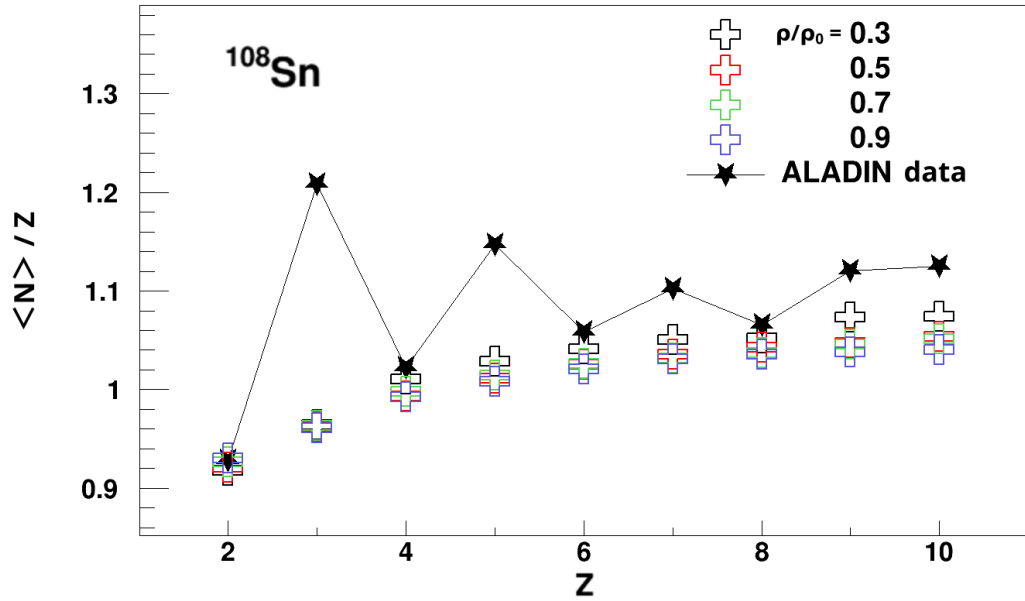


Figure 57: Measured mean neutron-to-proton ratios $\langle N \rangle / Z$ of fragments with $Z \leq 10$ (stars) produced in the fragmentation of ^{108}Sn for $Z_{\text{bound}}/Z_0 = 0.5$ (from Fig. 20, middle panel, of Ref. [11]). The crosses represent the LGM results for $Z_{\text{bound}} = 25 \pm 2$ and for the four indicated system densities.

Figure 57 shows the mean neutron-to-proton ratios $\langle N \rangle / Z$ for fragments with $2 \leq Z \leq 10$ produced in the multifragmentation of ^{108}Sn . Experimental data show so called “odd-even staggering” effect, which has been attributed to the pairing energy [109], while the LGM predictions show the smooth dependence. Its subtle contribution to the binding energy is missing in the model and thus the effect is not reproduced. Therefore, the contribution of the non existing isotopes as seen in Figs 55, 56 can be clearly seen as the lower mean number of neutrons for odd Z values. For example, overestimation of ^4Li and ^5Li leads to lower mean neutron number for $Z = 3$.

6.5 Isoscaling

A prominent feature observed in multifragmentation [91–95] that fragment yields of similar, but different in isospin asymmetry, reactions depend exponentially on the neutron N and proton Z numbers. This scaling law, called isoscaling, relates ratios $R_{21}(N, Z)$ of isotope yields $Y_1(N, Z)$ and $Y_2(N, Z)$ as:

$$R_{21}(N, Z) = \frac{Y_2(N, Z)}{Y_1(N, Z)} = C \cdot \exp(\alpha N + \beta Z) \quad (11)$$

where: $Y_1(N, Z)$ and $Y_2(N, Z)$ denotes the yield of the isotope (N, Z) for reactions 1 and 2, respectively, α and β are the isoscaling parameters, and C is a normalization factor. Conventionally the two reactions are denoted by 1 and 2, where reaction 2 corresponds to the system richer in neutrons (neutron-rich system).

The feature of the isoscaling can be derived and explained in the grand canonical approximation [110–112] however it should be noted as in [113] that for the considered systems more appropriate would be microcanonical or the canonical ensemble in which the neutron and proton numbers are conserved.

As an example, let us consider the following fragmenting sources:

$$\begin{aligned} A = 168, Z = 75, & \quad \text{for reaction 1} \\ A = 186, Z = 75, & \quad \text{for reaction 2} \end{aligned}$$

The cubic lattices of 7^3 and 8^3 sites are used for the systems, which gives the density of about $0.3\rho_0$ and $0.5\rho_0$, respectively. Pairs of equilibrated sources are considered at a two temperatures: 4 and 5 MeV. A number of 100 000 events were simulated for each temperature-density combination for each source.

Figure 58 shows the appearance of isoscaling behavior for this example. It plots the $R_{21}(N, Z)$ as a function of fragment neutron number N from which the slope parameter (isoscaling parameter) can be extracted. Results of calculations reported by C. B. Das and S. Das Gupta [97] for the same systems are shown in the Figure for comparison. Results were also confronted with experimental results and presented in [92]. The experimental results of isoscaling yielded $\alpha \approx 0.34$, while the LGM simulation significantly underestimated this value.

In the following, the analysis will be performed for systems studied in Ref. [11], i. e. ^{108}Sn and ^{124}Sn for events selected for critical value of Z_{bound} . The isoscaling for the largest fragments also will be investigated, aiming to obtain more realistic results, as it was suggested in Ref. [96].

The calculated yield ratios for isotopes in a range of Z from 3 to 10 as a function of N are plotted in Fig. 59. The slopes are fitted with equation (11) to extract the isoscaling parameter α . Averaged parameter α , which is calculated over the range of Z , is presented in Fig. 60. The α parameter is almost constant in the range of Z . These results significantly underestimate the experimental ones, almost by factor of two.

To obtain more realistic results, it was proposed to investigate the isotopic properties of the largest fragments, as shown in Fig. 61. Exponential relationship of yield ratios confirms the presence of isoscaling for the largest fragment in the model. As in the previous case, mean values of the α parameter are shown in Fig. 62. The spread of the values is noticeably larger, mainly due to the lower statistics, however, a slight increase of α values for larger Z_{max} can be observed. The isoscaling parameter exhibits a weak dependence on density, although for the density $0.9\rho_0$, the mean value is noticeably smaller.

Another way to estimate the isoscaling parameters is to consider the isotopic ratio R_{21} in a $N - Z$ plane. In this representation the isoscaling manifests when the R_{21} values are located in a plane, therefore a so-called global fit of Eq. (11) can be employed, eliminating the necessity of averaging the parameters. Alternatively, one can consider the R_{21} ratio in the $N - Z$ vs $N + Z$ plane i. e. the neutron excess vs mass number. In this representation, the R_{21} values are fitted with the following recipe:

$$R_{21}(N, Z) = C \exp[\alpha'(N - Z) + \beta'(N + Z)] \equiv C \exp[(\alpha' + \beta')N + (\beta' - \alpha')Z] \quad (12)$$

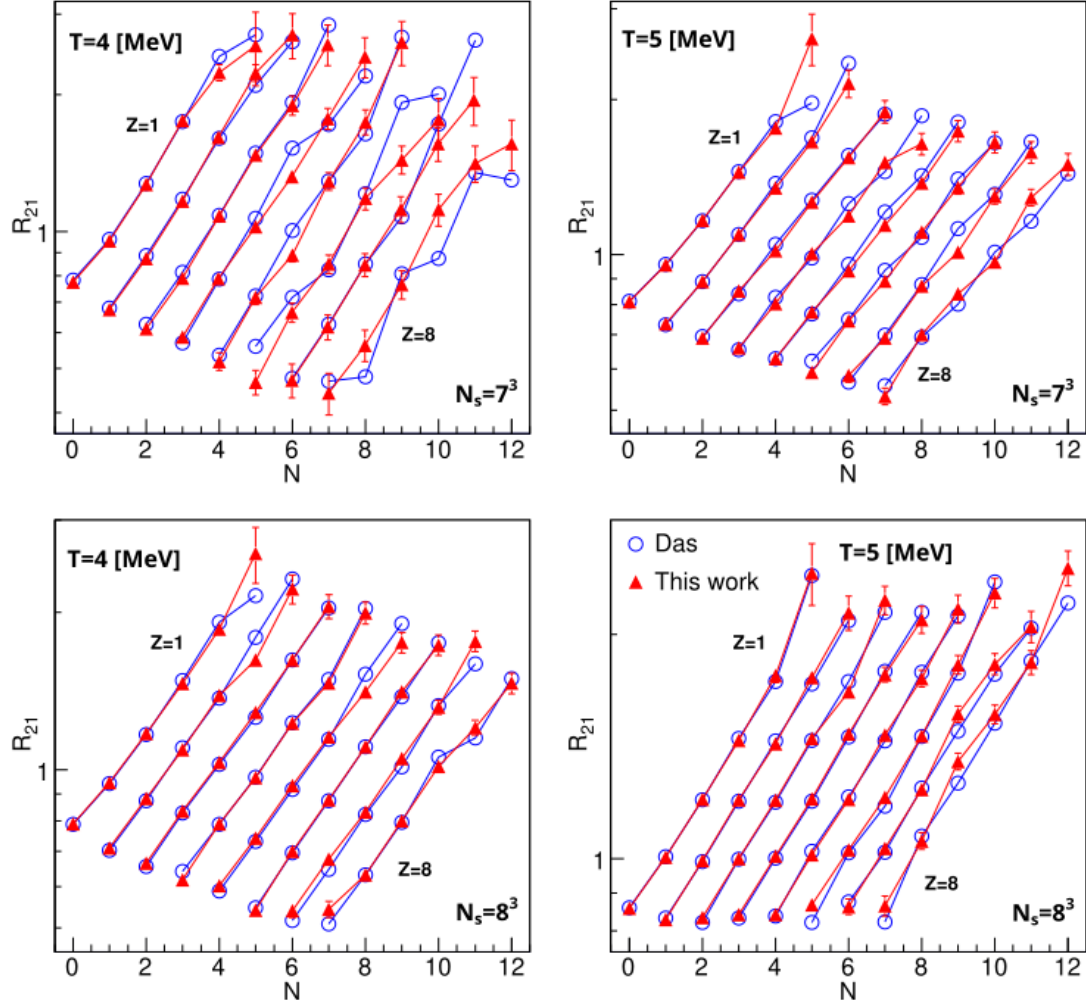


Figure 58: Isotopic yield ratio, R_{21} , as a function of the neutron number, N , of fragments with Z from 1 to 8, calculated for $(A = 186, Z = 75)$ and $(A = 168, Z = 75)$ at temperatures 4 MeV (left panels) and 5 MeV (right panels) and for lattice size 7^3 (top panels) and 8^3 (bottom panels). The triangles show results obtained in this work while the circles are from Ref. [97].

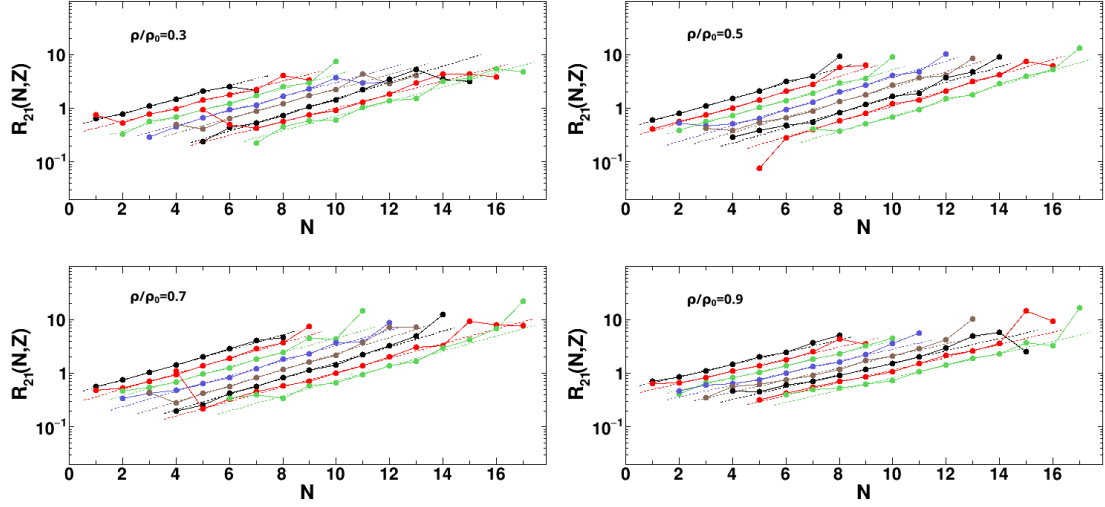


Figure 59: Isotopic yield ratio R_{21} for ^{124}Sn and ^{108}Sn systems as a function of the fragment neutron number N . The dots connected by a line are for a given fragment charge, ranging from 3 to 10. The dotted straight lines represent the individual exponential fits.

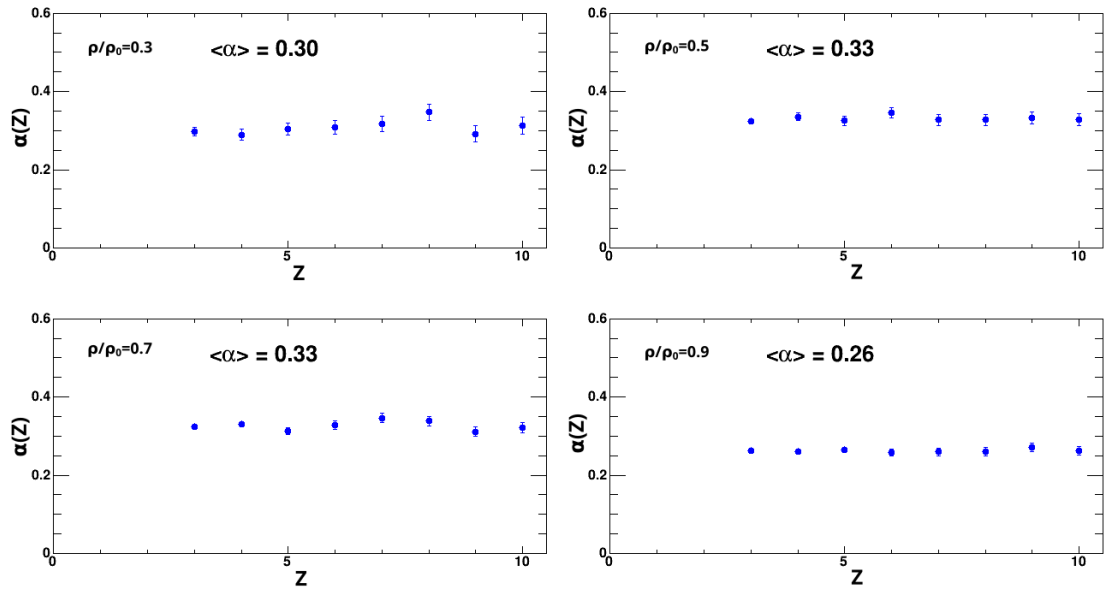


Figure 60: Isoscaling α parameter as a function of fragment charge obtained from the fits to the isotopic yield ratios shown in Fig. 59. The panels are for different system densities. The mean value of α is indicated in each panel.

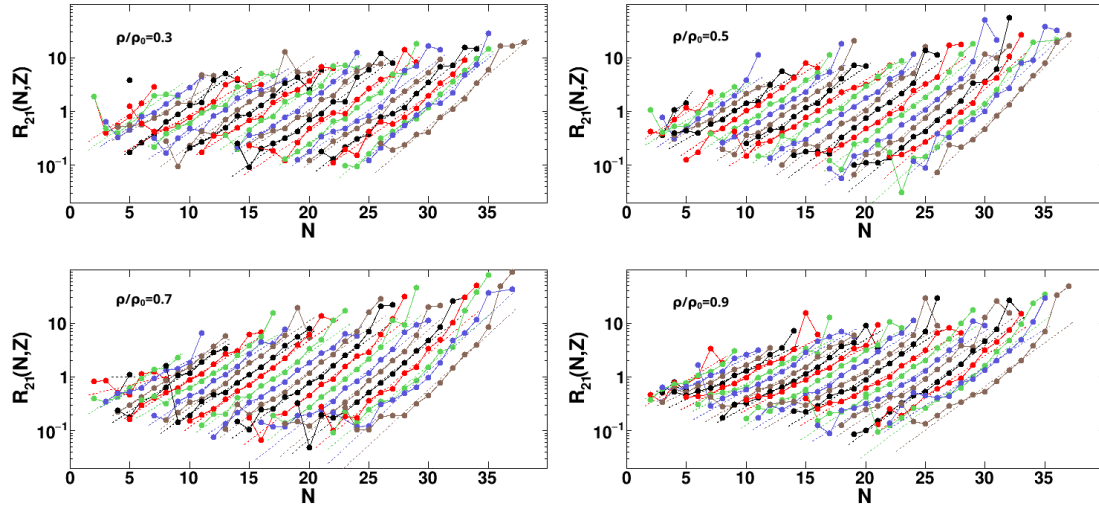


Figure 61: Isotopic yield ratios R_{21} for the largest fragment in ^{124}Sn and ^{108}Sn systems. The ratios are plotted as a function of N for a range of $Z_{\max}$ values. The dots connected by a line are for a given $Z_{\max}$. The dotted straight lines represent the individual exponential fits.

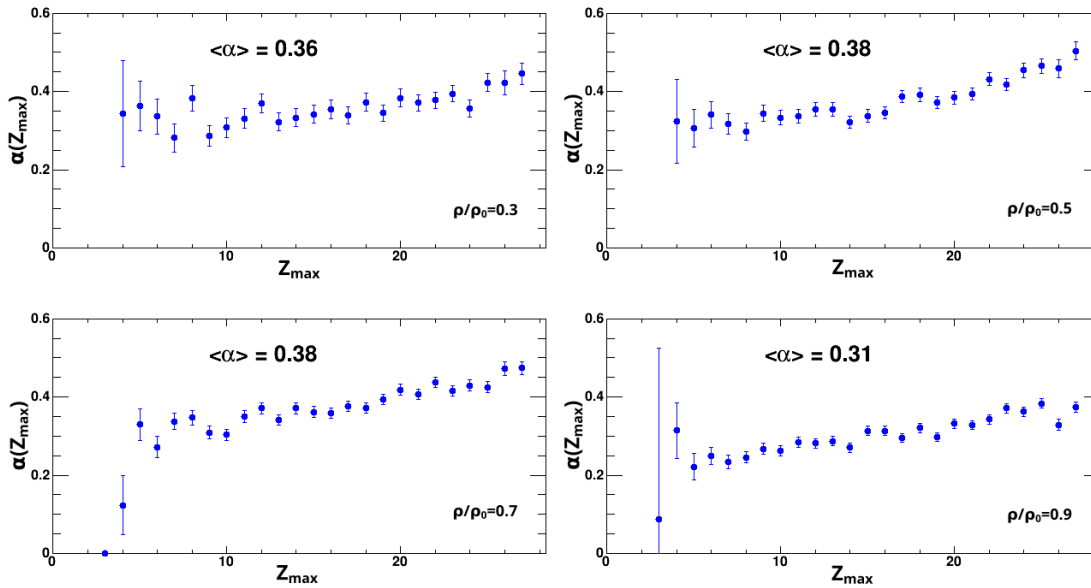


Figure 62: Isoscaling α parameter as a function of the largest fragment $Z_{\max}$ obtained from the fits to the isotopic yield ratios shown in Fig. 61.

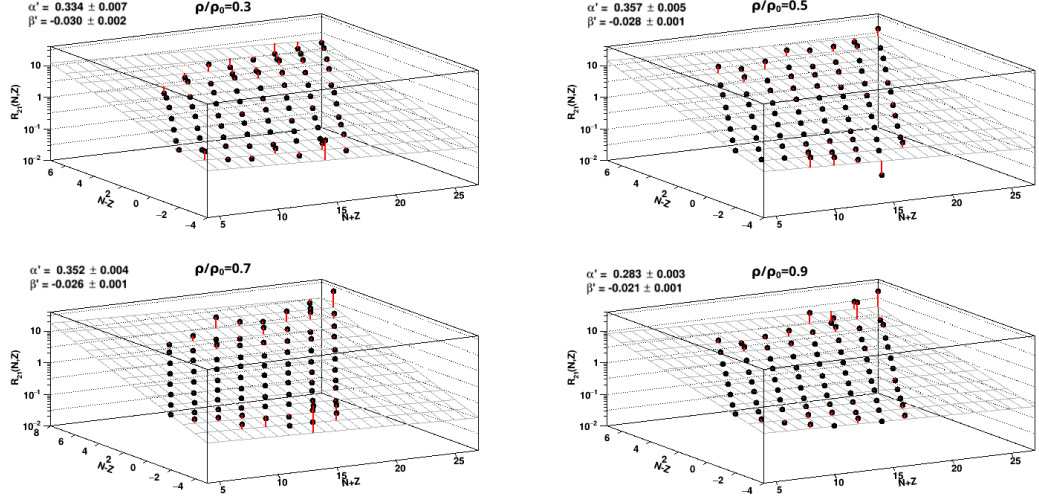


Figure 63: Isotopic ratios R_{21} for fragments with Z from 3 to 10 in ^{124}Sn and ^{108}Sn systems plotted in the $N - Z$ vs $N + Z$ plane. The grids indicate the fitted planes. Red line segments depict the vertical distance of a point from the fitted plane. The system densities and fit results (see text for the meaning of parameters) are indicated in each panel.

from which the isoscaling parameters can be extracted as:

$$\begin{aligned}\alpha &= \alpha' + \beta' \\ \beta &= \beta' - \alpha'\end{aligned}\tag{13}$$

Results of such a procedure are shown in Fig. 63 for all investigated densities, including fitted α' and β' parameters. In this representation one can notice that R_{21} weakly depends on the mass of the fragment (slowly decreases with the mass, which is indicated by a small, negative β') and relatively strongly depends on the neutron excess of the fragment (larger, positive α').

The parameters obtained in this alternative procedure are consistent with the previous approach. As for the light fragments, such an analysis has been also performed for the largest fragments Z_{max} and illustrated in Fig. 64. The values of α' and β' parameters obtained from the fit are given in the Figure. In this case, obtained isoscaling parameters are also similar as for the averaged ones from the previously described procedure.

Figure 65 shows results of the isoscaling analysis from the global fits of Figs 63 and 64. The line represents the experimental result at $Z_{\text{bound}} = 25$) [11]. The circles de-

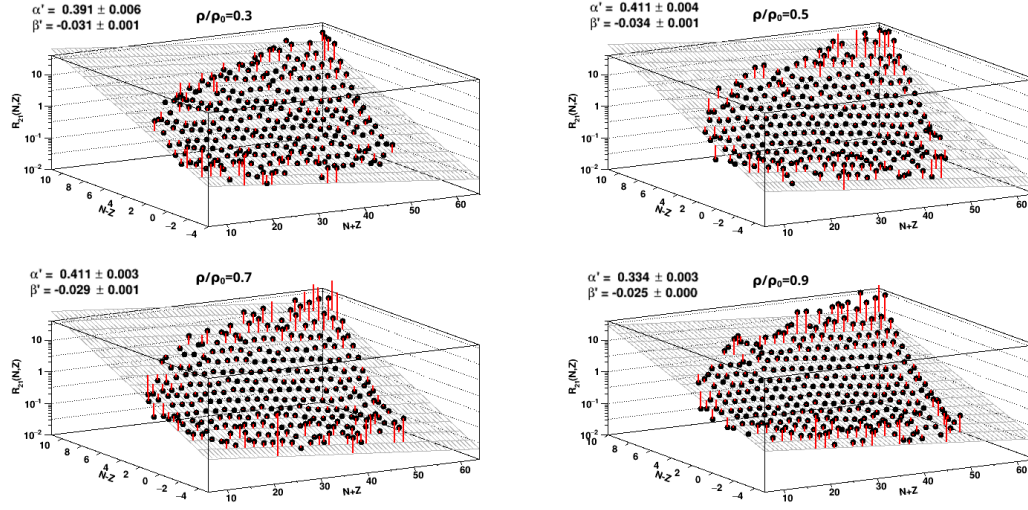


Figure 64: Same as Fig.63 but for the largest fragment.

pict the simulated results for the light fragments. The squares show the results for the largest fragment, and the crosses are for the largest fragment with $Z_{\max} > 20$.

Isoscaling applied to the largest fragments enhance the α parameter. Additional criterion regarding the size of the largest fragment ($Z_{\max} > 20$) further increases this parameter. However, it is important to note, that a direct comparison of the isoscaling results for the largest fragment with the experiment is not justified, as the experimental data were obtained for light fragments ($3 \leq Z \leq 10$). Thus, only the results represented by circles can be directly compared with the experimental results.

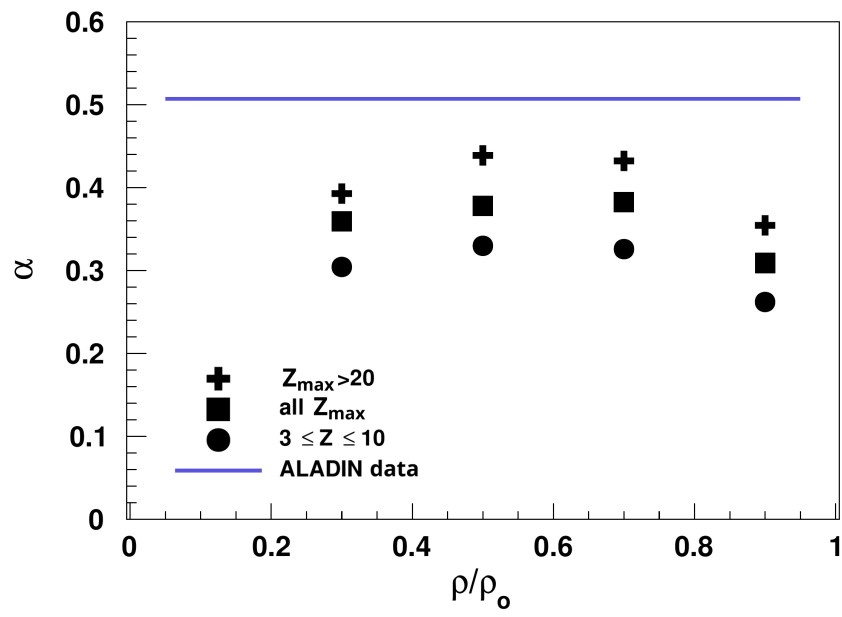


Figure 65: Isoscaling parameter α as a function of the system density obtained from the LGM analysis (symbols) compared with the experimental value (line) for semi-peripheral collisions ($Z_{\text{bound}}/Z_0 = 0.5$) [11].

7 Summary

A major motivation for studying nuclear multifragmentation was the expectation that this phenomenon is linked to a liquid-gas phase transition predicted for nuclear matter [1, 3, 12]. In this thesis the clustering in nuclear multifragmentation processes have been studied with the isospin-dependent Lattice Gas Model, focusing on the phase transition signatures. The model incorporates the distinction between protons and neutrons allowing for investigation of the isotopic effects as represented by the system neutron-proton asymmetry and the fragment isotopic compositions. It also allows to include the Coulomb term to the model Hamiltonian, providing more realistic description of nuclear fragmentation. The calculations were performed on simple cubic lattices in the canonical ensemble, at constant particle numbers (A, Z), volume and temperature, using free periodic boundary conditions to account for the presence of surface in finite systems. Events were generated using a Monte Carlo method with the Metropolis algorithm. Two alternative clusterization procedures were employed: E_{REL} - with the relative kinetic energy approach, and the Coniglio-Klein (C-K) prescription.

Recent studies [8, 9, 62] have shown that an analysis of the skewness, K_3 , and the kurtosis excess, K_4 , of the largest fragment size distribution is a valuable method to reveal the presence of a phase transition or critical behavior. The zero-crossing of K_3 coinciding with a minimum of K_4 is a robust indication of the percolation critical behavior in finite systems (the pseudocritical point), but also is observed at a liquid-gas phase transition in statistical multifragmentation models. This method was used in the present work to identify and trace phase transitions with the Lattice Gas Model in wide ranges of the system density, temperature and isospin asymmetry.

In order to obtain reference results, a simplified one-component LGM was examined, whose phase diagram is known in the thermodynamic limit. It contains a liquid-gas phase coexistence region with critical point located at one-half of the normal nuclear density, $\rho/\rho_0 = 0.5$, and a critical-like behavior is observed for supercritical densities and temperatures along the Kertész line. The LGM simulations with finite systems have shown that the applied cumulant signature ($K_3 = 0$ and minimum of K_4) indicates a percolation-like critical behavior along a line, which approximately follows the Kertész line but extends to lower, subcritical densities and temperatures, penetrating the liquid-gas coexistence zone. This corroborates earlier results obtained with using other criticality signatures [75, 76] and the conclusion that first-order liquid-gas transition in small systems may mimic critical behavior. The analysis performed with the isospin-dependent LGM gives

qualitatively similar results. The extracted transition temperatures exhibit a weak dependence on the system size and neutron-proton asymmetry, but a strong dependence on the system density. The temperature decreases with decreasing density, from about 6 MeV for $0.9\rho_0$ to about 3.5 MeV for $0.3\rho_0$. This results suggest that the determining the breakup temperatures can provide information on the system density at the freeze-out stage, which is of particular interest, since it can verify various multifragmentation scenarios represented by different models.

The Lattice Gas Model predictions have been widely confronted with the ALADIN experimental data on fragmentation of ^{197}Au spectator projectiles in heavy-ion collisions at energies $600 \div 1000$ AMeV [10]. These experimental data were reproduced with a high accuracy by the bond percolation model [8]. The fragment charge distributions and correlations were examined using for sorting events the quantity Z_{bound} (the total charge of fragments with $Z \geq 2$), which is correlated with the spectator excitation energy, and thus, playing the role of the control parameter. The cumulant analysis of the largest fragment size distributions revealed the presence of the pseudocritical point at $Z_{\text{bound}} = 54$, corresponding to a breakup temperature of about 5.5 MeV. Similar analysis performed in the present work shows that the more realistic Lattice Gas Model also very well reproduces the experimental data for the system densities $\rho \gtrsim 0.5\rho_0$. Significant discrepancies are observed for a lower density of $0.3\rho_0$. Derived from the model temperatures of the fragmenting system (the breakup temperatures) for several Z_{bound} values depend significantly on the system density. An approximate agreement with the experimental estimates obtained from measured isotopic ratios T_{HeLi} is observed for the highest investigated densities $0.7\rho_0$ and $0.9\rho_0$.

The further analysis concerned the isotopic dependence of nuclear multifragmentation using the ALADIN S254 experimental data on fragmentation of neutron-rich ^{124}Sn and radioactive neutron-poor ^{107}Sn and ^{124}La beams of energy 600 AMeV, directed on natural Sn targets [11]. The cumulant analysis of the largest fragment charge distributions has been applied to identify the transition (pseudocritical) points. They are observed at $Z_{\text{bound}} = 25$ for ^{107}Sn , and at $Z_{\text{bound}} = 27$ for ^{124}Sn and ^{124}La projectiles, nearly independent of the projectile A/Z ratio. The associated breakup temperatures derived from the Lattice Gas Model for different system densities were compared with the experimental T_{HeLi} temperatures that are about 5.5 MeV for all systems. The model results depend on the density, and the best agreement with the data is observed for the density about $0.7\rho_0$, similarly as for the fragmentation of Au projectiles.

The predicted by model the fragment isotopic distributions are nearly independ-

ent on the system density. Comparisons with the data performed for the mass distributions of Li and Ne fragments shows a good agreement in range of higher mass numbers but overestimates the contribution from fragments of the lowest mass number. Similarly, the mean fragment neutron number N as a function of Z is also nearly independent on the system density. The experimental data shows a typical odd-even effect while the model shows, as expected, a smooth dependence following the data for even Z .

The calculated isotopic compositions of light fragments ($Z \leq 10$) exhibit the isoscaling properties characterized by an exponent α , which has been determined for different system densities. The model shows a weak dependence of α on the density, and significantly underestimate its experimental value, about by a factor of two, which was also observed in other model studies. According to suggestion given in Ref. [96], the isoscaling was examined for the largest fragments showing a better agreement, however it should be stressed that the experimental α was determined only for the light fragments.

Overall, the isospin-dependent Lattice Gas Model very well reproduces the fragment charge partitions observed in the ALADIN measurements, in particular, the distributions of the largest fragment charge that reveal the presence of a critical behavior. The obtained results indicate that clusters are formed in the dense medium, in accordance with the “Little big bang” scenario of multifragmentation [108].

Appendix - Monte Carlo simulations of the Lattice Gas Model

The Lattice Gas Model model can be numerically solved using the Monte Carlo Metropolis algorithm. This approach achieves a desired probability distribution for a stochastic process through the Monte Carlo Markov Chain (MCMC) technique [114]. If the chain of stochastic events satisfies the Markov property, which means that the transition probability from the current state to any other state depends only on the current state and not on the history, the chain is called a Markov Chain (MC) [115]. To apply an algorithm that gives samples from a specific distribution, an additional rejection algorithm is used to discard samples that do not fit the desired distribution. This rejection algorithm is implemented through the Metropolis algorithm. The Metropolis algorithm operates on each sample, one by one, and each process can be referred to as a “step.” By iterating a large number of steps, the chain of stochastic events converges to the desired distribution in the limit. The method is ergodic i. e. any configuration of the system can be achieved with non-zero probability in a large enough number of trials.

The practical implementation of the Metropolis algorithm in the Lattice Gas Model code is described as follows: We assume canonical distribution. Let us define a parameter for rejection algorithm as a change in potential energy between consecutive steps $\Delta E = E(x_{i+1}) - E(x_i)$. Now we can describe transition probability explicitly:

$$P(x_i \rightarrow x_{i+1}) = \begin{cases} \exp\left(\frac{-\Delta E}{k_B T}\right) & \text{if } \Delta E > 0 \\ 1 & \text{if } \Delta E \leq 0 \end{cases} \quad (14)$$

The application of the Lattice Gas Model have a few input parameters and options. First of all, the lattice size has to be defined - in three dimensions. The second step is to define a number of particles (protons and neutrons in case of two-component LGM) to be placed in the lattice sites.

The model allows for diverse initial nucleon configurations. As depicted in Fig. 66, three noteworthy configurations are illustrated in a 6x6x6 (216 sites) lattice filled with 32 protons and 32 neutrons. The configuration on the left side is the most commonly used, wherein nucleons are randomly distributed throughout the lattice volume with a homogeneous distribution. This configuration is associated with a high potential energy of the system. Configuration in the middle demonstrates the possibility of defining the filling volume. In this case, the volume is limited to a 4x4x4 cube located at the lattice's center. Although the arrangement of nucleons is random, the potential energy is lower than in the previous case. The last config-

uration arranges the protons and neutrons alternately in each dimension, resulting in absolutely minimal potential energy.

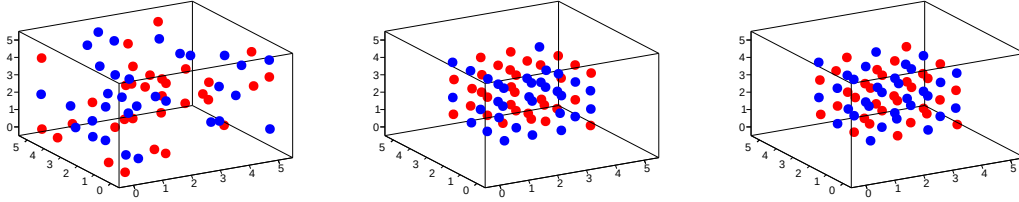


Figure 66: Examples of insertion methods (initial configurations). Left: nucleons spread over the whole lattice space; middle: nucleons create a random pattern in a one, compact cluster; right: nucleons create a compact cluster and are arranged in an alternating pattern.

The starting configuration with randomly distributed nucleons over the whole lattice can be called a hot start, while the configuration providing minimum of potential energy can be assigned as a cold start. It is necessary then to provide a method for thermalization of the system to desired temperature. At this point a questions arise: What is the sufficient “time” to equilibrate? Which starting configuration is better - hot or cold? Before moving to a proper tests, one more parameter has to be explained: the Metropolis step, which is, by default, considered as a move of a particle to adjacent site; in the applied algorithm an alternative was implemented - a global (or long range) step, which means, that the particle can be moved to any site of a lattice. In the following, a proper tests of those options are provided.

The duration of the thermalization process is measured in terms of the number of Metropolis steps. The potential energy of the system is calculated at each step. Two initial configurations have been tested: with the lowest and with the highest potential energy. A system consisting of 32 neutrons and 32 protons was placed in a 6x6x6 lattice. To achieve the lowest potential energy configuration, neutrons and protons were alternately arranged in a 4x4x4 cube positioned in the center of the lattice volume. Conversely, to achieve the highest potential energy, nucleons were randomly distributed throughout the entire lattice volume, using a homogeneous distribution.

Figures [67-69] display changes in potential energy at each Metropolis step in the following sequence: Fig. 67 depicts results of the Metropolis method with a minimum range step. The plots in the left column correspond to system without Coulomb interaction, while those in the right column include Coulomb interac-

tion. The blue curves represent the minimum potential energy configuration at the start, and the green curve represents the configuration near its maximum potential energy at the start. The top row shows system thermalizing to a temperature of 2 MeV. Each subsequent row indicates other (increased by 1 MeV) temperatures to be achieved in the thermalization process. In the last row, the thermalization temperature reaches 8 MeV. Figure 68 presents data generated using global range step, organized in the same manner as in the Fig. 67.

From Figs [67-69] it can be seen, that after reaching equilibrium, energies oscillate over the same level. Also the “time” to equilibrate is similar. It can be concluded, that the thermalization process is independent of the starting configuration.

Figure 69 facilitate a direct comparison between the minimum and global range step algorithms. In this plot, blue curves correspond to the global range step method, while the green curves represent the minimum range step method. The system includes Coulomb interaction. Results show, that the global range step method leads to a faster equilibration. Another conclusion drawn from the Fig. 69 is that, at the lowest temperature (2 MeV) starting from minimum potential energy (top left plot of the figure) the short range step algorithm can stuck at the minimum of the energy. On the other hand, the use of the global range is more likely to destroy the compact configuration with low potential energy typical for nucleus at low temperature by non-physical ejection of nucleons. This observation prompted the choice of a mixture of methods in the calculations following the thermalization process. The “time” to equilibrate set to 10^5 steps seems to be sufficient.

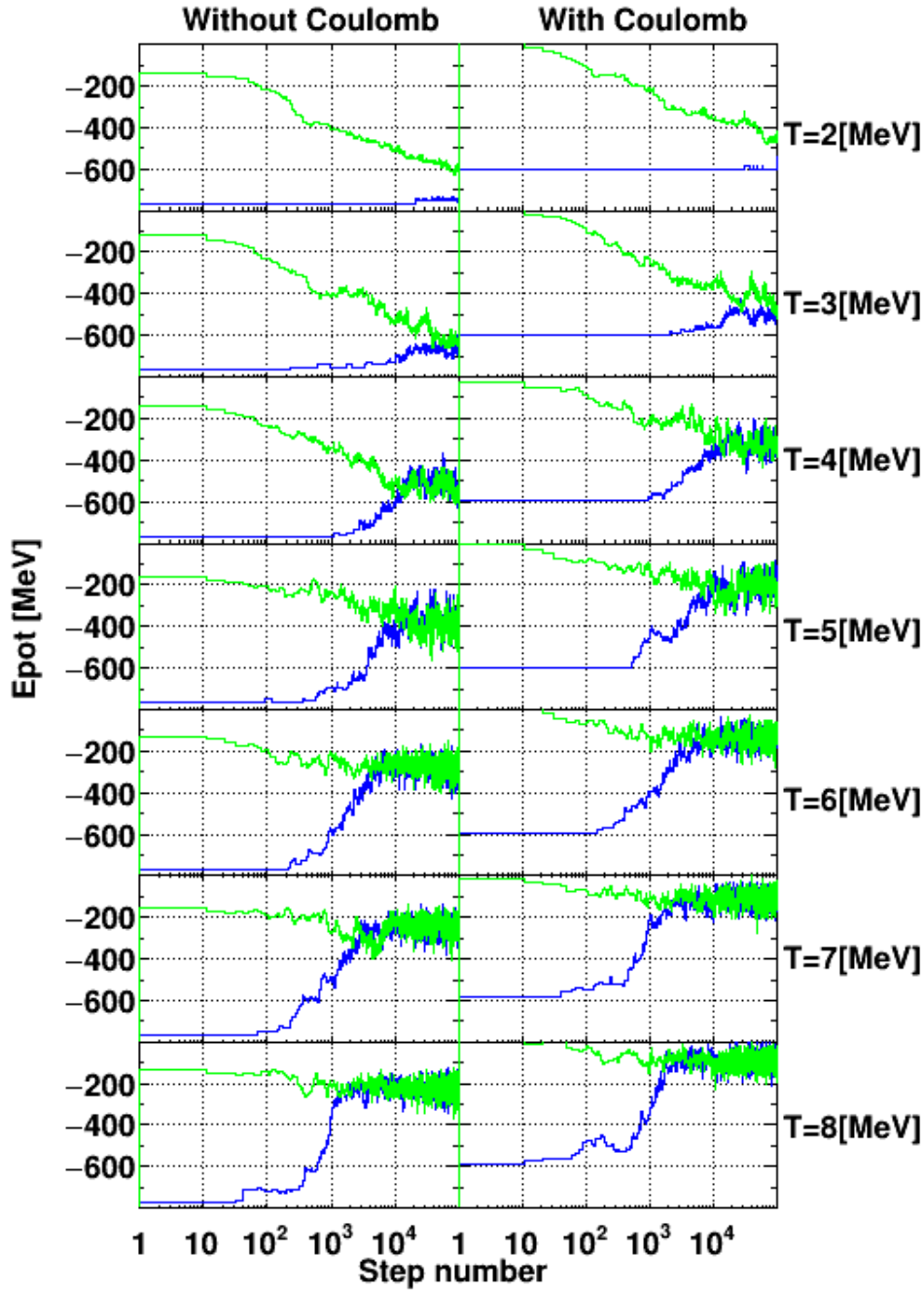


Figure 67: Minimum range step Metropolis method. Process of 100 000 events initialized with: the minimum possible potential energy - blue curve; the potential energy about its maximum - green curve. Right column data include Coulomb interaction energy while left column not. Temperatures applied in the ensemble vary from 2 MeV for the top row to 8 MeV for the bottom row with 1 MeV step.

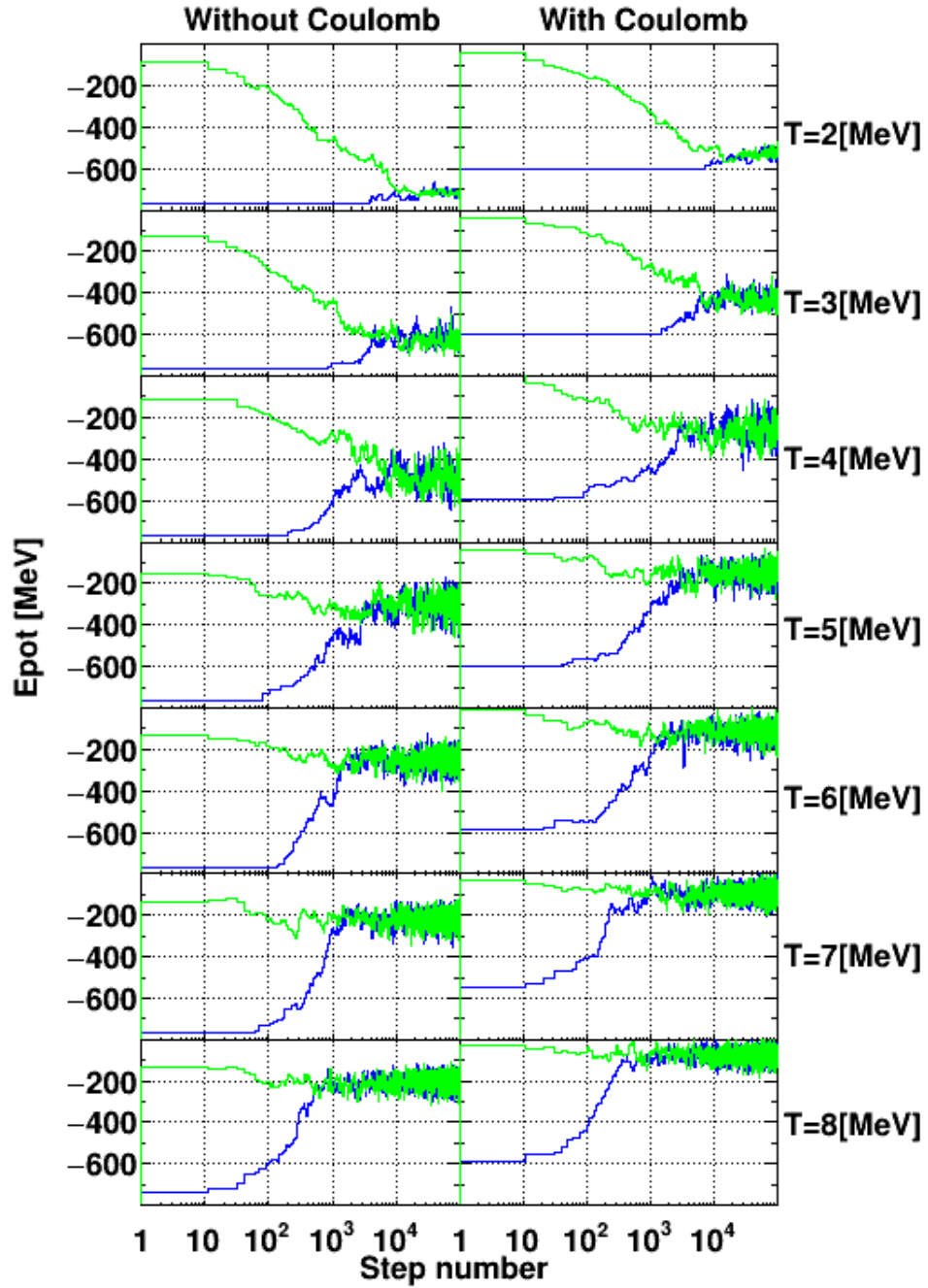


Figure 68: Global range step Metropolis method. Process of 100 000 events initialized with: the minimum possible potential energy - blue curve; the potential energy about its maximum - green curve. Right column data include Coulomb interaction energy while left column not. Temperatures applied in the ensemble vary from 2 MeV for the top row to 8 MeV for the bottom row with 1 MeV step.

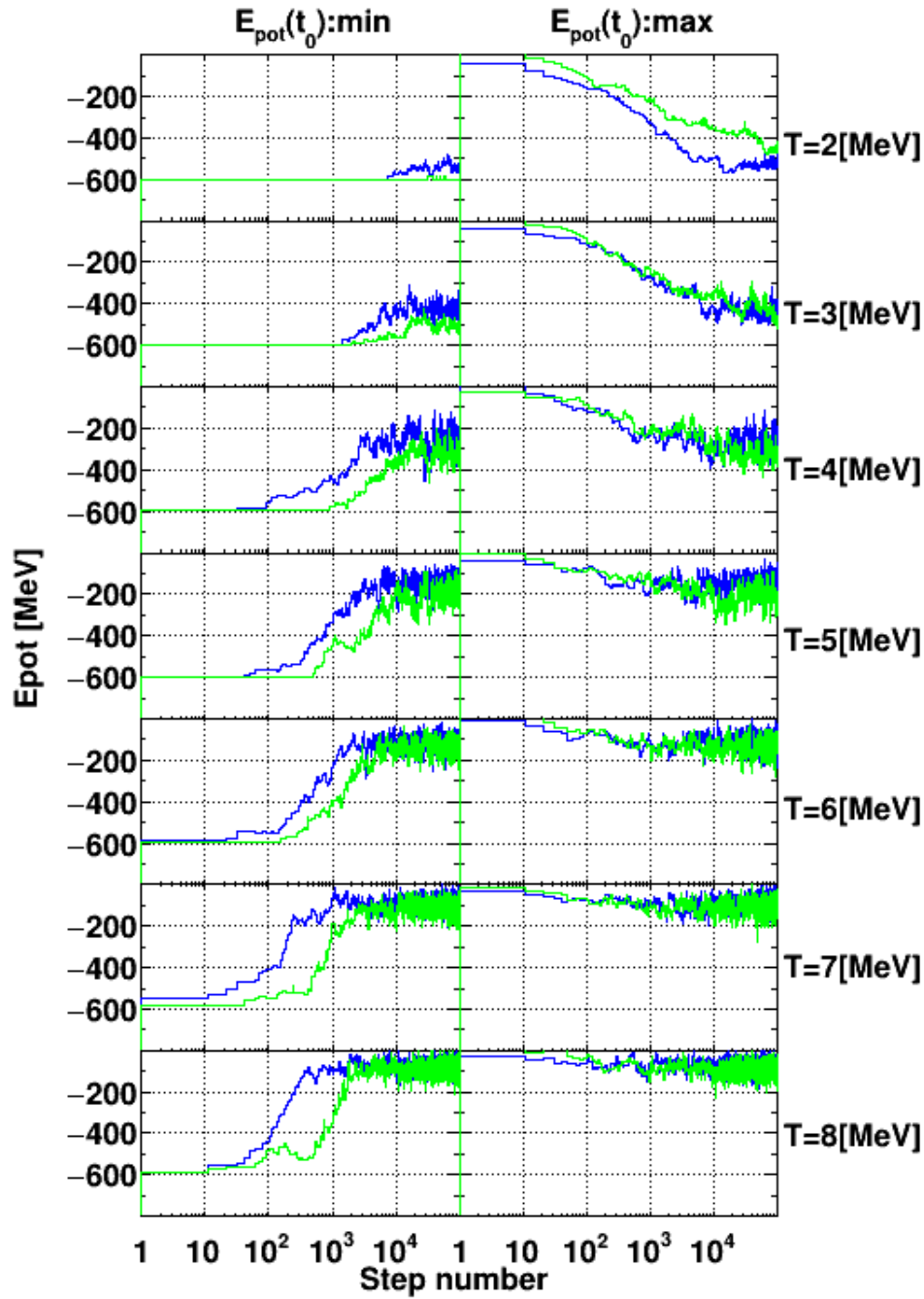


Figure 69: Global range step (blue curve) and minimum range step (green curve) Metropolis method comparison. Process of 100 000 events initialized with: minimum possible potential energy - left column; the potential energy about its maximum - right column. Temperatures applied in the ensemble vary from 2 MeV for the top row to 8 MeV for the bottom row with 1 MeV step. Coulomb interaction included.

The following description provides a comprehensive overview of the algorithm that can serve as a guide for developing a functional application. The algorithm is depicted in Figs 70 and 71, which illustrate the main function and its central component - the Metropolis method.

The practical implementation of the Metropolis algorithm in the LGM code involves setting two parameters, namely, the “number of events” and the “interspace”. The latter parameter denotes the number of Metropolis steps between events to be saved. During the simulation, a counter “i” is used to keep track of the actual number of Metropolis steps. The constant “tog time” is, by default, set to a half of the interspace, meaning that a half of the Metropolis steps are “short-range” and a half are “long-range”.

The program enters the main loop, in which the Metropolis function is called repeatedly. If the counter “i” reaches value set by the “nr of events” multiplied by the “interspace”, the program terminates. Otherwise, the Metropolis function is called with the parameter “tog”, which can take value “0” or “1” for “short-range” or “long-range” moves, respectively. After each Metropolis step, the program checks the condition $i \% \text{tog_time} = 0$. If this condition is satisfied, the variable “tog” toggles its value.

The final criterion determines whether a sufficient number of Metropolis steps have been performed to treat the nucleon configuration as an uncorrelated event. In other words, it checks if the number of Metropolis steps is equal to the value specified by the “interspace” variable. If the condition is satisfied, the program executes the subsequent procedures:

- **ChangeNucMom - Change Nucleons Momenta:** Nucleons momenta are updated by assigning new values drawn from a normal distribution implemented with the Box-Muller transform [116, 117]. The following equation is used to generate a random value for each momentum component of each particle:

$$p_i = \sqrt{mT}R \cos \theta = \sqrt{mT} \sqrt{-2 \ln(U_1)} \cos(2\pi U_2) \quad (15)$$

where m represents the nucleon’s mass, T represents the system’s temperature, and U_1 and U_2 are random numbers drawn from a uniform distribution in range of $(0, 1)$.

- **Break bonds and get clusters -** by calling “GetClusters” method, which can be considered as a “clusterization process”. The procedure examines each site of the lattice and its relationship to its neighboring sites. Each identified cluster is labeled, and the relevant data recorded.

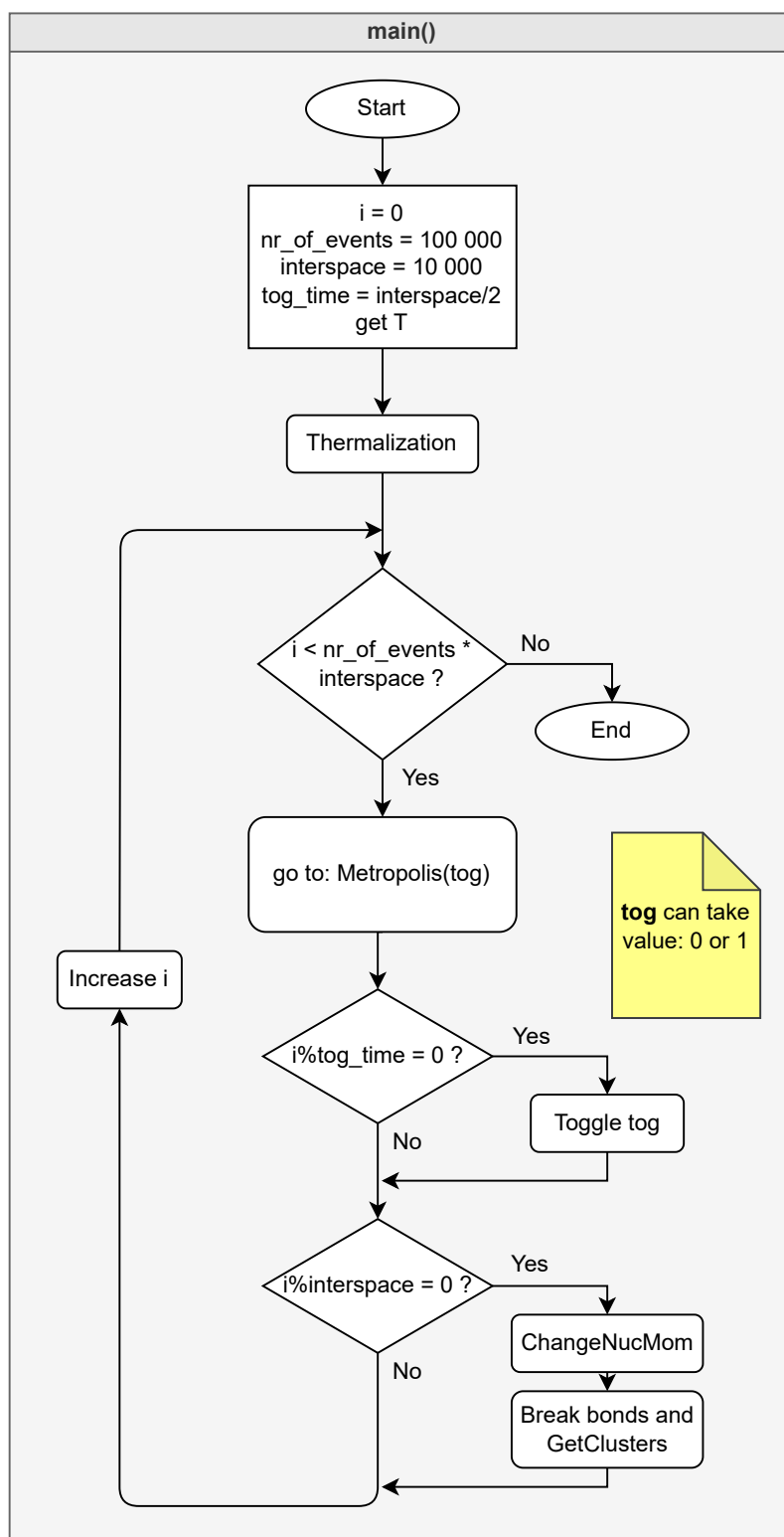


Figure 70: LGM algorithm: the main function diagram

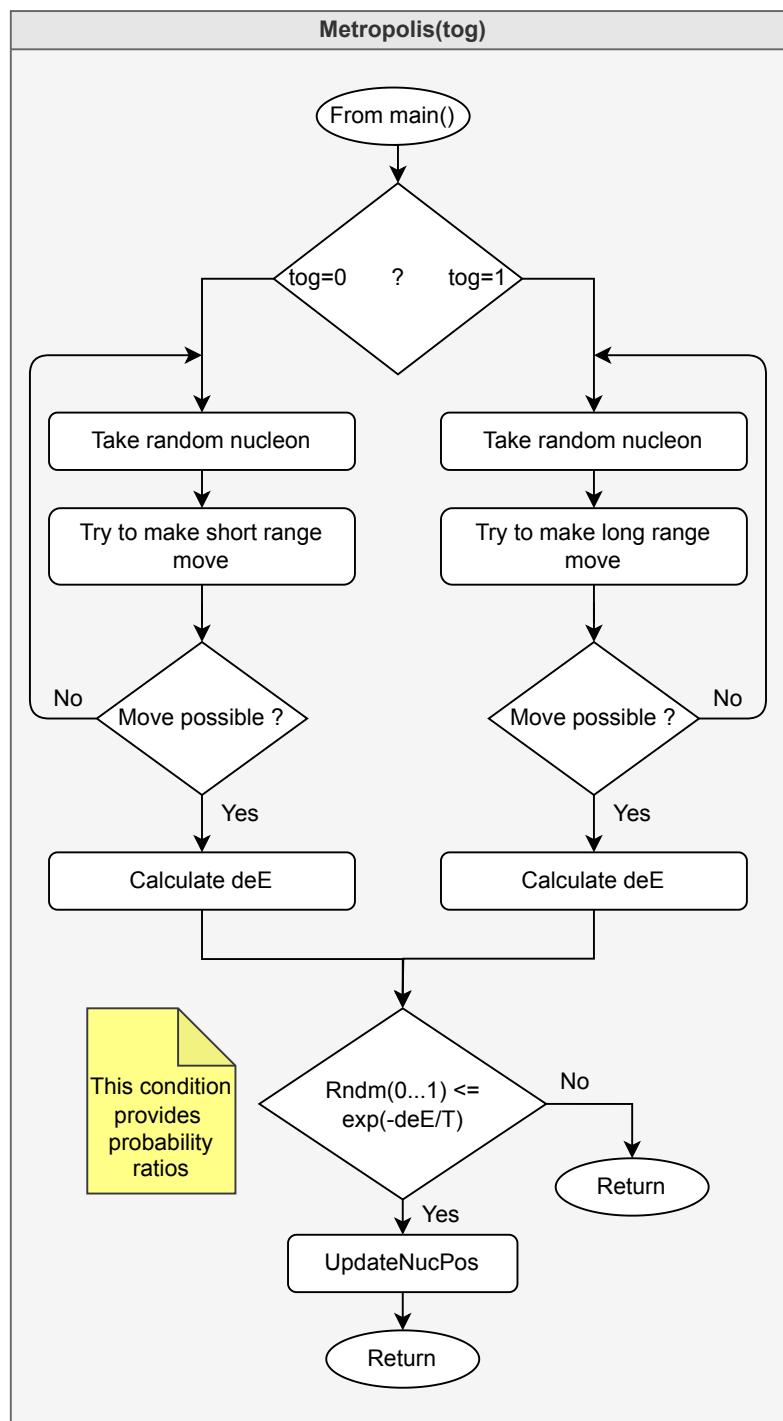


Figure 71: LGM algorithm: the Metropolis function diagram

The nucleon-cluster identification criterion depends on the chosen method: E_{REL} (relative energy) or C-K (Coniglio-Klein). Both methods share the common requirement that a nucleon must be adjacent to at least one nucleon from the cluster, where adjacency is defined as being connected by a side, not a corner. The additional conditions for each method differ: for the E_{REL} nucleon belongs to the neighboring cluster if the following equation is satisfied:

$$E_{\text{kinetic}} + E_{\text{nuclear}} + E_{\text{coulomb}} \leq 0 \quad (16)$$

where E_{kinetic} is the relative energy associated with the nucleon momenta; E_{nuclear} is associated with the nuclear interaction; E_{coulomb} is a Coulomb component.

The relative energy associated with the nucleon momenta can be calculated as follows:

$$E_{\text{kinetic}} = \frac{p_{12}^2}{2m_{\text{red}}} \quad (17)$$

where $p_{12} = \frac{p_1 - p_2}{2}$, thus:

$$E_{\text{kinetic}} = \frac{(p_1 - p_2)^2}{8m_{\text{red}}} \quad (18)$$

The isospin dependence use the following parameters (unless otherwise specified): $\epsilon_{nn} = \epsilon_{pp} = 0$ MeV - for like nucleons; $\epsilon_{pn} = -5.33$ MeV - for unlike nucleons.

The Coulomb energy between neighboring protons is given by: $E_{\text{coulomb}} = I_c/r_0 = 0.8[\text{MeV}]$, where I_c represents the Coulomb energy for two protons at a distance of 1 fm, which is 1.44 MeV and r_0 is a distance between adjacent sites, which is 1.8 fm.

Coniglio and Klein [72] proposed an alternative approach in which the bonds between neighboring particles are defined, incorporating a probability factor that reflects a form of thermal equilibrium [73,118,119]. The probability of an unbroken bond is determined by:

$$p = 1 - \exp\left(\frac{-|\epsilon|}{2T}\right) \quad (19)$$

where T is a temperature and $\epsilon = -5.33$ MeV.

The algorithm for the Metropolis function is depicted in Fig. 71. The function is invoked with the input parameter “tog”. A brief summary of how the algorithm operates for the two possible values of “tog” is presented below.

- $\text{tog}=0 \rightarrow$ short range move
 - 1) Choose a random nucleon. Denote it's position by x_i .
 - 2) Make an attempt step from x_i to the nearest lattice site x_{i+1} . There are six nearest sites around x_i site inside. To check for the availability of sites, first, determine whether there are like nucleons around x_i . If a site is occupied by a like nucleon, it is excluded (the step is ignored, because the resulting configuration is the same as the previous one). Choose randomly a site from available. If the random site is empty, make an attempted step. If the random site is occupied (by an unlike nucleon), exchange nucleons.
 - 3) Calculate the change in potential energy ΔE
 - 4) Use the rejection algorithm. To decide whether the step is accepted or not, generate a random number $r \in [0, 1)$ using a uniform distribution and accept the change if $r < \exp\left(\frac{-\Delta E}{k_B T}\right)$.
 - 5) Save (update) the new configuration of nucleons in the lattice using the function "UpdateNucPos" - Update nucleons positions.
- $\text{tog}=1 \rightarrow$ long-range move

In contrast to the short-range move, the algorithm differs only in step 2, where instead of attempting a step to the nearest lattice site, a step is attempted to any available site in the lattice. Here also "available" refers to sites that are either unoccupied or occupied by an unlike nucleon.

As an example of the model performance, Fig. 72 shows the temperature dependence of the kinetic, potential and total energy distributions for system consisting of 62 protons and 62 neutrons placed in a lattice of size $5 \times 5 \times 5$, thus the density is about the normal nuclear density. The temperature was set to 5 MeV. According to the energy equipartition theorem, the mean kinetic energy is expected to exhibit a linear relationship with the temperature. Furthermore, the mean kinetic energy should remain unaffected by the Coulomb interaction. The model successfully fulfills these requirements, as evidenced in the plot.

The energy equipartition theorem allow to calculate the mean kinetic energy:

$$\langle E_{kin} \rangle = \frac{1}{3} k_B T \quad (20)$$

where k_B is a Boltzman constant.

The Coulomb energy for the considered system can be estimated using the third component of the semi-empirical mass formula [120–122]:

$$E_B(Z, N) = \alpha_1 A - \alpha_2 A^{2/3} - \alpha_3 \frac{Z(Z-1)}{A^{1/3}} - \alpha_4 \frac{(N-Z)^2}{A} \quad (21)$$

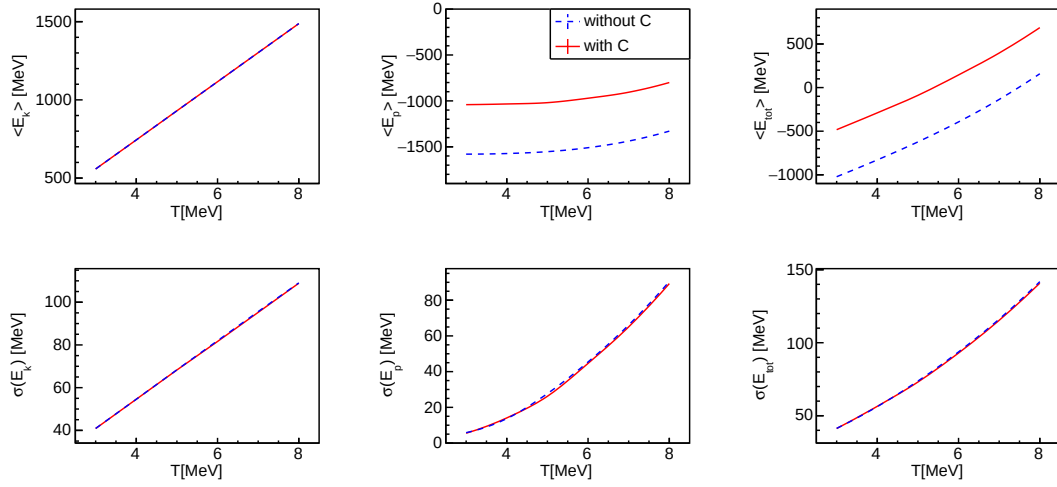


Figure 72: Upper panel: mean energy distribution; lower panel: mean energy distribution standard deviation. From the left: kinetic, potential and total mean energy distribution. Two series of data on each plot: with (red line) and without (blue dashed line) Coulomb interaction.

where α_1 gives a value of binding energy for a nucleon in infinite symmetrical nuclear matter at normal density, which is -16 MeV, the second part of the equation stays for the surface term and α_3 represents the electrostatic Coulomb constant (approximately 0.691 MeV). The last part is connected to the symmetry energy.

The equation yields an estimated energy of approximately 524 MeV. This value is manifested in the plots of mean potential and total energy as a shift between the dashed blue line (representing simulations without Coulomb interaction) and the solid red line (representing simulations with Coulomb interaction).

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