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DOCTORAL THESIS

Neutrino and electron scattering off the lightest nuclei



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Oświadczenie

Ja niżej podpisany ALESSANDRO GRASSI (nr indeksu: 1142426) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „Neutrino and electron scattering off the lightest nuclei” jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr. hab. Jacka Golaka. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawnioną w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia 12 czerwca 2023 r.

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podpis doktoranta

Abstract

In this thesis tools to perform consistent relativistic calculations of unpolarized cross sections and polarization observables in electroweak scattering experiments on deuteron targets are developed. Following investigations of the three-nucleon system (see for example H. Witała *et al.*, Phys. Rev. C **71**, 054001 (2005), H. Witała *et al.*, Phys. Rev. C **77**, 034004 (2008)), the strong interaction dynamics is defined by an explicit dynamical unitary representation of the Poincaré group.

The dynamical Poincaré generators are constructed using a relativistic re-interpretation of the Argonne V18 nucleon-nucleon potential. The dynamical representation is chosen so representations of space translations and rotations are identical in the interacting and non-interacting representations.

The predictions are shown for elastic and breakup electron and neutrino induced reactions and in some cases compared to experimental data. The current framework can be best applied to kinematics, where the internal two-nucleon energy remains below the pion production threshold but the magnitude of the three-momentum transfer extends at least to several GeV.

The constructed framework is incomplete in two ways. First, a pion production channel is not included. The second limitation is the lack of the dynamical two-body contributions to the current operators.

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*“I am not old
I am ancient.”*

Notation and conventions

Throughout this work we adopt natural units, *i.e.*

$$\hbar = c = 1.$$

Moreover, the magnitude of the electron charge e is fixed by requiring (Heaviside-Lorentz units) that the fine-structure constant $\alpha \approx \frac{1}{137}$ can be written as

$$\alpha = \frac{e^2}{4\pi}.$$

Vectors

Three dimensional vectors will be written in bold, their magnitude (where it is not ambiguous) in italic and four-vectors with an upper or lower Greek index. Unit vectors will be written bold and with a hat. As an example

- $\mathbf{p} = (p_x, p_y, p_z)$ is a three-dimensional vector;
- $p^\mu = (p^0, p^1, p^2, p^3) = (p^0, \mathbf{p})$ is a contravariant four-vector;
- $p_\mu = (p_0, p_1, p_2, p_3)$ is a covariant four-vector;
- $p = |\mathbf{p}|$ is the magnitude of the vector $\mathbf{p}$;
- $\hat{\mathbf{p}} = \mathbf{p}/p$ is the unit vector pointing in the direction of $\mathbf{p}$.

Scalar products are expressed as follows

- $\mathbf{a} \cdot \mathbf{b}$ is the scalar product of two three-dimensional vectors;
- $a \cdot b = \sum_\mu a^\mu b_\mu$ is the scalar product of two four-vectors.

Covariant and contravariant four-vectors are connected by the metric tensor $g_{\mu\nu} = g^{\mu\nu}$. In this work the Minkowski metric has been taken with the signature $(+, -, -, -)$ as in Ref. [1]. Therefore, $g^{\mu\nu}$ is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and contravariant (a^μ) and covariant (a_μ) four-vectors are connected by the relation

$$a_\mu = \sum_{\nu} g_{\mu\nu} a^\nu. \quad (1)$$

We adopt the Einstein notation: repeated indices are implicitly summed over, *i.e.*

$$a^\mu b_\mu \equiv \sum_{\mu=0}^3 a^\mu b_\mu$$

and

$$a_i b_i \equiv \sum_{i=1}^3 a_i b_i .$$

The relativistic four-divergence of a four-vector field A^μ reads

$$\partial_\mu A^\mu ,$$

where

$$\partial_\mu = (\partial_t \quad \partial_x \quad \partial_y \quad \partial_z) = (\partial_t \quad \nabla) .$$

Elements of quantum mechanics

Orbital angular momentum in momentum space

In this section we denote explicitly operators with the subscript “op”. The orbital angular operator is the cross product

$$\mathbf{L}_{\text{op}} = \mathbf{r}_{\text{op}} \times \mathbf{p}_{\text{op}} = -\mathbf{p}_{\text{op}} \times \mathbf{r}_{\text{op}} , \quad (2)$$

where $\mathbf{r}_{\text{op}}$ and $\mathbf{p}_{\text{op}}$ are the position and momentum operators, respectively. The definition (2) is valid for a single-particle angular momentum but also for the total and relative two-particle orbital angular momentum, as long as the commutation relations

$$[\mathbf{r}_{\text{op},k}, \mathbf{r}_{\text{op},l}] = 0 , \quad (3)$$

$$[\mathbf{p}_{\text{op},k}, \mathbf{p}_{\text{op},l}] = 0 , \quad (4)$$

and

$$[\mathbf{r}_{\text{op},k}, \mathbf{p}_{\text{op},l}] = i \delta_{kl} \quad (5)$$

hold. Usually, the orbital angular momentum is studied in coordinate space, where [2]

$$\begin{aligned} \mathbf{L}_{\text{op},x} &= -iy \frac{\partial}{\partial z} + iz \frac{\partial}{\partial y} , \\ \mathbf{L}_{\text{op},y} &= -iz \frac{\partial}{\partial x} + ix \frac{\partial}{\partial z} , \\ \mathbf{L}_{\text{op},z} &= -ix \frac{\partial}{\partial y} + iy \frac{\partial}{\partial x} , \end{aligned} \quad (6)$$

but we work in momentum space, where $\mathbf{r}_{\text{op}} = i \nabla_{\mathbf{p}}$, so we use

$$\begin{aligned} \mathbf{L}_{\text{op},x} &= -ip_y \frac{\partial}{\partial p_z} + ip_z \frac{\partial}{\partial p_y} , \\ \mathbf{L}_{\text{op},y} &= -ip_z \frac{\partial}{\partial p_x} + ip_x \frac{\partial}{\partial p_z} , \\ \mathbf{L}_{\text{op},z} &= -ip_x \frac{\partial}{\partial p_y} + ip_y \frac{\partial}{\partial p_x} . \end{aligned} \quad (7)$$

If we employ the spherical coordinates to express the position vector in coordinate space

$$\mathbf{r} = r (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \equiv r \hat{\mathbf{n}}$$

or the spherical variables to write the momentum vector in momentum space,

$$\mathbf{p} = p (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \equiv p \hat{\mathbf{n}},$$

in either space we arrive at *the same* functional form of the $\mathbf{L}_{\text{op}}$ components:

$$L_{\text{op},x} = -i \left(\sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right), \quad (8)$$

$$L_{\text{op},y} = -i \left(\cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right), \quad (9)$$

$$L_{\text{op},z} = -i \frac{\partial}{\partial \varphi}. \quad (10)$$

The spherical harmonics [2] are simultaneous eigenfunctions of the operators $\mathbf{L}_{\text{op}}^2$ and $L_{\text{op},z}$, with eigenvalues

$$\begin{aligned} \mathbf{L}_{\text{op}}^2 Y_{\ell m}(\theta, \varphi) &= \ell(\ell+1) Y_{\ell m}(\theta, \varphi), \\ L_{\text{op},z} Y_{\ell m}(\theta, \varphi) &= m Y_{\ell m}(\theta, \varphi), \end{aligned}$$

where ℓ is a positive integer and $m = -\ell, -\ell+1, \dots, \ell-1, \ell$. The spherical harmonics are orthonormal

$$\int d\hat{\mathbf{n}} Y_{\ell' m'}^*(\hat{\mathbf{n}}) Y_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell' \ell} \delta_{m' m}, \quad (11)$$

where δ_{ij} is the Kronecker delta. In Eq. (11) we introduced notation, which will be often used in this thesis:

$$\int d\hat{\mathbf{n}} \equiv \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta, \quad (12)$$

$$Y_{\ell m}(\hat{\mathbf{n}}) \equiv Y_{\ell m}(\theta, \varphi). \quad (13)$$

Coupling of two angular momenta [2, 3]

Let $|j_1 m_1\rangle$ be eigenstates of the angular momentum operators $\mathbf{J}_1^2$ and $J_{1,z}$. Similarly, let $|j_2 m_2\rangle$ be eigenstates of the $\mathbf{J}_2^2$ and $J_{2,z}$ operators. The product of two states

$$|j_1 m_1 j_2 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle \quad (14)$$

can be expressed in terms of simultaneous eigenstates of the square of the total angular momentum operator $\mathbf{J}^2$ and its z -component J_z , where

$$\mathbf{J} = \mathbf{J}_1 \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{J}_2.$$

Here $\mathbf{I}_i$ is the identity operator in the space spanned by the $|j_i m_i\rangle$ states. To this end the Clebsch-Gordan coefficients $\langle (j_1, j_2) j m | j_1 m_1 j_2 m_2 \rangle$ for the $SU(2)$ group are constructed and the eigenstates of $\mathbf{J}^2$ and J_z are given as

$$|(j_1, j_2) j m\rangle = \sum_{m_1 m_2} \langle (j_1, j_2) j m | j_1 m_1 j_2 m_2 \rangle |j_1 m_1 j_2 m_2\rangle. \quad (15)$$

Vice versa, we can express the states $|j_1 m_1 j_2 m_2\rangle$ as a linear combination of the $|(j_1, j_2) j m\rangle$ states

$$|j_1 m_1 j_2 m_2\rangle = \sum_{m_1 m_2} \langle(j_1, j_2) j m | j_1 m_1 j_2 m_2\rangle |(j_1, j_2) j m\rangle. \quad (16)$$

The Clebsch-Gordan coefficients will be also written in a slightly different form:

$$\langle j_1 m_1 j_2 m_2 | j m \rangle \equiv \langle(j_1, j_2) j m | j_1 m_1 j_2 m_2\rangle. \quad (17)$$

Useful matrix elements

We outline here some matrix elements that are important for the thesis. They belong to the nonrelativistic formulation of quantum mechanics, but the relativistic extensions are difficult to understand without them.

We start with the eigenstates of the position operator $|\mathbf{r}\rangle$, which are normalized to the three-dimensional Dirac delta function

$$\langle \mathbf{r}' | \mathbf{r} \rangle = \delta^3(\mathbf{r}' - \mathbf{r}). \quad (18)$$

The position operator can be the single-particle operator, the two-particle relative position operator or the two-particle center-of-mass position operator.

The overlap of the corresponding conjugate momentum eigenstate $|\mathbf{p}\rangle$ with the position operator eigenstate is chosen as

$$\langle \mathbf{r} | \mathbf{p} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{p} \cdot \mathbf{r}} \quad (19)$$

in order to fix the normalization of the momentum eigenstates in the following way:

$$\langle \mathbf{p}' | \mathbf{p} \rangle = \delta^3(\mathbf{p}' - \mathbf{p}). \quad (20)$$

The overlap (19) can be expanded as

$$\langle \mathbf{r} | \mathbf{p} \rangle = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} j_{\ell}(pr) Y_{\ell m}^*(\hat{\mathbf{p}}) Y_{\ell m}(\hat{\mathbf{r}}), \quad (21)$$

where $j_{\ell}(pr)$ are the spherical Bessel functions $j_{\ell}(z)$ (regular at $z = 0$) and $Y_{\ell m}(\hat{\mathbf{r}})$ and $Y_{\ell m}(\hat{\mathbf{p}})$ are the spherical harmonics. Further, we introduce partial wave states, which are eigenstates of the orbital angular momentum operator squared and its z -component in the coordinate ($|r\ell m\rangle$) and momentum ($|p\ell m\rangle$) space. Accordingly, they carry information about the magnitudes of the $\mathbf{r}$ or $\mathbf{p}$ vector and are defined through the conditions

$$\begin{aligned} \langle \mathbf{p}' | p\ell m \rangle &= \frac{\delta(|\mathbf{p}'| - p)}{p^2} Y_{\ell m}(\hat{\mathbf{p}}'), \\ \langle \mathbf{r}' | r\ell m \rangle &= \frac{\delta(|\mathbf{r}'| - r)}{r^2} Y_{\ell m}(\hat{\mathbf{r}}'), \end{aligned} \quad (22)$$

which lead to

$$\langle r\ell' m' | p\ell m \rangle = \sqrt{\frac{2}{\pi}} i^{\ell} j_{\ell}(pr) \delta_{\ell\ell'} \delta_{mm'}. \quad (23)$$

Equation (23) is used for example when matrix elements of a nucleon-nucleon potential in the momentum space are obtained from the matrix elements prepared in the coordinate space, as is the case for the Argonne V18 potential [4].

In the following we restrict ourselves to the two-nucleon system treated in the momentum space and additionally ignore the difference between the proton and neutron mass. We build the eigenstates of the total two-nucleon spin and isospin

$$|s\sigma\rangle = \sum_{\mu_1\mu_2} \langle 1/2 \mu_1 1/2 \mu_2 | s\sigma \rangle |1/2 \mu_1\rangle |1/2 \mu_2\rangle, \quad (24)$$

$$|T\tau\rangle = \sum_{\tau_1\tau_2} \langle 1/2 \tau_1 1/2 \tau_2 | T\tau \rangle |1/2 \tau_1\rangle |1/2 \tau_2\rangle. \quad (25)$$

Next we couple the partial wave states $|p\ell m\rangle$ with the spin states (24) to obtain the eigenstates of the *total* two-nucleon angular momentum, taking also the isospin degrees of freedom into account

$$|p(\ell, s)j\mu; T\tau\rangle = \sum_{m\sigma} \langle \ell m s\sigma | j\mu \rangle |p\ell m\rangle |s\sigma\rangle |T\tau\rangle. \quad (26)$$

The internal deuteron state with the total angular momentum $j_d = 1$, the total angular momentum projection μ_d and the fixed spin $s_d = 1$ and isospin $T_d = 0$ is then easily represented through the states (26), if p is understood as the magnitude of the *relative* two-nucleon momentum

$$\mathbf{p} = \frac{1}{2} (\mathbf{p}_1 - \mathbf{p}_2). \quad (27)$$

Namely,

$$|\phi_d \mu_d\rangle = \sum_{\ell=\{0,2\}} \int_0^\infty dp p^2 \phi_d^{(\ell)}(p) |p(\ell, s_d)j_d\mu_d; T_d\tau_d\rangle \quad (28)$$

and the deuteron S and D components are given as

$$\phi_d^{(\ell)}(p) = \langle p(\ell, s_d)j_d\mu_d; T_d\tau_d | \phi_d \mu_d \rangle, \quad (29)$$

taking $\ell = 0$ and $\ell = 2$, respectively.

In order to represent the general two-nucleon system the two-nucleon states (26) are supplemented with the eigenstate of the total two-nucleon momentum

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \quad (30)$$

and assume their final form:

$$|p(\ell, s)j\mu; T\tau; \mathbf{P}\rangle. \quad (31)$$

Now we can express the (nonrelativistic) deuteron state with the total momentum $\mathbf{P}_d$ through the single-nucleon states. To this end we use the overlap between the product of individual momentum states and the product of the relative and total momentum:

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \mathbf{p} \mathbf{P} \rangle = \delta^3(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \delta^3\left(\mathbf{p} - \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)\right) \quad (32)$$

as well as the completeness relation

$$\int d\mathbf{p} \int d\mathbf{P} |\mathbf{pP}\rangle \langle \mathbf{pP}| = \mathbf{I}, \quad (33)$$

where $\mathbf{I}$ is the identity operator. In this way we arrive at the crucial matrix element

$$\begin{aligned} \langle \mathbf{p}_1 \mu_1 \tau_1 \mathbf{p}_2 \mu_2 \tau_2 | \phi_d \mu_d \mathbf{P}_d \rangle &= \delta^3(\mathbf{P}_d - \mathbf{p}_1 - \mathbf{p}_2) \langle {}^{1/2} \tau_1 {}^{1/2} \tau_2 | T_d \tau_d \rangle \\ &\times \sum_{\ell=\{0,2\}} \phi_d^{(\ell)}(p) \sum_{m\sigma} \langle \ell m s_d \sigma | j_d \mu_d \rangle \langle {}^{1/2} \mu_1 {}^{1/2} \mu_2 | s_d \sigma \rangle Y_{\ell m}(\hat{\mathbf{p}}). \end{aligned} \quad (34)$$

The two-nucleon propagator in coordinate space

The relative kinetic energy operator for the two-nucleon system in terms of the relative momentum operator $\mathbf{p}_{\text{op}}$ reads

$$H_0 = \frac{1}{M} \mathbf{p}_{\text{op}}^2, \quad (35)$$

where M is the (average) nucleon mass. The matrix element of the free two-nucleon propagator between the eigenstates of the relative position operator $\mathbf{x}'$ and $\mathbf{x}$ are

$$\langle \mathbf{x}' | G_0(E \pm i\varepsilon) | \mathbf{x} \rangle = \langle \mathbf{x}' | \frac{1}{E \pm i\varepsilon - H_0} | \mathbf{x} \rangle, \quad (36)$$

for $E \equiv \frac{1}{M} p_0^2 > 0$, requires considering the limits $\pm i\varepsilon \rightarrow 0$, which give different results. Employing a contour integration method, one gets (see for example Ref. [5])

$$\langle \mathbf{x}' | G_0(E \pm i\varepsilon) | \mathbf{x} \rangle = -\frac{M}{4\pi} \frac{e^{\pm i\sqrt{ME}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \equiv -\frac{M}{4\pi r} e^{\pm ip_0 r}. \quad (37)$$

It is then clear that the $+i\varepsilon$ prescription gives rise to an asymptotic spherical outgoing wave.

Two-dimensional spinors

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obey the following commutation and anticommutation relations

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i \varepsilon_{ijk} \sigma_k, \\ \{\sigma_i, \sigma_j\} &= 2\delta_{ij}, \end{aligned}$$

where ε_{ijk} is the three-dimensional totally antisymmetric (Levi-Civita) symbol with $\varepsilon_{123} = 1$. We build the three-component vector of the Pauli matrices $\boldsymbol{\sigma}$ and consider a general unit vector

$$\hat{\mathbf{s}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta). \quad (38)$$

Let $\chi_{\pm s}$ denote a two-component spinor, which is an eigenvector of the $\hat{\mathbf{s}} \cdot \boldsymbol{\sigma}$ matrix with an eigenvalue $\lambda = \pm 1$:

$$\begin{aligned} (\hat{\mathbf{s}} \cdot \boldsymbol{\sigma}) \chi_s &= \chi_s, \\ (\hat{\mathbf{s}} \cdot \boldsymbol{\sigma}) \chi_{-s} &= -\chi_{-s}. \end{aligned}$$

Then normalized and mutually orthogonal $\chi_{\pm s}$ spinors ($\chi_{s'}^\dagger \chi_s = \delta_{s's}$) can be taken in the following form

$$\chi_s = \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 e^{i\varphi} \end{pmatrix} \quad \text{and} \quad \chi_{-s} = \begin{pmatrix} -\sin \theta/2 \\ \cos \theta/2 e^{i\varphi} \end{pmatrix}. \quad (39)$$

Gamma matrices

Gamma matrices appear in the Dirac equation (40). They have no unique form but satisfy the Clifford algebra given by the anticommutation rule

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4,$$

where I_4 is the four-by-four identity matrix in the spinor space. With our choice of the metric tensor

$$\begin{aligned} \gamma^0 &= (\gamma^0)^\dagger, & \gamma^i &= -(\gamma^i)^\dagger, \\ (\gamma^0)^2 &= I_4, & (\gamma^i)^2 &= -I_4, \\ \gamma^\mu \gamma_\mu &= 4I_4. \end{aligned}$$

We need also another gamma matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

which has the properties

$$(\gamma^5)^2 = I_4, \quad (\gamma^5)^\dagger = \gamma^5$$

and anticommutes with the gamma matrices γ^μ :

$$\{\gamma^5, \gamma^\mu\} = 0.$$

The Dirac sigma matrix is defined as

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

Trace theorems utilized in evaluation of leptonic and nucleonic matrix elements include:

$$\begin{aligned} \text{Tr} [\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu}, \\ \text{Tr} [\gamma^5] &= \text{Tr} [\gamma^\mu \gamma^\nu \gamma^5] = 0, \\ \text{Tr} [\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}), \\ \text{Tr} [\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] &= -4i\varepsilon^{\mu\nu\rho\sigma}, \end{aligned}$$

where $\varepsilon^{\mu\nu\sigma\rho}$ is the four-dimensional Levi-Civita symbol. We use the convention [6]

$$\varepsilon^{0123} = 1.$$

The trace of the product of any odd number of gamma matrices is zero.

Dirac representation In this work we adopt the Dirac representation of gamma matrices:

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

where I_2 is the two-by-two identity matrix, σ_i are the Pauli matrices and the index i runs through 1, 2 and 3.

In this representation

$$\gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

Dirac equation

The Dirac equation for a free particle with mass m reads

$$(i\cancel{\partial} - m)\Psi(x) = 0, \quad (40)$$

where the slash notation $\cancel{\partial}$ means

$$\cancel{\partial} \equiv \gamma^\mu \partial_\mu.$$

We restrict ourselves to plane-wave solutions, so we write $\Psi(x)$ as

$$\Psi(x) = u(\mathbf{p}, s) e^{-ip \cdot x}$$

and

$$\Psi(x) = v(\mathbf{p}, s) e^{ip \cdot x},$$

where $p^0 \equiv E = \sqrt{m^2 + \mathbf{p}^2}$. Then from Eq. (40) we arrive at

$$(\cancel{\partial} - m)u(\mathbf{p}, s) = 0$$

and

$$(\cancel{\partial} + m)v(\mathbf{p}, s) = 0.$$

The Dirac spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ can be written explicitly as

$$u(\mathbf{p}, s) = \sqrt{E + m} \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E + m} \chi_s \\ \chi_s \end{pmatrix},$$

$$v(\mathbf{p}, s) = \sqrt{E + m} \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E + m} \chi_s \\ \chi_s \end{pmatrix}, \quad (41)$$

with the two-component spinors χ_s defined in Eqs.(39). Introducing the adjoint spinors $\bar{u}(\mathbf{p}, s)$ and $\bar{v}(\mathbf{p}, s)$ defined as

$$\bar{u}(\mathbf{p}, s) = u^\dagger(\mathbf{p}, s) \gamma^0 \quad \text{and} \quad \bar{v}(\mathbf{p}, s) = v^\dagger(\mathbf{p}, s) \gamma^0$$

we obtain the following normalization conditions

$$\begin{aligned}\bar{u}(\mathbf{p}, s)u(\mathbf{p}, s) &= 2m, \\ u^\dagger(\mathbf{p}, s)u(\mathbf{p}, s) &= 2E \\ \bar{v}(\mathbf{p}, s)v(\mathbf{p}, s) &= -2m, \\ v^\dagger(\mathbf{p}, s)v(\mathbf{p}, s) &= 2E.\end{aligned}\tag{42}$$

Some other useful relations include

$$u(\mathbf{p}, s)\bar{u}(\mathbf{p}, s) = (\not{p} + m)\frac{1 + \gamma^5 \not{s}}{2},\tag{43}$$

$$v(\mathbf{p}, s)\bar{v}(\mathbf{p}, s) = (\not{p} - m)\frac{1 - \gamma^5 \not{s}}{2},\tag{44}$$

where the four-vector s^μ is constructed from the unit vector $\hat{\mathbf{s}}$ given in Eq. (38) and the momentum $\mathbf{p}$:

$$s^\mu = \left(\frac{\mathbf{p} \cdot \hat{\mathbf{s}}}{m}, \hat{\mathbf{s}} + \frac{(\mathbf{p} \cdot \hat{\mathbf{s}})\mathbf{p}}{m(E + m)} \right)\tag{45}$$

and results from the Lorentz transformation of $\hat{\mathbf{s}}$ from the particle rest frame to the frame where the particle momentum is $\mathbf{p}$ and total energy E . For the unpolarized case we sum over the two opposite unit vectors $\hat{\mathbf{s}}$ to obtain

$$\sum_s u(\mathbf{p}, s)\bar{u}(\mathbf{p}, s) = (\not{p} + m),\tag{46}$$

$$\sum_s v(\mathbf{p}, s)\bar{v}(\mathbf{p}, s) = (\not{p} - m).\tag{47}$$

The Dirac spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ defined in Eq. (41) are eigenvectors of the spin projection operator $P(s) \equiv \frac{1 + \gamma^5 \not{s}}{2}$:

$$\begin{aligned}P(s)u(\mathbf{p}, s) &= u(\mathbf{p}, s), \\ P(-s)u(\mathbf{p}, s) &= 0, \\ P(s)v(\mathbf{p}, -s) &= v(\mathbf{p}, -s), \\ P(-s)v(-\mathbf{p}, s) &= 0,\end{aligned}$$

so the particle's spin direction is defined in its rest frame. We will need also Dirac spinors which are eigenstates of the helicity operator – the (doubled) projection of the spin onto the direction of the momentum $\mathbf{p}$:

$$\hat{\mathbf{p}} \cdot \boldsymbol{\Sigma} \equiv \hat{\mathbf{p}} \cdot \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Such spinors will be denoted by $u(\mathbf{p}, h)$ and $v(\mathbf{p}, h)$; their functional form reads:

$$\begin{aligned}u(\mathbf{p}, h) &= \sqrt{E + m} \begin{pmatrix} \chi_h \\ \frac{h p}{E + m} \chi_h \end{pmatrix}, \\ v(\mathbf{p}, h) &= \sqrt{E + m} \begin{pmatrix} \frac{h p}{E + m} \chi_h \\ \chi_h \end{pmatrix},\end{aligned}\tag{48}$$

where for $h = \pm 1$ two-component spinors χ_h fulfill $(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \chi_h = h \chi_h$. They can be taken in the following form:

$$\chi_h = \begin{pmatrix} \frac{p_z + hp}{p_x + ip_y} \frac{\sqrt{p - hp_z}}{\sqrt{2p}} \\ \frac{\sqrt{p - hp_z}}{\sqrt{2p}} \end{pmatrix} \quad \text{for} \quad p_x^2 + p_y^2 > 0 \quad (49)$$

and

$$\chi_h = \delta_{h,-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \delta_{h,1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for} \quad p_x^2 + p_y^2 = 0 \quad \text{and} \quad p_z \neq 0. \quad (50)$$

Then it is easy to check that indeed

$$\hat{\mathbf{p}} \cdot \boldsymbol{\Sigma} u(\mathbf{p}, h) = h u(\mathbf{p}, h) \quad \text{and} \quad \hat{\mathbf{p}} \cdot \boldsymbol{\Sigma} v(\mathbf{p}, h) = h v(\mathbf{p}, h). \quad (51)$$

Finally, we construct the helicity projection operator $\tilde{P}(h)$. We define first

$$\tilde{s}^\mu(h) = \frac{h}{m} \left(p, \frac{E}{p} \mathbf{p} \right) \quad (52)$$

and construct $\tilde{P}(h)$ in the same way as previously $P(s)$:

$$\tilde{P}(h) = \frac{1 + \gamma^5 \not{\tilde{s}}}{2}. \quad (53)$$

Then it holds in particular:

$$\begin{aligned} \tilde{P}(h) u(\mathbf{p}, h) &= u(\mathbf{p}, h), \\ \tilde{P}(h) u(\mathbf{p}, -h) &= 0, \\ \tilde{P}(h) v(\mathbf{p}, -h) &= v(\mathbf{p}, -h), \\ \tilde{P}(h) v(\mathbf{p}, h) &= 0. \end{aligned}$$

Finally we note that for particles whose mass is much smaller than their total energy

$$m \tilde{s}^\mu(h) \approx h p^\mu. \quad (54)$$

General expression for the cross section

The general formula for the differential cross section in the case when two particles (A and B) collide and produce N particles ($k = 1, 2, \dots, N$) in the final state can be written as a product of three Lorentz invariant factors and reads [1]:

$$d\sigma = (2\pi)^4 \frac{1}{4I} |M_{fi}|^2 d\Phi^{(N)}, \quad (55)$$

where I is the flux factor, $|M_{fi}|^2$ is the square of the transition matrix element of the process and $d\Phi^{(N)}$ is the N -body invariant phase space. The latter two quantities are defined as

$$I = \sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2} \quad (56)$$

and

$$d\Phi^{(N)} = \delta^4(P_i^\mu - P_f^\mu) \prod_k \frac{d\mathbf{p}_k}{(2\pi)^3 2p_k^0}. \quad (57)$$

Here P_i^μ (P_f^μ) is the initial (final) total four-momentum of the system and $p_k^0 = E_{\mathbf{p}_k} = \sqrt{m_k^2 + p_k^2}$.
If the beam particles (A) are (anti)neutrinos or relativistic electrons, then

$$I \simeq p_A \cdot p_B. \quad (58)$$

Transition matrix elements M_{fi} for (anti)neutrino reactions and electron reactions with nucleons are calculated from the so-called Feynman rules and are analytically known. Transition matrix elements for the reactions with atomic nuclei are central quantities of the thesis and will be discussed in next chapters.

Chapter 1

Introduction

Neutrino scattering on nuclei has been investigated for many years. Information about early theoretical treatments of neutrino-nucleus reactions can be found in Ref. [7]. At the beginning of the century the establishment of the Sudbury Neutrino Observatory (SNO) triggered theoretical calculations needed to analyze emerging experimental data. Nakamura *et al.* [8, 9] calculated cross sections for both charged-current and neutral-current induced reactions, for incoming neutrino energies up to 150 MeV. The predictions in Ref. [8] and most of the results published in Ref. [9] were obtained using the Argonne V18 (AV18) nucleon-nucleon (NN) force [4] and augmenting the single-nucleon current operator with two-nucleon (2N) current operators linked to this potential. Some calculations in Ref. [9] were performed with the CD-Bonn NN potential [10] and used additionally input from chiral effective field theory (χ EFT) in order to estimate theoretical uncertainties of the results, which were later used to analyze the SNO data [11, 12].

The calculations by Shen *et al.* [13] also employed the AV18 potential and the corresponding nuclear weak single-nucleon and 2N current operators. In Ref. [13] inclusive neutrino scattering on the deuteron was studied up to neutrino energies of 1 GeV with coordinate space methods. The authors introduced some adjustments in the 2N current operator used by Nakamura *et al.*, but these modifications proved to be of minor importance and the results obtained by Shen *et al.* fully confirmed those of Nakamura *et al.* in the energy range up to 150 MeV. More importantly, Ref. [13] provided insights into the role of 2N currents and final state interactions for the whole considered neutrino energy range, even though pion production channels were fully ignored.

More recent calculations are often attempted within χ EFT. The pursuit of a complete theoretical framework offering consistent “chiral” 2N and many-nucleon forces as well as electroweak current operators has a long history. Already in 2001 Butler *et al.* [14] studied the neutrino-deuteron breakup reactions at next-to-next-to-leading order (N²LO) in pionless χ EFT, in the energy range up to 20 MeV. Their work agreed very well with the previous potential model calculations from Refs. [8, 9].

A construction of the chiral NN potential was pioneered by Weinberg [15, 16] more than thirty years ago and continued by several groups. Epelbaum *et al.* have constructed three generations of chiral potentials. In their first version the non-local regularization in momentum space was used [17, 18, 19]. The

next generation of the chiral NN potential by Epelbaum *et al.* with low-energy constants taken directly from pion-nucleon scattering employed a coordinate space regularization [20, 21]. These forces were used to study nucleon-deuteron scattering [22] and various electroweak processes in 2N and 3N systems [23]. The newest version of the Bochum-Bonn chiral NN potential was constructed up to fifth order in the chiral expansion (N4LO) [24]. This version returned to regularization in momentum space and contained further improvements, which led to a very good description of the NN scattering data from the self-consistent Granada-2013 database [25]. The Bochum-Bonn group has also been working on the chiral electromagnetic [26, 27] and weak (axial) [28] current operators, but consistent calculations with the Bochum-Bonn potentials and current operators are not yet possible, since the work on the regularization of the current operators is not completed.

The Moscow(Idaho)-Salamanca group derived another family of non-local chiral NN potentials [29, 30]. The most recent version of this potential, generated also up to fifth order in the chiral expansion was published in Ref. [31]. Many modern calculations of electromagnetic reactions employ these chiral potentials from Refs. [29, 30] and involve chiral current operators. The construction of the chiral currents was started by Park *et al.* [32]. The results of Refs. [32, 33] were later re-analyzed and augmented by many groups [34, 35, 36, 37]. Also the problem of unknown parameters of the effective theory was solved in various ways [38, 39, 40]. The current operators were used initially together with the wave functions calculated with the older potentials and later eventually with the potentials constructed by the Moscow(Idaho)-Salamanca group. Various groups studied processes relevant to astrophysics [41, 42], muon capture reactions [43, 44, 45, 46] and neutrino induced processes [47]. The results of Ref. [47] for inclusive neutrino scattering off the deuteron were obtained within the χ EFT approach. The predictions for the cross sections turned out to be very close to the ones obtained in the conventional formulation [9, 13]. This means that in the low-energy region the theoretical cross sections for neutrino-deuteron scattering have very small uncertainties.

Neutrino induced reactions with heavier than $A = 2$ nuclei have been studied theoretically and experimentally actively for many decades. Recent preprints available at the arXiv (<https://arxiv.org/>) comprise for example [48, 49, 50] and nearly every day brings a publication concerning neutrino physics. From the point of view of the calculations in the few-nucleon sector the work by Gazit *et al.* should be mentioned, as it reports calculations for neutrino induced breakup reactions with the ^3H , ^3He and ^4He nuclei [51, 52, 53]. Weak inclusive response functions of heavier light nuclei, including ^{12}C , were obtained with the Green's function Monte Carlo method [54, 55]. These calculations were important for the determination of the nucleon isovector axial form factor [56].

The momentum-space approach offers an independent possibility to perform calculations for the neutrino reactions with the ^2H , ^3H and ^3He nuclei. In Refs. [57, 58] cross sections and closely related nuclear response functions for several such reactions were calculated within the strictly nonrelativistic approach, using the AV18 NN potential. Moreover, the nuclear current operator contained only the single-nucleon part. Nevertheless, for low neutrino energies the momentum-space framework predictions agreed very well with the corresponding results by Shen *et al.* [13]. The framework of Refs. [57, 58] can be used with an input generated by χ EFT.

The momentum space calculations have indeed several advantages. The results obtained within χ EFT are usually provided in momentum space and can be easily used in momentum-space calculations. In the so-called “three-dimensional” calculations, carried out without any resort to partial wave representation of nuclear states and operators [59, 60] χ EFT potentials and current operators could be used directly, avoiding convergence problems inherently bound with partial wave decomposition. Momentum space framework allows one to systematically account for relativistic effects not only in the kinematics but also in the reaction dynamics. Such problems are very difficult in coordinate space, but are easier to solve in momentum space. A good example is the argument of the nucleon form factors in the single-nucleon current, which should be actually, taken as the four-momentum transfer *to* the nucleon squared but is usually replaced by the four-momentum transfer *to* the whole nuclear system squared. In momentum space the proper values of the form factor arguments pose no problems.

Relativity plays definitely an important role for higher neutrino energies and its approximate treatment increases the total theoretical uncertainty of calculations. Theoretical calculations should take into account complementary roles of kinematic and various dynamical contributions to the Poincaré invariant formulation of reaction theory. Important examples of such studies are given in Refs. [61, 62, 63, 64, 65, 66, 67, 68, 69]. These investigations proved that relativistic effects in kinematics and dynamics must be taken consistently and prompted us to construct a relativistic model to perform calculations of unpolarized cross sections and various polarization observables in electron and neutrino scattering experiments with deuteron targets [70]. In this model the strong interaction dynamics is defined by an explicit dynamical unitary representation of the Poincaré group. The AV18 NN potential is used to construct a relativistic nucleon-nucleon interaction. This choice was made deliberately in order to compare the present relativistic results with the former ones calculated in [57, 58] with the same NN force. Our formalism has important limitations: it does not include the pion-production channels and is restricted to the single-nucleon contributions in the electromagnetic as well as in the weak nuclear current operator. It is best applicable to reactions, where the internal two-nucleon energy remains below the pion-production threshold, but the magnitude of the three-momentum transfer reaches several GeV. Moreover, we neglect the radiative corrections in neutrino and electron scattering processes, as also Shen *et al.* do. The reader interested in how these corrections affect neutrino scattering at energies around 1 GeV can find more information in Ref. [71]. Another interesting study of this kind of corrections can be found in [72], where radiative corrections to low-energy charged-current processes involving nucleons, such as neutron beta decay and (anti)neutrino-nucleon scattering, are studied.

Finally, why electron and neutrino scattering are studied together? The answer can be found for example in the abstract of a very recent arXiv preprint [73]:

“A thorough understanding of neutrino-nucleus scattering physics is crucial for the successful execution of the entire US neutrino physics program. Neutrino-nucleus interaction constitutes one of the biggest systematic uncertainties in neutrino experiments - both at intermediate energies affecting long-baseline Deep Underground Neutrino

Experiment (DUNE), as well as at low energies affecting coherent scattering neutrino program - and could well be the difference between achieving or missing discovery level precision. To this end, electron-nucleus scattering experiments provide vital information to test, assess and validate different nuclear models and event generators intended to be used in neutrino experiments. In this white paper, we highlight connections between electron- and neutrino-nucleus scattering physics at energies ranging from 10s of MeV to a few GeV, review the status of ongoing and planned electron scattering experiments, identify gaps, and layout a path forward that benefits the neutrino community. We also highlight the systemic challenges with respect to the divide between the nuclear and high-energy physics communities and funding that presents additional hurdle in mobilizing these connections to the benefit of neutrino programs.”

The thesis is organized in the following way. In Chapter 2 we discuss the most general form of the single-nucleon weak and electromagnetic current operators. In Chapters 3 and 4 we take a step back and consider reactions involving only single nucleons. We invoke the properties of two-body kinematics and cross sections. We also derive the general forms of the cross sections in the laboratory frame for elastic electron-nucleon scattering as well as for quasielastic and elastic (anti)neutrino-nucleon scattering under the one photon or (heavy) boson exchange approximation, respectively. These forms are given in terms of purely kinematic v -functions and the response functions, which contain essential dynamical information. We will use them also for reactions with nuclei. On the other hand, the cross sections for the reactions with nucleons can be expressed independently in terms of invariant quantities (known analytically) so we can test various approximations in kinematics and in the single-nucleon current. Chapter 5 introduces the nonrelativistic description of the deuteron and two-nucleon scattering states in momentum space. Chapter 6 presents kinematics of the elastic and breakup reactions and demonstrates the limitations of the nonrelativistic formulas.

Chapters 7 and 8 are most important. In Chapter 7 the structure of the relativistic theory, in particular the construction of the dynamical unitary representation of the Poincaré group, is explained.

Results of numerical calculations for unpolarized cross sections and spin dependent observables are shown in Chapter 8 .

An excursion into reactions with three-nucleon systems is given in Chapter 9. Here some older results of the nonrelativistic calculations of the inclusive cross sections and the so-called response functions are recalled. The summary and conclusions follow in Chapter 10. The thesis contains additionally introductory material and several appendices, which explain the notation and collect formulas frequently used in the main text and in the calculations. In particular, the reader can find here examples concerning the Wigner spin rotations. Finally, a list of references is provided.

Chapter 2

Consequences of the electroweak theory for the single-nucleon weak and electromagnetic current operators

The electroweak interaction describes the electromagnetic and the weak interactions in an unified picture [74]. At low energies electromagnetism and the weak interactions appear to behave independently, but at energies of the order of hundreds of GeV they unify into a single interaction.

Mathematically speaking the electroweak theory is described by a Yang-Mills field with an $SU(2) \times U(1)$ symmetry group. The generators of the symmetry are called weak isospin and weak hypercharge and give rise to the interaction bosons of the theory: W_1 , W_2 and W_3 for the weak isospin and B for the weak hypercharge. These bosons are still not physical ones and are massless. The Higgs mechanism gives rise to the spontaneous symmetry breaking, giving masses to some of the physical bosons and generating the Z^0 and γ bosons via a superposition of W_3 and B . This superposition can be expressed as

$$\begin{pmatrix} \gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ -\sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} B \\ W_3 \end{pmatrix}, \quad (2.1)$$

where θ_W is the Weinberg angle, also known as the weak mixing angle.

This mixture generates a difference between the massive bosons of the theory, written as

$$M_Z = \frac{M_W}{\cos \theta_W}, \quad (2.2)$$

where M_W is the mass of the two charged bosons $W^\pm$, which are defined as

$$W^\pm = \frac{1}{\sqrt{2}}(W_1 \mp iW_2). \quad (2.3)$$

2.1 Weak single-nucleon current

The most general form of the weak nucleon current operator in the $V - A$ (vector minus axial vector) theory for weak interactions is [75]:

$$J_W^\mu = \bar{u}_{\mathbf{p}'} \left(\gamma^\mu F_1 + \frac{i}{2M} \sigma^{\mu\nu} q_\nu F_2 + \gamma^\mu \gamma^5 F_A + \frac{q^\mu}{M} \gamma^5 F_P \right) u_{\mathbf{p}}, \quad (2.4)$$

where $u_{\mathbf{p}}$ ($\bar{u}_{\mathbf{p}'}$) is the initial (final) nucleon Dirac spinor and the $F_i = F_i(Q^2)$ functions are the weak nucleon form factors. $F_{1,2}$ are the vector form factors, F_A is the axial form factor and F_P is the pseudoscalar form factor. Further, M is the average nucleon mass, p^μ (p'^μ) is the initial (final) nucleon four-momentum, $q^\mu = p'^\mu - p^\mu$ is the transferred four-momentum and

$$Q^2 = -q^\mu q_\mu > 0. \quad (2.5)$$

Using the Dirac equation and assuming that the initial and final nucleon masses are the same, the so-called *Gordon decomposition* leads to a simpler form of Eq. (2.4):

$$J_W^\mu = \bar{u}_{\mathbf{p}'} \left(\gamma^\mu (F_1 + F_2) - (p + p')^\mu \frac{F_2}{2M} + \gamma^\mu \gamma^5 F_A + \frac{q^\mu}{M} \gamma^5 F_P \right) u_{\mathbf{p}}. \quad (2.6)$$

Due to the time reversal symmetry all the form factors are real. The same *functional* form holds both for the charged current (CC) and neutral current (NC) nucleon currents, but the sets of form factors must be considered separately.

The CC reactions involve F_A and F_P essentially by construction. The other two CC nucleon form factors are simply related to the proton (F_i^p) and neutron (F_i^n) electromagnetic form factors:

$$F_i = F_i^V(Q^2) = F_i^p(Q^2) - F_i^n(Q^2), \quad \text{where } (i = 1, 2). \quad (2.7)$$

Very often the Sachs electromagnetic nucleon form factors, $G_E^{p,n}$ and $G_M^{p,n}$, are introduced as linear combinations of $F_1^{p,n}$ and $F_2^{p,n}$:

$$F_1^{p,n} = \frac{G_E^{p,n} + \tau G_M^{p,n}}{1 + \tau}, \quad (2.8)$$

$$F_2^{p,n} = \frac{-G_E^{p,n} + G_M^{p,n}}{1 + \tau}, \quad (2.9)$$

where

$$\tau = \frac{Q^2}{4M^2}. \quad (2.10)$$

Equation (2.6) has to be multiplied by the corresponding isospin operator which ensures a nucleon transition from a proton to neutron or vice versa.

The weak NC nucleon form factors have to be considered separately for the proton and neutron (just like for the electromagnetic reactions) and the previously defined CC form factors as well as the Weinberg angle θ_W appear in the construction of the NC form factors. For the proton we replace in Eq. (2.6)

$$F_i \rightarrow \tilde{F}_i^p(Q^2) = \frac{1}{2} \left((1 - 4 \sin^2 \theta_W) F_i^p - F_i^n \right), \quad \text{where } (i = 1, 2), \quad (2.11)$$

$$F_{A,P} \rightarrow \tilde{F}_{A,P}^p = \frac{1}{2} F_{A,P}, \quad (2.12)$$

while for the neutron we take

$$F_i \rightarrow \tilde{F}_i^n(Q^2) = \frac{1}{2} \left((1 - 4 \sin^2 \theta_W) F_i^n - F_i^p \right), \quad \text{where } (i = 1, 2), \quad (2.13)$$

and

$$F_{A,P} \rightarrow \tilde{F}_{A,P}^n = -\frac{1}{2} F_{A,P}. \quad (2.14)$$

2.2 Electromagnetic single-nucleon current

As in the case of the weak single-nucleon current, it is possible to write a general form for the electromagnetic single-nucleon current. It is expressed using the Dirac $F_1^{p,n}$ and Pauli $F_2^{p,n}$ form factors, which have to be taken separately for the proton and neutron:

$$J_{em}^{\mu(p,n)} = \bar{u}_{\mathbf{p}'} \left(\gamma^\mu F_1^{p,n} + \frac{i}{2M} \sigma^{\mu\nu} q_\nu F_2^{p,n} \right) u_{\mathbf{p}}. \quad (2.15)$$

This form corresponds to the vector part of the weak nucleon current. With the Gordon decomposition it can be again simplified as

$$J_{em}^{\mu(p,n)} = \bar{u}_{\mathbf{p}'} \left(\gamma^\mu (F_1^{p,n} + F_2^{p,n}) - (p'^\mu + p^\mu) \frac{F_2^{p,n}}{2M} \right) u_{\mathbf{p}}. \quad (2.16)$$

The Dirac and Pauli form factors account for the internal structure of nucleons and are related to their charge and anomalous magnetic moment distributions, respectively [1].

2.3 Form factors parametrizations

The form factors can be parameterized in different ways; in this thesis we adopted four different choices [76, 77, 78, 13]. A comparison of all the electromagnetic form factor parametrizations is shown in Figs. 2.1, 2.2, 2.3 and 2.4. From these figures it is clear that differences among these various parametrizations, at least for $Q^2 < 3 \text{ GeV}^2$, are rather small for G_E^p , G_M^p and G_M^n , since these quantities are more easily experimentally accessible. On the other hand, due to the lack of a free neutron target and smallness of G_E^n the information about this form factor is less accurate. Thus, the parametrizations for G_E^n have a much wider spread.

Among the parametrizations used in this thesis only Budd *et al.* [77] and Shen *et al.* [13] provide a description of all the form factors needed for the weak interactions. A comparison of these two sets is shown in Fig. 2.5.

	a_0	a_2	a_4	a_6	a_8	a_{10}	a_{12}
G_E^p	1	3.253	1.422	0.08582	0.3318	-0.09371	0.01076
G_M^p	1	3.104	1.428	0.1112	-0.006981	0.0003705	-0.7063×10^{-5}
G_M^n	1	3.043	0.8548	0.6806	-0.1287	0.008912	0
M_V	M_A	g_A	$G_E^p(0)$	$G_M^p(0)$	$G_M^n(0)$	a	b
0.843	1.012	-1.267	1	$\mu_p = 2.793$	$\mu_n = -1.913$	0.942	4.61

Table 2.1: Budd parametrization values. The masses M_A and M_V are expressed in GeV, while all other parameters are dimensionless.

The Budd *et al.* parametrization

This parametrization has been taken from Refs. [77, 79].

$$G_D(Q^2) = \frac{1}{(1 + Q^2/M_V^2)^2} \quad F_A(Q^2) = \frac{g_A}{(1 + Q^2/M_A^2)^2} \quad (2.17)$$

$$G_{M,E}^{(i)}(Q^2) = \frac{G_{M,E}^{(i)}(0)}{\sum_{i=0}^6 a_{2i} Q^{2i}} \quad G_E^n = -\mu_n \frac{a\tau}{1 + b\tau} G_D(Q^2). \quad (2.18)$$

The values of the constants are summarized in Tab. 2.1. These values of the electromagnetic $G_{E,M}$ have been extracted using the data from the JLAB electron scattering experiment [80, 81] and the value of M_A is changed from 1 GeV to 1.012 GeV. The $F_P(Q^2)$ form factor is parameterized as

$$F_P(Q^2) = -\frac{2m_\mu M}{m_\pi^2 + Q^2} F_A(Q^2). \quad (2.19)$$

The Shen *et al.* parametrization

This parametrization has been taken from Ref. [13] and it is based on the dipole parametrization.

$$G_D(Q^2) = \frac{1}{(1 + Q^2/\Lambda^2)^2} \quad G_A(Q^2) = \frac{g_A}{(1 + Q^2/\Lambda_A^2)^2} \quad (2.20)$$

$$G_E^p(Q^2) = G_D(Q^2) \quad G_E^n(Q^2) = -\mu_n \frac{Q^2}{4M^2} \frac{G_D(Q^2)}{1 + Q^2/M^2} \quad (2.21)$$

$$G_M^p(Q^2) = \mu_p G_D(Q^2) \quad G_M^n(Q^2) = \mu_n G_D(Q^2) \quad (2.22)$$

For the axial-form factors it is assumed

$$F_A(Q^2) = G_A(Q^2) \quad (2.23)$$

and

$$F_{PS}(Q^2) = -\frac{2m_\mu M}{m_\pi^2 + Q^2} G_A(Q^2), \quad (2.24)$$

where $\Lambda = 0.833$ GeV, $\Lambda_A = 1.00$ GeV, $m_\pi = 138.13$ MeV, $m_\mu = 105.65837$ MeV, $\mu_n = -1.913$, $\mu_p = 2.793$ and $g_A = 1.2694$.

	a_1	b_1	b_2	b_3
G_E^p	-0.24	10.98	12.82	21.97
G_M^p/μ_p	0.12	10.97	18.86	6.55
G_M^n/μ_n	2.33	14.72	24.20	84.1

Table 2.2: Kelly parametrization values.

The Kelly parametrization

This parametrization has been taken from Ref. [78]. Apart from G_E^n , the other form factors take the form

$$G(Q^2) = \frac{\sum_{i=0}^1 a_i \tau^i}{1 + \sum_{i=0}^3 b_i \tau^i}, \quad (2.25)$$

where

$$\tau = \frac{Q^2}{4M_p^2}. \quad (2.26)$$

Here M_p is the proton mass. In every case $a_0 = 1$ and the other parameters are listed in Tab. 2.2. G_E^n is parameterized as

$$G_E^n(Q^2) = \frac{A\tau}{1 + B\tau} G_D(Q^2), \quad (2.27)$$

here G_D is the same as in Eq. (2.20) but with $\Lambda^2 = 0.71 \text{ GeV}^2$, $A = 1.70$ and $B = 3.30$.

The Lomon parametrization

This parametrization has been taken from Ref. [76]. The Sachs form factors are defined from the Dirac and Pauli ones, as given in Eqs. (2.8)-(2.9). F_1 and F_2 are connected to the isoscalar and isovector electromagnetic currents through the relations

$$F_n^p = 1/2 (F_n^{is} + F_n^{iv}) \quad (2.28)$$

$$F_n^n = 1/2 (F_n^{is} - F_n^{iv}), \quad (2.29)$$

with $n = \{1, 2\}$. These form factors are defined as follows:

$$F_1^{iv} = \frac{N}{2} \frac{1.0317 + 0.0875(1 + Q^2/0.3176)^{-2}}{1 + Q^2/0.5495} F_1^\rho(Q^2) + \frac{g_{\rho'}}{f_{\rho'}} \frac{m_{\rho'}^2}{m_{\rho'}^2 + Q^2} F_1^\rho(Q^2) + \left(1 - 1.1192 \frac{N}{2} - \frac{g_{\rho'}}{f_{\rho'}}\right) F_1^D(Q^2), \quad (2.30)$$

$$F_2^{iv}(Q^2) = \frac{N}{2} \frac{5.7824 + 0.3907(1 + Q^2/0.1422)^{-1}}{1 + Q^2/0.5362} F_2^\rho(Q^2) + \kappa_{\rho'} \frac{g_{\rho'}}{f_{\rho'}} \frac{m_{\rho'}^2}{m_{\rho'}^2 + Q^2} F_2^\rho(Q^2) + \left(\kappa_\nu - 6.1731 \frac{N}{2} - \kappa_{\rho'} \frac{g_{\rho'}}{f_{\rho'}}\right) F_2^D(Q^2), \quad (2.31)$$

	GKex(01)	GKex(01-)	GKex(02L)	GKex(02S)
$g_{\rho'}/f_{\rho'}$	0.0636	0.0598	0.0608	0.0401
$\kappa_{\rho'}$	-0.4175	-15.9227	5.3038	6.8190
g_{ω}/f_{ω}	0.7918	0.6981	0.6896	0.6739
κ_{ω}	5.1109	1.9333	-2.8585	0.8762
g_{ϕ}/f_{ϕ}	-0.3011	-0.5270	-0.1852	-0.1676
κ_{ϕ}	13.8345	2.3241	13.0037	7.0172
μ_{ϕ}	1.1915	1.5113	0.6848	0.8544
$g_{\omega'}/f_{\omega'}$	0	0	0.2346	0.2552
$\kappa_{\omega'}$	0	0	18.2284	1.4916
Λ_1	0.9660	1.1276	0.9441	0.9407
Λ_D	1.3406	1.8598	1.2350	1.2111
Λ_2	2.1382	1.2255	2.8268	2.7891
Λ_{QCD}	0.1163	0.1315	0.150	0.150
N	1.0	0.8709	1.0	1.0

Table 2.3: Lomon parametrization values of four different models GKex; in this thesis the GKex(02S) model has been used.

$$F_1^{is} = \left(\frac{g_{\omega}}{f_{\omega}} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} + \frac{g_{\omega'}}{f_{\omega'}} \frac{m_{\omega'}^2}{m_{\omega'}^2 + Q^2} \right) F_1^{\omega}(Q^2) + \frac{g_{\phi}}{f_{\phi}} \frac{m_{\phi}^2}{m_{\phi}^2 + Q^2} F_1^{\phi}(Q^2) + \left(1 - \frac{g_{\omega}}{f_{\omega}} - \frac{g_{\omega'}}{f_{\omega'}} \right) F_1^D(Q^2), \quad (2.32)$$

$$F_2^{is} = \left(\kappa_{\omega} \frac{g_{\omega}}{f_{\omega}} \frac{m_{\omega}^2}{m_{\omega}^2 + Q^2} + \kappa_{\omega'} \frac{g_{\omega'}}{f_{\omega'}} \frac{m_{\omega'}^2}{m_{\omega'}^2 + Q^2} \right) F_2^{\omega}(Q^2) + \kappa_{\phi} \frac{g_{\phi}}{f_{\phi}} \frac{m_{\phi}^2}{m_{\phi}^2 + Q^2} F_2^{\phi}(Q^2) + \left(\kappa_s - \kappa_{\omega} \frac{g_{\omega}}{f_{\omega}} - \kappa_{\omega'} \frac{g_{\omega'}}{f_{\omega'}} - \kappa_{\phi} \frac{g_{\phi}}{f_{\phi}} \right) F_2^D(Q^2). \quad (2.33)$$

Here the F_n^{α} with $\alpha = \{\rho, \omega, \phi\}$ are the meson-nucleon form factors and F_n^D the effective quark-gluon form factors. These are parameterized as

$$F_n^{\rho, \omega, D}(Q^2) = \frac{\Lambda_{1,D}^2}{\Lambda_{1,D}^2 + \tilde{Q}^2} \left(\frac{\Lambda_2^2}{\Lambda_2^2 + \tilde{Q}^2} \right)^n \quad (2.34)$$

where $n = \{1, 2\}$. While Λ_1 is taken for ρ and ω , Λ_D is taken for D.

$$\tilde{Q}^2 = Q^2 \frac{\ln[(\Lambda_D^2 + Q^2)/\Lambda_{\text{QCD}}]}{\ln[\Lambda_D^2/\Lambda_{\text{QCD}}]}. \quad (2.35)$$

For F_n^{ϕ} one has

$$F_n^{\phi}(Q^2) = F_n^{\alpha} \left(\frac{\Lambda_1^{2(n-1)}}{\mu_{\phi}^{2(n-1)}} \frac{Q^2 + \mu_{\phi}^{2(n-1)}}{\Lambda_1^2 + Q^2} \right)^{1.5}. \quad (2.36)$$

All the parameters adopted for these form factors are gathered in Tab. 2.3.

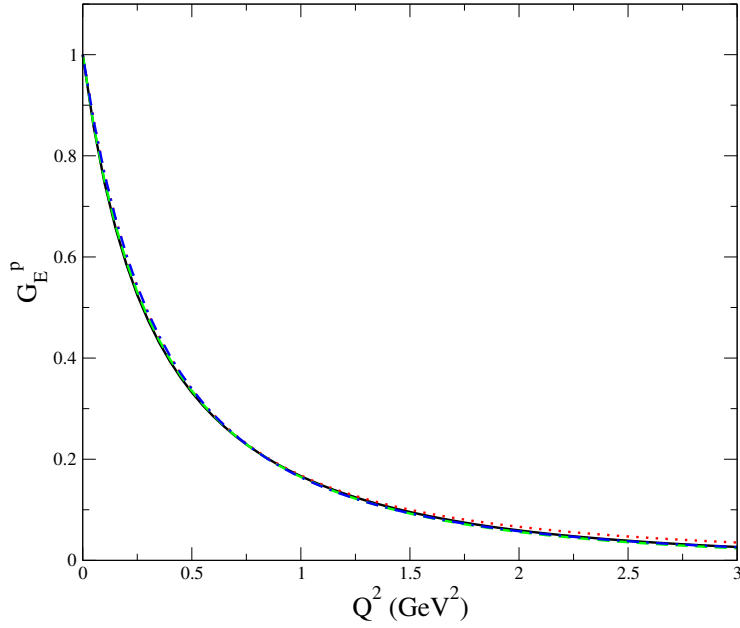


Figure 2.1: A comparison between all the parametrizations of the G_E^p form factor. The solid black line represents the Budd parametrization [77], the dotted red line the Shen parametrization [13], the dashed green line the Kelly parametrization [78] and the dotted-dashed blue line the Lomon parametrization [76].

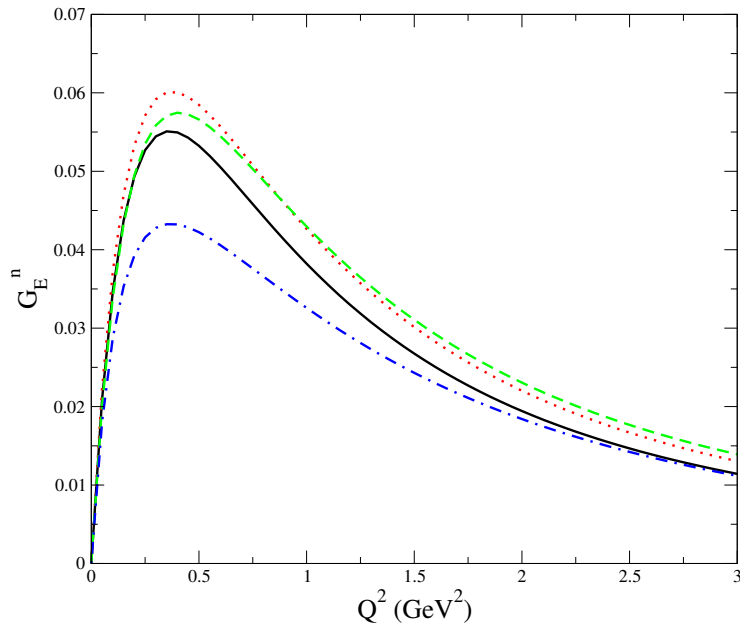


Figure 2.2: The same as in Fig. 2.1 but for G_E^n . Note the substantial difference between the parametrizations for this observable.

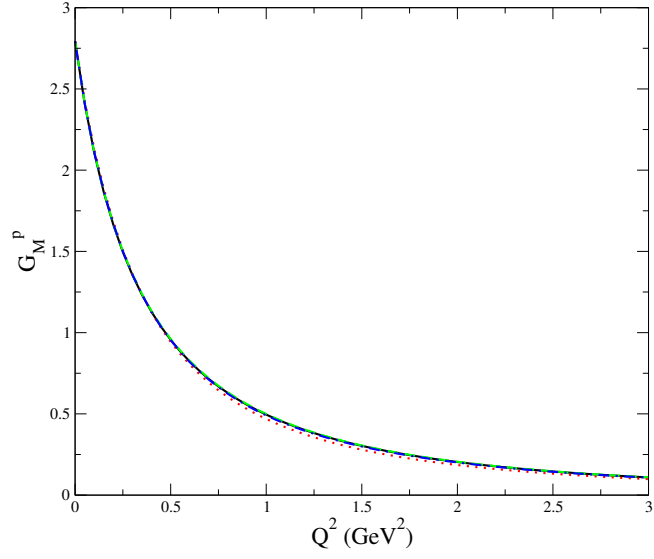


Figure 2.3: The same as in Fig. 2.1 but for G_M^p .

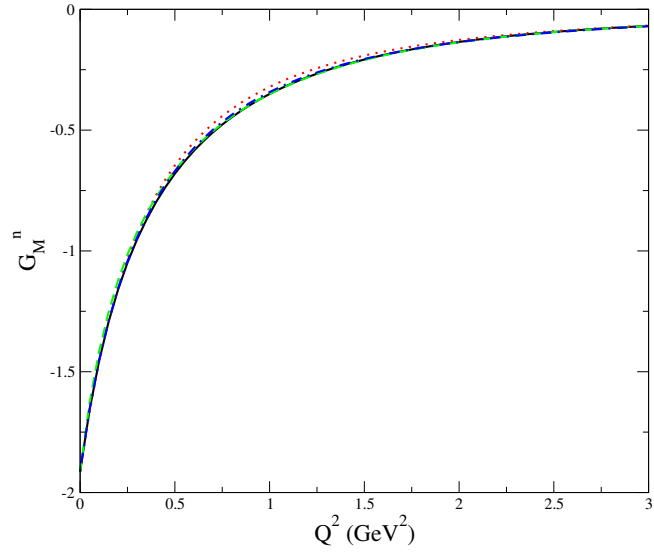


Figure 2.4: The same as in Fig. 2.1 but for G_M^n .

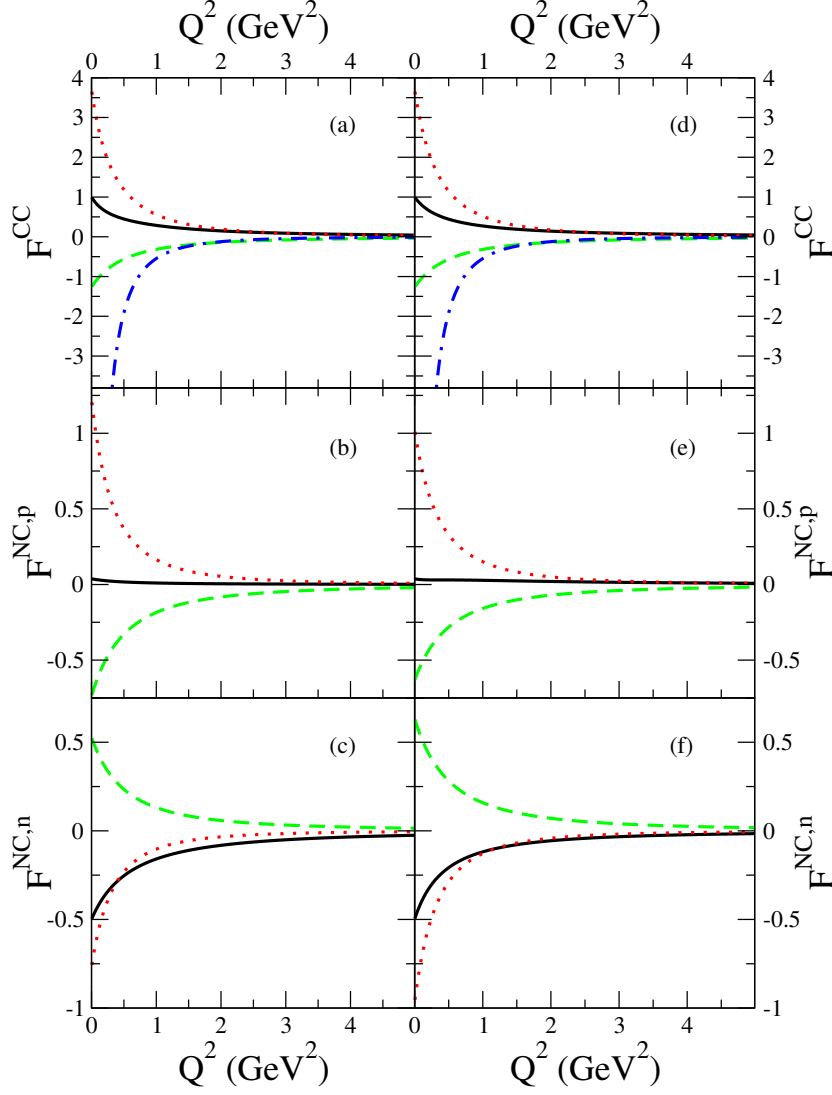


Figure 2.5: Weak form factors from Budd *et al.* [77] ((a), (b), (c)) and Shen *et al.* [13] ((d), (e), (f)). The plots for the CC case are (a) and (d), the plots for the NC case for the proton are (b) and (e) and the plots for the NC case for the neutron are (c) and (f). The solid black lines represent the form factors F_1 , dotted red lines the form factors F_2 , dashed green lines the form factors F_A and dotted-dashed blue lines the form factors F_P . The latter do not contribute for the NC reactions and thus are shown only in panels (a) and (d).

Chapter 3

Elastic and quasielastic neutrino reactions on single nucleons

Investigations of neutrino scattering on nucleons as well as on nuclei are necessary to understand the Standard Model limitations and to improve the knowledge about the universe we live in. For decades, these reactions were motivated by the so-called *solar neutrino problem*, a discrepancy between the predicted and measured electron neutrino fluxes produced by the Sun. This problem was ultimately solved by the discovery of neutrino oscillations.

Neutrino interactions include many possible processes: nuclear capture, inverse beta-decay, quasielastic scattering, resonant pion production, deep inelastic scattering and ultra-high energy interactions. These reactions are beyond the scope of this thesis. We refer the reader to highly informative recent review articles, which gathered information about neutrinos' properties and their interactions [82, 83, 84, 85, 86].

In this chapter we restrict ourselves to elastic and quasielastic neutrino-nucleon scattering, while in the next chapters we will deal with elastic (anti)neutrino or electron scattering off the deuteron as well as with the neutrino and electron induced deuteron breakup reactions. Specifically, in this chapter we study the following reactions

$$\nu_l + p \rightarrow \nu_l + p, \quad \text{NC} \quad (3.1)$$

$$\nu_l + n \rightarrow \nu_l + n, \quad \text{NC} \quad (3.2)$$

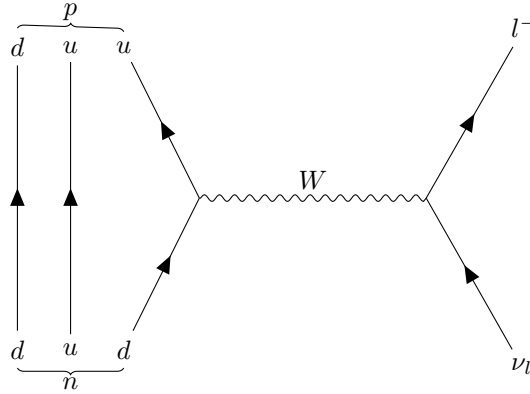
$$\bar{\nu}_l + p \rightarrow \bar{\nu}_l + p, \quad \text{NC} \quad (3.3)$$

$$\bar{\nu}_l + n \rightarrow \bar{\nu}_l + n, \quad \text{NC} \quad (3.4)$$

$$\nu_l + n \rightarrow l^- + p, \quad \text{CC} \quad (3.5)$$

$$\bar{\nu}_l + p \rightarrow l^+ + n, \quad \text{CC} \quad (3.6)$$

where l is the leptonic flavor, *i.e.* e , μ or τ . In this work we will consider only the first family. In the reactions (3.1)-(3.4) the interaction term is due to the coupling of the neutral weak-boson Z^0 , to the nucleon current, so we will call them neutral-current induced reactions. On the other hand, in the reactions

Figure 3.1: Diagram of the reaction $\nu + n \rightarrow l^- + p$.

(3.5)-(3.6) the interaction is mediated by the charged weak-boson $W^\pm$, hence the name charged-current induced reactions.

Already these reactions allow us to introduce important ingredients, which will be of use later for reactions with nuclei. We encounter here a general formula for the cross section and learn two-body kinematics necessary to evaluate the flux and phase space factors. The lepton weak and electromagnetic currents are defined in this chapter as well, leading to the evaluation of the so-called “lepton arm”. In turn, the so-called “nucleon arm” requires the nucleon weak (charged and neutral) and electromagnetic currents. The cross sections for neutrino-nucleon scattering under the assumed one-boson-exchange approximation are analytically known and are derived in a fully relativistic framework. In this way we can carefully test the usual approximations employed in nonrelativistic frameworks for neutrino and electron reactions with nuclei (nonrelativistic kinematics in the nuclear arm, nonrelativistic form of the single-nucleon weak as well electromagnetic current operators). In particular, the fully relativistic differential cross section in the laboratory frame not only can be derived using the properties of the gamma matrices, but it can be given in terms of the so-called leptonic v -functions and the nuclear (here nucleonic) response functions. Finally, it is easily demonstrated that the total cross sections for the weak processes can be evaluated equally well in the laboratory and in the total-momentum-zero frames.

3.1 Kinematics for (quasi)elastic neutrino induced reactions

We will denote by $\mathbf{k}$ ($\mathbf{k}'$) the initial (final) lepton momentum, while $\mathbf{p}$ ($\mathbf{p}'$) stands for the initial (final) nucleon momentum. Conservation of energy and momentum imposes

$$k + \sqrt{M^2 + p^2} = \sqrt{m_l^2 + k'^2} + \sqrt{M'^2 + p'^2} \quad (3.7)$$

and

$$\mathbf{k} + \mathbf{p} = \mathbf{k}' + \mathbf{p}' . \quad (3.8)$$

In the center-of-mass reference frame the solution for k' does not depend on the lepton scattering angle θ' , while in the laboratory reference frame $k'_L = k'_L(\theta')$. For the NC induced reactions, the final lepton mass is zero and $M = M'$, thus the solutions are

$$k'_{\text{CM}} = k \quad (3.9)$$

and

$$k'_L = \frac{k}{1 + k/M(1 - \cos \theta')} . \quad (3.10)$$

For the CC induced reactions, the solutions are more complicated but still analytically given:

$$k'_{\text{CM}} = \frac{\sqrt{(E_{\text{CM}} - m_l - M')(E_{\text{CM}} + m_l - M')(E_{\text{CM}} - m_l + M')(E_{\text{CM}} + m_l + M')}}{2E_{\text{CM}}} , \quad (3.11)$$

where here $E_{\text{CM}} = k + \sqrt{M^2 + k^2}$ is the initial energy in the center-of-mass reference frame. In the laboratory reference frame we find

$$k'_L = \frac{1}{2(k^2 \sin^2 \theta' + 2kM + M^2)} \left\{ k(2kM + m_l^2) \cos \theta' + \left[(k + M)^2 \left(2k^2 m_l^2 \cos 2\theta' - 2k^2(m_l^2 - 2M^2) - 4km_l^2 M + m_l^4 - 4m_l^2 M^2 \right) \right]^{1/2} \right\} , \quad (3.12)$$

where now k is the magnitude of the momentum of the initial neutrino in the laboratory reference frame.

3.1.1 Reaction threshold

For the CC reactions the neutrino needs to have a minimum energy in order to trigger the reaction, this is the energy threshold for the reaction. In the center-of-mass reference frame, from Eq. (3.7), if we set $k' = p' = 0$ we get

$$k + \sqrt{M^2 + k^2} = \widetilde{M} , \quad (3.13)$$

where $\widetilde{M}$ is the total mass of the product of the reaction. By solving the last equation with respect to k

$$k_{\text{min}}^{\text{CM}} = \frac{\widetilde{M}^2 - M^2}{2\widetilde{M}} , \quad (3.14)$$

therefore if $\widetilde{M} > M$ we have $k_{\text{min}}^{\text{CM}} > 0$.

In the laboratory reference frame this is replaced by

$$k_{\text{min}}^{\text{L}} = \frac{\widetilde{M}^2 - M^2}{2M} . \quad (3.15)$$

The values of these threshold energies are summarized in Tab. 3.1.

	$k_{\min}^{\text{CM}}$ (MeV)	$k_{\min}^{\text{L}}$ (MeV)
$\nu_e + n \rightarrow e^- + p$	0	0
$\bar{\nu}_e + p \rightarrow e^+ + n$	1.8026	1.8061
$\nu_\mu + n \rightarrow \mu^- + p$	99.1	110.2
$\bar{\nu}_\mu + p \rightarrow \mu^+ + n$	101.5	113.0

Table 3.1: Threshold neutrino energies for the CC induced reactions in the center-of-mass reference frame and in the laboratory reference frame.

3.1.2 Center of mass reference frame differential cross section

In the center-of-mass reference frame we have $\mathbf{p} = -\mathbf{k}$ and $\mathbf{p}' = -\mathbf{k}'$ so Eq. (3.7) becomes

$$k + \sqrt{M^2 + k^2} = \sqrt{m_l^2 + k'^2} + \sqrt{M'^2 + k'^2}. \quad (3.16)$$

The flux factor in Eq. (56) becomes

$$I = k \cdot p = k(\sqrt{M^2 + k^2} + k) = k\sqrt{s}, \quad (3.17)$$

where s is the Lorentz invariant Mandelstam variable

$$s = (p^\mu + k^\mu)^2. \quad (3.18)$$

The phase space factor in Eq. (57) is written as

$$d\Phi^{(2)} = \delta^4(p'^\mu + k'^\mu - p^\mu - k^\mu) \frac{d\mathbf{p}'}{(2\pi)^3 2\sqrt{M'^2 + p'^2}} \frac{d\mathbf{k}'}{(2\pi)^3 2\sqrt{m_l^2 + k'^2}}. \quad (3.19)$$

Using

$$d\mathbf{k}' = k'^2 dk' d\hat{\mathbf{k}}' \quad (3.20)$$

the differential cross section of Eq. (55) becomes

$$\begin{aligned} d\sigma = \frac{1}{64\pi^2} \frac{1}{k\sqrt{s}} \delta(k + \sqrt{M^2 + k^2} - \sqrt{m_l^2 + k'^2} - \sqrt{M'^2 + k'^2}) \\ \times |M_{fi}|^2 \frac{1}{\sqrt{M'^2 + k'^2}} \frac{k'^2}{\sqrt{m_l^2 + k'^2}} dk' d\hat{\mathbf{k}}' \end{aligned} \quad (3.21)$$

which becomes

$$\frac{d^2\sigma}{d\hat{\mathbf{k}}'} = \frac{1}{64\pi^2} \frac{1}{s} \frac{k'}{k} |M_{fi}|^2, \quad (3.22)$$

where k' is the only physical solution of Eq. (3.9) or Eq. (3.11).

3.1.3 Laboratory reference frame differential cross section

In the laboratory reference frame we have $\mathbf{p} = 0$ and Eq. (3.7) and Eq. (3.8) become

$$k + M = \sqrt{m_l^2 + k'^2} + \sqrt{M'^2 + p'^2}, \quad (3.23)$$

$$\mathbf{k} = \mathbf{k}' + \mathbf{p}'. \quad (3.24)$$

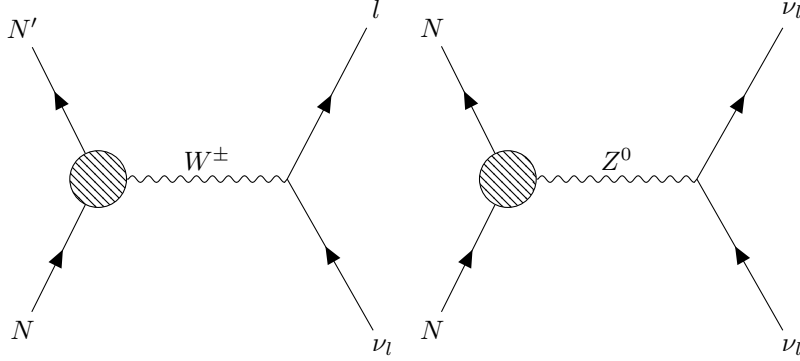


Figure 3.2: Diagram for reactions of the kind $\nu_l + N \rightarrow l + N'$ (left diagram) and $\nu_l + N \rightarrow \nu_l + N$ (right diagram).

That means that the three momentum transfer $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$ is just the final nucleon momentum $\mathbf{p}'$ and we arrive at

$$k + M = \sqrt{m_l^2 + k'^2} + \sqrt{M'^2 + k^2 + k'^2 - 2kk' \cos \theta'}. \quad (3.25)$$

The flux factor in Eq. (56) becomes

$$I = k \cdot p = Mk. \quad (3.26)$$

The differential cross section of Eq. (55) thus becomes

$$\begin{aligned} d\sigma = \frac{1}{64\pi^2} \frac{1}{kM} \delta\left(k + M - \sqrt{m_l^2 + k'^2} - \sqrt{M'^2 + k^2 + k'^2 - 2kk' \cos \theta'}\right) \\ \times |M_{fi}|^2 \frac{k'^2}{E_{\mathbf{k}'} E_{\mathbf{p}'}} d\mathbf{k}' d\hat{\mathbf{k}}' \end{aligned} \quad (3.27)$$

where $E_{\mathbf{k}'} = \sqrt{m_l^2 + k'^2}$ and $E_{\mathbf{p}'} = \sqrt{M'^2 + q^2}$. After some algebra we obtain

$$\frac{d^2\sigma}{d\hat{\mathbf{k}}'} = \frac{1}{64\pi^2} \frac{1}{kM} \frac{k'^2}{E_{\mathbf{k}'} E_{\mathbf{p}'}} \frac{1}{\left| \frac{k'}{E_{\mathbf{k}'}} + \frac{k' - k \cos \theta'}{E_{\mathbf{p}'}} \right|} |M_{fi}|^2, \quad (3.28)$$

where k' is the only positive solution either of Eq. (3.10) or of Eq. (3.12). Equation (3.28) can be further simplified, assuming $m_l = 0$.

3.2 Transition matrix element

At tree level, the reactions treated in this chapter can be represented by a single diagram of the form showed in Fig. 3.2. If we consider neutrino-induced reactions, we can write the transition matrix element (ignoring phases which will disappear for $|M_{fi}|^2$) as

$$M_{fi} = \bar{u}(\mathbf{k}', \sigma'_\nu) \left(\frac{g}{2} \gamma^\mu (1 - \gamma^5) \right) u(\mathbf{k}, \sigma_\nu) \frac{g_{\mu\nu} - q_\mu q_\nu / M_B^2}{q^2 - M_B^2} \left(\frac{g}{2} \right) N^\nu, \quad (3.29)$$

while for antineutrino-induced reactions

$$M_{fi} = \bar{v}(\mathbf{k}, \sigma_\nu) \left(\frac{g}{2} \gamma^\mu (1 - \gamma^5) \right) v(\mathbf{k}', \sigma'_\nu) \frac{g_{\mu\nu} - q_\mu q_\nu / M_B^2}{q^2 - M_B^2} \left(\frac{g}{2} \right) N^\nu. \quad (3.30)$$

Here σ_ν (σ'_ν) is the spin of initial (final) neutrino/antineutrino, M_B is the mass of the exchanged boson, $g = |e|/\sin \theta_W$ is the weak coupling constant for the $W^\pm$ bosons [87] and N^μ is the nucleon arm matrix element. The latter is written as

$$N^\mu = \bar{u}(\mathbf{p}', \sigma'_N) J^\mu u(\mathbf{p}, \sigma_N), \quad (3.31)$$

where J^μ is the single-nucleon current operator and σ_N (σ'_N) is the spin of the initial (final) nucleon.

The typical energies involved in the reaction that are studied in this work are much smaller than the energy scale for weak interactions, which is set by M_W and M_Z . In this case the boson propagator can be approximated as

$$i \frac{1}{q^2 - M_B^2} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_B^2} \right) \rightarrow i \frac{g_{\mu\nu}}{M_B^2} \quad (3.32)$$

and the transition matrix element can be written as

$$M_{fi} = \left(\frac{g^2}{4M_B^2} \right) \bar{u}(\mathbf{k}', \sigma'_\nu) (\gamma_\mu (1 - \gamma^5)) u(\mathbf{k}, \sigma_\nu) N^\mu \quad (3.33)$$

and

$$M_{fi} = \left(\frac{g^2}{4M_B^2} \right) \bar{v}(\mathbf{k}, \sigma_\nu) (\gamma_\mu (1 - \gamma^5)) v(\mathbf{k}', \sigma'_\nu) N^\mu. \quad (3.34)$$

By identifying L_μ with the leptonic arm these become

$$M_{fi} = \left(\frac{g^2}{4M_B^2} \right) L_\mu N^\mu. \quad (3.35)$$

For the so-called “unpolarized” cross section the square of the transition matrix element is summed over final spin projections and averaged over initial nucleon spin projections

$$\overline{|M_{fi}|^2} \equiv \frac{1}{2} \sum_{\sigma_N \sigma_\nu} \sum_{\sigma'_N \sigma'_\nu} |M_{fi}|^2 = \left(\frac{g^2}{4M_B^2} \right)^2 \sum_{\sigma_\nu \sigma'_\nu} (L_\mu)^\dagger L_\nu \frac{1}{2} \sum_{\sigma_N \sigma'_N} (N^\mu)^\dagger N^\nu. \quad (3.36)$$

Using trace identities we can rewrite $\sum (L_\mu)^\dagger L_\nu$ as

$$L_{\mu\nu}^{\nu/\bar{\nu}} \equiv \sum_{\sigma_\nu \sigma'_\nu} (L_\mu)^\dagger L_\nu = 8 \left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} (k \cdot k') \mp i \varepsilon_{\mu\nu\rho\sigma} k^\rho k'^\sigma \right], \quad (3.37)$$

where $\varepsilon^{0123} = 1$ as in Ref. [6].

A more complicated but still analytical expression can be obtained for the nucleon arm $N^{\mu\nu} \equiv 1/2 \sum (N^\mu)^\dagger N^\nu$ and the reader can find it in Appendix A.

By using these two new four-tensors we get to the expression

$$\overline{|M_{fi}|^2} = \frac{1}{2} \left(\frac{g^2}{4M_B^2} \right)^2 L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}. \quad (3.38)$$

We will use spherical components for the hadronic matrix elements, which are defined as

$$N_{\pm 1} = \mp \frac{(N^1 \pm i N^2)}{\sqrt{2}}. \quad (3.39)$$

Inserting the inverse relation for N^1 and N^2 , defining $R_{ab} = N^a N^b$ with $a, b = \{0, +, -, 3\}$ we can write

$$L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu} = \sum_{ab} \mathcal{L}_{ab}^{\nu/\bar{\nu}} R_{ab} \quad (3.40)$$

with

$$\mathcal{L}_{00}^{\nu/\bar{\nu}} = L_{00}^{\nu/\bar{\nu}} \quad (3.41)$$

$$\mathcal{L}_{0-}^{\nu/\bar{\nu}} = (L_{01}^{\nu/\bar{\nu}} + i L_{02}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.42)$$

$$\mathcal{L}_{0+}^{\nu/\bar{\nu}} = -(L_{01}^{\nu/\bar{\nu}} - i L_{02}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.43)$$

$$\mathcal{L}_{03}^{\nu/\bar{\nu}} = L_{03}^{\nu/\bar{\nu}} \quad (3.44)$$

$$\mathcal{L}_{-0}^{\nu/\bar{\nu}} = (L_{10}^{\nu/\bar{\nu}} - i L_{20}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.45)$$

$$\mathcal{L}_{--}^{\nu/\bar{\nu}} = (L_{11}^{\nu/\bar{\nu}} + i L_{12}^{\nu/\bar{\nu}} - i L_{21}^{\nu/\bar{\nu}} + L_{22}^{\nu/\bar{\nu}})/2 \quad (3.46)$$

$$\mathcal{L}_{-+}^{\nu/\bar{\nu}} = (-L_{11}^{\nu/\bar{\nu}} + i L_{12}^{\nu/\bar{\nu}} + i L_{21}^{\nu/\bar{\nu}} + L_{22}^{\nu/\bar{\nu}})/2 \quad (3.47)$$

$$\mathcal{L}_{-3}^{\nu/\bar{\nu}} = (L_{13}^{\nu/\bar{\nu}} - i L_{23}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.48)$$

$$\mathcal{L}_{+0}^{\nu/\bar{\nu}} = -(L_{10}^{\nu/\bar{\nu}} + i L_{20}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.49)$$

$$\mathcal{L}_{+-}^{\nu/\bar{\nu}} = (-L_{11}^{\nu/\bar{\nu}} - i L_{12}^{\nu/\bar{\nu}} - i L_{21}^{\nu/\bar{\nu}} + L_{22}^{\nu/\bar{\nu}})/2 \quad (3.50)$$

$$\mathcal{L}_{++}^{\nu/\bar{\nu}} = (L_{11}^{\nu/\bar{\nu}} - i L_{12}^{\nu/\bar{\nu}} + i L_{21}^{\nu/\bar{\nu}} + L_{22}^{\nu/\bar{\nu}})/2 \quad (3.51)$$

$$\mathcal{L}_{+3}^{\nu/\bar{\nu}} = -(L_{13}^{\nu/\bar{\nu}} + i L_{23}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.52)$$

$$\mathcal{L}_{30}^{\nu/\bar{\nu}} = L_{30}^{\nu/\bar{\nu}} \quad (3.53)$$

$$\mathcal{L}_{3-}^{\nu/\bar{\nu}} = (L_{31}^{\nu/\bar{\nu}} + i L_{32}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.54)$$

$$\mathcal{L}_{3+}^{\nu/\bar{\nu}} = -(L_{31}^{\nu/\bar{\nu}} - i L_{32}^{\nu/\bar{\nu}})/\sqrt{2} \quad (3.55)$$

$$\mathcal{L}_{33}^{\nu/\bar{\nu}} = L_{33}^{\nu/\bar{\nu}} \quad (3.56)$$

If the transferred momentum $\mathbf{k} - \mathbf{k}'$ is parallel to the z -axis and the reaction plane is the xy -plane, then the only nonzero cases are

$$\mathcal{L}_{00}^{\nu/\bar{\nu}} = 8(E_{\mathbf{k}} E_{\mathbf{k}'} + \mathbf{k} \cdot \mathbf{k}') \quad (3.57)$$

$$\mathcal{L}_{--}^{\nu/\bar{\nu}} = 8(E_{\mathbf{k}'} \mp k'_z)(E_{\mathbf{k}} \pm k_z) \quad (3.58)$$

$$\mathcal{L}_{++}^{\nu/\bar{\nu}} = 8(E_{\mathbf{k}'} \pm k'_z)(E_{\mathbf{k}} \mp k_z) \quad (3.59)$$

$$\mathcal{L}_{33}^{\nu/\bar{\nu}} = 8(E_{\mathbf{k}} E'_{\mathbf{k}} + 2 k'_z k_z - \mathbf{k} \cdot \mathbf{k}') \quad (3.60)$$

$$\mathcal{L}_{30}^{\nu/\bar{\nu}} = \mathcal{L}_{03}^{\nu/\bar{\nu}} = 8(-E_{\mathbf{k}} k'_z - E'_{\mathbf{k}} k_z) \quad (3.61)$$

This statement is true not only for elastic or quasielastic neutrino-nucleon reactions but applies also to elastic neutrino scattering on nuclei and to reactions

(including breakup reactions) where only the final lepton is measured. In fact, the contraction (3.40) is a general one and applies to any neutrino induced reactions under the one-boson exchange assumption.

3.3 Cross section for CC induced reactions

We can rewrite Eq. (3.38) for CC induced reactions by replacing M_B with M_W and

$$M_W = M_Z \cos \theta_W \quad (3.62)$$

$$g_Z = g \cos \theta_W \quad (3.63)$$

and

$$G_F = \frac{\sqrt{2}}{8} \frac{g^2}{M_W^2}, \quad (3.64)$$

where $G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2}$ [88] is the Fermi constant, obtaining

$$|\overline{M_{fi}}|^2 = \frac{G_F^2}{2} L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}. \quad (3.65)$$

Inserting Eq. (3.65) in Eqs. (3.22, 3.28) we get

$$\frac{d^2 \sigma_{\text{CM}}^{CC}}{d\hat{\mathbf{k}}'} = \frac{G_F^2 \cos^2 \theta_C}{128\pi^2} \frac{1}{s} \frac{k'}{k} L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}, \quad (3.66)$$

and

$$\frac{d^2 \sigma_{\text{L}}^{CC}}{d\hat{\mathbf{k}}'} = \frac{G_F^2 \cos^2 \theta_C}{128\pi^2} \frac{1}{kM} \frac{k'^2}{E_{\mathbf{k}'} E_{\mathbf{p}'}} \frac{1}{\left| \frac{k'}{E_{\mathbf{k}'}} + \frac{k' - k \cos \theta}{E_{\mathbf{p}'}} \right|} L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}, \quad (3.67)$$

where $\cos \theta_C = 0.974$ is the cosine of the Cabibbo angle θ_C .

3.4 Cross section for NC induced reactions

In NC induced reactions $E_{\mathbf{k}'} = k'$ and in the center-of-mass reference frame $k' = k$, hence

$$\frac{d^2 \sigma_{\text{CM}}^{NC}}{d\hat{\mathbf{k}}'} = \frac{G_F^2}{128\pi^2} \frac{1}{s} L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}. \quad (3.68)$$

In the laboratory reference frame we find

$$\frac{d^2 \sigma_{\text{L}}^{NC}}{d\hat{\mathbf{k}}'} = \frac{G_F^2}{128\pi^2} \frac{1}{kM} \frac{k'}{E_{\mathbf{p}'}} \frac{1}{\left| 1 + \frac{k' - k \cos \theta}{E_{\mathbf{p}'}} \right|} L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}, \quad (3.69)$$

which is usually written in a simpler form:

$$\frac{d^2 \sigma_{\text{L}}^{NC}}{d\hat{\mathbf{k}}'} = \frac{G_F^2}{128\pi^2} \frac{1}{M^2} \left(\frac{k'}{k} \right)^2 L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu}. \quad (3.70)$$

3.5 Fully relativistic form of the transition matrix element squared

Contracting tensors in Eq. (3.65) and assuming that the masses of the initial and final nucleon are the same, $|M_{fi}|^2$ can be expressed in terms of the Lorentz invariant quantities Q^2 and s , defined in Eq. (2.5) and 3.18 respectively. After the contraction we obtain

$$L_{\mu\nu}^{\nu/\bar{\nu}} N^{\mu\nu} = \sum_{i=1,2,A,P} \sum_{j=1,2,A,P} C_{i,j}(Q^2, s) F_i(Q^2) F_j(Q^2), \quad (3.71)$$

where

$$C_{1,1} = 8 (m_l^2 (Q^2 - 2s) + 2M^4 - 4M^2 s + Q^4 - 2Q^2 s + 2s^2), \quad (3.72)$$

$$C_{1,2} = 8(-m_l^4 + m_l^2 Q^2 + 2Q^4), \quad (3.73)$$

$$C_{1,A} = C_{2,A} = \pm 16Q^2 (m_l^2 + 2M^2 + Q^2 - 2s), \quad (3.74)$$

$$C_{2,2} = \frac{1}{M^2} \left(m_l^4 (Q^2 - 4M^2) + m_l^2 Q^2 (4M^2 + Q^2 - 4s) + 4Q^2 (M^4 + 2M^2 (Q^2 - s) + s (s - Q^2)) \right), \quad (3.75)$$

$$C_{A,A} = 8 (m_l^2 (4M^2 + Q^2 - 2s) + 2M^4 + 4M^2 (Q^2 - s) + Q^4 - 2Q^2 s + 2s^2), \quad (3.76)$$

$$C_{A,P} = 16m_l^2 (m_l^2 + Q^2), \quad (3.77)$$

$$C_{P,P} = \frac{4m_l^2 Q^2 (m_l^2 + Q^2)}{M^2} \quad (3.78)$$

and all the other coefficients are zero. The upper plus (lower minus) sign in front of $C_{1,A}$ and $C_{2,A}$ is for neutrino (antineutrino). For NC induced reactions $m_l = 0$ and therefore all contributions from F_P are zero. The contractions of the fully relativistic $|M_{fi}|^2$ in the case of reactions on the single nucleon were carried out with the help of the Mathematica[®] [89] package *FeynCalc* [90][91][92].

Inserting Eq. (3.71) in Eqs. (3.67) and (3.69), and integrating over the whole solid angle $\mathbf{k}'$, we evaluate the total cross section. The results for both sets of form factors are shown in Figs. 3.3-3.6. These results have been checked also using the formulas in Eqs. (3.66) and (3.68), and evaluating the total cross section in the center-of-mass reference frame for consistency.

As we can see from Figs. 3.3-3.4 the difference for CC induced reactions between the parametrization from Shen *et al.* and from Budd *et al.* are very small. It is worth noting that the cross section for these two reactions evaluated with the Budd *et al.* parametrization is slightly bigger than the one evaluated with the Shen *et al.* parametrization.

On the other hand, for NC induced reactions, the difference between the two parametrizations is striking and it is even more pronounced for the antineutrino reactions. Moreover, in the case of CC reactions on the neutron the cross sections evaluated with the Shen *et al.* parametrization is the biggest.

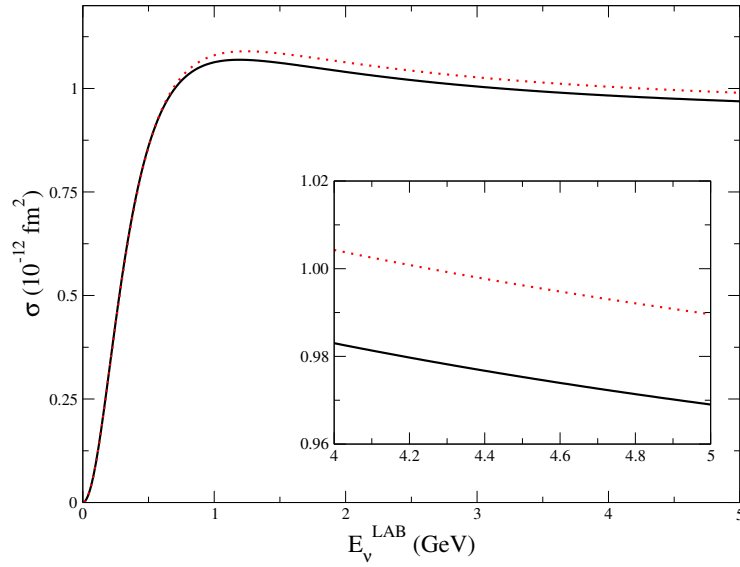


Figure 3.3: Total cross section for the CC induced reaction $\nu_e + n \rightarrow e^- + p$. The solid black line is the results calculated with the form factors from Shen *et al.*, while the dotted red line is the result calculated with the form factors from Budd *et al.*.

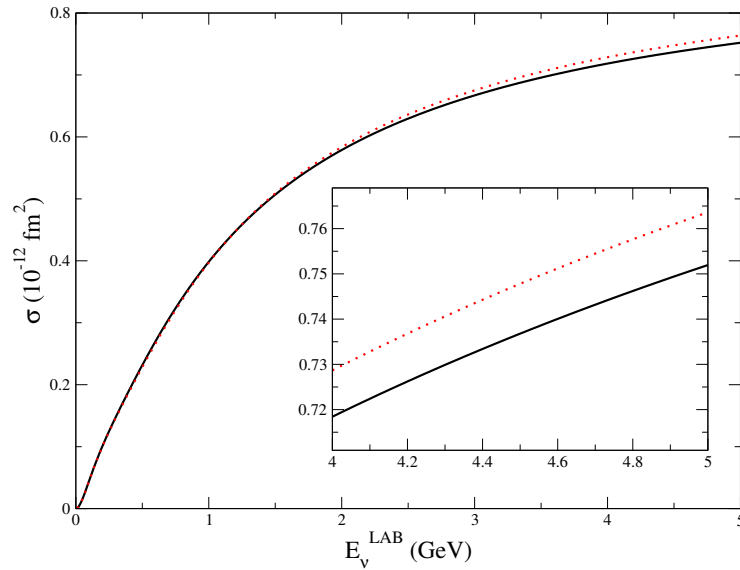


Figure 3.4: The same as Fig. 3.3 but for the CC induced reaction $\bar{\nu}_e + p \rightarrow e^+ + p$.

3.6 Nonrelativistic expansion of the single-nucleon weak current operator

The nonrelativistic form of the nuclear weak or electromagnetic current operator is required by all calculations which involve nonrelativistic nuclear wave

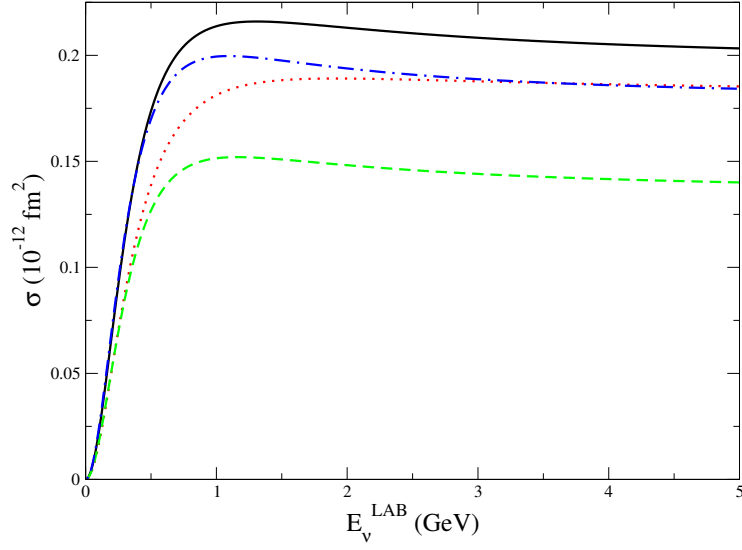


Figure 3.5: Total cross section for neutrino NC induced reactions on nucleons. The solid black (dashed green) line is the results calculated with the form factors from Shen *et al.* for the reaction $\nu + n \rightarrow \nu + n$ ($\nu + p \rightarrow \nu + p$), while the dotted red (dash-dotted blue) line is the result calculated with the form factors from Budd *et al.* for the reaction $\nu + n \rightarrow \nu + n$ ($\nu + p \rightarrow \nu + p$). Notice that the differences for the reaction on the proton are bigger than the ones for the reaction on the neutron.

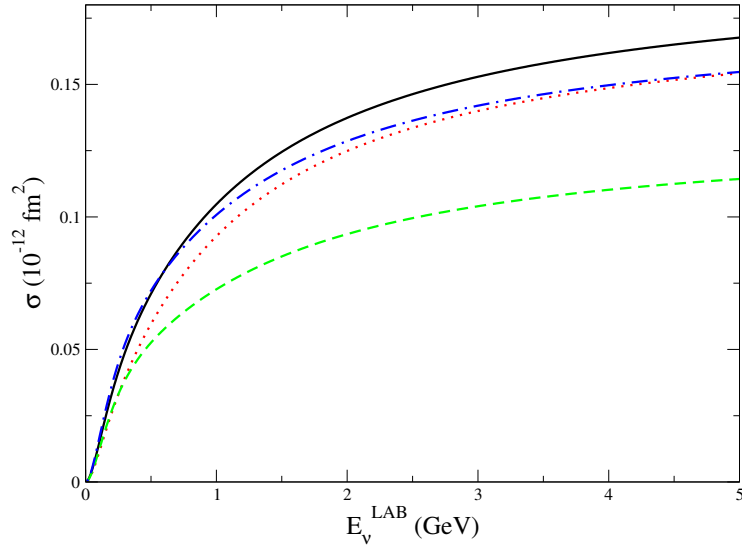


Figure 3.6: The same as Fig. 3.5 but for antineutrino NC induced reactions. Here as well the differences for the reaction on the proton are bigger than the ones for the reaction on the neutron.

functions in a description of interactions of the electroweak probes with nuclei. In many processes the dominant role in the nuclear current operator is played by the single-nucleon contributions. Processes on a single nucleon offer a unique opportunity to study frequently used nonrelativistic reductions of the relativistic current operators and to compare results of various approximations (also in the nucleon kinematics) with the *exact* results of the fully relativistic calculations.

In this section we show the results of the $O(p/M)$ expansion of the nucleon current operator. These results will be valid not only for CC and NC driven neutrino induced reactions but also for electron-nucleon scattering. Since in the relativistic calculation we encounter the nucleon matrix element divided by the initial and final nucleon energies, we expand

$$\tilde{N}^\mu = \frac{N^\mu}{\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{p}'}}} . \quad (3.79)$$

If we retain only terms up to $O(p/M)$ we get

$$\tilde{N}_{NR}^0 = \chi_{\sigma'}^\dagger \left(F_1 + F_A \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{p}')}{2M} \right) \chi_\sigma \quad (3.80)$$

and

$$\tilde{\mathbf{N}}_{NR} = \chi_{\sigma'}^\dagger \left[F_1 \frac{\mathbf{p}' + \mathbf{p}}{2M} + \frac{1}{2M} (F_1 + F_2) i \boldsymbol{\sigma} \times (\mathbf{p}' - \mathbf{p}) + F_A \boldsymbol{\sigma} \right] \chi_\sigma . \quad (3.81)$$

If we keep the relativistic corrections up to terms of the order $O(p^2/M^2)$, we have more terms:

$$\begin{aligned} \tilde{N}_{NR+RC}^0 = \chi_{\sigma'}^\dagger & \left[F_1 - (F_1 + 2F_2) \frac{(\mathbf{p}' - \mathbf{p})^2}{8M^2} \right. \\ & \left. + (F_1 + 2F_2) i \frac{(\mathbf{p}' \times \mathbf{p}) \cdot \boldsymbol{\sigma}}{4M^2} + F_A \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{p}')}{2M} \right] \chi_\sigma \end{aligned} \quad (3.82)$$

and

$$\begin{aligned} \tilde{\mathbf{N}}_{NR+RC} = \chi_{\sigma'}^\dagger & \left\{ F_1 \frac{\mathbf{p}' + \mathbf{p}}{2M} + \frac{1}{2M} (F_1 + F_2) i \boldsymbol{\sigma} \times (\mathbf{p}' - \mathbf{p}) \right. \\ & + F_A \left[1 - \frac{(\mathbf{p}' + \mathbf{p})^2}{8M^2} \right] \boldsymbol{\sigma} + \frac{F_A}{4M^2} \left[(\mathbf{p} \cdot \boldsymbol{\sigma}) \mathbf{p}' + (\mathbf{p}' \cdot \boldsymbol{\sigma}) \mathbf{p} \right. \\ & \left. \left. + i(\mathbf{p} \times \mathbf{p}') \right] + F_P \frac{(\mathbf{p}' - \mathbf{p})}{M} \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathbf{p}')}{2M} \right\} \chi_\sigma . \end{aligned} \quad (3.83)$$

Results for CC (NC) reactions are shown in Fig. 3.7 (Figs. 3.8-3.9).

It is important to notice that the nonrelativistic results are obtained by adopting nonrelativistic solutions for the kinematics as well.

Another interesting feature of the results is that, contrary to what one might expect, the nonrelativistic results obtained with the nonrelativistic current containing higher order corrections give worse approximations with respect to the fully relativistic results than the “totally” nonrelativistic counterpart in all the cases studied, except for the reaction $\bar{\nu} + p \rightarrow \bar{\nu} + p$. We will see in the next chapter that this feature appears also in the results for the elastic electron-nucleon scattering.

Finally, in Figs. 3.10 and 3.11 we show the differential cross section $d^2\sigma/d\hat{\mathbf{k}}'$ for the neutrino induced reaction on proton $\nu + p \rightarrow \nu + p$ at two different initial neutrino energies: 150 MeV and 1 GeV. The cross sections have been evaluated fully relativistically both in the center-of-mass reference frame and in the laboratory reference as well as with the two nonrelativistic expansions in the laboratory reference frame. Despite the difference between the angular distribution in the fully relativistic case, once integrated they yield the same result.

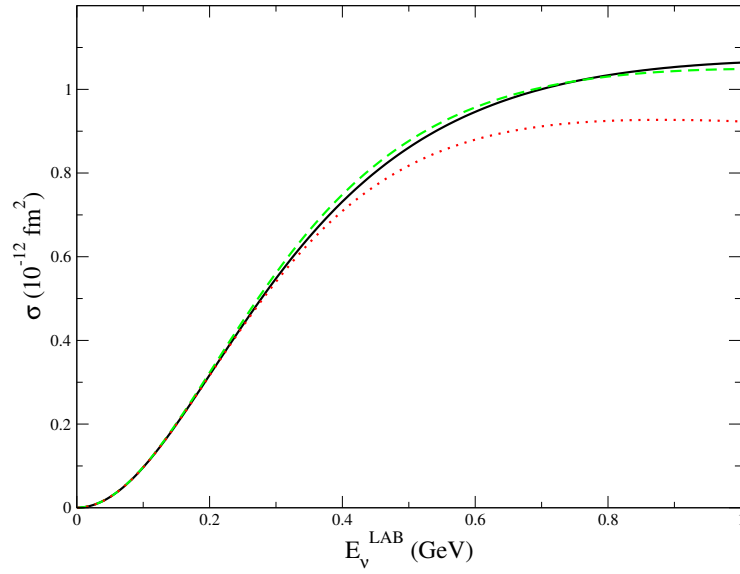
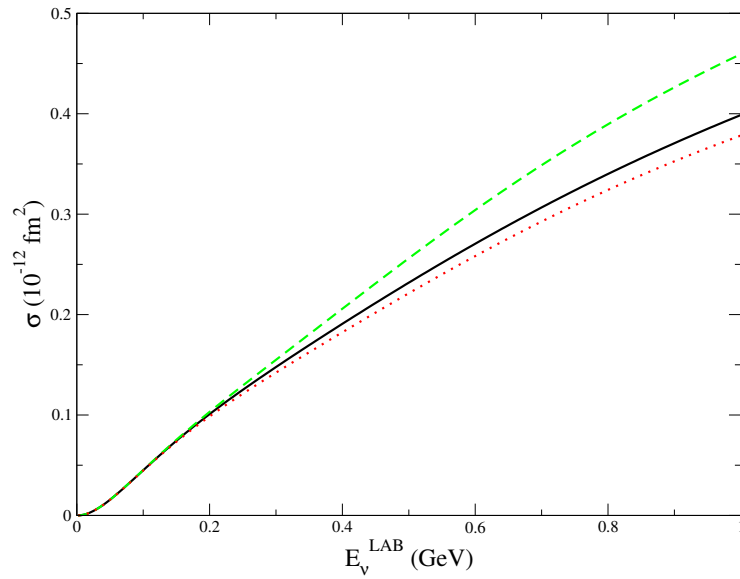
(a) $\nu_e + n \rightarrow e^- + p$ (b) $\bar{\nu}_e + p \rightarrow e^+ + n$

Figure 3.7: Cross section comparison up to 1 GeV for CC reactions evaluated with the Shen *et al.* parametrization of the form factors. Fully relativistic results are shown with a solid black line, nonrelativistic results up to order $O(p/M)$ are shown with a dashed green line and nonrelativistic results up to order $O(p^2/M^2)$ are shown with a dotted red line, finally fully relativistic results evaluated in the center-of-mass reference frame are shown with a dash-dotted black line. They completely overlap with the relativistic results evaluated in the laboratory reference frame. The results for the nonrelativistic expansion treat also the kinematics in the nonrelativistically fashion and employ also nonrelativistic kinematics. Notice that for all the considered reactions, only in the CC reaction on proton the fully relativistic results have a better agreement with the nonrelativistic results up to order $O(p^2/M^2)$ than the nonrelativistic results up to order $O(p/M)$.

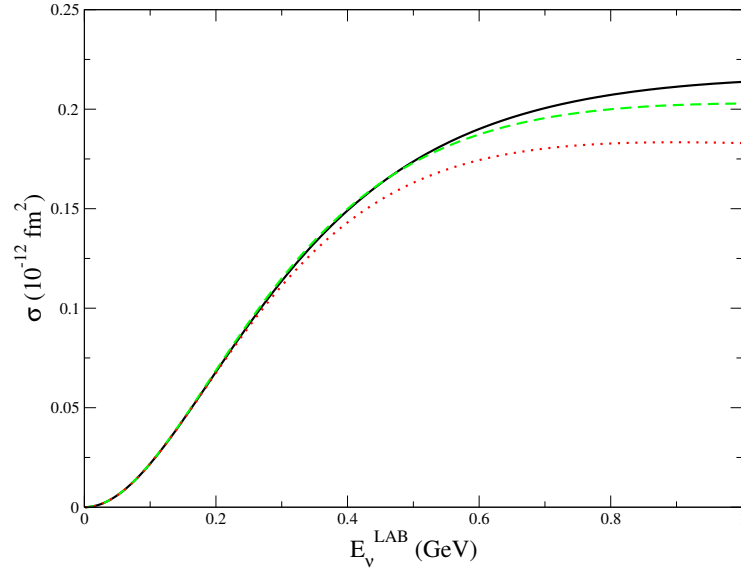
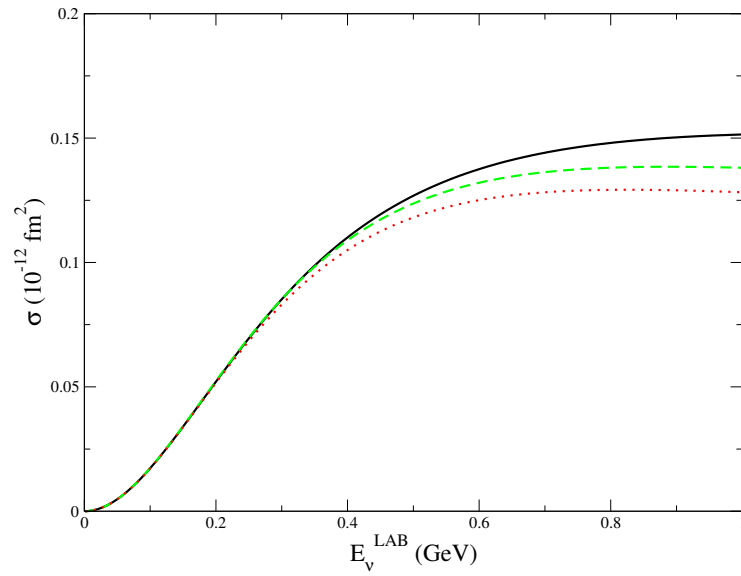
(a) $\nu + n \rightarrow \nu + n$ (b) $\nu + p \rightarrow \nu + p$

Figure 3.8: Cross section comparison up to 1 GeV for neutrino NC reactions. Line color and style are the same as in Fig. 3.7.

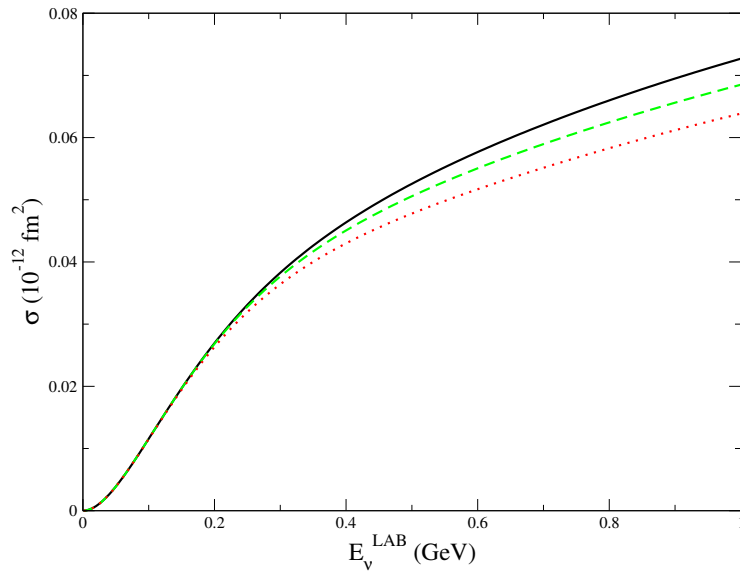
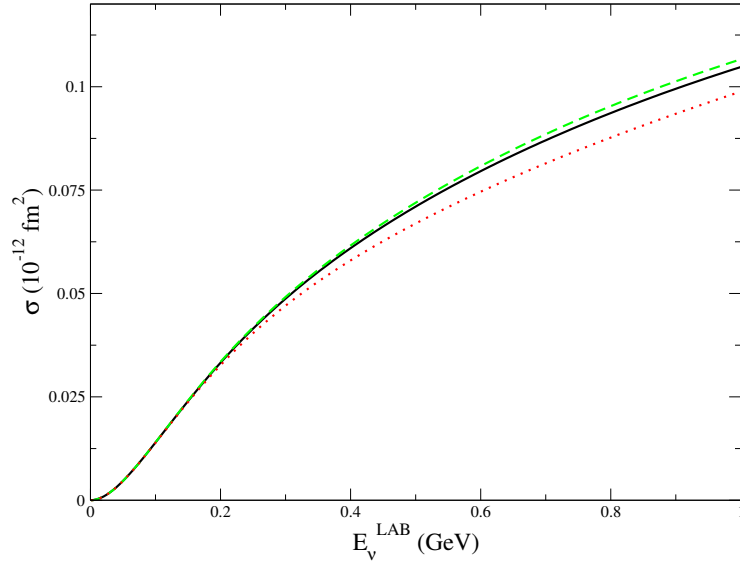


Figure 3.9: Cross section comparison up to 1 GeV for antineutrino NC reactions. Line color and style are the same as in Fig. 3.7.

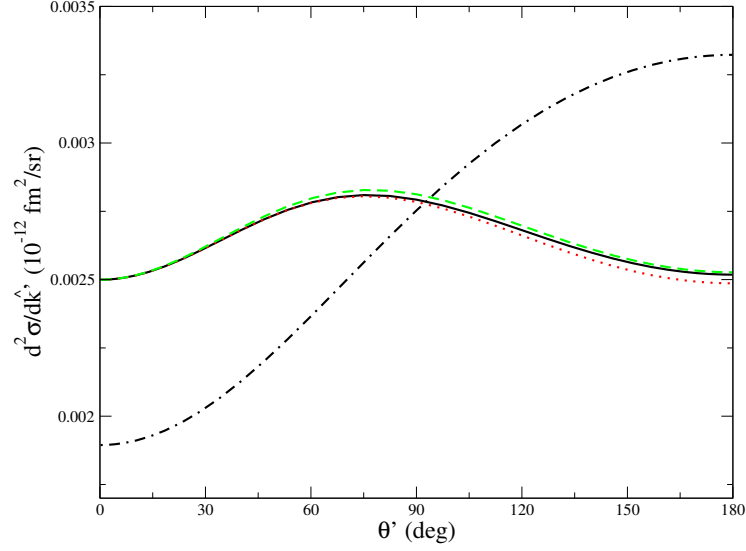


Figure 3.10: The differential cross section $\frac{d^2\sigma}{dk'}$ for the NC induced reaction $\nu_e + p \rightarrow \nu_e + p$ at the initial neutrino energy in the laboratory reference frame of 150 MeV in the laboratory and center-of-mass frame as a function of the corresponding neutrino scattering angle θ' . The lines are the same as in Fig. 3.7. Notice that, once integrated, the fully relativistic differential cross sections in the laboratory and in the center-of-mass reference frame will yield the same value.

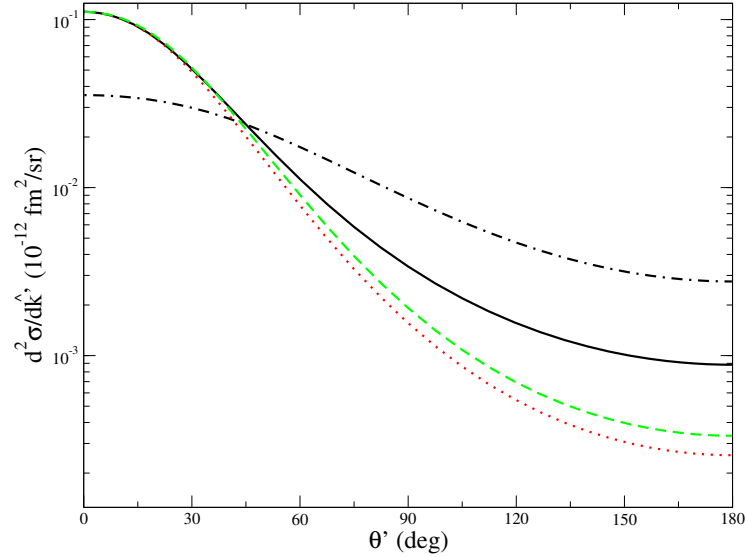


Figure 3.11: The same as Fig. 3.10 but for the initial neutrino energy of 1 GeV and on logarithmic scale.

Chapter 4

Elastic electron-nucleon scattering

The study of lepton reaction on single nucleons has many important applications in physics, since nucleons are bound states of quarks and gluons. The study of these bound states with *ab-initio* techniques involves the non-perturbative regime of QCD, which nowadays still poses a big challenge to physicists.

Quantum electrodynamics on the other hand is much better understood compared to QCD and has been proved a powerful tool to investigate the internal structure of nucleons.

As an example, many efforts have been made to explain the so-called root-mean-square (rms) radius of the proton discrepancy, that is a difference between values of this radius measured in muonic hydrogen spectroscopy and those stemming from elastic ep scattering experiments [93]. One of the ways to solve this problem is to investigate low-energy lepton-proton scattering and study the proton electric form factor G_E^p [94, 95, 96].

The experiments connected with the study of the proton electric form factor rely on the Rosenbluth formula and separation, with which it is possible to extract G_E^p from specific configurations of the measured cross section. In this chapter we obtain the Rosenbluth cross section as a special result for electron scattering.

Specifically, we will focus on the reactions

$$e^- + n \rightarrow e^- + n, \quad (4.1)$$

$$e^- + p \rightarrow e^- + p. \quad (4.2)$$

We show results for the differential cross section evaluated relativistically as well as with nonrelativistic approximations of the order $O(p/M)$ and $O(p^2/M^2)$ for the kinematics and the dynamics. Here p is the typical scale of the nucleon momentum in the laboratory reference frame.

Moreover, most of the formulas obtained in this chapter is very general and applies almost exactly to elastic deuteron electrodisintegration, which is studied in Chapters 6 and 7.

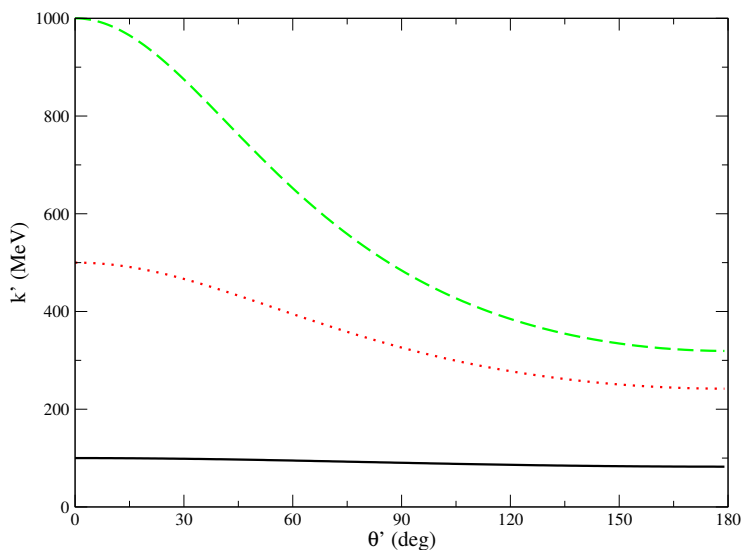


Figure 4.1: Solutions for k' with respect to the scattering angle θ' at 3 different values of k : 100 MeV (solid black line), 500 MeV (dotted red line) and 1000 MeV (dashed green line).

4.1 Kinematics

We denote by $\mathbf{k}$ ($\mathbf{k}'$) the initial (final) electron momentum and with $\mathbf{p}$ ($\mathbf{p}'$) the initial (final) nucleon momentum. As in the previous chapter θ' is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Now the equations describing conservation of energy and momentum read

$$\sqrt{m^2 + k^2} + \sqrt{M^2 + p^2} = \sqrt{m^2 + k'^2} + \sqrt{M^2 + p'^2} \quad (4.3)$$

and

$$\mathbf{k} + \mathbf{p} = \mathbf{k}' + \mathbf{p}'. \quad (4.4)$$

In the center-of-mass reference frame we obviously have $k = p$, $k' = p'$ and therefore $k = k'$.

From now on we will work exclusively in the laboratory reference frame. Here we have

$$\mathbf{p}' = \mathbf{k} - \mathbf{k}' \quad (4.5)$$

and the solutions for k' are

$$k' = \frac{k}{k^2(1 - \cos 2\theta') + 2(2ME + m^2 + M^2)} \times \left(\sqrt{2(2ME + E^2 + M^2)(m^2 \cos 2\theta' - m^2 + 2M^2)} + 2 \cos \theta' (ME + m^2) \right), \quad (4.6)$$

Here $E = \sqrt{m^2 + k^2}$ is the initial electron energy. A plot of the solution for three different values of k (100, 500 and 1000 MeV) is shown in Fig. 4.1.

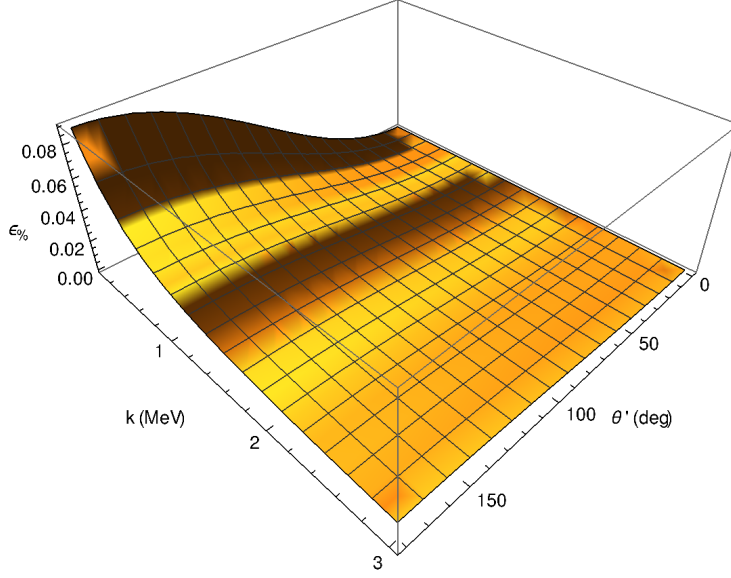


Figure 4.2: Percentage difference between k' evaluated considering and neglecting the electron mass, with respect to the initial electron momentum k and the scattering angle θ' .

We can expand the solution for k' to the second order in the variable $\eta = m/M \sim 5.4 \times 10^{-4}$, obtaining

$$k' \simeq \frac{k}{1 - k/M(1 + \cos \theta')} - \eta^2 \frac{M \sin^2 \left(\frac{\theta'}{2} \right) (k^2 - k \cos \theta' (k + M) + kM + M^2)}{(k(1 - \cos \theta') + M)^2} \quad (4.7)$$

The first term is the Compton scattering result, while the correction becomes more and more important for $k \rightarrow 0$. We can see it by evaluating

$$\epsilon_{\%} = 100 \left| \frac{k' - k'_0}{k'} \right| \quad (4.8)$$

with k'_0 being k' evaluated with $m = 0$. In the limit $k \rightarrow 0$ this becomes

$$\epsilon_{\%} = 100 \eta^2 \frac{M \sin^2 \frac{\theta'}{2}}{k}. \quad (4.9)$$

The percentage difference is shown in Fig. 4.2. From the figure we can see that the percentage difference has a peak around the point $k = 0$ and $\theta = 0$ and, even at the peak, it does not exceed 0.08%. In the limit in which we ignore the electron mass, Eq. (4.6) reduces to the expression in Eq. (3.10), *i.e.*

$$k' = \frac{k}{1 + k/M(1 - \cos \theta')}. \quad (4.10)$$

A comparison between this solution and the one in Eq. (4.6) for $k = 0.5$ MeV is shown in Fig. 4.3. We can see that the approximation in Eq. (4.10) works

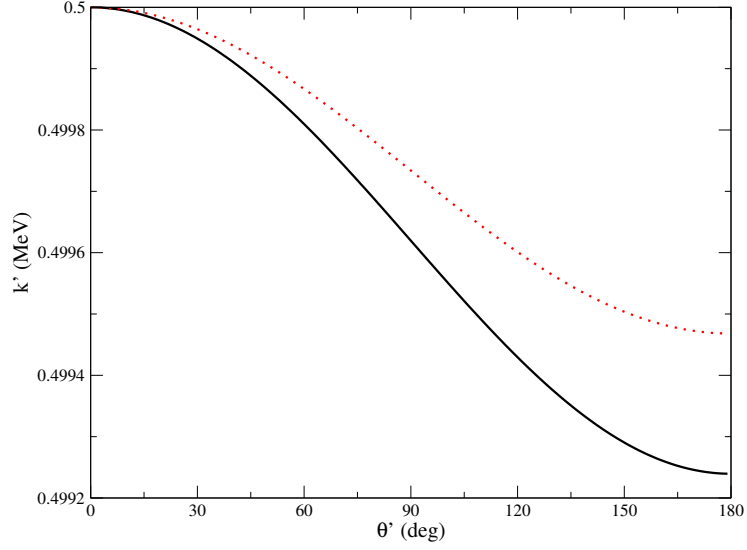


Figure 4.3: Solution for k' with respect to the scattering angles θ' at fixed initial electron energy $E = 0.5$ MeV. The solid black (dotted red) line is the solution evaluated by considering the electron mass equal to $m = 0.511$ ($m = 0$) MeV.

well already at these small energies and thus in our calculations we can set the electron mass to zero.

The momentum transfer squared Q^2 is

$$Q^2 = 2 \left(\sqrt{k^2 + m^2} \sqrt{k'^2 + m^2} - k k' \cos \theta' - m^2 \right) \quad (4.11)$$

which, if we ignore the electron mass, reduces to

$$Q^2 = 4 k k' \sin^2 \frac{\theta'}{2} = \frac{4 k^2 \sin^2 \frac{\theta'}{2}}{1 + 2 \frac{k}{M} \sin^4 \frac{\theta'}{2}}. \quad (4.12)$$

A plot for Q^2 with respect to θ' for three different values of k is shown in Fig. 4.4.

4.2 Cross section in the laboratory reference frame

By ignoring the electron mass the cross section in Eq. (55) can be written as

$$d\sigma = \frac{1}{64\pi^2} \frac{1}{M} \frac{k'}{E_{\mathbf{p}}' k} \frac{1}{\left| 1 + \frac{k' - k \cos \theta'}{E_{\mathbf{p}}'} \right|} |M_{fi}|^2 d\hat{\mathbf{k}}'. \quad (4.13)$$

and using

$$E_{\mathbf{p}}' = M + k - k' \quad (4.14)$$

we get the simpler expression

$$d\sigma = \frac{1}{64\pi^2} \frac{1}{M^2} \frac{k'^2}{k^2} |M_{fi}|^2 d\hat{\mathbf{k}}'. \quad (4.15)$$

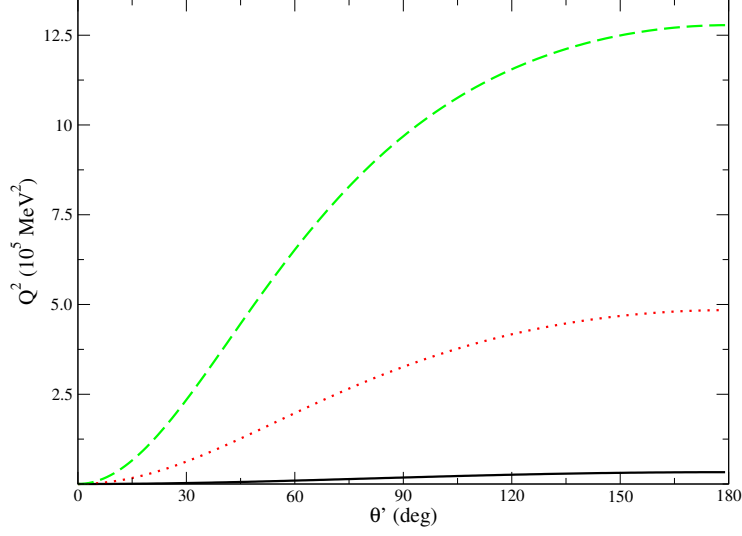


Figure 4.4: The four-momentum squared Q^2 with respect to the scattering angle θ' at $k = 100$ MeV (solid black line), $k = 500$ MeV (dotted red line) and $k = 1000$ MeV (dashed green line).

The transition matrix element squared $|M_{fi}|^2$ under the one-photon-exchange approximation stems from the photon propagator and the contraction of the electron L_μ and nucleon N^μ parts. Thus, we can write

$$|M_{fi}|^2 = \frac{e^4}{(Q^2)^2} |L_\mu N^\mu|^2, \quad (4.16)$$

where

$$L_\mu = \bar{u}(\mathbf{k}', \sigma'_e) \gamma_\mu u(\mathbf{k}, \sigma_e) \quad (4.17)$$

and the nucleon part N^μ contains the nucleon electromagnetic form factors $F_i = F_i(Q^2)$:

$$N^\mu = \bar{u}(\mathbf{p}', \sigma'_N) \left(\gamma^\mu F_1 + \frac{i}{2M} \sigma^{\mu\nu} q_\nu F_2 \right) u(\mathbf{p}, \sigma_N). \quad (4.18)$$

Using

$$\alpha = \frac{e^2}{4\pi} \quad \text{and} \quad Q^2 = 4kk' \sin^2 \frac{\theta'}{2} \quad (4.19)$$

we can write

$$\frac{d^2\sigma}{d\hat{\mathbf{k}}'} = \frac{\alpha^2}{256k^2k'^2 \sin^4 \frac{\theta'}{2}} \frac{1}{M^2} \frac{k'^2}{k^2} |L_\mu N^\mu|^2. \quad (4.20)$$

4.2.1 The transition matrix element

We consider first the situation where the spin direction of outgoing electrons is not measured, but initial electrons have helicity h . Then we use Eq. (53) to

obtain

$$L_{\mu\nu}^h = 2 \left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} (k \cdot k' - m^2) + i \epsilon_{\nu\mu\rho\sigma} m \tilde{s}^\rho(h) (k - k')^\sigma \right). \quad (4.21)$$

Neglecting the electron mass $m \tilde{s}^\rho(h)$ is replaced by $h k^\rho$. Note that the nucleon polarization in the initial and final states can still be chosen at will. If we further restrict ourselves to processes with unpolarized electron beams, we average $L_{\mu\nu}^h$ over the initial electron helicity h .

For fully unpolarized experiments, where we also average over the initial and sum over the final nucleon spin directions, we substitute

$$|L_\mu N^\mu|^2 \rightarrow \frac{1}{2} \sum_h \frac{1}{2} \sum_{\sigma_N \sigma'_N} L_{\mu\nu}^h N^{\mu\nu}, \quad (4.22)$$

where

$$N^{\mu\nu} = N^{*\mu} N^\nu. \quad (4.23)$$

By simplifying Eq. (4.22) we get

$$\begin{aligned} \frac{1}{4} L_{\mu\nu} N^{\mu\nu} = & 2F_1^2 \left[4m^2 (M^2 - s) + 2m^4 - 4M^2 s + 2M^4 - 2Q^2 s + (Q^2)^2 + 2s^2 \right] \\ & + 4F_1 F_2 Q^2 (Q^2 - 2m^2) + \frac{1}{M^2} F_2^2 \left\{ Q^2 \left[m^2 (-2M^2 + Q^2 - 2s) \right. \right. \\ & \left. \left. + m^4 + 2M^2 (Q^2 - s) + M^4 + s (s - Q^2) \right] \right\}, \quad (4.24) \end{aligned}$$

which, since we neglect the electron mass, becomes

$$\begin{aligned} \frac{1}{4} L_{\mu\nu} N^{\mu\nu} \Big|_{m=0} = & 2F_1^2 \left(-4M^2 s + 2M^4 - 2Q^2 s + (Q^2)^2 + 2s^2 \right) \\ & + 4F_1 F_2 (Q^2)^2 + F_2^2 \frac{Q^2 (2M^2 (Q^2 - s) + M^4 + s (s - Q^2))}{M^2}. \quad (4.25) \end{aligned}$$

This contraction corresponds to the one in Chapter 3.5 but for the electromagnetic elastic scattering on nucleons. For these reactions we can express s in terms of the initial electron energy in the laboratory reference frame k as

$$s = M \left(M + 2\sqrt{k^2 + m^2} \right) + m^2 \xrightarrow{m \rightarrow 0} M(M + 2k), \quad (4.26)$$

while Q^2 has been derived in Eq. (4.12).

By considering the Sachs form factors in place of F_1 and F_2 we arrive at the so-called Rosenbluth cross section

$$\left(\frac{d^2\sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Ros}} = \left(\frac{d^2\sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{1}{1 + \tau} \left(G_E^2 + \frac{\tau}{\epsilon} G_M^2 \right), \quad (4.27)$$

where

$$\epsilon = \left[1 + 2(1 + \tau) \tan^2 \frac{\theta'}{2} \right] \quad \text{and} \quad \tau = \frac{Q^2}{4M^2} \quad (4.28)$$

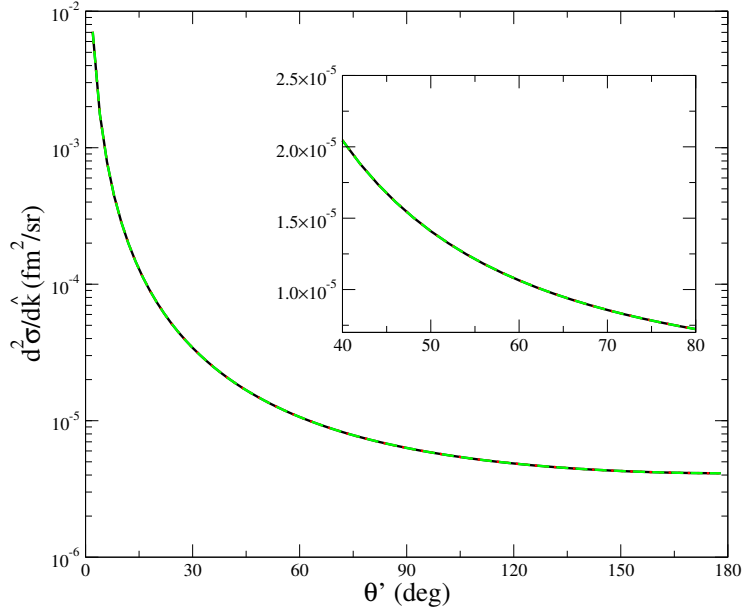


Figure 4.5: Differential cross section for the reaction $e^- + n \rightarrow e^- + n$ in the laboratory reference frame. The electron initial energy is 10 MeV. The solid black line is the relativistic cross section, the dotted red line is the nonrelativistic cross section with relativistic corrections and the dashed green line is the nonrelativistic cross section without relativistic corrections. In the box we focus on a smaller angle range in order to appreciate better the difference between the three lines. This is done by also adopting a linear scale for the insight.

and the Mott cross section is defined as

$$\left(\frac{d^2\sigma}{d\mathbf{k}'} \right)_{\text{Mott}} \equiv \frac{\alpha^2}{4k^2 \sin^4 \frac{\theta'}{2}} \cos^2 \frac{\theta'}{2}. \quad (4.29)$$

Some examples of the differential cross section for a set of chosen energies are shown in Figs. 4.5-4.7, compared to the results obtained adopting the relativistic expansion explained in the next section.

4.2.2 The $L_{\mu\nu}N^{\mu\nu}$ contraction and the v functions

We derive a general form of $L_{\mu\nu}^h N^{\mu\nu}$ in the reference frame where the transferred momentum is parallel to the z -axis and with the use of the current conservation relation. The last states that

$$q_\mu N^\mu = 0 \quad \rightarrow \quad \mathbf{q} \cdot \mathbf{N} = \omega N^0, \quad (4.30)$$

where $\omega = q^0$. Since we have q parallel to the z -axis we can substitute the z -component of $\mathbf{N}$, N^3 , with

$$N^3 = \frac{\omega}{q} N^0. \quad (4.31)$$

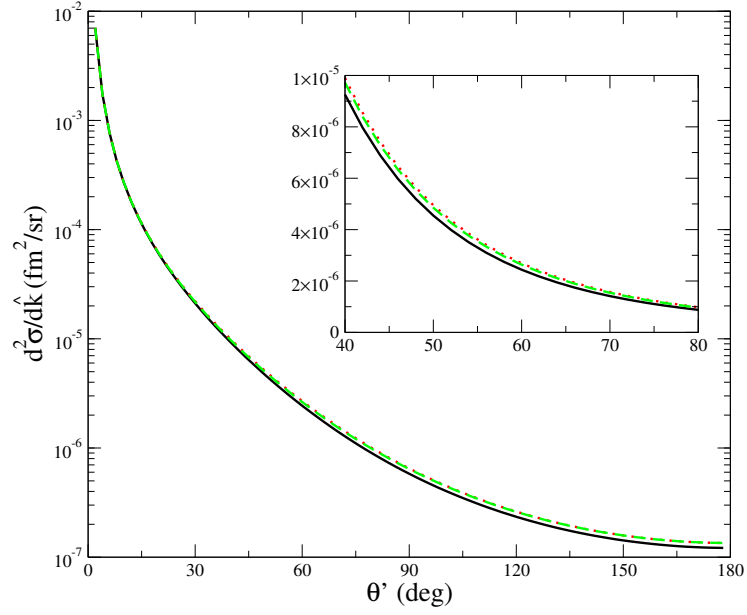


Figure 4.6: The same as Fig. 4.5 but for initial electron energy equal to 500 MeV.

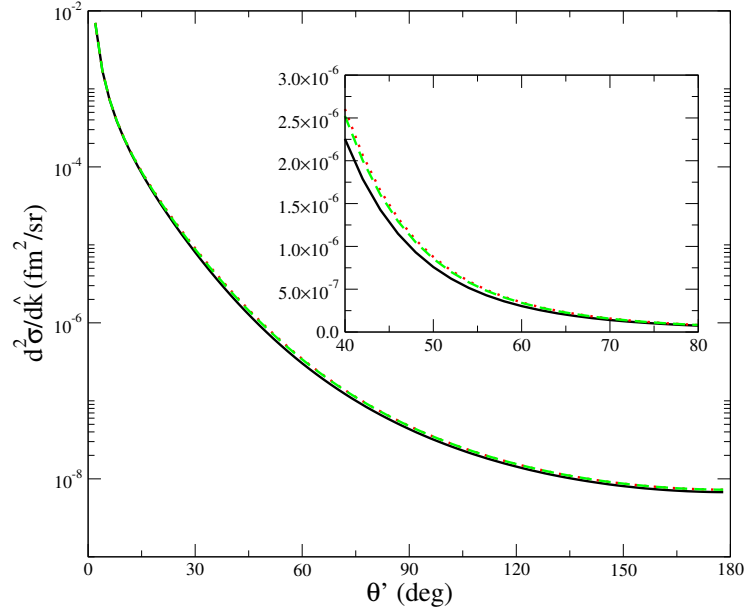


Figure 4.7: The same as Fig. 4.5 but for initial electron energy equal to 1000 MeV.

Repeating the steps taken in Chapter 3.2, *i.e.* switching to spherical components of N^μ , defining R_{ab} and additionally introducing new quantities

$$R_L = R_{00} , \quad R_T = R_{++} + R_{--} , \quad (4.32)$$

$$R_{TT} = 2\Re(R_{+-}) , \quad R_{TL} = -2\Re(R_{0+} - R_{0-}) , \quad (4.33)$$

$$R_{T'} = R_{++} - R_{--} , \quad R_{TL'} = R_{TL} = -2\Re(R_{0+} + R_{0-}) , \quad (4.34)$$

4.3. NONRELATIVISTIC EXPANSION OF THE SINGLE-NUCLEON ELECTROMAGNETIC CURRENT OPERATOR

we get [97]

$$L_{\mu\nu}N^{\mu\nu} = v_0 \left(v_L R_L + v_T R_T + v_{TL} R_{TL} + v_{TT} R_{TT} + h(v_{T'} R_{T'} + v_{TL'} R_{TL'}) \right). \quad (4.35)$$

Here

$$v_0 = (E + E')^2 - q^2 = 2kk'(1 + \cos \theta') \quad (4.36)$$

and

$$v_L = \frac{1}{v_0} \left(L_{00}^u + \frac{\omega}{q} (L_{03}^u + L_{30}^u) + \frac{\omega^2}{q^2} L_{33}^u \right) = \left(\frac{Q^2}{q^2} \right)^2 \quad (4.37)$$

$$v_{TL} = \frac{1}{\sqrt{2}v_0} \left(L_{01}^u + \frac{\omega}{q} L_{31}^u \right) = \frac{1}{\sqrt{2}} \frac{Q^2}{q^2} \sqrt{\frac{Q^2}{q^2} + \tan^2 \frac{\theta'}{2}} \quad (4.38)$$

$$v_T = \frac{1}{2v_0} \left(L_{11}^u + L_{22}^u \right) = \frac{1}{2} \frac{Q^2}{q^2} + \tan^2 \frac{\theta'}{2} \quad (4.39)$$

$$v_{TT} = \frac{1}{2v_0} \left(-L_{11}^u + L_{22}^u \right) = -\frac{1}{2} \frac{Q^2}{q^2} \quad (4.40)$$

$$v_{T'} = -\frac{i}{v_0} L_{12}^p = \sqrt{\frac{Q^2}{q^2} + \tan^2 \frac{\theta'}{2}} \tan \frac{\theta'}{2} \quad (4.41)$$

$$v_{TL'} = -\frac{i}{\sqrt{2}v_0} \left(L_{02}^p - \frac{\omega}{q} L_{23}^p \right) = -\frac{1}{\sqrt{2}} \frac{Q^2}{q^2} \tan \frac{\theta'}{2} \quad (4.42)$$

Equation 4.35 contains a general form for the $L_{\mu\nu}$ function. This shows that all the electromagnetic reactions can be described by these formulas in the laboratory reference frame and having $\mathbf{q}$ parallel to the z -axis. This will be true also for the elastic and breakup reactions on nuclei.

4.3 Nonrelativistic expansion of the single-nucleon electromagnetic current operator

We can perform the same expansion we have seen in Chapter 3.6 for the weak nucleon current, obtaining

$$\tilde{N}_{NR}^0 = F_1 \chi_{\sigma'}^\dagger \chi_\sigma = F_1 \delta_{\sigma'\sigma} \quad (4.43)$$

and

$$\tilde{\mathbf{N}}_{NR} = \chi_{\sigma'}^\dagger \left[F_1 \frac{\mathbf{p}' + \mathbf{p}}{2M} + \frac{1}{2M} (F_1 + F_2) i \boldsymbol{\sigma} \times (\mathbf{p}' - \mathbf{p}) \right] \chi_\sigma. \quad (4.44)$$

at order $O(p/M)$. If we consider the relativistic corrections up to $O(p^2/M^2)$ only $\tilde{N}_{NR}^0$ gets extra terms, becoming

$$\begin{aligned} \tilde{N}_{NR+RC}^0 = \chi_{\sigma'}^\dagger & \left[F_1 - (F_1 + 2F_2) \frac{(\mathbf{p}' - \mathbf{p})^2}{8M^2} \right. \\ & \left. + (F_1 + 2F_2) i \frac{(\mathbf{p}' \times \mathbf{p}) \cdot \boldsymbol{\sigma}}{4M^2} \right] \chi_\sigma. \end{aligned} \quad (4.45)$$

Keeping the expansion given in Eq. (4.35) in mind, the differential cross section in the laboratory reference frame can be written as

$$\frac{d^2\sigma}{d\hat{\mathbf{k}}'} = \left(\frac{d^2\sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{1}{16k'^2 \cos^2 \frac{\theta'}{2}} \frac{k'^2}{k^2} \left| L_\alpha \tilde{N}^\alpha \right|^2. \quad (4.46)$$

In Figs. 4.5,4.6,4.7 we show the results for the differential cross section $d^2\sigma/d\hat{\mathbf{k}}'$ at three different initial electron energies (10, 500 and 1000 MeV) obtained adopting this relativistic expansion, compared with the same obtained the full-fledged relativistic approach for the reaction $e^- + n \rightarrow e^- + n$. The results on proton have similar features so we will not show them in this work. In these calculations the relativistic expansion has been carried out also in the kinematic quantities, such as the value for k' , s , Q^2 and the phase space.

As it is possible to see from these figures the nonrelativistic expansion converges to the relativistic cross section for smaller energies as expected. Another interesting feature, which we have seen hold for neutrino induced reactions as well, is that the results which not include relativistic corrections are closer to the fully relativistic results.

4.4 Spin-dependent helicity asymmetry

Up to now we have considered only unpolarized cross sections. An interesting feature of the elastic electron-nucleon scattering (at least in one-photon-exchange approximation) is that, in order to see polarization effects arising from the initial state, it is not sufficient to polarize only the electron beam or only the nucleon target. If only one particle is polarized in the initial state, and no polarization is measured in the final state, then the cross section remains the same as in the unpolarized case.

Let us consider the case in which the electron beam has a well-defined helicity, $h \pm 1$, and the nucleon spin has a definite direction in its rest frame given by the unit vector $\hat{\sigma}_N$. Then the following asymmetry $A(\sigma_N)$ is built:

$$A(\sigma_N) = \frac{\sigma(h = +1, \sigma_N) - \sigma(h = -1, \sigma_N)}{\sigma(h = +1, \sigma_N) + \sigma(h = -1, \sigma_N)}, \quad (4.47)$$

where $\sigma(h, \sigma_N)$ is the exclusive or (partially) inclusive cross section defined for specific h and σ_N .

In order to evaluate $A(\sigma_N)$ in the simplest case of the exclusive kinematics for the fixed electron scattering angle θ' it is sufficient to replace $\sigma(h, \sigma_N)$ by the contraction of $L_{\mu\nu}^h$ with $N_{\sigma_N}^{\mu\nu} \equiv \sum_{\sigma'_N} (N^\mu)^* N^\nu$. We neglect the electron mass and simplify Eq. (4.21), which now reads

$$L_{\mu\nu}^h = 2 \left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} (k \cdot k') - i h \epsilon_{\nu\mu\rho\sigma} k^\rho k'^\sigma \right). \quad (4.48)$$

Using the properties of the Dirac spinors we write $N_{\sigma_N}^{\mu\nu}$ as

$$N_{\sigma_N}^{\mu\nu} = \text{Tr} \left[\frac{1 + \gamma^5 \not{\phi}_N}{2} (\not{p} + M) \left(\gamma^\nu (F_1 + F_2) - \frac{(p + p')^\nu}{2M} F_2 \right) \right. \\ \left. \times (\not{p}' + M) \left(\gamma^\mu (F_1 + F_2) - \frac{(p + p')^\mu}{2M} F_2 \right) \right], \quad (4.49)$$

where (see Eq. (4.50))

$$\sigma_N^\mu = \left(\frac{\mathbf{p} \cdot \hat{\sigma}_N}{M}, \hat{\sigma}_N + \frac{(\mathbf{p} \cdot \hat{\sigma}_N) \mathbf{p}}{M(p^0 + M)} \right) \quad (4.50)$$

and reduces to $(0, \hat{\sigma}_N)$ in the laboratory reference frame. If the electric G_E and magnetic G_M (Sachs) form factors are introduced

$$\begin{aligned} G_E(Q^2) &= F_1(Q^2) - \tau F_2(Q^2), \\ G_M(Q^2) &= F_1(Q^2) + F_2(Q^2), \\ \tau &\equiv \frac{Q^2}{4M^2}, \end{aligned} \quad (4.51)$$

then $A(\sigma_N)$ can be given in the following invariant form:

$$A(\sigma_N) = \frac{c_1 G_M^2 + c_2 G_M G_E}{d_1 G_M^2 + d_2 G_E^2} \quad (4.52)$$

with

$$\begin{aligned} c_1 &= 4M(p' \cdot \sigma_N) ((k' \cdot p)^2 - (k \cdot p)^2) \\ c_2 &= 2M((k \cdot p) - (k' \cdot p)) \\ &\quad \times \left(2(p' \cdot \sigma_N)((k \cdot p) + (k' \cdot p)) - (4M^2 + Q^2)((k \cdot \sigma_N) + (k' \cdot \sigma_N)) \right), \\ d_1 &= -2 \left((2M^2 + Q^2)(k \cdot p)^2 + (2M^2 + Q^2)(k' \cdot p)^2 - 4M^2(k \cdot p)(k' \cdot p) \right), \\ d_2 &= M^2 \left(Q^2(4M^2 + Q^2) - 4((k \cdot p) + (k' \cdot p))^2 \right). \end{aligned} \quad (4.53)$$

The helicity asymmetry is very often studied in the laboratory reference frame with axes orientation chosen in the following way:

$$\begin{aligned} \hat{\mathbf{z}} &= \frac{\mathbf{k} - \mathbf{k}'}{|\mathbf{k} - \mathbf{k}'|} \equiv \frac{\mathbf{q}}{q} \\ \hat{\mathbf{y}} &= \frac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|} \\ \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{z}}. \end{aligned} \quad (4.54)$$

It then simplifies to [98]

$$A(\hat{\sigma}_N) = \frac{-\frac{Q^2}{2M^2} \left[\sqrt{\frac{Q^2}{q^2} + \tan^2 \frac{\theta'}{2}} G_M^2 (\hat{\sigma}_N)_z + \frac{2M}{q} G_E G_M (\hat{\sigma}_N)_x \right]}{\left[G_E^2 + \frac{Q^2}{4M^2} G_M^2 \left(1 + 2 \left(1 + \frac{Q^2}{4M^2} \right) \tan^2 \frac{\theta'}{2} \right) \right] \left(1 + \frac{Q^2}{4M^2} \right)}. \quad (4.55)$$

It follows from Eq. (4.55) that

$$\begin{aligned} A^\parallel &\equiv A(\hat{\sigma}_N = \hat{\mathbf{z}}) \simeq G_M^2 \\ A^\perp &\equiv A(\hat{\sigma}_N = \hat{\mathbf{x}}) \simeq G_M G_E \\ \frac{A^\perp}{A^\parallel} &\simeq G_E. \end{aligned} \quad (4.56)$$

Thus, the helicity asymmetry gives access to the electromagnetic nucleon form factors and can be measured directly for the proton. Since there is no free neutron target in nature, very often polarized ^3He is used as an effective polarized neutron target and the asymmetries measured in $e^- - ^3\text{He}$ scattering experiments yield information about the electromagnetic neutron form factors [99].

Examples for the cross section asymmetry at the initial electron energy 1000 MeV are shown in Fig. 4.8.

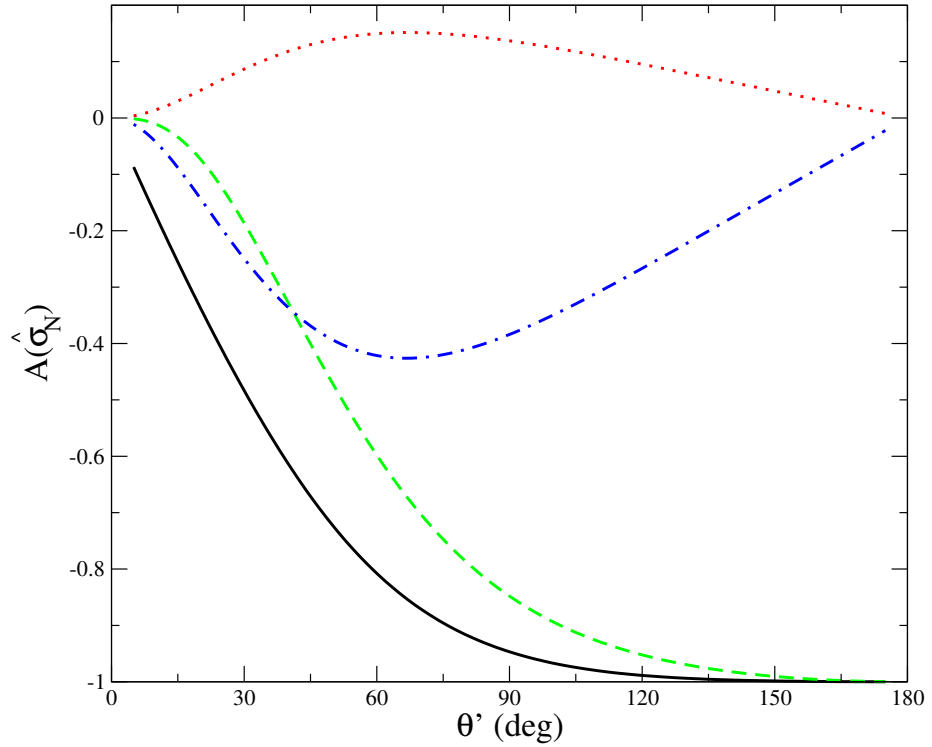


Figure 4.8: Asymmetries $A^{\parallel}$ and $A^{\perp}$ for the electron elastic scattering on proton and neutron. The solid black (dashed green) line represents $A^{\parallel}$ for the proton (neutron), while the dotted red (dash-dotted blue) line represents $A^{\perp}$ for the proton (neutron).

Chapter 5

Nonrelativistic nucleon-nucleon system

As mentioned in Chapter 3 there have been many studies on neutrino-induced reactions on light nuclei. The results from Refs. [8, 13] and the majority from Ref. [9] were obtained using the so-called “standard nuclear physics approach”, which means adopting the Argonne v_{18} potential (AV18) as the nucleon-nucleon potential and adding the contribution from two-nucleon currents to the single-nucleon one. In this work we also adopt the AV18 potential as our nucleon-nucleon interaction model. This potential contains 14 terms which are charge-independent, three charge-dependent and one charge-asymmetric term, and it is provided in the coordinate space [4]. We transform the AV18 potential to momentum space and obtain the binding energy and deuteron wave function both in position and in momentum space.

5.1 Nonrelativistic deuteron wave function

In order to test our usage of the potential we reproduce the deuteron binding energy and wave function. In Tab. 5.1 the essential data for the deuteron nucleus are summarized.

The deuteron is a bound neutron-proton system. According to experimental data, its total spin-parity is $J^\pi = 1^+$, therefore the proton-neutron relative orbital angular momentum must be even. The total spin of its constituents is either $S = 0$ or $S = 1$, due to the fact that each nucleon has spin $s_i = 1/2$. The deuteron has also a small, but non vanishing, electric quadrupole moment and this tells us that its wave function cannot be a pure s -wave state. Adding the information that the deuteron magnetic moment is very close to the sum of the proton and the neutron magnetic moments, leads to the conclusion that the

M_d (MeV)	J^π	B (keV)	μ (μ_N)	Q ($e\text{ fm}^2$)
1875.61	1^+	2224.57	0.857	0.28

Table 5.1: Deuteron mass M_d [100], spin/parity J^π , binding energy B [101], electric dipole and magnetic quadrupole moments (μ and Q) [102].

spins of the nucleons add to $S = 1$ and therefore the orbital angular momentum can be $\ell = 0$ or $\ell = 2$.

5.1.1 Configuration space approach

In order to solve the Schrödinger equation for the bound state, we work in the relative reference frame where the total momentum of the system is zero and introduce the relative position $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_n$, which is the difference between the proton ($\mathbf{r}_p$) and the neutron ($\mathbf{r}_n$) coordinates. The time-independent Schrödinger equation reads

$$H|\phi_d \mu_d\rangle = E|\phi_d \mu_d\rangle, \quad (5.1)$$

where E is the eigenvalue of the Hamiltonian operator H , which comprises the kinetic energy H_0 and potential energy V parts, and $|\phi_d \mu_d\rangle$ are the Hamiltonian eigenstates, which should not depend on the deuteron total angular momentum projection μ_d for rotationally invariant potentials. We can thus rewrite it as

$$(H_0 + V)|\phi_d \mu_d\rangle = E|\phi_d \mu_d\rangle. \quad (5.2)$$

We represent this equation in the coordinate space partial wave basis, obtaining

$$\langle r\alpha|(H_0 + V)|\phi_d \mu_d\rangle = E\langle r\alpha|\phi_d \mu_d\rangle. \quad (5.3)$$

The states $|r\alpha\rangle$ contain information about the magnitude of the relative position of the two nucleons, about their orbital angular momentum ℓ , spin S , the total angular momentum J with its projection μ . Since we treat nucleons as identical particles, we introduce additionally the total two-nucleon isospin T and its projection τ

$$|r\alpha\rangle \equiv |r(\ell S)J\mu; T\tau\rangle. \quad (5.4)$$

These states are normalized as

$$\langle r\alpha|r'\alpha'\rangle = \frac{\delta(r - r')}{r^2} \delta_{\alpha'\alpha}. \quad (5.5)$$

Using the completeness relation

$$\sum_{\alpha} \int_0^{\infty} dr' r'^2 |r'\alpha'\rangle \langle r'\alpha'| = 1 \quad (5.6)$$

we can write

$$\sum_{\alpha'} \int dr' r'^2 \langle r\alpha|(H_0 + V)|r'\alpha'\rangle \varphi^{(\ell')}(r') = E\varphi^{(\ell)}(r), \quad (5.7)$$

where we defined

$$\varphi^{(\ell)}(r) \equiv \langle r\alpha|\phi_d \mu_d\rangle. \quad (5.8)$$

Thanks to the property

$$\langle r\alpha|H_0|r'\alpha'\rangle = \delta_{\alpha\alpha'} \frac{\delta(r - r')}{r^2} \frac{1}{M} \left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\ell(\ell+1)}{r^2} \right], \quad (5.9)$$

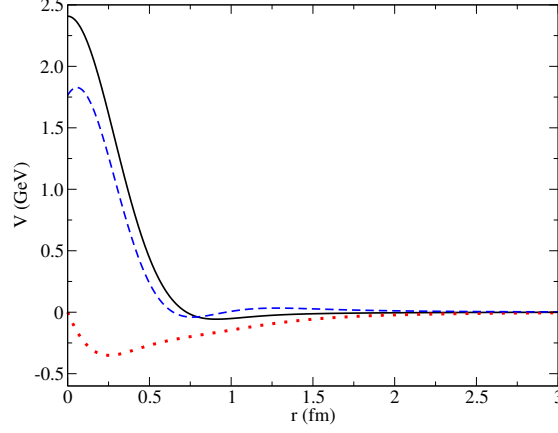


Figure 5.1: The AV18 potential for the deuteron channels. The solid black line is the $V_{\ell=0, \ell'=0}$ component, the dotted red line is the $V_{02} = V_{20}$ component and the dashed blue line is the V_{22} component.

where M is the average nucleon mass and due to the fact that for local potentials we have

$$\langle r\alpha|V|r'\alpha'\rangle = \frac{\delta(r-r')}{r^2} V_{\alpha\alpha'}(r), \quad (5.10)$$

we can write the Schödinger equation as

$$\sum_{\alpha'} \left[\left(-\frac{1}{M} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\ell(\ell+1)}{Mr^2} \right) \delta_{\alpha\alpha'} + V_{\alpha\alpha'}(r) \right] \varphi_d^{(\ell')}(r) = E \varphi_d^{(\ell)}(r). \quad (5.11)$$

The potential $V_{\alpha\alpha'}$ is shown in Fig. 5.1, where we plot the AV18 potential up to $r = 3$ fm for the deuteron channels, *i.e.* the ${}^3S_1$ and ${}^3D_1$ channels. The potential couples the two different channels. By introducing the reduced radial wave function

$$u_\ell(r) \equiv r \varphi_d^{(\ell)}(r), \quad (5.12)$$

Eq. (5.11) simplifies to

$$\sum_{\alpha'} \left[\left(-\frac{1}{M} \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{Mr^2} \right) \delta_{\alpha\alpha'} + V_{\alpha\alpha'}(r) \right] u_{\ell'}(r) = E u_\ell(r). \quad (5.13)$$

We solved this equation with the renormalized Numerov method presented in Ref. [103]. The resulting wave function components $\varphi_d^{(0)}(r)$ and $\varphi_d^{(2)}(r)$ are shown in Fig. 5.2a. The deuteron radial wave function is normalized as

$$\sum_\ell \int_0^\infty dr r^2 (\varphi_d^{(\ell)}(r))^2 = \sum_\ell \int_0^\infty dr u_\ell^2(r) = 1. \quad (5.14)$$

It is possible to transform the resulting wave function to its momentum space counterpart using Eq. (23)

$$\langle r\ell'm'|p\ell m\rangle = \sqrt{\frac{2}{\pi}} i^\ell j_\ell(pr) \delta_{\ell\ell'} \delta_{mm'} \quad (5.15)$$

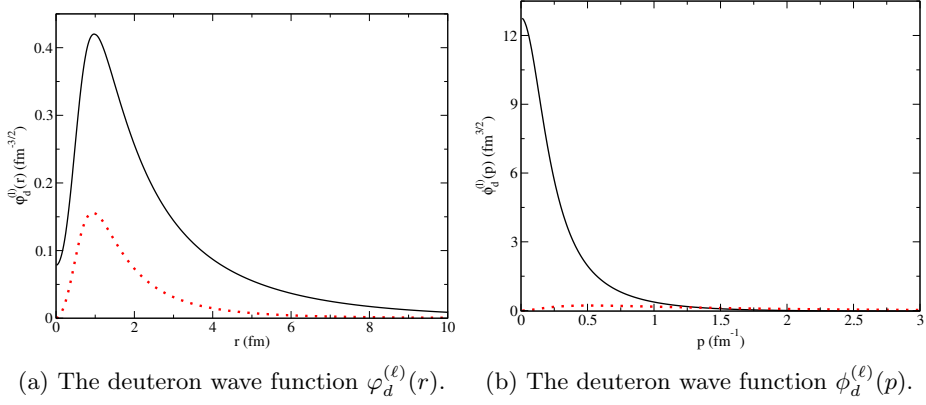


Figure 5.2: Deuteron wave function in coordinate (a) and momentum space (b). The solid black line (dotted red line) is the ${}^3S_1$ (${}^3D_1$) component of the wave function. The sign of the ${}^3D_1$ component in momentum space has been inverted.

where $j_\ell(x)$ is the spherical Bessel function regular at $x = 0$. From this equation it follows that

$$\langle p\alpha|r\alpha'\rangle = \sqrt{\frac{2}{\pi}}(-i)^\ell j_\ell(pr)\delta_{\alpha\alpha'} \quad (5.16)$$

and therefore if we are interested in $\phi_d^{(\ell)}(p)$ we have to calculate

$$\phi_d^{(\ell)}(p) = \langle p\alpha|\phi_d\mu_d\rangle = \sum_{\alpha'} \int dr r^2 \langle p\alpha|r\alpha'\rangle \varphi_d^{\ell'}(r), \quad (5.17)$$

here we used the completeness relation Eq. (5.6). By inserting Eq. (5.16) in the last equation we obtain

$$\phi_d^{(\ell)}(p) = \sqrt{\frac{2}{\pi}}(-i)^\ell \int dr r^2 j_\ell(pr) \varphi_d^{(\ell)}(r). \quad (5.18)$$

The resulting wave function components are shown in Fig. 5.2b. The normalization is

$$\sum_\ell \int dp p^2 (\phi_d^{(\ell)})^2 = 1. \quad (5.19)$$

5.1.2 Momentum space approach

In order to evaluate the deuteron wave function in momentum space we need the matrix elements of the potential operator in momentum space. To obtain them we insert the completeness relation for the coordinate space partial wave states

$$\begin{aligned} \langle p'\alpha'|V|p\alpha\rangle &= \sum_{\beta'} \sum_{\beta} \int dr' r'^2 \int dr r^2 \\ &\quad \times \langle p'\alpha'|r'\beta'\rangle \langle r'\beta'|V|r\beta\rangle \langle r\beta|p\alpha\rangle. \end{aligned} \quad (5.20)$$

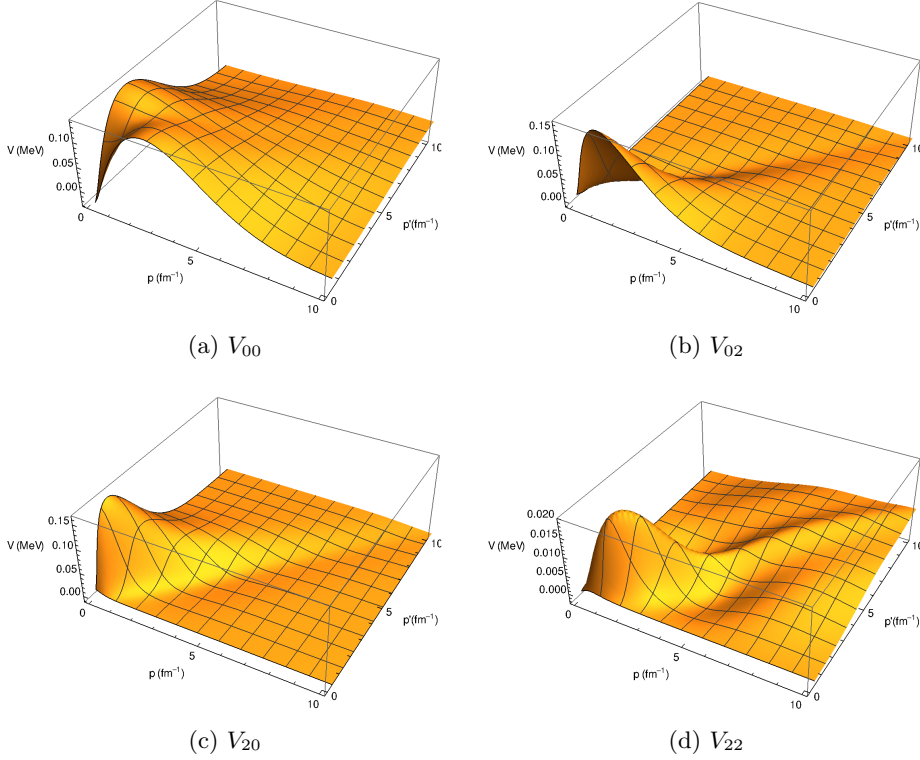


Figure 5.3: The AV18 potential for the coupled channel $J^\pi = 1^+$.

Using Eq. (5.16) we can rewrite it as

$$\langle p'\alpha' | V | p\alpha \rangle = \frac{2}{\pi} (-i)^{\ell' - \ell} \int dr' r'^2 \int dr r^2 j_{\ell'}(p'r') j_{\ell}(pr) \langle r'\alpha' | V | r\alpha \rangle. \quad (5.21)$$

For local potentials simplifies to

$$\langle p'\alpha' | V | p\alpha \rangle = \frac{2}{\pi} (-i)^{\ell' - \ell} \int dr r^2 j_{\ell'}(p'r) j_{\ell}(pr) \langle r\alpha' | V | r\alpha \rangle, \quad (5.22)$$

or, simplifying the notation,

$$V_{\alpha'\alpha}(p', p) = \frac{2}{\pi} (-i)^{\ell' - \ell} \int dr r^2 j_{\ell'}(p'r) j_{\ell}(pr) V_{\alpha'\alpha}(r). \quad (5.23)$$

The momentum space potential matrix elements for the deuteron channels are shown in Fig. 5.3. Due to the fact that the potential we use does not change the total angular momentum, its projection and the total spin, we use the following shorthand notation

$$V_{\ell'\ell}(p', p) \equiv \langle p'(\ell'S)J\mu'; T'\tau' | V | p(\ell S)J\mu; T\tau \rangle. \quad (5.24)$$

Having obtained the potential in this basis we evaluate the deuteron wave function by projecting the Schrödinger equation in the momentum partial wave

basis. We start from the Schrödinger equation in the center-of-mass reference frame

$$(H_0 + V) |\phi_d \mu_d\rangle = E |\phi_d \mu_d\rangle, \quad (5.25)$$

where $H_0 = \hat{p}^2/M$ is the operator of the relative kinetic energy. We project on the left of the Schrödinger equation the state $\langle p\alpha|$, obtaining

$$\left\langle p\alpha \left| \left(\frac{\hat{p}^2}{M} + V \right) \right| \phi_d \mu_d \right\rangle = E \langle p\alpha | \phi_d \mu_d \rangle. \quad (5.26)$$

We then insert the completeness relation

$$\sum_{\alpha'} \int_0^\infty dp' p'^2 |p' \alpha'\rangle \langle p' \alpha'| = 1 \quad (5.27)$$

obtaining

$$\frac{p^2}{M} \phi_d^{(\ell)}(p) + \sum_{\ell'} \int_0^\infty dp' p'^2 V_{\ell'\ell}(p', p) \phi_d^{(\ell')}(p') = E \phi_d^{(\ell)}(p). \quad (5.28)$$

We discretize this equation introducing p_i points, distributed from 0 to a certain value $p_{\max}$. We can then rewrite the equation as

$$\sum_{\ell=\{0,2\}} \sum_{j=1}^N \left(\frac{p_i^2}{M} \delta_{ij} \delta_{\ell'\ell} + w_j p_j^2 V_{ij}^{\ell'\ell} \right) \phi_j^{(\ell')} = E \phi_i^{(\ell)} \quad (5.29)$$

where $V_{ij}^{\ell'\ell} \equiv V_{\ell'\ell}(p_i, p_j)$, $\phi_i^{(\ell)} = \phi_d^{(\ell)}(p_i)$ and w_j are the integration weights. This turns to an eigenvalue problem for the matrix

$$H_{ij}^{\ell'\ell} = \frac{p_i^2}{M} \delta_{ij} \delta_{\ell'\ell} + w_j p_j^2 V_{ij}^{\ell'\ell}. \quad (5.30)$$

The deuteron binding energy corresponds to the only negative eigenvalue of the $H_{ij}^{\ell'\ell}$ matrix.

5.2 Nonrelativistic Lippmann-Schwinger equation and T -matrix for scattering states

The Schrödinger equation for an arbitrary positive energy E

$$(H_0 + V) |\Psi\rangle = E |\Psi\rangle \quad (5.31)$$

can be rearranged as

$$(E - H_0) |\Psi\rangle = V |\Psi\rangle. \quad (5.32)$$

This equation has two types of solutions $|\Psi\rangle^{(\pm)}$ (*Lippmann-Schwinger equation*)

$$|\Psi\rangle^{(\pm)} = |\phi\rangle + G_0(E \pm i\varepsilon) V |\Psi\rangle^{(\pm)}, \quad (5.33)$$

where $|\phi\rangle$ satisfies the free Schrödinger equation

$$H_0 |\phi\rangle = E |\phi\rangle \quad (5.34)$$

5.2. NONRELATIVISTIC LIPPMANN-SCHWINGER EQUATION AND T-MATRIX FOR SCATTERING STATES

and $G_0(E)$ is the free 2N propagator

$$G_0(E) = \frac{1}{E - H_0}. \quad (5.35)$$

As we discussed in the Notation section, the regularization $\pm i\varepsilon$ is necessary to deal with the singularity appearing in the denominator of G_0 . By evaluating the matrix elements of G_0 between two position eigenstates we find (for example see Ref. [5])

$$\langle \mathbf{x}' | G_0(E \pm i\varepsilon) | \mathbf{x} \rangle = -\frac{M}{4\pi} \frac{e^{\pm i\sqrt{ME}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \equiv -\frac{M}{4\pi} \frac{e^{\pm i\sqrt{ME}r}}{r}. \quad (5.36)$$

This exhibits the difference between an outgoing and incoming spherical waves if we choose the $+$ ($-$) sign in front of $i\varepsilon$.

5.2.1 Definition of the T operator

One way to determine the two-nucleon scattering state is by introducing the T operator, which is defined as

$$T|\phi\rangle \equiv V|\Psi\rangle^{(\pm)}. \quad (5.37)$$

Applying V on the left of Eq. (5.33) we obtain

$$V|\Psi\rangle^{(\pm)} = V|\phi\rangle + V G_0(E \pm i\varepsilon) V|\Psi\rangle^{(\pm)}$$

and therefore

$$T|\phi\rangle = V|\phi\rangle + V G_0(E \pm i\varepsilon) T|\phi\rangle.$$

From the last equation we get the so-called *right*-Lippmann-Schwinger equation for the T -matrix as

$$T = V + V G_0 T \quad (5.38)$$

Applying T on the left of Eq. (5.33) we obtain

$$T|\Psi\rangle^{(\pm)} = T|\phi\rangle + T G_0(E \pm i\varepsilon) V|\Psi\rangle^{(\pm)}$$

and therefore

$$T|\Psi\rangle^{(\pm)} = V|\Psi\rangle^{(\pm)} + T G_0(E \pm i\varepsilon) V|\Psi\rangle^{(\pm)},$$

leads to the so-called *left*-equation for T

$$T = V + T G_0 V. \quad (5.39)$$

By using the T operator we can write Eq. (5.33) as

$$|\Psi\rangle^{(\pm)} = \left(1 + G_0(E \pm i\varepsilon)T\right)|\phi\rangle \quad (5.40)$$

and express the scattering wave function as an addition to the plane wave solution. Similarly

$${}^{(\pm)}\langle\Psi| = \langle\phi|\left(1 + T G_0(E \mp i\varepsilon)\right). \quad (5.41)$$

5.2.2 T -matrix for two-nucleon states

In order to use Eq. (5.40) we need to evaluate matrix elements of the T operator in the partial wave basis $|p\alpha\rangle$

$$\langle p'\alpha'|T|p\alpha\rangle = \langle p'\alpha'|V|p\alpha\rangle + \langle p'\alpha'|T G_0 V|p\alpha\rangle. \quad (5.42)$$

By inserting the completeness relation

$$\sum_{\alpha''} \int_0^\infty dp'' p''^2 |p''\alpha''\rangle \langle p''\alpha''| = 1 \quad (5.43)$$

we get

$$\langle p'\alpha'|T|p\alpha\rangle = \langle p'\alpha'|V|p\alpha\rangle + \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{\langle p'\alpha'|V|p''\alpha''\rangle \langle p''\alpha''|T|p\alpha\rangle}{E - p''^2/M + i\varepsilon}. \quad (5.44)$$

We define $A_{p'p}^{\alpha'\alpha}$ as

$$A_{p'p}^{\alpha'\alpha} \equiv \langle p'\alpha'|A|p\alpha\rangle \quad (5.45)$$

so we can rewrite Eq. (5.44):

$$T_{p'p}^{\alpha'\alpha} = V_{p'p}^{\alpha'\alpha} + M \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{T_{p'p''}^{\alpha'\alpha''} V_{p''p}^{\alpha''\alpha}}{p_0^2 - p''^2 + i\varepsilon}, \quad (5.46)$$

where p_0 is defined by the relation $E = p_0^2/M$.

In practice the integral in Eq. (5.46) is limited to some maximum value for the momentum ($\bar{p}$) and we also need to take care of the pole in the denominator at $p'' = p_0$. In order to do so we write Eq. (5.46) as

$$\begin{aligned} T_{p'p}^{\alpha'\alpha} = V_{p'p}^{\alpha'\alpha} + M \sum_{\alpha''} \int_0^{\bar{p}} dp'' \frac{p''^2 T_{p'p''}^{\alpha'\alpha''} V_{p''p}^{\alpha''\alpha} - p_0^2 T_{p'p_0}^{\alpha'\alpha''} V_{p_0p}^{\alpha''\alpha}}{p_0^2 - p''^2} \\ + \frac{1}{2} M p_0 \left(i\pi - \ln \frac{\bar{p} + p_0}{\bar{p} - p_0} \right) \sum_{\alpha''} T_{p'p_0}^{\alpha'\alpha''} V_{p_0p}^{\alpha''\alpha}. \end{aligned} \quad (5.47)$$

By discretizing the integral over p'' we arrive at a system of linear equations that we can solve using standard techniques [104].

Chapter 6

Electron and neutrino scattering off deuteron

In this chapter we describe the theory and methods used in our study on the neutrino and electron induced reactions on the deuteron. The elastic or breakup electron and neutrino induced reactions on deuteron are

$$\nu_l (\bar{\nu}_l) + d \rightarrow \nu_l (\bar{\nu}_l) + d \quad \text{NC} \quad (6.1)$$

$$\nu_l (\bar{\nu}_l) + d \rightarrow \nu_l (\bar{\nu}_l) + p + n \quad \text{NC} \quad (6.2)$$

$$\nu_l + d \rightarrow l^- + p + p \quad \text{CC} \quad (6.3)$$

$$\bar{\nu}_l + d \rightarrow l^+ + n + n \quad \text{CC} \quad (6.4)$$

$$e^- + d \rightarrow e^- + d \quad \text{EMR} \quad (6.5)$$

$$e^- + d \rightarrow e^- + p + n \quad \text{EMR} \quad (6.6)$$

The nuclear matrix element for these reactions for the hadronic arm is

$$N^\mu = \langle \Psi_f | j^\mu | \phi_d \mu_d \mathbf{P}_i \rangle, \quad (6.7)$$

where, for the elastic reactions (6.1) and (6.5)

$$|\Psi_f\rangle = |\phi_d \mu'_d \mathbf{P}_f\rangle, \quad (6.8)$$

while for the reactions (6.2)-(6.4) and (6.6)

$$|\Psi_f\rangle = |\mathbf{p}'_1 \mu'_1 \tau'_1 \mathbf{p}'_2 \mu'_2 \tau'_2\rangle^{(-)}. \quad (6.9)$$

Here ϕ_d is the deuteron wave function, μ_d (μ'_d) is the initial (final) deuteron spin projection, $\mathbf{P}_i$ ($\mathbf{P}_f$) is the initial (final) deuteron momentum, $\mathbf{p}'_i$ are the final nucleon momenta, μ'_i are the final nucleon spins and τ'_i , the final nucleon isospin and the $(-)$ in Eq. (6.9) indicates the type of the outgoing nucleon scattering state.

The current operator for these reactions can be written as

$$j^\mu = j^\mu(1) + j^\mu(2) + j^\mu(1, 2), \quad (6.10)$$

where $j^\mu(1, 2)$ is the two-body current operator. Thanks to the symmetry of the nuclear states, the matrix element can be written as

$$N^\mu = 2 \langle \Psi_f | j^\mu(1) | \phi_d \mu_d \mathbf{P}_i \rangle + \langle \Psi_f | j^\mu_{12} | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (6.11)$$

From now on, we will ignore the contribution of the two-body component of the current and work with one-body currents only. For the part of the single-nucleon current acting on the particle number 1 we have

$$\langle \mathbf{p}'_1 \mu'_1 \tau'_1 | \mathbf{p}'_2 \mu'_2 \tau'_2 | j^\mu(1) | \mathbf{p}_1 \mu_1 \tau_1 | \mathbf{p}_2 \mu_2 \tau_2 \rangle = \delta^3(\mathbf{p}'_2 - \mathbf{p}_2) \delta_{\mu'_2 \mu_2} \delta_{\tau'_2 \tau_2} \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j^\mu(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle, \quad (6.12)$$

where $\langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j^\mu(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle$ is the single-nucleon current matrix element defined in Chapter 2 (relativistic form) and in Chapters 3 and 4 (nonrelativistic expansion).

In this chapter we study the reactions in a nonrelativistic framework. The relativistic effects on these reactions are investigated in the next chapter.

6.1 Elastic scattering reactions on deuteron

Let $\mathbf{k}$ ($\mathbf{k}'$) be the initial (final) lepton momentum and $\mathbf{P}_i$ ($\mathbf{P}_f$) the initial (final) deuteron momentum. We work in the laboratory reference frame, so $\mathbf{P}_i = 0$ and, due to momentum conservation, $\mathbf{P}_f = \mathbf{k} - \mathbf{k}'$. Nonrelativistically, the energy-momentum conservation implies

$$k + M_d = k' + M_d + \frac{k^2 + k'^2 - 2kk' \cos \theta'}{2M_d}, \quad (6.13)$$

where θ' is the angle between the initial and final neutrinos. The solution for k' is

$$k' = \sqrt{4k \sin^2 \frac{\theta'}{2} \left(M - k \cos^2 \frac{\theta'}{2} \right) - M_d + k \cos \theta'}. \quad (6.14)$$

In the following we consider only the unpolarized cross section.

6.1.1 Electron elastic scattering

The reaction taken into consideration is

$$e^- + d \rightarrow e^- + d. \quad (6.15)$$

Our result for the contraction of the electron and hadronic tensors in Chapter 4.2.2 was quite general, so we can use it now for this elastic reaction. The cross section formula needs to be supplemented by the corresponding phase space factor. We obtain

$$\frac{d^2 \sigma_{h,dd'}}{d\hat{\mathbf{k}}'} = \left(\frac{d^2 \sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{E_{\mathbf{p}'}}{M_d} \frac{k'}{k} \left(v_L R_L + v_T R_T + v_{TL} R_{TL} + v_{TT} R_{TT} + h(v_{T'} R_{T'} + v_{TL'} R_{TL'}) \right). \quad (6.16)$$

The indices dd' reminds the reader that the cross section still depends on the initial and final deuteron spin projections.

If we assume that the beam is unpolarized, we sum over h and divide by two, obtaining

$$|L_\alpha N^\alpha|^2 = v_0 \left(v_L R_L + v_T R_T + v_{TL} R_{TL} + v_{TT} R_{TT} \right) \quad (6.17)$$

and the cross section can be written as

$$\frac{d^2 \sigma_{dd'}}{d\hat{\mathbf{k}}'} = \left(\frac{d^2 \sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{E_{\mathbf{p}'}}{M_d} \frac{k'}{k} \left(v_L R_L + v_T R_T + v_{TL} R_{TL} + v_{TT} R_{TT} \right) \quad (6.18)$$

Furthermore, for electron-nucleon elastic scattering, only R_L and R_T are not zero and we have

$$\frac{d^2 \sigma_{dd'}}{d\hat{\mathbf{k}}'} = \left(\frac{d^2 \sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{E_{\mathbf{p}'}}{M_d} \frac{k'}{k} \left(v_L R_L + v_T R_T \right). \quad (6.19)$$

Due to covariance, current conservation and discrete symmetries only three of the deuteron current matrix elements are linearly independent [105, 106]. That implies that all deuteron elastic-scattering observables can be written in terms of three independent quantities for a given Q^2 . Standard choice comprises the structure functions $A(Q^2)$ and $B(Q^2)$ and the tensor analyzing power $T_{20}(Q^2, \theta')$ at $\theta' = 70^\circ$ [107, 105].

The structure functions are obtained through a Rosenbluth separation of the differential cross section [108], in this case neither the initial particles are polarized nor we care to measure the final particles spin. We write in this case

$$\frac{d^2 \sigma}{d\hat{\mathbf{k}}'} = \left(\frac{d^2 \sigma}{d\hat{\mathbf{k}}'} \right)_{\text{Mott}} \frac{k'}{k} \left(A(Q^2) + B(Q^2) \tan^2 \frac{\theta'}{2} \right). \quad (6.20)$$

It is possible to evaluate $A(Q^2)$, $B(Q^2)$ by comparing Eq. (6.20) with Eq. (6.19) and averaging over the initial deuteron spin and summing over its final spin, obtaining

$$A(Q^2) = \frac{E_{\mathbf{p}'}}{M_d} \frac{1}{3} \sum_{\mu'_d \mu_d} \left(v_L R_L + \frac{1}{2} \frac{Q^2}{q^2} R_T \right), \quad (6.21)$$

$$B(Q^2) = \frac{E_{\mathbf{p}'}}{M_d} \frac{1}{3} \sum_{\mu'_d \mu_d} R_T. \quad (6.22)$$

The deuteron tensor analyzing power T_{20} is calculated by considering different initial polarization of the deuteron target,

$$T_{20}(Q^2, \theta') = \sqrt{2} \frac{\sum_{d'} \frac{d^2 \sigma_{dd'}}{d\hat{\mathbf{k}}'} \Big|_{\mu_d=1} - \sum_{d'} \frac{d^2 \sigma_{dd'}}{d\hat{\mathbf{k}}'} \Big|_{\mu_d=0}}{\frac{d^2 \sigma}{d\hat{\mathbf{k}}'}}. \quad (6.23)$$

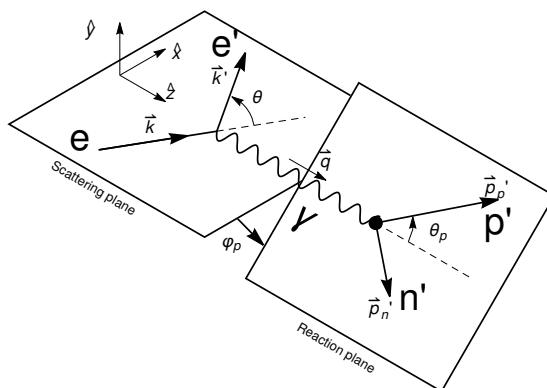


Figure 6.1: Kinematics of the reaction $e^- + d \rightarrow e^- + p + n$. The same applies to neutral-current neutrino induced deuteron disintegration with the exchange $e(e') \rightarrow \nu(\nu')$ and $\gamma \rightarrow Z_0$.

	$k_{\min}^{\text{CM}}$ (MeV)	$k_{\min}^{\text{L}}$ (MeV)
$\nu(\bar{\nu}) + d \rightarrow \nu(\bar{\nu}) + p + n$	2.223	2.226
$\nu_e + d \rightarrow e^- + p + p$	1.442	1.443
$\bar{\nu}_e + d \rightarrow e^+ + n + n$	4.025	4.033
$e^- + d \rightarrow e^- + p + n$	2.685	2.689

Table 6.1: Threshold lepton energies for reactions in Eqs. (6.2-6.4,6.6) in the center-of-mass reference frame and in the laboratory reference frame.

6.2 Deuteron breakup reactions

The reactions taken into consideration in this section are

$$\nu(\bar{\nu}) + d \rightarrow \nu(\bar{\nu}) + p + n \quad \text{NC} \quad (6.24)$$

$$\nu_e + d \rightarrow e^- + p + p \quad \text{CC} \quad (6.25)$$

$$\bar{\nu}_e + d \rightarrow e^+ + n + n \quad \text{CC} \quad (6.26)$$

$$e^- + d \rightarrow e^- + p + n \quad \text{EMR} \quad (6.27)$$

We choose $\mathbf{k}$ ($\mathbf{k}'$) to be the initial (final) lepton momentum, $\mathbf{P}_i$ to be the deuteron momentum and $\mathbf{p}'_1$ and $\mathbf{p}'_2$ to be the momentum of the two final nucleons. A schematic representation of the kinematics is shown in Fig. 6.1. Since we work in the laboratory reference frame we have $\mathbf{P}_i = \mathbf{0}$ and we define $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$. Thanks to momentum conservation we find that $\mathbf{q} = \mathbf{p}'_1 + \mathbf{p}'_2$. The thresholds for the initial (anti)neutrino energy for these reactions are summarized in Tab. 6.1.

Let us now study the kinematics needed for these reactions in the laboratory reference frame. In order to evaluate the cross section we want to understand the dependence of the maximum final lepton energy with respect to the scattering angle θ' .

To do so we evaluate the invariant quantity $(p_1'^\mu + p_2'^\mu)^2$. Being this quantity invariant, we can calculate it in the two final nucleons' center-of-mass reference frame, where $\mathbf{q} = \mathbf{0}$. In this reference frame, the lepton has maximum energy

when the final nucleons are at rest, so

$$(p_1'^\mu + p_2'^\mu)^2 = (E_{\mathbf{p}'_1} + E_{\mathbf{p}'_2})^2 - (\mathbf{p}'_1 + \mathbf{p}'_2)^2 = (M_1 + M_2)^2. \quad (6.28)$$

On the other hand $p_1'^\mu + p_2'^\mu = P_i^\mu + k^\mu - k'^\mu$. Squaring this equation and evaluating it in the laboratory frame we obtain

$$(P^\mu + k^\mu - k'^\mu)^2 = (E_i - \sqrt{m^2 + k'^2})^2 - (\mathbf{k} - \mathbf{k}')^2 = (M_1 + M_2)^2, \quad (6.29)$$

where $E_i = M_d + k$.

From this equation we can calculate analytically $k' = k'_{\max}(\theta')$ and therefore the maximum final lepton energy, which depends on the scattering angle θ' . The solution is

$$k'_{\max}(\theta') = \frac{A(k) \cos(\theta') + (k + M_d) \sqrt{2k^2 m^2 \cos 2\theta' + B(k)}}{k^2(1 - \cos 2\theta') + 4kM_d + 2M_d^2}, \quad (6.30)$$

with

$$A(k) = k(2kM_d + m^2 - B_d^2 - 2B_dM_d) \quad (6.31)$$

and

$$B(k) = -2k^2(m^2 - 2M_d^2) - 4kM_d(B_d(B_d + 2M_d) + m^2) + (m^2 - B_d^2)(m^2 - (B_d + 2M_d)^2), \quad (6.32)$$

where $B_d \equiv M_1 + M_2 - M_d$.

As usual we ignore the electron mass, therefore $k'_{\max}(\theta')$ becomes

$$k'_{\max}(\theta') = \frac{2(k + B_d)M_d - B_d^2}{2(M_d - k(1 - \cos \theta'))}. \quad (6.33)$$

This formula very well agrees with Eq. (6.30) down to energies of around 5 MeV, where the percentage difference becomes of the order of 1%.

The same calculation can be done in a nonrelativistic approach. The energy conservation states that

$$E_i = M_1 + M_2 + k' + \frac{q^2}{2(M_1 + M_2)} + E_{\text{CM}}, \quad (6.34)$$

where E_{CM} is the two nucleon relative kinetic energy. In order to find $k'_{\max}(\theta')$ nonrelativistically we impose $E_{\text{CM}} = 0$ and solve Eq. (6.34) numerically for $k' = k'_{\max}$, starting to look for the numerical solution from the relativistic result for $k'_{\max}$.

We will see in Chapter 8 that nonrelativistic kinematics has severe limitations and does not allow one to access observables investigated in experiments.

6.2.1 Deuteron electrodisintegration

The differential cross section can be written as Eq. (55), with $I = M_d k$ and with the substitution

$$d\Phi^{(2)} \rightarrow d\Phi^{(3)} = \delta^4(k + p - k' - p'_1 - p'_2) \frac{d\mathbf{k}'}{(2\pi)^3 2E_{\mathbf{k}'}} \frac{d\mathbf{p}'_1}{(2\pi)^3 2E_{\mathbf{p}'_1}} \frac{d\mathbf{p}'_2}{(2\pi)^3 2E_{\mathbf{p}'_2}}. \quad (6.35)$$

To evaluate the differential cross section we have to simplify the expression for the phase space factor $d\Phi^{(3)}$. We have different choices, one of them is to evaluate the differential cross section as

$$\begin{aligned} \frac{d^5\sigma_h}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'_1} &= \sigma_{\text{Mott}} \sum_{p'_1} \left[\frac{p_1'^2}{\frac{p'_1}{E'_1} + \frac{p'_1 - q \cos \theta'_1}{E'_2}} \right] \\ &\times (v_L R_L + v_T R_T + v_{TT} R_{TT} + v_{TL} R_{TL} + h(v_{T'} R_{T'} + v_{TL'} R_{TL'})) , \end{aligned} \quad (6.36)$$

where $h = 1(-1)$ is the helicity of the initial electron and we sum over the final electron spin configurations, p'_i (E'_i) is the momentum (energy) of the i -th nucleon and the sum over p'_1 is due to the fact that for some combination of the initial neutrino energies and scattering angle we might find more than one value of p'_1 .

Another useful factorization of the five-fold differential cross section is [109]

$$\begin{aligned} \frac{d^5\sigma_h}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} &= \sigma_{\text{Mott}} \frac{1}{4} k k' \\ &\times (v_L R_L + v_T R_T + v_{TT} R_{TT} + v_{TL} R_{TL} + h(v_{T'} R_{T'} + v_{TL'} R_{TL'})) , \end{aligned} \quad (6.37)$$

where $\mathbf{p}'$ is the relative momentum between the final nucleons in their center-of-mass reference frame. If we do not have a polarized beam we can average over the initial electron helicity, obtaining

$$\begin{aligned} \frac{d^5\sigma}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'_1} &= \sigma_{\text{Mott}} \sum_{p'_1} \left[\frac{p_1'^2}{\frac{p'_1}{E'_1} + \frac{p'_1 - q \cos \theta'_1}{E'_2}} \right] \\ &\times (v_L R_L + v_T R_T + v_{TT} R_{TT} + v_{TL} R_{TL}) \end{aligned} \quad (6.38)$$

and

$$\frac{d^5\sigma}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} = \sigma_{\text{Mott}} \frac{1}{4} k k' (v_L R_L + v_T R_T + v_{TT} R_{TT} + v_{TL} R_{TL}) . \quad (6.39)$$

Finally, we can evaluate the semi-exclusive cross section

$$\frac{d^3\sigma}{dk'd\hat{\mathbf{k}}'} = \int d\hat{\mathbf{p}}' \frac{d^5\sigma}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} . \quad (6.40)$$

6.2.2 Neutrino-induced deuteron breakup

The formulas for neutrino induced reactions are similar to the ones derived in the previous paragraph. The inclusive differential cross section can be written as

$$\frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'} = \frac{G_F^2}{64\pi^2} \frac{M p_0 k'}{k} \mathcal{L}_{ij}^{\nu/\bar{\nu}} \frac{1}{3} \sum_{\mu_d, \mu'_1, \mu'_2} \int R_{ij} d\hat{\mathbf{p}}' \quad (6.41)$$

with $\mathcal{L}_{ij}^{\nu/\bar{\nu}}$ introduced in Sec. 3.2. The total cross section is therefore

$$\sigma = \int d\hat{\mathbf{k}}' \int_0^{E'^{\text{max}}} dE' \frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'} , \quad (6.42)$$

where E'^{max} has been calculated in Eq. (6.33) since, for massless final lepton, we have $E'^{\text{max}} = k'_{\text{max}}(\theta')$.

Chapter 7

Fully relativistic framework for electron and neutrino scattering off deuteron

In this chapter we introduce a relativistic framework to perform calculations of exclusive, semi-exclusive and inclusive unpolarized cross sections and various polarization observables in electron and neutrino scattering experiments with deuteron targets. The study builds on research by W. Glöckle, H. Witała, H. Kamada, W. Polyzou and others [61, 62, 63, 64, 65, 66, 67, 68, 69], especially following investigations of the three-nucleon system. We restrict ourselves to the two-nucleon system but consider its interactions with electrons and neutrinos, as described in [70]. The strong interaction dynamics is defined by an explicit dynamical unitary representation of the Poincaré group, where representations of space translations and rotations in the interacting and noninteracting representations are the same. The Argonne V18 potential is used to construct a relativistic nucleon-nucleon interaction reproducing the experimental deuteron binding energy and nucleon-nucleon scattering observables. Our formalism does not include the pion production channel and neglects two-body contributions in the electromagnetic as well as in the weak nuclear current operator. We show that this approach is applicable in situations where the internal two-nucleon energy remains below the pion production threshold, but the magnitude of the three-momentum transfer can extend at least to several GeV.

7.1 The Poincaré group and single-particle representations

Our relativistic invariant model is defined by a representation $U(\Lambda, a)$ of the Poincaré group on the chosen Hilbert space $\mathcal{H}$. Observables will be therefore independent of the reference frame choice, as long as it is an inertial one.

We start by constructing one-particle irreducible representations. The Poincaré group has ten infinitesimal generators, the four components of the four-momentum P^μ and the six independent components of the anti-symmetric angular momentum tensor $J^{\mu\nu}$. These operators transform as four-vectors and as

anti-symmetric tensors under Lorentz transformations, respectively. It follows from the transformation properties of these operators that

$$M^2 = g_{\mu\nu} P^\mu P^\nu \quad (7.1)$$

and

$$W^2 = g_{\mu\nu} W^\mu W^\nu, \quad (7.2)$$

where

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} P_\nu J_{\alpha\beta} \quad (7.3)$$

are invariant. (W^μ is called the Pauli-Lubanski vector.) When the eigenvalue of the M operator is nonzero, W^2 can be expressed by the spin operator squared (S^2) and M^2 as

$$M^2 S^2 = W^2. \quad (7.4)$$

Since M^2 and S^2 are invariant, they commute with all ten generators. Their eigenvalues are identified with the mass and the spin of a particle.

A basis for the one-particle Hilbert space is chosen to be simultaneous eigenstates of a maximal set of commuting Hermitian functions of the generators. The commuting observables include the mass and spin operators, and four additional functions of the ten generators. In this work the four additional commuting observables are chosen to be the three components of the linear momentum and a projection of the spin operator on the z -axis. This basis is labeled as

$$|(j, m) \mathbf{p}, \mu\rangle, \quad (7.5)$$

where m is the mass of the particle, j its spin, $\mathbf{p}$ its momentum and μ its spin projection. In this space we use a non-covariant normalization for the single-particle wave function

$$\langle \Psi | \Psi \rangle = \sum_{\mu=-j}^j \int d\mathbf{p} |\langle (j, m), \mathbf{p}, \mu | \Psi \rangle|^2 = 1. \quad (7.6)$$

To construct a unitary representation of the Poincaré group on this representation of the Hilbert space we note that when $\mathbf{p} = \mathbf{0}$, the values of $j, m, \mathbf{p}$ do not change under rotations. It follows that under any rotation R we have

$$U(R) |(j, m) \mathbf{0}, \mu\rangle = \sum_{\nu} |(j, m) \mathbf{0}, \nu\rangle \langle (j, m) \mathbf{0}, \nu | U(R) |(j, m) \mathbf{0}, \mu\rangle = \sum_{\nu} D_{\nu\mu}^j [R] |(j, m) \mathbf{0}, \nu\rangle, \quad (7.7)$$

where $U(R) \equiv U(\Lambda = R, 0)$ and $D_{\nu\mu}^j [R]$ is the $(2j + 1)$ -dimensional unitary representation of rotation group in the $|(j, m) \mathbf{0}, \mu\rangle$ basis. The Wigner D -matrix is defined as

$$D_{\mu'\mu}^j [U(R)] \equiv \langle (j, m) \mathbf{0}', \mu' | U(R) |(j, m) \mathbf{0}, \mu\rangle. \quad (7.8)$$

To complete the definition of the spin, note that if Λ is a Lorentz transformation that transforms the momentum from $\mathbf{0}$ to $\mathbf{p}$, so does ΛR for any rotation R . The result is that both transformations lead to the same momentum but different

spin projections. In order to compare spins in different reference frames it is necessary to transform back to the rest frame with a rotationless Lorentz transformation (*i.e.* a Lorentz boost) $B^{-1}(\mathbf{p}/m)$ and then compare the spins in the rest frame. We denote by $B(\mathbf{p}/m)$ a boost where the boost velocity is $\boldsymbol{\beta} = -\mathbf{p}/\sqrt{m^2 + p^2}$. This boost maps $(m, 0, 0, 0)$ to $(\sqrt{m^2 + p^2}, \mathbf{p})$. The four-vector representation of the boost is

$$B(\mathbf{p}/m)^0_0 = \frac{\sqrt{m^2 + p^2}}{m}, \quad (7.9)$$

$$B(\mathbf{p}/m)^i_0 = B(\mathbf{p}/m)^0_i = \frac{p^i}{m}, \quad (7.10)$$

$$B(\mathbf{p}/m)^i_j = \delta^{ij} + \frac{p^i p^j}{m(m + \sqrt{m^2 + p^2})}. \quad (7.11)$$

We defined a spin observable in such a way that it does not change when it is transformed to and from the rest frame with the boost $B^{-1}(\mathbf{p}/m)$ or $B(\mathbf{p}/m)$. In this case

$$U\left(B\left(\frac{\mathbf{p}}{m}\right)\right) |(j, m)\mathbf{0}, \nu\rangle = \sqrt{J(p)} |(j, m)\mathbf{p}, \nu\rangle, \quad (7.12)$$

where $J(p)$ is a Jacobian factor that makes $U(B(\mathbf{p}/m))$ unitary. The Jacobian depends on the choice of normalization of the states in Eq. (7.5). Our choice for the normalization is

$$\langle (j, m)\mathbf{p}, \nu | (j, m)\mathbf{p}', \nu' \rangle = \delta(\mathbf{p} - \mathbf{p}') \delta_{\nu\nu'} \quad (7.13)$$

with

$$J(p) = \frac{\sqrt{m^2 + p^2}}{m}. \quad (7.14)$$

Our choice of standard boost is the one generated by J^{0i} , called the canonical or rotationless boost. The construction of the unitary representation of the Poincaré group is fully accomplished by requiring that space-time translations of the rest state yield

$$U(I, a) |(j, m)\mathbf{0}, \mu\rangle = e^{-ima^0} |(j, m)\mathbf{0}, \mu\rangle. \quad (7.15)$$

Using the fact that any Poincaré transformation can be expressed as a series of transformations applied to the rest frame, it results in the following unitary representation of the Poincaré group on the single-particle Hilbert space

$$\begin{aligned} U(\Lambda, a) |(j, m)\mathbf{p}, \mu\rangle &= e^{-ip' \cdot a} \sqrt{\frac{\sqrt{m^2 + \mathbf{p}'^2}}{\sqrt{m^2 + \mathbf{p}^2}}} \\ &\times \sum_{\mu'} D^j_{\mu'\mu} \left[B^{-1}\left(\frac{\mathbf{p}'}{m}\right) \Lambda B\left(\frac{\mathbf{p}}{m}\right) \right] |(j, m)\mathbf{p}', \mu'\rangle \\ &= e^{-ip' \cdot a} \sqrt{\frac{\sqrt{m^2 + \mathbf{p}'^2}}{\sqrt{m^2 + \mathbf{p}^2}}} \sum_{\mu'} D^j_{\mu'\mu} [R_W(\Lambda, p)] |(j, m)\mathbf{p}', \mu'\rangle, \end{aligned} \quad (7.16)$$

where $p'^\mu = \Lambda^\mu_{\nu} p^\nu = (\sqrt{m^2 + \mathbf{p}'^2}, \mathbf{p}')$. The argument of the Wigner D -matrix is a $SU(2)$ Wigner rotation, which depends on the choice of the standard boost,

$B(\mathbf{p}/m)$. Wigner rotations are the result of two non-collinear Lorentz boosts, which result in the composition of a Lorentz boost and a rotation, called Wigner rotation. Moreover, if a third boost is applied to the other two in order to return to the first reference frame, a pure rotation results as the composite Lorentz transformation of these specifically chosen three boosts.

The argument of the D -matrix is a therefore a Wigner rotation, which we write as

$$R_W(\Lambda, p) \equiv B^{-1} \left(\frac{\mathbf{p}'}{m} \right) \Lambda B \left(\frac{\mathbf{p}}{m} \right). \quad (7.17)$$

An example of Wigner rotations for the four-vector representation and for the single-particle representation with $j = 1/2$ are given in Appendix C.2. In Appendix D we show a way to represent the Wigner rotation by $SL(2, \mathbb{C})$ matrices.

By construction $U(\Lambda, a)$ is an irreducible unitary representation of the Poincaré group on the single-particle Hilbert space. The irreducibility means that any operator that commutes with $U(\Lambda, a)$ for all Λ and a is a multiple of the identity operator in this space.

7.2 Two-nucleon systems

To study the kinematics of two-nucleon systems we neglect the difference between the proton and neutron mass and in the formulas discussed in the previous section we substitute the eigenvalue of the mass operator m with the nucleon mass M . We denote the individual nucleon momenta by $\mathbf{p}_1$ and $\mathbf{p}_2$, the relative momentum between the two particles in the total momentum zero frame by $\mathbf{p}$ and the total momentum of the system by $\mathbf{P}$. The new pairs of momenta are related in the following way

$$\begin{aligned} \mathbf{p} &= \frac{1}{2} \left(\mathbf{p}_1 - \mathbf{p}_2 - (\mathbf{p}_1 + \mathbf{p}_2) \frac{E_{\mathbf{p}_1} - E_{\mathbf{p}_2}}{E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \sqrt{(E_{\mathbf{p}_1} + E_{\mathbf{p}_2})^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2}} \right), \\ \mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2 \end{aligned} \quad (7.18)$$

and reduce to the nonrelativistic expressions

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad (7.19)$$

$$\mathbf{p} = \frac{1}{2} (\mathbf{p}_1 - \mathbf{p}_2), \quad (7.20)$$

when the momenta of the particles are much smaller than their rest mass.

With this choice the kinetic energy in the center-of-mass reference frame is simply

$$E = 2\sqrt{M^2 + p^2}. \quad (7.21)$$

The overlap between the momentum eigenstates is evaluated as

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \mathbf{p} \mathbf{P} \rangle = \frac{1}{\mathcal{N}(\mathbf{p}_1, \mathbf{p}_2)} \delta^3 \left(\mathbf{P} - (\mathbf{p}_1 + \mathbf{p}_2) \right) \delta^3 \left(\mathbf{p} - \mathbf{p}(\mathbf{p}_1, \mathbf{p}_2) \right), \quad (7.22)$$

where $\mathcal{N}(\mathbf{p}_1, \mathbf{p}_2)$ is a normalization factor that will be discussed later.

The inverse relations to evaluate $\mathbf{p}_1$ and $\mathbf{p}_2$ from $\mathbf{p}$ and $\mathbf{P}$ are

$$\mathbf{p}_1 = \mathbf{p} + (\gamma - 1)(\mathbf{p} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \gamma\sqrt{M^2 + p^2}\mathbf{u}, \quad (7.23)$$

$$\mathbf{p}_2 = -\mathbf{p} - (\gamma - 1)(\mathbf{p} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \gamma\sqrt{M^2 + p^2}\mathbf{u}, \quad (7.24)$$

where

$$\gamma = \frac{1}{\sqrt{1-u^2}} \quad \text{and} \quad \mathbf{u} = \frac{\mathbf{P}}{E_L^{NN}}, \quad (7.25)$$

with

$$E_L^{NN} = \sqrt{\left(2\sqrt{M^2 + p^2}\right)^2 + P^2} = \sqrt{M^2 + p_1^2} + \sqrt{M^2 + p_2^2}. \quad (7.26)$$

7.2.1 Two-particle representation

We extend our framework to include two-particle states. The Hilbert space in this case is the tensor product of the symmetrized single-particle Hilbert spaces $\mathcal{H}_i$

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \quad (7.27)$$

The free dynamics is then given by

$$U_0(\Lambda, a) = U_1(\Lambda, a) \otimes U_2(\Lambda, a), \quad (7.28)$$

where $U_i(\Lambda, a)$ is the single-particle unitary representation of the Poincaré group of the i^{th} particle. The infinitesimal generators of the group are

$$P_0^\mu = \sum_i P_{0,i}^\mu \quad (7.29)$$

and

$$J_0^{\mu\nu} = \sum_i J_{0,i}^{\mu\nu}, \quad (7.30)$$

while for the mass and canonical spin operator we take

$$M_0 = \sqrt{P_{0\mu} P_0^\mu} \quad (7.31)$$

and

$$j_0^k = \frac{1}{2} \epsilon^{ijk} \left[B^{-1} \left(\frac{\mathbf{P}_0}{M_0} \right) \right]^i_{\mu} \left[B^{-1} \left(\frac{\mathbf{P}_0}{M_0} \right) \right]^j_{\nu} J_0^{\mu\nu}. \quad (7.32)$$

The free spin represents the angular momentum of the two noninteracting particles in their total momentum zero frame. The free spin $\mathbf{j}_0$ is Hermitian, its components satisfy $SU(2)$ commutation relations and commute with P_0^μ . It represents the angular momentum of the system of particles in the rest frame of the noninteracting system, assuming that the system was transformed to the rest frame with a noninteracting rotationless (canonical) Lorentz boost.

This allows us to construct simultaneous eigenstates of the commuting observables M_0 , $\mathbf{P}_0$, $\mathbf{j}_0^2$, $j_{0,z}$ for the noninteracting system

$$|(j_0, M_0)\mathbf{P}, \mu, d\rangle, \quad (7.33)$$

where d are kinematically invariant degeneracy parameters. Using the Poincaré group Clebsch-Gordan coefficients it is possible to express these states as linear combinations of single-particles states.

Expressing two-nucleon states with one-nucleon states It is necessary to evaluate the overlap between noninteracting two nucleon states (7.33) and products of the single-particle states (7.5). We start by defining the two-nucleon state in the reference frame in which the total momentum is zero as [65]

$$|\mathbf{p} \mathbf{0} \mu_1 \mu_2\rangle \equiv |(j_1, M) \mathbf{p}, \mu_1\rangle \otimes |(j_2, M)(-\mathbf{p}), \mu_2\rangle \quad (7.34)$$

$$\equiv |(j_1, M) \mathbf{p}, \mu_1; (j_2, M)(-\mathbf{p}), \mu_2\rangle, \quad (7.35)$$

where for two nucleons $j_1 = j_2 = 1/2$. We apply a Lorentz boost to this state

$$U(\Lambda)|\mathbf{p} \mathbf{0} \mu_1 \mu_2\rangle = U_1(\Lambda)U_2(\Lambda) |(j_1, M) \mathbf{p}, \mu_1; (j_2, M)(-\mathbf{p}), \mu_2\rangle. \quad (7.36)$$

We choose Λ in to have the total momentum of the system $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}$, where $p_1^\mu = \Lambda^\mu{}_\nu p^\nu$ and $p_2^\mu = \Lambda^\mu{}_\nu (-p^\nu)$. We have that

$$\Lambda = U\left(\frac{\mathbf{P}}{M_0}\right), \quad (7.37)$$

where the eigenvalue of M_0 (M_0 for short) is $2\sqrt{M^2 + p^2}$. Using Eq. (7.16) on the single-particle states we get

$$\begin{aligned} U(\Lambda)|\mathbf{p} \mathbf{0} \mu_1 \mu_2\rangle &= \sqrt{\frac{\sqrt{M^2 + p_1^2}}{\sqrt{M^2 + p^2}}} \sum_{\mu'_1} D_{\mu'_1 \mu_1}^{j_1} [R_W(\Lambda, \mathbf{p})] |(j_1, M) \mathbf{p}_1, \mu'_1\rangle \\ &\times \sqrt{\frac{\sqrt{M^2 + p_2^2}}{\sqrt{M^2 + p^2}}} \sum_{\mu'_2} D_{\mu'_2 \mu_2}^{j_2} [R_W(\Lambda, -\mathbf{p})] |(j_2, M) \mathbf{p}_2, \mu'_2\rangle. \end{aligned} \quad (7.38)$$

On the other hand, we can also treat this system as a *single* object having mass M_0 , therefore

$$U(\Lambda)|\mathbf{p} \mathbf{0} \mu_1 \mu_2\rangle = \sqrt{\frac{\sqrt{M_0^2 + P^2}}{M_0}} |\mathbf{p} \mathbf{P} \mu_1 \mu_2\rangle. \quad (7.39)$$

Comparing Eq. (7.38) and (7.39) we find

$$\begin{aligned} |\mathbf{p} \mathbf{P} \mu_1 \mu_2\rangle &= \sqrt{\frac{M_0 \sqrt{M^2 + p_1^2} \sqrt{M^2 + p_2^2}}{\sqrt{M_0^2 + P^2} \sqrt{M^2 + p^2} \sqrt{M^2 + p^2}}} \\ &\times \sum_{\mu'_1} D_{\mu'_1 \mu_1}^{j_1} [R(\Lambda, \mathbf{p})] |(j_1, M) \mathbf{p}_1, \mu'_1\rangle \\ &\times \sum_{\mu'_2} D_{\mu'_2 \mu_2}^{j_2} [R(\Lambda, -\mathbf{p})] |(j_2, M) \mathbf{p}_2, \mu'_2\rangle. \end{aligned} \quad (7.40)$$

We can see that

$$\sqrt{\frac{M_0 \sqrt{M^2 + p_1^2} \sqrt{M^2 + p_2^2}}{\sqrt{M_0^2 + P^2} \sqrt{M^2 + p^2} \sqrt{M^2 + p^2}}} = \left| \frac{\partial(\mathbf{p}_1, \mathbf{p}_2)}{\partial(\mathbf{p}, \mathbf{P})} \right|^{1/2} \equiv \mathcal{N}(\mathbf{p}_1, \mathbf{p}_2) \quad (7.41)$$

is the Jacobian of the transformation (7.18).

We can now make a transition to partial-wave states. The definition of the two-nucleon orbital angular momentum eigenstate is

$$|p\ell\mu_\ell; \mu_1\mu_2; \mathbf{0}\rangle \equiv \int d\hat{\mathbf{p}} Y_{\ell\mu_\ell}(\hat{\mathbf{p}}) |\mathbf{p} \mathbf{0} \mu_1\mu_2\rangle. \quad (7.42)$$

We couple this with the total spin of the two nucleons s

$$|(j, p)\mathbf{0}, \mu; \ell s\rangle = \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \sum_{\mu_1 \mu_2} \langle j_1 \mu_1 j_2 \mu_2 | s \mu_s \rangle |p\ell\mu_\ell; \mu_1\mu_2; \mathbf{0}\rangle. \quad (7.43)$$

Applying the boost $U(\Lambda)$ to both sides we get

$$\begin{aligned} \sqrt{\frac{\sqrt{M_0^2 + P^2}}{M_0}} |(j, p)\mathbf{P}\mu; \ell s\rangle = \\ \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \sum_{\mu_1 \mu_2} \langle j_1 \mu_1 j_2 \mu_2 | s \mu_s \rangle \int d\hat{\mathbf{p}} Y_{\ell\mu_\ell}(\hat{\mathbf{p}}) |\mathbf{p} \mathbf{P} \mu_1\mu_2\rangle \end{aligned} \quad (7.44)$$

and using Eq. (7.40) we obtain

$$\begin{aligned} |(j, p)\mathbf{P}\mu; \ell s\rangle = \mathcal{N}(\mathbf{p}_1, \mathbf{p}_2) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \sum_{\mu_1 \mu_2} \langle j_1 \mu_1 j_2 \mu_2 | s \mu_s \rangle \int d\hat{\mathbf{p}} Y_{\ell\mu_\ell}(\hat{\mathbf{p}}) \\ \times \sum_{\mu'_1} D_{\mu'_1 \mu_1}^{j_1} [R(\Lambda, \mathbf{p})] |(j_1, M)\mathbf{p}_1, \mu'_1\rangle \\ \times \sum_{\mu'_2} D_{\mu'_2 \mu_2}^{j_2} [R(\Lambda, -\mathbf{p})] |(j_2, M)\mathbf{p}_2, \mu'_2\rangle \end{aligned} \quad (7.45)$$

We can now evaluate the matrix element $\langle (j_1, M)\mathbf{q}_1\mu_1; (j_2, M)\mathbf{q}_2\mu_2 | (p, j)\mathbf{P}\mu; \ell s \rangle$: we apply $\langle (j_1, M)\mathbf{q}_1\mu_1; (j_2, M)\mathbf{q}_2\mu_2 |$ from the left, obtaining

$$\begin{aligned} \langle (j_1, M)\mathbf{q}_1\mu_1; (j_2, M)\mathbf{q}_2\mu_2 | (j, p)\mathbf{P}\mu; \ell s \rangle = \mathcal{N}(\mathbf{p}_1, \mathbf{p}_2) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \\ \times \sum_{\mu'_1 \mu'_2} \langle j_1 \mu'_1 j_2 \mu'_2 | s \mu_s \rangle \int d\hat{\mathbf{p}} Y_{\ell\mu_\ell}(\hat{\mathbf{p}}) \\ \times \sum_{\mu'_1} D_{\mu'_1 \mu_1}^{j_1} [R(\Lambda, \mathbf{p})] \langle (j_1, M)\mathbf{q}_1, \mu_1 | (j_1, M)\mathbf{p}_1, \mu'_1 \rangle \\ \times \sum_{\mu'_2} D_{\mu'_2 \mu_2}^{j_2} [R(\Lambda, -\mathbf{p})] \langle (j_2, M)\mathbf{q}_2, \mu_2 | (j_2, M)\mathbf{p}_2, \mu'_2 \rangle. \end{aligned} \quad (7.46)$$

Here $\mathbf{p}_i = \mathbf{p}_i(\mathbf{p}, \mathbf{P})$, so

$$\langle (j_i, M)\mathbf{q}_i, \mu_i | (j_i, M)\mathbf{p}_i, \mu'_i \rangle = \delta_{\mu_i \mu'_i} \delta^3(\mathbf{q}_i - \mathbf{p}_i(\mathbf{p}, \mathbf{P})), \quad (7.47)$$

therefore the matrix element can be written as

$$\begin{aligned} \langle (j_1, M)\mathbf{q}_1\mu_1; (j_2, M)\mathbf{q}_2\mu_2 | (j, p)\mathbf{P}\mu; \ell s \rangle = \\ \int d\hat{\mathbf{p}} \delta^3(\mathbf{q}_1 - \mathbf{p}_1(\mathbf{p}, \mathbf{P})) \delta^3(\mathbf{q}_2 - \mathbf{p}_2(\mathbf{p}, \mathbf{P})) \mathcal{N}(\mathbf{q}_1, \mathbf{q}_2) \sum_{\mu_\ell \mu_s \mu'_1 \mu'_2} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \\ \times Y_{\ell\mu_\ell}(\hat{\mathbf{p}}) \langle j_1 \mu'_1 j_2 \mu'_2 | s \mu_s \rangle D_{\mu'_1 \mu_1}^{j_1} [R(\Lambda, \mathbf{p})] D_{\mu'_2 \mu_2}^{j_2} [R(\Lambda, -\mathbf{p})]. \end{aligned} \quad (7.48)$$

This agrees with Eq. (4) from Ref. [65]. With a change of integration variables

$$\begin{aligned} & \langle (j_1, M) \mathbf{q}_1 \mu_1; (j_2, M) \mathbf{q}_2 \mu_2 | (j_1, M) \mathbf{p}_1 \mu'_1; (j_2, M) \mathbf{p}_2 \mu'_2 \rangle = \\ & \times \delta_{\mu_1 \mu'_1} \delta_{\mu_2 \mu'_2} \frac{1}{\mathcal{N}^2(\mathbf{q}_1, \mathbf{q}_2)} \delta^3(\mathbf{p} - \mathbf{p}(\mathbf{q}_1, \mathbf{q}_2)) \delta^3(\mathbf{P} - \mathbf{q}_1 - \mathbf{q}_2) \end{aligned} \quad (7.49)$$

and introducing the direction of $\mathbf{p} = p \hat{\mathbf{p}}$ we can write

$$\begin{aligned} & \langle (j_1, M) \mathbf{q}_1 \mu_1; (j_2, M) \mathbf{q}_2 \mu_2 | (j_1, M) \mathbf{p}_1 \mu'_1; (j_2, M) \mathbf{p}_2 \mu'_2 \rangle = \delta_{\mu_1 \mu'_1} \delta_{\mu_2 \mu'_2} \\ & \times \frac{1}{\mathcal{N}^2(\mathbf{q}_1, \mathbf{q}_2)} \frac{\delta(p - p(\mathbf{q}_1, \mathbf{q}_2))}{p^2} \delta^2(\hat{\mathbf{p}} - \hat{\mathbf{p}}(\mathbf{q}_1, \mathbf{q}_2)) \delta^3(\mathbf{P} - \mathbf{q}_1 - \mathbf{q}_2) \end{aligned} \quad (7.50)$$

and by integrating over the solid angle $\hat{\mathbf{p}}$ we get

$$\begin{aligned} & \langle (j_1, M) \mathbf{q}_1 \mu_1; (j_2, M) \mathbf{q}_2 \mu_2 | (j, p) \mathbf{P} \mu; \ell s \rangle = \frac{1}{\mathcal{N}(\mathbf{q}_1, \mathbf{q}_2)} \sum_{\mu_\ell \mu_s \mu'_1 \mu'_2} \langle \ell \mu_\ell s \mu_s | j \mu \rangle \\ & \times Y_{\ell \mu_\ell}(\hat{\mathbf{p}}) \langle j_1 \mu'_1 j_2 \mu'_2 | s \mu_s \rangle D_{\mu_1 \mu'_1}^{j_1} [R(\Lambda, \mathbf{p})] D_{\mu_2 \mu'_2}^{j_2} [R(\Lambda, -\mathbf{p})] \\ & \times \frac{\delta(p - p(\mathbf{q}_1, \mathbf{q}_2))}{p^2} \delta^3(\mathbf{P} - \mathbf{q}_1 - \mathbf{q}_2), \end{aligned} \quad (7.51)$$

which agrees with Eq. (3) from Ref. [65]. Note that in Ref. [65] there is a typo and the indices in the D -matrix should be interchanged.

7.3 Nucleon-nucleon interactions

We add now the relativistic interaction v to the noninteracting mass Casimir operator M_0 . v must commute with and be independent of $\mathbf{P}_0$, and also commute with all the components of $\mathbf{j}_0$. This defines

$$M_I = M_0 + v, \quad (7.52)$$

which is the dynamical mass operator. The matrix elements of the interaction in the basis (7.33) take the form

$$\begin{aligned} & \langle (j, M'_0) \mathbf{P}' \mu'; d' | v | (j, M_0) \mathbf{P} \mu; d \rangle = \\ & \delta^3(\mathbf{P}' - \mathbf{P}) \delta_{j'j} \delta_{\mu'\mu} \langle M'_0 d' | v^{(j)} | M_0 d \rangle, \end{aligned} \quad (7.53)$$

where $\langle M'_0 d' | v^{(j)} | M_0 d \rangle$ is the kernel of the reduced interaction, analog to a partial wave potential, and $|(j, M_0) \mathbf{P} \mu; d\rangle$ are the same states as in 7.33. The kinematically invariant degeneracy quantum numbers d depend on the model chosen for the Hilbert space and usually involve operators of the subsystems, like invariant masses and squares of the angular momenta. In the dynamical representation they will be no longer invariant as the interaction is not diagonal in them. When the operator v satisfies Eq. (7.53), the operators M_I , $\mathbf{P}_0$, $\mathbf{j}_0^2$ and $j_{0,z}$ are mutually commuting self-adjoint operators. The dynamical mass operator M_I can be diagonalized on the basis 7.33, resulting in simultaneous eigenstates of M_0 , $\mathbf{P}_0$, $\mathbf{j}_0^2$ and $j_{0,z}$

$$|(j, M_0) \mathbf{P} \mu; \bar{d}\rangle, \quad (7.54)$$

where $\bar{d}$ represent new dynamically invariant degeneracy parameters. The states transform as 7.16, once we replace the single-particle mass M with the eigenstates of M_I , *i.e.*

$$U \left(B \left(\frac{\mathbf{P}}{M_I} \right) \right) |(j, M_I) \mathbf{0}\mu; \bar{d}\rangle = \sqrt{\frac{\sqrt{M_I^2 + \mathbf{P}^2}}{M_I}} |(j, M_I) \mathbf{P}\mu; \bar{d}\rangle, \quad (7.55)$$

and

$$U(\Lambda, a) |(j, M_I) \mathbf{P}\mu; \bar{d}\rangle = e^{-iP' \cdot a} \sqrt{\frac{\sqrt{M_I^2 + \mathbf{P}^2}}{\sqrt{M_I^2 + \mathbf{P}'^2}}} \sum_{\mu'} D_{\mu'\mu}^j [R_W(\Lambda, \mathbf{P})] |(j, M_I) \mathbf{P}'\mu'; \bar{d}\rangle. \quad (7.56)$$

This construction leads to an unitary representation of the Poincaré group and the construction was proposed by Bakamjian and Thomas [110]. One of its properties is that Lorentz boosts are interaction independent. This construction will be used to model the strong interaction, while for the weak and electromagnetic interactions we use the one-boson exchange approximation. In the one-boson exchange approximation we can factorize

$$U(\Lambda, a) \simeq U_B(\Lambda, a) \otimes U_L(\Lambda, a), \quad (7.57)$$

where $U_B(\Lambda, a)$ ($U_L(\Lambda, a)$) is a unitary representation of the Poincaré group for the strongly interacting bosons (leptons). The coupling current can be decomposed as

$$J^\mu(x) = J_B^\mu(x) + J_L^\mu(x) + J_I^\mu(x), \quad (7.58)$$

where $J_B^\mu(x)$ ($J_L^\mu(x)$) is the baryonic (leptonic) current operator and the interaction term $J_I^\mu(x)$ contains parts of the current that do not contribute to the one-boson exchange approximation and will be therefore ignored. In the following we use

$$J^\mu(x) = J_B^\mu(x) + J_L^\mu(x). \quad (7.59)$$

Both the baryonic and the leptonic currents transform covariantly with respect to Poincaré transformations, *i.e.*

$$U_{B/L}(\Lambda, a) J_{B/L}^\mu(x) U_{B/L}^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu_\nu J_{B/L}^\nu(\Lambda x + a). \quad (7.60)$$

Both currents can be expanded as

$$J_{B/L}^\mu(x) = \sum_i J_{B/L,i}^\mu(x) + \frac{1}{2} \sum_{ij} J_{B/L,ij}^\mu(x) + \dots \quad (7.61)$$

where each the first term is the one-body current operator, the second term is the two-body current operator and so on. Differently from the leptonic current, whose covariant form holds up to higher order corrections with respect to the tree level, baryonic currents must contain many-body contributions in order to satisfy covariance and current conservation. It is possible to check this statement by evaluating the commutation relations with the dynamical generators of the Poincaré group. We can write Eq. (7.61) as

$$J_B^\mu(0) = J_{B0}^\mu(0) + J_{BI}^\mu(0) \quad \text{with} \quad J_{B0}^\mu(0) = \sum_i J_{Bi}^\mu(0). \quad (7.62)$$

We have, for the boost generator $\mathbf{K}$ a cluster expansion of the type

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_I \quad \text{with} \quad \mathbf{K}_0 = \sum_i \mathbf{K}_i. \quad (7.63)$$

We recall that $J^\mu(0)$ and P^μ transform in the same way under Lorentz transformation and have the same commutation relations. Therefore,

$$[K_0^i + K_I^i, J^i(0)] = i J^0(0) \quad \text{and} \quad [K_0^i + K_I^i, J^0(0)] = i J^i(0). \quad (7.64)$$

When the particles are asymptotically separated, *i.e.* $\mathbf{K}_I \rightarrow \mathbf{K}_0$ and $J_I^\mu(0) \rightarrow J_0^\mu(0)$, Eq. (7.64) holds because of the cluster properties. By canceling the one-body term, the remaining interacting term must satisfy

$$[K_0^i, J_I^i(0)] + [K_I^i, J_I^i(0)] - i J_I^0(0) = [J_0^i(0), K_I^i], \quad (7.65)$$

$$[K_0^i, J_I^0(0)] + [K_I^i, J_I^0(0)] - i J_I^0(0) = [J_0^0(0), K_I^i]. \quad (7.66)$$

If the right side of either equation is nonzero then the current must have many-body parts in order to satisfy current covariance. Similar conditions follow if the current is conserved. In general, these many-body contributions to the current have to be supplemented by the many-body currents that arise from physical processes such as exchange of charged mesons. These many-body terms are not uniquely determined by current covariance.

It is possible to maintain covariance by working with current matrix elements, as they transform covariantly. This is done by evaluating a set of matrix elements and use covariance to calculate the rest of them. In this way we maintain covariance, but the matrix elements will depend on the choice of the set chosen to be evaluated. This would not happen if the current operator used to evaluate the matrix elements was exactly covariant. We could therefore investigate covariance violations by choosing different sets of matrix elements and comparing the results obtained with these different sets.

By choosing the interacting mass operator M_I we define the model of the strong interaction dynamics, which, in the case of the two-nucleon system, should provide predictions for the scattering observables and have a discrete eigenstate of the value of the deuteron mass. Phenomenological nonrelativistic interaction is constructed by fixing the predicted phase shifts $\delta = \delta(k)$, which are relativistically invariant, to the measured ones. Here k is the magnitude of the center of mass momentum for one of the particles. We find that the relativistic dynamical mass operator preserves the wave function and the phase shifts evaluated with the nonrelativistic interaction if this operator is a function of the nonrelativistic Hamiltonian.

The relationship between the relativistic and the nonrelativistic potentials

Let us find the relationship between the nonrelativistic potential, the relativistic potential and the boosted potential. We start from the nonrelativistic Schrödinger equation in the center-of-mass reference frame of the two nucleons. We consider them to have the same mass M . The Schrödinger equation reads

$$\left(\frac{\hat{p}^2}{M} + \hat{V} \right) \Psi = \frac{p_0^2}{M} \Psi, \quad (7.67)$$

where $\hat{\mathbf{p}}$ is the relative momentum operator between the two nucleons and $\hat{V}$ is the nonrelativistic potential between them. Here we want to make explicit the nature of the momentum and potential operators, which are denoted by a hat symbol.

A candidate for the relativistic form of this equation is

$$\left(2\sqrt{M^2 + \hat{p}^2} + \hat{v}_0\right) \Phi = 2\sqrt{M^2 + p_0^2} \Phi, \quad (7.68)$$

where $\hat{v}_0$ is the relativistic potential in the center-of-mass reference frame. To find the connection between the two equations let us apply $\left(2\sqrt{M^2 + \hat{p}^2} + \hat{v}_0\right)$ to both sides of the last equation, obtaining

$$\left(4(M^2 + \hat{p}^2) + 2\sqrt{M^2 + \hat{p}^2} \hat{v}_0 + 2\hat{v}_0 \sqrt{M^2 + \hat{p}^2} + \hat{v}_0^2\right) \Phi = 4\left(M^2 + p_0^2\right) \Phi. \quad (7.69)$$

This is the same as Eq. (7.67), if we identify $\hat{V}$ with

$$\hat{V} = \frac{1}{4M} \left(2\hat{v}_0 \sqrt{M^2 + \hat{p}^2} + 2\sqrt{M^2 + \hat{p}^2} \hat{v}_0 + \hat{v}_0^2\right). \quad (7.70)$$

By defining

$$\omega(\hat{p}) \equiv \sqrt{M^2 + \hat{p}^2}, \quad (7.71)$$

we can write Eq. (7.70) as

$$\hat{V} = \frac{1}{4M} \left(2\hat{v}_0 \omega(\hat{p}) + 2\omega(\hat{p}) \hat{v}_0 + \hat{v}_0^2\right). \quad (7.72)$$

A way to evaluate the matrix elements for $\hat{v}_0$ is to evaluate the matrix elements of Eq. (7.70) on states of the kind $|p\alpha\rangle \equiv |(j, p)\mathbf{0}\mu; \bar{d}\rangle$, obtaining

$$\begin{aligned} \langle p'\alpha' | \hat{V} | p\alpha \rangle &= \frac{1}{4M} \left(2(\omega(p) + \omega(p')) \langle p'\alpha' | \hat{v}_0 | p\alpha \rangle + \right. \\ &\quad \left. \sum_{\alpha''} \int_0^\infty dp'' p''^2 \langle p'\alpha' | \hat{v}_0 | p''\alpha'' \rangle \langle p''\alpha'' | \hat{v}_0 | p\alpha \rangle \right) \end{aligned} \quad (7.73)$$

or

$$\begin{aligned} \langle p'\alpha' | \hat{v}_0 | p\alpha \rangle &+ \frac{1}{2(\omega(p) + \omega(p'))} \sum_{\alpha''} \int_0^\infty dp'' p''^2 \langle p'\alpha' | \hat{v}_0 | p''\alpha'' \rangle \langle p''\alpha'' | \hat{v}_0 | p\alpha \rangle \\ &= \frac{2M}{\omega(p) + \omega(p')} \langle p'\alpha' | \hat{V} | p\alpha \rangle. \end{aligned} \quad (7.74)$$

This equation was solved in Ref. [111] using an iteration scheme, assuming that the part proportional to $\hat{v}_0^2$ is small with respect to the other terms and evaluating higher orders until the desired precision was reached. The 0th order reads

$$\langle p'\alpha' | \hat{v}_0 | p\alpha \rangle^{(0)} = \frac{2M}{\omega(p) + \omega(p')} \langle p'\alpha' | \hat{V} | p\alpha \rangle \quad (7.75)$$

and the $n^{th}+1$ order is related to the n^{th} order by

$$\begin{aligned} \langle p'\alpha' | \hat{v}_0 | p\alpha \rangle^{(n+1)} &= \frac{1}{2(\omega(p) + \omega(p'))} \left[4M \langle p'\alpha' | \hat{V} | p\alpha \rangle \right. \\ &\quad \left. - \sum_{\alpha''} \int_0^\infty dp'' p''^2 \langle p'\alpha' | \hat{v}_0 | p''\alpha'' \rangle^{(n)} \langle p''\alpha'' | \hat{v}_0 | p\alpha \rangle^{(n)} \right] \end{aligned} \quad (7.76)$$

More technical details can be found in Ref. [111].

7.3.1 Boosted potentials

If one considers a two-nucleon system with nonzero total momentum $\mathbf{P}$, the relativistic boosted potential $\hat{v}_{\mathbf{P}}$ can be introduced heuristically as [111]

$$\hat{v}_{\mathbf{P}} = \sqrt{(2\omega(\hat{p}) + \hat{v}_0)^2 + P^2} - \sqrt{(2\omega(\hat{p}))^2 + P^2}. \quad (7.77)$$

This formula is the starting point for constructing the boosted potential directly from the relativistic one.

However, to find the connection to the nonrelativistic potential V one can write the candidate for the relativistic Schrödinger equation for the boosted two-nucleon system as

$$\left(\sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + \hat{P}^2} + \hat{v}_{\mathbf{P}} \right) \Psi = \left(\sqrt{\left(2\sqrt{(M^2 + p_0^2)} \right)^2 + P^2} \right) \Psi. \quad (7.78)$$

As before, it is necessary to square this equation and write it in the form of the nonrelativistic Schrödinger equation, which is written as

$$\left(\frac{\hat{p}^2}{M} + \frac{\hat{P}^2}{4M} + \hat{V} \right) \phi = \left(\frac{p_0^2}{M} + \frac{P^2}{4M} \right) \phi. \quad (7.79)$$

Squaring Eq. (7.78) one obtains

$$\begin{aligned} &\left(4(M^2 + \hat{p}^2) + \hat{P}^2 + \sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + \hat{P}^2} \hat{v}_{\mathbf{P}} + \right. \\ &\quad \left. \hat{v}_{\mathbf{P}} \sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + \hat{P}^2} + \hat{v}_{\mathbf{P}}^2 \right) \phi = \left(4(M^2 + p_0^2) + P^2 \right) \phi. \end{aligned} \quad (7.80)$$

Dividing by $4M$ it is possible to rearrange it as

$$\begin{aligned} &\left(\frac{\hat{p}^2}{M} + \frac{\hat{P}^2}{4M} + \frac{1}{4M} \left(\sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + \hat{P}^2} \hat{v}_{\mathbf{P}} + \right. \right. \\ &\quad \left. \left. \hat{v}_{\mathbf{P}} \sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + \hat{P}^2} + \hat{v}_{\mathbf{P}}^2 \right) \right) \phi = \left(\frac{p_0^2}{M} + \frac{P^2}{4M} \right) \phi. \end{aligned} \quad (7.81)$$

By defining

$$\omega_{\mathbf{P}}(\hat{p}) \equiv \sqrt{M^2 + \hat{p}^2 + (P/2)^2} \quad (7.82)$$

one can write

$$\left(\frac{\hat{p}^2}{M} + \frac{\hat{P}^2}{4M} + \frac{1}{4M} (2\omega_{\mathbf{P}}(\hat{p}) \hat{v}_{\mathbf{P}} + 2\hat{v}_{\mathbf{P}} \omega_{\mathbf{P}}(\hat{p}) + \hat{v}_{\mathbf{P}}^2) \right) \phi = \left(\frac{p_0^2}{M} + \frac{P^2}{4M} \right) \phi \quad (7.83)$$

and identify

$$V = \frac{1}{4M} (\omega_{\mathbf{P}}(\hat{p}) \hat{v}_{\mathbf{P}} + \hat{v}_{\mathbf{P}} \omega_{\mathbf{P}}(\hat{p}) + \hat{v}_{\mathbf{P}}^2) . \quad (7.84)$$

Equation (7.84) is similar to Eq. (7.72). It is thus possible to evaluate the matrix elements of $\hat{v}_{\mathbf{P}}$ as before using

$$\begin{aligned} \langle p' \alpha' | \hat{v}_{\mathbf{P}} | p \alpha \rangle^{(n+1)} &= \frac{1}{2(\omega_{\mathbf{P}}(p) + \omega_{\mathbf{P}}(p'))} \left[4M \langle p' \alpha' | \hat{V} | p \alpha \rangle \right. \\ &\quad \left. - \sum_{\alpha''} \int_0^\infty dp'' p''^2 \langle p' \alpha' | \hat{v}_{\mathbf{P}} | p'' \alpha'' \rangle^{(n)} \langle p'' \alpha'' | \hat{v}_{\mathbf{P}} | p \alpha \rangle^{(n)} \right] . \end{aligned} \quad (7.85)$$

A discussion about the convergence of the potential found using this method is found in Ref. [111].

7.3.2 Relativistic deuteron wave function

As an immediate application we solve the relativistic deuteron Schrödinger equation for a deuteron nucleus having momentum $\mathbf{P}$

$$\left(\sqrt{\left(2\sqrt{M^2 + \hat{p}^2} \right)^2 + P^2} + \hat{v}_{\mathbf{P}} \right) |\phi_d \mu_d\rangle = E |\phi_d \mu_d\rangle , \quad (7.86)$$

where $\omega_{\mathbf{P}}(p)$ has been defined in Eq. (7.82) and $\hat{v}_{\mathbf{P}}$ is the relativistic nucleon-nucleon potential in the reference frame where the total momentum of the two nucleons is $\mathbf{P}$. This equation can be projected and discretized as it was shown in Sec. 5.1.2. Once we find the eigenvalues E we can retrieve the deuteron binding energy B from the relations

$$\begin{cases} \sqrt{4(M^2 + p_0^2) + P^2} = E \\ B = -\frac{p_0^2}{M} \end{cases} , \quad (7.87)$$

which leads to

$$B = \frac{E^2 - 4M^2 - P^2}{4M} . \quad (7.88)$$

By construction, the resulting wave function should be the same as in the non-relativistic case. We evaluated it up to $P = 1000 \text{ fm}^{-1}$ numerically and verified that it is indistinguishable from the one shown in Fig. 5.2b.

7.4 Relativistic two-body T -matrix

In order to deal with deuteron breakup reactions, we have to evaluate matrix elements of the current between the initial deuteron state and the final two-nucleon scattering state. The latter is expressed in terms of the relativistic

boosted T -matrix, T . To this end we solve the Lippmann-Schwinger equation including the boosted potential. Then the breakup matrix element becomes

$$\langle (j_1, M) \mathbf{p}'_1 \mu'_1 \tau'_1; (j_2, M) \mathbf{p}'_2 \mu'_2 \tau'_2 | (1 + TG_0) j^\mu | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.89)$$

In the following we explain in detail how the boosted two-body T -matrix is obtained.

We start from the free Hamiltonian, which can be written as

$$H_0 = \sqrt{\left(2\sqrt{M^2 + p^2}\right)^2 + P^2}, \quad (7.90)$$

where $\mathbf{p}$ is the relative momentum of the two nucleons and $\mathbf{P}$ is their total momentum. These quantities were defined in Eq. (7.18). Equation (5.42) is transformed to

$$T_{p'p}^{\alpha'\alpha} = v_{p'p}^{\alpha'\alpha} + \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{v_{p'p''}^{\alpha'\alpha''} T_{p''p}^{\alpha''\alpha}}{E - \sqrt{(2\sqrt{M^2 + p''^2})^2 + P^2} + i\varepsilon}, \quad (7.91)$$

where now $v = v(P)$ is the matrix element of the boosted potential $\hat{v}_{\mathbf{P}}$ and therefore $T = T(P)$. We want to rewrite this as Eq. (5.46). To accomplish that we start by defining the new matrix elements $t^{\alpha'\alpha}$ and $w_{p'p}^{\alpha'\alpha}$ as

$$t_{p'p}^{\alpha'\alpha} = f_P(p') T_{p'p}^{\alpha'\alpha} f_P(p), \quad (7.92)$$

$$w_{p'p}^{\alpha'\alpha} = f_P(p') v_{p'p}^{\alpha'\alpha} f_P(p). \quad (7.93)$$

Then we rewrite Eq. (7.91) as

$$t_{p'p}^{\alpha'\alpha} = w_{p'p}^{\alpha'\alpha} + \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{w_{p'p''}^{\alpha'\alpha''} t_{p''p}^{\alpha''\alpha}}{\left(E - \sqrt{(2\sqrt{M^2 + p''^2})^2 + P^2} + i\varepsilon\right) f_P^2(p'')}. \quad (7.94)$$

To get rid of the square-root in the denominator we introduce

$$f_P^2(p) = \gamma \left(E + \sqrt{\left(2\sqrt{M^2 + p^2}\right)^2 + P^2} \right), \quad (7.95)$$

where γ is a constant yet to be defined. In this way, Eq. (7.94) becomes

$$t_{p'p}^{\alpha'\alpha} = w_{p'p}^{\alpha'\alpha} + M \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{w_{p'p''}^{\alpha'\alpha''} t_{p''p}^{\alpha''\alpha}}{\gamma(E^2 - 4M^2 - P^2 - 4p''^2) + i\varepsilon}. \quad (7.96)$$

If we choose $\gamma = 1/4M$ and

$$p_0 = \sqrt{\frac{E^2 - 4M^2 - P^2}{4}},$$

we can rewrite Eq. (7.96) as

$$t_{p'p}^{\alpha'\alpha} = w_{p'p}^{\alpha'\alpha} + M \sum_{\alpha''} \int_0^\infty dp'' p''^2 \frac{w_{p'p''}^{\alpha'\alpha''} t_{p''p}^{\alpha''\alpha}}{p_0^2 - p''^2 + i\varepsilon} \quad (7.97)$$

and solve it for $t_{p'p}^{\alpha'\alpha}$ as we did in Chapter 5.2.2. Once $t_{p'p}^{\alpha'\alpha}$ is found we can calculate $T_{p'p}^{\alpha'\alpha}$ using Eq. (7.92).

7.5 Nuclear current matrix elements

In this section we show how we evaluated all the nuclear matrix elements. As we have already stated we restrict ourselves to the single-nucleon current and show the matrix elements for the breakup and elastic reaction channels. Actually, in order to define matrix elements for the elastic case it is convenient to start with the ones for the breakup reaction.

7.5.1 Nuclear matrix elements for the deuteron breakup reaction

In order to evaluate the breakup cross section we start by evaluating the transition matrix element

$$B \equiv \langle \mathbf{p}'_1 \mu'_1 \tau'_1 \mathbf{p}'_2 \mu'_2 \tau'_2 | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle, \quad (7.98)$$

where the quantum numbers were introduced in Eq. (6.8), 6.9. From now on we identify the single-particle states with $|\mathbf{p}_i \mu_i \tau_i\rangle \equiv |(j_i, M) \mathbf{p}_i \mu_i \tau_i\rangle$. By inserting the completeness relation for the single-nucleon states and remembering that $j(1)$ does not act on the second particle we obtain

$$B = \sum_{\mu_1 \tau_1} \int d\mathbf{p}_1 \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle \langle \mathbf{p}_1 \mu_1 \tau_1 \mathbf{p}'_2 \mu'_2 \tau'_2 | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.99)$$

Expanding the deuteron state in partial waves

$$\begin{aligned} B = \sum_{\mu_1 \tau_1} \int d\mathbf{p}_1 \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle \\ \times \sum_{j \ell s t \tau \mu} \int d\mathbf{P} \int dp p^2 \langle \mathbf{p}_1 \mu_1 \tau_1 \mathbf{p}'_2 \mu'_2 \tau'_2 | (j, p) \mathbf{P} \mu; \ell s t \tau \rangle \\ \times \langle (j, p) \mathbf{P} \mu; \ell s t \tau | \phi_d \mu_d \mathbf{P}_i \rangle. \end{aligned} \quad (7.100)$$

Using Eq. (7.51) and the relation

$$\langle (j, p) \mathbf{P} \mu; \ell s t \tau | \phi_d \mu_d \mathbf{P}_i \rangle = \delta^3(\mathbf{P} - \mathbf{P}_i) \delta_{j_d j} \delta_{\mu_d \mu} \delta_{s_d s} \delta_{t_d t} \delta_{\tau_d \tau} \phi_d^{(\ell)}(p) \quad (7.101)$$

we obtain

$$\begin{aligned} B = \mathcal{N}^{-1}(\mathbf{p}_1, \mathbf{p}'_2) \sum_{\mu_1 \tau_1} \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \\ \times \sum_{\ell=0, 2} \phi_d^{(\ell)}(p) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle Y_{\ell \mu_\ell}(\hat{\mathbf{p}}) \sum_{\mu'_1 \mu'_2} \langle j_1 \mu'_1 j_2 \mu'_2 | s_d \mu_s \rangle \\ \times D_{\mu_1 \mu'_1}^{j_1} [R(\beta(\mathbf{P}_i), \mathbf{p})] D_{\mu'_2 \mu_2}^{j_2} [R(\beta(\mathbf{P}_i), -\mathbf{p})], \end{aligned} \quad (7.102)$$

where now

$$\mathbf{p}_1 = \mathbf{P}_i - \mathbf{p}'_2, \quad (7.103)$$

$$\mathbf{p} = \mathbf{p}(\mathbf{p}_1, \mathbf{p}'_2) = \mathbf{p}(\mathbf{P}_i - \mathbf{p}'_2, \mathbf{p}'_2). \quad (7.104)$$

Using the Clebsch-Gordan selection rules we see that the summation is over four indexes, in fact

$$\begin{aligned}
B = \mathcal{N}^{-1}(\mathbf{p}_1, \mathbf{p}'_2) \sum_{\mu_1} \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \\
\times \sum_{\ell} \phi_d^{(\ell)}(p) \sum_{\mu''_1 \mu''_2} Y_{\ell \mu_\ell}(\hat{\mathbf{p}}) \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle \langle j_1 \mu''_1 j_2 \mu''_2 | s_d \mu_s \rangle \\
\times D_{\mu_1 \mu'_1}^{j_1} [R(\beta(\mathbf{P}_i), \mathbf{p})] D_{\mu''_2 \mu'_2}^{j_2} [R(\beta(\mathbf{P}_i), -\mathbf{p})] , \quad (7.105)
\end{aligned}$$

where now

$$\tau_1 = -\tau'_2, \quad (7.106)$$

$$\mu_s = \mu''_1 + \mu''_2, \quad (7.107)$$

$$\mu_\ell = \mu_d - \mu_s = \mu_d - \mu''_1 - \mu''_2. \quad (7.108)$$

B'-term for inclusive cross section For inclusive cross section calculations, it is advantageous to evaluate

$$B' \equiv \langle \mathbf{p}' \mathbf{P}' \mu'_1 \tau'_1 \mu'_2 \tau'_2 | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.109)$$

We use Eq. (7.40), which we can write [62]

$$\begin{aligned}
\langle \mathbf{p}''_1 \mu''_1 \tau''_1 \mu''_2 \tau''_2 | \mathbf{p}' \mathbf{P}' \mu'_1 \tau'_1 \mu'_2 \tau'_2 \rangle = \mathcal{N}(\mathbf{p}''_1, \mathbf{p}''_2) \delta_{\tau''_1 \tau'_1} \delta_{\tau''_2 \tau'_2} \\
\times D_{\mu''_1 \mu'_1}^{j_1} [R(\beta(\mathbf{P}'), \mathbf{p})] D_{\mu''_2 \mu'_2}^{j_2} [R(\beta(\mathbf{P}'), -\mathbf{p})] \\
\times \delta^3(\mathbf{p}''_1 - \mathbf{p}_1(\mathbf{P}', \mathbf{p}')) \delta^3(\mathbf{p}''_2 - \mathbf{p}_2(\mathbf{P}', \mathbf{p}')) \quad (7.110)
\end{aligned}$$

and therefore

$$\begin{aligned}
B' = \sum_{\mu''_1 \mu''_2} \int d\mathbf{p}''_1 d\mathbf{p}''_2 \langle \mathbf{p}' \mathbf{P}' \mu'_1 \mu'_2 | \mathbf{p}''_1 \mu''_1 \mathbf{p}''_2 \mu''_2 \rangle \\
\times \langle \mathbf{p}''_1 \mu''_1 \tau''_1 \mathbf{p}''_2 \mu''_2 \tau''_2 | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle, \quad (7.111)
\end{aligned}$$

can be written as

$$B' = \mathcal{N}(\mathbf{p}''_1, \mathbf{p}''_2) \sum_{\mu''_1 \mu''_2} D_{\mu''_1 \mu'_1}^{j_1*} [R(\beta(\mathbf{P}'), \mathbf{p}')] D_{\mu''_2 \mu'_2}^{j_2*} [R(\beta(\mathbf{P}'), -\mathbf{p}')] \times B, \quad (7.112)$$

with the B -term defined in Eq. (7.98) and

$$\mathbf{p}''_i = \mathbf{p}''_i(\mathbf{p}', \mathbf{P}'). \quad (7.113)$$

B'' term Another useful matrix element we need to consider is the element

$$B'' \equiv \langle (j', p') \mathbf{P}' \mu'; \ell' s' t' \tau' | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.114)$$

This will be used for elastic reactions on the deuteron, since the deuteron wave function is represented in partial waves. We write B'' as

$$\begin{aligned}
B'' = \sum_{\mu''_1 \mu''_2} \sum_{\tau''_1 \tau''_2} \int d\mathbf{p} d\mathbf{P} \langle (j' p') \mathbf{P}' \mu'; \ell' s' t' \tau' | \mathbf{p} \mathbf{P} \mu''_1 \tau''_1 \mu''_2 \tau''_2 \rangle \\
\times \langle \mathbf{p} \mathbf{P} \mu''_1 \tau''_1 \mu''_2 \tau''_2 | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.115)
\end{aligned}$$

We evaluate $\langle (j', p') \mathbf{P}' \mu'; \ell' s' t' \tau' | \mathbf{P} \mathbf{P} \mu_1'' \tau_1'' \mu_2'' \tau_2'' \rangle$:

$$\begin{aligned} \langle (j', p') \mathbf{P}' \mu'; \ell' s' t' \tau' | \mathbf{P} \mathbf{P} \mu_1'' \tau_1'' \mu_2'' \tau_2'' \rangle &= \delta^3(\mathbf{P}' - \mathbf{P}) \frac{\delta(p' - p)}{p^2} \langle t_1 \tau_1'' t_2 \tau_2'' | t \tau \rangle \\ &\times \sum_{\mu'_\ell \mu'_s} \langle \ell' \mu'_\ell s' \mu'_s | j' \mu' \rangle \langle j_1 \mu_1'' j_2 \mu_2'' | s' \mu'_s \rangle Y_{\ell' \mu'_\ell}^*(\hat{\mathbf{p}}') . \end{aligned} \quad (7.116)$$

Therefore B'' is, after changing $\hat{\mathbf{p}}$ to $\hat{\mathbf{p}}'$,

$$\begin{aligned} B'' &= \sum_{\mu_1'' \mu_2''} \sum_{\tau_1'' \tau_2''} \int d\hat{\mathbf{p}}' \langle t_1 \tau_1'' t_2 \tau_2'' | t \tau \rangle \sum_{\mu'_\ell \mu'_s} \langle \ell' \mu'_\ell s' \mu'_s | j' \mu' \rangle \\ &\times \langle j_1 \mu_1'' j_2 \mu_2'' | s' \mu'_s \rangle Y_{\ell' \mu'_\ell}^*(\hat{\mathbf{p}}') \langle \mathbf{P}' \mathbf{P} \mu_1'' \tau_1'' \mu_2'' \tau_2'' | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle . \end{aligned} \quad (7.117)$$

C-term The last important term in the deuteron breakup interaction is

$$C \equiv \langle \mathbf{P}'_1 \mu'_1 \tau'_1 \mathbf{P}'_2 \mu'_2 \tau'_2 | t G_0 j(1) | \phi_d \mu_d \mathbf{P}_i \rangle . \quad (7.118)$$

Inserting the completeness relation, we obtain

$$\begin{aligned} C &= \sum_{j' \ell' \mu' s' t' \tau'} \sum_{j \ell \mu s t \tau} \int d\mathbf{P}' \int dp' p'^2 \int d\mathbf{P} \int dp p^2 \\ &\times \langle \mathbf{P}'_1 \mu'_1 \tau'_1 \mathbf{P}'_2 \mu'_2 \tau'_2 | (j', p') \mathbf{P}' \mu'; s' t' \tau' \rangle \langle (j', p') \mathbf{P}' \mu'; s' t' \tau' | t G_0 | (j, p) \mathbf{P} \mu; s t \tau \rangle \\ &\times \langle (j, p) \mathbf{P} \mu; s t \tau | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle , \end{aligned} \quad (7.119)$$

where

$$\begin{aligned} \langle (j', p') \mathbf{P}' \mu'; s' t' \tau' | t G_0 | (j, p) \mathbf{P} \mu; s t \tau \rangle &= \\ &\times \delta^3(\mathbf{P}' - \mathbf{P}) \delta_{j' j} \delta_{\mu' \mu} \delta_{s' s} \delta_{t' t} \delta_{\tau' \tau} \frac{\langle (j' p') \mathbf{P}' \mu'; s' t' \tau' | t | (j p) \mathbf{P} \mu; s t \tau \rangle}{E + i\epsilon - \sqrt{4(M^2 + p^2) + P^2}} . \end{aligned} \quad (7.120)$$

This reduces C to

$$\begin{aligned} C &= \sum_{\ell' \ell} \sum_{j' \mu' s' t' \tau'} \int d\mathbf{P}' \int dp' p'^2 \int dp p^2 \\ &\times \langle \mathbf{P}'_1 \mu'_1 \tau'_1 \mathbf{P}'_2 \mu'_2 \tau'_2 | (j', p') \mathbf{P}' \mu'; s' t' \tau' \rangle \frac{\langle (j', p') \mathbf{P}' \mu'; s' t' \tau' | t | (j', p) \mathbf{P}' \mu'; s' t' \tau' \rangle}{E + i\epsilon - \sqrt{4(M^2 + p^2) + P'^2}} \\ &\times \langle (j', p) \mathbf{P}' \mu'; s' t' \tau' | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle . \end{aligned} \quad (7.121)$$

By multiplying and dividing by some auxiliary quantity

$$h(E, P', p) \equiv \frac{1}{4} \left(E + \sqrt{4(M^2 + p^2) + P'^2} \right) \quad (7.122)$$

we can rewrite C as

$$\begin{aligned} C &= \sum_{\ell' \ell} \sum_{j' \mu' s' t' \tau'} \int d\mathbf{P}' \int dp' p'^2 \int dp p^2 h(E, P', p) \\ &\times \langle \mathbf{P}'_1 \mu'_1 \tau'_1 \mathbf{P}'_2 \mu'_2 \tau'_2 | (j', p') \mathbf{P}' \mu'; s' t' \tau' \rangle \frac{\langle (j', p') \mathbf{P}' \mu'; s' t' \tau' | t | (j', p) \mathbf{P}' \mu'; s' t' \tau' \rangle}{p_0^2 - p^2 + i\epsilon} \\ &\times \langle (j', p) \mathbf{P}' \mu'; s' t' \tau' | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle , \end{aligned} \quad (7.123)$$

where

$$p_0 = \frac{1}{2} \sqrt{E^2 - P'^2 - 4M^2} \quad (7.124)$$

and the last matrix element is just the matrix element B'' . With these changes the pole in C can be evaluated using standard techniques known from nonrelativistic calculations [5].

7.5.2 Nuclear matrix elements for elastic scattering on deuteron

The elastic scattering term is

$$A \equiv \langle \phi_d \mu'_d \mathbf{P}_f | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.125)$$

Using the completeness relation we can write it as

$$A = \sum_{j' \mu' \ell' s' t' \tau'} \int d\mathbf{P}' \int dp' p'^2 \langle \phi_d \mu'_d \mathbf{P}_f | (j', p') \mathbf{P}' \mu'; \ell' s' t' \tau' \rangle \times \langle (j', p') \mathbf{P}' \mu'; \ell' s' t' \tau' | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle. \quad (7.126)$$

This can be further simplified using Eq. (7.101), obtaining

$$A = \sum_{\ell'} \int dp' p'^2 \phi_d^{(\ell)}(p') \langle (j_d, p') \mathbf{P}_f \mu'_d; \ell' s_d t_d \tau_d | j(1) | \phi_d \mu_d \mathbf{P}_i \rangle, \quad (7.127)$$

$$= \sum_{\ell'} \int dp' p'^2 \phi_d^{(\ell)}(p') \times B''. \quad (7.128)$$

7.5.3 Nonrelativistic nuclear matrix elements

A strong point of our relativistic approach is that it is easy to “reconcile” with the nonrelativistic one.

The corresponding nonrelativistic expressions can be easily obtained from the relativistic ones by taking

$$D_{\mu' \mu}^j(R) = \delta_{\mu' \mu}, \quad (7.129)$$

$$\mathcal{N}(\mathbf{p}_1, \mathbf{p}_2) = 1, \quad (7.130)$$

$$\mathbf{p}_1 = \frac{1}{2} \mathbf{P} + \mathbf{p}, \quad (7.131)$$

$$\mathbf{p}_2 = \frac{1}{2} \mathbf{P} - \mathbf{p}. \quad (7.132)$$

Example: nonrelativistic A matrix element To evaluate A we need to evaluate B'' , which in turn is defined from B' , which is defined from B . Let us start from the B term. In the fully relativistic framework it has the form

$$B = \mathcal{N}^{-1}(\mathbf{p}_1, \mathbf{p}_2') \sum_{\mu_1 \tau_1} \langle \mathbf{p}_1' \mu_1' \tau_1' | j(1) | \mathbf{p}_1 \mu_1 \tau_1 \rangle \langle t_1 \tau_1 t_2 \tau_2' | t_d \tau_d \rangle \times \sum_{\ell=0,2} \phi_d^{(\ell)}(p) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle Y_{\ell \mu_\ell}(\hat{\mathbf{p}}) \sum_{\mu_1'' \mu_2''} \langle j_1 \mu_1'' j_2 \mu_2'' | s_d \mu_s \rangle \times D_{\mu_1 \mu_1'}^{j_1} [R(\beta(\mathbf{P}_i), \mathbf{p})] D_{\mu_2 \mu_2'}^{j_2} [R(\beta(\mathbf{P}_i), -\mathbf{p})]. \quad (7.133)$$

Using the relations 7.129-(7.132) the nonrelativistic expansion of B is written as

$$B = \sum_{\mu_1 \tau_1} \langle \mathbf{p}'_1 \mu'_1 \tau'_1 | j(1) | (\mathbf{P}_i - \mathbf{p}'_2) \mu_1 \tau_1 \rangle \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \\ \times \sum_{\ell=0,2} \phi_d^{(\ell)} \left(\left| \frac{\mathbf{P}_i}{2} - \mathbf{p}'_2 \right| \right) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle \langle j_1 \mu_1 j_2 \mu'_2 | s_d \mu_s \rangle Y_{\ell \mu_\ell} \left(\widehat{\frac{\mathbf{P}_i}{2} - \mathbf{p}'_2} \right). \quad (7.134)$$

The nonrelativistic form of B' is therefore

$$B' = \sum_{\mu_1 \mu_2} \delta_{\mu_1 \mu'_1} \delta_{\mu_2 \mu'_2} \left\langle \left(\frac{\mathbf{P}'}{2} + \mathbf{p}' \right) \mu_1 \tau'_1 \left(\frac{\mathbf{P}'}{2} - \mathbf{p}' \right) \mu_2 \tau'_2 \right| j(1) \left| \phi_d \mu_d \mathbf{P}_i \right\rangle \quad (7.135)$$

$$= \sum_{\mu'_1 \tau_1} \left\langle \left(\frac{\mathbf{P}'}{2} + \mathbf{p}' \right) \mu'_1 \tau'_1 \right| j(1) \left| \left(\mathbf{P}_i - \frac{\mathbf{P}'}{2} + \mathbf{p}' \right) \mu_1 \tau_1 \right\rangle \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \\ \times \sum_{\ell=0,2} \phi_d^{(\ell)} \left(\left| \frac{\mathbf{P}_i - \mathbf{P}'}{2} + \mathbf{p}' \right| \right) \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle \\ \times \langle j_1 \mu_1 j_2 \mu'_2 | s_d \mu_s \rangle Y_{\ell \mu_\ell} \left(\widehat{\frac{\mathbf{P}_i - \mathbf{P}'}{2} + \mathbf{p}'} \right). \quad (7.136)$$

For B'' we have that we can use directly Eq. (7.117) and substitute the nonrelativistic B' , obtaining

$$B'' = \sum_{\mu'_1 \mu'_2} \sum_{\tau'_1 \tau'_2} \langle t_1 \tau'_1 t_2 \tau'_2 | t_d \tau_d \rangle \sum_{\mu'_\ell \mu'_s} \langle \ell' \mu'_\ell s' \mu'_s | j' \mu' \rangle \langle j_1 \mu'_1 j_2 \mu'_2 | s' \mu'_s \rangle \\ \times \sum_{\mu'_1 \tau_1} \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \sum_{\ell} \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle \langle j_1 \mu_1 j_2 \mu'_2 | s_d \mu_s \rangle \\ \times \int d\mathbf{p}' \left\langle \left(\frac{\mathbf{P}'}{2} + \mathbf{p}' \right) \mu'_1 \tau'_1 \right| j(1) \left| \left(\mathbf{P}_i - \frac{\mathbf{P}'}{2} + \mathbf{p}' \right) \mu_1 \tau_1 \right\rangle \\ \times \phi_d^{(\ell)} \left(\left| \frac{\mathbf{P}_i - \mathbf{P}'}{2} + \mathbf{p}' \right| \right) Y_{\ell' \mu'_\ell}^*(\mathbf{p}') Y_{\ell \mu_\ell} \left(\widehat{\frac{\mathbf{P}_i - \mathbf{P}'}{2} + \mathbf{p}'} \right). \quad (7.137)$$

For A we start from Eq. (7.127) and substitute B'' , getting

$$A = \sum_{\mu'_1 \mu'_2} \sum_{\tau'_1 \tau'_2} \langle t_1 \tau'_1 t_2 \tau'_2 | t_d \tau_d \rangle \sum_{\mu'_\ell \mu'_s} \sum_{\ell \ell'} \langle \ell' \mu'_\ell s_d \mu'_s | j_d \mu'_d \rangle \langle j_1 \mu'_1 j_2 \mu'_2 | s_d \mu'_s \rangle \\ \times \sum_{\mu'_1 \tau_1} \langle t_1 \tau_1 t_2 \tau'_2 | t_d \tau_d \rangle \sum_{\mu_\ell \mu_s} \langle \ell \mu_\ell s_d \mu_s | j_d \mu_d \rangle \langle j_1 \mu_1 j_2 \mu'_2 | s_d \mu_s \rangle \\ \times \int d\mathbf{p}' \left\langle \left(\frac{\mathbf{P}_f}{2} + \mathbf{p}' \right) \mu'_1 \tau'_1 \right| j(1) \left| \left(\mathbf{P}_i - \frac{\mathbf{P}_f}{2} + \mathbf{p}' \right) \mu_1 \tau_1 \right\rangle \\ \times \phi_d^{(\ell)}(p') \phi_d^{(\ell)} \left(\left| \mathbf{p}' - \frac{\mathbf{q}}{2} \right| \right) Y_{\ell' \mu'_\ell}^*(\mathbf{p}') Y_{\ell \mu_\ell} \left(\widehat{\mathbf{p}' - \frac{\mathbf{q}}{2}} \right), \quad (7.138)$$

where $\mathbf{q} = \mathbf{P}_f - \mathbf{P}_i$. In the sum over the twelve quantum numbers only six survive by using the properties of the Clebsch-Gordan coefficient, moreover, if we take into account the isospin dependence in the single-nucleon current, we can reduce the sum to five quantum numbers.

To get consistent nonrelativistic results one has to use additionally the non-relativistic form of the single-nucleon current and nonrelativistic kinematics.

Chapter 8

Results

As we have seen, the difference between the nonrelativistic and the fully relativistic framework can be (quite technically) attributed to the following features: the kinematics, the form of the single-nucleon current operator, the interpretation of the argument of the deuteron wave function, the use of the boosted potential in the construction of the two-nucleon scattering states and the appearance of the Wigner spin rotations in the nuclear matrix elements. We will show on some examples how important these ingredients are in the numerical results.

8.1 Elastic electron-deuteron scattering

Many authors have been studying the elastic scattering between electrons and deuterons; for example Refs. [105, 112, 113, 114]. There are some standard observables of interest, which depend on the invariant four-momentum transfer squared Q^2 . Among these are the structure functions $A(Q^2)$ and $B(Q^2)$, and the tensor analyzing power $T_{20}(Q^2, \theta')$, evaluated at the scattering angle $\theta' = 70^\circ$ [105]. These quantities have been introduced in Chapter 6.1.1

We want to compare our relativistic and nonrelativistic predictions for a wide range of four-momentum transfer, which brings us to higher and higher initial electron energies E . Already big discrepancies at the kinematic level appear for $E \sim 1 - 3$ GeV. That is why when we perform nonrelativistic calculations, we take only the nuclear wave functions and the single-nucleon current operator in the nonrelativistic form. We are forced, however, to use the relativistic kinematics. The only exception is shown in Fig. 8.1, where for $Q^2 \leq 0.4$ we can still safely use nonrelativistic kinematics. In this figure we see that the difference between relativistic and nonrelativistic single-nucleon currents is negligible. Furthermore, at these values of Q^2 , the non-relativistic kinematics is a good approximation of the relativistic one, allowing us to produce consistent nonrelativistic calculations. The data for the deuteron structure functions have been taken from different publications; for $A(Q^2)$ see Refs. [115, 117, 119, 120, 121, 122, 123, 124, 125, 126], for $B(Q^2)$ see Refs. [118, 116, 119, 124, 127, 128].

We display the structure function $A(Q^2)$ and $B(Q^2)$ for a broader range of the four-momentum transfer squared, on a logarithmic scale in Fig. 8.2. As men-

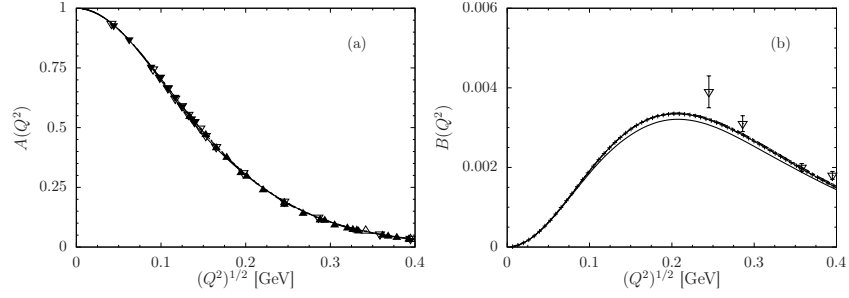


Figure 8.1: On the left (right) panel the structure function $A(Q^2)$ ($B(Q^2)$) as a function of the transferred momentum $(Q^2)^{1/2}$. The experimental data has been taken from Refs. [115] (white triangles), [116] (white down triangles), [117] (black down triangles) and [118] (black triangles). The dashed line (solid line with crosses) represents the structure function evaluated with nonrelativistic kinematics and nonrelativistic (nonrelativistic plus relativistic corrections) current operator, while the dotted (dash-dotted) line is the same quantity evaluated with relativistic kinematics and nonrelativistic (nonrelativistic plus relativistic corrections) current operator. The dashed line (solid line with crosses) represents the structure function evaluated with nonrelativistic kinematics and nonrelativistic (nonrelativistic plus relativistic corrections) current operator, while the dotted (dash-dotted) line is the same quantity evaluated with relativistic kinematics and nonrelativistic (nonrelativistic plus relativistic corrections) current operator. The solid line is the structure function evaluated fully relativistically.

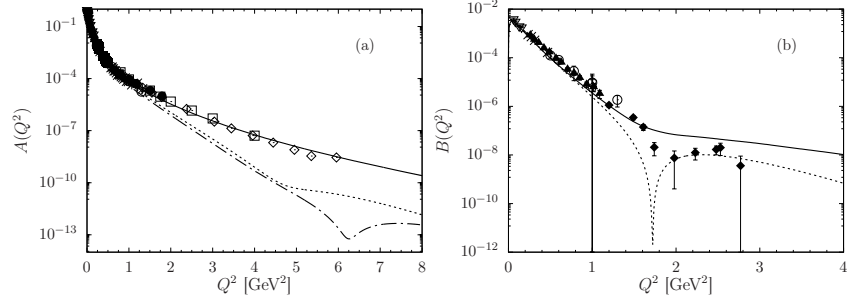


Figure 8.2: On the left (right) panel the structure function $A(Q^2)$ ($B(Q^2)$) as a function of the transferred momentum Q^2 . The experimental data that has not been already cited in Fig. 8.1 has been taken from Refs. [115] (white triangles), [116] (white down triangles), [117] (black down triangles) and [118] (black triangles). The lines are the same as Fig. 8.1.

tioned before, here for the nonrelativistic results the nonrelativistic kinematic is no longer applicable and we have to rely on the relativistic one, keeping only the nuclear wave function and the single-particle current operator nonrelativistic.

It is possible to see that the corrections to the nonrelativistic results make the discrepancy with the fully relativistic results larger. These discrepancies grow with Q^2 . Due to the fact that the corrections apply only to the density operator, they are visible only for $A(Q^2)$.

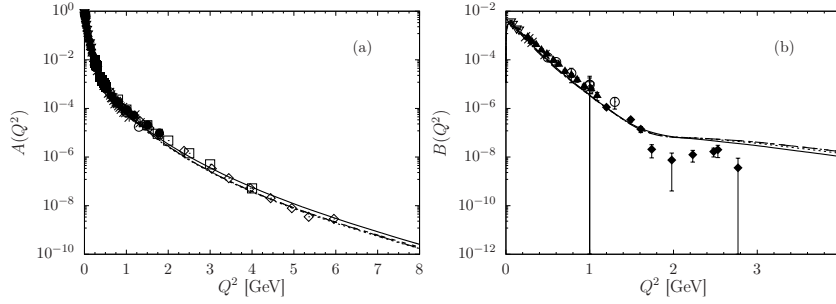


Figure 8.3: On the left (right) panel the structure function $A(Q^2)$ ($B(Q^2)$) as a function of the transferred momentum Q^2 . We evaluated the structure functions with four different parametrizations of the electromagnetic nucleon form factors. The solid line is the dipole parametrization from Ref. [13], the dashed line is from Ref. [78], the dotted line is from Ref. [76] and the dash-dotted line is from Ref. [77]. The symbols for the data points are the same as in Fig. 8.2.

The lack of contributions from the two-body current impairs the possibility to reproduce properly the data [105, 113, 114]. This is clearly visible even at low Q^2 for the structure function $B(Q^2)$.

For the bulk of our results, we use the dipole parametrization for the electromagnetic (and weak) form factors [13]. Nonetheless, we checked the effect of other parametrizations on the fully relativistic results. These are the parametrizations from Kelly [78], Lomon [76] and Budd, Bodek, and Arrington [77]. The results are shown in Fig. 8.3. Here some differences between the several parametrizations appear, especially at higher values of Q^2 .

Another effect taken into consideration is the influence of not including the Wigner spin rotations and replacing the Wigner D -functions with the Kronecker deltas. The results turned out to be almost indistinguishable from the original ones.

To generate Figs. 8.1-8.3 we fixed the initial energy of the electron to be 3 GeV and varied the scattering angle with steps of $1/8^\circ$, thus covering the needed Q^2 range.

When it comes to electron elastic scattering, another observable of interest is the deuteron tensor analyzing power $T_{20}(Q^2, \theta')$. It has been investigated at the laboratory scattering angle $\theta' = 70^\circ$. The results are shown in Fig. 8.4. We can see in the left panel that the discrepancy between the nonrelativistic and the relativistic results shows up already at $Q^2 = 0.4 \text{ GeV}^2$. In particular, the relativistic results for higher momentum transfer are smaller than their nonrelativistic counterpart. The nonrelativistic corrections start to play a role at around 2 GeV^2 . Finally, even at around 3 GeV^2 , the effect of the Wigner spin rotations is barely visible.

In the right panel we show the results with the aforementioned different parametrizations of the electromagnetic form factors. The results are indistinguishable up to 1.7 GeV^2 . For higher Q^2 values, the dipole parametrization deviates from the other three parametrizations, as noticed before for the structure functions.

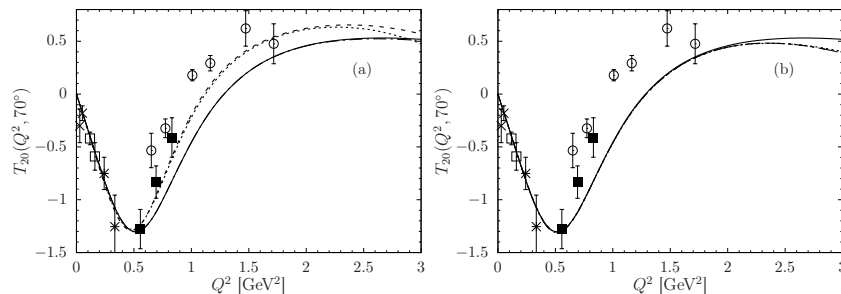


Figure 8.4: The deuteron tensor analyzing power T_{20} scattering angle $\theta' = 70^\circ$ in the laboratory reference frame as a function of the transferred momentum Q^2 . Two different nonrelativistic predictions are compared with the relativistic results in the left panel. The line types are the same as in Fig. 8.2. The effect of the Wigner spin rotation is hardly visible. It is possible to appreciate it only for large Q^2 values. In the right panel T_{20} is shown for four different parametrizations of the electromagnetic nucleon form factors [77, 13, 78, 76]. The lines are the same as in Fig. 8.3. Experimental data are from Refs. [129, 130] (x), [131] (*), [132] (white squares), [133] (black squares) and [134] (white circles).

8.2 Deuteron electrodisintegration

The study of the reaction

$$e^- + {}^2\text{H} \rightarrow e^- + p + n \quad (8.1)$$

is quite interesting because of the richness of its phase space. However, experimentally it is extremely difficult or just impossible to study the so-called pointlike geometry, where the momenta of all the outgoing particles are known with high accuracy. That is why we restrict ourselves only to showing some general features of this exclusive reaction.

Exclusive deuteron electrodisintegration In Fig. 8.5 we show the five-fold differential cross section for the exclusive electrodisintegration of the deuteron for two different sets of phase space quantities. In both figures we choose an arbitrary energy of the initial electron to be $E = 800$ MeV and we select two electron kinematics.

For the first one (Fig. 8.5a) we have $\theta' = 38.7^\circ$ and $E' = 635.3$ MeV. This gives $E_{\text{CM}} = 100$ MeV and $q = 500$ MeV, leaving no restriction on the angle θ'_p between $\mathbf{q}$ and $\mathbf{p}_p$. We therefore label the exclusive kinematics with θ'_p . We restrict ourselves to the cases $\phi_{pq} = 0$ and $\phi_{pq} = 180^\circ$. In the first case we choose θ'_p to be positive, while in the second case it is taken as negative. We see that in some regions the relativistic effects are very strong, for example in the proton knockout peak, where they reduce the cross section evaluated non-relativistically. In this peak rescattering effects are very small, but there are regions where they are clearly visible.

For the second kinematics, shown in Fig. 8.5b, we chose $\theta' = 106.5^\circ$ and $E' = 414.4$ MeV. The internal nucleon-nucleon energy is $E_{\text{CM}} = 150$ MeV,

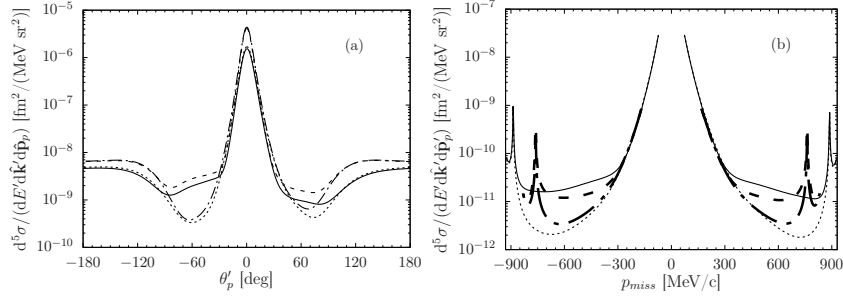


Figure 8.5: Some predictions for the five fold differential cross section $d^5\sigma/(dE'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}_p)$ for the exclusive ${}^2\text{H}(e, e'p)n$ reaction at fixed outgoing electron parameters. On the left (a) we show $E=800$ MeV, $\theta'=38.7^\circ$ $E'=635.3$ MeV, which give $E_{c.m.}=100$ MeV and $q=500$ MeV/c. On the right (b) we show $E=800$ MeV, $\theta'=106.5^\circ$ $E'=414.4$ MeV, which give $E_{c.m.}=150$ MeV and $q=1000$ MeV/c. In (a) the cross section is shown with respect to the polar angle of the outgoing proton momentum on a logarithmic scale, while in (b) with respect to the magnitude of the momentum of the undetected (here neutron) nucleon (“missing” momentum). Negative values of θ'_p or p_{miss} correspond to $\phi_p=0$ and their positive values to $\phi_p=180^\circ$. The dash-dotted (dashed) line represents the nonrelativistic results without (including) the rescattering contribution. The dotted (solid) line represents the relativistic results without (including) the rescattering contribution.

which is much smaller than the three-momentum transfer $q=1000$ MeV. In this case $0 \leq \theta'_p \leq 90^\circ$ and we have, for each θ'_p , two physical solutions. We choose therefore to label this kinematics with the missing momentum p_{miss} , the momentum of the scattered, unmeasured, neutron. We see a sharp peak, which corresponds to the point where θ'_p is maximal and the phase space factor becomes singular. Here the rescattering effects are very small for the proton knockout peak, which corresponds to small p_{miss} . It is possible to see that in some regions the rescattering contributions are more important than in others. In particular, for small scattering angles, the peak of the differential cross section is reduced by the rescattering. Here the striking differences between nonrelativistic and relativistic kinematics appear, pointing to the fact that non-relativistic calculations would not be able to describe the data.

As mentioned in the beginning of this chapter the fully exclusive measurements of the five-fold differential cross section are not feasible, therefore we move to the measurement which was pretty close to the exclusive kinematics [135].

The cross section in the deuteron electrodisintegration process at low energies and close to the threshold

Here we evaluate the deuteron electrodisintegration reaction at low initial electron energies and momentum transfer. In this way we can compare our results with the experimental data from Ref. [135]. In this and other papers by the same authors the representation for the data is given in a “mixed way”, where the electron scattering angle and energy are defined in the LAB reference frame,

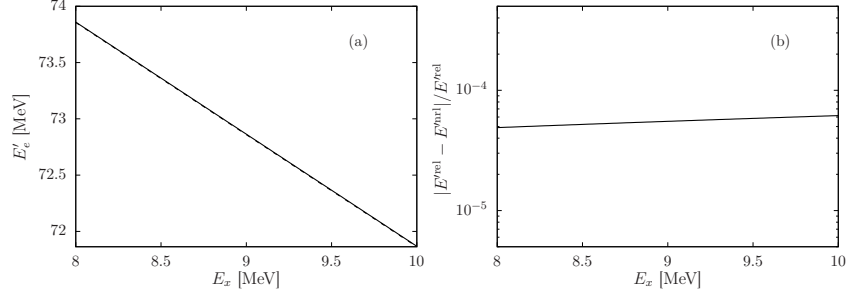


Figure 8.6: Comparison between $E' = E'(E_x)$ evaluated relativistically and nonrelativistically. In the left panel (a) the solid (dashed) line represents the result for E' evaluated with the relativistic (nonrelativistic) formula. In the right panel (b) the relative difference between the two results is shown. The values of E' have been evaluated for $E = 80$ MeV and $\theta' = 40^\circ$.

while the proton angle is given in the two-nucleon center of mass frame. In order to compare our results with the data, we start from Eq. (6.37) and we integrate the exclusive cross section over small final electron energy bins. In Ref. [135] the so-called electron excitation energy E_x is used. It is the kinetic energy in the two-nucleon system and it is related one-to-one to the final electron energy E' . We need to find the relation between E_x and E' . To do this we write the energy conservation in the laboratory reference frame

$$E_x = \sqrt{(E + M_D - E')^2 - q^2} - 2M, \quad (8.2)$$

$$q = \sqrt{E^2 + E'^2 - 2EE' \cos \theta'}, \quad (8.3)$$

therefore we obtain

$$E' = \frac{E_x^2 + 4E_x M - 2EM_D - M_D^2 + 4M^2}{2(E \cos \theta' - E - M_D)}. \quad (8.4)$$

We can do the same in the nonrelativistic approximation. Knowing that

$$E_x = E + M_D - 2M - \left(E' + \frac{q^2}{4M}\right), \quad (8.5)$$

we find

$$E' = \frac{1}{\sqrt{2}} \left(8EM - 8M(E_x - M_D + M) - E^2 + E(E \cos 2\theta' - 8M \cos \theta') \right)^{1/2} + E \cos 2\theta' - 2M. \quad (8.6)$$

For the considered low electron energies the nonrelativistic and relativistic relations between E_x and E' are completely equivalent (see Fig. 8.6). It is clear that for this kinematics relativistic effects are of no importance, so we just confirm the consistency between our present relativistic framework [70] and the older nonrelativistic calculations performed in Refs. [99, 57, 58].

Given the connection between E' and E_x we integrate the exclusive cross section over the required range of the final electron energies and arrive at the

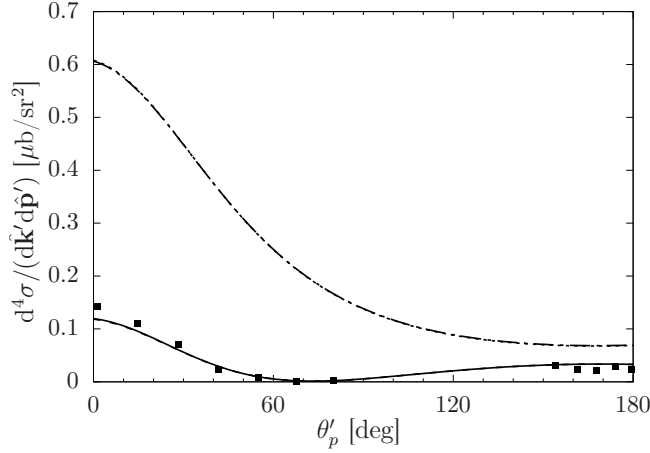


Figure 8.7: The four-fold cross section for the deuteron electrodisintegration $d^4\sigma/(d\hat{\mathbf{k}}'d\hat{\mathbf{p}}')$ at $E = 85$ MeV and for the excitation energy bin between 8 and 10 MeV. The line types are the same as in Fig. 8.5. The relativistic and nonrelativistic result overlap both for full and plane wave calculations. The data points are from Ref. [135].

four-fold differential cross section

$$\frac{d^4\sigma}{d\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} = \int_{E'_{\min}}^{E'_{\max}} dk' \frac{d^5\sigma}{dk'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} \quad (8.7)$$

studied in Ref. [135].

As usual, our results are calculated with the single-nucleon current and are shown in Fig. 8.7. It is known that for the small magnitudes of the three-momentum transfer the cross section is dominated by the charge density matrix element, where the two-body contributions are not very important (see for example [99]). Thus, it is not surprising that our calculations fairly well describe the experimental data. As we could expect, the relativistic and nonrelativistic predictions at these low energies are very close to each other. A striking feature of these results is the crucial role played by the rescattering term in the nuclear matrix element both in the nonrelativistic and relativistic case. The two-nucleon final state interactions reduce the cross section by nearly a factor of six.

The inclusive cross section for the deuteron electrodisintegration

We show the results for the differential cross section

$$\frac{d^3\sigma}{dE'd\hat{\mathbf{k}}'} = \int d\hat{\mathbf{p}}' \frac{d^5\sigma}{dE'd\hat{\mathbf{k}}'d\hat{\mathbf{p}}'} \quad (8.8)$$

along with data taken from Ref. [136]. In this research, several sets of data for the differential cross section and the corresponding response functions were compared with the theoretical predictions available at that time.

Studying the inclusive cross section still allows one to vary the magnitude of the energy and three-momentum transfer for the fixed initial electron energy.

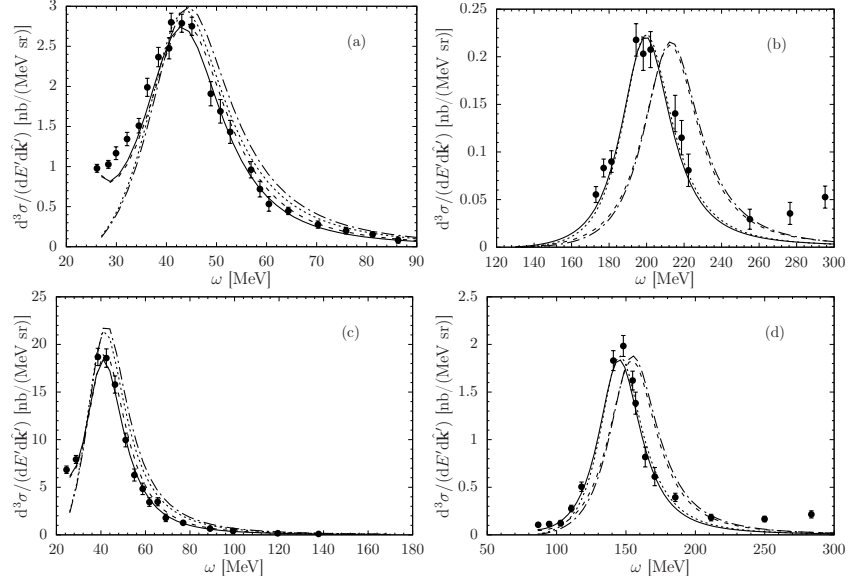


Figure 8.8: Results for the three-fold differential cross section $d^3\sigma/(dE'd\hat{\mathbf{k}}')$ for the deuteron electrodisintegration, shown as a function of the energy transfer ω . The sequence of subfigures follows Fig. 8 in Ref. [136]. The line types are the same as in Fig. 8.5.

Because of the integrated angle $\hat{\mathbf{p}}'$ the values of the cross section are higher than in the exclusive case, allowing many experiments to focus especially on broad maxima in the quasielastic scattering region in the (θ', E') plane [137, 138]. In this region we find the approximate relation

$$q^2 \sim 2M\omega, \quad (8.9)$$

where ω is the energy transferred to the two-nucleon system and M is the nucleon mass.

The results are shown in Figs. 8.8-8.10. Here we show nonrelativistic and relativistic calculations with and without the contribution of the rescattering term. These figures correspond to Figs. 8-10 in Ref. [136].

The figures concentrate on the quasielastic peak, where the contributions of the two-nucleon current operator, which we do not include in our calculations, should be less important. It is clear that the nonrelativistic and relativistic peaks are slightly shifted, which makes our relativistic calculations more realistic. We can also appreciate an enhancement of the cross section due to the inclusion of the rescattering term for energy transfers close to the threshold, where the energy of the two-nucleon system is very small. This is consistent with the data and incompatible with the plane-wave-based results. On the right slope with respect to the peak, the transferred energies involved are too big and new channels open, such as pion production and Δ particle excitation, not allowing us to describe the data within our theory.

It is encouraging to notice that generally our relativistic approach gives at least a fair description of the experimental data.

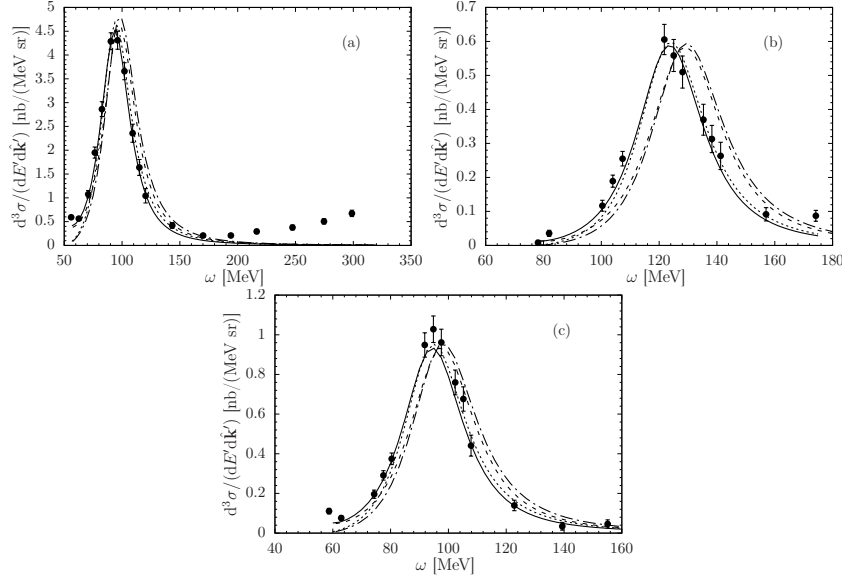


Figure 8.9: The same as in Fig. 8.8 but the results correspond to Fig. 9 in Ref. [136].

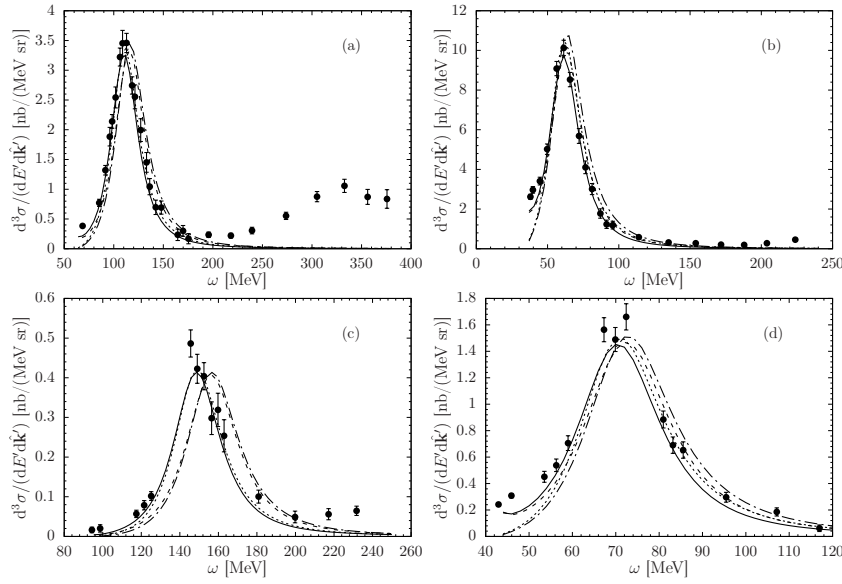


Figure 8.10: The same as in Fig. 8.8 but the results correspond to Fig. 10 in Ref. [136].

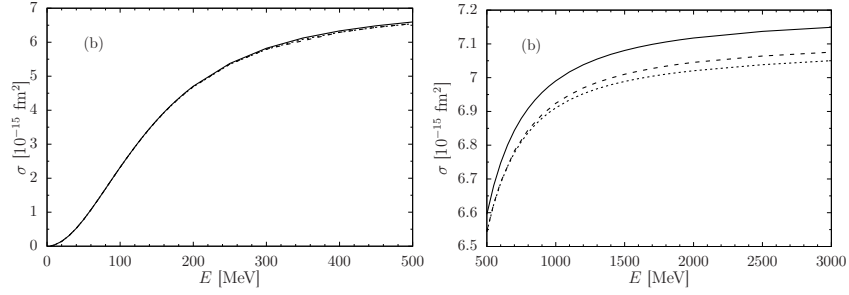


Figure 8.11: Predictions for the total elastic neutrino-deuteron cross section σ with respect to the neutrino energy in the laboratory reference frame, E . In (a) $E < 500$ MeV, while in (b) $E > 500$ MeV. The cross section was evaluated relativistically in the laboratory system (solid line) and in the center of mass system (dotted line), as well as nonrelativistically in the laboratory system (dashed line).

8.3 Neutrino elastic scattering on deuteron

The first reaction between neutrino and deuteron that we take into consideration is their elastic scattering: $\nu_e + d \rightarrow \nu_e + d$. We studied this NC reaction in both the laboratory reference frame and in the center-of-mass reference frame. The comparison is given in Fig. 8.11, where we show the results for the total cross section. The predictions are prepared for three different frameworks: nonrelativistically in the laboratory reference frame, relativistically in both the laboratory reference frame and in the center-of-mass reference frame. We can see that up to $E_\nu = 500$ MeV in the laboratory reference frame the predictions are very similar, while for higher energies clear discrepancies emerge. The incompleteness of the current operator causes some deviations between the relativistic results calculated in the laboratory and in the center-of-mass frame. In fact, the nonrelativistic results calculated in the laboratory frame are closer to the relativistic results calculated in the same frame than the other relativistic results obtained in the center-of-mass frame.

8.4 Neutrino-induced deuteron breakup

We calculated predictions for the deuteron breakup reactions: $\nu_e + d \rightarrow \nu_e + p + n$ and $\bar{\nu}_e + d \rightarrow e^+ + n + n$. For these weak reactions it is unlikely to obtain exclusive or semi-exclusive differential cross section and we focus on the total breakup cross section. It can be calculated as in Eq. (6.42)

$$\sigma = \int d\hat{\mathbf{k}}' \int_0^{E'^{\max}} dE' \frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'} \quad (8.10)$$

$$= 2\pi \int_0^\pi d\cos\theta' \int_0^{E'^{\max}} dE' \left. \frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'} \right|_{\phi'=0}, \quad (8.11)$$

where θ' and ϕ' are the spherical coordinates of the final lepton momentum in the system where the initial (anti)neutrino momentum is parallel to the z -

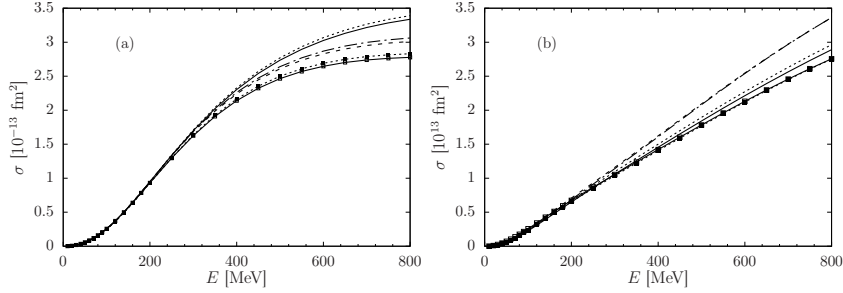


Figure 8.12: Predictions for the total breakup cross section for the (a) $\nu + d \rightarrow \nu + n + p$ and (b) $\bar{\nu}_e + d \rightarrow e^+ + n + n$ with respect to the (anti)neutrino energy E in the laboratory reference frame. The solid (dotted) line represents the differential cross section with (without) the contribution of the rescattering term calculated relativistically. The dashed (dash-dotted) line represents the differential cross section with (without) the contribution of the rescattering term and calculated within the nonrelativistic treatment. The solid (dashed) line with white (black) squares represents the result for the nonrelativistic calculations with (without) the contribution of the rescattering term and with the relativistic corrections to the nonrelativistic single-nucleon current operator.

axis. The results for these two reactions in the relativistic, nonrelativistic and nonrelativistic with relativistic correction frameworks are shown in Fig. 8.12, where we also show the results evaluated only with the plane-wave contribution to the cross section. The introduction of the rescattering effects is valid only for energies where the two-nucleons center of mass energy is smaller than the pion mass, therefore the “full” results are to be taken with great caution. We see that the rescattering contributions are small in all the cases, but we should keep in mind that other reaction channels are open.

Similarly to most of the elastic and quasielastic (anti)neutrino-nucleon scattering reactions, adding the relativistic corrections to the nonrelativistic current operator makes the difference between the relativistic and nonrelativistic results bigger compared to the case where these corrections are omitted.

As described above, the total cross section is calculated by integrating the three-fold differential cross section over the three variables defining the outgoing lepton momentum.

In Fig. 8.13 we show such differential cross sections for two different initial neutrino energies: 120 and 700 MeV. The differences between the relativistic and nonrelativistic results are general small and, obviously, bigger for the higher initial neutrino energy. The shape of the cross section angular distribution strongly depends on the neutrino energy.

The results for the antineutrino CC induced breakup reaction studied in this are displayed in Fig. 8.14. We see that the results differ from the ones for the neutrino NC induced reaction, where at low energies we do not observe a forward peak in the differential cross section. These peaks for the antineutrino reactions are broader than the corresponding peaks in the neutrino induced reactions. Lastly, we notice that the relativistic results differ much more from the nonrelativistic ones for the antineutrino case.

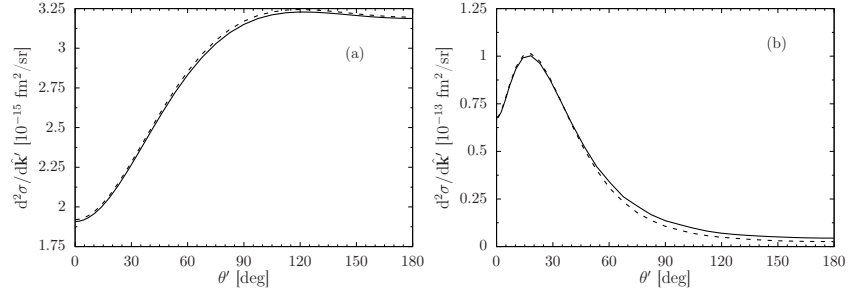


Figure 8.13: The differential breakup cross section $d^2\sigma/d\hat{k}'$ for the $\nu + d \rightarrow \nu + p + n$ process calculated in the laboratory reference frame for the initial neutrino energy $E = 120 \text{ MeV}$ (left panel) and 700 MeV (right panel). The lines are the same as in Fig. 8.11.

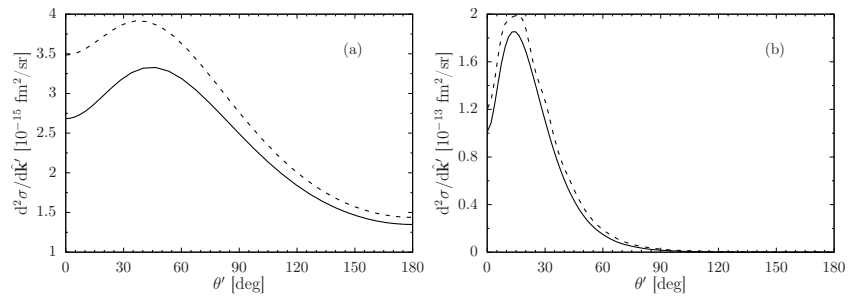


Figure 8.14: The same as Fig. 8.13 but for the $\bar{\nu}_e + d \rightarrow e^+ + n + n$ process.

Chapter 9

Selected neutrino induced reactions on ${}^3\text{H}$ and ${}^3\text{He}$

In this chapter we give a brief account of the calculations published in Refs. [57] and [58]. As in the rest of this work, these calculations were also performed in momentum space. They were based on the nonrelativistic Faddeev formalism for the 3N bound and scattering states and employed nonrelativistic kinematics. The only relativistic feature of the calculations were relativistic corrections in the nonrelativistic single-nucleon weak charged and neutral current operators. However, in Refs. [57, 58] the nonrelativistic single-nucleon weak current operator was taken from the Pisa group [13, 60, 44] and it is not quite consistent with the definitions used elsewhere in this thesis. The AV18 NN potential [4] was employed in the calculations of the 3N bound and scattering states in order to facilitate a comparison with earlier publications.

The same formalism which was built originally for electromagnetic (electron scattering and photodisintegration) reactions with 3N systems [139, 99] as well as for muon capture on the 3N bound states [60, 140] can be applied to elastic, quasielastic and inelastic (anti)neutrino scattering on ${}^3\text{He}$ and ${}^3\text{H}$. The most important ingredients of this formalism are presented below.

9.1 Elements of the nonrelativistic 3N Faddeev formalism

There are many possible reactions in which (anti)neutrinos interact with the 3N bound states, even if the internal 3N energy remains below the pion production threshold and only electron (anti)neutrinos are taken into consideration. There are elastic

$$\nu_l (\bar{\nu}_l) + {}^3\text{He} \rightarrow \nu_l (\bar{\nu}_l) + {}^3\text{He}, \quad (9.1)$$

$$\nu_l (\bar{\nu}_l) + {}^3\text{H} \rightarrow \nu_l (\bar{\nu}_l) + {}^3\text{H}, \quad (9.2)$$

and inelastic NC reactions

$$\nu_l (\bar{\nu}_l) + {}^3\text{He} \rightarrow \nu_l (\bar{\nu}_l) + {}^2\text{H} + p, \quad (9.3)$$

$$\nu_l (\bar{\nu}_l) + {}^3\text{H} \rightarrow \nu_l (\bar{\nu}_l) + {}^2\text{H} + n, \quad (9.4)$$

$$\nu_l (\bar{\nu}_l) + {}^3\text{He} \rightarrow \nu_l (\bar{\nu}_l) + p + p + n, \quad (9.5)$$

$$\nu_l (\bar{\nu}_l) + {}^3\text{H} \rightarrow \nu_l (\bar{\nu}_l) + p + n + n, \quad (9.6)$$

where l denotes any (anti)neutrino flavor. The possible CC reactions are

$$\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + {}^2\text{H} + n, \quad (9.7)$$

$$\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + p + n + n, \quad (9.8)$$

$$\bar{\nu}_e + {}^3\text{H} \rightarrow e^+ + n + n + n, \quad (9.9)$$

$$\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + {}^3\text{H}, \quad (9.10)$$

$$\nu_e + {}^3\text{He} \rightarrow e^- + p + p + p, \quad (9.11)$$

and

$$\nu_e + {}^3\text{H} \rightarrow e^- + p + p + n. \quad (9.12)$$

For many of these reactions two (or even three) protons emerge in the final state. Since our momentum-space framework was not able to treat exactly the Coulomb proton-proton force in the 3N system, we restricted ourselves in fact to very few reactions and investigated in the first place the non-breakup $\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + {}^3\text{H}$, $\bar{\nu}_l + {}^3\text{He} \rightarrow \bar{\nu}_l + {}^3\text{He}$, $\nu_l + {}^3\text{He} \rightarrow \nu_l + {}^3\text{He}$, $\bar{\nu}_l + {}^3\text{H} \rightarrow \bar{\nu}_l + {}^3\text{H}$, $\nu_l + {}^3\text{H} \rightarrow \nu_l + {}^3\text{H}$ processes.

We treated also several breakup reactions: $\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + n + d$, $\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + n + n + p$, $\nu_l + {}^3\text{H} \rightarrow \nu_l + n + d$, $\nu_l + {}^3\text{H} \rightarrow \nu_l + n + n + p$, $\bar{\nu}_l + {}^3\text{He} \rightarrow \bar{\nu}_l + p + d$, $\bar{\nu}_l + {}^3\text{He} \rightarrow \bar{\nu}_l + p + p + n$, but our predictions for the latter two should be treated with care, since we neglected the Coulomb force between the two protons in the final state.

The 3N Hamiltonian H contains the 3N kinetic energy (H_0), two-body subsystem potential energies (V_{12} , V_{23} and V_{31}) as well as the three-nucleon potential energy (V_{123}). The latter quantity is split into three terms

$$V_{123} = V^{(1)} + V^{(2)} + V^{(3)}, \quad (9.13)$$

where $V^{(i)}$ is symmetric under exchange of nucleons j and k ($i, j, k = 1, 2, 3$, $i \neq j \neq k \neq i$). The 3N bound state wave function is obtained using the method described in Ref. [141]. In the Faddeev framework the wave function is given in terms of three Faddeev components, where in the case of identical particles the Faddeev component $|\psi\rangle$ obeys the following equation

$$|\psi\rangle = G_0 t_{23} P |\psi\rangle + (1 + G_0 t_{23}) G_0 V^{(1)} (1 + P) |\psi\rangle. \quad (9.14)$$

Here $G_0 \equiv 1/(E_{3\text{N}} - H_0)$ is the free 3N propagator and $P \equiv P_{12}P_{23} + P_{13}P_{23}$ is the permutation operator constructed from transpositions P_{ij} , which interchange nucleons i and j . The internal 3N energy $E_{3\text{N}}$ is negative for the bound state. The two-nucleon t -operator t_{23} acts now in the 3N space. The full wave function $|\Psi\rangle$ is built from the Faddeev component as

$$|\Psi\rangle = (1 + P) |\psi\rangle. \quad (9.15)$$

The 3N current operator $j_{3\text{N}}^\mu$ contains the single-nucleon, 2N and, in principle, also the 3N contribution. It is thus given in the following way:

$$j_{3\text{N}}^\mu = j_1^\mu + j_2^\mu + j_3^\mu + j_{12}^\mu + j_{23}^\mu + j_{31}^\mu + j_{123}^\mu, \quad (9.16)$$

where the 3N part can be decomposed into three components $j_{123}^\mu = j^{\mu 1} + j^{\mu 2} + j^{\mu 3}$, just like the 3N potential energy in Eq. (9.13). It means that the *whole* 3N current operator can be split into three parts, $j^\mu(i)$ ($i = 1, 2, 3$), which exhibit the same symmetry properties as $V^{(i)}$:

$$j_{3N}^\mu = j^\mu(1) + j^\mu(2) + j^\mu(3). \quad (9.17)$$

In particular, $j^\mu(1) \equiv j_1^\mu + j_{23}^\mu + j^{\mu 1}$.

The crucial matrix elements for the nucleon-deuteron (Nd)

$$N_{Nd}^\mu = {}^{(-)}\langle \Psi_{Nd} | j_{3N}^\mu | \Psi \rangle \quad (9.18)$$

and the 3N breakup

$$N_{3N}^\mu = {}^{(-)}\langle \Psi_{3N} | j_{3N}^\mu | \Psi \rangle \quad (9.19)$$

channels comprise the corresponding channel scattering states. The matrix elements do not require that the scattering states are calculated explicitly. In order to obtain N_{Nd}^μ and N_{3N}^μ first a Faddeev-type equation is solved for an auxiliary state $|U^\mu\rangle$ [99],

$$\begin{aligned} |U^\mu\rangle &= \left(t_{23}G_0 + \frac{1}{2}(1+P)V^{(1)}G_0(1+t_{23}G_0) \right) (1+P)j^\mu(1)|\Psi\rangle \\ &+ \left(t_{23}G_0P + \frac{1}{2}(1+P)V^{(1)}G_0(1+t_{23}G_0)P \right) |U^\mu\rangle. \end{aligned} \quad (9.20)$$

It depends on the component of the current operator and two kinematical quantities, which are the 3N internal energy $E_{3N} > 0$ and the magnitude of the three-momentum transferred to the 3N system, $q \equiv |\mathbf{q}|$. For example, in the case of the antineutrino CC breakup of ${}^3\text{H}$ it reads

$$\begin{aligned} E_{3N} &= E - E' + M_{3H} - 3M_n - \frac{\mathbf{q}^2}{6M_n}, \\ \mathbf{q} &= \mathbf{k} - \mathbf{k}', \end{aligned} \quad (9.21)$$

where E (E') is the initial (final) lepton energy, $\mathbf{k}$ ($\mathbf{k}'$) is its initial (final) momentum, M_{3H} and M_n are the triton and neutron masses, respectively. It is important to note that $|U^\mu\rangle$ does not depend on the details of the final state kinematics. In practice we use the density operator $\rho \equiv j_{3N}^0$ as well as the spherical components of the current operator

$$\begin{aligned} j_{+1} &\equiv -\frac{1}{\sqrt{2}}(j_{3N}^1 + i j_{3N}^2) \equiv -\frac{1}{\sqrt{2}}(j_{x3N} + i j_{y3N}), \\ j_{-1} &\equiv \frac{1}{\sqrt{2}}(j_{3N}^1 - i j_{3N}^2) \equiv \frac{1}{\sqrt{2}}(j_{x3N} - i j_{y3N}) \end{aligned} \quad (9.22)$$

and choose $\mathbf{q}$ parallel to the z -axis. As in the 2N case, this guarantees the simplest relations between the total 3N angular momentum projections in the initial and final nuclear states.

The matrix elements N_{Nd}^μ and N_{3N}^μ for arbitrary exclusive kinematics are then calculated by quadratures:

$$N_{Nd}^\mu = \langle \phi_{Nd} | (1+P)j^\mu(1) | \Psi \rangle + \langle \phi_{Nd} | P | U^\mu \rangle, \quad (9.23)$$

$$\begin{aligned}
N_{3\text{N}}^\mu &= \langle \phi_{3\text{N}} | (1 + P) j^\mu(1) | \Psi \rangle + \langle \phi_{3\text{N}} | t_{23} G_0 (1 + P) j^\mu(1) | \Psi \rangle \\
&+ \langle \phi_{3\text{N}} | P | U^\mu \rangle + \langle \phi_{3\text{N}} | t_{23} G_0 P | U^\mu \rangle, \quad (9.24)
\end{aligned}$$

where $|\phi_{\text{Nd}}\rangle$ is a product of the internal deuteron state and the state describing the free relative motion of the third nucleon with respect to the deuteron and $|\phi_{3\text{N}}\rangle$ is a state (antisymmetrized in the $(2, 3)$ subsystem) representing the free motion of the three outgoing nucleons. If necessary, exclusive observables are integrated over suitable phase space domains to achieve the semi-exclusive or inclusive cross section or polarization observables.

The 3N inclusive observables can be, however, computed independently, without any explicit integration over the available 3N phase space [99, 139]. In such inclusive calculations carried out usually in the laboratory frame, the final 3N internal energy $E_{3\text{N}}$ and the magnitude of the three-momentum transfer q are known. The so-called “response functions” are defined by the following integral

$$R_{AB}^{\text{inc}} = \sum_{m_i, m_f} \int df \delta(E_{3\text{N}} - E_f) {}^{(-)}\langle \Psi_f | j_{3\text{N}}^A | \Psi \rangle \left({}^{(-)}\langle \Psi_f | j_{3\text{N}}^B | \Psi \rangle \right)^*, \quad (9.25)$$

and depend, in general, on two components of the nuclear current operator, A and B . In Eq. (9.25) m_i and m_f represent the whole sets of the initial and final spin magnetic quantum numbers. Additionally, the df integral denotes the sum and the integration over all final 3N states with the energy $E_{3\text{N}}$. Using closure, Eq. (9.25) can be rewritten as

$$R_{AB}^{\text{inc}} = \sum_{m_i} \int df \langle \Psi | (j_{3\text{N}}^B)^\dagger \delta(E_{3\text{N}} - H) j_{3\text{N}}^A | \Psi \rangle, \quad (9.26)$$

where H is the full 3N Hamiltonian. Note that the 3N bound state does not contribute to the df integration for $E_{3\text{N}} > 0$. Within the Faddeev scheme [139, 99], R_{AB}^{inc} can be expressed in terms of some auxiliary states as

$$\begin{aligned}
R_{AB}^{\text{inc}} &= \frac{1}{2\pi i} \sum_{m_i} \left(\langle \Psi | (j_{3\text{N}}^A)^\dagger | \Psi^B \rangle^* - \langle \Psi | (j_{3\text{N}}^B)^\dagger | \Psi^A \rangle \right) \\
&= \frac{3}{2\pi i} \sum_{m_i} \left(\langle \Psi | (j^A(1))^\dagger G_0(1 + P) | V^B \rangle^* \right. \\
&\quad \left. - \langle \Psi | (j^B(1))^\dagger G_0(1 + P) | V^A \rangle \right), \quad (9.27)
\end{aligned}$$

where in turn $(C = A, B)$

$$|\Psi^C\rangle = G_0(1 + P) | V^C \rangle. \quad (9.28)$$

The state $|V^C\rangle$ is a solution of the Faddeev-type equation

$$\begin{aligned}
|V^C\rangle &= (1 + t_{23} G_0) j^C(1) | \Psi \rangle \\
&+ \left(t G_0 P + (1 + t_{23} G_0) V^{(1)} G_0 (1 + P) \right) | V^C \rangle, \quad (9.29)
\end{aligned}$$

which has the same integral kernel as the equation solved in the 3N scattering calculations [142]. The connection between the auxiliary states defined in

9.2. SELECTED RESULTS FOR NEUTRINO SCATTERING ON ${}^3\text{He}$ AND ${}^3\text{H}$

Eqs. (9.20) and (9.29) is particularly simple when the 3N potential energy term in the 3N Hamiltonian is neglected and reads

$$|V^C\rangle = j^C(1)|\Psi\rangle + |U^C\rangle. \quad (9.30)$$

The fact that R_{AB}^{inc} can be computed both by direct integrations and using the method described above allows one to test the numerical accuracy.

All the quantities present in Eqs. (9.20) and (9.29) are represented in the momentum-space basis

$$|pw\alpha\rangle \equiv \left| pw(ls)j \left(\lambda \frac{1}{2} \right) I(jI)Jm_J \left(t \frac{1}{2} \right) Tm_T \right\rangle, \quad (9.31)$$

which is built upon the (2,3) subsystem basis $|p\alpha\rangle \equiv |p\alpha_2\rangle$ employed in Chapter 7.2. Thus, w is the magnitude of the Jacobi relative momentum, which describes the motion of the spectator nucleon 1 with respect to the center of mass of the (2,3) subsystem. Further, the orbital angular momentum λ of the spectator nucleon and its spin $1/2$ are coupled to the total spectator angular momentum I . The total angular momentum of the subsystem (j) and the total angular momentum of nucleon 1 (I) couple eventually to the total angular momentum of the 3N system J and its projection m_J . A corresponding coupling is introduced in the isospin space, where the (2,3) total subsystem isospin (t) and the isospin of nucleon 1 add to the total 3N isospin T with the projection m_T . In practice the calculations are restricted to a finite set of $|pw\alpha\rangle$ states, with $j \leq j_{\max}$ and $J \leq J_{\max}$. For example, the calculations reported in Refs. [57, 58] required $j_{\max} = 3$ and $J_{\max} = 15/2$ to achieve convergence.

In the following section we describe some results, which are obtained without any radiative corrections. Some information about possible changes due to these effects are discussed in Ref. [47] and references cited therein.

9.2 Selected results for neutrino scattering on ${}^3\text{He}$ and ${}^3\text{H}$

First, we consider quasielastic CC induced electron antineutrino scattering on ${}^3\text{He}$ with a positron and a ${}^3\text{H}$ nucleus in the final state. In Fig. 9.1 we show the total cross sections for the $\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + {}^3\text{H}$ reaction as a function of the initial antineutrino energy.

Further non-breakup reactions with trinucleons are possible only with the neutral current. In Fig. 9.2 we display the predictions for the total cross section of the four elastic $\bar{\nu}_l + {}^3\text{He} \rightarrow \bar{\nu}_l + {}^3\text{He}$, $\bar{\nu}_l + {}^3\text{H} \rightarrow \bar{\nu}_l + {}^3\text{H}$, $\nu_l + {}^3\text{He} \rightarrow \nu_l + {}^3\text{He}$, and $\nu_l + {}^3\text{H} \rightarrow \nu_l + {}^3\text{H}$ reactions as a function of the initial (anti)neutrino energy. In spite of very similar reaction kinematics, the cross sections take quite different values. The antineutrino- ${}^3\text{He}$ cross section is the smallest and the neutrino- ${}^3\text{H}$ cross section reaches the highest values. Lines representing neutrino- ${}^3\text{He}$ and antineutrino- ${}^3\text{H}$ cross sections cross at $E \approx 150$ MeV, with the former prevailing for the higher energies. All the four predictions do not depend on the (anti)neutrino flavor as long as the (anti)neutrinos are considered to be massless. The difference between the predictions for the neutrino and antineutrino induced reactions on either nucleus comes only from the functions defined already in Eqs. (3.57) and (3.61). On the other hand, the difference between the predictions

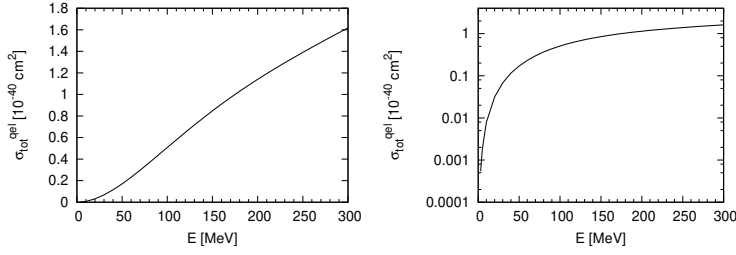


Figure 9.1: The total cross section for the quasielastic CC $\bar{\nu}_e + {}^3\text{He} \rightarrow e^+ + {}^3\text{H}$ process as a function of the initial antineutrino energy E shown on the linear (left panel) and logarithmic (right panel) scale. The results are obtained with the AV18 NN potential and with the single nucleon CC operator from Refs. [13, 60], [44], which contains the relativistic corrections.

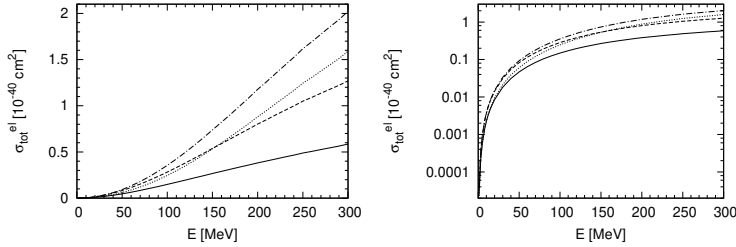


Figure 9.2: The total cross section for the elastic NC driven reactions $\bar{\nu}_l + {}^3\text{He} \rightarrow \bar{\nu}_l + {}^3\text{He}$ (solid line), $\bar{\nu}_l + {}^3\text{H} \rightarrow \bar{\nu}_l + {}^3\text{H}$ (dashed line), $\nu_l + {}^3\text{He} \rightarrow \nu_l + {}^3\text{He}$ (dotted line) and $\nu_l + {}^3\text{H} \rightarrow \nu_l + {}^3\text{H}$ (dash-dotted line) as a function of the initial (anti)neutrino energy E shown on a linear (left panel) and a logarithmic (right panel) scale. The results are obtained with the AV18 NN potential and the single nucleon NC operator, which contains the relativistic corrections.

for the $\nu_l + {}^3\text{He} \rightarrow \nu_l + {}^3\text{He}$ and $\nu_l + {}^3\text{H} \rightarrow \nu_l + {}^3\text{H}$ reactions is caused nearly exclusively by the difference between the proton and neutron weak NC form factors. Further effects introduced by the slightly different masses and wave functions of ${}^3\text{He}$ and ${}^3\text{H}$ are very small. The same holds for the two antineutrino induced elastic reactions.

Calculations of the CC and NC driven breakup reactions are more difficult due to the complicated kinematics, which has to take into account two- and three-body disintegration processes. In electron or (anti)neutrino scattering on the trinucleons one usually assumes that the initial lepton energy, the lepton scattering angle and the final lepton energy are known. This information allows one to study the 3N scattering states for a fixed 3N internal energy and three-momentum transfer. To this end the corresponding Faddeev-type equation (9.20) is solved separately for various values of the total 3N angular momenta, both parities and the total 3N isospins $T = 1/2$ and $T = 3/2$. In the case of the (anti)neutrino induced processes the total cross sections are of special interest, which in principle requires a consideration of at least hundreds of the neutrino kinematics. Fortunately, as already mentioned the essential dynamical quantities for inclusive reactions, the so-called response functions R_{AB}^{inc} , can be

9.2. SELECTED RESULTS FOR NEUTRINO SCATTERING ON ${}^3\text{He}$ AND ${}^3\text{H}$

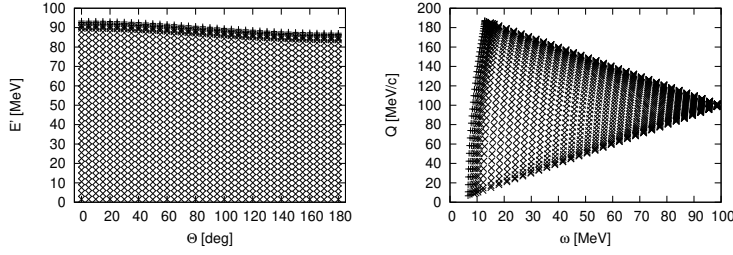


Figure 9.3: The ranges of various kinematical quantities describing the kinematics of the NC driven breakup of ${}^3\text{H}$ for the initial (anti)neutrino energy $E=100$ MeV. In the left panel the outgoing (anti)neutrino energy and the scattering angle are considered, while in the right panel the magnitude of the three-momentum transfer is plotted versus the energy transfer.

computed efficiently, that is without resort to explicit 3N kinematics. Moreover, they depend on two parameters, which are the internal 3N energy $E_{3\text{N}}$ and the magnitude of the three-momentum transfer q or, equivalently, the energy transfer $\omega \equiv E - E'$ and q . Of course the same statement is true for the CC induced reactions, so in both cases the total cross sections are built from the purely kinematical input and the response functions, calculated in the whole physical ω - q region. We show in Fig. 9.3 the ranges of these quantities for the NC neutrino inelastic scattering for the initial (anti)neutrino energy $E=100$ MeV.

What happens if the energy of the initial lepton is increased? The physical ω - q region for the higher energy is broader than for the smaller energy, but the values of the response functions calculated for the smaller energy can be also used for the calculation of the total cross section at the higher energy.

This idea was used in Ref. [58], where several thousands of Faddeev calculations were performed at various $(\omega$ - q) values on a sufficiently dense two-dimensional grid. The very time-consuming calculations of the response functions allowed later the authors to calculate, in a very short time, observables like the differential (with respect to the lepton arm) and total cross sections. Namely, response functions required for a specific initial lepton energy, lepton scattering angle and a final lepton energy were obtained by simple two-dimensional interpolations. This method was first carefully tested for the reactions on the deuteron, where the results of the direct calculations existed. Another advantage of storing response functions becomes clear for the NC induced processes, where *the same* response functions are used for the neutrino and antineutrino induced reactions.

Let us now discuss in detail the steps using the formulas in Eqs. (3.57) and

(3.61). We adopt a system of coordinates, where $\mathbf{q} \parallel \hat{\mathbf{z}}$ and $\hat{\mathbf{y}} = \frac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|}$, so

$$\begin{aligned} k'_x &= k_x = \frac{k k' \sin \theta'}{q}, \\ k'_y &= k_y = 0, \\ k_z &= \frac{k(k - k' \cos \theta')}{q}, \\ k'_z &= \frac{k'(-k' + k \cos \theta')}{q}, \\ q &= \sqrt{k^2 + k'^2 - 2kk' \cos \theta'}. \end{aligned} \quad (9.32)$$

Further, the part of the square of the transition matrix element introduced in Eq. (3.40), which contributes to the cross sections in the case, where all information from the nuclear sector is summed and only the final lepton momentum is known, is written as

$$\begin{aligned} |L_\lambda N^\lambda|^2 &= V_{00} |N^0|^2 + V_{MM} |N_{-1}|^2 + V_{PP} |N_{+1}|^2 \\ &+ V_{ZZ} |N_z|^2 + V_{Z0} \text{Re}(2 N_z (N^0)^*). \end{aligned} \quad (9.33)$$

For the neutrino induced reactions one obtains

$$\begin{aligned} V_{00} &= 8 (\mathbf{k}' \cdot \mathbf{k} + E E'), \\ V_{MM} &= 8 (E + k_z) (E' - k'_z), \\ V_{PP} &= 8 (E - k_z) (E' + k'_z), \\ V_{ZZ} &= 8 (-\mathbf{k}' \cdot \mathbf{k} + E E' + 2k_z k'_z), \\ V_{Z0} &= -8 (E k'_z + E' k_z) \end{aligned} \quad (9.34)$$

and for reactions with antineutrinos the corresponding $\bar{V}_{ij}$ functions are easily obtained from the previous set:

$$\begin{aligned} \bar{V}_{00} &= V_{00}, \\ \bar{V}_{MM} &= V_{PP}, \\ \bar{V}_{PP} &= V_{MM}, \\ \bar{V}_{ZZ} &= V_{ZZ}, \\ \bar{V}_{Z0} &= V_{Z0}. \end{aligned} \quad (9.35)$$

Taking into account the normalization of the nuclear states, one arrives at the final form of the cross section. For the CC neutrino induced reactions it reads

$$\begin{aligned} \frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'} &= \frac{G_F^2 \cos^2 \theta_C}{(2\pi)^2} F(Z, E') \frac{k'}{8E} \left(V_{00} R_{00,CC} + V_{MM} R_{MM,CC} + \right. \\ &\quad \left. V_{PP} R_{PP,CC} + V_{ZZ} R_{ZZ,CC} + V_{Z0} R_{Z0,CC} \right). \end{aligned} \quad (9.36)$$

The Fermi function $F(Z, E')$ [143] accounts for the change of the final lepton wave function by its Coulomb interaction with more than one proton in the final state and is not needed otherwise. For the NC driven processes the Fermi function $F(Z, E')$ and $\cos^2 \theta_C$ are absent in Eq. (9.36).

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The angular distributions of the cross sections, $d\sigma/d\theta'$, are easily obtained as

$$\frac{d\sigma}{d\theta'} = 2\pi \sin \theta' \int_{E'_{\min}}^{E'_{\max}} dE' \frac{d^3\sigma}{dE' d\hat{\mathbf{k}}'}, \quad (9.37)$$

where $(E')_{\min} = m_e(0)$ for the CC (NC) induced reactions and the factor 2π arises from the integration over the azimuthal angle ϕ . Finally, by the integration over the scattering angle θ' we calculate the total cross section

$$\sigma_{tot} = \int_0^\pi d\theta' \frac{d\sigma}{d\theta'}. \quad (9.38)$$

We restrict ourselves to presenting the response functions and the total cross sections, as these are the most important results from Ref. [58]. The 3N weak response functions are shown in Figs. 9.4–9.7. They are broader and extend to higher E_{3N} and q values than the corresponding 2N observables (not shown here). The differences between various response functions are not very big. The response functions for the CC electron antineutrino disintegration of ${}^3\text{He}$ and ${}^3\text{H}$ (Figs. 9.4–9.5) have similar shapes and depend on the number of protons in a nucleus. This is caused by the fact that the reactions described by the single nucleon charged current operators engage only protons.

In the case of the NC response functions both the proton and neutron contributions to the single nucleon NC operator are important. This leads to similar results for the ${}^3\text{He}$ and ${}^3\text{H}$ NC response functions displayed in Figs. 9.6–9.7.

The total cross sections for the selected (anti)neutrino reactions with the trinucleons are displayed in Fig. 9.8 for ${}^3\text{He}$ and in Fig. 9.9 for ${}^3\text{H}$. In the ${}^3\text{He}$ case the cross section for CC electron antineutrino disintegration takes the highest values in the whole considered energy range. The cross section values of NC neutrino disintegration are smaller but quite close to the predictions for the CC reaction. The cross section for NC electron antineutrino disintegration is approximately two times smaller than the other two cross sections.

The situation is different for ${}^3\text{H}$, where the cross section for NC electron neutrino disintegration is roughly two times bigger than the results for the two antineutrino induced reactions, which are close to each other for all the initial (anti)neutrino energies.

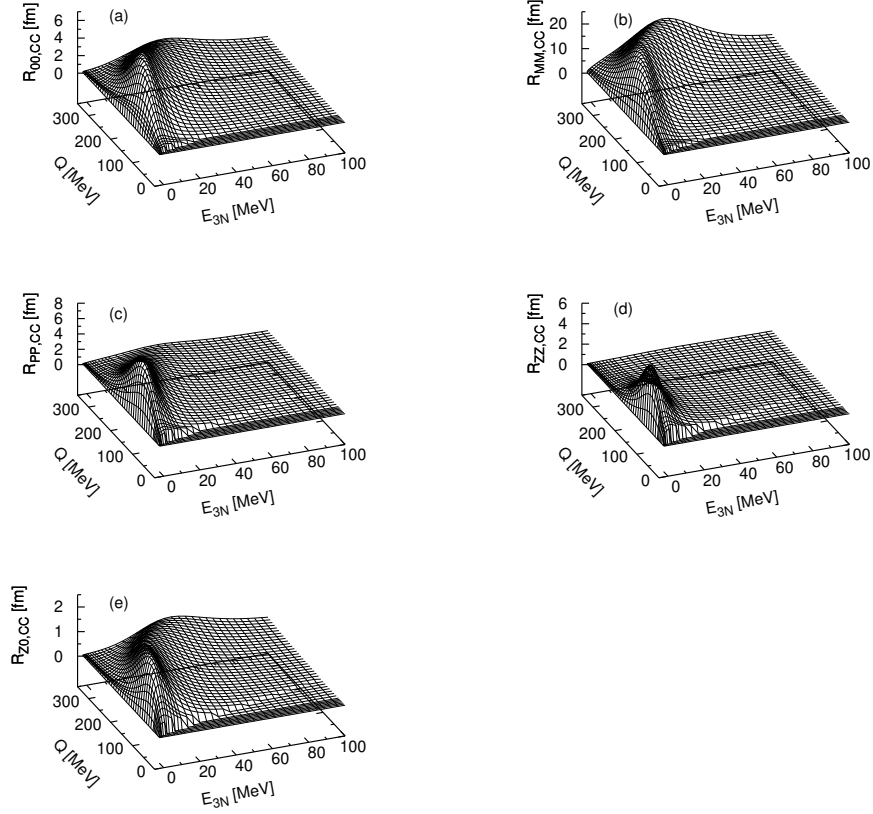


Figure 9.4: The total inclusive CC response functions $R_{00,CC}$ (a), $R_{MM,CC}$ (b), $R_{PP,CC}$ (c), $R_{ZZ,CC}$ (d) and $R_{Z0,CC}$ (e) for the CC electron antineutrino disintegration of ^3He as a function of the internal 3N energy E_{3N} and the magnitude of the three-momentum transfer Q . The results are obtained with the AV18 NN potential and the single nucleon CC operator, which contains the relativistic corrections.

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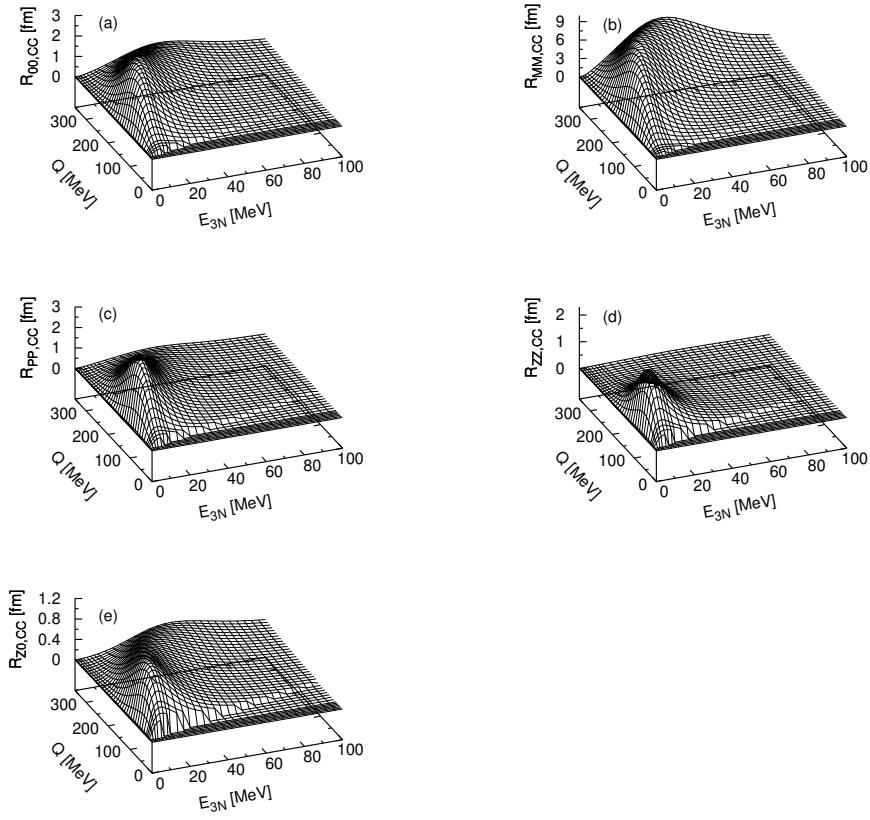


Figure 9.5: The same as in Fig. 9.4 but for the $\bar{\nu}_e + {}^3\text{H} \rightarrow e^+ + n + n + n$ reaction.

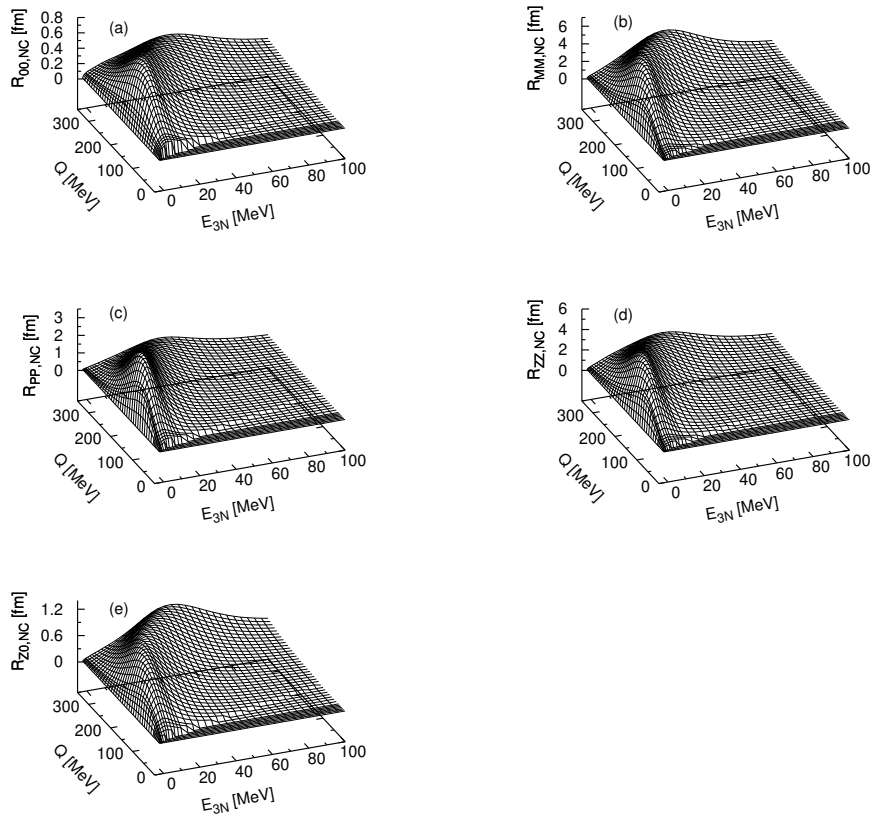


Figure 9.6: The same as in Figs. 9.4 and 9.5 for inclusive NC response functions for (anti)neutrino disintegration of ^3He .

9.2. SELECTED RESULTS FOR NEUTRINO SCATTERING ON ${}^3\text{He}$ AND ${}^3\text{H}$

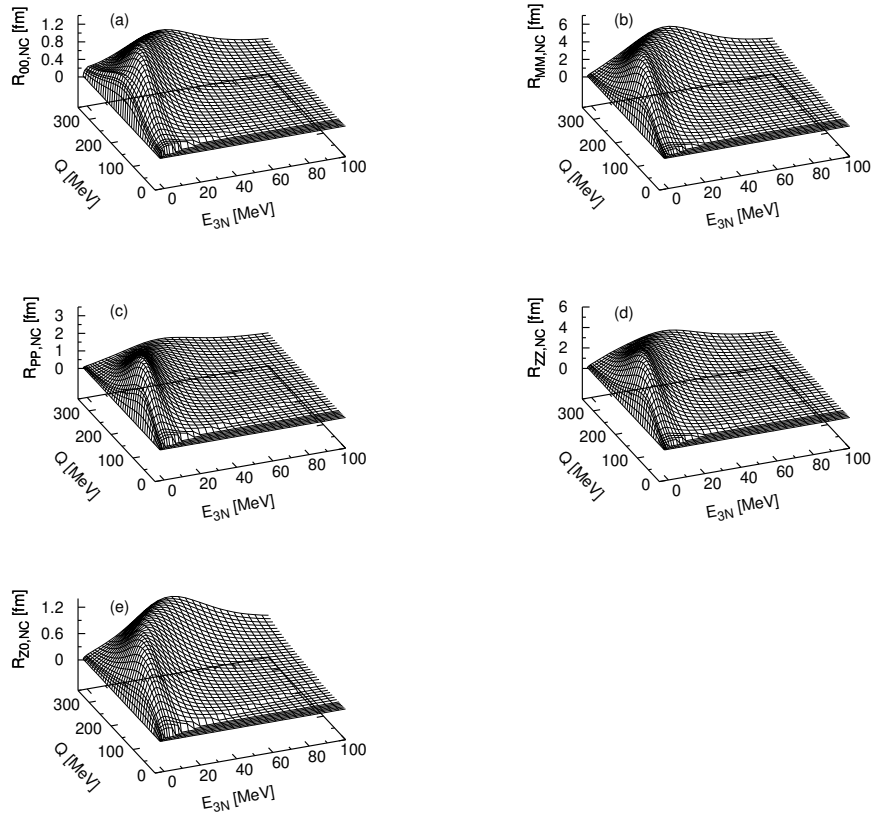


Figure 9.7: The same as in Fig. 9.6 for inclusive NC response functions for (anti)neutrino disintegration of ${}^3\text{H}$.

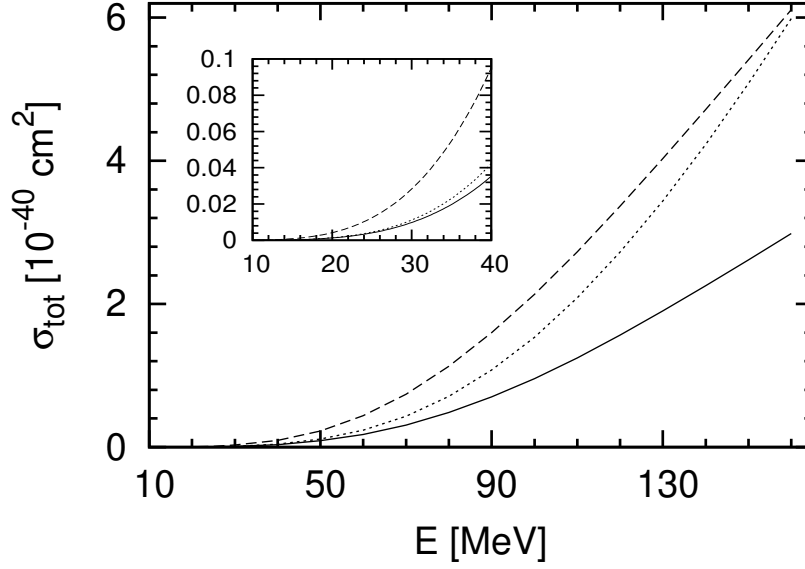


Figure 9.8: The total cross section σ_{tot} for three inclusive (anti)neutrino reactions with ^3He : CC electron antineutrino disintegration of ^3He (dashed line), NC electron antineutrino disintegration of ^3He (solid line), NC electron neutrino disintegration of ^3He (dotted line).

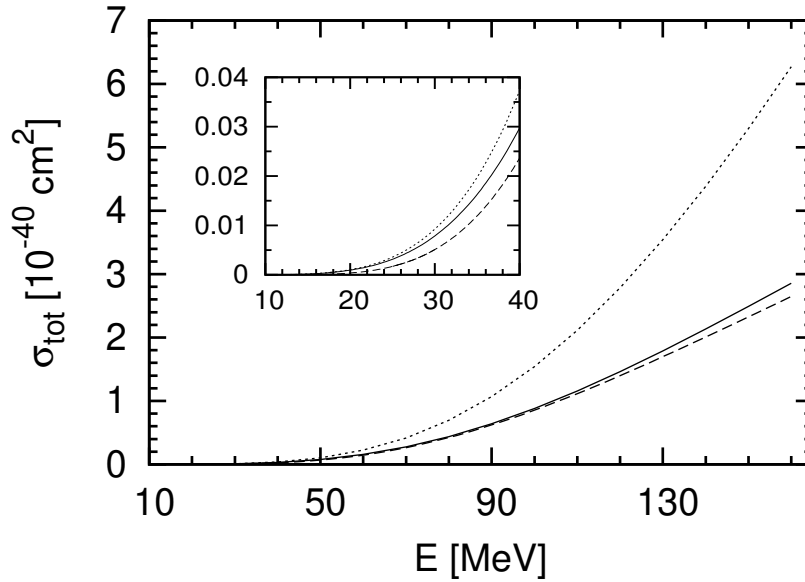


Figure 9.9: The same as in Fig. 9.8 for three inclusive (anti)neutrino reactions with ^3H .

Chapter 10

Summary

Neutrinos and electrons are standard probes of nuclear systems. The fundamental gauge theories describe their interactions with quarks, but nucleons and atomic nuclei interact with neutrinos and electrons in a much more complicated way. Thus, nuclei are usually modeled not in terms of sub-nucleon degrees of freedom but in terms of experimentally observable nucleons.

The deuteron is the smallest bound system of nucleons. Because of its relative simplicity it is an ideal system for investigations of strong interaction dynamics. One of the efficient ways to study the dynamics and structure of the deuteron is by scattering deuterons with photons, electrons or neutrinos. This is because the scattering reaction can be accurately approximated in the lowest non-trivial order in the electroweak interaction. In this approximation the scattering operator is linear in the matrix elements of the hadronic current operators. The hadronic current operators encode the density and motion of the strongly interacting charged constituent particles and comprise interaction-independent one-body operators and interaction-dependent many-body operators. In a scattering experiment involving electroweak probes the initial and final hadronic states are in different frames related by the momentum transferred to the target deuteron by the neutrino or electron. High resolution probes, sensitive to sub-nucleon degrees of freedom, must have high momentum transfers which require a relativistic description of the strong interaction dynamics.

The most detailed information about the hadronic current is contained in exclusive spin-dependent matrix elements of the current. Calculations using realistic interactions with controlled approximations are possible for spin-dependent elastic scattering and breakup reaction observables. The available phase space that can be explored is large and comparisons of detailed computations with experiment can put strong constraints on the model interactions, which have implications for larger nuclei.

The purpose of this thesis was to develop tools to perform consistent relativistic calculations of unpolarized cross sections and polarization observables in electroweak scattering experiments on deuteron targets. A relativistic framework can be constructed in many ways. Here, following the investigations of the three-nucleon system (see for example Refs. [65, 62]), the strong interaction dynamics is defined by an explicit dynamical unitary representation of the Poincaré group. The dynamical Poincaré generators are constructed using a relativistic re-interpretation of the AV18 nucleon-nucleon potential. The dy-

namical representation is chosen so that representations of space translations and rotations are identical in the interacting and non-interacting representations. The constructed framework is incomplete in two ways. First, a pion production channel is not included. The second limitation is the lack of the dynamical two-body contributions to the current operators.

Our results of numerical calculations for unpolarized cross sections and spin dependent observables are given in Chapter 4.4. The predictions are shown for elastic and breakup electron and neutrino induced reactions and, in some cases, compared to experimental data. For example, the description of the deuteron form factor $B(Q^2)$ and the deuteron tensor analyzing power T_{20} suggests the need for two-body contributions to the electromagnetic nuclear current operator. The current framework can be successfully applied to kinematics, where the internal two-nucleon energy remains below the pion production threshold, but the magnitude of the three-momentum transfer extends at least to several GeV. Here the final-state nucleon-nucleon interactions can be included exactly.

This notion is supported by fair agreement with the experimental data for deuteron electrodisintegration in the region of quasielastic peak is demonstrated, where the dynamics is governed predominantly by the single-nucleon current operator. Also, predictions for the total cross sections in the neutral-current and charged-current induced reactions are quite reliable, since the two-nucleon contributions in these reactions were found to be weak [13].

It is shown that the use of the relativistic kinematics is mandatory for the magnitudes of the three-momentum transfer comparable and higher than the nucleon mass. The results prove additionally that relativistic $(p/m)^2$ corrections to the nonrelativistic single-nucleon current are to be used with great caution. The frame dependence of the results for the total elastic neutrino-deuteron cross section points to a need for two-body parts in the nuclear weak neutral current operator.

We plan to improve the present framework by augmenting it with two-nucleon current contributions. The AV18 nucleon-nucleon potential can be easily replaced by the recently developed accurate chiral interaction [24]. We hope that the improved framework will be used to analyze experimental data.

Appendices

Appendix A

Analytical form of the single-nucleon tensor for neutrino-nucleon scattering

In the text we saw that it is possible to express the matrix element $L_{\mu\nu} \simeq \sum L_\mu^* L_\nu$ for the leptonic arm in a short and compact form by using trace identities. In this chapter we do the same but for the hadronic arm: we want to get to a simpler and more compact expression for

$$N^{\mu\nu} = \frac{1}{2} \sum (N^\mu)^* N^\nu, \quad (\text{A.1})$$

where

$$N^\mu = \bar{u}_{\mathbf{p}'} J^\mu u_{\mathbf{p}} \quad (\text{A.2})$$

and J^μ is the single-nucleon current operator. Let us report here the functional form in Eq. (2.4) for it, which reads

$$J^\mu = \left(\gamma^\mu F_1 + \frac{i}{2M} \sigma^{\mu\nu} q_\nu F_2 + \gamma^\mu \gamma^5 F_A + \frac{q^\mu}{M} \gamma^5 F_P \right). \quad (\text{A.3})$$

To evaluate the four-tensor $N^{\mu\nu}$ we also need to use the completeness relation reported in Eq. (46). We remind that the form factors F_i are in fact functions of $Q^2 = -(p - p')^\mu (p - p')_\mu$ and have different parametrizations for the NC and the CC induced reactions. We can also include EMR by choosing different parametrizations for F_1 and F_2 and by setting $F_A = F_P = 0$.

To study the impact of choosing different masses for the proton and the neutron in CC induced reactions, we assume the general case in which the mass of the final and initial hadron are different. We start with

$$M_{1,2} = M(1 \pm \alpha), \quad (\text{A.4})$$

where α is a dimensionless parameter, M_2 (M_1) is the initial (final) nucleon mass and M is the average nucleon mass. We find

$$\alpha = \frac{M_1 - M_2}{M_1 + M_2} \quad \Rightarrow \quad |\alpha| \simeq 6.9 \times 10^{-4}. \quad (\text{A.5})$$

We can rewrite the $N^{\mu\nu}$ tensor as

$$N^{\mu\nu} = \frac{1}{2} \sum_{i=\{1,2,A,P\}} \sum_{j=\{1,2,A,P\}} C_{i,j}^{\mu\nu} F_i F_j, \quad (\text{A.6})$$

where the $1/2$ accounts for the double counting of the form factors (for example $F_1 F_2$ is the same as $F_2 F_1$) and the coefficients $C_{i,j}^{\mu\nu} = C_{i,j}^{\mu\nu}(p, p')$ are

$$C_{1,1}^{\mu\nu} = 4 \left(p^\nu p'^\mu + p^\mu p'^\nu - g^{\mu\nu} (p \cdot p' - (1 - \alpha^2) M^2) \right), \quad (\text{A.7})$$

$$C_{1,2}^{\mu\nu} = 2 \left[\alpha (p^\mu p^\nu - p'^\mu p'^\nu) - q^\mu q^\nu - 2 g^{\mu\nu} (p \cdot p' - (1 - \alpha^2) M^2) \right], \quad (\text{A.8})$$

$$C_{1,A}^{\mu\nu} = 4 i \epsilon^{\mu\nu\rho\sigma} p_\rho p'_\sigma, \quad (\text{A.9})$$

$$C_{1,P}^{\mu\nu} = 0, \quad (\text{A.10})$$

$$\begin{aligned} C_{2,2}^{\mu\nu} = & \frac{1}{M^2} (p \cdot p') P^\mu P^\nu - 2 (p^\mu p^\nu + p'^\mu p'^\nu) - q^\mu q^\nu \\ & - 4 g^{\mu\nu} (p \cdot p' - (1 - \alpha^2) M^2) \\ & + 4 \alpha (p^\mu p^\nu - p'^\mu p'^\nu) - \alpha^2 P^\mu P^\nu, \end{aligned} \quad (\text{A.11})$$

$$C_{2,A}^{\mu\nu} = 4 i \epsilon^{\mu\nu\rho\sigma} p_\rho p'_\sigma, \quad (\text{A.12})$$

$$C_{2,P}^{\mu\nu} = 0, \quad (\text{A.13})$$

$$C_{A,A}^{\mu\nu} = 4 \left[p^\nu p'^\mu + p^\mu p'^\nu - g^{\mu\nu} ((1 - \alpha^2) M^2 + p \cdot p') \right], \quad (\text{A.14})$$

$$C_{A,P}^{\mu\nu} = 4 \left[\alpha (p^\mu p^\nu - p'^\mu p'^\nu) - q^\mu q^\nu \right], \quad (\text{A.15})$$

$$C_{P,P}^{\mu\nu} = \frac{4 q^\mu q^\nu (p \cdot p' - (1 - \alpha^2) M^2)}{M^2}, \quad (\text{A.16})$$

where $P^\mu = p^\mu + p'^\mu$ and $C_{i,j}^{\mu\nu} = C_{j,i}^{\mu\nu}$.

Since we checked that α is very small in all our calculations for the CC induced reactions, we will neglect it, thus having

$$C_{1,1}^{\mu\nu} = 4 \left(p^\nu p'^\mu + p^\mu p'^\nu - g^{\mu\nu} (p \cdot p' - M^2) \right), \quad (\text{A.17})$$

$$C_{1,2}^{\mu\nu} = -2 \left[q^\mu q^\nu + 2 g^{\mu\nu} (p \cdot p' - M^2) \right], \quad (\text{A.18})$$

$$C_{2,2}^{\mu\nu} = \frac{1}{M^2} (p \cdot p') P^\mu P^\nu - 2 (p^\mu p^\nu + p'^\mu p'^\nu) - q^\mu q^\nu - 4 g^{\mu\nu} (p \cdot p' - M^2), \quad (\text{A.19})$$

$$C_{A,A}^{\mu\nu} = 4 \left[p^\nu p'^\mu + p^\mu p'^\nu - g^{\mu\nu} (M^2 + p \cdot p') \right], \quad (\text{A.20})$$

$$C_{A,P}^{\mu\nu} = -4 q^\mu q^\nu, \quad (\text{A.21})$$

$$C_{P,P}^{\mu\nu} = \frac{4 q^\mu q^\nu (p \cdot p' - M^2)}{M^2}, \quad (\text{A.22})$$

while all other $C_{i,j}^{\mu\nu}$ are left unchanged.

Appendix B

Rotations in quantum mechanics

In quantum mechanics a rotation is represented by an unitary operator R which acts on the quantum state [2, 3]. If $\boldsymbol{\theta} = \theta \hat{\mathbf{n}}$ defines the rotation axis $\hat{\mathbf{n}}$ and the rotation angle θ , the rotation operator can be written as

$$R(\boldsymbol{\theta}) \equiv U(R) = e^{-i\mathbf{J}\cdot\boldsymbol{\theta}}, \quad (\text{B.1})$$

where $\mathbf{J}$ is the angular momentum operator, which is the generator of rotations. Any rotation can be expressed by the Euler angles (α, β, γ) , *i.e.* it can be written as three consecutive rotations along different axes. In this work we use the (z, y, z) axes convention; first the system is rotated by an angle θ around the z -axis, then by an angle β around the y -axis and finally by an angle α around the z -axis again. Each rotation is performed by keeping the rotation axes fixed. Thus, a rotation expressed with the Euler angles has the form

$$R(\alpha, \beta, \gamma) = e^{-iJ_z\alpha} e^{-iJ_y\beta} e^{-iJ_z\gamma}. \quad (\text{B.2})$$

Example For spin-1/2 particles the role of generators is fulfilled by the Pauli matrices $\boldsymbol{\sigma}$ and therefore the rotation can be represented by the matrix operator $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$.

We are interested in finding the eigenstates of the spin operator along a generic direction $\hat{\mathbf{n}} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$, expressed in the base of the eigenstates of the spin operator along the z -axis. Let us take as an example the state with projection $+1/2$ along $\hat{\mathbf{n}}$, which we call $|+\rangle_{\hat{\mathbf{n}}}$. We can construct this state directly from the eigensystem relation

$$\mathbf{S} \cdot \hat{\mathbf{n}} |s_z\rangle_{\hat{\mathbf{n}}} = \lambda |s_z\rangle_{\hat{\mathbf{n}}}. \quad (\text{B.3})$$

This is the same as finding the solution of the eigensystem of the operator

$$\mathbf{S} \cdot \hat{\mathbf{n}} = \frac{1}{2}\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}, \quad (\text{B.4})$$

which has solutions

$$|+\rangle_{\hat{\mathbf{n}}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad |-\rangle_{\hat{\mathbf{n}}} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi} \cos\frac{\theta}{2} \end{pmatrix}, \quad (\text{B.5})$$

with eigenvalues $\pm 1/2$ respectively. We see that for $\theta = \phi = 0$ we have the usual solutions for the problem when we choose z as the quantization axis, *i.e.*

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{B.6})$$

Let us now find this expression in a second way, by applying the rotation operator R to the state $|+\rangle = (1, 0)$, finally obtaining the solution (B.5). To do so, the needed Euler angles are $(\phi, \theta, 0)$. For spin- $1/2$ particles Eq. (B.1) can be written as

$$R(\boldsymbol{\theta}) = \cos \frac{\theta}{2} - i \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\theta}} \sin \frac{\theta}{2}. \quad (\text{B.7})$$

Using this equation we can write Eq. (B.2) as

$$R(\phi, \theta, 0) = e^{-i\sigma_3\phi/2} e^{-i\sigma_2\theta/2}, \quad (\text{B.8})$$

or

$$R(\phi, \theta, 0) = \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}. \quad (\text{B.9})$$

By applying this operator to the state $(1, 0)$ we obtain

$$R(\phi, \theta, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\phi}{2}} \cos \frac{\theta}{2} \\ e^{\frac{i\phi}{2}} \sin \frac{\theta}{2} \end{pmatrix} = e^{-\frac{i\phi}{2}} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}, \quad (\text{B.10})$$

which is, apart from the overall phase factor $e^{-\frac{i\phi}{2}}$, is what we found by directly solving the eigensystem for the operator $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$.

B.1 Wigner D -matrix

We can evaluate the spin rotation operator matrix elements in the eigenvector basis of the angular momentum and of its projection along the quantization axis. In order to do so, so-called Wigner D -matrix are used. Defined in Eq. (7.8), in the case of rotations defined by Euler angles, they get the form

$$D_{m'm}^j(\alpha, \beta, \gamma) = \langle jm' | R(\alpha, \beta, \gamma) | jm \rangle. \quad (\text{B.11})$$

Using Eq. (B.2) we get

$$D_{m'm}^j(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} d_{m'm}^j(\beta), \quad (\text{B.12})$$

where we defined the Wigner (small) d -matrix as

$$d_{m'm}^j(\beta) \equiv \langle jm' | e^{-iJ_y\beta} | jm \rangle = D_{m'm}^j(0, \beta, 0). \quad (\text{B.13})$$

The choice of the Euler angles with respect to z - y - z convention for rotations ensures the d -matrix to be real.

Using Eq. (B.11) we can express a rotated state as

$$R(\alpha, \beta, \gamma) | jm \rangle = \sum_{m'} | jm' \rangle \langle jm' | R(\alpha, \beta, \gamma) | jm \rangle = \sum_{m'} D_{m'm}^j(\alpha, \beta, \gamma) | jm' \rangle. \quad (\text{B.14})$$

The Wigner D -matrix has the properties

$$\sum_m D_{m'm}^{j*} [\alpha, \beta, \gamma] D_{m''m}^j [\alpha, \beta, \gamma] = \delta_{m'm''} , \quad (\text{B.15})$$

$$\sum_m D_{mm'}^{j*} [\alpha, \beta, \gamma] D_{mm''}^j [\alpha, \beta, \gamma] = \delta_{m'm''} , \quad (\text{B.16})$$

$$D_{mm'}^{j*} [\alpha, \beta, \gamma] = (-)^{m-m'} D_{-m, -m'}^j [\alpha, \beta, \gamma] . \quad (\text{B.17})$$

Wigner D -matrix for spin- $1/2$ particles We can easily evaluate the d -matrix for spin- $1/2$ particles. Using Eq. (B.7) we write

$$R_y(\beta) = \begin{pmatrix} \cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\ \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} , \quad (\text{B.18})$$

and therefore

$$d_{++}^{1/2}(\beta) = \cos \frac{\beta}{2} \quad d_{+-}^{1/2}(\beta) = -\sin \frac{\beta}{2} , \quad (\text{B.19})$$

$$d_{-+}^{1/2}(\beta) = \sin \frac{\beta}{2} \quad d_{--}^{1/2}(\beta) = \cos \frac{\beta}{2} , \quad (\text{B.20})$$

where $\pm$ stands for the plus or minus of the spin magnetic number $m = \pm 1/2$. An useful property for the d -matrix is

$$d_{m'm}^j(-\beta) = d_{mm'}^j(\beta) = (-)^{m'-m} d_{m'm}^j(\beta) . \quad (\text{B.21})$$

Appendix C

Relativistic examples

C.1 Lorentz transformations in quantum mechanics

Physics must be covariant with respect to Lorentz (and Poincaré) transformations, *i.e.* transformations leaving invariant the quantity

$$ds^2 = dx_\mu dx^\mu = dt^2 - d\mathbf{x}^2 \quad (\text{C.1})$$

called invariant interval, here dx^μ is the four-vector denoting the difference between the coordinates of two events. A generic Lorentz transformation $U(\Lambda)$ between two sets of coordinates is linear and homogeneous and thus can be expressed as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (\text{C.2})$$

where the Einstein summation convention is understood. $\Lambda^\mu{}_\nu$ is the form of the Lorentz boost in the four-vector representation. Eq. (C.1) implies that

$$\Lambda_\mu{}^\nu \Lambda^\mu{}_\sigma = \delta^\nu{}_\sigma \quad (\text{C.3})$$

The last two equations define a Lorentz transformation. For such transformations we have

$$\det \Lambda = \pm 1. \quad (\text{C.4})$$

Lorentz transformations for which $\det \Lambda = 1$ (-1) are called proper (improper) Lorentz transformations (PLT, (ILT)). PLT can be constructed by infinite infinitesimal transformations. PLT describe Lorentz boosts and rotations, while ILT are discrete and include space inversion and time inversions.

A general Lorentz boost is a Lorentz transformation in which rotations are not involved and can be written as

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta}^T \\ -\gamma\boldsymbol{\beta} & \mathbf{I}_3 + (\gamma - 1)\boldsymbol{\beta}\boldsymbol{\beta}^T/\beta^2 \end{pmatrix}, \quad (\text{C.5})$$

where $\mathbf{I}_3$ is the 3-dimensional identity operator,

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (\text{C.6})$$

is the so-called Lorentz factor and $\boldsymbol{\beta}$ is the boost velocity.

We can write the four-by-four boost operator in the four-vector representation as

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \quad (\text{C.7})$$

A rotation can be represented in the four-vector representation as

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}, \quad (\text{C.8})$$

where $\mathbf{R}$ is a three-dimensional rotation matrix, which can be expressed using the Euler angles. We talked about the Euler angles in Appendix B.

Example Let us write the transformation for the four-momentum of a particle from its rest frame to the frame in which the particle has momentum $p'^\mu = (\sqrt{M^2 + p^2}, p, 0, 0)$. The velocity of the particle in the second reference frame is

$$v = \frac{p}{\sqrt{M^2 + p^2}} \quad (\text{C.9})$$

Then the boost velocity $\boldsymbol{\beta} = (\beta, 0, 0)$ and the Lorentz factor γ we have to use in Lorentz boost operator $\Lambda^\mu{}_\nu$ are

$$\beta = -v = -\frac{p}{\sqrt{M^2 + p^2}} \quad \text{and} \quad \gamma = \frac{\sqrt{M^2 + p^2}}{M}, \quad (\text{C.10})$$

and from Eq. (C.5) we get

$$\Lambda = \begin{pmatrix} \frac{\sqrt{M^2 + p^2}}{M} & \frac{p}{M} & 0 & 0 \\ \frac{p}{M} & \frac{\sqrt{M^2 + p^2}}{M} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{C.11})$$

The equivalent formula for a generic Lorentz boost that boosts the rest frame of a particle of mass M to the reference frame where the particle has momentum $\mathbf{p}$ is

$$\Lambda = \begin{pmatrix} \frac{\sqrt{M^2 + p^2}}{M} & \frac{\mathbf{p}^\text{T}}{M} \\ \frac{\mathbf{p}}{M} & \mathbf{I}_3 + \frac{\mathbf{p}\mathbf{p}^\text{T}}{M(M + \sqrt{M^2 + p^2})} \end{pmatrix}. \quad (\text{C.12})$$

C.1.1 Covariance of the Dirac equation

We start from the Dirac equation

$$(\not{p} - m)\Psi(x) = 0 \quad (\text{C.13})$$

in a given reference frame. The covariance condition is that an observer in a second reference frame must be able, given $\Psi(x)$, to compute the corresponding wave function $\Psi'(x')$ in his own reference frame and $\Psi'(x')$ must obey

$$(\not{p}' - m)\Psi'(x') = 0, \quad (\text{C.14})$$

where $\not{p}' = \tilde{\gamma}_\mu p'^\mu$ and $p'^\mu = \Lambda^\mu{}_\nu p^\nu$. It is possible to prove that the new Dirac gamma matrices $\tilde{\gamma}_\mu$ are related to the old ones by a unitary transformation. We can therefore drop the distinction between the two and write $\not{p}' = \gamma_\mu p'^\mu$. We require the transformation from Ψ to Ψ' to be linear, since both the Dirac equation and the Lorentz transformation are linear. We write it as

$$\Psi'(x') = U(\Lambda)\Psi(x). \quad (\text{C.15})$$

$U(\Lambda)$ has the properties

$$U(\Lambda^{-1}) = U^{-1}(\Lambda) \quad (\text{C.16})$$

and

$$\Lambda^\nu{}_\mu \gamma^\mu = U^{-1}(\Lambda) \gamma^\nu U(\Lambda). \quad (\text{C.17})$$

For pure rotations with rotation angle $\boldsymbol{\theta}$ the operator $U(\Lambda)$ becomes

$$U(R) \equiv U(\Lambda) = A(\boldsymbol{\theta}) = \begin{pmatrix} R(\boldsymbol{\theta}) & 0 \\ 0 & R(\boldsymbol{\theta}) \end{pmatrix} \quad (\text{C.18})$$

where $R(\boldsymbol{\theta})$ is the rotation operator in the spinor representation. Here $R(\boldsymbol{\theta})$ is

$$R(\boldsymbol{\theta}) = e^{-i\mathbf{J}\cdot\boldsymbol{\theta}}, \quad (\text{C.19})$$

where $\mathbf{J} = 1/2\boldsymbol{\sigma}$ is the angular momentum operator.

For pure boosts with velocity $\boldsymbol{\beta}$ the operator $U(\Lambda)$ becomes (Eq. (3.7) from Ref. [1])

$$B(\boldsymbol{\beta}) \equiv U(\Lambda) = \sqrt{\frac{\gamma+1}{2}} - \sqrt{\frac{\gamma-1}{2}} \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\alpha}, \quad (\text{C.20})$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad (\text{C.21})$$

and $\gamma = \gamma(\boldsymbol{\beta})$ is the same as in Eq. (C.6).

Lorentz transformations do not exhaust all the type of transformations that leave the physics invariant. We want the physics to be invariant to translation as well. In this case we label the transformation $U(\Lambda, a)$, where $a = a^\mu$ is the four-vector defining the translation. The Lorentz group is therefore expanded and it is now called the Poincaré group.

C.2 Wigner rotations

Wigner rotations are the result of multiple noncollinear Lorentz boosts, whose result can be represented as a boost and a rotation. The rotation is called Thomas rotation, Thomas-Wigner rotation or simply Wigner-rotation. By combining two noncollinear Lorentz boosts and a third boost which returns to the original reference frame, the resulting transformation is a pure rotation, called Thomas precession.

C.2.1 Thomas precession for a massive particle

Let us take a particle at rest with mass M , the four-momentum in its rest frame it is simple $p^\mu = (M, \mathbf{0})$. After a boost to the reference frame in which the particle has a momentum $\mathbf{p}$ we have $p'^\mu = (\sqrt{M^2 + p^2}, \mathbf{p})$. The boost velocity in this case is $\beta' = -\mathbf{p}/\sqrt{M^2 + p^2}$. We perform another boost in a direction which is noncollinear to β' , let us call the new boost velocity β'' . The resulting four-momentum will be $p''^\mu = (\sqrt{M^2 + p''^2}, \mathbf{p}'')$. Finally we perform a boost with velocity $\beta''' = \mathbf{p}''/\sqrt{M^2 + p''^2}$, returning to the particle rest frame. We see that the composition of the Lorentz transformation matrices is not the identity matrix but rather a rotation in the space components, *i.e.*

$$\Lambda^\mu{}_\nu = \Lambda^\mu{}_\alpha(\beta''')\Lambda^\alpha{}_\beta(\beta'')\Lambda^\beta{}_\nu(\beta') = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{pmatrix}. \quad (\text{C.22})$$

Numerical example Let us take $M = 5 \text{ fm}^{-1}$, $\mathbf{p} = (0.5, 0.2, -0.3) \text{ fm}^{-1}$ and $\beta'' = (0.5, 0.2, 0.8)$. We first find the boost velocity β' needed to boost the particle from its rest frame to the reference frame where it has momentum $\mathbf{p}$. This velocity is

$$\beta' = (-0.0992486, -0.0396994, 0.0595491). \quad (\text{C.23})$$

Then we boost again along β'' . The particle now has a four-momentum

$$p''^\mu = (18.8523, -8.94593, -3.57837, -15.4135) \text{ fm}^{-1}. \quad (\text{C.24})$$

Finally we boost back to the particle rest frame. The boost velocity is found to be

$$\beta''' = (-0.474527, -0.189811, -0.817591). \quad (\text{C.25})$$

By combining the Lorentz boosts we find the four-by-four matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.996224 & -0.0015105 & 0.0868098 \\ 0 & -0.0015105 & 0.999396 & 0.0347239 \\ 0 & -0.0868098 & -0.0347239 & 0.99562 \end{pmatrix}, \quad (\text{C.26})$$

with the space-like rotation matrix being

$$R_{ij} = \begin{pmatrix} 0.996224 & -0.0015105 & 0.0868098 \\ -0.0015105 & 0.999396 & 0.0347239 \\ -0.0868098 & -0.0347239 & 0.99562 \end{pmatrix}. \quad (\text{C.27})$$

We can confront it with a generic rotation using the Euler angles (α, β, γ) , which is

$$R = \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\sin \alpha \cos \gamma - \cos \alpha \cos \beta \sin \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix} \quad (\text{C.28})$$

One can recover the Euler angles with simple combinations of the elements of R . Obviously if $\beta = 0$ (or $R_{33} = 1$) the rotation can be seen as a single rotation around the z -axis of an angle $\alpha + \gamma$. Otherwise we can see that

$$\frac{R_{11} + R_{22}}{R_{33} + 1} = \cos(\alpha + \gamma), \quad \frac{R_{21} - R_{12}}{R_{33} + 1} = \sin(\alpha + \gamma), \quad (\text{C.29})$$

$$\frac{R_{11} - R_{22}}{R_{33} - 1} = \cos(\alpha - \gamma), \quad \frac{R_{21} + R_{12}}{R_{33} - 1} = \sin(\alpha - \gamma), \quad (\text{C.30})$$

$$R_{33} = \cos \beta, \quad \frac{R_{13}}{\cos \alpha} = \sin \beta. \quad (\text{C.31})$$

In the example we find

$$\alpha = -\gamma = 21.8014^\circ \quad \text{and} \quad \beta = 5.36482^\circ. \quad (\text{C.32})$$

These angles can be then used to evaluate the precession of the spin of the particle, by using the rotation operators and the Wigner D -matrix seen above. The D -matrix found in this example is

$$D^{1/2} = \begin{pmatrix} 0.998904 & -0.0434525 + 0.017381 i \\ 0.0434525 + 0.017381 i & 0.998904 \end{pmatrix}. \quad (\text{C.33})$$

C.2.2 Example for the single-particle representation

We want now to see how single-particle states transform and check that it is consistent with what we have found so far. We take the same numerical example as in Chapter C.2.

Our convention for the Dirac bispinor of a particle moving with momentum $\mathbf{p}$ is

$$u(\mathbf{p}, \mu) = \sqrt{\frac{\sqrt{m^2 + p^2} + m}{2m}} \begin{pmatrix} 1 \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{m^2 + p^2} + m} \end{pmatrix} |\mu\rangle_{\hat{\mathbf{n}}}, \quad (\text{C.34})$$

where

$$\hat{\mathbf{n}} \langle \mu' | \mu \rangle_{\hat{\mathbf{n}}} = \delta_{\mu' \mu} \quad \text{and} \quad u(\mathbf{p}, \mu) \equiv |(1/2, m) \mathbf{p}, \mu\rangle. \quad (\text{C.35})$$

Note that this is the same representation we defined in Chapter 7. With this choice

$$\bar{u}u = u^\dagger \gamma^0 u = 1. \quad (\text{C.36})$$

We start our calculation with a particle with $\mu = 1/2$ and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $|1/2\rangle_{\hat{\mathbf{z}}} = (1, 0)$. Using the numerical values in the previous example we have

$$u_0 = u(\mathbf{0}, 1/2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{C.37})$$

Using Eq. (C.20) we boost to the reference frame where the particle has momentum $\mathbf{p}$ obtaining

$$u' \equiv |(1/2, m) \mathbf{p}, \mu\rangle = \sqrt{\frac{m}{\sqrt{m^2 + p^2}}} U \left(B \left(\frac{\mathbf{p}}{m} \right) \right) u_0 = \begin{pmatrix} 0.99812 \\ 0 \\ -0.0298307 \\ 0.0497178 + 0.0198871 i \end{pmatrix}. \quad (\text{C.38})$$

Boosting again with velocity β'' we get

$$u'' = \sqrt{\frac{\sqrt{m^2 + p^2}}{\sqrt{m^2 + p'^2}}} U \left(B(\beta'') \right) u' = \begin{pmatrix} 0.794496 \\ 0.0345607 + 0.0138243 i \\ -0.528443 \\ -0.275646 - 0.110259 i \end{pmatrix}, \quad (\text{C.39})$$

with $p' = \Lambda p = U(\beta'')p$. Finally, going back to the rest frame of the particle the bispinor becomes

$$u''' = \sqrt{\frac{\sqrt{m^2 + p'^2}}{m}} U^{-1} \left(B \left(\frac{\mathbf{p}'}{m} \right) \right) u'' = \begin{pmatrix} 0.998904 \\ 0.0434525 + 0.017381 i \\ 0 \\ 0 \end{pmatrix}, \quad (\text{C.40})$$

As we can see, this equivalent to rotating the spinor $|1/2\rangle_{\hat{\mathbf{z}}}$, in fact if we evaluate

$$U(B_{TOT}) = U \left(B \left(\frac{\mathbf{p}'}{m} \right) B(\beta'') B \left(\frac{\mathbf{p}}{m} \right) \right) = \begin{pmatrix} D^{1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix}, \quad (\text{C.41})$$

we see that

$$D^{1/2} = \begin{pmatrix} 0.998904 & -0.0434525 + 0.017381 i \\ 0.0434525 + 0.017381 i & 0.998904 \end{pmatrix} \quad (\text{C.42})$$

is exactly the D -matrix found in the example in Chapter C.2. This is what happens if we apply directly the boost operators $B(\beta)$, but it is the same as applying a rotation with the previously found Euler angles. u'' could be obtained from u_0 with that rotation.

Appendix D

Representation of the Wigner rotations by $SL(2, \mathbb{C})$ matrices

It has been shown in Ref. [65] that it is possible to use $SL(2, \mathbb{C})$ matrices to work with Wigner rotations and Lorentz transformations. We have shown the standard approach to evaluate the D -matrices in Appendix B, where we computed the Euler angles by using a set of two noncollinear Lorentz boost and another boost to switch back to the rest frame of the particle, thus obtaining the four-vector matrix representation of the transformation. From the matrix then we extracted the Euler angles and used them to rotate the spin state by means of the D -matrix.

Another way to obtain the same result is by the use of $SL(2, \mathbb{C})$ matrices, which allows us to avoid the Euler angles representations and find the Wigner rotations D -matrix without knowing the Euler angles.

Let us take a generic four-vector p^μ , we can construct the quantity

$$P = p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \quad (\text{D.1})$$

where $\sigma_\mu = (\sigma_0, \boldsymbol{\sigma})$ and σ_0 is the 2-dimensional identity matrix in spin space. It is possible to extract the components of p^μ by using

$$p^\mu = \frac{1}{2} \text{Tr}(\sigma_\mu P) = \frac{1}{2} \text{Tr}(P \sigma_\mu) \quad (\text{D.2})$$

We can prove that

$$\det(P) = m^2 \quad (\text{D.3})$$

and that

$$P^\dagger = P \quad (\text{D.4})$$

for any real p^μ . Because of this any transformation which is linear and which preserves the Hermiticity and determinant of P is a real Lorentz transformation. If A is a complex 2×2 matrix with $\det(A) = 1$, then the transformation

$$P \rightarrow P' = APA^\dagger \quad (\text{D.5})$$

preserves the aforementioned properties for P . We can write

$$A = e^{1/2 i \boldsymbol{\sigma} \cdot \mathbf{z}}, \quad (\text{D.6})$$

where

$$\mathbf{z} = \boldsymbol{\theta} - i \boldsymbol{\rho}. \quad (\text{D.7})$$

If $\boldsymbol{\rho} = \mathbf{0}$ then the transformation $A = U(\boldsymbol{\theta})$ represents a rotation by an angle θ around the axis $\hat{\boldsymbol{\theta}}$ in the SU(2) space. Such rotation can be expressed by the operator

$$U(\boldsymbol{\theta}) = \sigma_0 \cos \frac{\theta}{2} + i \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\theta}} \sin \frac{\theta}{2}, \quad (\text{D.8})$$

which is the same as Eq. (B.7) but with a difference in sign (active vs passive rotations).

If $\boldsymbol{\theta} = \mathbf{0}$ then $A = B(\mathbf{q})$ represents a pure Lorentz boost with velocity $\mathbf{q}$ and rapidity $\rho = \text{arctanh } q$ in the direction $\hat{\mathbf{q}}$. A particle at rest with mass m after the boost will acquire a momentum $\mathbf{p} = m\mathbf{q}$. $B(\mathbf{q})$ can be written as

$$B(\mathbf{q}) = \sigma_0 \cosh \frac{\rho}{2} + \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \sinh \frac{\rho}{2}. \quad (\text{D.9})$$

It is possible to see that the four-momentum p^μ from the rest frame to the frame where the particle has momentum $\mathbf{p}$ is correctly transformed using the transformation for P in Eq. (D.5), *i.e.*

$$P' = B P B^\dagger = B(\mathbf{q}) \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} B(\mathbf{q})^\dagger = \begin{pmatrix} p^0 + p^3 & p^1 - i p^2 \\ p^1 + i p^2 & p^0 - p^3 \end{pmatrix} \quad (\text{D.10})$$

where $p^0 = \sqrt{m^2 + p^2}$.

In Ref. [65] it has been shown the connection between the A operator and the Wigner D -matrix

$$D_{\mu\nu}^j(A) = \sum_{\alpha=0}^{2j} \frac{[(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!]^{1/2}}{(j+\mu-\alpha)!\alpha!(\alpha-\mu+\nu)!(j-\nu-\alpha)!} \times A_{11}^{j+\mu-\alpha} A_{12}^\alpha A_{21}^{\alpha-\mu+\nu} A_{22}^{j-\nu-\alpha} \quad (\text{D.11})$$

and, for spin-1/2 particles we have that $D_{ij}^{1/2} = A_{ij}$, where i and j take the values $\pm 1/2$ and $A = A_1 A_2 A_3$ is the operator matrix of the consecutive transformations.

Example Again, as an example let us take the values from Chapter C.2. A particle with spin 1/2, mass $M = 5 \text{ fm}^{-1}$. We saw that the three boost velocities were

$$\boldsymbol{\beta}' = (-0.0992486, -0.0396994, 0.0595491), \quad (\text{D.12})$$

$$\boldsymbol{\beta}'' = (0.5, 0.2, 0.8), \quad (\text{D.13})$$

$$\boldsymbol{\beta}''' = (-0.474527, -0.189811, -0.817591). \quad (\text{D.14})$$

Using these velocities in Eq. (D.9) we get

$$A_1 = B' = \begin{pmatrix} 1.03183 & -0.0499056 + 0.0199623 i \\ -0.0499056 - 0.0199623 i & 0.971948 \end{pmatrix}, \quad (\text{D.15})$$

$$A_2 = B'' = \begin{pmatrix} 2.52388 & 0.611235 - 0.244494 i \\ 0.611235 + 0.244494 i & 0.567929 \end{pmatrix} \quad (\text{D.16})$$

and

$$A_3 = B''' = \begin{pmatrix} 0.546408 & -0.579242 + 0.231697 i \\ -0.579242 - 0.231697 i & 2.54243 \end{pmatrix}. \quad (\text{D.17})$$

If we apply the transformations sequentially we find

$$D^{1/2} = A_3 A_2 A_1 = \begin{pmatrix} 0.998904 & -0.0434525 + 0.017381 i \\ 0.0434525 + 0.017381 i & 0.998904 \end{pmatrix}, \quad (\text{D.18})$$

which is exactly what we found using other methods in the previous sections.

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