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Symmetry and Classification of Multipartite
Entangled States

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*Separable states are all alike; every entangled state is
entangled in its own way.*

— Lew Tolstoy, Anna Karenina (Paraphrasis)

I am immensely grateful to all people who have been in my life for the last three years. First of all, to my supervisor, Karol Życzkowski, who introduced me to the world of quantum information, which I liked so much. Thank you for the scientific problems you presented to me, for the advice you gave me, and for the time you devoted to me.

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Abstract

One of the key manifestations of quantum mechanics is the phenomenon of quantum entanglement. While the entanglement of bipartite systems is already well understood, our knowledge of entanglement in multipartite systems is still limited. This dissertation covers various aspects of the quantification of entanglement in multipartite states and the role of symmetry in such systems. Firstly, we establish a connection between the classification of multipartite entanglement and knot theory and investigate the family of states that are resistant to particle loss. Furthermore, we construct several examples of such states using the Majorana representation as well as some combinatorial methods. Secondly, we introduce classes of highly-symmetric but not fully-symmetric states and investigate their entanglement properties. Thirdly, we study the well-established class of Absolutely Maximally Entangled (AME) quantum states. On one hand, we provide construction of new states belonging to this family, for instance an AME state of 4 subsystems with six levels each, on the other, we tackle the problem of equivalence of such states. Finally, we present a novel approach for the general problem of verification of the equivalence between any pair of arbitrary quantum states based on a single polynomial entanglement measure.

Streszczenie

Zjawisko splątania kwantowego jest jednym z kluczowych przejawów mechaniki kwantowej. Jakkolwiek splątanie kwantowe dwucząstkowych systemów jest dobrze poznanym zagadnieniem, nasze zrozumienie splątania w systemach wielocząściowych jest nadal niezadowalające. Niniejsza rozprawa dotyczy różnych aspektów kwantyfikacji splątania w stanach wielocząstkowych oraz roli symetrii w takich układach. Po pierwsze, prezentujemy związek między klasyfikacją wielocząstkowego splątania a teorią węzłów i badamy rodzinę stanów, w których splątanie jest odporne na utratę poszczególnych cząstek. Konstruujemy takie stany wykorzystując reprezentację Majorany, a także niektóre metody kombinatoryczne. Po drugie, wprowadzamy klasy stanów wysoce symetrycznych, ale nie w pełni symetrycznych i badamy własności ich splątania. Po trzecie, badamy szeroko dyskutowaną klasę stanów AME (ang. *Absolutely Maximally Entangled*). Z jednej strony pokazujemy konstrukcję nowych stanów należących do tej rodziny, ze stanem AME dla czterech podukładów, każdy z sześcioma poziomami, z drugiej zaś zajmujemy się problemem równoważności takich stanów. Na koniec przedstawiamy nowatorskie podejście do ogólnego problemu weryfikacji równoważności dwóch dowolnych stanów kwantowych w oparciu o pojedynczą miarę splątania.

This PhD Dissertation is based on the following publications and preprints available online

- [A] G. M. Quinta, R. André, **A. Burchardt**, and K. Życzkowski *Cut-resistant links and multipartite entanglement resistant to particle loss*, Phys. Rev. A 100, 062329 (2019).
- [B] **A. Burchardt**, Z. Raissi, *Stochastic local operations with classical communication of absolutely maximally entangled states*, Phys. Rev. A 102, 022413 (2020).
- [C] **A. Burchardt**, J. Czartowski, and K. Życzkowski, *Entanglement in highly symmetric multipartite quantum states*, Phys. Rev. A 104, 022426 (2021).
- [D] S. Rather*, **A. Burchardt***, W. Bruzda, G. Rajchel-Mieldzioć, A. Lakshminarayan, K. Życzkowski, *Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem*, ArXiv: 2104.05122 (2021).
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- [E] **A. Burchardt**, G. M. Quinta, R. André, *Entanglement Classification via Single Entanglement Measure*, ArXiv: 2106.00850 (2021).

Although publications and preprints mentioned above do not belong to the main body of this thesis, we attach them as appendices for the convenience of the reader.

List of Abbreviations

AME	Absolutely Maximally Entangled
LM	Local Monomial
LU	Local Unitary
MOLS	Mutually Orthogonal Latin Square
OA	Orthogonal Array
OLS	Orthogonal Latin Square
QECC	Quantum Error Correction Code
SL	Special linear
SLIP	SL-symmetric Invariant Polynomial
SLOCC	Stochastic Local Operations with Classical Communication

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Chapter 1

Introduction

In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen studied some counterintuitive predictions of quantum mechanics about strongly correlated systems, known today as Einstein–Podolsky–Rosen paradox (EPR paradox) [EPR35]. The EPR paradox aroused great interest among physicists. One of the concerned was Erwin Schrödinger, who in a later letter to Albert Einstein coined the word *entanglement* to describe this phenomenon. The phenomenon of quantum entanglement occurs when a collection of distinct particles interact in such a way that the resulting quantum state cannot be described independently for each particle, exactly as it is observed in the EPR paradox. The first experiment that verified entanglement was successfully corroborated by Chien-Shiung Wu and Irving Shakhnov in 1949 [WS50]. This result specifically proved the quantum correlations of a pair of photons. Since then, quantum entanglement has been demonstrated experimentally in many other systems, as with photons [KC67], neutrino,[FKMW16], electrons [HBD⁺15], large molecules as buckyballs [ANVA⁺99, NAZ03], and even small diamonds [LSS⁺11]. The EPR paradox and notion of entanglement became a milestone for the development of quantum mechanics and our understanding of the world. Moreover, quantum entanglement has become the heart of a completely new and dynamically developing field of science lying in the intersection of Quantum Physics and Information Theory: Quantum Information Theory.

Qubits. A central notion in Quantum Information Theory is *qubit*, an abstract term, distilled from various concrete physical realizations. Arguably, the simplest quantum-mechanical system is a two-level (or two-state) system. Such a system might be physically realized as the spin of the electron, which

in a given reference frame is either up or down, or as the polarization of a single photon, with distinguished vertical and horizontal polarization. In an abstract way, a pure qubit state is a coherent superposition of the aforementioned distinguished basis states $|0\rangle, |1\rangle$, i.e. a linear combination

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (1.1)$$

Such an object is nowadays referred as *qubit* - quantum binary unit [Sch95], a quantum analogue of a classical bit. In a classical system, a bit is always in precisely one of two states, either 0 or 1. However, quantum mechanics allows the qubit to be in a coherent superposition of both states simultaneously, a property that is fundamental to quantum mechanics and quantum computing.

Two-qubit system. The most common variant of the EPR paradox (formulated by Bohm [Boh51, BA57, RDB⁺09]) was expressed in terms of the quantum mechanical formulation of spin. In the more recent notations, it might be explained as a particular state of a two-qubit system, i.e. a linear combination of four vectors $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ representing directions of both spins. So-called EPR state might be thus written as

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (1.2)$$

Such state is strongly correlated in any reference frame, in a way that classical correlations do not allow.

While studying quantum entanglement, it is useful to abstract from certain local properties of a state which do not affect global entanglement and quantum correlations. An example of this type of abstraction is a Local Unitary (LU) equivalence of two given states. Having said that, it is known that any two-qubit state is LU-equivalent to the following system

$$|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle, \quad (1.3)$$

for some parameter $p \in [0, \frac{1}{2}]$. In particular, for $p = 0$ the state is separable, while for $p = \frac{1}{2}$ the state is maximally entangled, and coincides with the famous EPR state. In such a way, we obtained a satisfactory quantification of entanglement in two-qubit systems. Indeed, the exact amount of entanglement is measured by two related values $(p, 1-p)$, which are also known as Schmidt coefficients [HW08]. The closer the values p and $1-p$ are to each other, the more entangled the state is.

Furthermore, the number of non-vanishing values among $p, 1-p$ is known as Schmidt rank, which is equal to 1 for separable states and equal to 2 for entangled states. This results in the coarse-grained classification of entanglement in two-qubit systems. Indeed, with respect to the Schmidt rank two classes of states are distinguished: separable and entangled. This straightforward division into two classes of bipartite states coincides with the division under another class of local operations: Stochastic Local Operations with Classical Communications (SLOCC) [DVC00]. SLOCC operations include not only local unitary rotations but also additions of ancillas (i.e. enlarging the Hilbert space), measurements, and throwing away parts of the system, each performed locally on a given subsystem [DVC00, BPR⁺00]. As it was shown, mathematically SLOCC operations are represented via *invertible* operators [DVC00]. SLOCC operators cannot generate entanglement between subsystems, however, they might enhance or strengthen the existing entanglement with some non-vanishing probability of success. This is reflected in the fact that there are only two states which are not equivalent with respect to SLOCC transformations: separable $|00\rangle$ state and entangled $|EPR\rangle$ state.

An important feature of both types of local operations: LU and SLOCC are that each of them provides an equivalence relation on the state-space, and hence divides it into equivalence classes [Kra10a]. Any two states from one class are interconvertible by an adequate local operator, while such a transformation cannot be provided for states from different classes. In that way, quantum states which exhibit the same (for LU) or similar (for SLOCC) entanglement properties are grouped together, which gives the solution for the problem of *entanglement classification* of two-qubit systems.

As we already discussed, the problem of entanglement classification of two-qubit states is well-understood. Indeed, there are infinitely many LU-equivalence classes, indexed by one real parameter $p \in [0, \frac{1}{2}]$, see Eq. (1.3). On the other hand, there is only one SLOCC-equivalence class of genuinely entangled states, which might be represented by the EPR state.

In 1997 Scott Hill and William K. Wootters introduced the notion of *Concurrence* C , an entanglement measure, which for any two-qubit pure state $|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ reads

$$C(|\psi\rangle) := 2|c_{00}c_{11} - c_{01}c_{10}| \in [0, 1] \quad (1.4)$$

as an absolute value of the degree two polynomial in the state-coefficients [HW97]. As it was observed, the value of concurrence is not only independent of LU operation performed on any qubit, but also on any local Special

Linear operation (SL), i.e. local invertible matrices with determinant one. Any invertible operation might be presented as SL operation up to the global constant. In that way, local SL operations are relevant to SLOCC operations on qubits. The notion of concurrence was an important step towards a modern approach to the problem of entanglement quantification via *SL-polynomial invariant* (SLIP), measures. SLIP measures are at the same time polynomials in the state coefficients and invariant under SLOCC operations [ES14], which significantly facilitates the problem of quantifying entanglement resources.

Three-qubit system. In the late 1980s Daniel Greenberger, Michael Horne and Anton Zeilinger studied three-qubit entangled state the tripartite generalisation of the EPR pair [GHZ07]:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Bouwmeester et al. firstly presented the experimental observation of the state above as a polarization entanglement for three spatially separated photons [BPD⁺99]. Since then, many other experimental realisations of GHZ state were performed. From the theoretical point of view, there are two important features of GHZ state. On one hand it is *maximally entangled*, i.e each of its reductions to the one-particle subsystem is maximally mixed, which is often a desired property. On the other hand, however, state GHZ is not robust against particle loss. Indeed, all its reductions to the two-particle subsystem form an unentangled mixed state, in other words its two-particle correlations are of a classical nature.

There is yet another particular three-partite entangled state

$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

introduced by in 2000 by Wolfgang Dür, Guifre Vidal, and Ignacio Cirac in their seminal paper “Three qubits can be entangled in two inequivalent ways” [DVC00]. Contrary to the GHZ state, the W state is robust against the particle loss, indeed, each of its two-particle subsystems exhibits quantum correlations. On the other hand, the one-particle subsystems are not maximally-mixed, as it was for GHZ state.

As it was shown, the aforementioned GHZ and W states are the only two distinct SLOCC-non-equivalent states of genuinely entangled three-qubit states [DVC00]. Furthermore, there are infinitely many LU-equivalence

classes of genuinely entangled three qubit states, parametrized by three real parameters [DVC00, Sud01].

In 2000 Valerie Coffman, Joydip Kundu, and William K. Wootters introduced the first entanglement measure related to the 3-body quantum correlations in the system: the *three-tangle* $\tau^{(3)}$ [CKW00]. Three-tangle distinguishes between GHZ and W states and achieves two extreme values $\tau^{(3)}(\text{GHZ}) = 1$ and $\tau^{(3)}(\text{W}) = 0$ on both states. While the GHZ state exhibits maximal 3-body and vanishing 2-body quantum correlations, the quantum correlations in the W state are reversed. As it was shown, the three-tangle is a SLIP measure, hence invariant under SLOCC operations. In that way, the three-tangle provides a satisfactory method for distinguishing between different SLOCC classes of genuinely entangled three-qubit states.

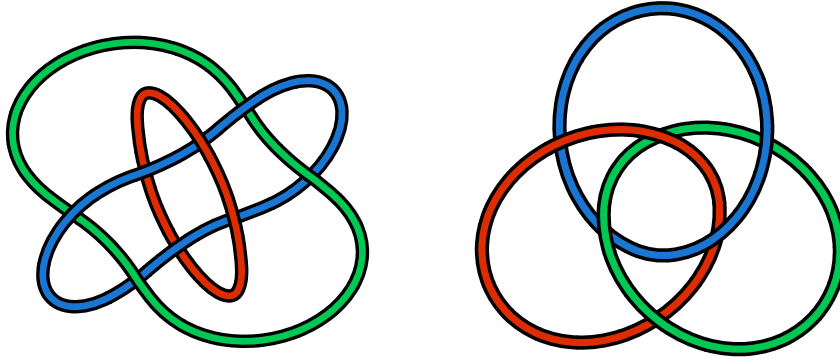


Figure 1.1: Two distinct three-partite qubit states: GHZ and W represented in the form of links. On the left *Borromean rings*, characterized by the fact that if one ring is cut, the remaining two become disconnected. This resembles entanglement in GHZ state, which is not robust against particle-loss. On the right another link of three knots, for which cut of any component does not disconnects the link itself. This is relevant to the robustness of the W state.

Four-qubit system. If the discussion of entangled in three-qubit systems did not convince the reader that entanglement in the multipartite system has a rich and complex form, we shall proceed to the four qubit states. As we shall see, a system of four qubits is already large enough to reveal two substantial problems concerning multipartite-entanglement: the existence of maximally entangled states and the classification of entanglement. In the following, I discuss briefly both problems.

Atsushi Higuchi and Anthony Sudbery in their seminal paper “How entangled can two couples get?” investigate entanglement in system of four qubits concerning bipartitions of the system [HS00]. In particular they showed that there is no four-partite qubit system, which is maximally entangled with respect to any of bipartitions for two-partite systems: 12|34, 13|24, and 14|23. Furthermore, they showed that the following state

$$|M_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |1100\rangle + e^{\frac{2\pi i}{3}} (|1010\rangle + |0101\rangle) + e^{\frac{4\pi i}{3}} (|1001\rangle + |0110\rangle) \right), \quad (1.5)$$

maximize the average entropy of entanglement for such bipartitions. As it was later observed, state $|M_4\rangle$, as well its conjugate transpose exhibits intriguing symmetric properties [LW11, CLSW10]. Contrary to tripartite states $|\text{GHZ}\rangle$ and $|\text{W}\rangle$, state $|M_4\rangle$ is not fully permutation-invariant. Nevertheless state $|M_4\rangle$ is invariant under any even permutation of qubits. Such states for which any element of an *alternating subgroup* of the permutation group, $A_4 < S_4$, leaves the state invariant, shall be called A_4 -symmetric.

Contrary to the case of three-qubits, there are infinitely many SLOCC classes of four-qubit states [Sud01]. Despite this obvious obstacle, the problem of classification of entanglement in four-qubit states arouses great interest [VDDMV01, CD07, CW07, LLHL09, VES11, GW13, SdVK16, GGM18]. The four-qubit states were successfully divided into nine families, most of which contain an infinite number of SLOCC-classes [VDDMV01, CD07, SdVK16]. In particular, the so called G_{abcd} family with the most degrees of freedom is represented by the following highly-symmetric states

$$\begin{aligned} |G_{abcd}\rangle = & \frac{a+d}{2} (|0000\rangle + |1111\rangle) + \frac{a-d}{2} (|0011\rangle + |1100\rangle) \\ & + \frac{b+c}{2} (|0101\rangle + |1010\rangle) + \frac{b-c}{2} (|0110\rangle + |1001\rangle). \end{aligned}$$

Furthermore, non-SLOCC-equivalent four qubit states can be effectively distinguished by three *independent* SLIP measures [VES11, LT03].

General N -qubit system. Quantum states with non-equivalent entanglement represent distinct resources and hence may be useful for different protocols. The idea of clustering states into classes exhibiting different qualities under quantum information processing tasks resulted in their classification under SLOCC. Such a classification was successfully presented for two, three and four qubits [DVC00, VDDMV01, GW13, LLHL09]. However, the full classification of larger systems is completely unknown. Even the much

simpler problem of detecting if two N -qubit states ($N > 4$) are SLOCC-equivalent is, in general, quite demanding [VES11, ZZH16, GW11, BR20, SHKS20].

It was shown that for multipartite $N \geq 4$ systems there exist infinitely many SLOCC classes [DVC00]. In particular, the set of equivalence classes under SLOCC operations depends at least on $2(2^N - 1) - 6N$ parameters, which grows exponentially with the number of qubits [DVC00]. Even larger number of LU-equivalence classes was also investigated [Kra10b, GRB97, dVCSK12].

So far, great effort has been made to reduce this verification problem to the problem of computing entanglement measures, which by definition take the same value for the equivalent states [LT03, GT09, DO09, VES11, LL13, Sza12, JYKH19, HLT13, HLT16]. Nevertheless, starting from four-partite states, one needs to compute values of at least three independent measures, to decide with certainty on local equivalence. Furthermore, the number of independent measures grows exponentially with the number of qubits [DVC00], making it intractable to use this approach to discriminate locally equivalent states [LBS⁺07]. In fact, this rapid increasing difficulty to verify the entanglement equivalence between two states is common to any procedure [LBS⁺07].

Qutrits and beyond. So far, in our study of entanglement we focused on the multipartite qubit systems, consisting of two-level subsystems. These theoretical considerations go hand in hand with experiments, since manipulation and relative control over several qubits have already become a standard task [MSB⁺11, RGF⁺17]. Even though two-level subsystems are the most common multipartite states, they are certainly not the most general from both a theoretical and experimental point of view. Recently, much attention has been paid to the *qutrits*. Qutrits are realized by a 3-level quantum system, that may be in a superposition of three mutually orthogonal quantum states [NJDH⁺13]. Similarly to Eq. (1.1), a qutrit state might be written in the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

Although precise manipulation of such states turned out to be a demanding task, there are some successful indirect methods for it [LWL⁺08]. Furthermore, there is an ongoing development of quantum computers using qutrits and qubits with multiple states [NJDH⁺13].

Overall, there is no reason to restrict the number of levels of each subsystem in the general considerations of multipartite quantum states [WHSK20].

There are several advantages of using larger dimensional subsystems in quantum computations, including the increasing the variety of quantum gates available, applications to adiabatic quantum computing devices [ADS13, ZE12, PLX⁺08], or to topological quantum systems [CHW15, BCRS16].

In this thesis, beside of multipartite qubit systems, I consider homogeneous multipartite qudit systems, i.e. systems of N particles in which each particle has the same number of levels d ($d = 2$ for qubits, $d = 3$ for qutrits, etc.). The heterogenous multipartite systems are also being investigated [HESG18], and some effort has been made to classify them [Miy03, MV04, MW03].

Absolutely Maximally Entangled states. Besides the difficult problem of the entanglement classification in the N qudit system, we prompt a simpler question about which states represent maximum entanglement in such a system. This question is ambiguous, there is no unique way of generalizing the maximally entangled state of two-qubit into any other system. In particular, according to different entanglement measures for multipartite states (like the tangle, the Schmidt measure, the localizable entanglement, or geometric measure of entanglement), the states with the largest entanglement do not overlap in general [Kra10a]. One of the possible interpretations of states with largest entanglement, however, was signaled by the work of Higuchi and Sudbery [HS00]. It resulted in the well-established notion of *Absolutely Maximally Entangled* (AME) states [HCR⁺12]. AME states are those multipartite quantum states that carry absolute maximum entanglement for all possible partitions. AME states are being applied in several branches of quantum information theory: in quantum secret sharing protocols [HCR⁺12], in parallel open-destination teleportation [HC13], in holographic quantum error correcting codes [PYHP15], among many others. Different families of AME states have been introduced [Rai99, Hel13] and the problem of their existence is being investigated [Sco04, HS00, HGS17]. It has been demonstrated that the simplest class of AME states, namely AME states with the *minimal support*, is in one-to-one correspondence with the classical error correction codes [RGRA17] and combinatorial designs known as *Orthogonal Arrays* [GRDMZ18]. Henceforward, the two-way interaction with combinatorial designs and quantum error correction codes is observed [Sco04, GZ14]. AME states are special cases of *k-uniform* states characterized by the property that all of their reductions to k parties are maximally mixed [AC13].

Structure and aims of the thesis. As we have demonstrated so far, the quantification and classification of entanglement for multipartite states is an involving long-distance project. Characterization of different classes of entanglement in multipartite quantum systems remains a major issue relevant for various quantum information tasks and is interesting from the point of view of foundations of quantum theory. In this dissertation, we will show the progress in several directions of these complex issues. The main goals and results of the thesis are presented in the six next chapters of the dissertation. We briefly present the scope of the proceeding chapters and indicate the main results below. If not specified differently, the author's contributions to the work covered by this chapter were significant.

Chapter 2 constitutes a link between the classification of multipartite entanglement and knot theory. We introduce the notion of *m-resistant states*, states which remains entangled after losing an arbitrary subset of m particles, but becomes separable after losing any number of particles larger than m . I construct several families of N -particle states with the desired property of entanglement resistance to particle loss.

Chapter 3 discusses highly-symmetric states and their properties. Inspired by remarkable entanglement properties of a state $|M_4\rangle$, I introduce the notion of G -symmetric states of N -qubits, for any subgroup of the permutation group symmetric $G < S_N$. I present two methods for constructing such states: first based on the group and second on graph theory. I propose quantum circuits efficiently generating such states, and Hamiltonians with the ground states related to them. In addition, these states were experimentally simulated on available quantum computers: IBM – Santiago, Vigo, and Athens.

Chapter 4 investigates a class of AME states and k -uniform states. I briefly recall correspondence between AME states and classical combinatorial designs focusing attention on the different linear structures of classical designs.

Chapter 5 settles a long-standing problem in multipartite quantum entanglement: of whether absolutely maximally entangled states exist for all 4 party states, with local dimensions greater than 2. We settle this positively and I provide an explicit analytical example for local dimension 6, which was the sticking point. Presented work is connected to the famous Euler problem

of the non-existence of orthogonal Latin squares in dimension 6, involving 36 officers from 6 different regiments and ranks. This result shows that if officers of different ranks and regiments can be entangled, the classically impossible problem has a quantum solution. I discuss how presented solution opens doors to the new area dubbed as quantum combinatorics. The aforementioned issue concerning the existence of the state presented in this thesis appears on at least two open-problem lists of quantum information.

Chapter 6 concerns the local equivalence of AME states. All reduced density matrices of AME states are maximally mixed. Therefore, the classical method for verification of local equivalence, which is comparison of Schmidt rank and coefficients, fails. I present general techniques for verifying either two AME states are locally equivalent. I falsify the conjecture that for a given multipartite quantum system all AME states are locally equivalent. I also show that the existence of AME states with minimal support of 6 or more particles results in the existence of infinitely many such non-equivalent states. As an immediate consequence, I show that not all AME states belong to the class of stabilizer states, which was supposed by several experts in the field. Moreover, I present AME states which are not locally equivalent to the existing AME states with minimal support.

Chapter 7 tackles a particularly discernible problem of discrimination and classification of multipartite entanglement. So far, states were compared by computing values of independent entanglement measures. Nevertheless, the number of independent measures grows exponentially with the number of qubits, making it intractable to use this approach to discriminate locally equivalent states. I provide a complementary approach and use a single entanglement measure for verification if generic multipartite states are locally equivalent. This approach can be used independently on the number of qubits and bypasses the exponential difficulty of standard procedures. In essence, I investigate the roots of an entanglement measure and show that the roots of locally equivalent states must be related by a Möbius transformation, which is straightforward to verify. Moreover, I apply the presented approach on 4-qubit states and show that roots of the 3-tangle measure of the related reduced 3-qubit states, might be used for the more demanding problem of classifying entanglement in 4-qubit states. This gives hope to apply this method to classify entanglement in larger ($N > 4$) systems, for which the classification is completely unknown. In addition, I have developed a novel method to obtain the normal form of a 4-qubit state which bypasses the

possibly infinite iterative procedure.

Chapter 8 summarizes results obtained in the dissertation and outlines the open problems and perspectives for further research.

Appendices contain original research papers, that do not belong to the dissertation, but have been included for the convenience of the reader.

Chapter 2

Quantum states and topological links

In this chapter, I introduce the notion of *m-resistant states*. We say that an entangled quantum state of N subsystems is *m-resistant* if and only if it remains entangled after losing an arbitrary subset of m particles, and becomes separable after losing any number of particles larger than m . Furthermore, I establish an analogy to the problem of designing a topological link consisting of N rings characterized by the fact that by cutting any $(m + 1)$ of them, the remaining link becomes unknotted. This analogy allows us to exhibit several N -particle states with the desired property of entanglement resistance to particle loss. An extension of some parts of this chapter, in which the author's contribution was not substantial can be found in the joint work [QABZ19], attached to the dissertation in Appendix A.

2.1 Motivation

An effort to study multipartite entanglement using the knot theory was proposed by Aravind, who related some N -partite quantum states with a link composed of N closed rings [Ara97]. The $|\text{GHZ}\rangle$ state, which becomes separable after any measurement in the computational basis performed on its subsystem, was an initial inspiration. In such a way $|\text{GHZ}\rangle$ state can be associated with a particular configuration of three rings, called *Borromean rings*, see Figure 1.1. Indeed, Borromean link is characterized by the fact that if one ring is cut, the remaining two become disconnected. Usage of other tools borrowed from knot theory to analyze multipartite entanglement was further advocated in [KL04, KM19].

The aforementioned analogy is, however, not basis independent. On the one hand, measuring the $|\text{GHZ}\rangle$ state in the computational basis collapses always in a separable state, on the other hand, measurement in the rotated basis $|+\rangle, |-\rangle$ basis results into an entangled state. This motivated Sugita [Sug07] to revise the above analogy by proposing the partial trace of a subsystem as an alternative interpretation for cutting related ring. In that way, the physical process corresponding to cutting the ring becomes basis independent [MSS13]. This analogy between quantum entanglement and linked rings, was later explored [QA18], and will be used here. Partial trace over given subsystems can be physically interpreted as not registering them by a measurement device or as the loss of related particles. Studying quantum entanglement resistant to a particle loss thus corresponds to the question, whether a given reduced density matrix represents an entangled state [NMB18, OB18, KNM02, KZM02].

2.2 Links and quantum states

Recently, Neven et al. introduced the notion of m -resistant quantum N -qubit states which entanglement of the reduced state of $N - m$ subsystems is fragile with respect to loss of any additional subsystem [NMB18].

Definition 1. An entangled state $|\psi\rangle$ of N parties is called m -resistant if:

- $|\psi\rangle$ remains entangled as any m of its N subsystems are traced away;
- $|\psi\rangle$ becomes separable if a partial trace is performed over an arbitrary set of $m + 1$ subsystems.

The first non-trivial examples of m -resistance arise for $N = 3$ qubits, which is simple enough to study on intuitive grounds. The $m = 0$ resistance is exemplified by the GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad (2.1)$$

since tracing out any party of GHZ state results in a separable state. The corresponding link has the properties of the well-know Borromean link, depicted in Figure 1.1. In particular, if any ring in Borromean link is cut, the remaining two rings become separated. Such a connection between Borromean link and the GHZ state was already noticed by Aravind [Ara97]. For arbitrary number N of rings, the natural generalization of Borromean link is called Brunnian link [Bru92]. Brunnian link is characterized by the same

property that cut of any of its rings disconnect the link, see Figure 2.3. On the other hand, the three-qubit state [DVC00],

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \quad (2.2)$$

possesses property of being $m = 1$ resistant. Indeed, a partial trace over any of its subsystems will result in an entangled mixed state. A corresponding link which mimics this property is represented on Figure 1.1. It was first considered as a representation of the W state in [Ara97].

2.3 In search for m -resistant states of N -qubit system

In [QABZ19], we presented the general form of the symmetric mixed state of N qubits with the property of being m resistant. This form was obtained by relating to a given m -resistant link to the associated quantum state. Unlike the case of mixed states, similar identification between links and pure states is more intricate, if at all possible. It seems that the problem of finding pure m -resistant states for a given m and the arbitrary number N of qubits is not trivial. Obviously, the desired property is fully symmetric with respect to the exchange of subsystems. Therefore a search for such states among symmetric states seems to be a reasonable approach. Below, we present the partial solution to this problem, which is based on the Majorana representation [AMM10, MGB⁺10].

The *stellar representation* of Majorana [MM10, AMM10, BZ17] provides an alternative intuition on the geometry of symmetric states. Any permutation-invariant state of N qubits might be written in the following form

$$|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{\sigma \in S_N} |\eta_1\rangle_{i_1} \cdots |\eta_N\rangle_{i_N} \quad (2.3)$$

where K is a suitable normalization constant, the sum runs over all permutations $\sigma \in S_N$ of particles indices $i_1, i_2, \dots, i_N$, and

$$|\eta_j\rangle_X = \cos\left(\frac{\theta_j}{2}\right) |0\rangle_X + e^{i\phi_j} \sin\left(\frac{\theta_j}{2}\right) |1\rangle_X. \quad (2.4)$$

The pairs of angles (θ_j, ϕ_j) represent a point on the sphere, and are called Majorana stars. Thus, one may define a fully symmetric N -qubit state by

fixing N points on the sphere. For this reason, the name stellar representation is also common, as each point represents a star in the sky, while a group of stars forms a constellation.

Among many advantages, the Majorana representation ascribes some geometrical intuition of entanglement of a symmetric state. If one chooses all of the stars at a single point the corresponding state is separable. For example, N degenerated stars on the North pole represent the separable state $|0 \cdots 0\rangle$. Nevertheless, as the degeneracy is lifted, most of the non-trivial constellation of stars corresponds to an entangled state of N qubits. Furthermore, the distance between the stars is related to the degree of entanglement [ZS01a, GKZ12], although the precise criterium to quantify entanglement is not uniquely defined.

We introduce the following family of states:

$$|\psi_m^N\rangle = \frac{1}{\sqrt{1 + \binom{N}{m}}} \left(\sqrt{\binom{N}{m}} |0\rangle^{\otimes N} - (-1)^{N+m} |D_m^N\rangle \right) \quad (2.5)$$

where $|D_m^N\rangle$ are so-called *Dicke states* [Dic54]:

$$|D_N^m\rangle \propto \sum_{\sigma \in \mathcal{S}_N} \sigma \left(\underbrace{|1 \cdots 1\rangle}_{k \text{ times}} \underbrace{|0 \cdots 0\rangle}_{N-m \text{ times}} \right), \quad (2.6)$$

with the summation going over all permutations σ . This family of states was motivated by investigation of $N = 3, 4$ qubit states. Indeed, for $N = 3, 4$ and any non-trivial value of m , the state $|\psi_m^N\rangle$ provides example of m -resistant state. In particular in the case of $N = 3$ qubits presented constellations give us $m = 0$, and $m = 1$ resistant states respectively:

$$|\psi_0^3\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad (2.7)$$

$$|\psi_1^3\rangle = \frac{1}{\sqrt{12}} (3|000\rangle + |011\rangle + |101\rangle + |110\rangle). \quad (2.8)$$

It is straightforward to show that the constellations presented on Figure 2.1 give the desired answer for both non-trivial values of m . Similarly, states $|\psi_m^4\rangle$ are $m = 0, 1, 2$ resistant states of four qubits for respective values of m . Related constellations are presented on Figure 2.2. Note that $|\psi_m^N\rangle$ may be directly related to the distribution of m stars on the North pole, with all others evenly distributed along the equator.

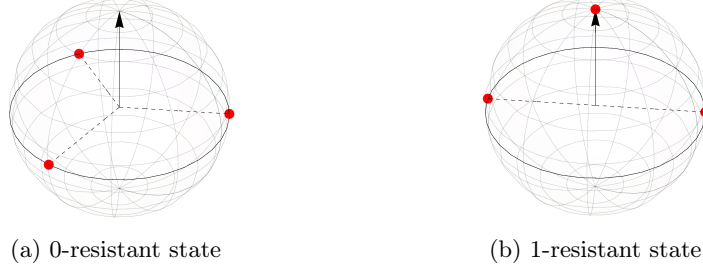


Figure 2.1: Constellations defining m -resistant states of 3 qubits. The middle arrow serves as a reference pointing to the North pole. The constellation in (a) corresponds to the state $|\text{GHZ}\rangle$ and is illustrated by the Borromean link, see Figure 1.1, while (b) presents an exemplary 1-resistant state of three qubits.

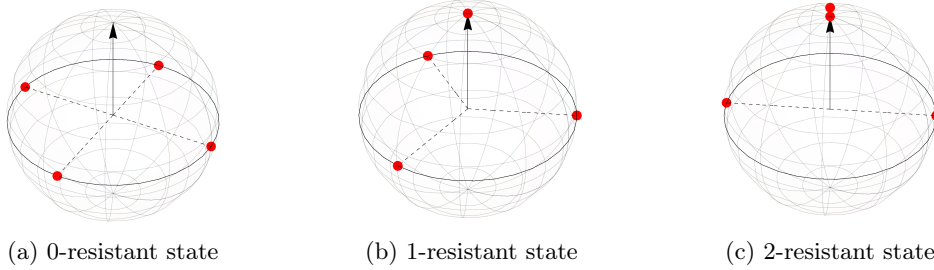


Figure 2.2: Constellations defining m -resistant states of 4 qubits for $m = 0, 1, 2$. The middle arrow serves as a reference pointing to the North pole. Presented constellations are related to three links presented on Figure 2.3.

The agreement between the above construction and m -resistant states breaks down for $N = 5$ parties. Indeed, the state $|\psi_1^5\rangle$ is in fact 2-resistant. In all other cases, i.e. $m = 0, 2$ and 3 , related pure states are 0, 2, and 3-resistant respectively. One may naturally inquire whether another symmetric state of five qubits might be in fact 1-resistant. We have tested all combinations where two stars are lifted by a general latitude, as well as four stars, and it is possible to show that no 1-resistant state exists for these families of states. Obviously, this does not prove that a pure 1-resistant state of 5 qubits does not exist, although it is tantalizing to conjecture so. In general, we have verified the following.

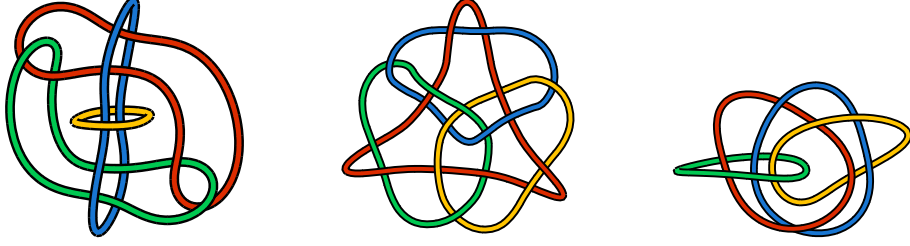


Figure 2.3: On the left, an example of a 0-resistant link of 4 rings, which is a 4-component Brunnian link. Notice that after cutting a single ring all three remaining rings become disconnected. In the center, an example of a 1-resistant link of 4 rings. After cutting any single ring one arrives at the Borromean link. On the right, an example of a 2-resistant link of 4 rings. After cutting any two rings the remaining two are still connected.

Proposition 1. *For any number of particles $N \geq 3$, the family of states Eq. (2.5):*

$$|\psi_m^N\rangle = \frac{1}{\sqrt{1 + \binom{N}{m}}} \left(\sqrt{\binom{N}{m}} |0\rangle^{\otimes N} - (-1)^{N+m} |D_m^N\rangle \right) \quad (2.9)$$

provides examples of $m = 0, (N-3)$, and $(N-2)$ -resistant states respectively.

Proof. Observe that $|\psi_0^N\rangle$ is zero resistant. Indeed, by (2.5), we have

$$|\psi_0^N\rangle = \frac{1}{2} \left(|0\rangle^{\otimes N} - (-1)^N |1\rangle^{\otimes N} \right). \quad (2.10)$$

This state is a generalization of the GHZ state for N qubits, which, after any partial trace, returns a density matrix of the form

$$\text{tr}_k [\hat{\rho}_0^N] = \frac{1}{2} \left(|0\rangle \langle 0|^{\otimes (N-k)} + |1\rangle \langle 1|^{\otimes (N-k)} \right), \quad (2.11)$$

where tr_k denotes the trace over any set consisting of k qubits. This density matrix is separable for any $k > 0$, and thus the state $|\psi_0^N\rangle$ is 0-resistant for any N .

Regarding the state $|\psi_{N-2}^N\rangle$, after simple algebraic operations, one finds that any $N - 2$ partial traces will result in the density matrix proportional to

$$\text{tr}_{N-2} [\hat{\rho}_{N-2}^N] \propto \begin{pmatrix} \alpha_{22} + (\alpha_{02})^2 & 0 & 0 & -\alpha_{02} \\ 0 & \alpha_{21} & \alpha_{21} & 0 \\ 0 & \alpha_{21} & \alpha_{21} & 0 \\ -\alpha_{02} & 0 & 0 & 1 \end{pmatrix}, \quad (2.12)$$

where we define $\alpha_{ij} \equiv \binom{N-i}{j}$. The partially transposed matrix has one of its eigenvalues equal to

$$\lambda_1 = -N^2 + 3N - 4 \quad (2.13)$$

which is negative for any N . Thus, by the positive partial transpose (PPT) test, we confirm that each 2-qubit reduced density matrix is always entangled, and the state $|\psi_{N-2}^N\rangle$ is $(N-2)$ -resistant for any N .

Finally, focusing on the state $|\psi_{N-3}^N\rangle$, we perform the partial trace over any set of $N-2$ qubits and obtain a two-qubit state

$$\text{tr}_{N-2} [\hat{\rho}_{N-3}^N] \propto \begin{pmatrix} \alpha_{23} + (\alpha_{03})^2 & 0 & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{22} & 0 \\ 0 & \alpha_{22} & \alpha_{22} & 0 \\ 0 & 0 & 0 & \alpha_{21} \end{pmatrix}. \quad (2.14)$$

The partially transposed matrix of size four has all positive eigenvalues for $N > 1$. In this case the PPT test guarantees that the resulting state is always separable. We must then look into the reduced density matrix with one less partial trace, in order to check if it is entangled. Such a 3-qubit density matrix is proportional to

$$\text{tr}_{N-3} [\hat{\rho}_{N-3}^N] \propto \begin{pmatrix} \alpha_{33} + (\alpha_{03})^2 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{03} \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ \alpha_{03} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.15)$$

By partially transposing any qubit, we will obtain a matrix which possesses an eigenvalue equal to

$$\lambda_1 = \left(13 - 7N + N^2 - \sqrt{193 - 202N + 79N^2 - 14N^3 + N^4} \right) \quad (2.16)$$

which is negative for all N . PPT criterion implies that all the subsystems are entangled. This proves that the state $|\psi_{N-3}^N\rangle$ is $(N-3)$ -resistant. $\square$

As a final remark to the section, we note that the constellations for all 0-resistant states follow the same rule, which is a regular N -sided polygon. This result has been previously found in [NMB18].

2.4 In search for m -resistant states of N -qudit system

As we mentioned in Section 2.3, we did not find a general construction of an m -resistant pure qubit state of N qubit system. Hence, we have expanded our search for non-symmetric states with subsystems of a larger local dimension $d \geq 3$. We present the general formula for a m -resistant N -qudit pure state for any $N \geq 2m$. Our construction is based on a particular family of combinatorial designs and their connection to quantum states. In particular, we use the notion of *orthogonal arrays* (OA), and the established link [GZ14] between multipartite quantum states and OA. This connection will be expanded and further use in Chapter 4 for designing another family of quantum states.

Orthogonal arrays [ASH99] are combinatorial arrangements, tables with entries satisfying given orthogonal properties. A close connection between OA and codes, entangled states, error-correcting codes, uniform states has been established [ASH99]. Therefore, investigation of the connections between OA and resistant states seems to be a natural approach. Firstly, we briefly present the concept of OA, secondly, we demonstrate relations between OA and resistant states.

An orthogonal array $\text{OA}(r, N, d, k)$ is a table composed by r rows, N columns with entries taken from $0, \dots, d-1$ in such a way that each subset of k columns contains all possible combination of symbols with the same amount of repetitions. The number of such repetitions is called *the index* of the OA and denoted by λ . One may observe, that the index of OA is related to the other parameters:

$$\lambda = \frac{r}{d^k}. \quad (2.17)$$

Figure 2.4 presents an example of an OA. A pure quantum state consisting of r terms might be associated with $\text{OA}(r, N, d, k)$, simply by reading all rows of OA [GZ14, ASH99]. The state of N qudits associated with the orthogonal array $\text{OA}(r, N, d, k)$ will be denoted as by $|\phi_{(N,k)}\rangle_d$.

The crucial quantity for our purpose, related to OA, is its index. It preserves the following information: how many repetitions of any sequence $i_1, \dots, i_k$ there are for each subsystem of k rows. For $\lambda = 1$, any sequence appears only once, and such an array is called *index unity array*. We emphasize their remarkable role in the search for resistant quantum states.

Proposition 2. *For any orthogonal array of index unity $\text{OA}(d^k, N, d, k)$, where $N \geq 2k$, the relevant quantum state $|(N, m)_d\rangle$ is $k-1$ -resistant.*

0 0	0 0 0	0	0 0	0 0	$ (5,1)_4\rangle \propto$	$ 00000\rangle$
0 1	1 1 1	0	1 1	1 1	+	$ 01111\rangle$
0 2	2 2 2	0	2 2	2 2	+	$ 02222\rangle$
0 3	3 3 3	0	3 3	3 3	+	$ 03333\rangle$
1 0	1 2 3	1	0 1	2 3	+	$ 10123\rangle$
1 1	0 3 2	1	1 0	3 2	+	$ 11032\rangle$
1 2	3 0 1	1	2 3	0 1	+	$ 12301\rangle$
1 3	2 1 0	1	3 2	1 0	+	$ 13210\rangle$
2 0	2 3 1	2	0 2	3 1	+	$ 20231\rangle$
2 1	3 2 0	2	1 3	2 0	+	$ 21320\rangle$
2 2	0 1 3	2	2 0	1 3	+	$ 22013\rangle$
2 3	1 0 2	2	3 1	0 2	+	$ 23102\rangle$
3 0	3 1 2	3	0 3	1 2	+	$ 30312\rangle$
3 1	2 0 3	3	1 2	0 3	+	$ 31203\rangle$
3 2	1 3 0	3	2 1	3 0	+	$ 32130\rangle$
3 3	0 2 1	3	3 0	2 1	+	$ 33021\rangle$

Figure 2.4: Orthogonal array of unity index $OA(4^2, 5, 4, 2)$ obtained from the Reed-Solomon code of length 5 over Galois field $GF(4)$. Each subset consisting of two columns contains all possible combination of symbols. Here, two such subsets are highlighted. The relevant quantum state is obtained by forming a superposition of states corresponding to consecutive rows of the array – see expression on the right-hand side.

For the proof of above statement, we refer to [QABZ19]. From the OA presented in Figure 2.4, we obtain, for example, the following five-ququart, 1-resistant state:

$$\begin{aligned}
|(5,1)_4\rangle \propto & |00000\rangle + |01111\rangle + |02222\rangle + |03333\rangle + \\
& |10123\rangle + |11032\rangle + |12301\rangle + |13210\rangle + \\
& |20231\rangle + |21320\rangle + |22013\rangle + |23102\rangle + \\
& |30312\rangle + |31203\rangle + |32130\rangle + |33021\rangle. \quad (2.18)
\end{aligned}$$

Bush provided the general method for constructing OAs of index unity [Bus52].

Theorem 1 (Bush, 53'). *If d is a prime power, i.e $d = p^n$ for some prime number p and natural number n , then we can construct the array $OA(d^k, d+1, d, k)$.*

Note that combining Bush's result with Proposition 2 provides the existence of m -resistant N -qudit states for $N \geq 2(m+1)$.

Proposition 3. *For any $N \geq 2(m+1)$ there exists the N -qudit state which is m -resistant. The local dimension d is the smallest prime power larger than $N-1$.*

Eq. (2.18) presents an example of a m -resistant qudit state obtained by reading consecutive rows of OA. A more interested reader might easily reproduce other resistant states with the help of available OA tables [Slo]. We organize all obtained results in Table 2.1. They encourage us to pose the following conjecture.

Conjecture 1. *For any N and m , there exists an m -resistant N -qudit state in some local dimension d .*

Furthermore, we leave as a list of related open problems.

1. Investigate whether there exist pure states of N qubits with the m -resistance property, for any $m = 0, 1, \dots, N-2$.
2. If such a state does not exist, for each N find the minimal local dimension d such that there exist m -resistant states of N qudits.
3. For any class of m -resistant states of N qubits find a state for which its average entanglement after partial trace over any set of m parties is the largest, if measured with respect to a given measure of entanglement.

2.5 Asymptotic case

Consider a system consisting of N subsystems with d levels each and assume that N is large. It is well known that a generic pure state of such a system is typically strongly entangled [ZS01b, HLW06]. Therefore the partial trace of the corresponding N -party projector over any choice of subsystems traced away becomes mixed. Typically, the more parties k are traced out, the more mixed is the state describing the remaining $N-k$ subsystems.

If the number k of subsystems removed is equal to $N/2$ the reduced state typically has a full rank so it can belong to the separable ball around the maximally mixed state [ZHSL98, GB02]. It was indeed observed [KNM02, KZM02, HLW06] that for $k \approx N/2$ a transition from entangled reduced states to the reductions with positive partial transpose takes place. Furthermore, if

$m \backslash N$	4	5	6	7	8
0	$ \psi_0^4\rangle$	$ \psi_0^5\rangle$	$ \psi_0^6\rangle$	$ \psi_0^7\rangle$	$ \psi_0^8\rangle$
1	$ \psi_1^4\rangle$	$ \phi_{1,5}\rangle_4$	$ \phi_{1,6}\rangle_5$	$ \phi_{1,7}\rangle_7$	$ \phi_{1,8}\rangle_7$
2	$ \psi_2^4\rangle$	$ \psi_2^5\rangle$	AME	$ \phi_{2,7}\rangle_7$	$ \phi_{2,8}\rangle_7$
3		$ \psi_3^5\rangle$	$ \psi_3^6\rangle$		$ \phi_{3,8}\rangle_7$
4			$ \psi_4^6\rangle$	$ \psi_4^7\rangle$	
5				$ \psi_5^7\rangle$	$ \psi_5^8\rangle$
6					$ \psi_6^8\rangle$

Table 2.1: Collection of m -resistant states of N parties obtained so far. Three families: $|\psi_0^N\rangle$, $|\psi_{N-3}^N\rangle$ and $|\psi_{N-2}^N\rangle$ of m -resistant N -qubit states discussed in Section 2.3 are presented on the differently shaded blue background. Furthermore, m -resistant qudit states $|\phi_{m,N}\rangle_d$ constructed by virtue of orthogonal arrays are demonstrated on the green background. The number in subscript is relevant to the local dimension d of a state. Observe that the local dimension d of qudit states is usually (but not always) equal to $N - 1$. Finally, a six-qubit state AME(6,2) provides an example of a 2-resistant state.

$k \approx 3N/5$ the remaining subsystem consisting of $2N/5$ particles is typically separable [ASY12, ASY14] and this transition becomes sharp if $N \rightarrow \infty$. Thus one can expect that a generic pure state of a N -system state is m -resistant with m typically of the order of $3N/5$ independently of the local dimension d .

2.6 Conclusions

In this chapter, I presented an analogy between quantum states and topological links, in which the operation of losing a subsystem is related to neglecting an associated ring. Furthermore, I introduced the notion of m -resistant states, which entanglement is resistant for loss of any m particles but fragile for loss of any larger subsystem (Definition 1). I presented two methods of constructing m -resistant states: Section 2.3 is based on the Majorana representation of symmetric states, while Section 2.4 on the combinatorial notion of Orthogonal Arrays. Related results are summarized in Proposition 1 and Propositions 2 and 3 respectively. Table 2.1 presents both aforementioned families of states at glance. Finally, Section 2.5 discusses the typical resistance property of large systems.

Chapter 3

Highly symmetric states and groups

In this chapter, I present an approach for constructing highly-entangled quantum states which is based on the group theory. Introduced states resemble the Dicke states, whereas the interactions occur only between specific subsystems related by the action of the chosen group. The states constructed by this technique exhibit desired symmetry properties and form a natural resource for less-symmetric quantum informational tasks. Furthermore, I introduce a larger class of genuinely entangled states based on an arbitrary network structure, graphically represented as a (hyper)graph, with excitations appearing only in particular subsystems represented by (hyper)edges. I investigate the entanglement properties of both families of states and show an interesting phenomenon: most of the entanglement is concentrated between nodes of distance two and is absent between immediate neighbours. In addition I present two different methods of constructing introduced states: firstly, I propose quantum circuits whose complexity is comparable with the complexity of quantum circuits proposed for Dicke states, secondly, I present such states as a ground states of Hamiltonians with 3-body interactions. I demonstrate the viability of the provided constructions by simulating one of the considered states on available quantum computers: IBM - Santiago and Athens. An extension of some parts, in which the author's part was not substantial, as well as proofs of presented results, can be found in the joint paper [BCZ21], attached to the dissertation in Appendix C.

3.1 Motivation

Among different types of entangled states, permutation invariant states attracted a lot of attention in both continuous [SAI05, AI07] and discrete [DPR07, BG13] variable systems. A remarkable example of such states is due to Dicke [Dic54], *Dicke state* with k excitations in a system of N -qubits is defined as [SGDM03],

$$|D_N^k\rangle \propto \sum_{\sigma \in \mathcal{S}_N} \sigma \left(\underbrace{|1 \cdots 1\rangle}_{k \text{ times}} \underbrace{|0 \cdots 0\rangle}_{N-k \text{ times}} \right), \quad (3.1)$$

where the summation runs over all permutations in the symmetric group $\mathcal{S}_N$. On one hand, symmetry of the Dicke states simplifies their theoretical [LYZZ19] and experimental [LPV⁺14, WKK⁺09] detection, furthermore facilitates the tasks of quantum tomography [MAT⁺15]. On the other hand, the entanglement of Dicke states turned out to be maximally persistent and robust for the particle loss [Dic54, BR01], such states provide inherent resources in numerous quantum information contexts, including quantum secret sharing protocols [GKBW07], open destination teleportation [MJPV99], quantum metrology [PSO⁺18], and decoherence free quantum communication [BEG⁺04].

So far most of the scientific attention was focused on fully symmetric tasks, for example, parallel teleportation [HC13], or symmetric quantum secret sharing protocols [HCR⁺12]. In various realistic situations, however, such a full symmetry between collaborating systems is not possible, required or even desirable. As an example, it was shown that there is no four-qubit state which is maximally entangled with respect to all possible symmetric partitions [HS00]. Such a state would allow for the parallel teleportation protocol of two qubits between any two two-qubit subsystems. Nevertheless, the following state:

$$\frac{1}{2} \left(|1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle \right)$$

allows for the teleportation of a single qubit to an arbitrary subsystem, and additionally for the parallel teleportation across the partition 13|24. As we already discussed in Introduction, the following state

$$|M_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |1100\rangle + e^{\frac{2\pi i}{3}} (|1010\rangle + |0101\rangle) + e^{\frac{4\pi i}{3}} (|1001\rangle + |0110\rangle) \right), \quad (3.2)$$

turned out to maximize the average entropy of entanglement over all bipartitions of four-qubit state. As it was later observed, state $|M_4\rangle$ is not fully

permutational invariant, but invariant under any even permutation of qubits [LW11, CLSW10].

On one hand, it is especially reasonable to share resources in a not fully symmetric way in variants of quantum secrets sharing schemes, allowing only some parties for cooperation. On the other hand, various molecules in Nature (like benzene) stand out with remarkable symmetries, the general investigations of entanglement in highly symmetric systems may shed some light on the nature of correlation in relevant chemical molecules. In recent years, correlations and the entanglement contained in chemical bonds was investigated [SBS⁺17, SPM⁺15, DMD⁺21], and a special attention was dedicated to highly symmetrical molecules [SPM⁺15]. Although for most molecules the total correlation between orbitals seems to be classical, the general significance of entanglement in chemical bonds seems to be present [DMD⁺21].

3.2 Group of symmetry of a quantum state

A pure state $|\psi\rangle$ is called *symmetric* if it is invariant under any permutation of its subsystems, i.e. $\sigma|\psi\rangle = |\psi\rangle$ for any element σ from the permutation group $\mathcal{S}_N$, where N denotes number of subsystems. This might be generalized to a very natural definition of the group of symmetry of a quantum state.

Definition 2. A state $|\psi\rangle$ of N subsystems is called H -symmetric, where H is a subgroup of the permutation group, $H < \mathcal{S}_N$, iff it is permutation invariant for any $\sigma \in H$, and only for such permutations.

We begin with several examples of states with restricted group of symmetry. Firstly, consider a system consisting of three qubits. The celebrated states [DVC00]:

$$\begin{aligned} |\text{GHZ}\rangle &\propto |000\rangle + |111\rangle, \\ |\text{W}\rangle &\propto |001\rangle + |010\rangle + |100\rangle. \end{aligned}$$

are fully symmetric, see Figure 3.1. We also may construct three-qubit quantum states with other types of symmetry. For example

$$|\chi_1\rangle \propto |001\rangle + |010\rangle$$

is $\mathcal{S}_2$ -symmetric state. This state is, however, bi-separable with respect to the partition $A|BC$. Nevertheless, a similar example $|\chi\rangle$ can be found among genuinely entangled states:

$$|\chi_2\rangle \propto |001\rangle + |010\rangle + 2|100\rangle + 2|111\rangle.$$

Furthermore the following state

$$|\chi_3\rangle \propto |001\rangle + e^{\frac{2\pi i}{3}} |010\rangle + e^{\frac{4\pi i}{3}} |100\rangle$$

exhibits the alternating $\mathcal{A}_3$ -symmetry, symmetry of all even permutations of three qubits. Note that the above state is entangled and locally equivalent to the celebrated $|W\rangle$ state. In that way, we constructed states with all possible types of discrete symmetries in a three-qubit setting.

Observe that there exists an easy recipe for the construction of an H -symmetric state, where $H < S_N$ on N qudits.

Proposition 4. *Consider a subgroup $H < S_N$. The following state $|\psi\rangle \in \mathcal{H}_N^{\otimes N}$ of the local dimension N :*

$$|\psi\rangle \propto \sum_{\sigma \in H} |\sigma_0(0) \cdots \sigma_{N-1}(N-1)\rangle$$

is H symmetric.

Proof. Clearly, that the group H stabilizes $|\psi\rangle$. Suppose there is a larger group H' , which contains H , i.e. $H < H'$, and at the same time stabilize $|\psi\rangle$. Take any element $\sigma \in H' \setminus H$. Observe, that there is no term $|\sigma_0(0) \cdots \sigma_{N-1}(N-1)\rangle$ in $|\psi\rangle$, hence H' cannot stabilize $|\psi\rangle$. $\square$

As an example, the three qutrits state

$$|\psi\rangle \propto |012\rangle + |201\rangle + |120\rangle \quad (3.3)$$

is $\mathcal{A}_3$ -symmetric. Indeed, one can see, that the cyclic permutation of the last three qutrits does not change the state. This encourages us to pose the following question.

Question 1. *Consider any group of symmetry $H < S_N$. What is the minimal local dimension d for which there exists a H -symmetric state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$?*

Consider now a natural basis for symmetric states. Any symmetric state $|\psi\rangle$ is a superposition of Dicke states [Dic54, SGDM03]. A similar decomposition for a H -symmetric state is also possible. We proceed with the following definition.

Definition 3. For a given subgroup $H < S_N$, we define the N -qubit *Dicke-like* state with k excitations is the following way:

$$|D_N^k\rangle_H \propto \sum_{\sigma \in H} \sigma \left(\underbrace{|1 \cdots 1\rangle}_{k \text{ times}} \underbrace{|0 \cdots 0\rangle}_{N-k \text{ times}} \right) \quad (3.4)$$

where the summation runs over all permutations from the group $H < S_N$.

As we shall see, the normalization constant in Eq. (3.4) highly depends on the structure of the subgroup H and the number of excitations, and is difficult to present in a consistent way. Note that the group symmetry of the Dicke-like state $|D_N^k\rangle_H$ is not necessarily given by H . The general analysis of the group of symmetries is tightly connected with the *partially ordered set* (poset) of all subgroups of $\mathcal{S}_N$, which has rather complicated structure [Bra01].

Any symmetric state $|\psi\rangle$ can be written in the computational basis as:

$$|\psi\rangle \propto \sum_{\sigma \in \mathcal{S}_N} |\phi_{\sigma(1)}\rangle \cdots |\phi_{\sigma(N)}\rangle, \quad (3.5)$$

where the sum is taken over all permutations $\sigma \in \mathcal{S}_N$ in the symmetric group. In such a way, symmetric states have an effective representation, called *Stellar representation* [MM10], as N points (stars) on the Bloch sphere, corresponding to vectors $|\phi_i\rangle, \dots, |\phi_N\rangle$, as we already discussed in Chapter 2. Stellar representation turned out to play a role by classification of entanglement in symmetric quantum states [MM10, RM11, GKZ12]. Furthermore, special symmetry conditions imposed on the stars, are related with highly entanglement properties of resulted states [MGB⁺10, BDGM15, DGM17].

I introduce a natural generalization of this approach – the *generalized stellar representation*, which is suitable for quantum states exhibiting modes symmetries, i.e. for which summation in Eq. (3.5) runs over a subgroup H of the symmetric group $\mathcal{S}_N$. This might possibly restrict the group of symmetries in the resulting state. As for symmetric states, consider N points on the Bloch sphere: $|\phi_1\rangle, \dots, |\phi_N\rangle$, and the following product

$$|\psi\rangle \propto \sum_{\sigma \in H} |\phi_{\sigma(1)}\rangle \cdots |\phi_{\sigma(N)}\rangle, \quad (3.6)$$

where the sum runs over all permutations σ from the group H . We might represent the state $|\psi\rangle$ as a collection of N points on the Bloch sphere relevant to vectors $|\phi_i\rangle, \dots, |\phi_N\rangle$ with indicated action of the group H , see Figure 3.1.

Notice that a given constellation of ‘stars’ at the Bloch sphere with the selected symmetry group H do not represent uniquely the quantum state. The important information is carried in how the group H is contained in $\mathcal{S}_N$, which mathematically might be expressed by immersion $H \hookrightarrow \mathcal{S}_N$ of the group H into the symmetric group $\mathcal{S}_N$.

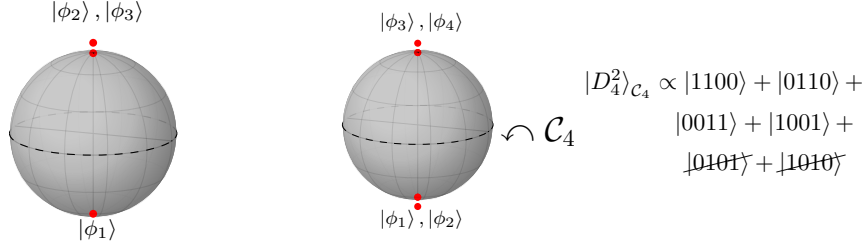


Figure 3.1: On the left, stellar representation of three qubits $|W\rangle$ state, by two stars at the North pole and a single star at the South pole. Stellar representation of a Dicke state $|D_N^k\rangle$ consists of k stars at the South Pole and $N - k$ stars at the North Pole in general [GKZ12]. Similar holds true for the Dicke-like states. On the right, $|D_4^2\rangle_{C_4}$ defined in Example 1 is represented by four stars on the Bloch sphere, with indicated action of the Cyclic group C_4 .

3.3 Symmetric states related to (hyper)graphs

In this section, I propose a scheme to associate to a given graph with N vertices a single pure quantum state of an N -party system. Such a representation reflects not only the symmetry, but also the structure of a quantum circuit under which presented families of quantum states can be constructed. Let us emphasize that states constructed in that way are completely different form, the so-called *graph states*, known also as *cluster states* [BR01, HEB04, AB06, HC13].

A *graph* G is a pair (V, E) where V is a finite set, and E is a collection of two-element subsets of V . We refer to elements of V as *vertices*, and elements of E as *edges* respectively. A *hypergraph* is a natural generalisation of a graph, in which edges are arbitrary (not necessarily 2-element) subsets of V . A hypergraph is called *uniform* if its all edges consist of the same number of elements equal to k , we refer to such an object as k -hypergraph. In particular, a 2-hypergraph is simply a graph. We shall denote the number of vertices by N . Moreover, we assume that vertices of the (hyper)graph are labeled by numbers $1, \dots, N$.

Definition 4. With a given (hyper)graph $G = (V, E)$, we associate a quantum state of N qubits in the following way:

$$|G\rangle := \frac{1}{\sqrt{|E|}} \sum_{e \in E} |\psi_e\rangle \quad (3.7)$$

where $|\psi_e\rangle$ is a tensor product of $|1\rangle$ on positions labelled by indices which form the (hyper)edge e and $|0\rangle$ on other positions. We shall refer to such states as *excitation-states*. Figure 3.2 illustrates this definition.

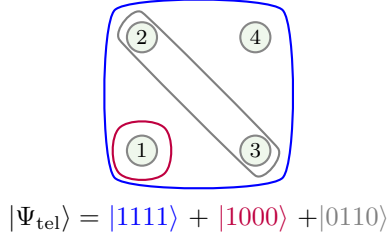


Figure 3.2: So-called telescope state $|\Psi_{\text{tel}}\rangle$ [EOS09] represented as an excitation-state. Related hypergraph is not uniform. Telescope state is simply obtained by reading over all edges. Note that as the empty set cannot form the hyperedge in the hypergraph, the ket $|0000\rangle$ cannot appear in the superposition which forms the excitation state.

There is the following relation between the *automorphisms group* of a complete k -hypergraph and the group of symmetry of related state.

Observation 1. *Consider an excitation-state $|G\rangle$ related to the hypergraph G . The group of symmetries of a state $|G\rangle$ is an automorphisms group of the related hypergraph G .*

3.4 Entanglement properties

In this section, I present results concerning entanglement properties of general excitation states and Dike-like states. In particular, Proposition 6 and Corollary 1 provides separability criterion, while Propositions 5, 7 and 8 provides formulas for exact value of a concurrence for regular graphs. As we shall see, entanglement properties of excitation states reflect the structure of the (hyper)graph.

We begin by recalling the notion of *Concurrence*, denoted by C [HW97]. Concurrence is an entanglement measure, for any two-qubit mixed state ρ its concurrence reads,

$$C(\rho) := \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (3.8)$$

where $\lambda_1, \dots, \lambda_4$ denote square roots of the eigenvalues of a Hermitian matrix $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ ordered decreasingly, where

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$$

is, the so-called, spin-flipped form of ρ . Note that the above definition coincides with Eq. (1.4) for all pure two-qubit states.

The *generalized concurrence* $C_{v|\text{rest}}$ measures the entanglement between the subsystem v and the rest of the system. For pure states, this quantity is determined by the relevant reduced density matrix [HW97], $C_{v|\text{rest}} = 2\sqrt{\det \rho_v}$, hence its values belong to the range $[0, 1]$.

The distribution of bipartite entanglement, measured by the concurrence satisfies so-called monogamy inequality [CKW00, OV06]:

$$C_{v_1|v_2\dots v_N}^2 \geq C_{v_1v_2}^2 + \dots C_{v_1v_N}^2, \quad (3.9)$$

where $v_2 \dots v_N$ are vertices relevant to subsystems.

In further considerations, we restrict our analysis to the k -uniform hypergraphs, and assume our graphs to be connected. Disconnected graphs are relevant for the tensor product of two excitation-states, and hence might be analyzed separately. We use the following notation. By *distance* d between vertices v_0 and w , we understand the minimal number of vertices $v_1, \dots, v_d$, such that there exist (hyper)edges $e_1, \dots, e_d$:

$$v_{i-1}, v_i \in e_i$$

and $v_d = w$. The *degree* d_v of the vertex v is the number of edges on which v is incident. The *joint neighborhood* n_{vw} of two vertices is the number of sets W such that both: $W \cup v$ and $W \cup w$ constitute an edge. Finally, the *section* s_{vw} is the number of edges on which v and w are incident. Notice that for graphs, the joint neighborhood is simply the number of vertices adjacent to both: v and w , while section $s_{vw} = 1$ or $s_{vw} = 0$ depending if vertices v and w are connected. Furthermore, to simplify the notation, for a given excitation state $|G\rangle$ and two selected subsystems v and w , the concurrence of the reduced state ρ_{vw}^G , will be denoted as

$$C_{vw} := C(\rho_{vw}^G). \quad (3.10)$$

I obtained the following separability criterion.

Proposition 5. *For a k -uniform hypergraph, the two-party concurrence C_{vw} reads*

$$C_{vw} = \max\left\{0, \frac{2}{|E|}(n_{vw} - \sqrt{s_{vw}\lambda})\right\} \quad (3.11)$$

where $\lambda = |E| - d_v - d_w + s_{vw}$.

Furthermore, I obtain the following result relating factorisation of the hypergraph with separability of the related state. The *product* of two disjoint hypergraphs $(V_1, E_1) \sqcup (V_2, E_2)$ is a hypergraph $(V_1 \cup V_2, E_1 \sqcup E_2)$ with vertices being the union $V_1 \cup V_2$ of vertices sets and with edges of the following form:

$$E_1 \sqcup E_2 := \sum_{e_1 \in E_1, e_2 \in E_2} e_1 \cup e_2.$$

Proposition 6. *The excitation-state $|G\rangle$ corresponding to a k -hypergraph is separable, $|G\rangle = |G\rangle_{V_1} \otimes |G\rangle_{V_2}$, iff it is a product hypergraph with respect to the division $V_1|V_2$.*

Corollary 1. *(Only for graphs) The excitation-state $|G\rangle$ is separable $|G\rangle = |G\rangle_{V_1} \otimes |G\rangle_{V_2}$ iff the relevant graph G is complete bipartite graph, $G = K_{V_1 V_2}$. In fact, such a state forms a tensor product of two $|W\rangle$ -like states:*

$$|G\rangle = |W\rangle_{V_1} \otimes |W\rangle_{V_2}.$$

We might reformulate separability criteria in terms of symmetries of the Dicke-like states.

Corollary 2. *A Dicke-like state $|D_N^2\rangle_H$ with two excitations is separable across the bipartition $N_1|N_2$ iff its group of symmetry is equal to $\mathcal{S}_{N_1} \times \mathcal{S}_{N_2}$.*

Recall that the graph is called *regular* if the degree d_v is constant for any vertex v . Moreover, it is called *distance-1 regular graph*, if it satisfies an additional condition: If vertices v and w are connected, they are connected with the same number of hyperedges. In other words, the section s_{vw} takes the same values depending on the distance between vertices:

$$s_{vw} = \begin{cases} s & \text{for } d(v, w) = 1, \\ 0 & \text{for } d(v, w) > 1. \end{cases}$$

From Proposition 5, I derive an expression for the concurrence C_{vw} between subsystems corresponding to vertices of a distance-1 regular graph G .

Proposition 7. *For connected, and distance-1 regular graph G the concurrence C_{vw} between two nodes v and w reads,*

$$C_{vw}(G) = \frac{2}{|E|} \cdot \begin{cases} \max\{0, C\} & \text{for } d(v, w) = 1 \\ n_{vw} & \text{for } d(v, w) = 2 \\ 0 & \text{for } d(v, w) > 2 \end{cases} \quad (3.12)$$

where $C = n_{vw} - \sqrt{s(|E| - 2d + s)}$, d is the degree of each node, and s is a section for each adjacent vertices.

Furthermore, an elementary calculations (presented in a detailed way in [BCZ21]) lead to the following result.

Proposition 8. *Square of the generalized concurrence $C_{v|rest}^2$ between the particle v and the rest of the system, expressed as a function of the number of edges $|E|$ and the number of vertices N , reads*

$$C_{v|rest}^2 = 4 \frac{d_v(|E| - d_v)}{|E|^2} = 4 \frac{k(N - k)}{N^2}. \quad (3.13)$$

The second equation is valid under the assumption of the regularity of a k -hypergraph, i.e. the degrees of vertices are the same.

3.5 Examples of Dicke-like states

In Sections 3.2 and 3.3, I presented two similar, but different, constructions of genuinely entangled states: excitation-states $|G\rangle$, related to a graph G , and Dicke-like states, determined by a subgroup. A Dicke-like state can be considered as a special case of the excitation-states, which exhibits a certain symmetry structure. We combine both representations and introduce examples of excitation-states related to the graphs given by highly symmetric objects, such as *regular polygons*, *Platonic solids*, and *regular plane tilings*. Such symmetric objects were already used in various contexts concerning multipartite entanglement including: quantification of entanglement of permutation-symmetric states [Mar11, BDGM15], identification of quantumness of a state [GKG⁺20], search for the maximally entangled symmetric state [AMM10], or general geometrical quantification of entanglement [GKZ12, GBB10] especially among states with imposed symmetries on the roots of Majorana polynomial [MGB⁺10, BDGM15].

Results from Section 3.4, allow us to discuss their entanglement properties. We computed the concurrence in two-partite subsystems for all presented examples. For excitation-states, the entanglement shared between a particular node v and the rest of the system depends only on the number of parties N , as it was shown in Proposition 8. Hence, we may define the *entanglement ratio* Γ_v for the node v as:

$$\Gamma_v := \frac{\sum_{i \neq v} C_{v|i}^2}{C_{v|rest}^2} \in [0, 1], \quad (3.14)$$

which measures the ratio of entanglement shared between particular parties in a two-partite way in comparison to the amount of entanglement shared in

the multi-partite way. Note that concurrence satisfies monogamy inequality (3.9), hence the parameter Γ_v takes values in the range $[0, 1]$.

Example 1. Dicke states. The Dicke states $|D_N^k\rangle$ are excitation-states for complete k -regular hypergraphs on N vertices. Their group of symmetry is the entire permutation group $\mathcal{S}_N$. By Eqs. (3.12) and (3.13), and elementary calculation, we show that the concurrence in two-partite subsystems reads,

$$C_{v|w} = 2 \binom{N}{k}^{-1} \left(\binom{N-2}{k-1} - \sqrt{\binom{N-2}{k} \binom{N-2}{k-2}} \right).$$

The entanglement ratio Γ_v for a given node v is equal to

$$\Gamma_v(D_N^k) = \frac{N-1}{\binom{N-1}{k} \binom{N-1}{k-1}} \left(\binom{N-2}{k-1} - \sqrt{\binom{N-2}{k} \binom{N-2}{k-2}} \right)^2.$$

By Propositions 7 and 8, for the Dicke states $|D_N^k\rangle$ the entanglement ratio at infinite dimension is nonzero,

$$\lim_{N \rightarrow \infty} \Gamma_v(D_N^k) = 2k - 1 - \sqrt{k(k-1)}.$$

In particular, for states related to graphs, $k = 1$, we find

$$\lim_{N \rightarrow \infty} \Gamma_v(D_N^1) = 3 - 2\sqrt{2}.$$

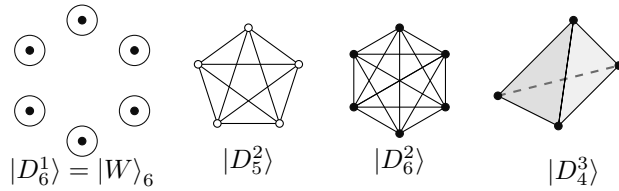


Figure 3.3: Dicke state $|D_N^k\rangle$ are associated with a complete k -regular hypergraph with N vertices. The graph and the corresponding state is completely symmetric, so the labels of the nodes can be omitted. Dicke states $|D_N^2\rangle$ with two excitations are related to complete graphs on N vertices. A tetrahedron represents the Dicke states $|D_4^1\rangle$, $|D_4^2\rangle$ and $|D_4^3\rangle$, depending on whether we consider the vertices, the edges or the faces of the tetrahedron.

Example 2. Cyclic states. The simplest non-trivial subgroup of the permutation group $\mathcal{S}_N$ is a cyclic group $\mathcal{C}_N$. In general, the states $|D_N^k\rangle_{\mathcal{C}_N}$ are *translationally invariant*, i.e. invariant under a cyclic permutation of qubits [VC06, WBG20]. Such a family of states is widely considered in several 1D models in condensed-matter physics, like XY model or the Heisenberg model. Note that such states $|D_N^2\rangle_{\mathcal{C}_N}$ with the number of excitations is equal to $k = 2$ can be constructed as an excitation-state. Indeed, consider the cyclic graph on N vertices. The relevant excitation-state, denoted by $|C_N\rangle$, matches perfectly $|D_N^2\rangle_{\mathcal{C}_N}$. Note that the group of automorphisms of a cyclic graph C_N is a *Dihedral* group $\mathcal{D}_{2N}$, where the lower index stands for the order of the group $|\mathcal{D}_{2N}| = 2N$. Hence we conclude that the group of symmetries of $|D_N^2\rangle_{\mathcal{C}_N} = |C_N\rangle$ is a dihedral group $\mathcal{D}_{2N}$. Systems with dihedral $\mathcal{D}_{2N}$ symmetries were considered in a context of correlation theory of the chemical bond [DMD⁺21]. Molecules invariant under a rotation and inversion were investigated *ibidem*. The concurrence in two-partite subsystems takes the following value:

$$C_{v|w}(C_N) = \begin{cases} 2/N & \text{for } d(v, w) = 2 \\ 0 & \text{otherwise} \end{cases},$$

with a small correction for $N = 4$, where $C_{v|w}^2 = 1$ for distance two vertices. The entanglement ratio Γ_v for a given node reads

$$\Gamma_v(C_N) = \frac{1}{N-2},$$

with the same correction in the case $N = 4$, for which $\Gamma_v = 1$.

Example 3. Platonic states. Platonic solids were used to construct quantum states in various contexts [GGB10, GKZ12, GKG⁺20]. Here, with any Platonic solid we associate two quantum excitation-states, by looking at the edges and faces of related solid. For instance, the tetrahedron is linked to the Dicke state $|D_4^2\rangle$ (by reading edges) and $|D_4^3\rangle$ (by reading faces), see Figure 3.3. We denote such state by $|P^e\rangle_N$ and $|P^f\rangle_N$ respectively. An elementary argument from the representation theory shows that group of symmetries of the Platonic solid, determines the symmetry of related states. In such a way, we constructed states with symmetries $\mathcal{S}_4$, $\mathcal{S}_4 \times \mathcal{S}_2$, and $\mathcal{A}_5 \times \mathcal{S}_2$ respectively. In particular, in the case of dodecahedron the alternating symmetry A_5 is observed, which is not easy to achieve. The concurrence C_{12} and the entanglement ratio Γ_v for two-qubit systems obtained by partial trace of $(N-2)$ subsystems of distance-two of Platonic states determined by the edges of the solid is compared below.

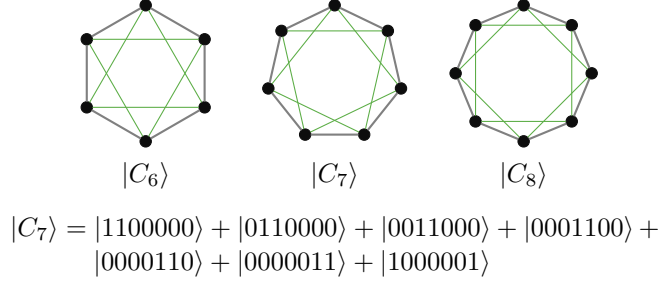


Figure 3.4: Three cycle graphs: C_6 , C_7 and C_8 indicated by gray edges. The form of the state $|C_7\rangle$, corresponding to C_7 , is listed below. Entanglement on two-parties subsystem is vanishing except of parties of distance two. Non-vanishing entanglement is indicated by green lines connecting relevant parties.

	$ P^e\rangle_4$	$ P^e\rangle_6$	$ P^e\rangle_8$	$ P^e\rangle_{12}$	$ P^e\rangle_{20}$
Concurrence C_{12}	0.333	0.667	0.333	0.133	0.067
Ent. Ratio Γ_v	0.333	0.500	0.444	0.160	0.074

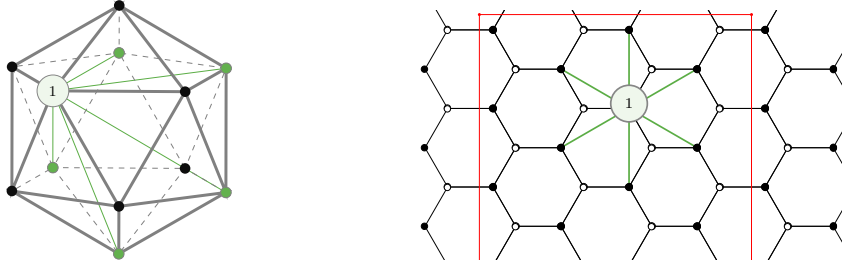


Figure 3.5: On the left, Icosahedron, one of five Platonic solids. Its edges (indicated by gray) determine the excitation state $|P^e\rangle_{20}$ on 20 qubits. On the right, hexagonal tiling of a plane. The red rectangle indicates a finite region for which the boundary left-right and up-down is glued together. Such a region determines the excitation state $|H\rangle_{20}$ on 20 qubits. On both pictures, for a chosen node 1, the entanglement in the two-party subsystem C_{1v} is vanishing except for subsystems corresponding to distance-two nodes, indicated by green lines and green nodes.

Example 4. Regular m -polytope families. There are three natural generalizations of Platonic solids in higher dimensions: the self-dual m -simplices

and m -hypercubes with dual m -orthoplexes. Each of these polytopes provides a set of k -uniform hypergraphs for $1 \leq k \leq m-1$ defined by the set of their k -dimensional hyperedges. We denote the states corresponding to the k -dimensional hyperedges of m -simplex as $|S_m^k\rangle$ and analogously $|B_{2m}^k\rangle$ for m -hypercubes and $|O_{2m}^k\rangle$ for m -orthoplexes, where the lower index stands for the number of subsystems N in the state, note that $N = m, 2^m, 2m$ for m -simplex, m -hypercube, and m -orthoplexe respectively. The states related to the m -simplices are equivalent to the Dicke states $|S_m^k\rangle = |D_m^k\rangle$ and hence exhibit the full symmetry. On the other hand, the symmetry group of $|B_{2m}^k\rangle$ and $|O_{2m}^k\rangle$ is given by the non-trivial hyperoctahedral group B_m . The states $|O_{2m}^{m-1}\rangle$ are separable with respect to the partition $12|34|\dots|(2m-1)2m$, with each pair of vertices being 2-distance or, in other words, lying on a common diagonal of the related m -orthoplex. Furthermore, one can easily calculate the concurrence C_{vw} for the states $|O_{2m}^2\rangle$ connected to the 2-edges of the m -orthoplex,

$$C_{vw}(O_{2m}^2) = \begin{cases} \max\{0, \frac{\sqrt{2m^2-4m+3}-2m+4}{2m-2m^2}\} & \text{for } d(v, w) = 1 \\ 2/m & \text{for } d(v, w) = 2 \\ 0 & \text{for } d(v, w) > 2, \end{cases} \quad (3.15)$$

and similarly the entanglement ratio

$$\Gamma_v(O_{2m}^2) = \begin{cases} 1/(m-1) & \text{for } m \leq 3 \\ \frac{6m^2-4m(\sqrt{2m^2-6m+5}+5)+8\sqrt{2m^2-6m+5}+19}{2(m-1)^2}, & \text{for } m > 3, \end{cases} \quad (3.16)$$

where the first case holds for $m \leq 3$ and the second one for $m > 3$. With Propositions 7 and 8 at hand, one may show that the entanglement ratio for the states $|O_{2m}^2\rangle$ converges to a nonzero value,

$$\lim_{n \rightarrow \infty} \Gamma_v(O_{2m}^2) = 3 - 2\sqrt{2} \approx 0.1716. \quad (3.17)$$

The situation is simpler for the hypercubic states $|B_{2m}^2\rangle$, where the concurrence occurs only between the distance-2 vertices, $C_{12}(B_{2m}^2) = 2^{3-m}/m$, while the entanglement ratio reads

$$\Gamma_v(B_{2m}^2) = \frac{4(m-1)}{(2^m-2)m},$$

which asymptotically tends to zero.

Example 5. Plane regular tilings. Regular and semi-regular tessellation of the plane are yet another highly symmetrical objects. Regular tilings correspond to Dicke-like states. The group of symmetry is given by a relevant wallpaper group, restricted to the chosen size of the tiling. Figure 3.5 presents the hexagonal tiling of the plane and relevant excitation-state $|H_N\rangle$. For such a tiling, the concurrence in bipartite subsystems takes the following value:

$$C_{v|w}(H_N) = \begin{cases} 2/N & \text{for } d(v, w) = 2 \\ 0 & \text{otherwise} \end{cases},$$

for the tiling restricted to N nodes (minimal size of a cut is 3×3). Furthermore, the entanglement ratio Γ_v for a given node v reads,

$$\Gamma_v(H_N) = \frac{4}{3} \frac{1}{N-2}.$$

Although the value of the two-partite concurrence $C_{v|w}$ is the same as in the cyclic case, the parameter Γ_v takes a larger value.

3.6 Quantum circuits

We present below a quantum circuit efficiently transforming a separable tensor product state into the excitations-state, related to any graph G . Presented construction was inspired by similar circuits for Dicke states recently developed in [PBcv10, BE19]. As we showed in [BCZ21], our scheme uses between $\sim 4|V|$ and $\sim 10|E|$ CNOT gates, depending on the structure of a graph.

First step. Choose arbitrary vertex v , and consider all adjacent vertices $v_1, \dots, v_d$, where d is the degree of v . Suppose that each v_i is related to the i th particle, while v is related to $d+1$ particle. For $i = 1, \dots, d-2$, we consecutively apply the following three-qubit gates $U_{\{i,d,d+1\}}^{(1)}$ on parties $i, d, d+1$:

$$\begin{aligned} \frac{1}{\sqrt{i}} (|101\rangle + \sqrt{i-1}|011\rangle) &\mapsto |011\rangle \\ |110\rangle &\mapsto |110\rangle \\ |100\rangle &\mapsto |100\rangle \\ |010\rangle &\mapsto |010\rangle \\ |001\rangle &\mapsto |001\rangle \\ |000\rangle &\mapsto |000\rangle, \end{aligned} \tag{3.18}$$

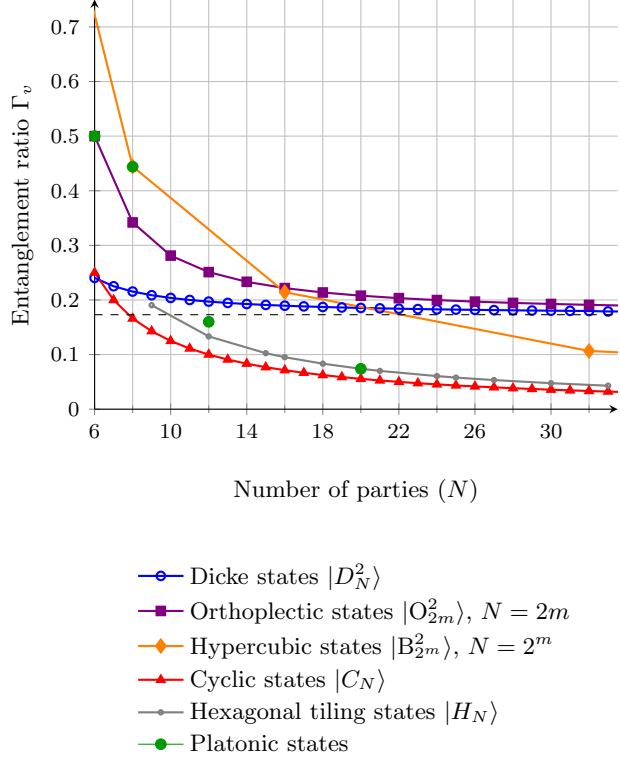


Figure 3.6: Distribution of entanglement for Dicke-like states with $k = 2$ excitations is compared. The distribution of entanglement depends on the density of related graph. For states related to dense graphs, $|D_N^2\rangle$ and $|O_{2m}^2\rangle$, the ratio Γ_v tends to a nonzero value $3 - 2\sqrt{2} \approx 0.17$ (indicated by the horizontal dashed line). For families of states in which the local degree does not scale with the size of a graph ($|B_{2^m}^2\rangle$, $|C_N\rangle$, $|H_N\rangle$), the complexity of Γ_v is $\mathcal{O}(1/N)$.

where the action on the remaining subspace is arbitrary. Application of $U_{\{i,d,d+1\}}^{(1)}$ operation is relevant to the graphical operation of deleting an edge $e = \{i, d+1\}$, see Figure 3.7.

Secondly, we apply the following three-qubit operator $U_{\{d-1,d,d+1\}}^{(2)}$ on

parties $d-1, d, d+1$

$$\begin{aligned}
\frac{1}{\sqrt{d-1}} (|101\rangle + \sqrt{d-2} |001\rangle) &\mapsto |001\rangle \\
|011\rangle &\mapsto |011\rangle \\
|100\rangle &\mapsto |100\rangle \\
|010\rangle &\mapsto |010\rangle \\
|000\rangle &\mapsto |000\rangle,
\end{aligned} \tag{3.19}$$

where the action on the remaining subspace is arbitrary. This is related to the graphical operation of deleting an edge $e = \{d-1, d+1\}$, see Figure 3.7.

Finally we apply the following two-qubit operator $U_{\{d, d+1\}}^{(3)}$ on parties $d, d+1$:

$$\begin{aligned}
\frac{1}{\sqrt{d}} (|11\rangle + \sqrt{d-1} |01\rangle) &\mapsto |01\rangle \\
|00\rangle &\mapsto |00\rangle \\
|10\rangle &\mapsto |10\rangle
\end{aligned} \tag{3.20}$$

where the action on the remaining subspace is uniquely determined. This is relevant to deleting the only remaining edge: $e = \{d, d+1\}$, see Figure 3.7. There is a simple logic behind those three operations. We combine all terms having excitations on position $d+1$ into a single term with an excitation on this position. After applying these operations, the state takes the form of the superposition:

$$\frac{1}{\sqrt{|E|}} \left(\sqrt{d} |1\rangle_v \otimes |0 \dots 0\rangle + |0\rangle_v \otimes \sum_{e \in E \setminus v} |\psi_e\rangle \right),$$

where $E \setminus v$ denotes a set of edges which do not contain vertex v .

Second and the next steps. Consider the graph $G' = (V \setminus v, E \setminus v)$ with deleted vertex v and all adjacent edges. We repeat the procedure from the first step for an arbitrary vertex w from the graph G' . We repeat this procedure further, for $G'' = (V \setminus v, w; E \setminus v, w)$, until we delete $N-1$ vertices, which fully separates the initial graph.

Final step. After applying presented procedure iteratively $N-1$ times, the state takes the form:

$$\frac{1}{\sqrt{|E|}} \sum_{v \in V} \sqrt{d'_v} |1\rangle_v \otimes |0 \dots 0\rangle_{v^c},$$

where d'_v denotes the degree of the vertex v of the graph reduced according to the procedure. The state above is similar to the state

$$|W_N\rangle := \frac{1}{\sqrt{N}} (|10\cdots 0\rangle + \cdots + |0\cdots 01\rangle).$$

The separation of $|W_N\rangle$ state is a well-known procedure [JHJ⁺08], and might be obtained by performing two-qubit gates $U_{1i}^{(4)}$:

$$\begin{aligned} \sqrt{d'_{v_1} + \cdots + d'_{v_{i-1}}} |10\rangle + \sqrt{d'_{v_i}} |01\rangle &\longmapsto \sqrt{d'_{v_1} + \cdots + d'_{v_i}} |10\rangle \\ |00\rangle &\longmapsto |00\rangle \end{aligned}$$

on particles 1 and i . We refer to [BCZ21] for the precise estimation of the computational cost for the presented procedure.

We simulated the simplest nontrivial cyclic state $|C_5\rangle$, in order to demonstrate the viability of the provided construction. We refer to [BCZ21] for a detailed description. The overall circuit requires 22 CNOT operations, where the topology of the quantum computer is not taken into account. Such a circuit can be realised on the state-of-the-art 5-qubit quantum computers provided by IBM – linear-topology Santiago and Athens with quantum volumes (QV) of 32 and Vigo with QV of 16 with T topology. In total we used more than 740 000 samples over all three computers, which gives distribution with significant values for all expected computational states proceeding from cyclic permutations of the state $|00011\rangle$ with the probability 0.487 of finding the system in one of them, see Figure 3.8. Further analyses and the comparison of obtained results to a model of noise might be found in [BCZ21].

3.7 Hamiltonians

The Dicke states were introduced as a ground states of the Hamiltonians with a well-defined number of excitations. The excitation-states can be also considered as ground states of analogous Hamiltonians, with the same property of a well defined number of excitations. The single-mode Dicke Hamiltonian (known also as Tavis-Cummings model or generalized Jaynes-Cummings model) has the following form [EB03, Gar11]:

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}} \underbrace{(a^\dagger + a)(J_- + J_+)}_{\text{interaction}},$$

where J_α are the collective operators:

$$J_z \equiv \sum_{i=1}^N \sigma_z^i, \quad J_\pm \equiv \sum_{i=1}^N \sigma_\pm^i.$$

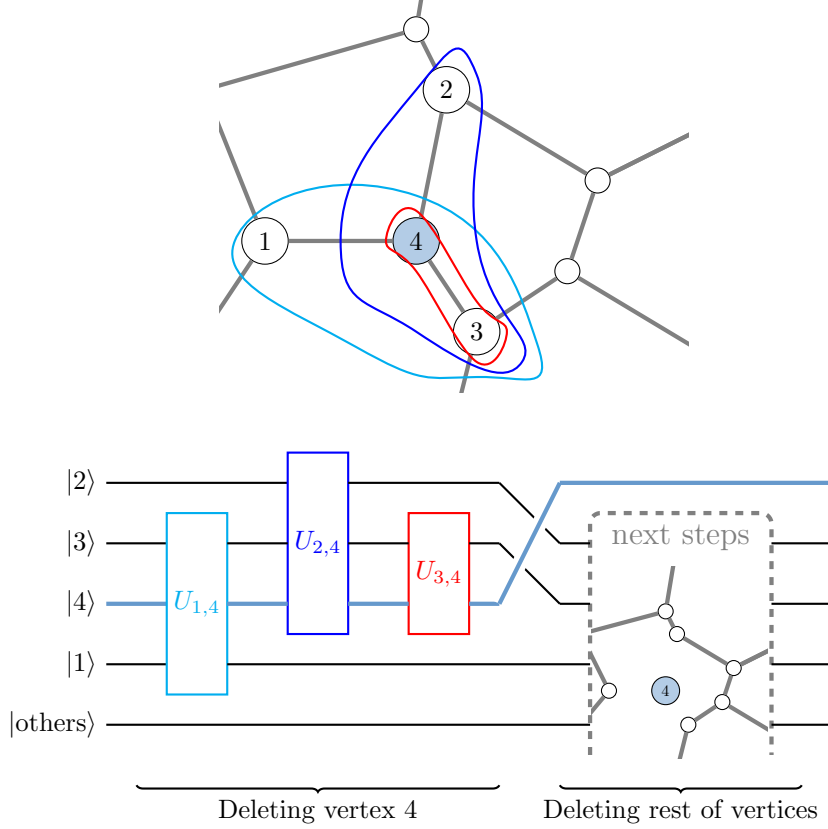


Figure 3.7: Procedure sketched for vertex 4 of degree $d = 3$. Three operations described in the first step of the procedure: firstly, we apply unitary three-qubit operations $U_{\{1,3,4\}}^{(1)}$ on parties 1, 3, 4, secondly $U_{\{2,3,4\}}^{(2)}$ on parties 2, 3, 4, and finally the two-qubit gate $U_{\{3,4\}}^{(3)}$ on parties 3, 4. Resulting state is separable with respect to the 4-th particle. This is graphically represented by deleting all adjacent edges. In the consecutive steps, we apply this procedure iteratively to the remaining vertices.

By fixing interaction, $\lambda = 0$, the coupling term vanishes. Thus eigenstates have the simple tensor product form with a factor representing the Fock states of the field and the other factor as an eigenstate of the collective angular momentum operator J_z . Operator J_z has the following degenerated eigenvalues: $-N/2, -N/2 + 1, \dots, N/2$. The further partition of its eigenspaces might be performed by the square of the total angular momentum operator J^2 , which can be written as $J^2 \equiv J_+ J_- + J_z(J_z + \mathbb{I})$, in terms

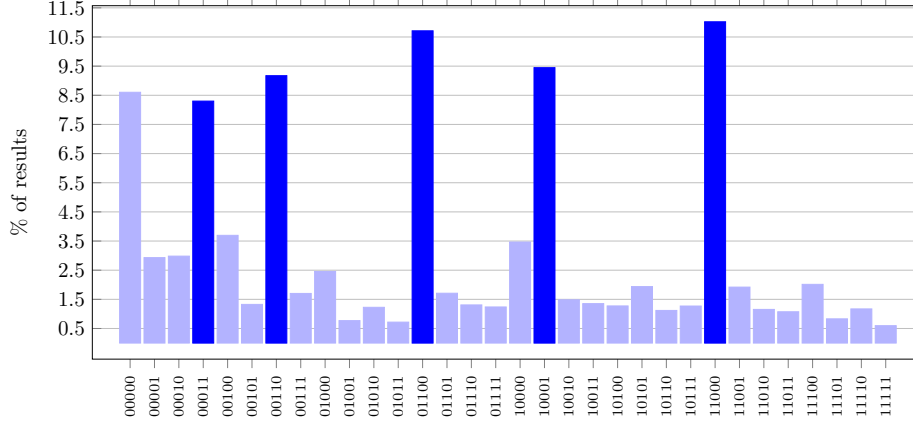


Figure 3.8: Mean results obtained over total 740 000 samples acquired on IBM Vigo, Athens and Santiago Computers. Prominent peaks at all cyclic permutations of $|00011\rangle$ are plotted in dark, with an additional significant peak at $|00000\rangle$. Due to effects of decoherence we observe a background of all remaining states represented by light bars.

of the raising and lowering operators. The Dicke states $|j, m\rangle_N$ (in the more classical notation) are eigenstates of both operators J_z and J^2 :

$$\begin{aligned} J_z |j, m\rangle_N &= m |j, m\rangle_N, \\ J^2 |j, m\rangle_N &= j(j+1) |j, m\rangle_N \end{aligned}$$

with $m = -j, -j+1, \dots, j$ corresponding to the number of excitations, while $j = 1/2, 3/2, \dots, N/2$ (for N odd) and $j = 0, 1, \dots, N/2$ (for N even) related to the cooperation number. In general, atomic configurations for $N > 2$ contain entanglement [WM02] and are degenerated for $j < N/2$ [Gar11]. By taking the maximal value $j = N/2$, we obtain fully symmetric Dicke states $|D_N^k\rangle$,

$$|D_N^k\rangle := |N/2, k + N/2\rangle_N.$$

Independently of the number of excitations k , the Dicke states $|D_N^k\rangle$ are eigenstates of the operator $J_+ J_-$. Such an operator can be rewritten as a fully symmetric, two-body Hamiltonian,

$$H := J_+ J_- = \sum_{i=1}^N \sigma_-^{(i)} \sigma_+^{(i)} + \sum_{i \neq j}^N \sigma_-^{(i)} \sigma_+^{(j)},$$

in which any two parties might exchange excitations. Note that the Dicke states $|D_N^k\rangle$ are eigenvectors corresponding to the non-degenerated maximal

eigenvalue $k(N + 1 - k)$. In various physical scenarios, the particles are not distributed symmetrically as some interactions are more likely to occur. For instance, the two-body Hamiltonian of the system of N particles with spin-1/2 considered in [HG16] generalizes the Heisenberg model.

We shall see that the excitation-states, and hence Dicke-like states, are ground states for Hamiltonians with an interaction term analogous to $J_+ J_-$. As in the Dicke case they have a well-defined excitation number. We restrict ourselves to the subspace of the operator J_z relevant to two-excitations, i.e. we set $k = 2$. For any graph G , consider the four-body Hamiltonian, where any pair of edges might exchange excitations:

$$H_G := J_+^G J_-^G, \quad \text{where} \quad J_\pm^G \equiv \sum_{vv' \in E} \sigma_\pm^{(v)} \sigma_\pm^{(v')}. \quad (3.21)$$

Notice the similarity between operators J_+^G, J_-^G and $J_+ J_-$. The maximal eigenvalue of the Hamiltonian H_G is equal to the squared number of edges $|E|^2$. This energy level is non-degenerated and the corresponding eigenstate is exactly given by the excitation-state $|G\rangle$. Indeed, the Hamiltonian H_G is simply a sum of $|E|^2$ projections, which justifies the bound of the energy. The state, $|G\rangle$ saturates this bound, contrary to any other state with specified number of excitations, $k = 2$.

Furthermore, the excitation-state $|G\rangle$ might be seen as a ground state of the three-body interaction Hamiltonian,

$$H_G = \sum_{v \in V} \sum_{\substack{vv' \in E \\ v'v'' \in E}} \sigma_-^{(v)} (\sigma_- \sigma_+)^{(v')} \sigma_+^{(v'')}. \quad (3.22)$$

Above Hamiltonian describes situation, in which the pairs of excitations might be exchanged only between connected edges. The maximal eigenvalue is equal to $\binom{d}{2}|V|$, where d denotes the degree of a vertex. In analogy to the Hamiltonian (3.21) the maximal eigenvalue is not degenerated and the state $|G\rangle$ forms the relevant eigenvector. Observe that the Dicke state $|D_N^2\rangle$ is a ground state of the Hamiltonian associated with the fully connected graph G on N vertices.

It is worth mentioning that $\sigma_- \sigma_+ = (\mathbb{I} - \sigma_z)/2 = |1\rangle\langle 1|$ and the Hamiltonian can be written in an alternative way:

$$H_G = \sum_{v \in V} \sum_{\substack{vv' \in E \\ v'v'' \in E}} \sigma_-^{(v)} |1\rangle\langle 1|^{(v')} \sigma_+^{(v'')}. \quad (3.23)$$

In this form the term $\sigma_- \sigma_+$ might be interpreted as conditional hopping interaction - if the site v' is occupied, the hopping from v to v'' is effected,

otherwise no interaction happens. Such a behaviour is very much akin to what is called the quantum transistor. One of the simplest models of such a quantum transistor [LKAZ18] involve left and right qubit and a two qubit gate, an interaction very similar to observed here.

Furthermore, the excitation-state $|G\rangle$ related to the k -uniform hypergraphs G is described as the singular ground state of the Hamiltonian with $2k$ -body interaction, $H = J_+^G J_-^G$. The excitation-state $|G\rangle$ is a ground state of the Hamiltonian (3.21), where the operators $J_\pm^G$ are defined as

$$J_\pm^G \equiv \sum_{\{v^1, \dots, v^k\} \in E} \sigma_\pm^{(v^1)} \cdots \sigma_\pm^{(v^k)}.$$

In summary, as the Dicke states $|D_N^m\rangle$ are uniquely determined as eigenvectors of operators J_z and the two-body interaction Hamiltonian $H = J_+ J_-$, the excitation-states are also eigenvectors of an operator J_z , nevertheless the further division of its subspaces differs from the one corresponding to the Dicke states. For any excitation state $|G\rangle$ one may construct a $(k+1)$ -body interaction Hamiltonian with the state $|G\rangle$ being its non-degenerated ground state. Notice that the construction of the Hamiltonian depends on the number of excitations.

3.8 Conclusions

In this chapter, I introduced the notion of H -symmetric states, where H is an arbitrary subgroup of permutation group (Definition 2). I show a natural example of H symmetric state for any subgroup of symmetric group $H < \mathcal{S}_N$ (Proposition 4). Note that any group H might be realized as a subgroup of the symmetric group S_N for N arbitrary large. Furthermore, I introduce two related families of states with remarkable symmetric properties: in Definition 4 excitation-states $|G\rangle$, related to a graph G , and in Definition 3 Dicke-like states, determined by an embedding of a subgroup $H < \mathcal{S}_N$. Section 3.4 presents results concerning entanglement properties of general excitation states and Dike-like states (Propositions 5 and 6 and Corollary 1 and 2). With some additional assumptions on the regularity of the graph, I obtained the form of 2-partite concurrence (Proposition 7), and generalized concurrence (Proposition 8) of the related excitation states. In particular, I have shown that:

1. Generalized concurrence $C_{v|\text{rest}}$ describing entanglement between given subsystem v and the rest of the system depends only on the total number of nodes and uniformity of the graph.

2. Concurrence C_{vw} between nodes v and w is positive only for distance-two nodes, and in states related to almost complete hypergraphs, i.e. $|E| \sim \binom{N}{k}$, also for distance-one nodes.
3. Concurrence C_{vw} between nodes of distance two is proportional to the number of shared neighbors.

Furthermore, Section 3.5 introduces several examples of excitation and Dicke-like states of given symmetries: cyclic, dihedral, symmetry of Platonic solids. Figure 3.6 compares entanglement properties of introduced families of states. Moreover, I show two different methods of constructing introduced families of states: Section 3.7 presents excitation states as ground states of Hamiltonians with 3-body interactions, while Section 3.6 proposes relevant quantum circuits. The complexity of introduced circuits is comparable with the complexity of quantum circuits proposed for Dicke states known in the literature. A simple example of a cyclic state of five qubits is successfully simulated on available quantum computers: IBM - Santiago and Athens – see Figure 3.8.

Chapter 4

Absolutely maximally entangled states

Absolutely Maximally Entangled (AME) states of a multipartite quantum system are maximally entangled for every bipartition of the system. AME states are special cases of *k-uniform* states characterized by the property that all of their reductions to k parties are maximally mixed. Both classes of states are crucial resources for various quantum information protocols. In this chapter, I briefly recall correspondence between AME states and classical combinatorial designs. I focus my attention on the different linear structures of classical designs that affect the structure of the related AME and k -uniform states.

4.1 AME and k -uniform states

A multipartite quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties with a local dimension d each is called AME, if it is maximally entangled for every of its bipartition, i.e. the partial trace

$$\text{tr}_S |\psi\rangle \langle \psi| \propto \text{Id}, \quad (4.1)$$

for any subsystem S of $|S| = \lceil N/2 \rceil$ parties [Sco04]. AME states of N parties with the local dimension d are denoted as $\text{AME}(N, d)$. The class of AME states is being applied in several branches of quantum information theory: in quantum secret sharing protocols [HCR⁺12], in parallel open-destination teleportation [HC13], in holographic quantum error correcting codes [PYHP15, MFG⁺20], among many others. The state $\text{AME}(N, d)$ allows one to construct a *pure quantum error correction code* (pure QECC), which saturates the Singleton bound [HCR⁺12]. Particular attention is paid

to AME states of an even number of parties, those are equivalent to notions as perfect tensors [PYHP15] or multiunitary matrices [GAR⁺15]. AME states are special type of k -uniform states.

Definition 5. A quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N is k -uniform if its reduced density matrices are maximally mixed, i.e.

$$\rho_S(\psi) := \text{tr}_{\bar{S}} |\psi\rangle \langle \psi| \propto \text{Id}$$

for any subsystem S of k parties ($|\bar{S}|=k$) and the complementary subsystem $\bar{S}$. It is known that the uniformity k cannot exceed $\lfloor N/2 \rfloor$ [Sco04]. States which saturate this bound, i.e. $\lfloor N/2 \rfloor$ -uniform states, are called AME states.

Example 6. Greenberger–Horne–Zeilinger (GHZ) state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle \right)$$

is a 1-uniform state. Its natural generalization to N parties with d distinguishable energy levels:

$$|\text{GHZ}_d^N\rangle = \frac{1}{\sqrt{d}} \left(|0 \cdots 0\rangle + \cdots + |d-1 \cdots d-1\rangle \right)$$

is also 1-uniform.

Example 7. The following state of four qutrits

$$\begin{aligned} |\text{AME}(4,3)\rangle = \frac{1}{3} \Big(& |0000\rangle + |0121\rangle + |0212\rangle + \\ & |1110\rangle + |1201\rangle + |1022\rangle + \\ & |2220\rangle + |2011\rangle + |2102\rangle \Big) \end{aligned}$$

is 2-uniform, so it is an AME state of 4 qutrits [Hel13]. It reveals larger entanglement properties than the corresponding $|\text{GHZ}_3^4\rangle$ state of four qutrits.

Note that both states presented in Examples 6 and 7 might be written in simple closed formulas:

$$\begin{aligned} |\text{GHZ}_d^N\rangle &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i, \dots, i\rangle, \\ |\text{AME}(4,d)\rangle &= \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j\rangle. \end{aligned} \tag{4.2}$$

where the summation is understood modulo d .

AME(N, d) states are maximizing entanglement properties among all N -parties states, each with d levels [HCR⁺12]. There is no general construction of AME(N, d) state, for an arbitrary number of parties N and an arbitrary number of energy levels d . Surprisingly, AME states do not exist for any numbers N and d . Indeed, it was first observed by Higuchi and Sudbery in their study of bipartite entanglement that AME state of four qubits does not exist [HS00]. Until today, more of such negative results are known [HGS17, HESG18].

4.2 Orthogonal arrays

Combinatorial mathematics deals with the existence and properties of designs composed of elements of a finite set and arranged with certain symmetry and balance [Ce07]. A simple example of a combinatorial design is given by a single Orthogonal array [ASH99]. Orthogonal array is a combinatorial arrangements, tables with entries satisfying given orthogonal properties [Kos96]. A tight connection between OAs and (quantum) error-correcting codes [ASH99], and maximally entangled states [GZ14] brought a new life for these combinatorial objects.

As we already presented in Section 2.4, an orthogonal array OA(r, N, d, k) is a table composed by r rows, N columns with entries taken from $0, \dots, d-1$ in such a way that each subset of k columns contains all possible combinations of symbols with the same amount of repetitions, see Figure 4.1. The number k is known as the *strength* of the OA. We assume that OAs are simple, i.e. all rows are distinct. Notice, that the minimal number of rows in OA is equal to $r = d^k$, OAs saturating this bound are called of *Index Unity*. In general *the index* of the OA

$$\lambda = \frac{r}{d^k}$$

is always a natural number. An OA(r, N, d, k) is said to be an *irredundant* orthogonal array (IrOA) if in any subset of $N - k$ columns all combinations of symbols are different [GZ14].

Usually, an orthogonal array is called *linear* if the set of r rows form a vector space over the Galois field $\text{GF}(d)$. The linearity condition is equivalent to the following: for each two rows R_1 and R_2 of an OA, and any two elements $c_1, c_2 \in \text{GF}(d)$,

$$c_1 R_1 + c_2 R_2 \tag{4.3}$$

is also a row in the OA. Such an OA can be represented by a $N \times s$ *generator matrix*, whose rows form a basis of aforementioned vector space, see Figure 4.1. Up to the isomorphism of a linear space, the generator matrix G can always be written in the standard form $G = [\text{Id}_s | A]$, where A is an $(N - s) \times s$ matrix [RGRA17].

On the one hand, the generator matrix greatly suppresses the notation of an OA but also indicates the internal structure of OA. In fact, except for linear OAs, there is another class of OAs that might be written in a form of a generator matrix. Indeed, in place of the finite field $\text{GF}(d)$, one may consider a finite ring $\mathcal{R}$ of order $|\mathcal{R}| = d$. In such a way all linear combinations of rows of the generator matrix forms a *module* over the ring $\mathcal{R}$. A practical selection for a ring $\mathcal{R}$ is cyclic group $\mathbb{Z}_d$ or, more generally, simple sum of cyclic groups $\mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_s}$. In fact those are the only possibilities for commutative rings $\mathcal{R}$.

As we already have show in Chapter 2, OAs might be successfully used to construct m -resistant states of N -qudits, for which the entanglement of the reduced state of $N - m$ subsystems is fragile for the loss of any additional subsystem. In fact, states constructed in this method exhibit more exceptional property of being k -uniform, as we describe below.

For any OA (r, N, d, k) one may associate a pure quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ by reading all rows of OA and creating a superposition of r terms in the computational basis [GZ14, ASH99], see Figure 4.2. Moreover, for IrOAa of a strength k , the corresponding quantum state is k -uniform [GZ14, ASH99].

Consider now a linear OAs over the module $\mathcal{R}$, $|\mathcal{R}| = d$ with the $N \times s$ generator matrix of the form $G = [\text{Id}_s | A]$, where $A = (a_{ij})$ is an $(N - s) \times s$ array with elements from $\mathcal{R}$. The corresponding quantum state might be conveniently presented as follows:

$$|\psi\rangle := \frac{1}{\sqrt{d^s}} \sum_{\vec{v} \in \mathcal{R}^{\otimes s}} |G\vec{v}\rangle \quad (4.4)$$

where the multiplication and addition hidden inside an expression $G\vec{v}$ are determined by the structure of a ring $\mathcal{R}$, and vector $\vec{v} \in \mathcal{R}^{\otimes s}$ has exactly d^s elements [RGRA17].

Example 8. Consider the following generator matrix

$$G = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right], \quad (4.5)$$

0 0 0 0		0 0 0 0
0 1 1 1		0 1 1 1
0 2 2 2		0 2 2 2
0 3 3 3		0 3 3 3
1 0 1 2		1 0 1 2
1 1 2 3		1 1 0 3
1 2 3 0		1 2 3 0
1 3 0 1		1 3 2 1
2 0 2 0	$\xleftarrow[\mathbb{Z}_4]{\text{over}} \left[\begin{array}{cc cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right] \xrightarrow[\text{GF}(4)]{\text{over}}$	2 0 2 3
2 1 3 1		2 1 3 2
2 2 0 2		2 2 0 1
2 3 1 3		2 3 1 0
3 0 3 2		3 0 3 1
3 1 0 3		3 1 2 0
3 2 1 0		3 2 1 3
3 3 2 1		3 3 0 2
OA(16,4,3,1)		OA(16,4,3,2)

Figure 4.1: In the middle, a generator matrix of an orthogonal array $\text{OA}(d^2, 4, d, k)$ for a prime power dimension $= 4d = 2^2$. Rows of an OA are given by all linear combinations $c_1(1, 0, 1, 2) + c_2(0, 1, 1, 1)$ for $c_1, c_2 \in \text{GF}(4)$ and $\mathbb{Z}_4$ respectively. Note a difference in OAs strength, $k = 1$ and $k = 2$ for $\text{GF}(4)$ and $\mathbb{Z}_4$ respectively. Indeed, pairs of symbols 00 and 02 repeats twice on the second and fourth column in $\text{OA}(d^2, 4, d, 1)$, while combinations 01 and 03 do not appear in those two columns. Both OAs are irredundant.

with entries from the ring $\mathcal{R} = \mathbb{Z}_d$. Eq. (4.4) associates the quantum state

$$|\psi\rangle = \frac{1}{d} \sum_{i_1, i_2=0}^{d-1} |i_1\rangle |i_2\rangle |i_1 + i_2\rangle |2i_1 + i_2\rangle, \quad (4.6)$$

where addition and multiplication are considered over the ring $\mathbb{Z}_d$. For any odd dimension d ($2 \nmid d$), the corresponding state $|\psi\rangle \in \mathcal{H}_d^{\otimes 4}$ is an AME(4, d) state, as the corresponding $\text{OA}(d^2, 4, d, 2)$ has strength $k = 2$, see Figure 4.2.

$$\begin{array}{ccc}
\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right] & \xrightarrow[\text{GF}(3)]{\text{over}} & \begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 2 & 0 \\ 2 & 1 \\ 2 & 2 \end{array} \begin{array}{cc} 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 1 & 2 \\ 2 & 0 \\ 0 & 1 \\ 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{array} & \xrightarrow[\text{AME}(4,3)\rangle_{\infty}]{\text{corresponding state}} & \begin{array}{l} |0000\rangle + \\ |0111\rangle + \\ |0222\rangle + \\ |1012\rangle + \\ |1120\rangle + \\ |1201\rangle + \\ |2021\rangle + \\ |2102\rangle + \\ |2210\rangle \end{array}
\end{array}$$

Figure 4.2: The orthogonal array of unity index OA $(9, 4, 3, 2)$ on the left and repeated in the center. Each subset consisting of two columns contains all possible combinations of symbols. Here, two such subsets are highlighted. The relevant quantum state is obtained as a superposition of states corresponding to consecutive rows of the array.

4.3 Different linear structures

For dimension $d = 4$, Figure 4.1 presents two OAs corresponding to the ring structures $\mathcal{R} = \text{GF}(4), \mathbb{Z}_4$. Two quantum states $|\psi\rangle_{\text{GF}(4)}$, and $|\psi\rangle_{\mathbb{Z}_4}$ might be constructed simply by reading consecutive rows of respective OAs. As the corresponding OAs has different strength, quantum states $|\psi\rangle_{\text{GF}(4)}$, and $|\psi\rangle_{\mathbb{Z}_4}$ are 1-, and 2-uniform respectively, see Figure 4.1. In general, states obtained by the linear action of a Galois field $\text{GF}(d)$ exhibit larger uniformity k then those based on the other ring structures $\mathcal{R}$, $|\mathcal{R}| = d$.

Once more, consider the generator matrix G defined by Eq. (4.5). In dimension $d = 9$, we may consider three different ring structures: $\mathcal{R} = \text{GF}(9), \mathbb{Z}_9$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. The ring $\mathbb{Z}_9$ consists of nine numbers $0, \dots, 8$ with the usual multiplication and addition modulo 9. We denote the related quantum state (4.6) by $|\psi\rangle_{\mathbb{Z}_9}$.

Elements of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are given by pairs of numbers (a, b) , where $a, b = 0, 1, 2$, while the multiplication and addition is considered on respective positions

$$\begin{aligned}
(a, b) + (a', b') &= (a + a', b + b'), \\
(a, b) \cdot (a', b') &= (a \cdot a', b \cdot b'),
\end{aligned}$$

modulo 3. Note that a pair (a, b) might be canonically associated with the number $3a + b \in \mathbb{Z}_9$, which we further use for linking with a pure quantum

	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
2	0	2	1	$2x$	$2x+2$	$2x+1$	x	$x+2$	$x+1$
x	0	x	$2x$	2	$x+2$	$2x+2$	1	$x+1$	$2x+1$
$x+1$	0	$x+1$	$2x+2$	$x+2$	$2x$	1	$2x+1$	2	x
$x+2$	0	$x+2$	$2x+1$	$2x+2$	1	x	$x+1$	$2x$	2
$2x$	0	$2x$	x	1	$2x+1$	$x+1$	2	$2x+2$	$x+2$
$2x+1$	0	$2x+1$	$x+2$	$x+1$	2	$2x$	$2x+2$	x	1
$2x+2$	0	$2x+2$	$x+1$	$2x+1$	x	2	$x+2$	1	$2x$

Figure 4.3: Elements of the Galois field $\text{GF}(9)$ are given by nine polynomials $ax + b$, where $a, b = 0, 1, 2$. Their multiplication is considered modulo an irreducible polynomial $x^2 + 1$. Exact structure of a multiplication is presented in the table, while the addition structure is the same as for the ring $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Note that there are no zero divisors in $\text{GF}(9)$, i.e pairs of non-zero elements which product is zero. Each polynomial $ax + b$ might be later associated with a number $3a + b = 0, \dots, 8$.

state (4.6) denoted by $|\psi\rangle_{\mathbb{Z}_3 \oplus \mathbb{Z}_3} \in \mathcal{H}_9^{\otimes 4}$ written in the computational basis, $(a, b) \rightarrow 3a + b \rightarrow |3a + b\rangle$.

The structure of a Galois field $\text{GF}(9)$ might be presented as an addition and multiplication of nine polynomials in x variable $ax + b$, where $a, b = 0, 1, 2$. Addition and multiplication of aforementioned polynomials is considered modulo an irreducible degree 2 polynomial $x^2 + 1$. Although addition of polynomials $ax + b$ matches the addition of pairs (a, b) , the multiplication structure is different, see Figure 4.3. Same as before, each polynomial $ax + b$ might be associated with the natural number $3a + b \in \mathbb{Z}_9$, which defines quantum state $|\psi\rangle_{\text{GF}(9)}$ via Eq. (4.6) written in the computational basis.

Unlike as for rings $\mathcal{R} = \text{GF}(4), \mathbb{Z}_4$, three arrays OA(81, 4, 9, 2) corresponding to three different ring structures $\mathcal{R} = \text{GF}(9), \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are all of strength 2. Therefore, the related quantum states $|\psi\rangle_{\mathbb{Z}_9}$, $|\psi\rangle_{\mathbb{Z}_3 \oplus \mathbb{Z}_3}$, and $|\psi\rangle_{\text{GF}(9)}$, are all AME. In Section 6.8, we discuss the problem of equivalence of such states.

4.4 Conclusions

In this chapter, I recall the well-established notion of Absolutely Maximally Entangled and k -uniform states (Definition 5) and their correspondence to

classical combinatorial designs. I focus my attention on the different linear structures of classical designs that affect the structure of the related AME and k -uniform states. This chapter does not contain any significant contribution by the author and is intended to introduce some notions used in the consecutive chapters.

Chapter 5

Thirty-six entangled officers of Euler

In this chapter, I discuss a quantum variant of the famous problem of 36 officers of Euler. While the classical problem of Euler is known to have no solutions, I present an analytical form of an $\text{AME}(4, 6)$ state, which might be seen as a quantum solution to the Euler’s problem. Moreover, I show a coarse-grained combinatorial structure behind constructed state. Considerations of such a coarse-grained combinatorial structures might lead to a successful approach for constructing genuinely entangled states beyond the stabilizer approach. An extension of some parts of this chapter, in which the author’s contribution was not substantial can be found in the joint work [RBB⁺21], attached to the dissertation in Appendix D.

5.1 Thirty-six officers of Euler

In 1779, Euler examined the now-famous officer problem [Eul82]: “Six different regiments have six officers, each one belonging to different ranks. Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one of each regiment?” As Euler observed, the possibility of such an arrangement is equivalent to existence of *Graeco-Latin squares* of order 6. Single Latin square of order d is filled with d copies of d symbols arranged in a square in such a way that no row or column of the square contains the same symbol twice. Two orthogonal Latin squares (OLS), also called Graeco-Latin square, is an arrangement of ordered pairs of symbols, for instance, one Greek character and one Latin, into $d \times d$ square. Each symbol appears exactly once in each row and column, while

each pair of symbols appears exactly once in the entire OLS, see Figure 5.1.

$A\alpha$	$B\beta$	$C\gamma$		$A\spadesuit$	$K\clubsuit$	$Q\heartsuit$		0, 0	2, 1	1, 2
$C\beta$	$A\gamma$	$B\alpha$	=	$Q\clubsuit$	$A\heartsuit$	$K\spadesuit$	=	1, 1	0, 2	2, 0
$B\gamma$	$C\alpha$	$A\beta$		$K\heartsuit$	$Q\spadesuit$	$A\clubsuit$		2, 2	1, 0	0, 1

Figure 5.1: An example of Graeco-Latin square of order $d = 3$, relevant to AME(4,3) state. In the middle, Greek and Latin letters are replaced by ranks and suits of cards, on the right by pair of numbers.

In Chapter 4 we discussed a connection between combinatorial designs, such as OLS, and multipartite entangled states, and error-correcting codes. In particular, to any OLS, one may associate a quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes 4}$

$$|\psi\rangle = \sum_{i,j,k,\ell=0}^{d-1} T_{ijkl} \frac{1}{d} |i\rangle |j\rangle |k\rangle |\ell\rangle, \quad (5.1)$$

where $T_{ijkl} = 1$ if the pair (k, ℓ) is an entry in i -th row and j -th column, while $T_{ijkl} = 0$ otherwise. Conditions imposed on OLS translates onto fact that for the following bipartition of four indices into pairs: $ij|k\ell$, $ik|j\ell$, $il|jk$, the corresponding matrices are permutations, i.e. $T_{ij}^{k\ell}, T_{ik}^{j\ell}, T_{il}^{jk}$ are permutation matrices.

Unitary matrices are a natural generalization of permutation matrices. A tensor T_{ijkl} for which each of matrices $T_{ij}^{k\ell}, T_{ik}^{j\ell}, T_{il}^{jk}$ is a unitary matrix is called *perfect*. Furthermore, such a square matrix $U = T_{ij}^{k\ell}$ is called 2-unitary, as the three matrices: $U = T_{ij}^{k\ell}$, and related *reshuffled* $\mathcal{U}^R = T_{ik}^{j\ell}$, and *partially transposed* $(\mathcal{U}^R)^\Gamma = T_{il}^{jk}$ matrices are all unitary. Perfect tensor provides an *isometry* between any pair of its indices. The partial trace of the state $|\psi\rangle$ related to a perfect tensor is maximally mixed for any bipartition of the system, i.e. $\rho_S \propto \text{Id}$ for $|S| = 2$, and hence is related to an AME(4, d) state [Sco04].

Two OLS of order d exist for $d = 3, 4, 5$ and any natural number $d \geq 7$ [JH01]. On the other hand, OLS of order 2 and 6 do not exist, which can be easily observed for order 2, and what was proven by an exhaustive case study for order 6 [Tar00]. The only local dimension d for which the existence of a quantum version of OLS was not decided is six [HS00, AME, HW, YSW⁺21].

5.2 AME state of four quhex

In [RBB⁺21], we present a solution for the quantum version of the Euler's problem of 36 officers. More specifically, we provide there an analytical form of an AME(4, 6) state, which can be associated with an example of orthogonal quantum Latin squares of order six [MV16, GRDMZ18, Ric20, RBB⁺21]. In order to find a 2-unitary matrix of order 36, we used an iterative numerical technique based on nonlinear maps in the space of unitary matrices $\mathbb{U}(d^2)$ [RAL20]. One of key challenges was to find an appropriate seed matrix, which generates a numerical 2-unitary matrix of order $d^2 = 36$. Details of the procedure and an example of a seed that leads to the 2-unitary solution might be find in [RBB⁺21]

The author's main contribution to the solution of this problem, was finding an analytical form of the presented state, obtained with relation (5.1) from a 2-unitary matrix U_{36} , which determines the tensor T_{ijkl} . Any 2-unitary matrix remains 2-unitary under a multiplication by local unitary operators. Using this freedom, we applied a search algorithm over the group $\mathbb{U}(6) \otimes \mathbb{U}(6)$ of local unitary operations, to orthogonalize certain rows and columns in a given numerical 2-unitary matrix U and its rearrangements U^R , $(U^R)^T$. The particular choice of the orthogonality relations corresponds to the block structure of the eventually obtained analytical solution. These tools can be generalized to construct multi-unitary operators and corresponding AME states in other local dimensions and number of parties, and can potentially yield states that are not created by presently known techniques.

In order to present our solution for the problem of 36 entangled officers of Euler, we express coefficients of a quantum state $|\psi\rangle \in \mathcal{H}_6^{\otimes 4}$ (four qudits state) via a four-index tensor T_{ijkl} as it is presented on Eq. 5.1. Non-vanishing terms of T_{ijkl} might be conveniently written in form of a table, see Fig. 5.2. Provided construction is based on root of unity of order 20, denoted by $\omega = \exp(i\pi/10)$. There are three non-zero amplitudes:

$$\begin{aligned} a &= [\sqrt{2}(\omega + \bar{\omega})]^{-1} = [5 + \sqrt{5}]^{-1/2} \simeq 0.3717 \\ b &= [\sqrt{2}(\omega^3 + \bar{\omega}^3)]^{-1} = [(5 + \sqrt{5})/20]^{1/2} \simeq 0.6015 \\ c &= 1/\sqrt{2} \simeq 0.7071 \end{aligned}$$

which might be expressed in terms of ω . Two relations $a^2 + b^2 = c^2 = 1/2$ and $b/a = \varphi = (1 + \sqrt{5})/2$ the golden ratio, determine uniquely amplitudes appearing in the solution and explain why the constructed AME state deserves to be called the *golden AME state*. Checking the property of being an AME state comes down to verification of several equations involving roots

of unity of order 20, which I discuss in a detailed way in Section 5.4.

(1,1)	(2,2)	(1,2)	(2,1)		(1,3)	(2,4)	(1,4)	(2,3)		(1,5)	(2,6)	(1,6)	(2,5)	
$a \omega^{10}$	a	$b \omega^{15}$	$b \omega^5$	(6,3)	$a \omega^2$	$a \omega^{14}$	$b \omega$	$b \omega^5$	(2,5)	$a \omega$	$a \omega^{19}$	$b \omega^{14}$	$b \omega^{16}$	(4,1)
		c	c	(1,1)			$c \omega^5$	$c \omega^{19}$	(3,3)	$a \omega$	$a \omega^3$	$b \omega^{10}$	$b \omega^4$	(3,4)
c	c			(5,6)	$c \omega^{17}$	$c \omega^{19}$			(1,2)	$b \omega^4$	$b \omega^{18}$	$a \omega^3$	$a \omega^9$	(2,6)
$b \omega^{10}$	b	$a \omega^5$	$a \omega^{15}$	(4,2)	$b \omega^{14}$	$b \omega^6$	$a \omega^3$	$a \omega^7$	(6,4)	$b \omega^2$	$b \omega^8$	$a \omega^5$	$a \omega^{15}$	(5,5)
(3,1)	(4,2)	(3,2)	(4,1)		(3,3)	(4,4)	(3,4)	(4,3)		(3,5)	(4,6)	(3,6)	(4,5)	
$a \omega^4$	$a \omega^{10}$	$b \omega^{17}$	$b \omega^7$	(4,5)	a	a	$b \omega^{15}$	$b \omega^{15}$	(4,6)	$a \omega^2$	a	$b \omega^{19}$	$b \omega^{13}$	(2,3)
		$c \omega^2$	$c \omega^2$	(3,2)			c	$c \omega^{10}$	(6,1)			$c \omega^{16}$	c	(6,2)
$c \omega^{10}$	$c \omega^6$			(2,4)	c	$c \omega^{10}$			(5,4)	$c \omega^8$	$c \omega^{16}$			(3,1)
$b \omega^7$	$b \omega^{13}$	$a \omega^{10}$	a	(5,3)	b	b	$a \omega^5$	$a \omega^5$	(1,5)	$b \omega^{14}$	$b \omega^{12}$	$a \omega$	$a \omega^{15}$	(1,6)
(5,1)	(6,2)	(5,2)	(6,1)		(5,3)	(6,4)	(5,4)	(6,3)		(5,5)	(6,6)	(5,6)	(6,5)	
$a \omega^3$	$a \omega^7$	b	b	(1,4)	$a \omega^{12}$	$a \omega^{14}$	$b \omega^{15}$	$b \omega$	(3,6)	$a \omega^{18}$	$a \omega^{18}$	$b \omega^3$	$b \omega^3$	(1,3)
		c	$c \omega^{10}$	(2,1)			$c \omega^{14}$	$c \omega^{10}$	(5,1)			c	$c \omega^{10}$	(5,2)
$c \omega^{13}$	$c \omega^7$			(3,5)	$c \omega^7$	$c \omega^{19}$			(2,2)	$c \omega$	$c \omega^{11}$			(6,5)
$b \omega^9$	$b \omega^{13}$	$a \omega^{16}$	$a \omega^{16}$	(6,6)	$b \omega^{14}$	$b \omega^{16}$	$a \omega^7$	$a \omega^{13}$	(4,3)	$b \omega^{10}$	$b \omega^{10}$	$a \omega^5$	$a \omega^5$	(4,4)

Figure 5.2: Non-vanishing elements of a perfect tensor T_{ijkl} related to the AME(4,6) state. A pair of indices (i, j) are indicated in rows, (k, l) in columns respectively. Treating each coordinates (i, j) as a position (row and column) of Euler's officer, its rank and regiment are in superposition of two or four canonical ranks and regiments.

5.3 Structure of the AME(4,6) state

Recall that a classical OLS corresponds to a 2-unitary permutation matrix. Since there is no solution to the original Euler's problem, the 2-unitary permutation matrix of size 36 does not exist. Nonetheless, the 2-unitary matrix presented on Figure 5.2 has the structure of nine 4×4 blocks. Moreover, the block structure is preserved under reshuffling and the partial transpose of the matrix. In that sense, we found the *invariant* block structure in the original 36×36 with respect to reshuffling and partial transpose operations. In fact, the problem of finding 36 entangled officers of Euler splits into two sub-problems: finding such a block invariant structure at first, and then select adequate non-zero elements inside the chosen structure. Such an approach indicates the plausible direction of search for other AME and k -uniform states. In fact, the notion of such an invariant structure might be formalized via certain combinatorial designs. Grouping symbols of indices k and ℓ in the presented perfect tensor T_{ijkl} in pairs: $1, 2 \rightarrow A/\alpha$, and $3, 4 \rightarrow B/\beta$, and $5, 6 \rightarrow C/\gamma$ for k/ℓ respectively, results in coarse-grained OLS, which reveals the described block structure, see Figure 5.3.

$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$	$B\beta$	$B\gamma$
$C\alpha$	$C\beta$	$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$
$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$	$C\alpha$	$C\beta$
$A\gamma$	$A\alpha$	$C\beta$	$C\gamma$	$B\alpha$	$B\beta$
$C\beta$	$C\gamma$	$B\alpha$	$B\beta$	$A\gamma$	$A\alpha$
$B\beta$	$B\gamma$	$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$

Figure 5.3: A coarse-grained OLS of order 6, which reveals the block structure of a perfect tensor T_{ijkl} . Indices of non-vanishing elements of the tensor T_{ijkl} are presented: i in row, j in column, while a pair of coarse-grained indices k, ℓ in relevant entry. Each pair of symbols repeats exactly four times on the grid. Moreover, each symbol on each position repeats exactly twice in each row and column.

Furthermore, presented AME(4, 6) state might be written in the following form, which reveals its block structure

$$|\Psi\rangle = \sum_{i,j=1}^6 |i, j\rangle \otimes |\psi_{i,j}\rangle, \quad (5.2)$$

where each of states $|\psi_{i,j}\rangle \in \mathcal{H}_{36} \cong \mathcal{H}_3 \otimes \mathcal{H}_3 \otimes \mathcal{H}_{2,2}$ has the following form:

$$|\psi_{i,j}\rangle = \left(\underbrace{\left| 2\left\lfloor \frac{j}{2} \right\rfloor + 2i + 1 \right\rangle}_{\mathcal{H}_3} \otimes \underbrace{\left| j + 2\left\lfloor \frac{i}{2} \right\rfloor + 1 \right\rangle}_{\mathcal{H}_3} \right) \otimes \text{Id} \otimes U_{ij} \left(\underbrace{\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle}_{\mathcal{H}_{2,2}} \right) \quad (5.3)$$

where $i, j = 1, \dots, 6$ and U_{ij} are unitary matrices of order 2, which form might be read from Figure 5.4. Note that pairs of indices

$$\left(2\left\lfloor \frac{j}{2} \right\rfloor + 2i + 1, j + 2\left\lfloor \frac{i}{2} \right\rfloor + 1 \right) \quad (5.4)$$

are in accordance with the entries of 6×6 coarse-grained OLS presented on Figure 5.3 after associating Greek character and one Latin characters with numbers: $A, \alpha \rightarrow 0$; $B, \beta \rightarrow 1$; $C, \gamma \rightarrow 2$.

Most of the examples of quantum error correction codes belong to the class of, so-called, additive (stabilizer) states [CSSZ09, AR21]. In particular all hitherto known AME states, are either stabilizer states [AR21], or might

$j \backslash i$	1	2	3	4	5	6
1	$\begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}$	$\begin{pmatrix} \gamma\omega^{17} & 0 \\ 0 & \gamma\omega^{19} \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^{18} & \beta\omega^3 \\ \beta\omega^3 & \alpha\omega^{18} \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^3 & \beta \\ \beta & \alpha\omega^7 \end{pmatrix}$	$\begin{pmatrix} \beta & \alpha\omega^5 \\ \alpha\omega^5 & \beta \end{pmatrix}$	$\begin{pmatrix} \beta\omega^{14} & \alpha\omega \\ \alpha\omega^{15} & \beta\omega^{12} \end{pmatrix}$
2	$\begin{pmatrix} 0 & \gamma \\ \gamma\omega^{10} & 0 \end{pmatrix}$	$\begin{pmatrix} \gamma\omega^7 & 0 \\ 0 & \gamma\omega^{19} \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^2 & \beta\omega^{19} \\ \beta\omega^{13} & \alpha \end{pmatrix}$	$\begin{pmatrix} \gamma\omega^{10} & 0 \\ 0 & \gamma\omega^6 \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^2 & \beta\omega \\ \beta\omega^5 & \alpha\omega^{14} \end{pmatrix}$	$\begin{pmatrix} \beta\omega^4 & \alpha\omega^3 \\ \alpha\omega^9 & \beta\omega^{18} \end{pmatrix}$
3	$\begin{pmatrix} \gamma\omega^8 & 0 \\ 0 & \gamma\omega^{16} \end{pmatrix}$	$\begin{pmatrix} 0 & \gamma\omega^2 \\ \gamma\omega^2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \gamma\omega^5 \\ \gamma\omega^{19} & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^1 & \beta\omega^{10} \\ \beta\omega^4 & \alpha\omega^3 \end{pmatrix}$	$\begin{pmatrix} \gamma\omega^{13} & 0 \\ 0 & \gamma\omega^7 \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^{12} & \beta\omega^{15} \\ \beta\omega & \alpha\omega^{14} \end{pmatrix}$
4	$\begin{pmatrix} \alpha\omega & \beta\omega^{14} \\ \beta\omega^{16} & \alpha\omega^{19} \end{pmatrix}$	$\begin{pmatrix} \beta\omega^{10} & \alpha\omega^5 \\ \alpha\omega^{15} & \beta \end{pmatrix}$	$\begin{pmatrix} \beta\omega^{14} & \alpha\omega^7 \\ \alpha\omega^{13} & \beta\omega^{16} \end{pmatrix}$	$\begin{pmatrix} \beta\omega^{10} & \alpha\omega^5 \\ \alpha\omega^5 & \beta\omega^{10} \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^4 & \beta\omega^{17} \\ \beta\omega^7 & \alpha\omega^{10} \end{pmatrix}$	$\begin{pmatrix} \alpha & \beta\omega^{15} \\ \beta\omega^{15} & \alpha \end{pmatrix}$
5	$\begin{pmatrix} 0 & \gamma\omega^{14} \\ \gamma\omega^{10} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \gamma \\ \gamma\omega^{10} & 0 \end{pmatrix}$	$\begin{pmatrix} \beta\omega^7 & \alpha\omega^{10} \\ \alpha & \beta\omega^{13} \end{pmatrix}$	$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma\omega^{10} \end{pmatrix}$	$\begin{pmatrix} \beta\omega^2 & \alpha\omega^5 \\ \alpha\omega^{15} & \beta\omega^8 \end{pmatrix}$	$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$
6	$\begin{pmatrix} 0 & \gamma \\ \gamma\omega^{10} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \gamma\omega^{16} \\ \gamma & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha\omega^{10} & \beta\omega^{15} \\ \beta\omega^5 & \alpha \end{pmatrix}$	$\begin{pmatrix} \beta\omega^{14} & \alpha\omega^3 \\ \alpha\omega^7 & \beta\omega^6 \end{pmatrix}$	$\begin{pmatrix} \gamma\omega & 0 \\ 0 & \gamma\omega^{11} \end{pmatrix}$	$\begin{pmatrix} \beta\omega^9 & \alpha\omega^{16} \\ \alpha\omega^{16} & \beta\omega^{13} \end{pmatrix}$

Figure 5.4: An exact form of 2×2 unitary matrices U_{ij} for $i, j = 1, \dots, 6$. For the sake of visibility, the correct matrix notation has been replaced only by indication of matrix entries. For example, for $i = 1, j = 2$ the related matrix reads $U_{12} = \begin{pmatrix} 0 & \gamma \\ \gamma\omega^{10} & 0 \end{pmatrix}$.

be derived from the stabilizer construction [BR20]. We examined all stabilizer sets of four quhex in their standard form and did not find an AME(4, 6) state, hence the presented state is not a stabilizer state.

Nonadditive quantum codes are in general more difficult to construct, however, in many cases, they outperform the stabilizer codes [YCLO08]. Thus far, the stabilizer approach practically contained the combinatorial approach to constructing AME and k -uniform states. As we demonstrated, the consideration of coarse-grained combinatorial structures might be successful in constructing genuinely entangled states and have advantages over the stabilizer approach. It is tempting to believe that the quantum design presented here will trigger further research on quantum combinatorics.

Even though OLS of order six does not exist, we present a coarse-grained OLS of that order, which structure stands behind the provided AME state. This sheds some light on how to construct quantum nonadditive AME states, and more generally QECCs, when the stabilizer approach fails.

5.4 Verification of 2-unitarity

In this section we show that each of three matrices $T_{ij}^{k\ell}, T_{ik}^{j\ell}, T_{il}^{jk}$ related to the tensor T_{ijkl} is unitary. In order to simplify the notation, we introduce a matrix $\mathcal{U}$ of size 36 with entries, determined by the tensor T_{il}^{jk} , in the following way $\mathcal{U}_{p,s} = T_{ijkl}$, with $p = j + 6(i - 1)$ and $s = \ell(k - 1)$, i.e. the matrix corresponding to the flattening T_{il}^{jk} . Similarly, by $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$, we denote matrices corresponding to the flattening $T_{ik}^{j\ell}$ and $T_{ij}^{k\ell}$ respectively. Operations R and Γ are known as *reshuffling* and the *partial transpose* respectively, compare with Section 5.1.

To show the tensor T_{ijkl} is perfect, we shall verify that three related matrices: $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ are unitary. In fact, each of those matrices has the block structure with nine 4×4 sub-matrices. Hence our task simplifies to verification that constituent blocks are unitary matrices.

Consider the matrix $\mathcal{U}$. Interestingly, except for one block component in the matrix $\mathcal{U}$, all eight remaining blocks are equivalent (up to a multiplication of rows and columns by adequate phases) to the following 4×4 unitary matrix:

$$V = \begin{bmatrix} a & a & b & b \\ 0 & 0 & c & -c \\ c & -c & 0 & 0 \\ b & b & -a & -a \end{bmatrix}.$$

Orthogonality between rows in the matrix above might be presented as pairs

of antipodal points on the complex plane, for example, the orthogonality between the first two rows reads

$$bc(1 - 1) = 0. \quad (5.5)$$

The right top corner in Figure 5.6 presents the exceptional block of matrix $\mathcal{U}$. Six orthogonality relations between rows read

$$a^2(\omega^8 + \omega^{-8}) + b^2(\omega^4 + \omega^{-4}) = 0, \quad (5.6)$$

$$ab(1 + \omega^2 + \omega^{-8} - 1) = 0, \quad (5.7)$$

$$ab(\omega^{-2} + \omega^2 + \omega^{-8} + \omega^8) = 0,$$

up to a phase factor, where $\omega = \exp(i\pi/10)$. Each equation might be presented as a *unitarity rectangle* - a constellation of four points on the complex plane which sum up to zero, as it is shown on Fig. 5.6. Observe that the second and third equations above are both related to two pairs of antipodal points on the complex plane. Geometric interpretation of the amplitudes a and b is shown in Fig. 5.5.

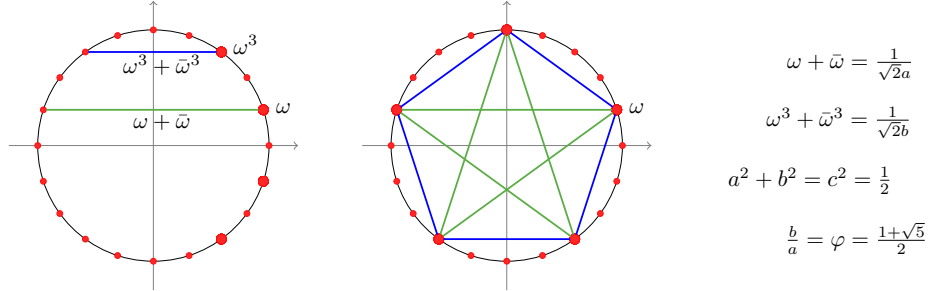


Figure 5.5: Geometric interpretation of the three amplitudes a, b and c appearing in the AME(4, 6) state with use of a regular pentagon. Amplitudes a, b contain the phase $\omega = \exp(i\pi/10)$ and are related to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

Although both matrices $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ enjoy the structure of nine 4×4 blocks, similar to $\mathcal{U}$, the particular arrangement inside their blocks is significantly different from the $\mathcal{U}$ matrix. Blocks in $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ are of four distinct types up to multiplication of their rows and columns by phase factors, see Fig. 5.6. Orthogonality relations between rows of both matrices reflect their complex structure. In particular, we distinguished five addi-

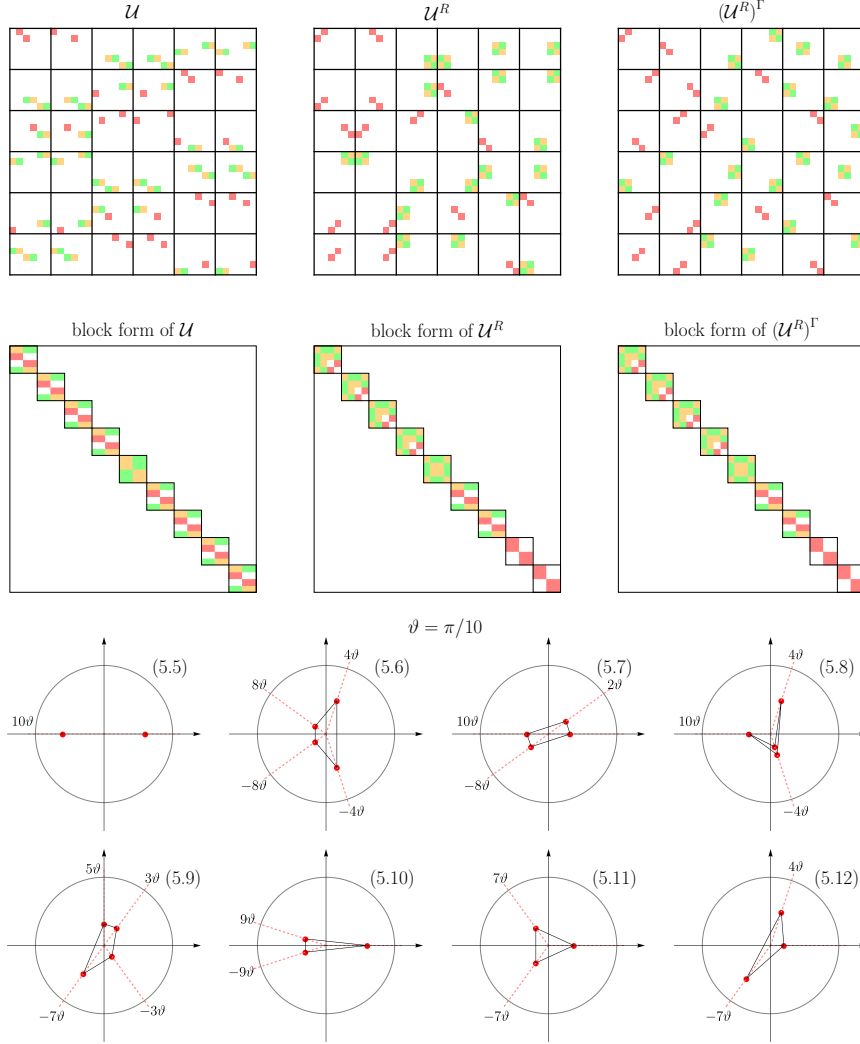


Figure 5.6: Non-vanishing elements of three unitary matrices $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ of order 36 are presented on top. The modulus of a non-vanishing element is represented by the intensity of the background color, where green, orange and red are related to constants a , b , and c respectively. Each matrix has the structure of nine 4×4 blocks, the sub-structure within the blocks is different. Orthogonality relations between pairs of rows in block matrices may be presented as a constellation of points on the complex plane which sum up to zero. Constellations related to Eqs. (5.5-5.12) are indicated.

tional orthogonality relations given by the following equations:

$$a^2\omega^4 + ab(\omega^{10} + \omega^{-4}) + b^2\omega^{-4} = 0, \quad (5.8)$$

$$a^2\omega^{-3} + ab(\omega^5 + \omega^3) + b^2\omega^{-7} = 0, \quad (5.9)$$

$$ab(\omega^{-4} + \omega^{-6}) + bc\omega^5 = 0, \quad (5.10)$$

$$ab(\omega^{-8} + \omega^{-2}) + ac\omega^5 = 0, \quad (5.11)$$

$$a^2 + b^2\omega^4 + bc\omega^{-7} = 0. \quad (5.12)$$

Related constellations are presented on Fig. 5.6. The above-listed equalities provide orthogonality between rows in the three matrices $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$, hence they imply that the matrix $\mathcal{U}$ is 2-unitary.

Interestingly, in the presented solution each officer is entangled with at most three other officers (out of remaining 35). This implies the matrix $\mathcal{U}$ is sparse.

Amplitudes a, b, c which appear in the presented construction might be defined as the unique solution of the following three algebraic equations: $a^2 + b^2 = c^2 = 1/2$ and $b/a = \varphi = (1 + \sqrt{5})/2$, see Fig. 5.5. Notice the similarities between algebraic equations which lead to values a, b, c , and the algebraic equations which lead to the amplitudes in an another AME state, in the heterogenous $2 \times 3 \times 3 \times 3$ system presented in [HESG18].

The phases of the coefficients shown in Figure 5.2, being multiples of $\omega = \exp(i\pi/10)$, are chosen in such a way that all 36 quantum states $|\psi_{ij}\rangle$, each represented by a single row of the matrix $\mathcal{U}$, are equivalent to the standard, two-qubit Bell state. This is easy to see, for any state formed by two coefficients of moduli, $|c| = 1/\sqrt{2}$, as states $|\psi_{11}\rangle$ or $|\psi_{56}\rangle$ represented in the second and the third row in the upper left block in Figure 5.2. One can show that this property holds also for other states. For example, the state $|\psi_{63}\rangle$ corresponding to the first line of the aforementioned block can be written in the product basis as

$$|\psi_{63}\rangle = a\omega^{10} |11\rangle + b\omega^{15} |12\rangle + b\omega^5 |21\rangle + a |22\rangle.$$

Thus the partial trace reads, $\text{Tr}_B |\psi_{63}\rangle \langle \psi_{63}| = \text{diag}(a^2 + b^2, a^2 + b^2) = \mathbb{I}/2$, which proves that $|\psi_{63}\rangle$ is locally equivalent to the maximally entangled Bell state. A similar reasoning works for all other states consisting of four terms and represented in Figure 5.2 by green and yellow elements. In such a way, all 36 states, corresponding to 36 entangled officers of Euler, can be considered as maximally entangled, two qubit states.

5.5 Conclusions

In this chapter I presented the famous combinatorial problem of 36 officers was posed by Euler, and its quantum version. I present an analytical form of an $\text{AME}(4, 6)$ state of four subsystems with six levels each, which might be seen as a quantum solution to Euler's problem of finding two orthogonal Latin squares of order six. The existence of such a state is equivalent to the existence of a 2-unitary matrix of order 36 or a perfect tensor with four indices, each running from one to six. This result allows us to optimally encode a single quhex into a three quhex state. Furthermore, I show a coarse-grained combinatorial structure behind the constructed state. Further analysis of such coarse-grained combinatorial structures might potentially lead to the construction of other genuinely entangled states beyond the stabilizer approach.

Chapter 6

Classification of absolutely maximally entangled states

In this chapter of the thesis, I present techniques for verifying whether two AME states are equivalent with respect to Stochastic Local Operations and Classical Communication (SLOCC). I falsify the conjecture that for a given multipartite quantum system all AME states are SLOCC-equivalent. I also show that the existence of AME states with minimal support of 6 or more particles results in the existence of infinitely many such non-SLOCC-equivalent states. Moreover, I present $\text{AME}(5,d)$ states which are not SLOCC-equivalent to the existing AME states with minimal support. Proofs of the statements presented in this Chapter, as well as an extension of some parts, in which the author's part was not substantial can be found in the joint work [BR20], attached to the dissertation in Appendix B.

6.1 Local equivalence of AME and k -uniform states

The initial motivation addressed in this chapter is the question whether different constructions of AME states are equivalent by any local transformation. Note, that for AME states all reduced density matrices are maximally mixed. Therefore, the classical method for verification of local equivalence, which is comparison of Schmidt rank and coefficients, will obviously fail [HW08].

Contrary to the AME states, it was already shown that some k -uniform states are not locally equivalent [RGA20], this result was indeed based on a comparison of Schmidt ranks of reduced density matrices. Nevertheless, the aforementioned rank argument is never conclusive for SLOCC-verification of

two k -uniform states of minimal support.

In this Chapter, we tackle those two cases, i.e k -uniform states of the minimal support and AME states, for which the bound on the number k is saturated, and provide general techniques of SLOCC-equivalence verification between such states.

We begin with the formal definition of SLOCC- and LU-equivalence. Two N -qudit states $|\psi\rangle$ and $|\phi\rangle$ are LU-equivalent if one can be transformed into another by local unitary operators, i.e

$$|\phi\rangle = U_1 \otimes \cdots \otimes U_N |\psi\rangle. \quad (6.1)$$

Similarly, two states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC-equivalent if and only if there exists a *local invertible* operator connecting those states [DVC00]:

$$|\phi\rangle = O_1 \otimes \cdots \otimes O_N |\psi\rangle. \quad (6.2)$$

Since LU- and SLOCC-equivalences are equivalence relations, the state space might be naturally partitioned into *LU classes* and *SLOCC classes* respectively. As a consequence of Kempf-Ness theorem [KN06], two AME states, or more generally k -uniform states are SLOCC-equivalent if and only if they are LU-equivalent [GW10].

6.2 Local equivalences, case $2k < N$

We begin with introducing the notion of a *support* of a state and specific class of local unitary operations, namely *monomial matrices*. Both are essential for the further investigations of the classification problem of k -uniform states.

The *support* of a state $|\psi\rangle$ of N subsystems with d levels each is the number of non-zero coefficients in the computational basis. Note, that the support of k -uniform state is at least d^k , since the partial trace over $N - k$ particles is an identity matrix Id_{d^k} . The k -uniform state with support achieving its lower bound d^k is called a *minimal support* state. Note that examples of AME and 1-uniform states presented in Examples 6 and 7, Chapter 4 are all of the minimal support.

Definition 6. A unitary matrix M is called *monomial* if one of the following holds:

1. M has exactly one nonzero element in each row and each column,
2. M is a product of a permutation and an invertible diagonal matrix,

3. M does not change the support of any quantum state.

The local monomial operation of the multipartite quantum states are denoted as LM -eoperations.

It is straightforward to verify that conditions 1-3 are equivalent. Obviously, any LM -operation provides the LU -equivalence between two states of minimal support. Interestingly, the reverse statement holds true for k -uniform states for which $2k < N$. This simplifies a search for LU -equivalence between two k -uniform states of minimal support to the search within the LM class. In [BR20], I showed the following.

Proposition 9. *For $2k < N$, each LU - or $SLOCC$ -equivalency between two k -uniform states of minimal support is in fact LM -equivalency.*

Corollary 3. *For $2k < N$, two k -uniform states of minimal support are LU - or $SLOCC$ -equivalent if and only if they are LM -equivalent.*

The above result turned to be useful in a more refined problem of $SLOCC$ -verification between two AME states, with not necessary minimal support. Below, we present examples of two families of $AME(5, d)$ states.

Example 9. The following state

$$|AME'(5, d)\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j, 3i+j\rangle$$

is an AME state with minimal support for all prime dimensions $d \geq 5$ [RGRA17].

It is known that AME states with minimal support of five or six qubits do not exist [HESG18]. Nevertheless, the construction of $AME(5, 2)$ state with non-minimal support is known [Rai99, GRDMZ18].

Example 10. The following state

$$|AME(5, d)\rangle = \frac{1}{\sqrt{d^3}} \sum_{i,j,\ell=0}^{d-1} \omega^{(3i+j)\ell} |i, j, i+j, 2i+j+\ell, \ell\rangle,$$

where ω is d th root of unity, is $|AME(5, d)\rangle$ state for any integer number $d \geq 2$ [GRDMZ18, Rai99].

In [BR20], I showed that states presented in Example 10 and Example 9 are not locally equivalent. In such a way, I falsified the conjecture that for a given multipartite quantum system all AME states are SLOCC-equivalent.

Proposition 10. *For any prime local dimension d , two families of $\text{AME}(5, d)$ states: $|\text{AME}(5, d)\rangle$ and $|\text{AME}'(5, d)\rangle$ presented in Example 10 and Example 9 respectively, are not SLOCC-/LU-equivalent.*

6.3 Local equivalences, case $2k=N$

Interestingly, for $\text{AME}(2k, d)$ states of even number of parties $N = 2k$, the statement of Proposition 9 does not hold anymore. We illustrate it on the example below.

Example 11. Matrix proportional to the tensor product of the Fourier transform F_3 :

$$(F_3)^{\otimes 4} = \frac{1}{9} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix}^{\otimes 4} \quad (6.3)$$

provides an automorphism of $\text{AME}(4, 3)$ state presented in Example 7 in Chapter 4.

One can suspect that Fourier matrices are not the only non-monomial matrices which may provide the LU-equivalence between $\text{AME}(2k, d)$ states of minimal support. This turned out not to be true, if we consider AME states in composed dimension [BR20]. For sufficiently small dimensions, however, I derived some further conclusions on the form of matrices which can provide LU-equivalence between AME states. We define the class of matrices beyond the monomial class.

Definition 7. Let d, q be arbitrary positive integers. A *Butson-type* complex Hadamard matrix of order d and complexity q is a unitary matrix in which each entry is a complex q th root of unity scaled by the factor $1/\sqrt{d}$. The set of Butson-type matrices is denoted by $\text{BH}(d, q)$.

Note that, in literature, [But62, TZ06], Butson-type matrices are usually defined without scaling factor $1/\sqrt{d}$. In such a way, they are not unitary, but proportional to the unitary matrices. In particular, the class of matrices $\text{BH}(d, 2)$ is simply the class of real Hadamard matrices of order d . In our analysis, we focus on matrices of the type $\text{BH}(d, d)$. Those matrices together with the monomial matrices provide all possible LU-equivalences of AME

states. Note that the Fourier matrices belong to this class of matrices, $F_d \in \text{BH}(d, d)$.

Proposition 11. *Consider two $\text{AME}(2k, d)$ states of minimal support, where k and d are sufficiently small (see Remark 1). Each LU-equivalency between them is of one of the following forms:*

1. *tensor product of Butson-type matrices $B_i \in \text{BH}(d, d)$ multiplied by LM matrices from each side;*
2. *or LM matrices itself.*

Remark 1. There is the following restriction on numbers d and k imposed in the statement of Proposition 11:

$$(k+1) \left(1 + \sqrt[k-1]{k}\right) \leq d.$$

for $k > 1$ and $2 < d$ for $k = 1$. This bound is related to the necessary condition for existence and extension of combinatorial designs called *mutually orthogonal hypercubes*. We discuss them in detail in [BR20]. For example, for $k = 2, 3, 4, 5, 6$ the smallest value of $d_{\min}$ which does not satisfy the bound above is presented in a Section 6.3. However, the given bound is not tight. In addition, Section 6.3 presents $d_{\max}$ value, which is the maximal value of a local dimension d for which Proposition 11 does not hold (we found a counterexample). We conjecture, that the asymptotic behavior of these values is $(k-1)^2$.

k	AME(2k,d)	$d_{\min}$	$d_{\max}$
1	AME(2,d)	3	3
2	AME(4,d)	9	9
3	AME(6,d)	11	16
4	AME(8,d)	13	25

Table 6.1: Values $d_{\min}$ and $d_{\max}$ for respective values of uniformity k . Note that the statement of Proposition 11 holds true for related $\text{AME}(2k, d)$ states in any local dimension $d < d_{\min}$.

Note that for small values of $d = 3, \dots, 12$, the Butson matrices are classified and contains 1, 2, 1, 4, 1, 143, 23, 51, 1, 4497733 matrices (classified up to monomial matrices) equivalently [LÖS17]. Tables of Butson matrices are

available in [Lam, WBZ]. Although, the class of Butson matrices $\text{BH}(d, d)$ grows rapidly with the dimension d ; it seems that the subclass of such matrices that might be involved in LU-equivalences of AME states is significantly smaller, and more specific classification of matrices providing eventual LU-equivalences of AME states is needed. We conjecture that all such matrices are Fourier transform and tensor product of such.

6.4 One-uniform states

In three consecutive section, we apply results described in Section 6.2 and Section 6.3 for various classes of k -uniform states for $k = 1, 2, 3$. As we shall see the analysis of their LU/SLOCC-equivalences differs substantially.

We begin with 1-uniform states. All 1-uniform states of N -qudit system with the minimal support have the following form:

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega_i |j_i^1, \dots, j_i^N\rangle,$$

where indices j_i^ℓ runs over all levels $0, \dots, d-1$ for all indices ℓ . One can observe that they are pairwise LU-equivalent, in particular equivalent to the generalized GHZ state $|\text{GHZ}_d^N\rangle$, presented in Example 6 in Chapter 4. Indeed, the following local transformation

$$U_1(|j_i^1\rangle) = (\omega_i^{-1} |j_i^1\rangle), \quad U_\ell(|j_i^\ell\rangle) = (|j_i^1\rangle)$$

for systems $\ell = 2, \dots, N$ gives aforementioned LU-equivalence. In that way, we conclude that all 1-uniform states of minimal support are LU-equivalent.

6.5 Two-uniform states

Consider a family of 2-uniform states of N -qudit system with minimal support of the following form:

$$|\phi_\alpha\rangle := \frac{1}{d} \left(\alpha |0, \dots, 0\rangle + \sum_{i,j \neq (0,0)} |i, j\rangle \otimes |\phi_{i,j}\rangle \right)$$

which is indexed by a complex numbers α , $|\alpha| = 1$. Each of such a state is LU-equivalent to $|\phi_{\alpha=0}\rangle$ by the following unitary transformations:

$$U_1 = \text{diag}\left((\overline{\omega_\alpha})^{n-1}, 1, \dots, 1\right),$$

$$U_i = \text{diag}\left((\overline{\omega_\alpha})^{d-1}, \omega_\alpha, \dots, \omega_\alpha\right),$$

for $i = 2, \dots, N$, where $\omega_\alpha = \sqrt[d(N-1)]{\overline{\alpha}}$ is an arbitrary root. Therefore, all states $|\phi_\alpha\rangle$ are pairwise equivalent. Furthermore, if the exceptional phase stands by a different term, the similar transformation of such a state onto $|\phi_{\alpha=0}\rangle$ might be obviously given. Therefore, all 2-uniform states of the form:

$$|\phi_\omega\rangle = \frac{1}{d} \sum_{i,j} \omega_{i,j} |i, j\rangle \otimes |\phi_{i,j}\rangle$$

are equivalent. We conclude this discussion in the following corollary.

Corollary 4. *Two 2-uniform states of N -qudit system with the minimal support which differs only by phases,*

$$|\psi\rangle = \sum_{I \in \mathcal{I}} \omega_I |I\rangle,$$

$$|\psi'\rangle = \sum_{I \in \mathcal{I}} \omega'_I |I\rangle,$$

where the sum runs over multi-index set $\mathcal{I} \subset [d]^N$ of size $|\mathcal{I}| = d^2$, are always LU-equivalent (and hence belong to the same SLOCC class).

From Corollary 4, it becomes clear that the diversity of possible phases in front of each term in the 2-uniform state with minimal support does not reflect in the number of SLOCC classes. In fact, each 2-uniform state with minimal support is equivalent to the one with all equal phases.

Henceforward, the enumeration of SLOCC classes of 2-uniform states with the minimal support might be restricted only to states with all phases equal. In that way, it coincides with the classification of related orthogonal array.

Corollary 5. *Classification of 2-uniform states with minimal support for $N > 4$ particles is equivalent to the classification of relevant orthogonal arrays, $OA(d^2, N, d, 2)$, up to permutation of indices on each position. Potentially, for $N = 4$ two $AME(4, d)$ states of minimal support might be in the same SLOCC class, even though the corresponding OA are not equivalent.*

In literature, the classification of OAs is usually considered up to permutations of rows and columns [HSS97, YMJ08]. Note that permutation of columns resembles the physical operation of exchanging subsystems. Hence, while verifying the SLOCC-equivalence one should always indicate whether such operations are considered [VDDMV01, GGM18]. In particular, by the classification of OAs, there is at most one $OA(d^k, N, d, k)$ for dimensions $d = 2, \dots, 17$ and for any strength k and any number N [Slo]. Therefore, 2-uniform state with minimal support and the local dimension $d = 2, \dots, 17$ are always SLOCC equivalent or SLOCC equivalent after permutation of parties.

Conjecture 2. *All 2-uniform states of minimal support are LU-equivalent, and hence represent the same SLOCC class.*

6.6 Three-uniform states

In Section 6.5, we have shown that the number of LU/SLOCC classes for 2-uniform states of minimal support greatly coincides with the number of related OAs which are non-isomorphic. In particular, two states which differ only by phases are LU- and SLOCC-equivalent. As we shall see, this is in a strong contrast to the 3-uniform states.

We shall illustrate it on the example of $AME(6, d)$ states. Note, that for any $d \geq 4$ there exists an $AME(6, d)$ state with minimal support [RGRA17], its precise form might be obtained by reading consecutive rows of the corresponding $OA(d^3, 6, d, 3)$ from the OAs table [Slo]. One may enhance successive terms of an $AME(6, d)$ state with any phase factor $|\omega| = 1$. Such family of states:

$$|AME(6, d)_\omega\rangle = \frac{1}{d\sqrt{d}} \left(\sum_{i,j,k=0}^{d-1} \omega_{i,j,k} |i, j, k\rangle \otimes |\psi_{i,j,k}\rangle \right)$$

is obviously a family of $AME(6, d)$ states. We focus our attention on states with all phases are equal to unity $\omega_{i,j,k} = 1$ with one exception: $\omega_{0,0,0} = \alpha$. We denote them as $|\psi_\alpha\rangle$. In [BR20], I showed that states $|\psi_{e^{i\phi_1}}\rangle$ and $|\psi_{e^{i\phi_2}}\rangle$ are not LU-equivalent unless $\phi_1 = \phi_2 + \pi t$ for some integer t . In such a way, I obtained a continuous family of non-LU-equivalent $AME(6, d)$ states with minimal support.

Corollary 6. *With the above notation, the $AME(6, d)$ states, $d \geq 4$:*

$$|AME(6, d)_{e^{i\phi}}\rangle := \frac{1}{d\sqrt{d}} \left(e^{i\phi} |000000\rangle + \sum_{i,j,k \neq (0,0,0)} |i, j, k\rangle \otimes |\psi_{i,j,k}\rangle \right) \quad (6.4)$$

are pairwise in different LU- and SLOCC-classes for all phases $\phi \in [0, \pi)$.

In fact, for any k -uniform state with minimal support where $k > 2$, the similar construction of continuous non LU-equivalent family might be provided.

Corollary 7. *If there exists a k -uniform state with minimal support $|\psi\rangle$ where $k > 2$, then there are infinitely many pairwise non LU- and SLOCC-equivalent k -uniform states*

6.7 Number of non SLOCC-equivalent AME orbits

In this section, we shortly summarize the number of non-SLOCC-equivalent AME states (the number of orbits) and AME states with minimal support.

The existence of AME states with minimal support for $N, d < 8$ was analyzed [Ber17] based on the table of OAs and similar combinatorial designs. According to the discussion presented in the previous sections, if $N \geq 6$ existence of AME(N, d) state with minimal support persuade to infinitely many non-SLOCC-equivalent such states, see Corollary 7. in Table 6.3 and Table 6.2 respectively.

Note, that verification of the existence of AME states which are not necessarily of the minimal support is more complex problem. Several results concerning this problem [HS00, PPGB09, HGS17, Ber17, HESG18] are summarizes in the tables of AME states [HW]. Although the exact classification of AME states up to SLOCC-equivalence is not known, in some specific cases non-trivial lower bound is given.

6.8 Further discussion and open problems

As we have shown, the LU- and SLOCC-classification of k -uniform states, even of minimal support, is in fact a complex project, which involves many open mathematical problems, in particular:

1. Existence and extension of mutually orthogonal Latin hypercubes.
2. Classification of Hadamard matrices of Butson type $B(d, d)$.
3. Classification/uniqueness of OAs of index unity (without permutation).

among others. In addition, we discuss some open problems regarding LU- and SLOCC-classification of k -uniform states with minimal support in a detailed way.

local dimension	2 qubits	3 qutrits	4	5	6	7
AME(3,d)	1	1	1	1	1	1
AME(4,d)	0	1	1	1	0	1
AME(5,d)	0	0	1	1	0	1
AME(6,d)	0	0	∞	∞	0	∞
AME(7,d)	0	0	0	0	0	∞

Table 6.2: The exact number of not SLOCC-equivalent AME orbits with minimal support presented on a differently shaded blue background. Note that classification is done up to the permutation of qudits.

local dimension	2 qubits	3 qutrits	4	5	6	7
AME(3,d)	1	1	1	1	1	1
AME(4,d)	0	1	1	1	1	1
AME(5,d)	1	1	1	2	1	2
AME(6,d)	1	1	∞	∞	1	∞
AME(7,d)	0	1	1	1	0?	∞

Table 6.3: The minimal number of non-SLOCC-equivalent AME orbits. The question mark by zero value suggests that the existence of the relevant state is dubitative, while 0 itself emphasizes that the relevant state certainly does not exist. Note that the AME(4,6) was constructed in Chapter 5.

Firstly, consider any two k -uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$ with all phases equal to 1. We have showed that both states are LU-equivalent if and only if there exist local permutation matrices relating $|\psi\rangle$ and $|\psi'\rangle$. On the other hand, both states are in one-to-one correspondence with OAs of index unity. In such a way, the existence of local permutation is equivalent to an isomorphism between two OAs of index unity, and hence LU-classification is equivalent to the classification of OAs of index unity. Classification of OAs of index unity is an open mathematical problem. For small number of parties N , uniformity k , and the local dimension d is small $d < 9$, it is known that all OAs of index unity are isomorphic [BM08, WW92, ST07, SGZ18].

Conjecture 3. *All OAs of index unity with small local dimension $d < 9$ are isomorphic by permutations of symbols on each level. Equivalently, all k -uniform states with minimal support and all terms phases equal are LU-equivalent.*

Secondly, in Proposition 9 shows that any LU-operator between two k -uniform states of minimal support consists of products of phase (diagonal) and permutation matrices (for $2k < N$). Furthermore, considering states such states with various phases, we showed that not all of them are LU-equivalent for $k > 2$. The precise description of SLOCC classes, however, is not given.

Thirdly, Section 6.3 discusses the significant difference between k -uniform states of minimal support with $2k < N$ and $2k = N$ particles. We showed that LU-equivalence between two k -uniform states with $2k = N$ might be decomposed into multiplication of Butson-type matrices and local monomial matrices. From our analysis, it is not yet clear whether such LU-equivalences are beyond local monomial equivalences. Indeed, in all examples where LU-equivalence involves Butson-type matrices, related states were always LM-equivalent. Therefore we conjecture that Corollary 3 holds true in the case $2k = N$ (even though Proposition 9 does not hold anymore).

Conjecture 4. *AME($2k, d$) states with minimal support are LU-equivalent if and only if they are LM-equivalent*

Finally, we address the problem of equivalence of AME($2k, d$) states in composite dimensions. It turned out, that in many cases composite dimensions allows for the construction of AME states based on OAs of different linear structures. Chapter 4 presents quantum states related to OAs with different linear structures. In particular, in Section 4.3, we show two states

$|\psi\rangle_{\text{GF}(4)}$, and $|\psi\rangle_{\mathbb{Z}_4}$ corresponding to OAs of different strength. It turned out that quantum states $|\psi\rangle_{\text{GF}(4)}$, and $|\psi\rangle_{\mathbb{Z}_4}$ are 1-, and 2-uniform respectively, see Figure 4.1. Notice that an immediate consequence is that both states are not SLOCC-equivalent. Indeed, one may compare the ranks of reduced density matrices, which agree for SLOCC-equivalent states [HW08].

Nevertheless, Section 4.3 presents three AME states $|\psi\rangle_{\mathbb{Z}_9}$, $|\psi\rangle_{\mathbb{Z}_3 \oplus \mathbb{Z}_3}$, and $|\psi\rangle_{\text{GF}(9)}$ of four particles with the local dimension $d = 9$. They are related to OAs of a linear structures over rings $\mathcal{R} = \text{GF}(9), \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ respectively. Notice that OA $(81, 4, 9, 2)$ corresponding to those three different ring structures are not equivalent by relabelling symbols. This might be scrutinized by analysis of existing tables of OAs [Slo]. This, however, does not imply that the corresponding quantum states are not LU-equivalent [BR20], however, the numerical analysis suggests so. Note that results obtained in Proposition 11 do not allow for the verification of SLOCC-equivalence of the aforementioned states.

Conjecture 5. *Consider three rings $\mathcal{R} = \text{GF}(9), \mathbb{Z}_9$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ with nine elements presented in Figure 4.3. Three states constructed via*

$$|\psi\rangle = \frac{1}{d} \sum_{i_1, i_2=0}^8 |i_1\rangle |i_2\rangle |i_1 + i_2\rangle |2i_1 + i_2\rangle, \quad (6.5)$$

where addition and multiplication are considered over the relevant ring $\mathcal{R}$, are pairwise non SLOCC-equivalent $\text{AME}(4, 9)$ states.

6.9 Conclusions

In this chapter, I develop general techniques of SLOCC-verification between k -uniform and AME states. In particular, I show that two k -uniform states of N particles with the minimal support, where $k < 2N$, are SLOCC-equivalent iff they are locally monomial (LM)-equivalent (Proposition 9 and Corollary 3). Furthermore, when uniformity achieves its upper bound, i.e. $k = 2N$, such equivalence is always provided by a Butson-type matrix or monomial matrix (Proposition 11). This restriction is valid, however, only for small local dimensions d and number of parties N (Remark 1), in particular for arbitrary N and $d < 9$. I illustrate the usefulness of the provided criteria on various examples. Firstly, I show that the existence of AME states with minimal support of 6 or more particles yields the existence of infinitely many such non-SLOCC-equivalent states (Corollary 7). Secondly, I show that some AME states cannot be locally transformed into existing AME states of minimal support (Proposition 10).

Chapter 7

Roots of polynomial invariants

In this chapter, I provide necessary and sufficient conditions for generic N -qubit states to be equivalent under SLOCC operations using a single polynomial entanglement measure. More precisely, I investigate how the roots of the entanglement measure behave under SLOCC operations. I demonstrate that SLOCC operations may be represented geometrically by Möbius transformations on the roots of the entanglement measure on the Bloch sphere. I show that if the states are SLOCC-equivalent, then the roots of the polynomial entanglement measure for each state must be related by a Möbius transformation, which is straightforward to verify. I use this procedure to show that the roots of the 3-tangle measure classify 4-qubit generic states. Moreover, I propose an alternative method to obtain the normal form of a 4-qubit state which bypasses the possibly infinite iterative procedure. An extension of presented results in which the author's part was not substantial can be found in the joint work [BQA21], attached to the dissertation in Appendix E.

7.1 Polynomial Invariant Measures

An *entanglement measure* is any function $E(|\psi\rangle)$ defined for all pure states of N qubits which vanishes for all separable states. As we discussed in Introduction, a particularly desired feature of an entanglement measure is invariance under SLOCC operations. Apart from the normalization of states, a SLOCC operation acting on N -qubit state $|\psi\rangle$ can be represented by the action of a local invertible operator with a determinant equal to one, $\mathcal{O}_{\vec{N}} \in \text{SL}(2, \mathbb{C})^{\otimes N}$, where $\text{SL}(2, \mathbb{C})$ is the special linear group of complex matrices of order 2. In this chapter, we consider only the subclass of invertible matrices

with determinant one, unlike in Chapter 6, see Eq. (6.2). To distinguish the subclass of invertible matrices with determinant one from the class of all invertible matrices, we shall denote them with the calligraphic font $\mathcal{O}$ in contrast to the notation O used in Chapter 6 for all invertible matrices.

A given entanglement measure $E(|\psi\rangle)$ defined on the system of N qubits is called a *SL-invariant polynomial of homogeneous degree h* if it is polynomial in the coefficients of a pure state $|\psi\rangle$ and satisfies

$$E(\kappa \mathcal{O}_{\bar{N}} |\psi\rangle) = \kappa^h E(|\psi\rangle) \quad (7.1)$$

for each real constant $\kappa > 0$ and operator $\mathcal{O}_{\bar{N}} \in \text{SL}(2, \mathbb{C})^{\otimes N}$ [DVC00, GW13, ES14]. Note that Eq. (7.1) relates the entanglement measure of any non-normalized state with the entanglement measure of the related normalized state by fixing operator $\mathcal{O}_{\bar{N}}$ as an identity operator and choosing an appropriate constant κ . A *SL-invariant polynomial* (SLIP) measure of homogeneous degree h will be denoted as $\text{SLIP}_{\bar{N}}^h$, where the upper index indicates the degree h of the polynomial and the lower index is related to the number of qubits N .

Among various approaches to the problem of quantification and classification of entanglement, the one via SLIP measures turned out to be a particularly useful. The most famous examples of such measures are concurrence and three-tangle, which measure the 2-body and 3-body quantum correlations of the system respectively [Woo98, CKW00]. SLIP measures provide a convenient method for entanglement classification and its practical detection at the same time. For example, it was demonstrated that almost all SLOCC equivalence classes can be distinguished by ratios of SLIP measures [GW13]. Moreover, any given two states are SLOCC-equivalent if a *complete set of SLIP measures* achieves the same values for both of them [VES11]. However, the size of such a set grows exponentially with the number of qubits N , making it intractable to use this approach to decide SLOCC-equivalence between states with more than four qubits [LBS⁺07].

Any SLIP measure E might be further extended to the set of mixed states by determining the largest convex function on such set which coincides with E on the set of pure states [Uhl98]. Despite the simple definition of a *convex roof extension*, its evaluation requires non-linear minimization procedure, and through this, is a challenging task for a general density matrix [OV06, RDMLA14, TMG15, ROA16, SG18]. An auspicious attempt to address this task was carried out by introducing the so-called *zero-polytope*, the convex hull of pure states with vanishing E measure [OSU08, LOSU06, Ost16, GO18]. In the simplest case of rank-2 density matrices ρ , the zero-polytope can be represented as a convex polytope inscribed in a Bloch sphere,

spanned by the roots of E [LOSU06, RA16a]. I adapt this approach, and focus only on the vertices of the zero-polytope, equivalently the roots of polynomial invariant.

7.2 System of roots

We begin with the general discussion of roots of a SLIP measure, which adapts approach presented in [LOSU06, RA16a]. Firstly, consider a $(N+1)$ -partite qubit state $|\psi\rangle$. Such a state can be uniquely written as

$$|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle, \quad (7.2)$$

which provides the canonical decomposition of its reduced density matrix

$$\rho = |\psi_0\rangle \langle \psi_0| + |\psi_1\rangle \langle \psi_1|$$

obtained by tracing out the first qubit. Notice that both states $|\psi_0\rangle$ and $|\psi_1\rangle$ are in general neither normalized nor orthogonal. Secondly, consider the following family of non-normalized states

$$|\psi_z\rangle = z |\psi_0\rangle + |\psi_1\rangle, \quad (7.3)$$

where $z \in \hat{\mathbb{C}}$ is taken from the extended complex plane $\hat{\mathbb{C}}$, which contains all complex numbers plus infinity. We shall refer to such a representation as the *extended plane representation*.

Furthermore, consider any SLIP_N^h measure E defined on the set of N -partite pure qubit states. By the definition, E is polynomial in the coefficients of $|\psi_z\rangle$, hence it is also polynomial in the complex variable z [OSU08]. In such a way, the polynomial

$$E(z |\psi_0\rangle + |\psi_1\rangle)$$

has exactly h roots: $\zeta_1, \dots, \zeta_h$ (which may be degenerated and/or at infinity), in accordance with the degree of E .

Using the complex number z , the states $|\psi_z\rangle$ can be mapped to the surface of a sphere via the standard *stereographic projection*

$$(\theta, \phi) := (\arctan 1/|z|, -\arg z)$$

written in spherical coordinates. Furthermore, a point on the unit 2-sphere (θ, ϕ) can be associated with the following quantum state

$$|\tilde{\psi}_z\rangle := \cos \frac{\theta}{2} |\psi_0\rangle + \sin \frac{\theta}{2} e^{i\phi} |\psi_1\rangle \quad (7.4)$$

with $z = \text{ctg}(\theta/2) e^{-i\phi}$. Note that $|\psi_0\rangle$ lies in the North pole and $|\psi_1\rangle$ lies in the South pole, see Figure 7.1. We shall refer to such a representation as the *Bloch sphere representation*. Note the following proportionality of states: $|\tilde{\psi}_z\rangle \propto |\psi_z\rangle$, and that neither of these states is normalized, since $|\psi_0\rangle$ and $|\psi_1\rangle$ are not normalized in general either.

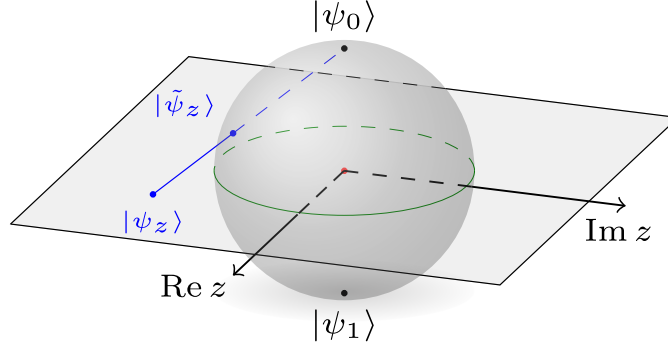


Figure 7.1: The stereographic projection relates the family of states $|\psi_z\rangle$ on the extended complex plane with the family of states $|\tilde{\psi}_z\rangle$ on the Bloch sphere. The spherical coordinates (θ, ϕ) and the complex coordinate z are related by the stereographic projection $z = \text{ctg}(\theta/2) e^{-i\phi}$.

7.3 Local operations on the system of roots

Each linear invertible operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, might be associated with a *Möbius transformation* $z \mapsto z' := \frac{az+b}{cz+d}$, which maps the extended complex plane $\hat{\mathbb{C}}$ into itself [BZ17, Aul12]. The composition of two such transformations is related to the multiplication of the associated operators. Furthermore, $z \mapsto z' := \frac{dz-b}{-cz+a}$ is an inverse Möbius transformation related with $\mathcal{O}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Although Möbius transformations are typically represented on the extended complex plane, one may also represent them as transformations on the Bloch sphere via the stereographic projection. This correspondence between invertible operators and Möbius transformations represented on the Bloch sphere was already successfully used for SLOCC classification of permutation-symmetric states [BKM⁺09, RM11, KL20]. Studies of effects of SLOCC operations on the system of roots might be summarized in the theorem below.

Theorem 2. Consider an $(N+1)$ -partite pure quantum state $|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle$. The roots ζ_i of any $SLIP_N^h$ entanglement measure associated to the partial trace of the first qubit:

1. are invariant under invertible operators, i.e. invariant under $\mathbf{1} \otimes \mathcal{O}_{\vec{N}} \in SL(2, \mathbb{C})^{\otimes N}$ operators;
2. transform via an inverse Möbius transformation $\zeta'_i = \frac{d\zeta_i - b}{-c\zeta_i + a}$ w.r.t the $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \mathbf{1}^N \in SL(2, \mathbb{C})$ operator.

Before we proceed with the proof of Theorem 2, we shall emphasize that normalizing the states $|\psi_0\rangle$ and $|\psi_1\rangle$ after the action of the local operator $\mathcal{O}_{N+1} \in SL(2, \mathbb{C})^{\otimes N+1}$, as is the case in existing related works [OSU08, LOSU06, GO18, RA16a, RA16b], spoils the consistency of the stereographic projection with the roots transformations. As a consequence, the action of SLOCC operators on the states $|\psi_z\rangle$ would no longer be given by the associated Möbius transformation, and the statements presented in Theorem 2 would no longer hold true.

Proof of Theorem 2. As we discussed, a $N + 1$ partite qubit state $|\psi\rangle \in \mathcal{H}_2^{\otimes(N+1)}$ might be written as

$$|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle. \quad (7.5)$$

This form provides the canonical decomposition of the reduced density matrix

$$\rho = |\psi_0\rangle \langle \psi_0| + |\psi_1\rangle \langle \psi_1|$$

for the two non-normalized states $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}_2^{\otimes N}$. Consider now a reversible operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ acting on the first qubit. Under the action of above operator, the state $|\psi\rangle$ is transformed into

$$|\psi'\rangle := \mathcal{O} |\psi\rangle = |0\rangle (a |\psi_0\rangle + b |\psi_1\rangle) + |1\rangle (c |\psi_0\rangle + d |\psi_1\rangle) = |0\rangle |\psi'_0\rangle + |1\rangle |\psi'_1\rangle \quad (7.6)$$

where

$$|\psi'_0\rangle := a |\psi_0\rangle + b |\psi_1\rangle, \quad (7.7)$$

$$|\psi'_1\rangle := c |\psi_0\rangle + d |\psi_1\rangle. \quad (7.8)$$

Consider now any superposition of states $|\psi'_0\rangle$ and $|\psi'_1\rangle$. In particular, observe that

$$\begin{aligned} |\psi'_z\rangle &:= z|\psi'_0\rangle + |\psi'_1\rangle = z\left(a|\psi_0\rangle + b|\psi_1\rangle\right) + c|\psi_0\rangle + d|\psi_1\rangle \\ &= (az + b)|\psi_0\rangle + (cz + d)|\psi_1\rangle \\ &\propto \frac{az + b}{cz + d}|\psi_0\rangle + |\psi_1\rangle. \end{aligned}$$

In other words, we have

$$\mathcal{O}|\psi_z\rangle = |\psi_{z'}\rangle, \quad z' = \frac{az + b}{cz + d}, \quad (7.9)$$

i.e. the operator $\mathcal{O}$ transforms states in the extended plane representation by applying a Möbius transformation on the index z . Suppose that ζ_i is a complex root of a polynomial function E , i.e. $E(\zeta_i|\psi_0\rangle + |\psi_1\rangle) = 0$. Acting on the first qubit with operator $\mathcal{O}$, the density matrix obtained by tracing out the first qubit becomes $|\psi'_0\rangle\langle\psi'_0| + |\psi'_1\rangle\langle\psi'_1|$, and hence the entanglement measure E will vanish for a roots ζ'_i , such that $E(\zeta'_i|\psi'_0\rangle + |\psi'_1\rangle) = 0$. Using Eqs. (7.7)-(7.8), the later equation transforms into

$$E\left((c\zeta'_i + d)\left(\frac{a\zeta'_i + b}{c\zeta'_i + d}|\psi_0\rangle + |\psi_1\rangle\right)\right) = 0 \quad (7.10)$$

where the factor $(c\zeta'_i + d)$ is not relevant. Comparing with the equation for the roots before the action of $\mathcal{O}$, we conclude that the roots transform according to the inverse Möbius transformation as

$$\zeta'_i = \frac{d\zeta_i - b}{-c\zeta_i + a}, \quad (7.11)$$

under the action of the operator $\mathcal{O}$. Consequently, the roots of the zero-polytope transform with respect to the inverse Möbius transformation associated to the operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Furthermore, consider multi-local operators $\mathcal{O}_{\vec{N}} = \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_N$ acting on the remaining qubits of the state $|\psi\rangle$ from Eq. (7.5). The state $|\psi\rangle$ transforms accordingly as

$$|\psi'\rangle := \mathcal{O}_{\vec{N}}|\psi\rangle = |0\rangle \underbrace{\mathcal{O}_{\vec{N}}|\psi_0\rangle}_{:=|\psi'_0\rangle} + |1\rangle \underbrace{\mathcal{O}_{\vec{N}}|\psi_1\rangle}_{:=|\psi'_1\rangle}. \quad (7.12)$$

After the action of $\mathcal{O}_{\vec{N}}$, a value of entanglement measure E reads

$$E\left(z|\psi'_0\rangle + |\psi'_1\rangle\right) = E\left(\mathcal{O}_{\vec{N}}(z|\psi_0\rangle + |\psi_1\rangle)\right).$$

However, since E is $\text{SL}(2, \mathbb{C})^{\otimes N}$ invariant function, one may conclude that $E(z|\psi_0\rangle + |\psi_1\rangle) = 0$ iff $E(z|\psi'_0\rangle + |\psi'_1\rangle) = 0$. Hence the roots of both polynomial equations are the same. As a consequence, the roots of the zero-polytope remain unchanged under the action of $\mathcal{O}_{\vec{N}}$. This concludes the proof of Theorem 2. $\square$

7.4 States discrimination

The decomposition (7.2) can be performed with respect to any other subsystem, each with its own system of roots. Any local operator $\mathcal{O}_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the k -th subsystem will influence independently the corresponding k -th system of roots and the associated zero-polytope according to the Möbius transformation $\zeta_i \mapsto \frac{d\zeta_i - b}{-c\zeta_i + a}$. A global action of a local operator $\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_{N+1}$ shall affect all roots and thus related zero-polytopes. Since a Möbius transformations are bijections of the Bloch sphere, the total number of roots will always be preserved [RA16b]. Furthermore, since Möbius transformations are fully classified, the existence of a local transformation between two given states becomes straightforward to verify.

In this way, Theorem 2 provides novel solution for the problem of discriminating $(N + 1)$ -qubit states up to the SLOCC-equivalence [VES11, ZZH16, GW11, BR20]. To verify if two pure states are SLOCC-equivalent, one can thus use the following procedure:

- 1) Choose any SLIP_N^h entanglement measure of degree $h \geq 3$ and calculate its roots for each subsystem for both states. Note that a generic state will always have h roots for each subsystem.
- 2) Focus on one subsystem i , where $1 \leq i \leq N + 1$ and choose 3 of the total h roots from each state.
- 3) Write the unique Möbius transformation between the two triplets of roots. Derive the local operator $\mathcal{O}_i$ associated to such transformation.
- 4) Choose a different set of 3 roots of the second state and repeat step 3). Repeat it for all $3!\binom{h}{3}$ possibilities.
- 5) Repeat steps 3) and 4) for all other subsystems. Consider the tensor products of all the local operators obtained. This results in a finite set of operators of the form $\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_{N+1}$.

- 6) If the two given $(N + 1)$ -qubit states are SLOCC-equivalent, one of obtained operators must transform one state into the other. Otherwise, they are not SLOCC-equivalent.

Presented procedure has two main important features. Firstly, it factorizes the problem of finding SLOCC-equivalence. Indeed, local operations are determined separately for each subsystem. Secondly, it discretizes the initial discrimination task. Indeed, there are at most $(3! \binom{h}{3})^{N+1}$ local operators which might provide SLOCC equivalence between initial states.

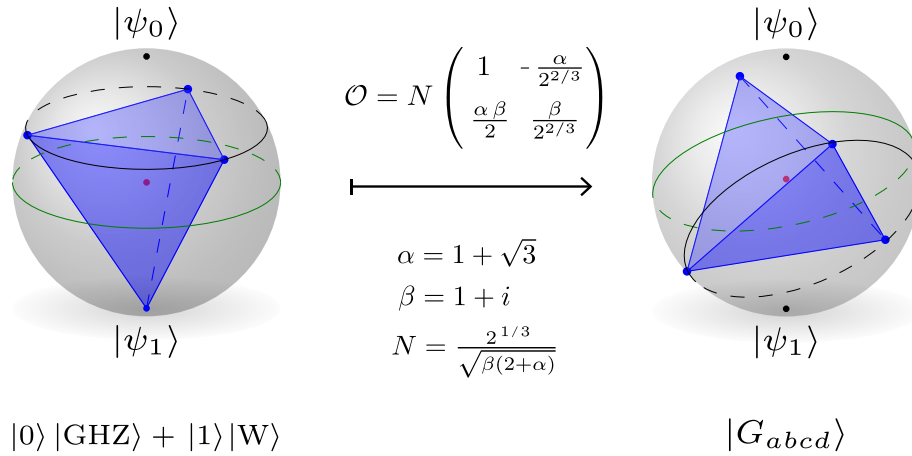


Figure 7.2: The system of four roots (represented as blue dots) related to the 3-tangle polynomial measure $\tau^{(3)}$ evaluated on the first subsystem of the state $1/\sqrt{2}(|0\rangle |\text{GHZ}\rangle + |1\rangle |\text{W}\rangle)$. This system of four points can be mapped into a normal system (i.e. symmetrically related points $z, -z, 1/z, -1/z$) by a Möbius transformation. Similar local transformations can be performed with respect to other subsystems, transforming the states into a state in the normal form.

7.5 Normal system of roots and cross-ratio

Any three distinct points on the sphere can be transformed onto any other three distinct points via a unique Möbius transformation. This is not the case for four points on the sphere. For a given four points on the extended complex plane z_1, z_2, z_3, z_4 , one may associate a so-called *cross-ratio*

$$\lambda(z_1, z_2, z_3, z_4) := \frac{z_3 - z_1}{z_3 - z_2} \frac{z_4 - z_2}{z_4 - z_1}, \quad (7.13)$$

which is preserved under any Möbius transformations [RM11, BZ17]. Two systems of four distinct points are related via Möbius transformations if and only if their associated cross-ratios are the same. The cross-ratio is not invariant under permutations of points, however, and depending on the ordering taken for the four points, it takes six related values [RM11]:

$$\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda - 1}{\lambda}, \frac{\lambda}{\lambda - 1}.$$

A particular interesting set of four points is one of the form $z, 1/z, -z, -1/z$, which we call a *normal system*.

Consider such a set of four related complex points $\Phi = \{z, \frac{1}{z}, -z, -\frac{1}{z}\}$. It is convenient to associate with them the cuboid spanned by eight points:

$$\Phi \cup \bar{\Phi} = \left\{ z, \frac{1}{z}, -z, -\frac{1}{z}, \bar{z}, \frac{1}{\bar{z}}, -\bar{z}, -\frac{1}{\bar{z}} \right\},$$

as it is presented on Figure 7.3. Notice that all six faces of the cuboid are parallel to one of the canonical planes: XZ , XY , or YZ . This property is equivalent to the initial assumption that the set of points Φ is in normal form. Clearly, all rotations of the Bloch ball (associated to the unitary operations) preserve the form of the cuboid. Nevertheless, only a special subgroup of all rotations leave faces of the cuboid parallel to XZ , XY , or YZ . This special subgroup $\mathcal{G}_{24}$ contains exactly 24 elements and is generated by three rotations of $\pi/2$ around X , Y , and Z axis:

$$\mathbf{R}_x(\pi/2) = \begin{pmatrix} \cos \pi/4 & -i \sin \pi/4 \\ -i \sin \pi/4 & \cos \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad (7.14)$$

$$\mathbf{R}_y(\pi/2) = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (7.15)$$

$$\mathbf{R}_z(\pi/2) = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}. \quad (7.16)$$

Notice that $\mathcal{G}_{24}$ is a group of rotations preserving the regular cube (the group of orientable cube symmetries). All rotations in the $\mathcal{G}_{24}$ group preserve the normal-form structure of Φ . Therefore, the normal form is uniquely determined up to 24 rotations in the $\mathcal{G}_{24}$ group.

We show that any set of four points may be mapped into a normal system, for which $z, 1/z, -z, -1/z$.

Proposition 12. *Any system of non-degenerated four points z_1, z_2, z_3, z_4 on the Bloch sphere can be transformed onto the normal form $z, \frac{1}{z}, -z, -\frac{1}{z}$ via some Möbius transformation T . Transformation T is uniquely defined up to 24 rotations in the group $\mathcal{G}_{24}$.*

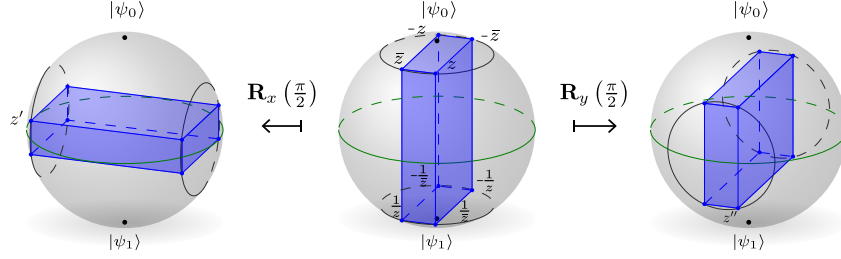


Figure 7.3: A normal system of four points $z, 1/z, -z, -1/z$ together with the conjugate points $\bar{z}, 1/\bar{z}, -\bar{z}, -1/\bar{z}$ span the cuboid whose faces are parallel to the XZ , XY and YZ planes. There are exactly 24 rotations of the Bloch sphere which preserve this property, composing the elements of the group $\mathcal{G}_{24}$. Two of them: rotation by a $\pi/2$ angle around X and Y axes are presented on the left and right respectively. The system of roots transforms according to Eqs. (7.14-7.16), which gives $z \mapsto z' := \frac{z-i}{-iz+1}$ and $z \mapsto z'' := \frac{z-1}{z+1}$ for the two presented rotations.

Proof. Consider a complex number λ . There exists a complex number z , such that the cross-ratio of the four points is equal to λ , i.e.

$$\left(z, \frac{1}{z}; -z, -\frac{1}{z}\right) = \lambda. \quad (7.17)$$

Indeed, the cross-ratio on the left side equals $4z^2/(1+z^2)^2$, and the equation $4z^2/(1+z^2)^2 = \lambda$ has exactly four complex solutions

$$z_0 = \frac{4 - 2\lambda + \sqrt{1 - \lambda}}{2\lambda}, \quad \frac{1}{z_0}, \quad -z_0, \quad -\frac{1}{z_0}. \quad (7.18)$$

In such a way, for a given value λ there exists a unique λ -normal system, such that the cross-ratio of its vertices is given by $(z_0, \frac{1}{z_0}; -z_0, -\frac{1}{z_0}) = \lambda$. Note that replacement of the vertex z_0 by any other vertex $z_0, 1/z_0, -z_0$, or $-1/z_0$ does not change the value of the cross-ratio $(z_0, \frac{1}{z_0}; -z_0, -\frac{1}{z_0}) = \lambda$. Furthermore, there is a unique Möbius transformation T which maps z_1, z_2, z_3 onto $z_0, 1/z_0, -z_0$, with the remaining z_4 mapped onto $-1/z_0$. Observe that the value z_0 is unique up to its inverse, opposite and opposite inverse elements, according to Eq. (7.18), with the corresponding Möbius transformations associated to the matrices $T, \sigma_x T, \sigma_y T$ and $\sigma_z T$. Each of those transformations maps the entire set of points $\{z_1, z_2, z_3, z_4\}$ onto the same set of points $\{z_0, 1/z_0, -z_0, -1/z_0\}$, (the exact bijection between those two sets of roots is different for each transformation).

Depending on the chosen order of four points $\{z_1, z_2, z_3, z_4\}$, the corresponding cross-ratio takes one of six values: $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}$ and $\frac{\lambda}{\lambda-1}$. For each value, there is a corresponding set of solutions of the form $\{z_0, 1/z_0, -z_0, -1/z_0\}$ via Eq. (7.18) with related four Möbius transformations. Hence, in total there are 24 Möbius transformations that map any four non-degenerated points onto a normal system. Each transformation is related to an element of the group $\mathcal{G}_{24}$ which has exactly 24 elements. $\square$

7.6 SLOCC classification of four qubit states

In this section I discuss how one may use the normal system of roots for SLOCC-classification of small systems.

Firstly, consider the three-qubit case. Genuinely entangled pure three-qubit states are SLOCC-equivalent to one of two states [Sud01]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

as we discussed in Introduction. Consider the 2-tangle $\tau^{(2)}$ [Woo98] as the entanglement measure, and its roots as we discussed so far. One may use the roots of $\tau^{(2)}$ to distinguish between the two classes. Indeed, all reduced density matrices of the $|W\rangle$ state have a single root, while there are always two distinct roots for the $|\text{GHZ}\rangle$ state [RA16a].

Secondly, consider the four-qubit case. Contrary to the case of three qubits, there are infinitely many SLOCC classes of four qubit states [Sud01]. Nevertheless, four-qubit states were successfully divided into nine families, most of which contains an infinite number of SLOCC-classes [VDDMV01, CD07, SdVK16]. The family with the most degrees of freedom, so called G_{abcd} family, is represented by states of the following form [VDDMV01]:

$$\begin{aligned} |G_{abcd}\rangle = & \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\ & + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle), \end{aligned}$$

where $a^2 \neq b^2 \neq c^2 \neq d^2$ are pairwise different. Note that any generic 4-qubit states, i.e. 4-qubit states with random coefficients belonging to the G_{abcd} family, since it has the most degrees of freedom.

Choosing the 3-tangle $\tau^{(3)}$ [CKW00] as the entanglement measure, we computed roots of states $|G_{abcd}\rangle$. As we shall see, each state has four non-degenerate roots already in the normal form. Indeed, the state $|G_{abcd}\rangle$ has the following decomposition with respect to the first subsystem $|G_{abcd}\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle$, where

$$\begin{aligned} |\psi_0\rangle &= \frac{a+d}{2} |000\rangle + \frac{a-d}{2} |011\rangle + \frac{b+c}{2} |101\rangle + \frac{b-c}{2} |110\rangle, \\ |\psi_1\rangle &= \frac{a+d}{2} |111\rangle + \frac{a-d}{2} |100\rangle + \frac{b+c}{2} |010\rangle + \frac{b-c}{2} |001\rangle. \end{aligned}$$

Suppose that the three-tangle vanishes, $\tau^{(3)}(\zeta |\psi_0\rangle + |\psi_1\rangle) = 0$. Since $\tau^{(3)}$ is a $\text{SL}(2, \mathbb{C})^{\otimes 3}$ invariant measure, for any local operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ we have

$$\tau^{(3)}\left((\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3)(\zeta |\psi_0\rangle + |\psi_1\rangle)\right) = 0.$$

Thus

$$\begin{aligned} |\psi_0\rangle &= (\sigma_x \otimes \sigma_x \otimes \sigma_x) |\psi_1\rangle, \\ |\psi_1\rangle &= (\sigma_x \otimes \sigma_x \otimes \sigma_x) |\psi_0\rangle, \end{aligned}$$

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices. Hence, taking all local operators $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_3$ equal to σ_x , we conclude that

$$0 = \tau^{(3)}\left((\sigma_x \otimes \sigma_x \otimes \sigma_x)(\zeta |\psi_0\rangle + |\psi_1\rangle)\right) = \zeta |\psi_1\rangle + |\psi_0\rangle \propto \frac{1}{\zeta} |\psi_0\rangle + |\psi_1\rangle, \quad (7.19)$$

and $1/\zeta$ is another root of $\tau^{(3)}$. Similarly, considering $(\sigma_y \otimes \sigma_y \otimes \sigma_y)$ and $(\sigma_z \otimes \sigma_z \otimes \sigma_z)$, one may find another two roots $-\zeta, -1/\zeta$ of a measure $\tau^{(3)}$. Furthermore, it shows that the roots of $\tau^{(3)}$ evaluated on any state from the G_{abcd} family are symmetrical with respect to rotations around X, Y, Z axes by the angle π . Writting $\tau^{(3)}(z |\psi_0\rangle + |\psi_1\rangle) = 0$ explicitly, we obtain the equation

$$\tau^{(3)}(z |\psi_0\rangle + |\psi_1\rangle) = Az^4 - 2(2B + A)z^2 + A = 0, \quad (7.20)$$

where $A = (b^2 - c^2)(a^2 - d^2)$ and $B = (c^2 - d^2)(a^2 - b^2)$. The above equation is non-degenerated iff $A, B, A + 2B \neq 0$, which happens iff $a^2 \neq b^2 \neq c^2 \neq d^2$ are pairwise different.

Recall that the normal form of roots is unique up to the group of rotations $\mathcal{G}_{24}$. In such a way, the problem of SLOCC-equivalence of states $|G_{abcd}\rangle$ becomes solvable, with a discrete amount of solutions. Indeed, two states in

the G_{abcd} class are SLOCC equivalent iff one can be obtained from the other by the action of an element of the finite class of operators $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. We thus find that exactly 192 states of the form $|G_{abcd}\rangle$ are SLOCC-equivalent.

Proposition 13. *Two states $|G_{abcd}\rangle$ and $|G_{a'b'c'd'}\rangle$ are SLOCC-equivalent iff their coefficients are related by the following three operations:*

1. *multiplication by a phase factor $(a', b', c', d') = e^{i\phi}(a, b, c, d)$,*
2. *and permutation $\sigma \in S_4$ of coefficients $(a', b', c', d') = \sigma(a, b, c, d)$,*
3. *and change of sign in front of two coefficients from a, b, c, d .*

Before we proceed with the proof, note that the symmetry in Proposition 13 is given by the Weyl group of Cartan type D_4 . Such symmetry has already been observed among generators of four-qubit polynomial invariants exhibit this type of symmetry [LT03, CD07, HLP14]. As a consequence, this result constitutes a new relation between 4-qubit invariants and the convex roof extension of 3-tangle $\tau^{(3)}$. This may shed some light on the problem of generalizing the CKW inequality [CKW00] for four qubit states [GW10, EOS09, RDMLA14, ES14, ROA16], and beyond [EOS09, GW10, ES15].

Lemma 1. *Any local operator $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 \otimes \mathcal{O}_4 \in SL(2, \mathbb{C})^{\otimes 4}$ which transforms states $|G_{a'b'c'd'}\rangle \propto \mathcal{O} |G_{abcd}\rangle$ with $a^2 \neq b^2 \neq c^2 \neq d^2$, is of the form $\mathcal{O}_i \in \mathcal{G}_{24}$.*

Proof. Consider a local operator $\mathcal{O}_1$ acting on the first qubit which transforms the state $|G_{abcd}\rangle$ onto $|G_{a'b'c'd'}\rangle$. Operator $\mathcal{O}_1$ also transforms corresponding systems of roots denoted as Λ and Λ' , respectively, via the action of the related Möbius transformation. Note that both systems Λ and Λ' are in the normal form, therefore, according to Proposition 12, we have that $\mathcal{O}_i \in \mathcal{G}_{24}$. Similar analysis with respect to all other qubits shows that $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4 \in \mathcal{G}_{24}$. $\square$

Proof of Proposition 13. Lemma 1 shows that the search for a general form of an SLOCC-equivalence between the states $|G_{abcd}\rangle$ and $|G_{a'b'c'd'}\rangle$ might be restricted to the search within the finite class of operators $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. Note that the group $\mathcal{G}_{24}$ has only 24 elements, and hence one may numerically verify that there are exactly $8 \times 24 = 192$ states in the G_{abcd} family which are SLOCC-equivalent to $|G_{abcd}\rangle$ by $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. For example, the following tensor operation

$$\mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \quad (7.21)$$

transforms state $|G_{abcd}\rangle$ into $|G_{-b-acd}\rangle$. This might be written as a transformation of a tuples of indices: the tuple (a, b, c, d) is transformed into the tuple $(-b, -a, c, b)$. Similarly, the operators showed on the following right hand sides provide the corresponding transformations of the tuple (a, b, c, d) on the left side:

$$\begin{aligned}
\mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) &\longleftrightarrow (a, \textcolor{blue}{d}, c, \textcolor{blue}{b}), \\
\mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) &\longleftrightarrow (-\textcolor{blue}{d}, b, c, -\textcolor{blue}{a}), \\
\mathbf{R}_y(\pi) \otimes \mathbf{R}_y(\pi) \otimes \mathbf{1} \otimes \mathbf{1} &\longleftrightarrow (a, -\textcolor{blue}{b}, -\textcolor{blue}{c}, d), \\
\mathbf{R}_x(\pi) \otimes \mathbf{R}_x(\pi) \otimes \mathbf{1} \otimes \mathbf{1} &\longleftrightarrow (a, b, -\textcolor{blue}{c}, -\textcolor{blue}{d}), \\
\mathbf{R}_y(\pi) \otimes \mathbf{1} \otimes \mathbf{R}_y(\pi) \otimes \mathbf{1} &\longleftrightarrow (\textcolor{blue}{d}, \textcolor{blue}{c}, \textcolor{blue}{b}, a), \\
\mathbf{R}_x(\pi) \otimes \mathbf{1} \otimes \mathbf{R}_x(\pi) \otimes \mathbf{1} &\longleftrightarrow (\textcolor{blue}{b}, \textcolor{blue}{a}, \textcolor{blue}{d}, c).
\end{aligned}$$

In addition, the tuples (a, b, c, d) and $(-a, -b, -c, -d)$ represent the same state, since states are defined up to the global phase. Note that any composition of the above operations also provides SLOCC equivalences between states of the form $|G_{abcd}\rangle$. The eight aforementioned transformations of tuples generate all possible permutations of the a, b, c, d indices, together with the change of a sign of any two or all four indices. Note that there are exactly 24 permutations in total and for each permutation the signs can be matched in exactly $1 + \binom{4}{2} + 1 = 8$ ways. This gives 192 tuples representing SLOCC equivalent states, which perfectly matches the numerical result.

Note yet another trivial manipulation with indices a, b, c, d comes from multiplying by a global phase. This operation transforms the indices as

$$(e^{i\theta}a, e^{i\theta}b, e^{i\theta}c, e^{i\theta}d) \sim (a, b, c, d),$$

resulting in the same quantum state for any real number $\theta \in [0, 2\pi)$. In particular, for $\theta = \pi$, the system of opposite indices determines the same state as the initial one, i.e. $(-a, -b, -c, -d) \sim (a, b, c, d)$. $\square$

Finally, we show a link between normal systems of roots and the normal form of pure states of four qubits in the G_{abcd} class. We say that a state is in its normal form if all its reductions are maximally mixed [VDDM03]. The process to determine the normal form of a state (if it exists) is straightforward, although, it may also turn out to be an infinite iterative process [VDDM03]. However, presented results of Theorem 2 applied to the four qubits states in the G_{abcd} family show that this difficulty can be avoided. Indeed, consider the representative state of the G_{abcd} family is in normal form

[VDDM03] and the corresponding roots associated with the three-tangle $\tau^{(3)}$ measure also form a normal system. One may calculate the roots associated with any state in the G_{abcd} class, and transform them into a normal system of roots using a Möbius transformation. Furthermore, the associated SLOCC operator to convert the initial state into a state in the normal form. We illustrate this procedure by transforming the widely discussed four-partite state $\frac{1}{\sqrt{2}}(|0\rangle|\text{GHZ}\rangle + |1\rangle|\text{W}\rangle)$ [GO18, LOSU06] into its normal form, see Figure 7.2. Without presented technique, the standard way of obtaining the normal form would result in an infinite iterative procedure.

7.7 Conclusions

In this chapter, I showed how a single SL-invariant polynomial entanglement measure SLIP_N^h defined in N -qubit system can be used to derive necessary and sufficient conditions for any two $(N+1)$ -qubit states to be equivalent under SLOCC operations. This was possible by showing that the roots of any SLIP_N^h measure transform via Möbius transformations under the SLOCC operations performed on the subsystems (Theorem 2). In that way, SLOCC equivalence between two states is implied by the easily verifiable existence of a Möbius transformation relating the aforementioned roots for each subsystem, as it is described in a detailed way in the procedure presented in Section 7.4. Furthermore, I demonstrated that the 3-tangle measure $\tau^{(3)}$ is enough to classify 4-qubit generic states (Proposition 13). Lastly, I present a procedure of determining the normal form of states in the G_{abcd} family that circumvents the possibility of an infinite iterative process of the standard procedure.

Chapter 8

Concluding remarks

8.1 Conclusions

This dissertation covers various aspects of the quantification and classification of entanglement in multipartite states and the role of symmetry in such systems. The main results of the research in this thesis can be summarized in the following.

1. Chapter 2 introduces the notion of m -resistant states of N -partite systems and presents it by analogy with topological links (Definition 1). I present two general methods of constructing m -resistant states, one based on the Majorana representation of symmetric states (Proposition 1), the other on the combinatorial objects known as orthogonal arrays (Propositions 2 and 3).
2. In Chapter 3, I introduce the notion of H -symmetric states, where H is an arbitrary subgroup of permutation group – see Definition 2. Proposition 4 provides an elementary example of a quantum state with the given group of symmetry.
3. I introduce two families of states with remarkable symmetric properties: in Definition 4 excitation-states $|G\rangle$, related to a graph G , and in Definition 3 Dicke-like states, determined by an embedding of a subgroup $H < \mathcal{S}_N$. Furthermore, I investigate entanglement properties of excitation states related to regular graphs, in particular the form of 2-partite concurrence (Proposition 7), and generalized concurrence (Proposition 8) of such states.

4. I present two different methods of constructing excitation states: in Section 3.7 excitation states are obtained as ground states of Hamiltonians with 3-body interactions, while in Section 3.6 as an outcome of a quantum circuit, with satisfactory complexity. A simple example of five qubit excitation is successfully simulated on available quantum computers: IBM - Santiago and Athens (Figure 3.8).
5. In Chapter 5, I present the famous combinatorial problem of 36 officers that was posed by Euler, and its quantum version. I show an analytical form of an AME(4, 6) state, which might be seen as a quantum solution to the Euler's problem. Moreover, I present a coarse-grained combinatorial structure behind constructed AME(4, 6) state, and elaborate how such coarse-grained combinatorial structures might lead to the construction of other genuinely entangled states beyond the stabilizer approach.
6. Chapter 6 presents general techniques of verification whether two k -uniform or AME states are SLOCC-equivalent. In particular, I show that SLOCC equivalence between two k -uniform states of N particles with the minimal support has one of the following forms: if $k < 2N$, it is provided by local monomial matrix (Proposition 9 and Corollary 3), while if $k = 2N$ by a Butson-type complex Hadamard matrix or monomial matrix (Proposition 11).
7. I apply general results concerning the SLOCC equivalence of AME states. In particular, I show that some AME states cannot be locally transformed into existing AME states of minimal support (Proposition 10), which falsifies the conjecture that for a given multipartite quantum system all AME states are locally equivalent. Furthermore, I show that the existence of AME states with minimal support of 6 or more particles yields the existence of infinitely many such non-SLOCC-equivalent states (Corollary 7). As an immediate consequence, I show that not all AME states belong to the class of stabilizer states.
8. In Chapter 7, I show how a single entanglement measure can be used to derive necessary and sufficient conditions for any two multipartite quantum qubit states to be equivalent under SLOCC operations. Precisely, I show that under an SLOCC operation the roots of any appropriate measure transform via related Möbius transformation (Theorem 2). SLOCC equivalence between two states is implied by the easily verifiable existence of a Möbius transformation relating the afore-

mentioned roots for each subsystem, as it is precisely described in the procedure presented in Section 7.4.

9. I apply general results described in Chapter 7 to 4-qubit systems. In particular, I demonstrate that the 3-tangle measure is enough to classify 4-qubit generic states (Proposition 13). Lastly, I present a procedure of determining the normal form of a generic 4-qubit state that circumvents the possibility of an infinite iterative process of the standard procedure.

8.2 Open problems

As we have demonstrated so far, the quantification and classification of entanglement for multipartite states is an ambitious long-term project. We hope that the results presented in this dissertation will contribute to the development of these complex issues. In addition, we outline related problems and important open questions for future research.

- A. As we elaborated in Chapter 2, it is not clear if m -resistant states of N -qubit do exists for any $m = 0, 1, \dots, N - 2$, especially among symmetric states. One may also pose the more general problem of verification whether, for any number of parties N and number m , there exists an m -resistant N -qudit state, see Conjecture 1.
- B. Admissible groups of symmetries in multipartite quantum systems of N -qudits. For any subgroup of symmetric group $H < \mathcal{S}_N$, what is the minimal local dimension d for which there exists a H -symmetric state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$, see Question 1? In particular, which groups of symmetries are admissible in systems of N qubits?
- C. Chapter 5 presents a coarse-grained combinatorial structure behind constructed AME(4, 6) state. Is there any general method for constructing such objects in any dimension? Based on such coarse-grained combinatorial structures, is it possible to provide a general construction of other genuinely entangled states beyond the stabilizer approach?
- D. Uniqueness of AME states in small dimensions. Verify if all OAs of index unity with small local dimension $d < 9$ are isomorphic by permutations of symbols on each level (Conjecture 3). From this would follow that all related k -uniform states with minimal support and all phases equal are LU-equivalent.

- E. Precise form of SLOCC-equivalence between $\text{AME}(2k, d)$ states. Which subclass of Butson-type complex Hadamard matrices might appear in the LU-equivalence between $\text{AME}(2k, d)$ states of the minimal support? Verify if all $\text{AME}(2k, d)$ states with minimal support are LU-equivalent if and only if they are LM-equivalent (Conjecture 4).
- F. Linear structures of orthogonal arrays and corresponding k -uniform and AME states. Verify if AME states related to different linear structures of classical combinatorial designs are not equivalent. In particular, verify whether three $\text{AME}(4, 9)$ states presented in Conjecture 5 are SLOCC-equivalent.
- G. CKW-equalities [CKW00]. Proposition 13 constitutes a new relation between 4-qubit invariants and the convex roof extension of 3-tangle. Is it possible to understand presented relation in context of generalized CKW (in)equalities of four qubit states?
- H. Classification of large multipartite systems. Chapter 7 shows how generic 4-qubit states may be classified by the 3-tangle measure. Is it possible to provide such classification for generic N -qubit states, $N > 4$, via a single entanglement measure?

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Appendices

Reprints of original publications and preprints

I attach below the following three publications and two preprints, which do not belong to the main body of this thesis. As they present some further results not discussed here, I attach them for the convenience of the reader.

- [A] G. M. Quinta, R. André, **A. Burchardt**, and K. Życzkowski *Cut-resistant links and multipartite entanglement resistant to particle loss*, Phys. Rev. A 100, 062329 (2019).
- [B] **A. Burchardt**, Z. Raissi, *Stochastic local operations with classical communication of absolutely maximally entangled states*, Phys. Rev. A 102, 022413 (2020).
- [C] **A. Burchardt**, J. Czartowski, and K. Życzkowski, *Entanglement in highly symmetric multipartite quantum states*, Phys. Rev. A 104, 022426 (2021).
- [D] S. Rather*, **A. Burchardt***, W. Bruzda, G. Rajchel-Mieldzioć, A. Lakshminarayan, K. Życzkowski, *Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem*, ArXiv: 2104.05122 (2021).
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- [E] **A. Burchardt**, G. M. Quinta, R. André, *Entanglement Classification via Single Entanglement Measure*, ArXiv: 2106.00850 (2021).

**A Cut-resistant links and multipartite entanglement
resistant to particle loss**

Cut-resistant links and multipartite entanglement resistant to particle loss

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In this work, we explore the space of quantum states composed of N particles. To investigate entanglement resistant to particle loss, we introduce the notion of m -resistant states. A quantum state is m resistant if it remains entangled after losing an arbitrary subset of m particles but becomes separable after losing any number of particles larger than m . We establish an analogy to the problem of designing a topological link consisting of N rings such that, after cutting any $(m + 1)$ of them, the remaining rings become disconnected. We present a constructive solution to this problem, which allows us to exhibit several N -particle states with the desired property of entanglement resistance to particle loss.

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I. INTRODUCTION

Entanglement is one of the most fundamental resources for quantum technologies. It is the pillar-stone of dense coding, quantum teleportation, quantum key distribution [1], error correcting codes [2], and quantum computation [3,4]. Therefore various aspects of quantum entanglement have been investigated [5] and different methods for its experimental detection proposed [6,7].

After more than two decades of intensive research, it is fair to say that entanglement in bipartite systems is well understood and quantified [5]. On the other hand, entanglement in the multipartite setup is much more complex and intricate. In particular, the topic of characterizing, classifying, and quantifying the entanglement of many-particle systems turned out to be an extremely rich subject and is still an active field of research [8,9]. Already for three or more subsystems there is no unique way to quantify entanglement, as different entanglement measures induce different orderings and are maximized by different states [10,11]. For instance, the three-qubit state $|\text{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ is the most entangled with respect to an important three-party entangle-

ment measure called the three-tangle [12], while the state $|\text{W}\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ maximizes the average entanglement contained in two-party reductions.

Therefore, it is reasonable to characterize entanglement depending on the applications considered. A possible example of such a classification comes through stochastic local operations assisted by classical communications (SLOCC). In this approach [13], one considers two states to be equivalent if they can be obtained from one another with some finite probability using only local operations assisted by classical communications. States belonging in the same SLOCC class may then be used to perform the same protocols, which motivates this scheme of classification. Although for three qubits there exist only two SLOCC classes [10], already for four qubits there are infinitely many such classes [14]. It is thus necessary to focus on different traits of quantum states in order to find alternative finite classification schemes.

Many useful and complementary features of quantum entanglement have been developed in recent years, shedding some light on different aspects of multipartite entanglement. Certain approaches proposed for two subsystems can be directly generalized for a larger number of parties. For instance, the robustness of entanglement, originally defined for bipartite systems [15] as the minimal amount of mixing with separable states needed to wipe out the entanglement, can be directly applied to multipartite systems.

Starting from the notion of the Bell state one can proceed to the multipartite domain by analyzing how a given state is

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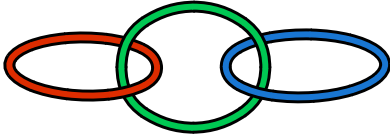


FIG. 1. An element of link class 3^3 , classified by the polynomial $ab + bc$, given in (1).

affected if some of its parts are altered or ignored. This procedure leads to the concept of absolutely maximally entangled (AME) states, which are maximally entangled with respect to all possible bipartitions of the system [16,17]. AME states are particularly relevant for multipartite teleportation and quantum secret sharing [18] and for construction of quantum error correction codes [19].

A natural generalization of AME states is the so-called k -uniform states of N parties, for which a partial trace over any $(N - k)$ subsystem results in a maximally mixed state [20,21]. Such states can be constructed with the help of combinatorial tools like orthogonal arrays (OAs) [22], which allow for a coarse-grained classification of multipartite entanglement [23]. Another related measure, called the persistence of entanglement [24], is defined by the minimal number of local measurements such that the state becomes completely disentangled for any measurement outcome. This property provides a somewhat intuitive notion of entanglement strength by looking into the minimal requirements for destroying it completely.

An effort to study multipartite entanglement using an entirely new intuition was pursued by Aravind [25], who proposed relating some N -partite quantum states with a link composed of N closed rings. The inspiration was that, after any measurement performed on any subsystem of the state $|\text{GHZ}\rangle$, the remaining bipartite state becomes separable. Thus this state can be associated with a particular configuration of three rings, called Borromean rings (see Fig. 2), characterized by the fact that if one ring is cut, the remaining two become disconnected. Usage of other tools borrowed from knot theory to analyze multipartite entanglement was further advocated in [26] and [27]. In fact, Borromean rings and their generalizations interact with various branches of physics, being both an inspiration for and a trigger of their development [28].

The analogy provided by Aravind is, however, not basis independent. Even though measuring the GHZ state in the

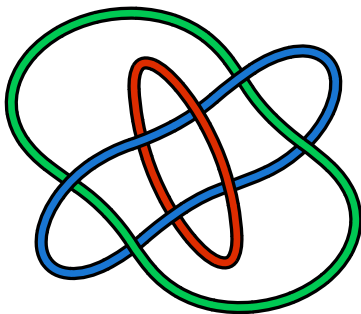


FIG. 2. The Borromean link, an example of a 0-resistant link of three rings, classified by the polynomial abc and given in Eq. (3).

computational basis results in a separable state, when measured in the $|+\rangle, |-\rangle$ basis the GHZ state collapses into an entangled state. This motivated Sugita [29] to revise the aforementioned analogy by proposing the partial trace of a subsystem as an alternative interpretation for cutting a ring. This way, the physical process corresponding to cutting the ring does not depend on the outcomes of a measurement and is defined uniquely [30]. This analogy between quantum entanglement and linked rings, used later in [31], is further explored here.

A partial trace over any given subsystems can be interpreted as not registering them by a measurement device or as the loss of the corresponding particles. We treat both these situations as being synonymous with a partial trace. A well-motivated physical problem in studying quantum entanglement resistant to particle loss [32,33] thus corresponds to the question whether a given reduced density matrix represents an entangled state [34,35]. Due to modern techniques of asymptotic geometric analysis it is possible to obtain rather precise estimations [36,37] for the relative ratio of lost subsystems such that a generic entangled pure state of a multipartite system becomes separable.

The aim of this work is to refine the classification of multipartite entanglement by introducing the notion of m -resistant states. Entanglement of these N -partite states is preserved even if any m subsystems are traced away, while removing any further subsystem makes the state completely separable. In terms recently proposed by Neven *et al.* [32], a quantum state is m resistant if the entanglement of the reduced state of $N - m$ subsystems is fragile with respect to the loss of any additional subsystem. Despite sharing the spirit of looking into what happens when certain parts of a state are ignored, as for AME states and k -uniform states, the above definition does not impose the restriction of maximal entanglement, which consequently increases the complexity of the problem.

In addition, this approach shares the attributes of persistence and robustness of entanglement, in that it measures how the entanglement of a given state resists ignoring a certain number of nonaccessible subsystems, which corresponds to taking a partial trace over them. This process is not as invasive as performing local measurements on selected subsystems or mixing the analyzed state with locally prepared separable states. Making use of the analogy between entangled states and linked rings [29,31], we investigate first the topological question of linking N closed rings in such a way that after cutting any m of them the remaining rings are connected, whereas cutting any additional ring separates the remaining rings. This problem is solved for any numbers N and m with the use of algebraic techniques developed in [31] which allow one to associate any link of N rings with a polynomial of N variables. A formalism developed for study of the links can be used not only to gain new intuition and pictorial visualization of the entanglement involved, but also to find representative states for each class of m -resistant quantum states, in a way which bypasses a great deal of algebra.

This work is organized as follows. In Sec. II the topological notion of an m -resistant link of N rings is introduced, while the analogy to multipartite quantum entanglement resistant to the loss of a subsystem is presented in Sec. III. Exemplary constructions of m -resistant states of three and four qubits

are provided in Sec. IV. In the case of pure states symmetric with respect to permutation of subsystems it is useful to use the stellar representation of Majorana [38–40]. The search for m -resistant states of N subsystems each of local dimension d is discussed in Sec. V, while Sec. VI presents concluding remarks and lists some open questions. In the Appendixes we list exemplary quantum states obtained with the use of the combinatorial notion of orthogonal arrays and prove a topological proposition on the existence of m -resistant links of N rings.

II. m -RESISTANT LINKS

The question of how many different ways one can connect any N rings has been addressed in [31]. We ignore (or cut) each subset of rings and ask which rings in the remaining link are connected. We do not take into account other superfluous details of the link. In this sense, two rings can only be connected in one way because they are either connected or not, while from a knot theory perspective this can be done in infinitely many ways.

To formalize the above idea, a particular configuration of any number of linked rings is characterized by a polynomial, denoted $\mathcal{P}$. The construction of this polynomial starts by associating a variable with each ring and then interpreting the product between variables to be equivalent to the associated rings being linked. Taking a variable to 0 is then interpreted as cutting or ignoring the associated ring. Thus, a given link can be characterized by the remaining links produced from all possible cuts. As an example, consider the polynomial

$$\mathcal{P}(3^3) = ab + bc. \quad (1)$$

We use the notation N^i , introduced in [31], where N is the number of rings and i is a cataloging index denoting the specific link class. Equation (1) implies that ring a is linked to ring b , which is also linked with ring c , while rings a and c are not directly connected. This can be seen by ignoring ring a , i.e., taking $a = 0$, which leaves us with bc , meaning that the two rings b and c are still linked. The same thing happens when ring c is ignored, which leaves the term ab , implying that rings a and b remain connected. The link associated with this configuration is represented in Fig. 1.

This line of interpretation naturally induces a set of rules for the construction of valid polynomials, which are summarized as follows:

- (1) There must not be any repeated terms, i.e., no ring variable can have a power greater than 1 (e.g., aab is the same as ab).
- (2) Each ring variable must appear at least once.
- (3) There must not be first-order terms (i.e., we want any ring to be linked with at least one other ring).
- (4) Relabeling of variables is irrelevant.
- (5) An N -variable monomial M is irrelevant if all of its variables are already present as an N -ring link of lesser-order monomials, built only with the variables of M (e.g., $abc + ac + ab$ gives the same results as $ac + ab$ after each variable is taken to 0, so we keep only $ac + ab$).

Rules 1 and 4 exist for simplicity of interpretation and to minimize unnecessary degrees of freedom. Rules 2 and

3 have a physical motivation, as we are interested only in investigations of entangled states. In this regard, first-order terms would correspond to a subsystem separate from the remaining system and, thus, are not of interest for this work. Rule 5 is the most intricate and the less obvious one, but it is necessary to reduce superfluous polynomials as well.

These rules for polynomial construction can then be used to count the number of distinct ways to link a given number of rings. For $N = 2, 3, 4$, and 5 rings, there are, respectively, 1, 4, 40, and 6900 ways to form a link [31], while for an arbitrary N the corresponding number has not been found yet.

A particularly interesting subclass of links with N rings is one where all rings remain connected after any $m < N$ cuts but become fully separated if $m + 1$ cuts are performed. Links with this property will be called m resistant, since they resist up to m cuts before becoming fully disconnected. Clearly, $0 \leq m < N - 2$, since a link cannot resist any cut or it can resist up to $N - 2$, in which case we are left with a link of two rings that evidently will not resist any other additional cut.

The problem of determining how many m -resistant links of N rings exist is straightforward to solve. There must always be exactly $m + 1$ cuts until the link is separated, i.e., until the associated polynomial becomes 0, and it does not matter which rings are chosen to be cut. This immediately implies that the polynomial must be the sum of all possible $N - m$ letter terms. For N rings, there are $N - 1$ different classes of resistance.

III. LINKS AND QUANTUM STATES

Picking up from the previous section, an analogy can be established by identifying a ring with a particle and attributing the act of ignoring a ring to the operation of tracing out the particle [29], as if that subsystem has not been detected. Linked rings are then associated with entangled particles, while separated rings represent separable particles. Distinct links thus represent distinct entanglement classes. Since the trace is used as the connecting ingredient, the entire analogy is basis independent. In addition, since the partial trace is defined for arbitrary Hilbert spaces, we may consider composite quantum systems of any dimensionality. In this paper, we start working with N -qubit systems, but in Sec. V we discuss also systems with a larger local dimension d .

The “link–quantum entanglement” analogy goes both ways: one may find the link class associated with a certain state or one may look for a state possessing the entanglement properties of a specific link. The former problem is straightforward to answer: one adds a term with k variables to the polynomial whenever the subsystem with the associated k particles is entangled. The latter problem, as we show, is much more intricate but can usually be solved using maps between polynomials and quantum states.

The notion of m -resistant links of N rings introduced in the previous section naturally motivates us to distinguish the following class of entangled quantum states of N subsystems.

Definition 1 (m resistance). An entangled state $|\psi\rangle$ of N parties is called m resistant if

- (i) $|\psi\rangle$ remains entangled when any m of its N subsystems are traced away or

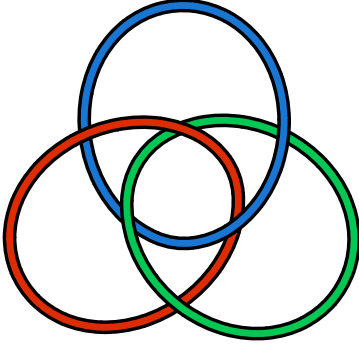


FIG. 3. An example of a 1-resistant link of three rings, classified by the polynomial $ab + ac + bc$, given in Eq. (6).

(ii) $|\psi\rangle$ becomes separable if a partial trace is performed over an arbitrary set of $m + 1$ subsystems.

The first nontrivial example of m resistance arises for $N = 3$, which is simple enough to study on intuitive grounds. The case $m = 0$ is exemplified by the GHZ state [10–12],

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (2)$$

since tracing out any party will result in a separable density matrix. The corresponding link has the properties of the well-known Borromean link, depicted in Fig. 2.

This link is constructed in such a way that, if any ring is cut, the remaining two rings become separated, and its connection to the GHZ state was first conjectured in [25]. We denote the associated link class 3^1 , and the polynomial that characterizes it is given by

$$\mathcal{P}(3^1) = abc. \quad (3)$$

For N rings, the generalization is called the Brunnian link, with the general polynomial

$$\mathcal{P}(N^1) = abcd \dots, \quad (4)$$

so if a single variable is set to 0, the polynomial vanishes.

The case $m = 1$ has different properties, since if any party is ignored, the remaining two parts are still connected. From the point of view of quantum entanglement a three-qubit state, called $|\text{W}\rangle$ (see [10]),

$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \quad (5)$$

possesses this property, since a partial trace over any of its subsystems will result in an entangled mixed state. A link which mimics this property is represented in Fig. 3, and it was first considered as a representation of the W state in [25].

The polynomial which describes this link is symmetric,

$$\mathcal{P}(3^4) = ab + ac + bc, \quad (6)$$

and the generalization to any number of rings is given by the sum of all two-variable terms.

IV. IN SEARCH OF m -RESISTANT STATES OF N QUBITS

The problem of finding m -resistant states for a general number N of qubits is not trivial. Since we are only

interested in whether or not a reduced state is entangled, for any partial trace, it is clear that at least these states can be symmetric regarding exchanges of subsystems. This is not strictly necessary, as symmetry under qubit permutation is not invariant under local operations, but it does provide one possible direction. However, the requirement of separability, or lack of it, is not simple to achieve using any standard algebraic techniques.

In this section we demonstrate how the analogy between links of rings and entangled quantum states may be used in order to find representative mixed states for all m -resistant cases up to $N = 7$. For pure states the problem has not been fully solved yet, but nevertheless we present a few examples as well as some geometric intuition behind them.

A. A general rule for mixed qubit states

The fundamental aspect of the analogy between links of rings and entangled states is that one can solve certain entanglement problems using the polynomials that characterize the links and apply a map converting polynomials into quantum states. As we show below, this heuristic reasoning proves useful for identifying interesting examples of quantum states resistant to loss of subsystems.

The method for obtaining m -resistant mixed states of N qubits starts from the polynomial of N variables that characterizes the corresponding m -resistant links and consists of J terms. For each term t_i of the polynomial, we associate a state $|t_i\rangle \in \mathcal{H}_2^{\otimes N} \otimes \mathcal{H}_d$ of the form

$$|t_i\rangle = |\phi_{\text{ent}}\rangle |\phi_{\text{sep}}\rangle |\chi_D\rangle, \quad (7)$$

where $|\phi_{\text{sep}}\rangle$ stands for a separable state of m qubits, while $|\phi_{\text{ent}}\rangle$ denotes an entangled state of the remaining $N - m$ qubits. Depending on the term t_i being considered, the qubits appearing in $|\phi_{\text{ent}}\rangle$ and $|\phi_{\text{sep}}\rangle$ will change. An additional state, $|\chi_D\rangle$, of a single system D with d levels plays an auxiliary role, as it will be traced over. An unnormalized superposition state, $|\psi_N, m\rangle = \sum_{i=1}^J |t_i\rangle$, allows us to generate the resulting mixed state $\hat{\rho}(N, m)$ of N qubits by a partial trace over the d -dimensional environment,

$$\hat{\rho}(N, m) = \frac{\text{tr}_D[|\psi_{N,m}\rangle \langle \psi_{N,m}|]}{\langle \psi_{N,m} | \psi_{N,m} \rangle}. \quad (8)$$

To form an m -resistant state we choose the separable m -partite state $|\phi_{\text{sep}}\rangle$ in (7) to be $|0 \dots 0\rangle$, where the qubit labels will depend on the term t_i considered. The extra qubit will assume the index value of the current term, and the entangled part $|\phi_{\text{ent}}\rangle$ will have the form

$$|E_m\rangle_{ijk\dots} = (m+1)|0 \dots 0\rangle_{ijk\dots} + |1 \dots 1\rangle_{ijk\dots}, \quad (9)$$

where $ijk\dots$ are the variables appearing explicitly in the corresponding term t_i of the polynomial. Note that for any $m > 0$ the above superposition is not symmetric, while for $m = 0$ the state $|E_0\rangle_{ijk}$ denotes just the standard $|\text{GHZ}\rangle$ state, (2), among subsystems labeled by indices i, j , and k .

Putting the above recipe into practice, we have obtained m -resistant representative mixed states for all m for the cases $N = 3$ up to $N = 7$. In order to test for entanglement or separability of the reduced density matrices after each partial trace, we used a code available in [41], which looks iteratively

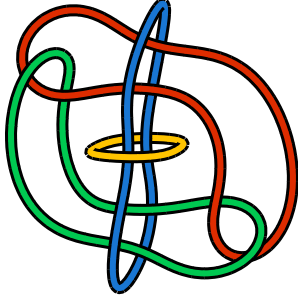


FIG. 4. An example of a 0-resistant link of four rings, which is a four-component Brunnian link, corresponding to the polynomial $abcd$; see Eq. (14). Note that after a single ring is cut all three remaining rings become disconnected.

for the nearest separable state and its decomposition into pure product states, returning the distance from the target matrix.

1. $N=3$

Although representative pure states have been provided in the previous section for $N=3$, we show here an alternative answer in terms of mixed states. They are of the general form

$$\hat{\rho}(3, m) = \frac{\text{tr}_d[|\psi_{3,m}\rangle\langle\psi_{3,m}|]}{\sqrt{\langle\psi_{3,m}|\psi_{3,m}\rangle}}, \quad (10)$$

with $|\psi_{3,m}\rangle$ specified case by case.

For $m=0$, we have a state with the entanglement properties under partial trace represented in the link in Fig. 2. Its polynomial given by Eq. (3) has only a single term, which leads to the product state

$$|\psi_{3,0}\rangle = |E_0\rangle_{abc} |0\rangle_d. \quad (11)$$

Hence the resulting state, (8), obtained by a partial trace is pure, and it is shown merely for completeness. The remaining case, $m=1$, is depicted in Fig. 3 and is characterized by the polynomial of Eq. (6). As such, we are led to the following superposition of three states, entangled with respect to different splittings:

$$|\psi_{3,1}\rangle = |E_1\rangle_{ab} |0\rangle_c |0\rangle_d + |E_1\rangle_{ac} |0\rangle_b |1\rangle_d + |E_1\rangle_{bc} |0\rangle_a |2\rangle_d. \quad (12)$$

2. $N=4$

Using exactly the same procedure as for $N=3$, we obtain representative m -resistant mixed states for $N=4$, written in general as

$$\hat{\rho}(4, m) = \frac{\text{tr}_e[|\psi_{4,m}\rangle\langle\psi_{4,m}|]}{\sqrt{\langle\psi_{4,m}|\psi_{4,m}\rangle}}, \quad (13)$$

with $|\psi_{4,m}\rangle$ specified case by case. It is also worth mentioning that the links for more than three rings have been drawn using a technique developed in [31] that makes direct use of the polynomials associated with them.

For $m=0$, the corresponding monomial reads

$$\mathcal{P}(4^1) = abcd, \quad (14)$$

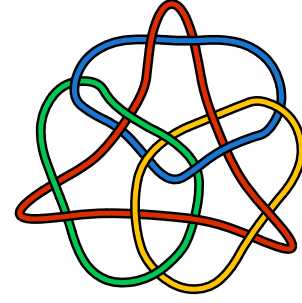


FIG. 5. An example of a 1-resistant link of four rings, classified by the polynomial $abc + abd + acd + bcd$, given in Eq. (16). After cutting any single ring one arrives at the Borromean link.

so the state associated with the link shown in Fig. 4 has the product structure

$$|\psi_{4,0}\rangle = |E_0\rangle_{abcd} |0\rangle_e. \quad (15)$$

The case $m=1$ is associated with the polynomial

$$\mathcal{P} = abc + abd + acd + bcd, \quad (16)$$

which leads to the link in Fig. 5. Polynomial (16) is then mapped onto the mixed state with

$$|\psi_{4,1}\rangle = |E_1\rangle_{abc} |0\rangle_d |0\rangle_e + \dots + |E_1\rangle_{bcd} |0\rangle_a |3\rangle_e, \quad (17)$$

where the ellipsis entails all the other states associated with the remaining terms of the polynomial. Finally, the case $m=2$ is described by

$$\mathcal{P} = ab + ac + ad + bc + bd + cd, \quad (18)$$

whose associated link class can be represented by the link in Fig. 6. The corresponding mixed state is readily obtained from

$$|\psi_{4,2}\rangle = |E_1\rangle_{ab} |00\rangle_{cd} |0\rangle_e + \dots + |E_1\rangle_{cd} |00\rangle_{ab} |5\rangle_e. \quad (19)$$

3. $N=5$

The case of $N=5$ can be analyzed analogously to the previous cases. The general m -resistant mixed state is written as

$$\hat{\rho}(5, m) = \frac{\text{tr}_f[|\psi_{5,m}\rangle\langle\psi_{5,m}|]}{\sqrt{\langle\psi_{5,m}|\psi_{5,m}\rangle}}, \quad (20)$$

with $|\psi_{5,m}\rangle$ specified case by case. For $m=0$ we have

$$\mathcal{P} = abcde \quad (21)$$

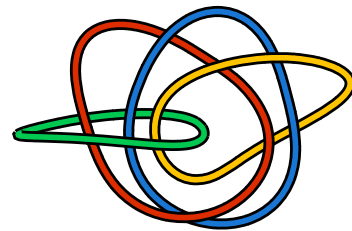


FIG. 6. An example of a 2-resistant link of four rings, classified by the polynomial $ab + ac + ad + bc + bd + cd$, given in Eq. (18). After any two rings are cut the remaining two are still connected.

so the respective state will be

$$|\psi_{5,0}\rangle = |E_0\rangle_{abcde} |0\rangle_f. \quad (22)$$

For $m = 1$, we have

$$\mathcal{P} = abcd + \dots + bcde, \quad (23)$$

which gives rise to the mixed state obtained from

$$|\psi_{5,1}\rangle = |E_1\rangle_{abcd} |0\rangle_e |0\rangle_f + \dots + |E_1\rangle_{bcde} |0\rangle_a |4\rangle_f. \quad (24)$$

For $m = 2$, the polynomial is

$$\mathcal{P} = abc + \dots + cde, \quad (25)$$

which is mapped onto the state

$$|\psi_{5,2}\rangle = |E_2\rangle_{abc} |00\rangle_{de} |0\rangle_f + \dots + |E_2\rangle_{cde} |00\rangle_{ab} |9\rangle_f. \quad (26)$$

Finally, for $m = 3$, we have

$$\mathcal{P} = ab + \dots + de, \quad (27)$$

which gives the state

$$|\psi_{5,3}\rangle = |E_3\rangle_{ab} |000\rangle_{cde} |0\rangle_f + \dots + |E_3\rangle_{de} |00\rangle_{abc} |9\rangle_f. \quad (28)$$

4. $N > 5$

Taking into account the examples presented in the previous sections, it becomes tentative to conjecture the following statement. Let S_{p-q} be the set containing all subsets of p letters with $p - q$ elements, and let $\mathcal{B}_p$ be the set with all p letters. If N and m are positive integers, with $N > m$, then the mixed state

$$\hat{\rho}(N, m) = \frac{\text{tr}_\Lambda[|\psi_{N,m}\rangle \langle \psi_{N,m}|]}{\sqrt{\langle \psi_{N,m} | \psi_{N,m} \rangle}} \quad (29)$$

with

$$|\psi_{N,m}\rangle = \sum_{g_i \in S_{N-m}} |E_m\rangle_{g_i} |0 \dots 0\rangle_{\mathcal{B}_N \setminus g_i} |i\rangle_\Lambda \quad (30)$$

is an m -resistant state of N qubits. This result has been checked for N up to 7. For a larger number of qubits the size of the system grows as 2^N so the computing time required to investigate the separability of reduced density matrices increases accordingly. Therefore studies of systems with a larger number of qubits, although possible, become impracticable.

Regarding the representation of links with more than six rings, it becomes less elucidating, so we refrain from providing them in those cases. Nevertheless, using the respective polynomials, it is a straightforward task to draw these links.

B. Pure qubit states

Unlike the case of mixed states, the identification of the pure state of an N -qubit system associated with a given m -resistant link of N rings is more intricate, if possible at all. We provide here some partial answers following two approaches: one which makes use of the polynomial formalism for links and another based on the Majorana representation [38,39] of symmetric states of several qubits.

In a similar spirit to the previous sections we may inquire whether there exists a straightforward map between the polynomials of m -resistant links of N rings and m -resistant pure states of N qubits. Taking into account Eqs. (29) and (30), the most immediate ansatz for a pure m -resistant N -qubit state, denoted $|(N, m)\rangle$, would be

$$|(N, m)\rangle = \sum_{g_i \in S_{N-m}} |E_m\rangle_{g_i} |0 \dots 0\rangle_{\mathcal{B}_N \setminus g_i}. \quad (31)$$

Hence, we now remove the extra qudit in Eq. (30) and do not need to perform a partial trace. The state still possesses the same symmetries as the corresponding polynomial under particle permutations, so it is at least intuitive as a candidate. This ansatz, tested up to $N = 7$, gives a correct answer only for $m = 0$, $N - 1$, and $N - 2$. The first failure thus occurs for $N = 5$ and $m = 2$, where no corresponding 2-resistant pure state of five qubits has yet been found. Other choices could be made for $|E_m\rangle$ in Eq. (31), for example, but as it turns out, no change in the coefficients of Eq. (9) will provide the correct answer.

Different approaches can be used to search for m -resistant pure qubit states. Since the states can be symmetric under particle permutation, one may use the stellar representation of Majorana [38,42], which provides an alternative intuition on the geometry of these states. The Majorana representation of an N -qubit pure state, fully symmetric under permutations of its $i_1, i_2, \dots, i_N$ qubits, is given by

$$|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{\sigma \in S_N} |\eta_1\rangle_{i_1} \dots |\eta_N\rangle_{i_N}, \quad (32)$$

where the sum runs over all permutations $\sigma \in S_N$ of particle indices, K is a suitable normalization constant, and

$$|\eta_j\rangle_X = \cos\left(\frac{\theta_j}{2}\right) |0\rangle_X + e^{i\phi_j} \sin\left(\frac{\theta_j}{2}\right) |1\rangle_X. \quad (33)$$

The N pairs (θ_j, ϕ_j) are called Majorana points and uniquely define a point in the Bloch sphere. In this way, one may define a fully symmetric N -qubit state by fixing N points on a sphere. For this reason, the term stellar representation is also common, as each point represents a star in the sky, while a group of stars forms a constellation.

One of the advantages of the Majorana representation is that it allows one to ascribe to entanglement a degree of geometrical intuition. If one fixes all the stars at a single point, the corresponding state is separable. For instance, N degenerated stars at the North Pole represent the state $|0 \dots 0\rangle$. However, as the degeneracy is lifted, any nontrivial constellation of stars corresponds to an entangled state of N qubits. Thus, in a sense, the distance between the stars is directly related to the degree of entanglement [42,43], although such a criterion to quantify entanglement is not uniquely defined.

1. $N = 3$

In order to investigate possible patterns, one should start with the lowest qubit case $N = 3$ and look for the simplest choices of constellations which give expected results. It is straightforward to show that the constellations in Fig. 7 give the desired answer for all m -resistant states of three qubits.

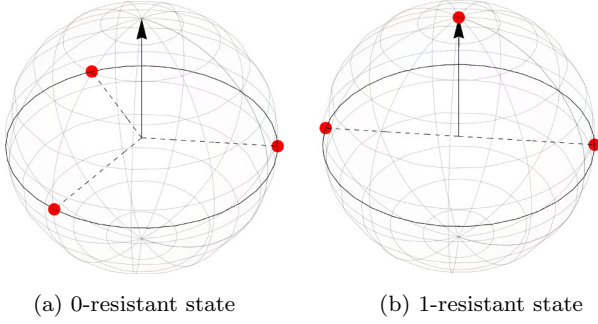


FIG. 7. Constellations defining m -resistant states of three qubits. The middle arrow serves as a reference pointing to the North Pole. (a) This constellation corresponds to the state $|\text{GHZ}\rangle$ and is illustrated by the Borromean link in Fig. 2. (b) This constellation is related to the link in Fig. 3.

The m -resistant states of N qubits constructed using the stellar representation and Majorana points are henceforth denoted $|(N, m)_{\text{MP}}\rangle$. The constellation corresponding to $m = 0$ results in the GHZ state

$$|(3, 0)_{\text{MP}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (34)$$

for which the distance between the points at the equator is the greatest. From this perspective, the $|\text{GHZ}\rangle$ state could be considered the most entangled, although from the point of view of resistance, entanglement of this 0-resistant state is not resistant to the loss of a subsystem. The case of $m = 1$ corresponds to the state

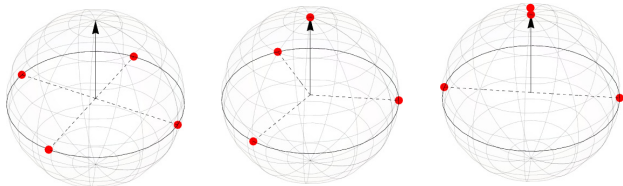
$$|(3, 1)_{\text{MP}}\rangle = \frac{1}{\sqrt{12}}(3|000\rangle + |011\rangle + |101\rangle + |110\rangle), \quad (35)$$

where the weight of the state $|000\rangle$ is increased with respect to Eq. (34), as one of the stars sits at the North Pole.

2. $N \geq 4$

The example of $N = 3$ suggests that m resistance may be directly related to the number of stars at the North Pole, with all others evenly distributed along the equator. Indeed, for $N = 4$ this is in fact the case, where pure qubit states for all m -resistant states can be constructed exactly in this manner, with the constellations depicted in Fig. 8.

The pattern is thus to place m stars at the North Pole and distribute the remaining stars evenly at the equator, where for



(a) 0-resistant state (b) 1-resistant state (c) 2-resistant state

FIG. 8. Constellations defining m -resistant states of four qubits. The middle arrow serves as a reference pointing to the North Pole. Constellations corresponding (a) to the four-ring Brunnian link in Fig. 4, (b) to the link in Fig. 5, and (c) to the link in Fig. 6.

convention we choose the first star at the equator with $(\frac{\pi}{2}, 0)$. This construction gives rise to a family of states which, after expansion of Eq. (32), can be expressed in general as

$$|\psi_m^N\rangle = \frac{1}{\sqrt{1 + \binom{N}{m}}} \left(\sqrt{\binom{N}{m}} |0\rangle^{\otimes N} - (-1)^{N+m} |D_m^N\rangle \right), \quad (36)$$

where we have used the notion of Dicke states [44],

$$|D_m^N\rangle = \frac{1}{\sqrt{\binom{N}{m}}} \sum_j P_j \{ |0\rangle^{\otimes m} |1\rangle^{\otimes (N-m)} \}, \quad (37)$$

with the sum going through all permutations P_j . For $N = 4$, we have

$$|\psi_m^4\rangle = |(4, m)\rangle. \quad (38)$$

The construction described above breaks down at $N = 5$. The first exception appears at $(5, 1)$, where it is found that $|\psi_1^5\rangle = |(5, 2)\rangle$. All other cases, i.e., $m = 0, 2$, and 3 , give pure states with the desired resistance. One may naturally inquire whether other symmetric positions of stars may give rise to $|(5, 2)\rangle$, other than one star at the North Pole and four stars distributed in a square at the equator. We have tested combinations where two stars are lifted by a general latitude, as well as four stars, and it is possible to show that no 1-resistant state exists for these families of states. Evidently, this does not prove that a pure 1-resistant state of five qubits does not exist, although it is tantalizing to conjecture it, given that pure states with certain entanglement properties under a partial trace do not exist. For instance, it is known [45] that there are no AME states for four qubits.

For $N > 5$ up to 7, we have verified that the ansatz in Eq. (36) fails in all situations except for $m = 0, N - 2$, and $N - 1$. In fact, it is possible to show that for any N these cases are always satisfied:

$$|\psi_0^N\rangle = |(N, 0)\rangle, \quad (39)$$

$$|\psi_{N-3}^N\rangle = |(N, N-3)\rangle, \quad (40)$$

$$|\psi_{N-2}^N\rangle = |(N, N-2)\rangle. \quad (41)$$

The proof that $|\psi_0^N\rangle$ is 0 resistant is straightforward. Indeed, by Eq. (36), we have

$$|\psi_0^N\rangle = \frac{1}{2}(|0\rangle^{\otimes N} - (-1)^N |1\rangle^{\otimes N}). \quad (42)$$

This state is a generalization of the GHZ state for N qubits, which, after any partial trace, returns a density matrix of the form

$$\text{tr}_k[\hat{\rho}_0^N] = \frac{1}{2}(|0\rangle\langle 0|^{\otimes (N-k)} + |1\rangle\langle 1|^{\otimes (N-k)}), \quad (43)$$

where tr_k denotes the trace over any set consisting of k qubits. This density matrix is separable for any $k > 0$, and thus the state $|\psi_0^N\rangle$ is 0 resistant for any N .

Regarding the state $|\psi_{N-2}^N\rangle$, after simple algebraic operations, one finds that any $N - 2$ partial traces will result in the

density matrix

$$\text{tr}_{N-2}[\hat{\rho}_{N-2}^N] \propto \begin{pmatrix} \alpha_{22} + (\alpha_{02})^2 & 0 & 0 & -\alpha_{02} \\ 0 & \alpha_{21} & \alpha_{21} & 0 \\ 0 & \alpha_{21} & \alpha_{21} & 0 \\ -\alpha_{02} & 0 & 0 & 1 \end{pmatrix}, \quad (44)$$

where we define $\alpha_{ij} \equiv \binom{N-i}{j}$. The partially transposed matrix has one eigenvalue equal to

$$-N^2 + 3N - 4, \quad (45)$$

which is negative for any N . Thus, by the positive partial transpose (PPT) test, we confirm that every two-qubit reduced density matrix is always entangled, and the state $|\psi_{N-2}^N\rangle$ is $(N-2)$ resistant for any N .

Finally, focusing on the state $|\psi_{N-3}^N\rangle$, we perform a partial trace over any set of $N-2$ qubits and obtain

$$\text{tr}_{N-2}[\hat{\rho}_{N-3}^N] \propto \begin{pmatrix} \alpha_{23} + (\alpha_{03})^2 & 0 & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{22} & 0 \\ 0 & \alpha_{22} & \alpha_{22} & 0 \\ 0 & 0 & 0 & \alpha_{21} \end{pmatrix}. \quad (46)$$

The partially transposed matrix of size 4 has all positive eigenvalues for $N > 1$. In this case the PPT test guarantees that the resulting state is always separable. We must then look into the reduced density matrix with one less partial trace, in order to check whether it is entangled. Such a density matrix is proportional to

$$\text{tr}_{N-3}[\hat{\rho}_{N-3}^N] \propto \begin{pmatrix} \alpha_{33} + (\alpha_{03})^2 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{03} \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ 0 & \alpha_{32} & \alpha_{32} & 0 & \alpha_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ 0 & 0 & 0 & \alpha_{31} & 0 & \alpha_{31} & \alpha_{31} & 0 \\ \alpha_{03} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (47)$$

By partially transposing any qubit, we obtain a matrix which possesses an eigenvalue equal to

$$(13 - 7N + N^2 - \sqrt{193 - 202N + 79N^2 - 14N^3 + N^4}) \quad (48)$$

which is negative for all N . The PPT criterion implies that all the subsystems are entangled. This proves that the state $|\psi_{N-3}^N\rangle$ is $(N-3)$ resistant.

As a final remark in this section, we note that the constellations for all 0-resistant states follow the same rule, which is a regular N -sided polygon. This result has been found in [32].

V. IN SEARCH OF m -RESISTANT QUDIT STATES

As mentioned in Sec. IV B, we were not able to establish the general construction of an m -resistant pure state of an N -qubit system. Therefore, we expanded our search to subsystems with a larger local dimension, $d \geq 3$, sometimes called qudits. We succeeded in providing formulas for an m -resistant N -qudit pure state for $N \geq 2m$.

The example of the AME state of six qubits [46] inspired us to search for states with diagonal density matrices after tracing the appropriate number of parts—more precisely, $m+1$ parts—in order to find an m -resistant state. Trivially such states are separable after tracing $m+1$ parts. Of course, the states we are looking for must remain entangled after tracing out a smaller number of parts. In order to find such states, we present a connection between a family of combinatorial designs and a family of quantum states with the required properties. In particular, we use the notion of *orthogonal arrays* and the established connection [22] between multipartite quantum states and OAs.

For an m -resistant state of N qudits, we use the notation $|(N, m)_d\rangle$.

A. Orthogonal arrays and quantum states

Orthogonal arrays [47] are combinatorial arrangements, tables with entries satisfying given orthogonal properties. A close connection between OAs and codes, entangled states, error-correcting codes, and uniform states has been established [48]. Therefore, investigation of the connections between OAs and resistant states seems to be a natural approach. First, the concept of the OA is briefly presented; second, relations between OAs and resistant states are given.

An orthogonal array $\text{OA}(r, N, d, k)$ is a table composed of r rows and N columns with entries taken from $0, \dots, d-1$ in such a way that each subset of k columns contains all possible combinations of symbols with the same number of repetitions. The number of such repetitions is called the *index* of the OA and denoted λ . One can observe that the index of the OA is related to the other parameters:

$$\lambda = \frac{r}{d^k}. \quad (49)$$

Figure 9 presents an example of an OA. A pure quantum state consisting of r terms might be associated with $\text{OA}(r, N, d, k)$, simply by reading all rows in the OA [22]. The state of N qudits associated with the orthogonal array $\text{OA}(r, N, d, k)$ is denoted $|\phi_{(N,k)}\rangle_d$.

B. Orthogonal arrays of index unity

The crucial quantity for our purpose, related to the OA, is its index. It preserves the following information: how many repetitions of any sequence $i_1, \dots, i_k$ there are for each subsystem of k rows. For $\lambda = 1$, any sequence appears only once, and such an array is called an *index unity array*. We emphasize their remarkable role in the search for resistant quantum states.

Proposition 1. For any orthogonal array of index unity, $\text{OA}(d^k, N, d, k)$, where $N \geq 2k$, the relevant quantum state $|\phi_{(N,k)}\rangle$ is $k-1$ resistant, i.e.,

$$|\phi_{(N,k)}\rangle = |(N, k-1)_d\rangle. \quad (50)$$

A proof of the latter statement is provided in Appendix A. From the OA presented in Fig. 9, we obtain, for example, the

0 0	0 0 0	0	0 0	0 0	$ (5, 1)_4\rangle =$	00000>
0 1	1 1 1	0	1 1	1 1	+	01111>
0 2	2 2 2	0	2 2	2 2	+	02222>
0 3	3 3 3	0	3 3	3 3	+	03333>
1 0	1 2 3	1	0 1	2 3	+	10123>
1 1	0 3 2	1	1 0	3 2	+	11032>
1 2	3 0 1	1	2 3	0 1	+	12301>
1 3	2 1 0	1	3 2	1 0	+	13210>
2 0	2 3 1	2	0 2	3 1	+	20231>
2 1	3 2 0	2	1 3	2 0	+	21320>
2 2	0 1 3	2	2 0	1 3	+	22013>
2 3	1 0 2	2	3 1	0 2	+	23102>
3 0	3 1 2	3	0 3	1 2	+	30312>
3 1	2 0 3	3	1 2	0 3	+	31203>
3 2	1 3 0	3	2 1	3 0	+	32130>
3 3	0 2 1	3	3 0	2 1	+	33021>

FIG. 9. Orthogonal array of index unity, $OA(4^2, 5, 4, 2)$, obtained from the Reed-Solomon code of length 5 over the Galois field $GF(4)$. Each subset consisting of two columns contains all possible combinations of symbols. Here, two such subsets are highlighted. The relevant quantum state is obtained by forming a superposition of states corresponding to consecutive rows of the array; see the expression on the right.

following five-ququart, 1-resistant state:

$$\begin{aligned}
 |(5, 1)_4\rangle = & |00000\rangle + |01111\rangle + |02222\rangle + |03333\rangle \\
 & + |10123\rangle + |11032\rangle + |12301\rangle + |13210\rangle \\
 & + |20231\rangle + |21320\rangle + |22013\rangle + |23102\rangle \\
 & + |30312\rangle + |31203\rangle + |32130\rangle + |33021\rangle. \quad (51)
 \end{aligned}$$

C. Existence of OAs and relevant m -resistant states

The construction of index unity OAs was provided by Bush [49] in 1952. He used methods based on Galois field theory in order to obtain the following result:

Theorem 1 [47]. If d is a prime power, i.e., $d = p^n$ for some prime number p and natural number n , then we can construct the array, $OA(d^k, d + 1, d, k)$.

Combining Bush's result with Sec. V B implies the existence of m -resistant N -qudit states for $N \geq 2(m + 1)$.

Corollary 1. For any $N \geq 2(m + 1)$ there exists an N -qudit state which is m resistant. The local dimension d is the smallest prime power larger than $N - 1$.

In Appendix B we present some m -resistant qudit states. A more interested reader might easily reproduce more of them with the help of available OA tables [50].

We organize all the results obtained so far in Table I. These results encourage us to pose the following conjecture:

Conjecture 1. For any N and m , there exists an m -resistant N -qudit state in some local dimension d .

VI. ASYMPTOTIC CASE

Consider a system consisting of N subsystems with d levels each and assume that N is large. It is well known that the generic pure state of such a system is typically strongly entangled [51,52]. Therefore the partial trace of the corre-

TABLE I. Search for m -resistant states of N parties at a glance. Note that $|\psi_0^4\rangle$ is equivalent to the $|\text{GHZ}_4\rangle$ state of four qubits. Three families, $|\psi_0^N\rangle$, $|\psi_{N-3}^N\rangle$, and $|\psi_{N-2}^N\rangle$, of m -resistant N -qubit states discussed in Sec. IV are presented on the differently shaded blue background. The six-qubit state $\text{AME}(6,2)$ is also signalized. Moreover, m -resistant qudit states $|\phi_{m,N}\rangle_d$ constructed by the use of orthogonal arrays are shown on the green background. The subscript number is relevant to the local dimension d of a state. Observe that the local dimension d of qudit states is usually (but not always) equal to $N - 1$.

$m \backslash N$	4	5	6	7	8
0	$ \psi_0^4\rangle$	$ \psi_0^5\rangle$	$ \psi_0^6\rangle$	$ \psi_0^7\rangle$	$ \psi_0^8\rangle$
1	$ \psi_1^4\rangle$	$ \phi_{1,5}\rangle_4$	$ \phi_{1,6}\rangle_5$	$ \phi_{1,7}\rangle_7$	$ \phi_{1,8}\rangle_7$
2	$ \psi_2^4\rangle$	$ \psi_2^5\rangle$	AME	$ \phi_{2,7}\rangle_7$	$ \phi_{2,8}\rangle_7$
3		$ \psi_3^5\rangle$	$ \psi_3^6\rangle$		$ \phi_{3,8}\rangle_7$
4			$ \psi_4^6\rangle$	$ \psi_4^7\rangle$	
5				$ \psi_5^7\rangle$	$ \psi_5^8\rangle$
6					$ \psi_6^8\rangle$

sponding N -party projector over any choice of subsystems traced away becomes mixed. Typically, the more parties k are traced out, the more mixed is the state describing the remaining $N - k$ subsystems.

If the number k of subsystems removed is equal to $N/2$, the reduced state typically has a full rank so it can belong to the separable ball around the maximally mixed state [53,54]. It was indeed observed [34,35,52] that for $k \approx N/2$ a transition from entangled reduced states to reductions with a positive partial transpose takes place. Furthermore, if $k \approx 3N/5$, the remaining subsystem, consisting of $2N/5$ particles, is typically separable [36,37], and this transition becomes sharp if $N \rightarrow \infty$. Thus one can expect that the generic pure state of an N -system state is m resistant, with m typically of the order of $3N/5$, independently of the local dimension d .

VII. CONCLUDING REMARKS

Any link of N closed rings can be associated with a polynomial of N variables. This technique, developed in [31], allowed us to construct a class of m -resistant links of N closed rings such that, after cutting any number m of them, the remaining rings are connected, while cutting any of the remaining rings separates them. Making use of the analogy between linked rings and entangled states [29] we provided examples of m -resistant entangled quantum states, such that their entanglement is resistant to the loss of any set of m subsystems.

An explicit construction involving a partial trace provides examples of m -resistant mixed states of N -qubit systems. An alternative method based on the stellar representation yields m -resistant pure states, but for $N \geq 5$ it does not produce all the required states for all values of $m = 0, 1, \dots, N - 2$. However, in some cases for which the above method fails (e.g., $N = 5$ and $m = 1$), we constructed m -resistant pure states of N parties with a higher local dimension d making use of the combinatorial notion of orthogonal arrays [48].

The reason that certain pure states, like pure 1-resistant states of five qubits, are difficult to find is not yet clear. It becomes even more elusive considering the fact that the mixed states for these same cases have been so straightforward to obtain. For more intuition concerning the problem, note that mixed states have many more degrees of freedom than pure states, which in turn provides more room for the required entanglement properties. In addition, the symmetry constraints regarding the interchange of qubits increase the difficulty of the problem. For example, it is rather easy to construct an entangled pure state such that it remains entangled after a specific qubit is traced out but becomes separable after any additional qubit is traced. The difficulty appears only when the aforementioned property is required for any choice of qubits.

Making use of mathematical results involving the effect of concentration of measure [36,37] one can formulate predictions concerning the generic properties of a multipart system with a large number of subsystems. In particular, we can infer that the generic pure state of an N -system state is typically m resistant with $m \approx 3N/5$ for a large N and any local dimension d .

It is important to emphasize two defining characteristics of any m -resistant quantum state. The first is the presence or absence of entanglement after partial traces, which is equivalent to fixing which subsystems of a quantum state are entangled, when the remaining part of the system is ignored or cannot be measured. In other words, it fixes which subsystems are able to successfully perform protocols between each other. The second defining property is symmetry under permutations of parties, which brings about a notion of equality between all subsystems.

These two characteristics of quantum resistance can be combined into applications which require the simultaneous action of any given number of parties in order to perform some concrete task. For instance, consider a digital locker belonging to a certain network of N parties, where each person is in possession of a qubit, being a part of a genuinely entangled N -qubit state. One may devise this locker in such a way that it can be opened through a protocol involving the cooperation of at least m people. In this case, an m -resistant state of N qubits might be used in the aforementioned protocol. Indeed, any system of $m - 1$ parties will always be in a separable state, which disallows forming correlations between them.

Finally, we report as a final remark a list of some relevant open problems: (a) investigate whether there exist pure states of N qubits with the m -resistance property, for any $m = 0, 1, \dots, N - 2$; (b) if this is not the case, for each N find the minimal local dimension d such that there exist m -resistant states of N qudits; and (c) for any class of m -resistant states of N qubits find the state for which its average entanglement after a partial trace over any set of m parties is the largest, when measured with respect to a given measure of entanglement.

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APPENDIX A: PROOF OF PROPOSITION 1

Proof of Proposition 1. In order to prove the $(k - 1)$ resistance of $|\phi_{(N,k)}\rangle$ we need to show that it becomes separable after a partial trace of any k particles, while it remains entangled as any $k - 1$ particles are traced away.

Observe that after any k particles are traced out, the density matrix becomes a diagonal matrix, hence it represents a separable state.

Checking the entanglement properties of the system after tracing out $k - 1$ particles is a bit trickier. We show that such a system after the partial transpose of one particle will always have a negative eigenvalue. Hence by the PPT criterion, it remains entangled.

We investigate the density matrix $\rho_a(\psi)$ after tracing out the first $(k - 1)$ particles. We show that the partial transpose of the last particle $\rho_{k-1}^{T_N}(\psi)$ has nonnegative eigenvalues. To clarify, we sketch an example in Fig. 10 of how the argument for the proof is developed.

To begin, take two rows of the OA, which have 0s in the first $(k - 1)$ positions. Since the OA has strength k and is of index unity, these rows differ at all other positions. Without loss of generality assume that they are of the form

$$\underbrace{0 \dots 0}_{k-1} \quad \underbrace{1 \dots 1}_{N-k+1}.$$

Consider now the density matrix $\rho_{k-1}(\psi)$ obtained by tracing out the first $(k - 1)$ particles and examine the eigenvalues of the partial transpose of the last particle $\rho_{k-1}^{T_i}(\psi)$. The aforementioned rows generate the following elements in the matrix $\rho_{k-1}^{T_i}(\psi)$:

$$\begin{aligned} &|0 \dots 01\rangle\langle 1 \dots 10|, \\ &|1 \dots 10\rangle\langle 0 \dots 01|. \end{aligned}$$

We show that these elements are the only ones interlocking the bras

$$\langle 1 \dots 10|, \langle 0 \dots 01|$$

and kets

$$|0 \dots 01\rangle, |1 \dots 10\rangle.$$

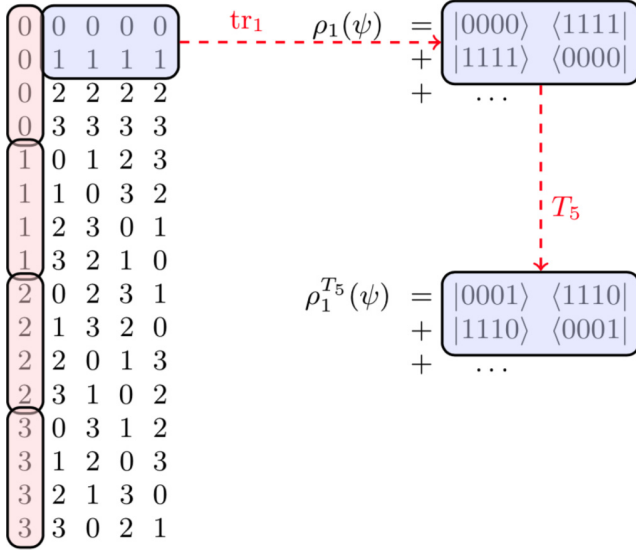


FIG. 10. Orthogonal array of index unity, $OA(4^2, 5, 4, 2)$. We investigate the matrix $\rho_1^{T_5}(\psi)$ obtained by tracing out the first particle and transposing the last one. The first two rows of presented OAs differ at all positions except the first one. They generate the elements $|0001\rangle\langle 1110|$ and $|1110\rangle\langle 0001|$ in $\rho_1^{T_5}(\psi)$. There are no other elements in $\rho_1^{T_5}(\psi)$ involving $|0001\rangle$, $|1110\rangle$ or $\langle 1110|$, $\langle 0001|$. Consequently, the matrix $\rho_1^{T_5}(\psi)$ has a negative eigenvalue. A similar argument holds for any other choice of the traced particle, and thus $\rho_1(\psi)$ is an entangled state.

With this observation at hand, one can see that the matrix $\rho_{k-1}^{T_i}(\psi)$ has a block structure:

$$\rho_{k-1}^{T_i}(\psi) \propto \begin{pmatrix} \boxed{} & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & \boxed{} & \\ 0 & 0 & & \end{pmatrix},$$

where we have changed the order of the computational basis in such a way that the first two vectors are

$$\begin{pmatrix} |0 \dots 0\rangle \\ |1 \dots 1\rangle \end{pmatrix} \quad \underbrace{}_{N-k+1}.$$

Consequently, $\rho_{k-1}^{T_i}(\psi)$ has at least one negative eigenvalue, namely, -1 .

It remains only to show the block form of the matrix $\rho_{k-1}^{T_i}(\psi)$. One can deduce it from the following two observations:

(1) The first row in the OA is the only one having 0s at positions $k, \dots, N$ among the first d rows. Indeed, each of the first d rows has 0s in the first $k-1$ position. Since the OA is of index unity at all other positions, the elements will differ between rows.

(2) There are no other rows having only 0s or only 1s at the positions $k, \dots, N-1$. Indeed, since we have assumed $N \geq 2k$, the number of these positions is $\geq k$. The first and second rows have 0s and 1s, respectively, in these positions. Since the OA is of index unity, no other row can inherit this property.

This ends the proof.

APPENDIX B: EXAMPLES OF RESISTANT STATES OBTAINED FROM OAs

We present some of the m -resistant qudit states obtained using the established connection with orthogonal arrays. More states can be straightforwardly found with the help of OA tables available in [50].

The state $|(7, 1)_7\rangle$ can be obtained from $|(8, 1)_7\rangle$ simply by deleting the last qudit from each summand. The number of summands increases rapidly as a function of N^{m+1} . For instance, the states $|(7, 2)_7\rangle$ and $|(8, 2)_7\rangle$ are superpositions of 7^3 elements each, while the state $|(8, 3)_7\rangle$ consists of 7^4 elements. The states can be quickly obtained using the method demonstrated in Sec. V A with the help of OA tables available in [50].

$$\begin{aligned} |(6, 1)_5\rangle = & |000000\rangle + |011234\rangle + |022341\rangle + |033412\rangle \\ & + |044123\rangle + |101111\rangle + |112403\rangle + |124032\rangle \\ & + |130324\rangle + |143240\rangle + |202222\rangle + |214310\rangle \\ & + |223104\rangle + |231043\rangle + |240431\rangle + |303333\rangle \\ & + |310142\rangle + |321420\rangle + |334201\rangle + |342014\rangle \\ & + |404444\rangle + |413021\rangle + |420213\rangle + |432130\rangle \\ & + |441302\rangle. \end{aligned} \quad (B1)$$

$$\begin{aligned} |(8, 1)_7\rangle = & |00000000\rangle + |01123456\rangle + |02234561\rangle + |03345612\rangle + |04456123\rangle + |05561234\rangle + |06612345\rangle \\ & + |10111111\rangle + |11352064\rangle + |12520643\rangle + |13206435\rangle + |14064352\rangle + |15643520\rangle + |16435206\rangle \\ & + |20222222\rangle + |21546301\rangle + |22463015\rangle + |23630154\rangle + |24301546\rangle + |25015463\rangle + |26154630\rangle \\ & + |30333333\rangle + |31265140\rangle + |32651402\rangle + |33514026\rangle + |34140265\rangle + |35402651\rangle + |36026514\rangle \\ & + |40444444\rangle + |41031625\rangle + |42316250\rangle + |43162503\rangle + |44625031\rangle + |45250316\rangle + |46503162\rangle \\ & + |50555555\rangle + |51604213\rangle + |52042136\rangle + |53421360\rangle + |54213604\rangle + |55136042\rangle + |56360421\rangle \\ & + |60666666\rangle + |61410532\rangle + |62105324\rangle + |63053241\rangle + |64532410\rangle + |65324105\rangle + |66241053\rangle. \end{aligned} \quad (B2)$$

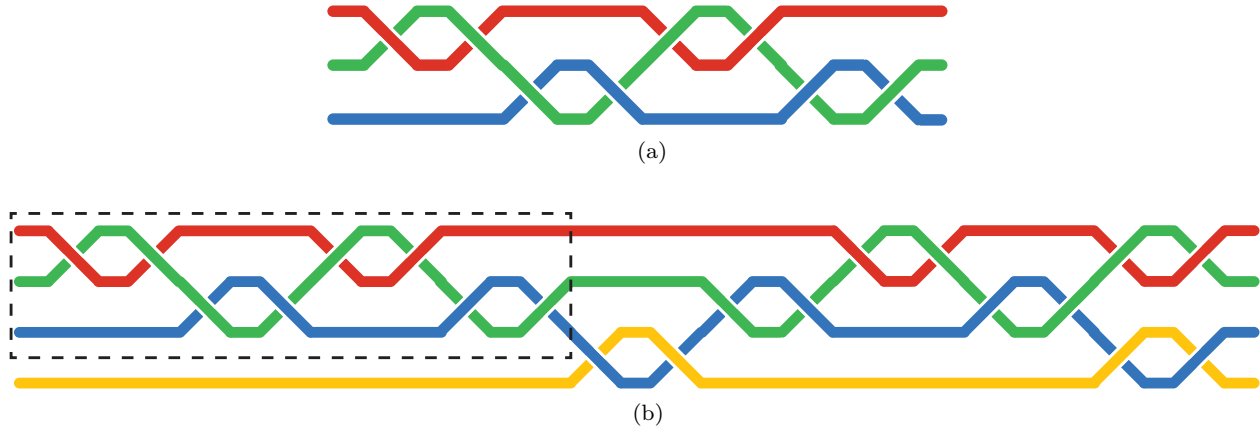


FIG. 11. Braid representations for Brunnian links of three rings (a) and four rings (b). These braids constitute the building blocks for $(N - 4)$ - and $(N - 3)$ -resistant links on N rings, respectively. In the braid representation, the links are obtained by joining each end of the braids. Interestingly, within the dashed box in (b), one has the braid representation of the three-ring Brunnian link in (a).

APPENDIX C: m -RESISTANT LINKS REVISITED

The notion of m -resistant links is shown to be useful for visualizing the entanglement properties of a multipartite quantum state. Furthermore, it can also be relevant in the context of knot theory. Even though from this perspective it is not common to analyze problems related to cutting the links or neglecting them, the literature contains some items on this topic [55].

We emphasize the relation between m resistance and n triviality of a link diagram. Let us begin with the precise definition of an m -resistant link.

Definition 2 (m -resistant link). A connected link consisting of N components is called m resistant if

- (i) it remains linked when any m components are neglected;
- (ii) it becomes unlinked after neglecting any $m + 1$ components.

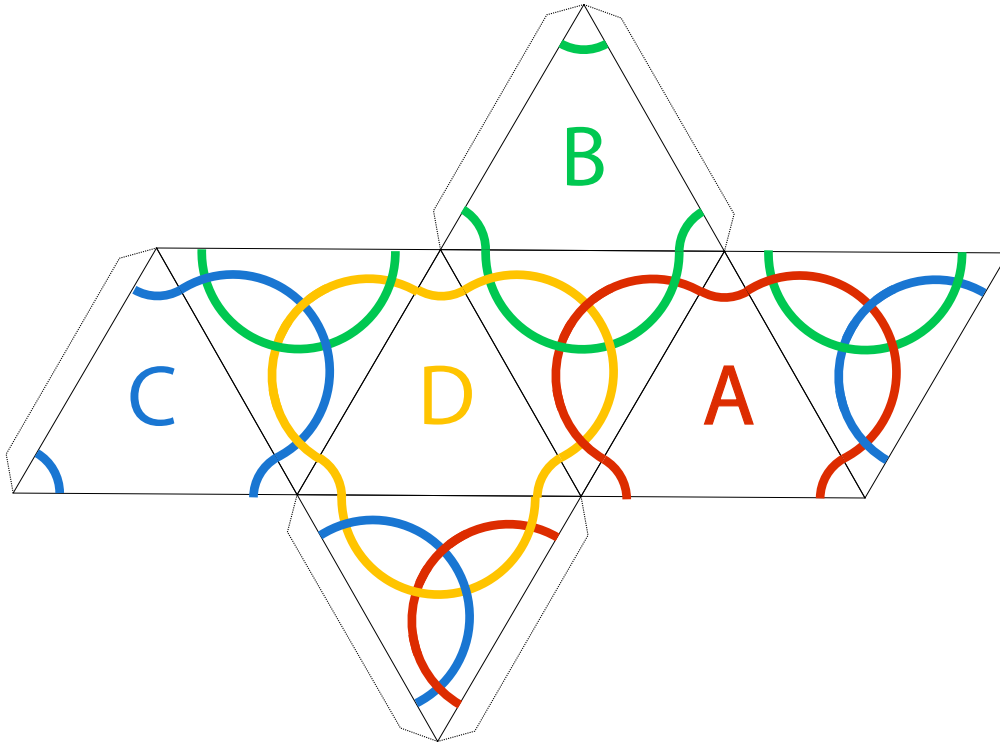


FIG. 12. Blueprint of an octahedron corresponding to the 1-resistant link of four rings (see Fig. 5), classified by the polynomial $abc + abd + acd + bcd$, given in Eq. (16). There is one face for each term of the polynomial with a Brunnian link of three rings, each with open edges. When the edges of the blueprint are glued together, the edges of each ring are connected according to their color, without any extra crossings.

The most common example of an m -resistant link is the Brunnian link, with $m = 0$. We recall that the Brunnian link is a nontrivial link that becomes a set of trivial unlinked rings if any single component is removed.

A Brunnian link consisting of N rings is by construction 0 resistant (see Fig. 4). There is a natural question, which is whether or not, for any N and $m < N$, there exists an m -resistant link. We show that this is in fact possible, as it should be on intuitive grounds.

Proposition 2. For any N and $m = 0, 1, \dots, N - 1$, there exists an m -resistant link of N rings.

To prove this statement we use the technique of associating rings with polynomials provided in [31]. Consider a polynomial of N variables, $P_m(x_1, x_2, \dots, x_N)$, with $\binom{N}{m}$ different terms, each being a product of $N - m$ variables. If $m + 1$ variables are set to 0, then P_m is equal to 0, which is taken to imply that the corresponding link becomes disjoint after cutting $m + 1$ links.

The procedure for constructing such links uses Brunnian braids spanned on the appropriate number of rings as basic building blocks. In particular, constructing an m -resistant link of N components amounts to constructing a polynomial of

N variables with $\binom{N}{N-m}$ terms of $N - m$ variables each and identifying each of these terms with the appropriate Brunnian building block (see Fig. 11). Each Brunnian building block is a pure braid, i.e., a braid where the lines begin and end in the same order. In addition, each Brunnian braid interlaces only the lines corresponding to the variables appearing in the specific term of the polynomial, with the remaining lines present but not interlaced. In the end, all braids are put together by connecting lines of the same color to form a single braid, corresponding to the specific m -resistant link obeying both conditions required in Definition 2. The order in which they are connected does not matter.

Finally, the procedure which is used to represent all m -resistant links in this work, developed in [31], can also be adapted in a way which is easier for three-dimensional visualization. For a given m -resistant link, this procedure begins by drawing one Brunnian link for each term of the polynomial that characterizes the link and then cutting one end of each ring. The ends of all rings of the same color are then connected in such a way that no additional crossings are performed. A specific way to systematize this process is by building blueprints like the one described in Fig. 12.

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B Stochastic local operations with classical communication of absolutely maximally entangled states

Stochastic local operations with classical communication of absolutely maximally entangled states

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Absolutely maximally entangled (AME) states are maximally entangled for every bipartition of the system. They are crucial resources for various quantum information protocols. We present techniques for verifying that two AME states are equivalent concerning stochastic local operations and classical communication (SLOCC). The conjecture that for a given multipartite quantum system all AME states are SLOCC equivalent is proven false. We also show that the existence of AME states with minimal support of six or more particles results in the existence of infinitely many such non-SLOCC-equivalent states. Moreover, we present AME states which are not SLOCC equivalent to the existing AME states with minimal support.

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I. INTRODUCTION

Entanglement of bipartite states is a widely discussed problem and, in fact, already very well understood [1]. However, quantification of entanglement for multipartite states remains a challenge [2]. In particular, according to different entanglement measures for multipartite states (like the tangle, the Schmidt measure, the localizable entanglement, or geometric measure of entanglement), the states with the largest entanglement do not overlap in general.

We discuss AME states which are maximally entangled for every bipartite of the system. AME states are being applied in several branches of quantum information theory: in quantum secret sharing protocols [3], in parallel open-destination teleportation [4], in holographic quantum error correcting codes [5], among many others. Different families of AME states have been introduced [6,7] and the problem of their existence is being investigated [8–10]. It has been demonstrated that the simplest class of AME states, namely, AME states with the *minimal support*, is in one-to-one correspondence with the classical error correction codes [11] and combinatorial designs known as *orthogonal arrays* (OAs) [12]. Henceforward, two-way interaction with combinatorial designs and quantum error correction codes is observed [8,13]. AME states are special cases of *k-uniform* states characterized by the property that all of their reductions to *k* parties are maximally mixed [14].

Since the entanglement quantification becomes an ambitious project while the number of parties in a system increases, the problem of satisfactory classification of states turns out to be essential [2]. The state space might be partitioned into equivalence classes with respect to a selected class of local operations [15]. Any two states from one class are interconvertible by an adequate local operator, while such a transformation cannot be provided for states from different classes. Nevertheless, it is not obvious which class of local operators provides the ultimate division of state space. One of the reasonable choices was the division according to local unitary (LU) operations [2]. Two states $|\psi\rangle$ and $|\phi\rangle$ belong to

the same LU class if and only if there exists a local unitary operator transforming one into the other: $|\phi\rangle = U_1 \otimes \cdots \otimes U_N |\psi\rangle$.

The fact that entanglement is used for the transmission of information between parties far apart restricts us to the LU operations. Nevertheless, we may also allow classical information to be transmitted between the distant parties. This leads us to supersede the class of LU operations with the local operators and classical communications (LOCC) [16,17]. It is known that if the state $|\psi\rangle$ can be transformed into $|\phi\rangle$ by using LOCC operations only, $|\psi\rangle$ possesses at least as much entanglement as $|\phi\rangle$. In general, this transformation cannot be inverted, and hence LOCC imposes the partial order on the state space. Nevertheless, one may study whether two states are LOCC equivalent with a nonvanishing probability of success. Such operations are known as stochastic LOCC (SLOCC). Mathematically, two states $|\psi\rangle$ and $|\phi\rangle$ belong to the same SLOCC class if and only if there exists a local invertible operator transforming one into the other: $|\phi\rangle = O_1 \otimes \cdots \otimes O_N |\psi\rangle$ [18]. Thus, the state space might be partitioned into SLOCC classes.

The number of SLOCC classes rapidly increases with the number of parties *N* and the local dimension *d* of a given system. For instance, all bipartite pure states are equivalent by SLOCC [15]. In the simplest multipartite system, namely, the three qubit system, there are precisely six distinct SLOCC classes (only two of them among fully entangled states, represented by Greenberger-Horne-Zeilinger (GHZ) and *W* states, respectively) [18]. Those systems are the last ones with a finite number of SLOCC classes; for *N* > 3 or *d* > 2 there are infinitely many SLOCC classes [18]. Despite this fact, all four-partite qubit states were classified into nine families, some of them with infinitely many SLOCC classes but of a similar structure [19]. This result was later corrected to eight such classes, while one of the proposed families turned out to be not fully entangled [20].

For larger quantum systems a comprehensive and satisfactory description of SLOCC-class structure has not been established yet. Several studies in this area lead us in

constructing invariants under LU and SLOCC transformations for three parties in [21], multipartite pure quantum systems in [22–24], and mixed multipartite states in [25]. It is known that that polynomial invariants are completely characterizing LU equivalence classes [26] and that the number of nonzero polynomials is a function of a local dimension d . SLOCC invariant polynomials were found for three qubit systems [27,28], for four qubit systems [29,30], and generally for multiqubit systems [31,32]. In the qudit case, an interesting attempt to provide polynomial invariants has been also made [33]. Despite many attempts of classification of these polynomials [34], and enhancing them with physical meaning, their structure remains inscrutable. Many efforts have also focused on LU and SLOCC equivalence of stabilizer states [35], matrix-product states and projected entangled pair states [36], Gaussian states [37,38], locally maximally entangleable states [39], or generalized Bell states [40].

The initial motivation for our paper was the question of whether different constructions of AME states are equivalent by any local transformation. It was already shown that some k -uniform states are not LU (and in fact not SLOCC) equivalent [41]. This result was based on a comparison of the ranks of reduced density matrices [42]. Nevertheless, in some specific cases, the aforementioned rank argument is never conclusive. This is the case when two k -uniform states are of minimal support, or when the bound on k is saturated, i.e., in the case of AME states. In this paper, we develop techniques of SLOCC verification between such states. We provide methods of SLOCC equivalence verification for all k -uniform states with minimal support. We show that the conjecture of all AME states being LU and SLOCC equivalent does not hold. In particular, we show that some AME states cannot be transformed into existing minimal support form by any local invertible operation. Moreover, the class of LU and SLOCC transformation is widely investigated and a systematic method for verification of LU and SLOCC equivalence of AME states and k uniformity with minimal support is provided. We expose the vital contrast between AME states of small systems (up to five parties) and larger systems. In particular, for larger systems there exist infinitely many non-LU- and non-SLOCC-equivalent AME states of minimal support differing only by phases. Additionally, we emphasize the essential difference between local transformation of k -uniform states or AME states of odd number of parties ($k < N/2$) and AME states of even number of parties ($k = N/2$). The structure of the latter is more complex and nonclassical in some sense. Despite the refined analysis of this case, obtained results are still intricate and dependent on specific cases.

The paper is organized as follows. In Sec. II, we recall construction methods of k -uniform and AME states known from the literature, and we provide several explicit examples of such states. Moreover, we discuss the relation between LU and SLOCC equivalences restricted to k -uniform and AME states. The main results of our paper are presented in the following three consecutive sections. Section III discusses local transformation of k -uniform states or AME states of odd number of parties ($k < N/2$). Similar results concerning AME states of even number of parties ($k = N/2$) are presented in Sec. IV. Section V applies general results obtained in the previous sections. We present several examples of LU and

SLOCC nonequivalent k -uniform states. The precise number of LU and SLOCC classes of AME states with minimal support is specified. Moreover, the nontrivial bounds on the number of such classes of general AME states are given. In Sec. VI, we discuss some classes of combinatorial designs directly related to our problem. Existence and extension of those designs turned out to be crucial to obtain the aforementioned results. This phenomenon is presented in detail. Further discussion and open problems are left for Sec. VII. Summary and conclusions are presented in Sec. VIII. Proofs of statements included in Sec. III might be found in Appendices A, B, and D, whereas claims presented in Sec. IV are justified in Appendix C.

II. NOTATION AND PRELIMINARIES

A. k -uniform states and AME states

Consider a multipartite quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties with a local dimension d each. We say that $|\psi\rangle$ is a k -uniform state if its reduced density matrices are maximally mixed, i.e.,

$$\rho_S(\psi) \propto \text{Id}$$

for any subsystem S of k parties ($|S| = k$). The uniformity k cannot exceed $\lfloor N/2 \rfloor$ [8]. States which saturate this bound, i.e., $\lfloor N/2 \rfloor$ -uniform states, are called AME states, and are denoted by $\text{AME}(N, d)$. Particular attention is paid to AME states of an even number of parties, which are equivalent to notions as perfect tensors [5] or multiunitary matrices [43].

The *support* of a state $|\psi\rangle$ is the number of nonzero coefficients when $|\psi\rangle$ is written in the computational basis. Note that the support of the k -uniform state is at least d^{N-k} . Indeed, the partial trace over $N - K$ particles is an identity matrix $\text{Id}_{d^{N-k}}$. k -uniform states with support equal to d^k are called *minimal support*. k -uniform states are a natural generalization of the well-established GHZ state.

Example 1. The GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

is a 1-uniform state of minimal support. Similarly, its natural generalization to N -party qudit states (each party has exactly d distinguishable energy levels)

$$|\text{GHZ}_d^N\rangle = \frac{1}{\sqrt{d}}(|0 \cdots 0\rangle + \cdots + |d-1 \cdots d-1\rangle)$$

is a 1-uniform state of minimal support.

It is worth mentioning that the GHZ state is maximizing entanglement properties among all three qudit states. This statement, however, is not true anymore for larger systems [44].

Example 2. The state of four qutrits

$$\begin{aligned} |\text{AME}(4,3)\rangle = & \frac{1}{3}(|0000\rangle + |0121\rangle + |0212\rangle \\ & + |1110\rangle + |1201\rangle + |1022\rangle \\ & + |2220\rangle + |2011\rangle + |2102\rangle) \end{aligned}$$

is an $\text{AME}(4,3)$ state of minimal support [7]. It reveals larger entanglement properties than a relevant GHZ state.

AME(N, d) states are maximizing entanglement properties among all N -party states, each with d levels [3]. There is no general construction of AME(N, d) states, and, in fact, they do not exist for any numbers N and d . Indeed, it was first observed that the AME state of four qubits does not exist [9]. Nowadays, more of such negative results are known [10]. Some cases, such as AME(4,6), are believed to not exist, despite the fact that the mathematical proof is still missing [12].

We would like to finish this section with two remarkable observations. First, all known k -uniform and AME states might be written by simple closed formulas. For instance,

$$\begin{aligned} |\text{GHZ}\rangle &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i, \dots, i\rangle, \\ |\text{AME}(4,3)\rangle &= \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j\rangle \end{aligned} \quad (1)$$

are relevant to the GHZ state and AME(4,3) presented in Examples 1 and 2

Secondly, not all k -uniform states are of minimal support. It is rather easy to verify that AME states with minimal support of five or six qubits do not exist. Nevertheless, the construction of AME(6,2) was provided [6,12]. We present one example of AME states with nonminimal support relevant to the future discussion.

Example 3. Consider the following states:

$$|\text{AME}(5,d)\rangle = \frac{1}{\sqrt{d^3}} \sum_{i,j,k=0}^{d-1} \omega^{(3i+j)k} |i, j, i+j, 2i+j+k, k\rangle,$$

where ω is the d th root of unity. $|\text{AME}(5,d)\rangle$ states satisfy all properties required from AME states for any integer number $d \geq 2$ [6,12]. They cannot be written, however, in the minimal support form.

B. Orthogonal arrays

Orthogonal arrays [45] are combinatorial arrangements, tables with entries satisfying given orthogonal properties. They were created in response to optimization problems in statistical analysis. Their most famous application can be summarized in one sentence: “Your automobile lasts longer today because of orthogonal arrays” [46].

A close connection between OAs and maximally entangled states [13], error-correcting codes [45], brought a new life for these combinatorial objects. As some OAs might be in one-to-one correspondence with k -uniform states, the concept of OA is briefly presented below.

An orthogonal array $\text{OA}(r, N, d, k)$ is a table composed by r rows and N columns with entries taken from $0, \dots, d-1$ in such a way that each subset of k columns contains all possible combinations of symbols with the same amount of repetitions (see Fig. 1). The number of such repetitions is called the *index* of the OA and denoted by λ . One may observe that the index of the OA is related to the other parameters:

$$\lambda = \frac{r}{d^k}.$$

0 0 0 1 0 2 1 0 1 1 1 2 2 0 2 1 2 2	0 0	0	0 0 1 1 2 2 0 1 1 0 2 0 2 1 2 2	0	$ \text{AME}(4,3)\rangle \propto \omega_{00} 0000\rangle$
	1 1	0		1	$+ \omega_{01} 0111\rangle$
	2 2	0		2	$+ \omega_{02} 0222\rangle$
	1 2	1		2	$+ \omega_{10} 1012\rangle$
	0 2	1		2	$+ \omega_{11} 1120\rangle$
	0 1	1		2	$+ \omega_{12} 1201\rangle$
	2 1	2		1	$+ \omega_{20} 2021\rangle$
	0 2	2		2	$+ \omega_{21} 2102\rangle$
	1 0	2		0	$+ \omega_{22} 2210\rangle$

FIG. 1. The orthogonal array of unity index OA (9, 4, 3, 2) on the left and repeated in the center. Each subset consisting of two columns contains all possible combinations of symbols. Here, two such subsets are highlighted. The relevant quantum state is obtained by forming a superposition of states corresponding to consecutive rows of the array enhanced by some phases (see the expression on the right).

The OA with $\lambda = 1$ is called the OA of *index unity*. Figure 1 presents an example of an index unity OA.

A pure quantum state consisting of r terms might be associated with $\text{OA}(r, N, d, k)$, simply by reading all rows of the OA [13,45]. With a little more effort, one may adjust phases $\omega_1, \dots, \omega_r$ in front of any term (see Example 4). Intriguingly, this relevance provides a one-to-one correspondence between k -uniform states of minimal support and OAs of index unity.

Proposition 1. There is one-to-one correspondence between k -uniform states with the minimal support of N qudits and OA $\text{OA}(d^k, N, d, k)$ enhanced with the phase vectors

$$(\omega_1, \dots, \omega_{d^k}),$$

where $|\omega_i| = 1$.

From $\text{OA}(d^k, N, d, k)$ the k -uniform state is created by reading all terms and adjusting them with the relevant phases ω_i . Conversely, from the AME state the OA of index unity might be built, simply by erasing phases and adjusting all terms one above the other.

Example 4. For any phases $|\omega_{i,j}| = 1$ the following state is 2-uniform:

$$|\text{AME}(4,3)_\omega\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \omega_{i,j} |i, j, i+j, 2i+j\rangle.$$

It is defined uniquely up to global phase.

It is known that for any number N and k there exists $\text{OA}(d^k, N, d, k)$ for a local dimension d being sufficiently large (in fact, such construction is given for d being a prime power satisfying $d > k$ and $N-1$) [47]. Hence, for any number of parties N the k -uniform state with minimal support might be created where the component systems have a sufficiently large number of levels. The problem of existence and classification of OAs (in particular OAs of index unity) has been extensively studied [48,49]. We refer to the web page of Sloane for tables of OAs [50].

Example 5. The following states

$$|\text{AME}(5,d)'\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j, 3i+j\rangle$$

are AME states with minimal support for all $d \geq 5$ being prime numbers [11].

In fact, with a little more effort such states might be constructed for all prime powers $d \geq 4$ [11]. For instance, the state

$$|\text{AME}(4,4)\rangle = \frac{1}{4} \sum_{i,j=0}^{d-1} |i, j, M_{i,j}^1, M_{i,j}^2\rangle$$

where

$$M^1 := \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \quad M^2 := \begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}$$

is an AME(4,4) state of minimal support (rows and columns of M^1 and M^2 are indexed by $i, j = 0, \dots, 3$) [11]. Construction of matrices M^1 and M^2 comes from the multiplication structure in the Galois field $\text{GF}(4)$, which might be seen as a multiplication of irreducible polynomials of degree 2 [11]. In fact, matrices M^1 and M^2 form a mutually orthogonal Latin square MOLS(4) (see Definition 3 for details).

As we already mentioned in Example 3, not all k -uniform states are of minimal support, which simply means that not all AME states are obtained from OAs of index unity.

C. Composed systems

For any two k -uniform states $|\psi_1\rangle$ and $|\psi_2\rangle$, one may consider the composed system $|\psi_1\rangle \otimes |\psi_2\rangle$, which inherits the property of being k uniform. For instance, the state

$$\begin{aligned} & |\text{AME}(4,9)_{3 \times 3}\rangle \\ &= \frac{1}{9} \sum_{\substack{i,j=0 \\ k,\ell=0}}^2 |(i,k), (j,\ell), (i+j, k+\ell), (i+2j, k+2\ell)\rangle \end{aligned} \quad (2)$$

is a composition of two $|\text{AME}(4,3)\rangle$ states from Eq. (1). Each pair (i, k) is identified with a number $0, \dots, 8$ written in the ternary numeral system, i.e., $(i, k) \cong 3i + j$.

D. Local transformations

Two N qudit states $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent if one can be transformed into another by local unitary operators, i.e.,

$$|\phi\rangle = U_1 \otimes \dots \otimes U_N |\psi\rangle.$$

The LU equivalence of $|\psi\rangle$ is referred to in the text as an *automorphism*.

Mathematically, two states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent if and only if there exists a *local invertible* operator connecting those states [18]:

$$|\phi\rangle = O_1 \otimes \dots \otimes O_N |\psi\rangle.$$

Since LU and SLOCC equivalences are equivalence relations, the state space might be naturally partitioned into *LU classes* and *SLOCC classes*, respectively.

E. The structure of SLOCC classes

We present a brief outline of some algebraic invariant methods interconnected to the SLOCC partition problem of multipartite entangled states. We introduce the notion of critical states and we discuss consequences of the Kempf-Ness theorem [51] for multipartite systems. We refer to [52] for more details.

The state ρ is called a *critical state* if all its reduced density matrices ρ_i are proportional to the identity. In particular, the class of critical states contains stabilizer states, cluster states, and all k -uniform states, among many others [52].

Notice that the critical states were initially defined differently, via an action of the Lie group associated with the state space. By applying the Kempf-Ness theorem [51] it was later observed that, indeed, states are critical if and only if they are maximally entangled [53].

The Kempf-Ness theorem has one more significant consequence for multipartite quantum states. It follows that within one SLOCC class the critical states are unique up to LU equivalences. Therefore, such classes possess the canonical representative.

Notice that not all SLOCC classes contain a critical state, and hence there is no one-to-one correspondence between SLOCC classes and maximally mixed states. More precisely, each SLOCC class is topologically closed (equivalently closed with respect to Zariski topology) if and only if it contains a critical state [54]. In fact, closed SLOCC classes are dense in a state space [55].

We conclude this discussion with the following corollary.

Corollary 1. Two critical states are in the same SLOCC class if and only if they are LU equivalent. Notice that all k -uniform states are critical states.

Therefore, verification of LU equivalence between two k -uniform states is equivalent to verification of SLOCC equivalence between them.

III. LOCAL EQUIVALENCES, CASE $2k < N$

We introduce another class of local unitary operations, essential for the classification problem of k -uniform states.

Definition 1. A unitary matrix M is called a *unitary monomial matrix* if one of the following holds.

(1) M has exactly one nonzero entry in each row and each column.

(2) M is a product of a permutation and diagonal matrix.

(3) M does not change the support of any quantum state.

For multipartite systems, the local monomial operation will be denoted as *LM* equivalency.

One can see that all conditions 1–3 are, indeed, equivalent. Obviously, each local monomial operation provides the LU equivalence between two states of minimal support. Indeed, since it is a local unitary operation, it does not change the entanglement properties of a state; moreover, it preserves the number of elements in the support of a state. As we shall see, the reverse statement is also true for k -uniform states where $2k < N$. In other words, searching for LU equivalence between two such k -uniform states of minimal support might be restricted to the search within the LM class.

Proposition 2. For $2k < N$, each LU or SLOCC equivalence between two k -uniform states of minimal support is in fact LM equivalency.

Corollary 2. For $2k < N$, two k -uniform states of minimal support are LU or SLOCC equivalent if and only if they are LM equivalent.

We shall prove the above statement in a slightly enhanced version in Appendix A.

With the strengthened version of Proposition 2 at hand (see Appendix A), we have shown that not all AME states are equivalent, which was an initial motivation for our research.

Proposition 3. Two families of AME(5, d) states, $|\text{AME}(5, d)\rangle$ and $|\text{AME}(5, d)'\rangle$ presented in Examples 3 and 5, respectively, are not LU equivalent for any prime local dimension d .

Therefore, as an immediate conclusion from Corollary 1, states $|\text{AME}(5, d)\rangle$ and $|\text{AME}(5, d)'\rangle$ belong to different SLOCC classes. We refer to Appendix D for the proof of the above statement.

Both states $|\text{AME}(5, d)\rangle$ and $|\text{AME}(5, d)'\rangle$ belong to special classes of quantum states: *stabilizer* states and *graph* states. A comprehensive introduction to this topic might be found in [56]. The class of natural local operation among stabilizer and graph states is called *local Clifford* (LC) operations. The verification of LC equivalence between two stabilizer states is rather a simple problem and might be the polynomial-time algorithm [57]. Nevertheless, LU- or SLOCC-equivalence verification of such states is generally a challenging problem. Surprisingly, it was shown that there are LU equivalences of graph and stabilizer states which are beyond LC class [58]. One can see our results as proof for lack of LU equivalence of two graph and stabilizer states.

Verification of LU equivalence

The problem of verification whether two different states are LU and SLOCC equivalent is of the most importance from the application point of view [59]. Monomial matrices are products of permutations and diagonal matrices. Although permutation matrices are easy to quantify, diagonal matrices are indexed by a real coefficient. The following statement shows that we can overcome this apparent difficulty. In fact, verification of LU equivalence between two states is restricted to testing a finite number of possible equivalences.

We introduce the following notation. Consider a k -uniform state with minimal support:

$$|\psi\rangle = \sum_{I \in \mathcal{I}} \omega_I |I\rangle,$$

where $I \in [d]^n$ are multi-indices running over the set $\mathcal{I}$ of the size $|\mathcal{I}| = d^k$. Denote by $\mathcal{I}_a^i$ all those indices with a on the i th position:

$$W_a^i := \prod_{I \in \mathcal{I}_a^i} \omega_I.$$

Similarly, $\mathcal{I}_{a_1, \dots, a_\ell}^{i_1, \dots, i_\ell}$ denotes the set of indices with a_i on the i th position:

$$W_{a_1, \dots, a_\ell}^{i_1, \dots, i_\ell} := \prod_{I \in \mathcal{I}_{a_1, \dots, a_\ell}^{i_1, \dots, i_\ell}} \omega_I.$$

Observe that any local permutation $\sigma = \sigma_1 \otimes \dots \otimes \sigma_n$ acts on $W_{a_1, \dots, a_\ell}^{i_1, \dots, i_\ell}$ by permuting relevant elements:

$$\sigma(W_{a_1, \dots, a_\ell}^{i_1, \dots, i_\ell}) := \prod_{I \in \mathcal{I}_{\sigma_{i_1}(a_1), \dots, \sigma_{i_\ell}(a_\ell)}^{i_1, \dots, i_\ell}} \omega_I.$$

We specify the diagonal matrices which might appear in local equivalences described in Proposition 2.

Proposition 4. Consider two k -uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$. The eventual LU equivalence between them is of the following form:

$$\omega(\sigma_1 D_1 \otimes \dots \otimes \sigma_n D_n)$$

where ω is a global phase, σ is a local permutation $\sigma = \sigma_1, \dots, \sigma_n$, and D_i are the following diagonal matrices:

$$D_i = \text{diag} \left(\sqrt[d]{\frac{(W_{0,I}^{i,S})'}{\sigma(W_{0,I}^{i,S})}}, \dots, \sqrt[d]{\frac{(W_{d-1,I}^{i,S})'}{\sigma(W_{d-1,I}^{i,S})}} \right),$$

with entries given by any d th root of a relevant complex number, where S is any subset of $k-2$ indices which do not contain i , and I is any multi-index $I = i_2, \dots, i_{k-1}$. In particular, for $k=2$ it is the empty set $S \equiv \emptyset$.

Moreover, for $k > 2$ there is the following necessary condition for existence of such LU equivalence. For any $S \subset [n] \setminus \{i\}$, such that $|S| = k-2$ and any symbol ℓ

$$\frac{(W_{\ell,I}^{i,S})'}{\sigma(W_{\ell,I}^{i,S})} = \frac{(W_{\ell,I'}^{i,S})'}{\sigma(W_{\ell,I'}^{i,S})}$$

for arbitrary multi-indices $I = i_2, \dots, i_{k-1}$ and $I' = i'_2, \dots, i'_{k-1}$.

The proof of Proposition 4 is given in Appendix B. We illustrate the usefulness of this criterion in two examples: 2- and 3-uniform states (see Sec. V).

IV. LOCAL EQUIVALENCES, CASE $2k = N$

For AME($2k, d$) states, the statement of Proposition 2 does not hold anymore. The following example illustrates this difference.

Example 6. The Fourier transform F_3 ,

$$(F_3)^{\otimes 4} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix}^{\otimes 4}, \quad (3)$$

provides an automorphism of the AME(4,3) state from Eq. (1).

As we shall see, Fourier matrices are not the only non-monomial matrices providing the LU equivalence between AME($2k, d$) states of minimal support. On the other hand, it is not generally true that Fourier matrices preserve all AME($2k, d$) states with minimal support. Despite the exhaustive analysis performed, the general structure of LU equivalences between AME($2k, d$) states is still puzzling and remains unknown. Nevertheless, for sufficiently small values of k and d , the complexity of the problem reduces significantly, and the analog of Proposition 2 might be stated. We begin with the definition of matrices beyond monomial class, which can provide LU equivalence between AME states.

Definition 2. Let d and q be positive integers. A *Butson-type* complex Hadamard matrix of order d and complexity q is a unitary matrix in which each entry is a complex q th root of unity scaled by the factor $1/d$. The set of Butson-type matrices is denoted by $\text{BH}(d, q)$.

In literature [60,61], Butson-type matrices are defined without scaling factor $1/d$, therefore they are proportional to the unitary matrices.

For our purpose, it is enough to focus on matrices of the type $\text{BH}(d, d)$. As we shall see, those matrices (up to monomial matrices) define all possible LU equivalences of AME states. The problem of existence and classification of such matrices is discussed later, in Sec. IV B.

Proposition 5. Consider two $\text{AME}(2k, d)$ states of minimal support, where k and d are sufficiently small (see Remark 1). Each LU equivalency between them is of one of the following forms: (1) the tensor product of Butson-type matrices $B_i \in \text{BH}(d, d)$ multiplied by LM matrices from each side or (2) the LM matrices themselves.

Similarly to the case $2k < n$, we can specify the class of diagonal matrices which might appear in LM equivalences from Proposition 5.

Proposition 6. LU equivalence between two $\text{AME}(2k, d)$ states of minimal support, where k and d are sufficiently small (see Remark 1), is of the following form: either

$$\omega(\vec{D}_1^{-1} B_1 \overleftarrow{D}_1^{-1} \otimes \cdots \otimes \vec{D}_n^{-1} B_n \overleftarrow{D}_n^{-1}) \quad (4)$$

or

$$\omega(\sigma_1 D_1 \otimes \cdots \otimes \sigma_n D_n) \quad (5)$$

where ω is a global phase; $B_i \in \text{BH}(d, d)$ are Butson-type matrices; σ_i are permutation matrices; and D , $\vec{D}_i$, and $\overleftarrow{D}_i$ are diagonal matrices.

The entries of diagonal matrices are the d th root of specified complex numbers:

$$D_i = \text{diag} \left(\sqrt[d]{\frac{(W_{0,I}^{i,S})'}{\sigma_i(W_{0,I}^{i,S})}}, \dots, \sqrt[d]{\frac{(W_{d-1,I}^{i,S})'}{\sigma_i(W_{d-1,I}^{i,S})}} \right),$$

$$\vec{D}_i = \text{diag} \left(\sqrt[d]{(W_{0,I}^{i,S})'}, \dots, \sqrt[d]{(W_{d-1,I}^{i,S})'} \right),$$

$$\overleftarrow{D}_i = \text{diag} \left(\sqrt[d]{W_{0,I}^{i,S}}, \dots, \sqrt[d]{W_{d-1,I}^{i,S}} \right),$$

where S is any subset of $k-2$ indices which do not contain i , I is any multi-index $I = i_2, \dots, i_{k-1}$, and σ is a local permutation $\sigma = \sigma_1, \dots, \sigma_n$ of levels. In particular, for $k=2$ it is the empty set $S \equiv \emptyset$.

Moreover, for $k > 2$ there are the following necessary conditions for existence of such equivalence. Consider any $S \subset [n] \setminus \{i\}$, such that $|S| = k-2$, and any symbols ℓ, ℓ' . For arbitrary multi-indices $I = i_2, \dots, i_{k-1}$ and $I' = i'_2, \dots, i'_{k-1}$,

$$\frac{(W_{\ell,I}^{i,S})'}{\sigma(W_{\ell,I}^{i,S})} = \frac{(W_{\ell,I'}^{i,S})'}{\sigma(W_{\ell,I'}^{i,S})}$$

TABLE I. Examples of $d_{\min}$ and $d_{\max}$ of AME states.

$\text{AME}(2k, d)$	$d_{\min}$	$d_{\max}$
$\text{AME}(2, d)$	3	3
$\text{AME}(4, d)$	9	9
$\text{AME}(6, d)$	11	16
$\text{AME}(8, d)$	13	25

if the equivalence is of the form Eq. (4), and

$$\frac{W_{\ell,I}^{i,S}}{W_{\ell',I}^{i,S}} = \frac{W_{\ell,I'}^{i,S}}{W_{\ell',I'}^{i,S}},$$

$$\frac{(W_{\ell,I}^{i,S})'}{(W_{\ell',I}^{i,S})'} = \frac{(W_{\ell,I'}^{i,S})'}{(W_{\ell',I'}^{i,S})'}$$

if the equivalence is of the form Eq. (5).

The statement above is important from the application point of view. The enormous class of diagonal matrices is significantly restricted. In fact, the classes of matrices from Proposition 5 which may provide the LU equivalence between two states is brought to be finite. Therefore, the LU-equivalence verification problem is discretized and made finite. Moreover, the second part of Proposition 6 imposes some necessary conditions for two $\text{AME}(2k, d)$ states to be LU equivalent for $k > 2$. As we shall see, such assumptions might be easily validated, which implies the existence of non-LU- and non-SLOCC-equivalent $\text{AME}(2k, d)$ states for all $k > 2$ (see Sec. V C). For the proof of Propositions 5 and 6, we refer to Appendix C.

Remark 1. There is the following restriction on numbers d and k imposed in the statement of Propositions 5 and 6:

$$(k+1)(1 + \sqrt[k-1]{k}) \leq d$$

for $k > 1$ and $2 < d$ for $k=1$. This bound is related to the necessary condition for existence and extension of combinatorial designs called *mutually orthogonal hypercubes*. We discuss them in detail in Sec. VI.

In particular, for $k=2, 3, 4, 5, 6$ the smallest value of $d_{\min}$ which does not satisfy the bound above is presented in a Table I. The given bound is not tight. Moreover, we present $d_{\max}$ as the maximal value of a local dimension d for which Proposition 5 does not hold (we found a counterexample). In particular, for the local dimension $d_{\max}$ we found LU equivalence which is not of the form presented in Proposition 5. The origin of $d_{\max}$ is presented in Sec. IV A.

We conjecture that those values behave asymptotically as $(k-1)^2$.

In fact, assumptions on values k and d are not restrictive from the application point of view. Indeed states outside the described class are far beyond current laboratory possibilities [62,63].

A. Composed systems

Consider the $\text{AME}(4,9)$ state being a product of two $\text{AME}(4,3)$ states as it was described in Eq. (2). Since the Fourier transform $(F_3)^{\otimes 4}$ and the identity $\text{Id}_3^{\otimes 4}$ are

automorphisms of AME(4,3), obviously

$$(F_3 \otimes \text{Id})^{\otimes 4} \quad (6)$$

provides an automorphism of the aforementioned AME(9,3) state. One immediately observes that Eq. (6) is not of the form postulated in Proposition 5. Indeed, according to Proposition 5 matrices providing LU equivalence between two AME states of minimal support have either 1 or d nonzero entries in each row and column, each entry of the same norm. Matrices from Eq. (6), however, have exactly three nonzero entries in each row and column. Nevertheless, k and d were assumed to be sufficiently small in Proposition 5; indeed, according to Remark 1, the statement was restricted to $d < 9$ for AME(4, d) states.

Notice that similar automorphisms might be potentially given for any product states. We conjecture that LU equivalences of such a product form are the only ones violating the statement of Proposition 5.

Conjecture 1. If the LU equivalence $U_1 \otimes \cdots \otimes U_{2k}$ of two AME states of minimal support is not of the form proposed in Proposition 5 for $k > 1$, those states are product states and U_i splits according to the composition of states into matrices postulated in Proposition 5.

The conjecture excludes AME(2, d) where the structure of LU equivalences is more abundant (see Sec. V A).

Even though Conjecture 1 seems reliable, the mathematical proof of it is out of reach at this stage of the research. In the most general case (without any assumptions on k and d), we showed that the following might be noted about LU equivalence between AME states.

Proposition 7. Consider two AME states $|\psi\rangle$ and $|\psi'\rangle$ of minimal support which are locally equivalent by $U := U_1 \otimes \cdots \otimes U_n$.

(1) Each row or column of each matrix U_i has the same number s of nonzero elements, all having the same norm $\sqrt{s}$.

(2) Consider a mutually orthogonal Latin hypercube (MOLH) and the number s that satisfies the extension property, there exists a MOLH(s) which can be further extended onto MOLH(d).

(3) Under the assumption that all phases are trivial, i.e., $\omega_I \equiv \omega'_I \equiv 1$ for all multi-indices I , all nonzero entries of matrices U_i are s th roots of unity (scaled by $\sqrt{s}$) up to global multiplication by a complex number.

For the notion of MOLH we refer to Sec. VI.

For the proof, we refer to Appendix C.

B. Classification of Butson-type matrices

Equation (4) provides a unitary equivalence of AME states with minimal support which is not a monomial matrix. We have shown that, under some restrictions on k and d (see Remark 1), such equivalences are local Butson matrices $B_i \in \text{BH}(d, d)$ up to local monomial transformations. It is not difficult to show that Fourier-transform and tensor products of such are elements of the class $\text{BH}(d, d)$. Nevertheless the class of Butson matrices is much larger and contains 1, 2, 1, 4, 1, 143, 23, 51, 1, and 449 773 3 matrices (classified up to monomial matrices) for $d = 3, \dots, 12$ [64]. Tables of Butson matrices are available in [65,66].

Even though the class of Butson matrices $\text{BH}(d, d)$ grows rapidly with d , it seems that the subclass of such matrices that might be involved in LU equivalences of AME states is significantly smaller. Therefore, more specific classification of matrices providing eventual LU equivalences of AME states is needed. We suspect that all such matrices are Fourier-transform and tensor products of such.

On the other hand, not all Fourier matrices might be involved in LU equivalences of AME states. For $d > 3$, we could not construct LU equivalence of AME(5, d) states from Example 5 based on Fourier matrices F_5 . It is worth mentioning that the Fourier transforms F_5 , F_7 , and F_{11} are the only matrices of type $\text{BH}(p, p)$ for $p = 5, 7, 11$. Hence, most probably all automorphisms of AME(p, p) states from Example 5 for $p = 5, 7, 11$ are within the LM class. Nevertheless, for $d = 4$ such an equivalence might be provided by tensor product $F_2 \otimes F_2$, which is illustrated below.

Example 7. The action of

$$(F_2 \otimes F_2)^{\otimes 4}$$

on the state $|\text{AME}(4,4)\rangle$ from Example 5 is equivalent to the following local permutation of indices:

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \otimes \text{Id} \\ \otimes \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

and hence provides the LU equivalence between AME states of minimal support.

We shall finish this section by linking the problem of decreasing the class of Butson matrices involved in LU equivalences of AME states with two mathematical problems.

First, it has been conjectured that for prime dimensions d the Fourier matrices F_d are the only matrices in $\text{BH}(d, d)$ [64].

Second, for some numbers n_1 and n_2 the tensor product of two Fourier matrices F_{n_1} and F_{n_2} is isomorphic to $F_{n_1 n_2}$. For instance, $F_2 \otimes F_3 \cong F_6$, while $F_2 \otimes F_2 \not\cong F_4$. The problem of determining those numbers has been solved [67].

V. EXISTENCE AND UNIQUENESS OF k -UNIFORM STATES

We apply so-far obtained results for various classes of AME states here. We present 1-, 2-, and 3-uniform state classes separately since the analysis of their LU and SLOCC equivalences differs greatly.

A. 1-uniform states

All 1-uniform states of minimal support are of the following form:

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega_i |j_i^1, \dots, j_i^N\rangle,$$

where j_i^ℓ runs over all levels $0, \dots, d-1$ for all indices ℓ . One can observe that they are equivalent to the generalized

Bell state $|\text{GHZ}_d^N\rangle$ (see Example 1) and hence pairwise LU equivalent. Indeed, the local transformation

$$U_1(|j_i^1\rangle) = (\omega_i^{-1}|j_i^1\rangle), \quad U_\ell(|j_i^\ell\rangle) = (|j_i^\ell\rangle)$$

for systems $\ell = 2, \dots, N$ provides aforementioned LU equivalence.

Observation 1. All 1-uniform states of minimal support are LU equivalent.

This straightforward observation suggests that the structure of one-uniform states is rather simple and, in fact, not interesting. However, we point out that an intriguing property is the automorphism group of $\text{AME}(2, d)$ states. One can see that the Fourier transform F_2 preserves the Bell state:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes 2} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Similarly, the Fourier transform $F_n \otimes \bar{F}_n$ preserves the generalized Bell state of two parties:

$$|\text{AME}(2, d)\rangle := \frac{1}{\sqrt{d}}[|00\rangle + \dots + |(d-1)(d-1)\rangle]. \quad (7)$$

Interestingly, the tensor product $U \otimes \bar{U}$ of unitary matrices preserves the generalized Bell state $\text{AME}(2, d)$. Indeed, for each i ,

$$U \otimes \bar{U}|ii\rangle = \sum_{j=0}^{d-1} |u_{ij}|^2 |jj\rangle + \text{others},$$

and hence

$$\begin{aligned} U \otimes \bar{U}|\text{AME}(2, d)\rangle &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} |u_{ij}|^2 |jj\rangle + \text{others} \\ &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \left(\sum_{i=0}^{d-1} |u_{ij}|^2 \right) |jj\rangle + \text{others} \\ &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle + \text{others}. \end{aligned}$$

Since the state was normalized, all other terms on the right-hand side disappear.

This is in contrast to $\text{AME}(2k, d)$ states for $k > 1$, where LU equivalences were provided only by appropriate Butson type matrices $B(d, d)$ for all d sufficiently small.

B. 2-uniform states

Consider 2-uniform states with minimal support

$$|\phi_\alpha\rangle := \frac{1}{d} \left(|\alpha|0, \dots, 0\rangle + \sum_{i,j \neq (0,0)} |i, j\rangle \otimes |\phi_{i,j}\rangle \right)$$

indexed by a complex numbers α , $|\alpha| = 1$. Each of such a state is LU equivalent to $|\phi_{\alpha=0}\rangle$ by the following:

$$\begin{aligned} U_1 &= \text{diag}[(\bar{\omega}_\alpha)^{n-1}, 1, \dots, 1], \\ U_i &= \text{diag}[(\bar{\omega}_\alpha)^{d-1}, \omega_\alpha, \dots, \omega_\alpha], \end{aligned}$$

for $i = 2, \dots, n$, where $\omega_\alpha = \sqrt[n]{\alpha}$ is an arbitrary root.

Since all states from the family $|\phi_\alpha\rangle$ are LU equivalent with $|\phi_{\alpha=0}\rangle$, they are also pairwise equivalent. If the exceptional phase stands by a different term, the similar transformation of such a state onto $|\phi_{\alpha=0}\rangle$ might be given. Therefore, all 2-uniform states

$$|\phi_\omega\rangle = \frac{1}{d} \sum_{i,j} \omega_{i,j} |i, j\rangle \otimes |\phi_{i,j}\rangle$$

are LU equivalent to $|\phi_{\alpha=0}\rangle$, and hence pairwise equivalent. Indeed, the LU equivalence is a composition of the aforementioned transformations. The matrices $U_1, \dots, U_N$ are the simplest matrices satisfying restrictions given in Proposition 4, which explains how they were found. Observe that the assumption $2k < N$ was irrelevant in the presented analysis. Therefore, we conclude this discussion in the following corollary.

Corollary 3. Two 2-uniform states of minimal support which differs only with phases, i.e.,

$$\begin{aligned} |\psi\rangle &= \sum_{I \in \mathcal{I}} \omega_I |I\rangle, \\ |\psi'\rangle &= \sum_{I \in \mathcal{I}} \omega'_I |I\rangle, \end{aligned}$$

where the sum runs over multi-index set $\mathcal{I} \subset [d]^N$ of size $|\mathcal{I}| = d^k$, are always LU equivalent (and hence belong to the same SLOCC class).

Example 8. All $\text{AME}(4, 3)$ states $|\text{AME}(4, 3)_\omega\rangle$ from Example 4 are LU equivalent, and hence they belong to the same SLOCC class. Similarly, the states

$$|\text{AME}(5, d)_\omega\rangle = \frac{1}{d} \left(\sum_{i,j=0}^{d-1} \omega_{i,j} |i, j, i+j, 2i+j, 3i+j\rangle \right)$$

for $|\omega_{i,j}| = 1$ are LU and SLOCC equivalent.

From Corollary 3, it becomes clear that the diversity of possible phases in front of each term in the 2-uniform state with minimal support does not reflect the number of SLOCC classes. In fact, each 2-uniform state with minimal support is equivalent to the one with all phases equal to 1.

Therefore, enumeration of SLOCC classes might be restricted to phase 1 states only. In fact, it is equivalent to the classification of the relevant OA.

Corollary 4. Classification of two-uniform states with minimal support for $N > 2k$ is equivalent to the classification of the relevant OA, i.e., $\text{OA}(d^k, N, d, k)$ up to permutation of indices on each position. Potentially, for $N = 2k$ two $\text{AME}(4, d)$ states of minimal support might be in the same SLOCC class, even though the corresponding OA is not equivalent.

In literature, the classification of OAs is considered up to permutations of rows and columns [48,49]. Permutation of columns resembles, however, the physical operation of exchanging subsystems. Therefore, by dividing the state space into SLOCC classes one should always indicate whether such operations are considered under the division [19,20].

In particular, by the classification of OA, there exists at most one $\text{OA}(d^k, N, d, k)$ for $d = 2, \dots, 17$ for any number N [50]. Hence, 2-uniform states with minimal support and the local dimension $d = 2, \dots, 17$ are always SLOCC

equivalent or SLOCC equivalent after permutation of parties. Nevertheless, we checked that for $N = 4, 5$ and $d = 4, 5, 6, 7$ permutation of parties is not necessary for being SLOCC equivalent. We suppose this is true in general.

Conjecture 2. All 2-uniform states of minimal support are LU equivalent, and hence represent the same SLOCC class.

In all verified cases there exists only one 2-uniform state of minimal support. Nevertheless, some two-uniform states are not equivalent to the above-mentioned one. In particular, for $d = 5, 7, 11, 13, \dots$ there are $\text{AME}(5, d)$ states of nonminimal support belonging to different SLOCC classes (see Proposition 3).

C. 3-uniform states

We have shown that the number of LU and SLOCC classes for 2-uniform states of minimal support coincides with the number of relevant OAs which are nonisomorphic. In particular, two states which differ only with phases are always LU and SLOCC equivalent. This is in a strong contrast to the 3-uniform states.

Example 9. There exists a $\text{AME}(6, 4)$ state with minimal support [11]. In fact, this state might be obtained by reading consecutive rows of $\text{OA}(64, 6, 4, 3)$ from the OA table [50]. Obviously, enhancing successive terms with any phase factor $|\omega| = 1$ also yields the $\text{AME}(6, 4)$ state:

$$|\text{AME}(6, d)_{\omega}\rangle = \frac{1}{d\sqrt{d}} \left(\sum_{i,j,k=0}^{d-1} \omega_{i,j,k} |i, j, k\rangle \otimes |\psi_{i,j,k}\rangle \right).$$

We focus our attention on states with all phases $\omega_{i,j,k} = 1$ with one exception: $\omega_{0,0,0} = \alpha$. Denote them as $|\psi_{\alpha}\rangle$ for simplicity. According to Proposition 6, the necessary condition for equivalence of such states $|\psi_{\alpha_1}\rangle$ and $|\psi_{\alpha_2}\rangle$ is

$$\frac{(W_{00}^{1,2})'}{W_{\sigma_1(0), \sigma_2(0)}^{1,2}} = \frac{(W_{0,1}^{1,2})'}{W_{\sigma_1(0), \sigma_2(1)}^{1,2}}$$

for any permutations σ_1 and σ_2 . According to the form of permutations, we have the following.

- (1) If $(\sigma_1(0), \sigma_2(0)) = (0, 0)$, then $\alpha_1 = \alpha_2$.
- (2) If $(\sigma_1(0), \sigma_2(1)) = (0, 0)$, then $\alpha_1 = \overline{\alpha_2}$.
- (3) Otherwise $\alpha_1, \alpha_2 = 1$.

Therefore, if none of those conditions is satisfied, states $|\psi_{\alpha_1}\rangle$ and $|\psi_{\alpha_2}\rangle$ cannot be LU equivalent. By simple analysis, all states $|\psi_{e^{i\phi}}\rangle$ are pairwise non-LU-equivalent for $\phi \in [0, \pi)$.

Observe that, in such a way, we obtained a continuous family of non-LU-equivalent $\text{AME}(6, 4)$ states with minimal support. We conclude this observation in the corollary below. In fact, if the necessary conditions from Proposition 6 are satisfied, the LU equivalence may be provided (similarly to the case of 2-uniform states).

Corollary 5. The $\text{AME}(6, 4)$ states

$$|\text{AME}(6, d)_{e^{i\phi}}\rangle := \frac{1}{d\sqrt{d}} \left(e^{i\phi} |000000\rangle + \sum_{i,j,k \neq (0,0,0)} |i, j, k\rangle \otimes |\psi_{i,j,k}\rangle \right)$$

are pairwise in different LU and SLOCC classes for all phases $\phi \in [0, \pi)$.

Notice that for any k -uniform state with minimal support where $k > 2$ a similar construction of a continuous non-LU-equivalent family might be provided.

Corollary 6. If there exists a k -uniform state with minimal support $|\psi\rangle$ where $k > 2$,

$$|\psi\rangle = \frac{1}{\sqrt{d^k}} \sum_{i_1, \dots, i_k=0}^{d-1} |i_1, \dots, i_k\rangle \otimes |\psi_{i_1, \dots, i_k}\rangle,$$

then the family of k -uniform states

$$|\psi_{e^{i\phi}}\rangle := \frac{1}{\sqrt{d^k}} \left(e^{i\phi} |0, \dots, 0\rangle \otimes |\psi_{0, \dots, 0}\rangle + \sum_{(i_1, \dots, i_k) \neq (0, \dots, 0)} |i_1, \dots, i_k\rangle \otimes |\psi_{i_1, \dots, i_k}\rangle \right)$$

is pairwise non-LU- and non-SLOCC-equivalent for all phases $\phi \in [0, \pi)$.

D. Number of non-SLOCC-equivalent AME states

We summarize shortly the number of non-SLOCC-equivalent AME states and AME states with minimal support in Tables III and II, respectively.

The existence of AME states with minimal support for $N, d < 8$ was analyzed [68] based on the table of OAs and similar combinatorial designs. According to the discussion presented in the previous sections, if $N \geq 6$, existence of the $\text{AME}(n, d)$ state with minimal support indicates the existence of infinitely many non-SLOCC-equivalent such states (see Corollary 6).

Verification of the existence of AME states (not necessarily with minimal support) is a far more complex problem. We refer to the tables of AME states [69], which summarize several results concerning this problem [9, 10, 68, 70, 71]. Even though the exact classification of AME states up to SLOCC equivalence is yet unobtainable, in some specific cases the nontrivial lower bound is given.

TABLE II. The exact number of not SLOCC-equivalent AME states with minimal support presented on a differently shaded blue background.

local dimension	2 qubits	3 qutrits	4	5	6	7
$\text{AME}(3, d)$	1	1	1	1	1	1
$\text{AME}(4, d)$	0	1	1	1	0	1
$\text{AME}(5, d)$	0	0	1	1	0	1
$\text{AME}(6, d)$	0	0	∞	∞	0	∞
$\text{AME}(7, d)$	0	0	0	0	0	∞

TABLE III. The minimal number of non-SLOCC-equivalent AME states. The question mark by zero value suggests that the existence of the relevant state is dubitative, while 0 itself emphasizes that the relevant state certainly does not exist.

local dimension	2 qubits	3 qutrits	4	5	6	7
AME(3,d)	1	1	1	1	1	1
AME(4,d)	0	1	1	1	0?	1
AME(5,d)	1	1	1	2	1	2
AME(6,d)	1	1	∞	∞	1	∞
AME(7,d)	0	1	1	1	0?	∞

VI. COMBINATORIAL DESIGNS

We discuss a class of combinatorial designs, known as mutually orthogonal Latin hypercubes. As we shall see, the existence and extension of those designs turned out to be crucial in the classification of AME states. In particular, we show the origin of restrictions on values of d and k imposed in Propositions 5 and 6. Exceeding those bounds potentially yields the existence of composed AME states. Automorphisms of such states have product form and are not described by Propositions 5 and 6. In that sense, we show that obtained bounds on d and k are tight.

In general, classical combinatorial designs (as orthogonal arrays: mutually orthogonal Latin squares, cubes, and hypercubes) are related to AME and k -uniform states of minimal support. Quantized versions of such combinatorial designs are related to arbitrary AME and k -uniform states. Our paper is restricted to minimal support states, hence the presentation of quantum combinatorial designs is not needed here. It is not our intention to provide a full picture of interactions between AME states and different combinatorial designs. For that purpose, we refer to Goyeneche and Życzkowski [13], where the comprehensive introduction to that topic is presented.

Consider a discrete hypercube $[d]^k$ of dimension k . One can relate to $[d]^k$ the lower dimensional hypercube $[d]^s$ in two natural ways. First, by choosing $k-s$ indices $S = \{s_1, \dots, s_{k-s}\} \subset [k]$ and their values $i_1, \dots, i_{k-s} \in [d]$, there is an injective map:

$$[d]^s \cong [d]_{s_1=i_1, \dots, s_{k-s}=i_{k-s}}^k \xrightarrow{i} [d]^d.$$

Second, for any subset $S' \subset [k]$ of indices where $|S'| = s$, one can simply forget about indices out of S' . This operation is relevant to the surjection

$$[d]^k \xrightarrow{\text{sur}} [d]_{S'}^k \cong [d]^s.$$

Definition 3. A k -MOLH of size d and dimension k is a bijection

$$L : [d]^k \longrightarrow [d]^k$$

such that by choosing any set $S = \{s_1, \dots, s_{k-s}\} \subset [k]$ of $k-s$ indices and their values $i_1, \dots, i_{k-s} \in [d]$, and any subset $S' \subset [k]$, the composition of L with above-defined injection i (on the left) and surjection sur (on the right) provides a bijection:

$$[d]^s \cong [d]_{s_1=i_1, \dots, s_{k-s}=i_{k-s}}^k \xrightarrow{i \circ L \circ \text{sur}} [d]_{S'}^k \cong [d]^s.$$

We denote such an object as k -MOLH(d).

Example 10. Bijection

$$L : [d]^2 \longrightarrow [d]^2$$

such that in each row and on each position all elements appear exactly once constitutes a mutually orthogonal Latin square MOLS(d). Here *square* stands for MOLH dimension $k = 2$.

In general, orthogonality and dimension of the MOLH might be indexed by different numbers (here both are equal and denoted by k) [12]. This distinction is, however, not needed for our purpose.

There is a one-to-one correspondence between k -MOLH(d) and orthogonal arrays $\text{OA}(d^k, k, 2k, k)$ and, hence, between them and AME($2k, d$) states of minimal support [12].

Proposition 8. Any AME($2k, d$) state of minimal support is equivalent to k -mutually orthogonal Latin hypercube L :

$$L(i_1, \dots, i_k) := (\phi_1^1, \dots, \phi_k^k).$$

Proof. Consider a k -MOLH(d) L . By adjusting

$$i_1, \dots, i_k, \phi_1^1, \dots, \phi_k^k \quad (8)$$

into d^k rows (with $2k$ elements each), one obtains $\text{OA}(d^k, k, 2k, k)$. Indeed, choose any set of k indices and split it into two: $S \cup S'$, where S is a $(k-s)$ -elementary subset of the first half of indices, and S' is an s -elementary subset of the second half of indices. For any choice of values $i_1, \dots, i_{k-s} \in [d]$, by Definition 3, there is a bijection

$$[d]_{s_1=i_1, \dots, s_{k-s}=i_{k-s}}^k \xrightarrow{i \circ L \circ \text{sur}} [d]_{S'}^k,$$

and hence the subset $S \cup S'$ of k columns in Eq. (8) contains all possible combinations of symbols. Since the choice of k -elementary subset $S \cup S'$ was unrestricted, this is a defining property of an OA of index unity. Overturning this argument provides the reverse statement. ■

Example 11. The AME(4,3) state from Eq. (1) is equivalent to the mutually orthogonal Latin square (2-MOLH):

As it is shown in Table IV, the entries of MOLS are pairs of numbers (k, ℓ) and considering this, the relevant quantum

TABLE IV. The entries of MOLS(3).

$i \setminus j$	0	1	2
0	00	11	22
1	12	20	01
2	21	02	10

state can be obtained by reading all entries as

$$|\text{AME}(4,3)\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, k, \ell\rangle.$$

The notion of MOLS was used for the construction of several AME states [70].

A. Existence of Latin designs

Proposition 9. If a mutually orthogonal Latin hypercube k -MOLH(d) for $k > 1$ exists, then indices d and k satisfy

$$k \leq d - 1.$$

Proof. The hyper-row $I := (i, 0, \dots, 0)$, $i \in d$ defines the following mapping:

$$L_I : i \mapsto (\phi_i^1, \dots, \phi_i^k) \in [d]^k$$

where $\phi_i^\ell := \phi_i^\ell$ for simplicity. Observe that on each position all symbols appear, i.e.,

$$\{\phi_i^\ell : i \in [d]\} = [d]. \quad (9)$$

Consider now the element

$$(j_1, \dots, j_k) := L(0, 1, 0, \dots, 0) \in [d]^k.$$

From Eq. (9), clearly

$$j_1 = \phi_{i_1}^1, \dots, j_k = \phi_{i_k}^k$$

for some indices $i_1, \dots, i_k$. Observe the following.

(1) $i_1, \dots, i_k \neq 0$. Suppose the contrary, $i_s = 0$. Then $j_s = \phi_0^s$, and hence

$$L(0, 0, 0, \dots, 0) = (\dots, j_s, \dots),$$

$$L(0, 1, 0, \dots, 0) = (\dots, j_s, \dots),$$

where dotted symbols on the right are not specified. This is in contradiction to Definition 3 for S given by $i_j = 0$ for all $j \neq 1$, and $S' = \{s\}$.

(2) Indices $i_1, \dots, i_k$ are pairwise different. Suppose the contrary, $i_{s_1} = i_{s_2}$. Then

$$L(i_{s_1}, 0, 0, \dots, 0) = (\dots, j_{s_1}, \dots, j_{s_2}, \dots),$$

$$L(0, 1, 0, \dots, 0) = (\dots, j_{s_1}, \dots, j_{s_2}, \dots).$$

This is in contradiction to Definition 3 for S given by $i_j = 0$ for all $j \neq 0, 1$, and $S' = \{s_1, s_2\}$.

Since all indices $i_1, \dots, i_k \in [d]$ are pairwise different and nonzero, $k \leq d - 1$. ■

It is worth mentioning that the condition given in Proposition 9 is only a necessary condition for the existence of the MOLH. If it is satisfied, the precise construction of the MOLH is known for all d being prime powers. This construction might be extended further by composing two MOLHs of a different size. Nevertheless, the aforementioned condition is not a sufficient one. For instance, construction of the MOLS (2-MOLH) of size $d = 6$ refers to the famous problem of 36 officers of Euler [72], which was proven to have no solution [73].

TABLE V. Extension of MOLS(3) into MOLS(9).

$i \backslash j$	0	1	2	3	4	5	6	7	8
0	00	11	22	33	44	55	66	77	88
1	12	20	01	45	53	34	78	86	67
2	21	02	10	54	35	43	87	68	76
3	36	47	58	60	71	82	03	14	25
4	48	56	37	72	80	61	15	23	04
5	57	38	46	81	62	70	24	05	13
6	63	74	85	06	17	28	30	41	52
7	75	83	64	18	26	07	42	50	31
8	84	65	73	27	08	16	51	32	40

B. Extension of Latin designs

As we shall see, Latin designs are not only related to the construction of AME states with minimal support, but also to the local unitary relations between such. In particular, the problem of existence and extension of the k -dimensional MOLH is relevant to the description of LU equivalences between AME($2k, d$) states. Therefore, a short outline of the extension problem is presented below.

Definition 4. A MOLH(s) of size k and dimension s ,

$$L : [s]^k \longrightarrow [s]^k,$$

might be extended to MOLH(d) if there exists MOLH(d):

$$L' : [d]^k \longrightarrow [d]^k,$$

which preserves the structure of L , i.e., $L'|_{[s]^k} \equiv L$. Moreover, we refer to L as a sub-MOLH(s) of MOLH(d).

Example 12. A MOLS(3) L might be extended into MOLS(9) presented in Table V. Indeed, one can see that entries in the square consisting of three first rows and columns are taken from 0, 1, and 2 only. In fact, this extension is relevant to a tensor product of two identical Latin squares L .

Remark 2. Consider a MOLH(d) L . If there exists a sub-hypercube $S = S_1 \times \dots \times S_k \subseteq [k]^d$ which is mapped by L on another hypercube $S' = S'_1 \times \dots \times S'_k \subseteq [k]^d$, then up to permutation of labels $L' := L|_{S'}$ is the sub-MOLH(s) of MOLH(d). Moreover, L cannot map hyper-rectangles onto hyper-rectangles except hypercubes into hypercubes. Hence, a sub-hyper-rectangle of L does not exist.

The problem of extension of Latin designs might be traced to Ryser's theorem [74], and is a plentiful scientific problem considered in several papers [75,76].

There are the following dimension bounds for extending MOLH(s) into MOLH(d) or, equivalently, finding sub-MOLH(s) of MOLH(d).

Proposition 10. Inequality

$$s \leq \frac{1}{1 + \frac{1}{\sqrt[k-1]{k}}} d$$

is a necessary condition for extending $\text{MOLH}(s)$ into $\text{MOLH}(d)$ for any $k > 1$.

Proof. Suppose that L' extends to L . Consider pairwise disjoint sets:

$$S_i := \underbrace{S \times S \times S^c}_{i-1} \times \underbrace{S \times S}_{k-i}$$

where $S := [s]$. Denote their sum by $S' = \cup_{i=1}^k S_i$. Observe that for any multi-index $I \in S'$

$$L(I) \in S^c \times \cdots \times S^c.$$

Indeed, it follows from the fact that $L(S) = S$, and hence on any position no indices from S might appear in $L(S')$. Since L is the bijection

$$|S'| \leq |S^c \times \cdots \times S^c|,$$

and hence

$$ks^{k-1} \leq (d-s)^{k-1},$$

which is equivalent to the statement of Proposition 10. ■

As we shall see, the existence of nontrivial sub-MOLH(s) in $\text{MOLH}(d)$ is directly related to the problem of describing the automorphisms of $\text{AME}(2k, d)$ states. More precisely, assumptions that either nontrivial sub-MOLH(s) does not exist or cannot be extended to $\text{MOLH}(d)$ allows us to provide a comprehensive description of LU equivalences of $\text{AME}(2k, d)$ states. Therefore, the necessary conditions of existence and extension of MOLH are limiting the statements of Propositions 5 and 6. This limitation is notified in Proposition 1, which follows directly from Propositions 9 and 10.

VII. FURTHER DISCUSSION AND OPEN PROBLEMS

Ultimate LU and SLOCC classification of k -uniform states, even of minimal support, is in fact a complex project involving many open mathematical problems, such as (1) existence and extension of mutually orthogonal Latin hypercubes (see Sec. VI), (2) classification of Hadamard matrices of Butson type $B(d, d)$, and (3) classification and uniqueness of OAs of index unity (without permutation), among others. Therefore, with great conviction, we claim it to be currently out of reach. Below, we discuss three open problems regarding LU and SLOCC classification of k -uniform states with minimal support in a detailed way. We show their connections with some open mathematical problems.

First, consider two k -uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$ with all phases equal to 1 for simplicity. With this constraint on phases, Proposition 2 shows that $|\psi\rangle$ and $|\psi'\rangle$ are LU equivalent if and only if there exist local permutation matrices relating $|\psi\rangle$ and $|\psi'\rangle$:

$$|\psi'\rangle = \sigma_1 \otimes \cdots \otimes \sigma_n |\psi\rangle.$$

States $|\psi\rangle$ and $|\psi'\rangle$ are in one-to-one correspondence with two OAs of index unity. The existence of local permutation matrices is equivalent to an isomorphism between two OAs of index unity. Hence LU classification of such states is equivalent to the classification of OAs of index unity. Such

a classification is, however, an open mathematical problem. In many situations, when number of parties N , uniformity k , and local dimension d are small, it is known that all OAs of index unity are isomorphic [77–79].

Conjecture 3. All OAs of index unity are isomorphic by permutations of symbols on each level. Equivalently, all k -uniform states with minimal support and all term phases equal are LU equivalent.

Second, in Propositions 5 and 6, the form of the arbitrary LU operator between two $\text{AME}(2k, d)$ states with minimal support is provided for small numbers k and d . It is given by a Butson-type matrix $B(d, d)$ or an identity matrix, multiplied by local monomial matrices from both sides. We have shown that for a composed system there are local operators beyond the provided formula. Indeed, in this case, the tensor product of the Butson-type matrix and the identity matrix may provide LU equivalence. We conjecture that it is a general form for LU equivalences for all $\text{AME}(2k, d)$ states, and it is tightly related to the possible decomposition of a system. This supposition is stated in Conjecture 1.

Third, Proposition 2 states that any LU operator between two k -uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$ is a local product of phase (diagonal) and permutation matrices (for $2k < N$). By considering states $|\psi\rangle$ and $|\psi'\rangle$ with terms of various phases, we showed that not all of them are LU equivalent for $k > 2$. Nevertheless, the precise description of SLOCC classes containing such states is not given. Therefore, the role of permutation matrices in LU classification is not yet absolutely clear.

Finally, in Sec. IV the basic difference between k -uniform states of minimal support where $2k < N$ and $2k = N$ is discussed. LU equivalence between two k -uniform states with $2k = N$ decomposes into multiplication of the Butson-type matrix and LM matrices from both sides. Obviously, Butson-type matrices significantly increase the class of LU equivalences between two states. Nevertheless, it is not known yet whether such LU equivalences are beyond local monomial equivalences. In fact, in all provided examples involving Butson-type matrices in LU equivalence, states were always LM equivalent. Therefore we conjecture that Corollary 2 holds true in the case $2k = N$ (even though Proposition 2 does not hold anymore).

Conjecture 4. All $\text{AME}(2k, d)$ states with minimal support are LU equivalent if and only if they are LM equivalent.

Notice that any attempt to prove the statement above makes sense only if Conjecture 3 is true. In such a case, the classification of Butson type $B(d, d)$, which is an open mathematical problem, and refined analysis of such are required.

VIII. CONCLUSIONS

In this paper, we develop techniques of SLOCC verification between k -uniform and AME states. In particular, we show that two k -uniform states are SLOCC equivalent if and only if they are LM equivalent. We further specify the matrices which might appear in such equivalences. These results significantly restrict the class of possible local transformations to a finite set, which makes SLOCC verification feasible.

For $\text{AME}(2k, d)$ states, the aforementioned statement is not true anymore. Intriguingly, SLOCC equivalences might

be provided by Fourier transforms and, in general, by Butson-type matrices. This restriction is valid, however, only for small local dimensions d and number of parties N (in particular for arbitrary N and $d < 9$). The exact bound on d and N is related to the necessary condition for existence and extension of combinatorial designs called mutually orthogonal hypercubes. Despite the exhaustive analysis performed, the general structure of SLOCC equivalences between $\text{AME}(2k, d)$ states is still puzzling and remains unknown. We present evidence that exceeding this class of equivalences is possible only in composed systems. General results concerning SLOCC equivalences of $\text{AME}(2k, d)$ states are also presented.

We illustrate the usefulness of the provided criteria on various examples. First, we show that the existence of AME states with minimal support of six or more particles yields the existence of infinitely many such non-SLOCC-equivalent states. The exact number of SLOCC classes containing AME states with minimal support is given. Second, we show that some AME states cannot be locally transformed into existing AME states of minimal support. This shows that the notion of support is relevant even for AME states.

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APPENDIX A: THE PROOF OF PROPOSITION 2

We shall prove Proposition 2 in a slightly enhanced version. Notice that LU and SLOCC equivalences coincide on the class of AME states, which is an immediate conclusion from Corollary 1. Therefore, we restrict our argument to LU equivalences only. We would like to emphasize the statement below as primary and more valuable for the LU-verification procedure than the claim of Proposition 2 itself. Indeed, this extended version is used later in Appendix D for the demonstration of nonequivalence of two families of $\text{AME}(5, d)$ states.

Proposition 11. Consider two k -uniform states $|\psi\rangle$ and $|\phi\rangle$ with minimal support. For any subsystem S consisting of $s > k$ parties, the reduced density matrices $\rho_S(\psi)$ and $\rho_S(\phi)$ are LU equivalent if and only if they are LM equivalent.

Observe that Proposition 2 is an immediate consequence of the statement above for subsystem S of all parties, i.e., $|S| = N$. Without loss of generality it is enough to prove the statement of Proposition 11 only for the smallest possible subsystems S , i.e., consisting of $k + 1$ parties. Indeed, assume that the reduced states $\rho_{S'}(\psi)$ and $\rho_{S'}(\phi)$ are equivalent by a local unitary matrix U . Consider any subsystem $S \subseteq S'$ of $k + 1$ parties. The local operator U splits:

$$U = U_S U_{S' \setminus S}$$

where U_S is the local operation on the S subsystem and $U_{S' \setminus S}$ is the local operation on the $S' \setminus S$ subsystem equivalently. Since the (partial) trace is invariant under cycling permutations, we have

$$\rho_S(\psi) = U_S[\rho_S(\phi)]$$

and, by Proposition 11, U_S is a local monomial operation. Since the subsystem S was chosen arbitrary, U is a local monomial operation.

Notice that the size $s > k$ of the subsystem S in Proposition 11 is the largest possible. Indeed, after taking the partial trace over the larger subsystem, both states $|\psi\rangle$ and $|\phi\rangle$ become proportional to the identity, and hence any local unitary operation provides their equivalence.

We introduce the following notation. Consider two LU-equivalent k -uniform states: $|\psi\rangle$ and $|\phi\rangle$ of minimal support form. We make use of the decomposition into support elements:

$$|\psi\rangle = \sum_{i=1}^{d^k} \alpha_i |\psi_i\rangle$$

where $|\psi_i\rangle$ are of unity support. Moreover, we denote elements of $|\psi_i\rangle$ as follows:

$$|\psi_i\rangle = |x_1^i \cdots x_N^i\rangle$$

where $x_j^i = 0, \dots, d - 1$. We use similar notation for the state $|\phi\rangle$:

$$|\phi\rangle = \sum_{i=1}^{d^k} \alpha_i |\phi_i\rangle, \quad |\phi_i\rangle = |y_1^i \cdots y_N^i\rangle.$$

Lemma 1. For a partial trace over any subsystem S of $|S| \geq k$ parties

$$\text{tr}_S |\psi\rangle\langle\psi| = \sum_{i=0}^{d^k} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$$

where $|\tilde{\psi}_i\rangle = \text{tr}_S |\psi_i\rangle$. Moreover, for any $i \neq j$, $|\psi_i\rangle$ and $|\psi_j\rangle$ coincide on at most $k - 1$ positions.

Proof. For any subsystem S such that $|S| = N - k$, $\text{tr}_S |\psi\rangle\langle\psi| = \text{Id}_{d^k}$. Hence, for any $i \neq j$, vectors $|\psi_i\rangle$ and $|\psi_j\rangle$ coincide on at most $k - 1$ positions. Indeed, suppose the contrary, i.e., they coincide on some k positions. Then, by tracing out the rest of parties, we get $\text{tr}_{N-k} |\psi_i\rangle\langle\psi_i| = \text{tr}_{N-k} |\psi_j\rangle\langle\psi_j|$ and by the minimality of the support $\text{tr}_{N-k} |\psi\rangle\langle\psi| \neq \text{Id}_{d^k}$. The statement of the lemma follows immediately from the presented observation. ■

Proof of Proposition 11. As we already discussed, it is enough to prove Proposition 11 for subsystems S of $k + 1$ parties. Without loss of generality, analyze the reduction to the subsystem S of the first $k + 1$ parties. Consider a local unitary operation $U := U_1 \otimes \cdots \otimes U_{k+1}$ transforming $\rho_S(\psi)$ into $\rho_S(\phi)$. Unitary operators U_i might be seen as the following change of basis:

$$|\widehat{i}_1 \cdots \widehat{i}_{k+1}\rangle := |U_1(i_1) \cdots U_{k+1}(i_{k+1})\rangle. \quad (\text{A1})$$

From Lemma 1 we have

$$\rho_S(\psi) = \sum_{i=1}^{d^k} |x_1^i \cdots x_{k+1}^i\rangle \langle x_1^i \cdots x_{k+1}^i| \quad (\text{A2})$$

and

$$\rho_S(\phi) = \sum_{i=1}^{d^k} |y_1^i \cdots y_{k+1}^i\rangle \langle y_1^i \cdots y_{k+1}^i|. \quad (\text{A3})$$

Observe that

$$U \rho_S(\psi) U^{-1} = \sum_{i=1}^{d^k} |\widehat{x_1^i \cdots x_{k+1}^i}\rangle \langle \widehat{x_1^i \cdots x_{k+1}^i}|,$$

and since U is the LU equivalence the two expressions above are equal. Consequently, we have equality of the following spaces:

$$\begin{aligned} & \text{span}\{|y_1^i \cdots y_{k+1}^i\rangle; i = 1, \dots, d^k\} \\ &= \text{span}\{|\widehat{x_1^i \cdots x_{k+1}^i}\rangle; i = 1, \dots, d^k\}. \end{aligned}$$

In general, each vector from the first space is a linear combination of vectors from the second space. We will show, however, that there is a one-to-one correspondence between vectors from both spaces; namely, for any index i there exists an index j_i such that

$$|\widehat{x_1^i \cdots x_{k+1}^i}\rangle = |y_1^{j_i} \cdots y_{k+1}^{j_i}\rangle. \quad (\text{A4})$$

With this observation at hand, and by Eq. (A1), the statement of Proposition 2 follows immediately.

What remains to be shown is that, indeed, Eq. (A4) holds. Consider the vector $|\widehat{x_1^i \cdots x_{k+1}^i}\rangle$ and present it as the following linear combination:

$$|\widehat{x_1^i \cdots x_{k+1}^i}\rangle = \sum_{i=1}^{d^k} \beta_i |y_1^i \cdots y_{k+1}^i\rangle. \quad (\text{A5})$$

On the other hand,

$$|\widehat{x_1^i \cdots x_{k+1}^i}\rangle = |U_1(x_1^i) \cdots U_{k+1}(x_{k+1}^i)\rangle \quad (\text{A6})$$

$$= \sum_{j_1, \dots, j_{k+1}=0}^{d-1} u_{x_1^i j_1}^1 \cdots u_{x_{k+1}^i j_{k+1}}^{k+1} |j_1 \cdots j_{k+1}\rangle \quad (\text{A7})$$

where $u_{lm}^k = (U_k)_{lm}$, and hence

$$\beta_i = u_{x_1^i y_1^{j_i}}^1 \cdots u_{x_{k+1}^i y_{k+1}^{j_i}}^{k+1}.$$

Suppose now that, for $i \neq j$, $\beta_i, \beta_j \neq 0$. Consequently,

$$u_{x_m^i y_m^{j_i}}^m \neq 0, \quad u_{x_m^j y_m^{j_j}}^m \neq 0,$$

for $m = 1, \dots, k+1$. By Lemma 1, $|y_1^{j_i} \cdots y_{k+1}^{j_i}\rangle$ and $|y_1^{j_j} \cdots y_{k+1}^{j_j}\rangle$ differ on at least two positions; without loss of generality suppose $y_1^{j_i} \neq y_1^{j_j}$. Observe that

$$u_{x_1^i y_1^{j_i}}^1 \neq 0, \quad u_{x_1^j y_1^{j_j}}^1 \neq 0, \dots, u_{x_{k+1}^i y_{k+1}^{j_i}}^{k+1} \neq 0$$

and hence the expression

$$|y_1^{j_i} y_2^{j_i} \cdots y_{k+1}^{j_i}\rangle \langle y_1^{j_j} y_2^{j_j} \cdots y_{k+1}^{j_j}|$$

appear on the right-hand side of Eq. (A3) with nonzero coefficient, which is in contradiction to Lemma 1.

Since there is at most one $\beta_i \neq 0$, the sum in Eq. (A5) collapses to the one term. Similar reasoning shows that $|\widehat{x_1^m \cdots x_{k+1}^m}\rangle$ is in general equal to $|y_1^i \cdots y_{k+1}^i\rangle$ for some $i = 1, \dots, d^k$, which should have been shown. ■

Notice that the presented argument does not hold if $2k = N$. Indeed, the smallest nontrivial reduced system of k -uniform states consists of $k+1$ parties. The proof is based on Lemma 1 to justify that vectors $|y_1^i \cdots y_{k+1}^i\rangle$ and $|y_1^j \cdots y_{k+1}^j\rangle$ differ on at least two positions. For $2k = N$, however, the argument of Lemma 1 might be used only for the trivial reduction to k parties.

APPENDIX B: THE PROOF OF PROPOSITION 4

Proof of Proposition 4. From Proposition 2 the LU equivalence between two states of minimal support is a product of permutation and diagonal matrices. Suppose that the local permutation σ was already applied to the state $|\psi\rangle$. Therefore, we may assume that $|\psi\rangle$ and $|\psi'\rangle$ are related only by diagonal operators.

Since both states $|\psi\rangle$ and $|\psi'\rangle$ are k -uniform states of minimal support related only by diagonal operators, they might be written in the following form:

$$|\psi\rangle = \sum_{I \in \mathcal{I}} \omega_I |I\rangle, \quad (\text{B1})$$

$$|\psi'\rangle = \sum_{I \in \mathcal{I}} \omega'_I |I\rangle \quad (\text{B2})$$

where $\mathcal{I} \subset [d]^n$ has dimension $|\mathcal{I}| = [d]^k$. Denote by

$$D^\ell = \text{diag}(u_1^\ell, \dots, u_d^\ell)$$

the diagonal operators relating $|\psi\rangle$ and $|\psi'\rangle$. Clearly,

$$\omega'_I = u_{i_1}^1 \cdots u_{i_n}^n \omega_I \quad (\text{B3})$$

for any index $I = i_1, \dots, i_n$. Denote by $U^\ell := \prod_{i \in [d]} u_i^\ell$ the product of all nonzero elements from the matrix D^ℓ .

For simplicity, let us choose $S = \{1, \dots, k-1\}$ as the set of first $k-1$ indices. We shall show the statement with respect to the first matrix D^1 . Consider any multi-index $I = i_2, \dots, i_{k-1}$. By multiplying adequate expressions from Eq. (B3) by sides, one may obtain

$$(W_{i,I}^S)' = (u_i^1)^d (u_{i_2}^2)^d \cdots (u_{i_{k-1}}^{k-1})^d (U^k \cdots U^n) W_{i,I}^S.$$

The fact that $U^k \cdots U^n$ appears on the right-hand side follows from the basic properties of OAs related to the states $|\psi\rangle$ and $|\psi'\rangle$. Hence

$$(u_0^1)^d \frac{W_{0,I}^S}{(W_{0,I}^S)'} = \cdots = (u_{d-1}^1)^d \frac{W_{d-1,I}^S}{(W_{d-1,I}^S)'}. \quad (\text{B4})$$

From this immediately follows that

$$D^1 = \omega_1 \text{diag} \left(\sqrt[d]{\frac{(W_{0,I}^S)'}{W_{0,I}^S}}, \dots, \sqrt[d]{\frac{(W_{d-1,I}^S)'}{W_{d-1,I}^S}} \right),$$

for some phase factor ω_1 .

Since the multi-index I is arbitrary, from Eq. (B4) follows that

$$\frac{(W_{0,I}^S)'}{W_{0,I}^S} = \frac{(W_{0,I'}^S)'}{W_{0,I'}^S} \quad (\text{B5})$$

for any other multi-index $I' = i'_2, \dots, i'_{k-1}$.

Since we assumed that the local permutation σ was already applied to the state $|\psi\rangle$, one has to consider the action of σ on the denominator in Eq. (B5) in the general case. This shows the statement of Proposition 4 for the set S of first $k-1$ indices. The same reasoning might be applied for other sets S , and further for any indices: $2, \dots, n$. The global phase is then a multiplication of obtained factors ω_i . ■

APPENDIX C: SEC. IV REVISITED

We discuss in detail LU equivalences of AME($2k, d$) states with minimal support. In particular, proofs of Propositions 5 to 7 are presented. We begin with the necessary notation. For simplicity, all AME($2k, d$) states considered in Appendix C are normalized to $\sqrt{d^k}$. In such a way, all terms in the computational basis of states with minimal support are normalized to one. For a local unitary operator $U := U_1 \otimes \dots \otimes U_\ell$ and a subset of indices $S \subseteq [\ell]$, we define its S part by

$$U_S := \bigotimes_{i \in S} U_i.$$

Observation 2. Let $U := U_1 \otimes \dots \otimes U_\ell$ be a local unitary operator transforming a state $\rho(\psi) \in \mathcal{H}^{\otimes \ell}$ onto $\rho(\psi')$, i.e.,

$$U[\rho(\psi)] = \rho(\psi').$$

Then, for any subset $S \subseteq [\ell]$ of indices,

$$U_S[\rho_S(\psi)] = \rho_S(\psi'). \quad (\text{C1})$$

Moreover, if

$$\rho_S(\psi) = |\psi_1\rangle\langle\psi_1| + \dots + |\psi_k\rangle\langle\psi_k|,$$

$$\rho_S(\psi') = |\psi'_1\rangle\langle\psi'_1| + \dots + |\psi'_k\rangle\langle\psi'_k|,$$

where vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$, respectively, are orthogonal, i.e., $\langle\psi_i|\psi_j\rangle = c_i\delta_{ij}$ and $\langle\psi'_i|\psi'_j\rangle = c'_i\delta_{ij}$, then

$$U_S|\psi_i\rangle = \sum_{j=1}^k v_{ij}|\psi'_j\rangle \quad (\text{C2})$$

for some elements v_{ij} , which form a unitary matrix $V := (v_{ij})$ if and only if all the vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$ have the same norm.

Proof. Equation (C1) follows immediately from basic properties of partial trace and unitary operations.

By imposing the orthogonality relations between vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$, respectively, one may extend both families (up to normalization of vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$) into the basis of the entire Hilbert space. Consider now the matrix U_S in this basis. Since Eq. (C1) holds and U_S is a unitary matrix, U_S has a block structure, which transfers subspace spanned by vectors $|\psi_i\rangle$ onto the subspace spanned by vectors $|\psi'_i\rangle$, which implies Eq. (C2).

Assume now that vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$ are normalized. Observe that, in the aforementioned basis, V is a block matrix of U_S , and hence is a unitary matrix. ■

Each AME($2k, d$) state $|\psi\rangle$ might be written in the following form:

$$|\psi\rangle = \sum_{I \in [d]^k} |I\rangle \otimes |\phi_I\rangle,$$

where I is multi-index $I = i_1, \dots, i_k$ which runs over the space $[d]^k$. If $|\psi\rangle$ is of minimal support, the vector $|\phi_I\rangle \in \mathcal{H}_d^{\otimes k}$ is separable in the computational basis, i.e.,

$$|\psi\rangle = \sum_{I \in [d]^k} \omega_I |I\rangle \otimes |\phi_I^1\rangle \otimes \dots \otimes |\phi_I^k\rangle, \quad (\text{C3})$$

where vectors $|\phi_I^j\rangle$ are from the computational basis, i.e., $|\phi_I^j\rangle = |0\rangle, \dots, |d-1\rangle$.

Lemma 2. Consider two AME states $|\psi\rangle$ and $|\psi'\rangle$ of the form

$$|\psi\rangle = \sum_{I=i_1, \dots, i_k} \omega_I |I\rangle \otimes |\phi_I\rangle, \\ |\psi'\rangle = \sum_{I=i_1, \dots, i_k} \omega'_I |I\rangle \otimes |\phi'_I\rangle,$$

which are local unitary equivalent by U . For any multi-index $I = i_1, \dots, i_k$

$$U(\omega_I |\phi_I\rangle) = \sum_{I' \in [d]^k} v_{i_1 i'_1}^1 v_{i_2 i'_2}^2 \dots v_{i_k i'_k}^k \omega_{I'} |\phi_{I'}\rangle,$$

where elements $v_{ij}^\ell(i_1, \dots, i_{\ell-1})$ form a unitary matrix

$$V^\ell(i_1, \dots, i_{\ell-1}) := [v_{ij}^\ell(i_1, \dots, i_{\ell-1})]$$

for any $\ell = 1, \dots, k$ and indices $i_1, \dots, i_{\ell-1}$.

Notice that structure constants $v_{ij}^\ell(i_1, \dots, i_{\ell-1})$ depend on indices $i_1, \dots, i_{\ell-1}$, which in fact is the main obstruction for obtaining more general results as those presented in this section.

Proof. We shall use Observation 2 repetitively k times, by tracing out parties $1, \dots, k$, respectively. In fact, the order of the procedure does not matter. In each step, the orthogonality of adequate vectors is fulfilled by relations $\langle\phi_{I'}|\phi_I\rangle = \delta_{I',I}$. We present the first two steps of the procedure in a more detailed way.

We know that $U(|\psi\rangle) = |\psi'\rangle$. Consider the partial traces over the first subsystem in both vectors $|\psi\rangle$ and $|\psi'\rangle$:

$$\rho_{1^c}(\psi) = \sum_{i_1=0}^{d-1} |\psi_{i_1}\rangle\langle\psi_{i_1}|, \\ \rho_{1^c}(\psi') = \sum_{i_1=0}^{d-1} |\psi'_{i_1}\rangle\langle\psi'_{i_1}|,$$

where

$$|\psi_{i_1}\rangle = \sum_{I=i_2, \dots, i_k} \omega_{i_1, I} |I\rangle \otimes |\phi_{i_1, I}\rangle, \quad (\text{C4})$$

and similarly

$$|\psi'_{i_1}\rangle = \sum_{I=i_2, \dots, i_k} \omega'_{i_1, I} |I\rangle \otimes |\phi'_{i_1, I}\rangle.$$

From Observation 2 follows that

$$U_{1^c}(|\psi_{i_1}\rangle) = \sum_{i'_1=0}^{d-1} v_{i_1 i'_1}^1 |\psi'_{i'_1}\rangle, \quad (\text{C5})$$

and the elements $v_{i_1 i'_1}^1$ form the unitary matrix $V^1 := (v_{i_1 i'_1}^1)$.

We shall consider vectors $|\psi_{i_1}\rangle$ separately. For an arbitrary index i_1 , consider a partial trace over the second party of $|\psi_{i_1}\rangle$. From Eq. (C4)

$$\rho_{2^c}(\psi_{i_1}) = \sum_{i_2=0}^{d-1} |\psi_{i_1, i_2}\rangle \langle \psi_{i_1, i_2}|,$$

where

$$|\psi_{i_1, i_2}\rangle = \sum_{I=i_3, \dots, i_k} \omega_{i_1, i_2, I} |I\rangle \otimes |\phi_{i_1, i_2, I}\rangle.$$

The partial trace over the second party of the right-hand side of Eq. (C5) is equal to

$$\sum_{i_2=0}^{d-1} |\psi'_{i_1, i_2}\rangle \langle \psi'_{i_1, i_2}|,$$

where

$$|\psi'_{i_1, i_2}\rangle = \sum_{i'_1} v_{i_1 i'_1}^1 \sum_{I=i_3, \dots, i_k} \omega'_{i_1, i_2, I} |I\rangle \otimes |\phi'_{i_1, i_2, I}\rangle,$$

and hence by Observation 2 applied to both sides of Eq. (C5) we have

$$U_{\{1,2\}^c}(|\psi_{i_1, i_2}\rangle) = \sum_{i'_2=0}^{d-1} v_{i_2 i'_2}^2(i_1) |\psi'_{i_1, i'_2}\rangle, \quad (\text{C6})$$

for elements $v_{i_2 i'_2}^2(i_1)$, which form a unitary matrix. Notice that the matrix $V^2(i_1) := [v_{i_2 i'_2}^2(i_1)]$ is in particular dependent on the chosen index i_1 .

We repeat the presented procedure for an arbitrary pair of indices i_1 and i_2 ; then for i_1, i_2 , and i_3 ; and in general k times up to $i_1, \dots, i_k$. Finally, we obtain

$$U_{\{1, \dots, k\}^c}(|\psi_{i_1, \dots, i_k}\rangle) = \sum_{i'_k=0}^{d-1} v_{i_k i'_k}^k(i_1, \dots, i_{k-1}) |\psi'_{i_1, \dots, i'_k}\rangle \quad (\text{C7})$$

where elements $v_{i_k i'_k}^k(i_1, \dots, i_{k-1})$ form a unitary matrix. Notice that $|\psi_{i_1, \dots, i_k}\rangle = \omega_I |\phi_{i_1, \dots, i_k}\rangle$. On the other hand

$$|\psi'_{i_1, \dots, i'_k}\rangle = \sum_{i'_1, \dots, i'_{k-1}} v_{i_1 i'_1}^1 \cdots v_{i_{k-1} i'_{k-1}}^{k-1}(i_1, \dots, i_{k-2}) \omega'_{i_1, \dots, i'_k} |\phi'_{i_1, \dots, i'_k}\rangle.$$

By the analysis of the recursion, substitution of this formula to Eq. (C7) proves the proposition. ■

Corollary 7. For two AME states of minimal support

$$|\psi\rangle = \sum_{I=i_1, \dots, i_k} \omega_I |I\rangle \otimes |\phi_I^1\rangle \otimes \cdots \otimes |\phi_I^k\rangle,$$

$$|\psi'\rangle = \sum_{I=i_1, \dots, i_k} \omega'_I |I\rangle \otimes |\phi_I^{1'}\rangle \otimes \cdots \otimes |\phi_I^{k'}\rangle,$$

which are local unitary equivalent by U , the equivalence is of the following form:

$$U_{[k]}(\omega_I |I\rangle) = \sum_{I'=\tilde{i}'_1, \dots, \tilde{i}'_k} v_{\phi_I^1 \phi_{I'}^1}^1 v_{\phi_I^2 \phi_{I'}^2}^2 (\phi_I^1) \cdots \times v_{\phi_I^k \phi_{I'}^k}^k (\phi_I^1, \dots, \phi_I^{k-1}) \omega_{I'} |I'\rangle. \quad (\text{C8})$$

Proof. The proof follows immediately from Lemma 2 applied to the second half of indices. ■

The above obtained statements are rather technical. We shall demonstrate their effectiveness.

Proof of Proposition 7. Fix a multi-index $I = i_1, \dots, i_k$. By the definition,

$$U_{[k]}(|I\rangle) = \sum_{I'=\tilde{i}'_1, \dots, \tilde{i}'_k} u_{i_1 i'_1}^1 \cdots u_{i_k i'_k}^k |I'\rangle. \quad (\text{C9})$$

On the other hand, $U_{[k]}$ might be expressed in the form Eq. (C8). For simplicity of the proof, we use the following notation:

$$u_{i'_\ell}^\ell := u_{i'_\ell}^\ell, \quad v_{\phi_I^\ell \phi_{I'}^\ell}^\ell := v_{\phi_I^\ell \phi_{I'}^\ell}^\ell (\phi_I^1, \dots, \phi_I^{\ell-1}),$$

which is correctly defined once the index $I = i_1, \dots, i_k$ is fixed. Furthermore, we define

$$\ell_{I'} := u_{i'_1}^1 \cdots u_{i'_k}^k, \quad p_{I'} := v_{i'_1}^1 \cdots v_{i'_k}^k,$$

for any multi-index $I' = i'_1, \dots, i'_k$.

By comparison of Eqs. (C8) and (C9), we have

$$\ell_{I'} = p_{\phi_{I'}'} \frac{\omega_{I'}}{\omega_I}, \quad (\text{C10})$$

where $\phi_{I'}' := (\phi_{I'}^{1'}, \dots, \phi_{I'}^{k'})$ for any $I' = i'_1, \dots, i'_k$.

Consider now the mutually orthogonal Latin hypercube MOLH(d) (see Proposition 8 for details):

$$L(i'_1 \dots i'_k) := (\phi_{I'}^{1'}, \dots, \phi_{I'}^{k'}).$$

On the one hand, the set of indices $I' = i'_1, \dots, i'_k$ for which $\ell_{I'} \neq 0$ forms a rectangle $S := S_1 \times \cdots \times S_k \subseteq [d]^k$. Indeed, it follows from the product form of $\ell_{I'}$. On the other hand, the set of indices $I'' = i''_1, \dots, i''_k$ for which $p_{I''} \neq 0$ also forms a rectangle $S' := S'_1 \times \cdots \times S'_k \subseteq [d]^k$, which follows from the product form of $p_{I''}$. From the equality in Eq. (C10) follows that L maps S onto S' . From Remark 2, S and S' are hypercubes and $L|_S$ is a MOLH(s), where $s := |S_1| = \cdots = |S_k| = |S'_1| = \cdots = |S'_k|$. This proves the second statement of Proposition 7.

So far, we showed that for any fixed multi-index $I = i_1, \dots, i_k$ the relevant rows of matrices $U_1, \dots, U_k$, i.e., vectors

$$(u_{i_1 i}^1)_{i=0}^{d-1}, \dots, (u_{i_k i}^k)_{i=0}^{d-1},$$

have the same number s of nonzero elements. Observe that this is also true for any other multi-index $I' = i'_1, \dots, i'_k$. Indeed, the analogous reasoning, for other multi-indices $I'_0 = i_1, i'_2, \dots, i'_k$ (I' and I'_0 are equal on the first position), ensures us that the vectors

$$(u_{i_1 i}^1)_{i=0}^{d-1}, (u_{i'_2 i}^2)_{i=0}^{d-1}, \dots, (u_{i'_k i}^k)_{i=0}^{d-1}$$

have the same number of nonzero elements, equal to s . From here, one can deduce it for arbitrary $I' = i'_1, \dots, i'_k$. Since the inverse of the unitary matrix is its conjugate transpose, the matrices $U_1^\dagger, \dots, U_k^\dagger, \dots$ provide the local unitary equivalence between $|\psi'\rangle$ and $|\psi\rangle$. Reasoning similar to the above proves that those matrices have the same number of nonzero elements in each row. This is equivalent to the fact that $U_1, \dots, U_k$ have the same number of nonzero elements in each column. This proves the first statement of Proposition 7 for matrices $U_1, \dots, U_k$ (without stating the equality of the element's norms). Observe that by taking another set of indices $S \subset [n]$ one may extend this reasoning to all matrices $U_1, \dots, U_k, \dots, U_{2k}$.

In fact, more detailed analysis of matrices $U_1, \dots, U_k$ might be performed. Once more, fix $I = i_1, \dots, i_k$, and keep the notation introduced before in the proof. For each $\tilde{I}' = i'_1, \dots, i'_{k-1} \in S_1 \times \dots \times S_{k-1}$, we have

$$\prod_{i'_k \in S_k} \ell_{\tilde{I}' i'_k} = (u_{i'_1}^1)^s \cdots (u_{i'_{k-1}}^{k-1})^s C_1 \quad (\text{C11})$$

where $C_1 = \prod_{i'_k \in S_k} u_{i'_k}^k$. By Eq. (C10), the left-hand side of Eq. (C11) is equal to

$$\prod_{i'_k \in S_k} \frac{\omega_{\tilde{I}' i'_k}'}{\omega_I} = \frac{1}{\omega_I} W_{\tilde{I}'}^{[k-1]} \prod_{i'_k \in S_k} v_{L_{\tilde{I}' i'_k}^1}^1 \cdots v_{L_{\tilde{I}' i'_k}^k}^k. \quad (\text{C12})$$

Here, we use the notation of $W_{\tilde{I}'}^{[k-1]}$ introduced in Sec. III A with a slight modification. Namely, the product

$$W_{\tilde{I}'}^{[k-1]} := \prod_{i'_k \in S_k} \omega_{\tilde{I}' i'_k}$$

runs only over all nonzero elements $\omega_{\tilde{I}' i'_k}$, which is exactly s . One of the basic properties of a MOLH is that in each row and on each position all elements appear exactly once (see Definition 3 for details). Hence, Eq. (C12) is equal to

$$\frac{1}{\omega_I} W_{\tilde{I}'}^{[k-1]} C_2, \quad \text{where} \quad C_2 = \prod_{I'' \in S'} v_{i'_1}^1 \cdots v_{i'_k}^k. \quad (\text{C13})$$

Equations (C11)–(C13) combine to the following:

$$(u_{i'_1}^1)^s \cdots (u_{i'_{k-1}}^{k-1})^s \frac{1}{W_{\tilde{I}'}^{[k-1]}} = \frac{1}{\omega_I} \frac{C_2}{C_1} \quad (\text{C14})$$

for each $\tilde{I}' = i'_1, \dots, i'_{k-1} \in S_1 \times \dots \times S_{k-1}$, where constants C_1 and C_2 are independent of $\tilde{I}'$. From Eq. (C14), one can deduce the proportions of nonzero elements in rows of U_1 . Indeed, choose a multi-index $\tilde{I}' \in S_2 \times \dots \times S_{k-1}$ and two indices $i'_1, i''_1 \in S_1$. From Eq. (C14) applied to $\tilde{I}' = i'_1 \tilde{I}'$ and $i''_1 \tilde{I}'$ follows that

$$(u_{i'_1}^1)^s \frac{1}{W_{i'_1 \tilde{I}'}^{[k-1]}} = (u_{i''_1}^1)^s \frac{1}{W_{i''_1 \tilde{I}'}^{[k-1]}}. \quad (\text{C15})$$

Since $W_{i'_1 \tilde{I}'}^{[k-1]}$ and $W_{i''_1 \tilde{I}'}^{[k-1]}$ have the same norms, there is the following equality: $|u_{i'_1}^1| = |u_{i''_1}^1|$. Those, however, under the introduced notation denote elements of the local unitary matrix: u_{i_1, i'_1} and u_{i_1, i''_1} . We conclude that in the i_1 th row of the matrix U_1 there are exactly s nonzero elements, all having the same norm. Since U_1 is a unitary matrix (in particular, it preserves the norms), all nonzero elements from the i_1 th row have norm equal to $1/\sqrt{s}$. Analogous reasoning might be performed with respect to each row of matrix U_1 and further to each matrix U_i , $i = 1, \dots, 2k$. Therefore, we affirm the equality of the element's norms in the first statement of Proposition 7.

In fact, Eq. (C15) is the last general result concerning the characteristic of local unitary matrices U_i . We restrict now to the case when $\omega_I \equiv \omega'_I \equiv 1$ for all multi-indices $I = i_1, \dots, i_k$. Obviously, $W_{i'_1 \tilde{I}'}^{[k-1]} = 1$ for all multi-indices $i'_1 \tilde{I}' \in S_1 \times \dots \times S_{k-1}$. Thus, from Eq. (C15) follows that all nonzero entries of the i_1 th row have not only equal norms but also equal s th powers. In other words, all nonzero entries of the i_1 th row are s th roots of unity up to some scaling complex number w_{i_1} . Analogous reasoning might be performed with respect to each row of matrix U_1 . In such a way we obtain scaling factors w_i for $i = 1, \dots, d$. Observe that matrix U_1 multiplied by the diagonal matrix

$$\sqrt{s} \text{diag}(w_1, \dots, w_d)$$

consists only of zeros and s th roots of unity. Similarly, one can show the same property for local matrices $U_2, \dots, U_{2k}$. This proves the third statement of Proposition 7. ■

We conjecture that the matrices $U_1, \dots, U_k, \dots, U_{2k}$ from Proposition 7 have the block structure

$$U_\ell = \begin{bmatrix} S_{\ell,1} & 0 & \cdots & 0 \\ 0 & S_{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\ell,d/s} \end{bmatrix},$$

where matrices $S_{\ell,j}$ are $s \times s$ unitary matrices with all entries of the same norm. Unfortunately, coefficients $v_{ij}^\ell(i_1, \dots, i_{\ell-1})$ depend on indices $i_1, \dots, i_{\ell-1}$, which impose selection of the index I at the beginning of the proof above. This enables us to deduce the general block structure of matrices U_i . Notice that not all unitary matrices having the same number of nonzero elements in each row and column of the same norm each are necessarily of the block structure.

Concluding the block structure of matrices U_ℓ is not within our reach yet. Nevertheless, in two specific cases when $s = 1$ or d the block structure is obvious. Interestingly, those two values of s are the only possibilities for most small dimensional AME($2k, d$) states (see Remark 1). This follows from the requirement of appropriate dimensional MOLH extension.

Proof of Propositions 5 and 6. Suppose that the matrix $U := U_1 \otimes \dots \otimes U_{2k}$ provides an LU equivalence between two AME($2k, d$) states with minimal support. Assumption of k and d being small enough is equivalent to the fact that the only possible values of s are simply 1 and d .

In the first case when $s = 1$, matrices U_i are, by the definition, monomial matrices. This gives the second possibility in Proposition 5. Each monomial matrix is a product of

permutation and the diagonal matrix, hence the form of Eq. (5) in Proposition 6 follows. Similarly to the proof of Proposition 4 presented in Appendix B, it may be shown that the form of diagonal matrices D_i from Eq. (5) are exactly as it is indicated in Proposition 6.

The second case, when $s = d$, is new and goes beyond the analysis performed so far. We investigate matrix U_1 . Let us recall the last general formula in our analysis, namely, Eq. (C15). By descrambling the notation introduced in the proof of Proposition 7, Eq. (C15) takes the following form:

$$(u_{ij}^1)^d \frac{1}{W_{jI}^{[k-1]}} = (u_{ij'}^1)^d \frac{1}{W_{j'I}^{[k-1]}}, \quad (\text{C16})$$

where $i \in [d]$ and $j, j' \in [d]$ are arbitrary indices, and $I \in [d]^{k-2}$ is an arbitrary multi-index. This equation describes the proportion of d th powers of elements in the i th row of the matrix U_1 . Observe that they are independent of the index i . Indeed, multiply the matrix U_1 on the right-hand side by the following diagonal matrix:

$$\overleftarrow{D}_1 = \text{diag}\left(\sqrt[d]{(W_{0,I}^{[k-1]})}, \dots, \sqrt[d]{(W_{d-1,I}^{[k-1]})}\right).$$

Observe that the entries of $\tilde{U}_1 := U_1 \overleftarrow{D}_1$ satisfy

$$(\tilde{u}_{ij}^1)^d = (\tilde{u}_{ij'}^1)^d,$$

for any number i indexing the rows and any pair of indices $j, j' \in [d]$. As we already discussed while proving Proposition 7, the Hermitian-conjugate matrix $U_1^\dagger$ provides the reverse LU equivalence. One can analyze the proportions of elements in rows of $U_1^\dagger$ in the same way as we did for U_1 . The analog of Eq. (C16) yields the following conclusions on the columns of matrix U_1 :

$$(u_{ij}^1)^d \frac{1}{(W_{iI}^{[k-1]})'} = (u_{i'j}^1)^d \frac{1}{(W_{i'I}^{[k-1]})'}, \quad (\text{C17})$$

where $i, i', j \in [d]$ are arbitrary indices and $I \in [d]^{k-2}$ is an arbitrary multi-index. Therefore, by multiplying the matrix $\tilde{U}_1$ from the left-hand side by the diagonal matrix

$$\overrightarrow{D}_1 = \text{diag}\left(\sqrt[d]{(W_{0,I}^{[k-1]})'}, \dots, \sqrt[d]{(W_{d-1,I}^{[k-1]})'}\right),$$

we obtain the matrix

$$\tilde{\tilde{U}}_1 := \overrightarrow{D}_1 \tilde{U}_1 \overleftarrow{D}_1$$

with the following property:

$$(\tilde{\tilde{u}}_{ij}^1)^d = (\tilde{\tilde{u}}_{i'j'}^1)^d,$$

for any indices $i, i', j, j' \in [d]$. Up to some global factor ω_1 , all entries of the matrix $\tilde{\tilde{U}}_1$ are d th roots of unity. By the definition $\omega_1 \tilde{\tilde{U}}_1$ is a Butson-type matrix.

We have shown that under the assumption $s = d$ the statement in Eq. (4) of Proposition 6 holds for the matrix U_1 and the set S of the consecutive next $k - 2$ indices. The same reasoning might be applied for another matrix U_i and set S . The global phase in Eq. (4) is then a multiplication of obtained factors ω_i .

What is left for the analysis is the necessary condition for existence of LU equivalence. Consider Eq. (C16). Since $I \in$

$[d]^{k-2}$ is an arbitrary multi-index and the ratio of two matrix elements $u_{ij}^1/u_{i'j'}^1$ is constant, we immediately obtain

$$\frac{W_{jI}^{[k-1]}}{W_{j'I}^{[k-1]}} = \frac{W_{j'I'}^{[k-1]}}{W_{j'I}^{[k-1]}}$$

for any multi-indices $I, I' \in [d]^{k-2}$. Similarly, from Eq. (C17), we have

$$\frac{(W_{jI}^{[k-1]})'}{(W_{j'I}^{[k-1]})'} = \frac{(W_{j'I'}^{[k-1]})'}{(W_{j'I}^{[k-1]})'}.$$

This ends the proof of Proposition 6. Obviously, Proposition 5 is an immediate consequence of Proposition 6. ■

APPENDIX D: THE PROOF OF PROPOSITION 3

We begin this section manifesting the problem of LU verification of two given AME states. In general, for two given k -uniform states, one can compare ranks of reduced density matrices in order to exclude a local equivalence between them [42]. We illustrate this phenomenon in the following example of two 1-uniform states of four qubits:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^1 |i, i, i, i\rangle, \quad |\psi_2\rangle = \frac{1}{2} \sum_{i,k=0}^1 |i, k, k, i+k\rangle.$$

Observe that $\text{rank}_{\rho_{12}}(\psi_1) \neq \text{rank}_{\rho_{12}}(\psi_2)$, hence the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are not LU equivalent. Nevertheless, this simple argument is never conclusive for both states being AME. Indeed, all reduced density matrices of AME state $|\psi\rangle$ have precisely determined ranks: $\text{rank}_{\rho_S}(\psi) = \min\{|S|, |S^c|\}$.

Our initial attempt for showing that states $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d)'\rangle$ presented in Examples 3 and 5 are not locally equivalent was to reduce this problem to smaller subsystems. We investigated the reduced density matrices:

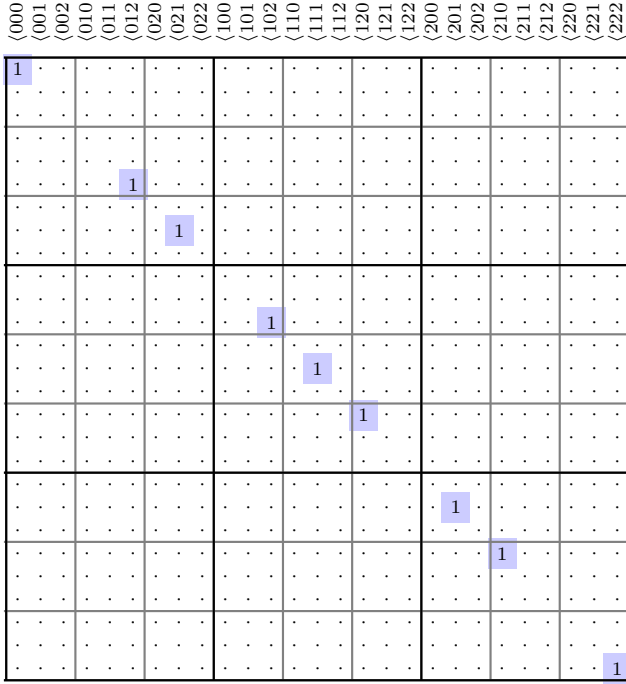
$$\begin{aligned} & \rho_{345}[\text{AME}(5,d)'] \\ &= \sum_{i,j=0}^{d-1} |i+j, i+2j, i+3j\rangle \langle i+j, i+2j, i+3j| \end{aligned}$$

and

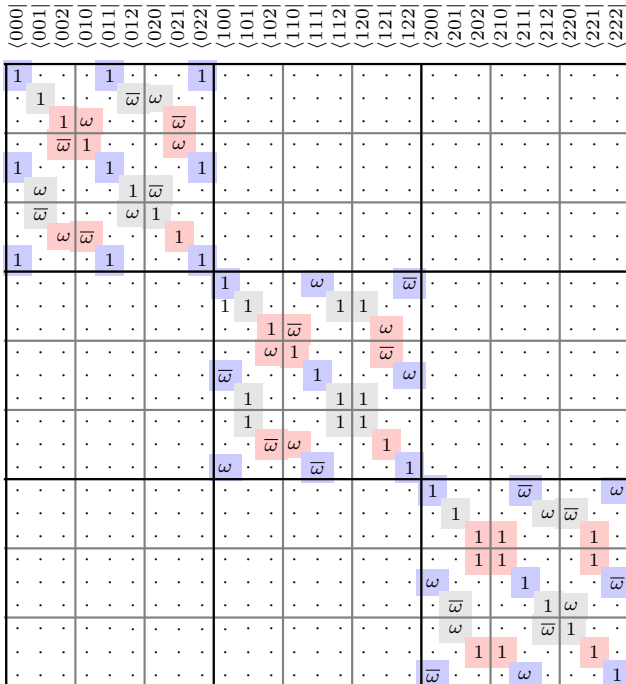
$$\begin{aligned} & \rho_{345}[\text{AME}(5,d)] \\ &= \sum_{i,j=0}^{d-1} \left(\sum_{k,k'=0}^{d-1} \omega^{(i+3j)(k-k')} |i+j, k+i+2j, k\rangle \right. \\ & \quad \left. \langle i+j, k'+i+2j, k'| \right), \end{aligned}$$

where ω is the d th root of unity. Even though $\rho_{345}[\text{AME}(5,d)']$ and $\rho_{345}[\text{AME}(5,d)]$ have the same rank, we attempt to show that they are not LU equivalent. Surprisingly, it turned out that they are. We present both reduced density matrices for a local dimension $d = 3$ in Figs. 2 and 3, respectively.

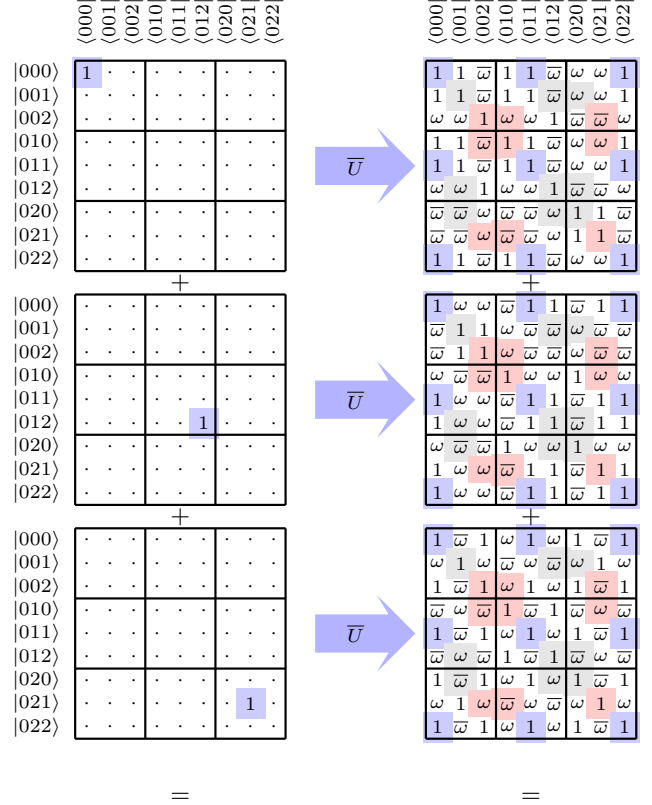
It is not a straightforward observation that both states are actually LU equivalent. Standard procedures of diagonalizing

FIG. 2. The density matrix $\rho_{345}[\text{AME}(5,d)]$.

the density matrix $\rho_{345}[\text{AME}(5,d)]$ lead to nonlocal operations. Attentive analysis of low local dimensions ($d = 3, 5$) showed that diagonalization might be performed in a local way. Indeed, for a local dimension $d = 3$, the unitary matrices

FIG. 3. The density matrix $\rho_{345}[\text{AME}(5,d)']$.

$$\tilde{U} = \text{Id} \otimes \begin{pmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ \omega & 1 & 1 \end{pmatrix}$$

FIG. 4. The matrix $\text{Id} \otimes \tilde{U}_4 \otimes \tilde{U}_5$ transforms $\rho_{cde}(\psi)$ into $\rho_{cde}(\phi)$. The figure depicts how it acts on the first block of Fig. 2.

$$\tilde{U}_3 = \text{Id}_3,$$

$$\tilde{U}_4 = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{pmatrix}, \quad \tilde{U}_5 = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ \omega & 1 & 1 \end{pmatrix}$$

provide a local equivalence between $\rho_{345}[\text{AME}(5,3)']$ and $\rho_{345}[\text{AME}(5,3)]$. This equivalence is illustrated in Fig. 4.

Similarly, for a local dimension $d = 5$, matrices $\tilde{U}_3 = \text{Id}_5$,

$$\tilde{U}_4 = \frac{1}{5} \begin{pmatrix} 1 & 1 & \omega^4 & \omega^2 & \omega^4 \\ \omega^4 & 1 & 1 & \omega^4 & \omega^2 \\ \omega^2 & \omega^4 & 1 & 1 & \omega^4 \\ \omega^4 & \omega^2 & \omega^4 & 1 & 1 \\ 1 & \omega^4 & \omega^2 & \omega^4 & 1 \end{pmatrix},$$

and

$$\tilde{U}_5 = \frac{1}{5} \begin{pmatrix} 1 & 1 & \omega & \omega^3 & \omega \\ \omega & \omega^3 & \omega & 1 & 1 \\ \omega & 1 & 1 & \omega & \omega^3 \\ 1 & \omega & \omega^3 & \omega & 1 \\ \omega^3 & \omega & 1 & 1 & \omega \end{pmatrix}$$

provide a local equivalence between $\rho_{345}[\text{AME}(5,5)']$ and $\rho_{345}[\text{AME}(5,5)]$. In order to provide the general formula, for any odd local dimension d , we introduce the following recursive construction of two $(d \times d)$ -dimensional matrices:

$$W = (w_{ij}) \text{ and } V = (v_{ij})$$

with coefficients in $\text{GF}(d)$.

(a) $w_{00} = 0, v_{00} = 0$.

(b) $w_{0(j+1)}$ is defined by the formula $w_{0(j+1)} = w_{0j} + 2j$; similarly, $v_{0(j+1)} = v_{0j} - 2j$ (definition of first rows).

(c) $w_{ij} := w_{(i-1)(j-1)}$; similarly, $v_{ij} := v_{(i-2)(j-1)}$ (definition of succeeding rows).

Notice that in order to define rows of the matrix V correctly we impose $2 \nmid d$. The matrices $\tilde{U}_4$ and $\tilde{U}_5$ are of the following form:

$$\tilde{U}_4 = (\omega^{w_{ij}}) \text{ and } \tilde{U}_5 = (\omega^{v_{ij}}).$$

This construction overlaps with the aforementioned constructions in dimensions $d = 3, 5$. In fact, there is a close formula for entries of matrices W and V (and hence for matrices $\tilde{U}_4$ and $\tilde{U}_5$) given in terms of triangular numbers:

$$w_{ij} = 2t_{j-i-1}, \quad v_{ij} = \begin{cases} -2t_{j-i/2-1} & \text{for } 2 \mid i \\ -2t_{j-(i+d)/2-1} & \text{for } 2 \nmid i \end{cases}$$

where $t_i = 0, 1, 3, 6, 10, 15, \dots$ are consecutive triangular numbers defined as

$$t_k = \sum_{i=0}^k i = \frac{(k+1)k}{2}. \quad (\text{D1})$$

Lemma 3. For any odd local dimension d , the matrices $\tilde{U}_4$ and $\tilde{U}_5$ provide the LU equivalence between $\rho_{345}[\text{AME}(5,d)']$ and $\rho_{345}[\text{AME}(5,d)]$, i.e.,

$$\rho_{345}[\text{AME}(5,d)'] = \text{Id} \otimes \tilde{U}_4 \otimes \tilde{U}_5 \{ \rho_{345}[\text{AME}(5,d)] \}.$$

Proof. Observe that

$$\sum_{s,j=0}^{d-1} |s, s+j, s+2j\rangle \langle s, s+j, s+2j| \xrightarrow{\text{Id}_3 \otimes \tilde{U}_4 \otimes \tilde{U}_5} \sum_{s=0}^{d-1} \sum_{\substack{m,m',k, \\ k'=0}}^{d-1} \left[\sum_{j=0}^{d-1} \omega^{(w_{(s+j)m} \overline{w_{(s+j)m'}} v_{(s+2j)k} \overline{v_{(s+2j)k'}})} |s, m, k\rangle \langle s, m', k'| \right].$$

We examine the coefficient by

$$|s, k+s+j, k\rangle \langle s, k'+s+j', k'|$$

in the expression above:

$$\begin{aligned} & \frac{1}{d} \sum_{i=0}^{d-1} \omega^{(w_{(s+i)(k+j+s)} \overline{w_{(s+i)(k'+j'+s)}} v_{(s+2i)k} \overline{v_{(s+2i)k'}})} \\ &= \frac{1}{d} \sum_{i=0}^{d-1} \omega^{2(t_{k+i-j-1-t_{k'+j'-i-1}} - t_{k-s/2-i-1} t_{k'-s/2-i-1})}, \end{aligned} \quad (\text{D2})$$

where we assumed $2 \mid s$; the argument for the opposite case, where $2 \nmid s$, is similar to the one we present. Using Eq. (D1), after elementary transformations Eq. (D2) is equal to

$$\frac{1}{d} \omega^{[s(k-k')+2jk-2j'k']} \omega^{[j(j-1)-j'(j'-1)]} \underbrace{\sum_{i=0}^{d-1} \omega^{[2i(j-j')]}_{d\delta_{jj'}} = \omega^{(s+2j)(k-k')},$$

which remains to be shown. $\blacksquare$

Proof of Proposition 3. We show the statement by contradiction. Assume that states $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d)'\rangle$ are LU equivalent by some unitary matrices:

$$U_1 \otimes U_2 \otimes U_3 \otimes U_4 \otimes U_5 =: U_{12} \otimes U_{345}.$$

We keep “ $\otimes$ ” in our notation in order to distinguish it from the matrix multiplication. Since the (partial) trace is invariant under cycling permutations, we have

$$\begin{aligned} & \rho_{345}[\text{AME}(5,d)'] \\ &= \text{tr}_{12} |\text{AME}(5,d)'\rangle \langle \text{AME}(5,d)'| \\ &= \text{tr}_{12} [U_{12} \otimes U_{345} |\text{AME}(5,d)\rangle \langle \text{AME}(5,d)| U_{12}^{-1} \otimes U_{345}^{-1}] \\ &= U_{345} \{ \text{tr}_{12} [U_{12} |\text{AME}(5,d)\rangle \langle \text{AME}(5,d)| U_{12}^{-1}] \} U_{345}^{-1} \\ &= U_{345} \{ \text{tr}_{12} [|\text{AME}(5,d)\rangle \langle \text{AME}(5,d)|] \} U_{345}^{-1} \\ &= U_{345} \{ \rho_{345}[\text{AME}(5,d)] \} U_{345}^{-1}. \end{aligned}$$

Hence, the operator $U_{345} := U_3 \otimes U_4 \otimes U_5$ provides the local equivalence between $\rho_{345}[\text{AME}(5,d)]$ and $\rho_{345}[\text{AME}(5,d)']$.

Notice that in Lemma 3 we pointed out that the earlier constructed matrices $\tilde{U}_4$ and $\tilde{U}_5$ provide the LU equivalence between $\rho_{345}[\text{AME}(5,d)']$ and $\rho_{345}[\text{AME}(5,d)]$ for any odd local dimension d . Precisely,

$$\rho_{345}[\text{AME}(5,d)] = \text{Id} \otimes \tilde{U}_4 \otimes \tilde{U}_5 \{ \rho_{345}[\text{AME}(5,d)'] \}.$$

Therefore from Proposition 2 we conclude that

$$U_3 = M_3, \quad U_4 = M_4 \tilde{U}_4, \quad U_5 = M_5 \tilde{U}_5,$$

for some monomial matrices M_3, M_4 , and M_5 . Indeed, $U_{345}(\text{Id} \otimes \tilde{U}_4 \otimes \tilde{U}_5)^{-1}$ constitutes an automorphism of $\rho_{345}[\text{AME}(5,d)']$ and hence, by Proposition 2, is a tensor product of monomial matrices. We shall prove that such restriction on matrices U_3, U_4 , and U_5 leads to a contradiction.

To sum up the discussion so far, LU equivalence between states $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d')\rangle$ has the following form:

$$U_1 \otimes U_2 \otimes M_3 \otimes M_4 \tilde{U}_4 \otimes M_5 \tilde{U}_5$$

where U_i are arbitrary unitary matrices, while M_i is the product of diagonal and permutation matrices. Therefore

$$|\text{AME}(5,d)\rangle = (U_1 \otimes U_2)|i, j\rangle \otimes M_3|i + j\rangle \otimes B_{ij} \quad (\text{D3})$$

where

$$B_{ij} := (M_4 \tilde{U}_4 \otimes M_5 \tilde{U}_5)|i + 2j, i + 3j\rangle.$$

Observe that B_{ij} are linearly independent. Indeed, they are unitary transformed linearly independent vectors $|i + 2j, i + 3j\rangle$.

We shall show that matrices U_1 and U_2 are monomial matrices. Suppose for simplicity that $M_3 = \text{Id}$. Recall that $|\text{AME}(5,d)\rangle$ has the following form:

$$|\text{AME}(5,d)\rangle = |i, j, i + j\rangle \otimes C_{ij}, \quad (\text{D4})$$

where

$$C_{ij} = \omega^{(i+3j)k} |k + 2j, k\rangle.$$

We compare this expression with Eq. (D3). Suppose now that in some column of the matrix U_1 there are at least two nonzero elements: $u_{lk}^1, u_{l'k}^1$ ($l \neq l'$); consider some nonzero element u_{nm}^2 of the matrix U_2 . Observe that it leads to the expressions

$$|l, n\rangle \otimes |k + m\rangle \otimes B_{km} \quad \text{and} \quad |l', n\rangle \otimes |k + m\rangle \otimes B_{km}$$

in Eq. (D3). Clearly, there is an additional contribution from other nonzero elements of matrices U_1 and U_2 . Since B_{ij} are linearly independent, there are the terms

$$|l, n\rangle \otimes |k + m\rangle \otimes D_{ln} \quad \text{and} \quad |l', n\rangle \otimes |k + m\rangle \otimes D_{l'n}$$

in Eq. (D3), where D_{ln} and $D_{l'n}$ are some nonzero elements. Observe that such terms might appear in Eq. (D4) only if $l + n = k + m$ and $l' + n = k + m$, which is contradictory to $l \neq l'$. Similarly, one can show that none of the matrix U_1 columns have two nonzero elements. We have shown that, indeed, matrices U_1 and U_2 are monomial under the assumption $M_3 = \text{Id}$. Nevertheless, the assumption $M_3 = \text{Id}$ is not essential here; a similar argument might be given for arbitrary monomial matrix M_3 . Hence U_1 and U_2 are monomial matrices in general.

Observe that $\text{supp}(B_{ij}) = d^2$. Indeed,

$$\text{supp}(\tilde{U}_4 \otimes \tilde{U}_5 |i + 2j, i + 3j\rangle) = d^2,$$

and the monomial operators M_4 and M_5 do not change the support. Since U_1 and U_2 are monomial matrices, the support of the right-hand side in Eq. (D3) is equal to d^4 . This is contradictory to the fact that $\text{supp}[|\text{AME}(5,d)\rangle] = d^3$. ■

We have shown that two families of $\text{AME}(5,d)$ states are not LU and SLOCC equivalent. Even though only a special family of states is considered here, analysis of the proof of Proposition 3 reveals the general method of LU verification of AME and k -uniform states where one of them is written with minimal support. First, the formula for LU equivalence between reduced $(k + 1)$ -dimensional systems should be provided. Second, based on Proposition 2, one can classify such equivalences between those reduced states. Finally, it should be shown that none of such equivalences can be extended to the local equivalence of initial states.

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C Entanglement in highly symmetric multipartite quantum states

Entanglement in highly symmetric multipartite quantum states

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A construction of genuinely entangled multipartite quantum states based on the group theory is presented. Analyzed states resemble the Dicke states, whereas the interactions occur only between specific subsystems related by the action of the selected group. The states constructed by this technique exhibit desired symmetry properties and form a natural resource for less-symmetric quantum informational tasks. We propose quantum circuits efficiently generating such states, which in general have smaller complexity than the circuits necessary to create fully symmetric entangled states. Exemplary five-qubit quantum states are simulated on quantum computers.

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I. INTRODUCTION

Characterization of different classes of entanglement in multipartite quantum systems remains a major issue relevant for various quantum information tasks and is interesting from the point of view of foundations of the quantum theory. Since manipulation and relative control over several qubits have already become a standard task [1,2], analysis of such systems in the context of resources for various information protocols is needed. Among the different types of entangled states, permutation-invariant states have attracted a lot of attention for both continuous [3,4] and discrete [5,6] variable systems.

A notable example of such states is due to Dicke [7]. The *Dicke states* of an N -qubit system with k excitations are defined as [8]

$$|D_N^k\rangle \propto \sum_{\sigma \in \mathcal{S}_N} \sigma \left(\underbrace{1 \dots 1}_{k \text{ times}} \underbrace{0 \dots 0}_{N-k \text{ times}} \right), \quad (1)$$

where the summation runs over the symmetric group $\mathcal{S}_N$. Permutational symmetry of the Dicke states simplifies their theoretical [9] and experimental [10] detection and facilitates the tasks of quantum tomography [11]. Multipartite Dicke states are experimentally accessible [12]. Since the entanglement of Dicke states turned out to be maximally persistent and robust for the particle loss [7,13], such states provide inherent resources in numerous quantum information contexts, including decoherence-free quantum communication [14], quantum secret sharing [15], open destination teleportation [16], and quantum metrology [17]. Entanglement properties and application of mixtures of Dicke states were also studied [18–20].

Until now the vast majority of scientific interest was focused on fully symmetric tasks, like parallel teleportation [21] and symmetric quantum secret sharing protocols [22]. In various realistic situations it seems natural that such a

full symmetry between collaborating systems is not possible, required, or even desirable. As an example, it was shown [23] that a four-qubit state maximally entangled with respect to all possible symmetric partitions does not exist. Such a state would allow for the parallel teleportation of two qubits between any two subsystems to the remaining pair of systems. Nevertheless, the following state,

$$\frac{1}{2}(|1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle),$$

allows for the teleportation of a single qubit to an arbitrary subsystem, and additionally for the parallel teleportation across the partition 13|24. It is especially reasonable to share resources in a not fully symmetric way in variants of quantum secret sharing schemes, allowing only some parties for cooperation. Alternatively, such schemes were already considered, modeled by quantum channels [24] or pairs of Einstein, Podolsky, and Rosen (EPR) states shared between nodes of a network [25–28]. Implementation of such communication networks is expected to be developed in the near future [24,29], with a variety of possible applications [30].

In this work we present an approach for constructing highly entangled quantum states with a given symmetry group. Their form is similar to Eq. (1), whereas the summation runs over elements of a specific subgroup H of the symmetric group $\mathcal{S}_N$. We refer to such states as *Dicke-like states* and analyze which types of symmetries are feasible. In particular, we present explicit constructions of quantum states based on highly symmetric geometrical objects, such as regular polygons, Platonic solids, and regular plane tilings. We compare them as entanglement resources with the original Dicke states, which were investigated as ground states of specific two-body Hamiltonians with a well-defined number of excitations [31,32]. We present the generalized, Dicke-like states as ground states of a wider class of Hamiltonians with three-body interactions.

Dicke-like states constitute a canonical basis of all pure quantum states with a given symmetry. Apart from the theoretical aspect of this statement, such a basis might be efficiently applied in the context of quantum chemistry.

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Various molecules in nature (like benzene) stand out with remarkable symmetries. In principle, the spin model of such states might be efficiently simulated by a proper superposition of Dicke-like states. In recent years, correlations and the entanglement contained in chemical bonds was investigated [33–35], and special attention was dedicated to highly symmetrical molecules [34]. Although for most molecules the total correlation between orbitals seems to be classical [35], the general significance of entanglement in chemical bonds is not yet clear. Therefore, general investigations of entanglement in highly symmetric systems may shed some light on the nature of correlation in relevant chemical molecules.

Furthermore, we introduce a larger class of genuinely entangled states based on an arbitrary network structure, graphically represented as a (hyper)graph. Once more, this construction generalizes the Dicke states (1); however, excitations appear only in particular subsystems represented by (hyper)edges of the graph. We do not impose any restrictions on the network structure; however, the detailed analysis of entanglement properties is performed under the assumption of regularity. In order to construct excitation states, we propose quantum circuits whose complexity is comparable with the complexity of quantum circuits proposed for Dicke states [36,37]. Joint unitary operations in the circuit are performed on the neighboring nodes.

We demonstrate entanglement properties of Dicke-like states and excitation states in terms of entanglement resources [38,39] contained in particular subsystems. In several cases an interesting phenomenon is observed: most of the entanglement is concentrated between nodes of distance 2 and is absent between immediate neighbors. This indicates a particular advantage of such states in the case, in which collaboration between neighboring nodes is not desired. In principle, constructed states are genuinely but not maximally entangled, persistent with respect to measurements performed locally on each subsystem and with respect to losses of certain subsystems [40].

Mixed-state entanglement is a valuable resource for quantum communication protocols [41,42]. Therefore, the Dicke-like states and excitation states might be distilled on particular subsystems [43] and further used in concrete protocols. Notice that once the excitation state is established, it does not require any particular actions on nodes (as entanglement swapping [44]), even for the collaboration of parties being far away in the network structure. This is in contrast with the case analyzed earlier, in which the nodes of the graph share pairs of maximally entangled states.

This work is organized as follows. In Sec. II, we introduce the notion of group symmetry of a multipartite state. We demonstrate that not all kinds of symmetries are feasible. Moreover, we define the Dicke-like states and discuss their separability criteria. In Sec. III, we generalize Dicke-like states in a less symmetric manner. Based on a given (hyper)graph, we introduce the excitation states. Such a graphical representation is later used in the construction of quantum circuits in Sec. VII, and Hamiltonians in Sec. IX, both relevant to the Dicke-like states and the excitation states. Section IV analyzes the entanglement properties of introduced families of states. In Sec. V particular examples of Dicke-like states and excitation states are presented. We observe a phase transition

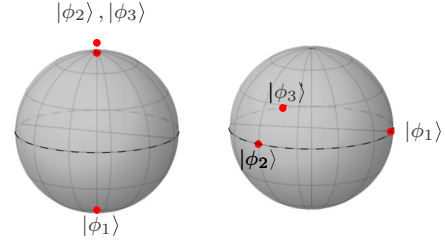


FIG. 1. Stellar representation of three qubits states. (a) The state $|W\rangle \propto |001\rangle + |010\rangle + |100\rangle$ is represented by two stars at the North Pole and a single star at the South Pole. (b) The state $|GHZ\rangle \propto |000\rangle + |111\rangle$ corresponds to three stars evenly distributed on the equatorial plane: $|0\rangle + e^{2ik/3}|1\rangle$ for $k = 0, 1, 2$. The stellar representation is not unique as any rotation of the Bloch sphere around the vertical axis yields the same state.

of bipartite entanglement in excitation states with respect to the average degree of a relevant (hyper)graph and present some more details in Sec. VI. In order to demonstrate the viability of the provided constructions, we have simulated one of the considered states on available quantum computers: IBM Santiago and Athens. In Sec. VIII these results are presented and discussed.

II. GROUP SYMMETRY OF A QUANTUM STATE

A pure state $|\psi\rangle$ on an N -fold product Hilbert space is called *symmetric* if it is invariant under permutation of N subsystems, i.e., $\sigma|\psi\rangle = |\psi\rangle$ for any element σ from the permutation group S_N . Each such state $|\psi\rangle$ can be written in the computational basis:

$$|\psi\rangle \propto \sum_{\sigma \in S_N} |\phi_{\sigma(1)}\rangle \cdots |\phi_{\sigma(N)}\rangle, \quad (2)$$

where the sum is taken over all permutations σ from the symmetric group S_N . Any symmetric state of the N -qubit system can be graphically represented as a collection of N points on the Bloch sphere corresponding to vectors $|\phi_i\rangle, \dots, |\phi_N\rangle$. Such a visualization scheme is called the *stellar* or *Majorana* representation (see Fig. 1). This representation, originally used for pure states of simple systems of a finite dimension, was later generalized for mixed states [45], symmetric states of systems consisting of several qubits [46–48], and symmetric states of higher-dimensional subsystems [49].

In this section, we discuss a natural generalization of this approach—the *generalized stellar representation*—suitable for quantum states exhibiting modes symmetries, i.e., for which summation in Eq. (2) runs over a subgroup H of the symmetric group S_N . This might possibly restrict the group of symmetries in the resulting state. Restricting the symmetries of a state to the subgroup $H < S_n$ was already considered in yet another context of continuous symmetries given by an irreducible representation of $SU(N)$ (see [50]). We begin with a very natural definition of the symmetry group of a quantum state.

Definition 1. A state $|\psi\rangle$ of N subsystems is called H symmetric, where H is a subgroup of the permutation group,

$H < S_N$, if and only if (iff) it is permutation invariant for any $\sigma \in H$, and only for such permutations.

We begin with examples of states with restricted group symmetry. Consider first a system consisting of three qubits. There exist two particular classes of states which should be distinguished [51]:

$$\begin{aligned} |\text{GHZ}\rangle &\propto |000\rangle + |111\rangle, \\ |W\rangle &\propto |001\rangle + |010\rangle + |100\rangle. \end{aligned}$$

Both states are symmetric with respect to any permutation of particles (see Fig. 1). We discuss briefly the problem of constructing three-qubit quantum states with other types of symmetry. It is easy to find an example of S_2 -type symmetry in a three-qubit setting, namely, $|001\rangle + |010\rangle$. The mentioned state is obviously separable, but a similar example $|\chi\rangle$ can be found also among genuinely entangled states:

$$|\chi\rangle \propto |001\rangle + |010\rangle + 2|100\rangle + 2|111\rangle.$$

We discuss the necessary and sufficient conditions for existence of a quantum state with a given symmetry. Let us begin with a simple observation that for any group $H < S_N$ there exists an H -symmetric state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ if the number of energy levels d in each subsystem is large enough.

Proposition 1. Consider the subgroup $H < S_N$. The following state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of the local dimension N ,

$$|\psi\rangle \propto \sum_{\sigma \in H} |\sigma_0(0) \cdots \sigma_{N-1}(N-1)\rangle,$$

is H symmetric.

Proof. It is easy to see that the group H stabilizes $|\psi\rangle$. Suppose there is a larger group H' containing H , i.e., $H < H'$, and stabilizing $|\psi\rangle$. Take any element $\sigma \in H' \setminus H$. Observe that there is no term $|\sigma_0(0) \cdots \sigma_{N-1}(N-1)\rangle$ in $|\psi\rangle$; hence H' does not stabilize $|\psi\rangle$. ■

For instance, the three-qutrit state

$$|\psi\rangle \propto |012\rangle + |201\rangle + |120\rangle \quad (3)$$

is A_3 symmetric. Indeed, one can see that the cyclic permutation of the last three qutrits does not change the entire state.

Question 1. Consider any group symmetry $H < S_N$. What is the minimal local dimension d for which there exists an H -symmetric state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$?

In order to answer the question above, consider a suitable basis for symmetric states. Any symmetric state $|\psi\rangle$ is a superposition of Dicke states [7,8]:

$$|\psi\rangle \propto \alpha_0 |D_N^0\rangle + \cdots + \alpha_N |D_N^N\rangle.$$

Therefore, the Dicke states, $|D_N^k\rangle$ with $k = 0, 1, \dots, N$, can be treated as a basis of the symmetric state space. In other words, the coefficient $\alpha_{i_1, \dots, i_k}$ in front of any term $e_{i_1, \dots, i_k}$ in the symmetric state is the same, and the symmetric state is uniquely determined by the coefficients $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_N$ in front of terms $|1 \cdots 10 \cdots 0\rangle$. A similar decomposition for any H -symmetric states is possible; however, its building blocks have to be suitably selected.

Definition 2. For a given subgroup $H < S_N$, we define the N -qubit Dicke-like H state with k excitations is the following

way:

$$|D_N^k\rangle_H \propto \sum_{\sigma \in H} \sigma \left(\underbrace{1 \cdots 1}_{k \text{ times}} \underbrace{0 \cdots 0}_{N-k \text{ times}} \right), \quad (4)$$

where the summation runs over all permutations belonging to the group H .

As we will see, the proper normalization constant in Eq. (4) highly depends on the structure of the subgroup H and the number of excitations, and it is difficult to present in a consistent way. Notice that the group symmetry of the Dicke-like states is not necessarily H . In general, usage of such states to construct H -symmetric states is not straightforward. The example presented below illustrates problems which might occur.

Suppose we are looking for the cyclic, C_4 -symmetric state $|\phi\rangle$ (invariant under cyclic permutations) among states of the following form:

$$\begin{aligned} |\phi\rangle &= \alpha_0 |D_4^0\rangle + \alpha_1 |D_4^1\rangle + \alpha_{2,1} |D_{4(1)}^2\rangle + \alpha_{2,2} |D_{4(2)}^2\rangle \\ &\quad + \alpha_3 |D_4^3\rangle + \alpha_4 |D_4^4\rangle, \end{aligned} \quad (5)$$

where

$$|D_{4(1)}^2\rangle = \frac{1}{2}(|1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle), \quad (6)$$

$$|D_{4(2)}^2\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle). \quad (7)$$

In each of them a separate coefficient might be adjusted. Observe that up to the normalization constant

$$|D_4^2\rangle \propto |D_{4(1)}^2\rangle + |D_{4(2)}^2\rangle, \quad (8)$$

which means that the fully symmetric term $|D_4^2\rangle$ splits into two classes: $|D_{4(1)}^2\rangle$ and $|D_{4(2)}^2\rangle$. This division was obtained by taking appropriate terms $|1 \cdots 10 \cdots 0\rangle$ and acting on them by the cyclic group C_4 , unless not all such terms belong to the prior class. This might be seen as a decomposition of the symmetric subspaces. Observe that the symmetry group of the state $|D_{4(1)}^2\rangle$ is the dihedral group $\mathcal{D}_8$ (in fact for $|D_{4(2)}^2\rangle$ it is the same). Hence, by taking $\alpha_{2,1} \neq \alpha_{2,2}$, we obtain the $\mathcal{D}_8$ -symmetric state, in any other case, the S_N -symmetric case. Therefore, among states of the form (5), the C_4 -symmetric state $|\psi\rangle \in \mathcal{H}_2^{\otimes 4}$ does not exist. In fact, the $|D_{4(1)}^2\rangle$ and $|D_{4(2)}^2\rangle$ are nontrivial examples of the dihedral $\mathcal{D}_8$ -symmetric states; nevertheless, both of them are separable with respect to the partition $(13|24)$.

On the other hand, the $\mathcal{D}_8$ -symmetric state can be constructed among genuinely entangled states. Indeed, a superposition of $|D_{4(1)}^2\rangle$ and $|1111\rangle$,

$$|\phi\rangle \propto |1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle + 2|1111\rangle,$$

leads to a $\mathcal{D}_8$ -symmetric state.

The general analysis of the group of symmetries is tightly connected with the *partially ordered set* (poset) of all subgroups of S_N , which has a rather complicated structure [52]. We conjecture that alternating groups A_N are rather difficult to realize as a group of symmetries of quantum states, impossible in the N -qubit setting.

Symmetric states of N -qubit systems have an effective representation, called the *stellar representation* [53], as N

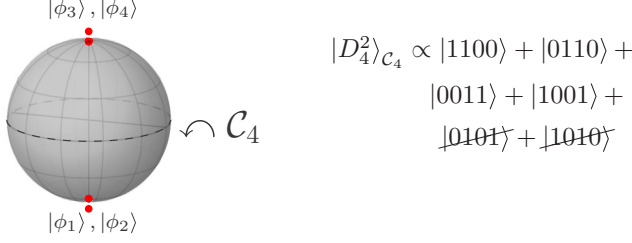


FIG. 2. Stellar representation of a Dicke state $|D_N^k\rangle$ that consists of k stars at the South Pole and $N - k$ stars at the North Pole [48]. A similar picture holds true for the Dicke-like states: $|D_4^2\rangle_{C_4}$ defined in Eq. (6) and not equivalent to the Dicke state $|D_4^2\rangle$ is represented by four stars on the Bloch sphere, with indicated action of the cyclic group C_4 . Instead of choosing this group one could consider the action of the dihedral group $\mathcal{D}_8$ (see Sec. VC). The resulting states are the same, $|D_4^2\rangle_{C_4} = |D_4^2\rangle_{\mathcal{D}_8}$, and they are based on the graph construction (see Fig. 4).

points (stars) on the Bloch sphere, related to the roots of the Majorana polynomial. It was recently generalized for a mixed-states setting [45]. The stellar representation turned out to be useful for classification of entanglement in symmetric quantum states [48,53,54]. Moreover, by imposing the special symmetry conditions on the stars, related states exhibits high entanglement properties [55–57]. The stellar representation might be also generalized to the H -symmetric states. As it was before, consider N points on the Bloch sphere, $|\phi_1\rangle, \dots, |\phi_N\rangle$, and the following product:

$$|\psi\rangle \propto \sum_{\sigma \in H} |\phi_{\sigma(1)}\rangle \cdots |\phi_{\sigma(N)}\rangle, \quad (9)$$

where the sum is taken over all permutations σ from the group H . We represent state $|\psi\rangle$ as a collection of N points on the Bloch sphere relevant to vectors $|\phi_i\rangle, \dots, |\phi_N\rangle$ with the indicated group H which is acting in Eq. (9) (see Fig. 2).

Note that a given constellation of “stars” at the Bloch sphere together with the selected symmetry group H do not represent uniquely the quantum state. The important information is carried in how the group H is contained in $\mathcal{S}_N$, which mathematically might be expressed by immersion $H \hookrightarrow \mathcal{S}_N$ of the group H into the symmetric group $\mathcal{S}_N$.

III. STATES FROM (HYPER)GRAPHS

In the previous section we discussed the problem of determining quantum states invariant under specified permutations $\sigma \in H$, where $H < \mathcal{S}_N$. Moreover, we presented the Dicke-like states, which are building blocks for all such modestly symmetric states. They were obtained by limiting the group of summation in the standard construction of the classical Dicke states. As specification of all symmetries of a given quantum state is a challenging task, we propose here an alternative—a more geometric approach to the problem. Certain quantum states can be associated with (hyper)graphs, whereas their symmetries are automorphisms of (hyper)graphs.

Let us emphasize here that graphs [58,59] and hypergraphs [60] were already successfully used in order to construct genuinely entangled states, and the notion of *graph states* is well

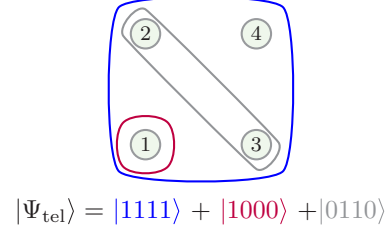


FIG. 3. Telescope state $|\Psi_{\text{tel}}\rangle$ [65] represented as an excitation state. The (hyper)graph is not uniform since (hyper)edges are of different capacities. The relevant state is simply obtained by reading over all edges. It has only one nontrivial symmetry, which exchanges parties 2 and 3.

established. Graph states (known also as cluster states [13]) exhibit genuine entanglement properties [21]. They are useful in the context of the one-way quantum computer or quantum error-correcting codes [61,62]. Another approach allows one to generate a family of random quantum states related to a given graph [63,64], the vertices of which describe the interaction between subsystems.

In this work we propose another scheme to associate to a given graph with N vertices a single pure quantum state of an N -party system. Such a representation reflects not only the symmetry, but also the structure of a quantum circuit under which presented families of quantum states can be constructed.

Recall that a *graph* G is a pair (V, E) where V is a finite set, and E is a collection of two-element subsets of V . We refer to elements of V as *vertices*, and elements of E as *edges*, respectively. A successful generalization of the notion of a graph is the *hypergraph* [60], for which edges are arbitrary (not necessarily two-element) subsets of V . A hypergraph is called *uniform* if all of its edges consist of the same number of elements equal to k ; we refer to such an object as a k -hypergraph. For example, a 2-hypergraph is simply a graph. We denote the number of vertices by N . For simplicity, we assume that vertices of the (hyper)graph are labeled by numbers $1, \dots, N$.

Definition 3. With a given (hyper)graph $G = (V, E)$, we associate a quantum state of N qubits in the following way:

$$|G\rangle := \frac{1}{\sqrt{|E|}} \sum_{e \in E} |\psi_e\rangle, \quad (10)$$

where $|\psi_e\rangle$ is a tensor product of $|1\rangle$ on positions labeled by indices which form the (hyper)edge e and $|0\rangle$ on other positions. We refer to such states as *excitation states*. Figure 3 illustrates this definition.

In the definition above, we could consider *weighted hypergraphs*, i.e., in which each edge $e \in E$ has a *weight* given by a complex number. Obviously, the normalization of a state should be changed. Interestingly, in such a way, we can obtain a general form of a quantum state. Indeed, a given quantum state $|\psi\rangle \in \mathcal{H}_d^N$ might be presented in a computational basis. From this form, we can construct a weighted hypergraph, where edges are relevant to the nonvanishing terms and weights are given by coefficients. Nevertheless, to keep the

presentation simple, in this work we do not consider weighted graphs.

Observe that the $|W\rangle$ state, the generalized $|W_N\rangle$ states, and the Dicke states $|D_N^k\rangle$ are associated with the complete k -hypergraphs under provided construction (see Fig. 7). All these states are fully symmetric. This reflects the fact that the *automorphisms group* of a complete k -hypergraph is the symmetric group S_N . This relation might be generalized for all excitation states.

Observation 1. Consider an excitation state $|G\rangle$ related to the hypergraph G . The group of symmetries of a state $|G\rangle$ is an automorphisms group of the related hypergraph G .

Even though determining the automorphisms group for a graph is an NP-hard problem, it is widely studied for several classes of graphs. Therefore, this graphical representation of quantum states may be conclusive for determining its group of symmetries. We use this relation directly on various examples of Dicke-like states.

The Dicke-like states introduced in Sec. II belong to the class of excitation states and form building blocks for all states of intermediate symmetry. Notice that the (hyper)graph construction of excitation states is far more general; in particular, it does not require any symmetries.

Observation 2. Each Dicke-like state $|D_N^k\rangle_H$ is equivalent to the excitation state of a 2-transitive k -hypergraph. Furthermore, such a hypergraph state is a Dicke-like state with the group symmetries determined by the group automorphisms of the graph.

Proof. Recall that the state $|D_N^k\rangle_H$ was obtained via the action of a group H on the vector $|1 \dots 10 \dots 0\rangle$ with k entries equal to unity. The group $H < S_N$, so it acts on the N -element set $\{1, \dots, N\}$. Consider now the following (hyper)graph. For any element $h \in H$, join $h(1), \dots, h(k)$ with an edge. The relevant excitation state coincides with $|D_N^k\rangle_H$. The group symmetry for a Dicke-like state corresponds to the group automorphisms of a relevant (hyper)graph. By the above construction, the related graph is *edge transitive*; i.e., for any two edges $e_1 = \{v_1, \dots, v_k\}$ and $e_2 = \{w_1, \dots, w_k\}$, there exists an element $h \in H$ such that $h(v_i) = w_i$ for any $i = 1, \dots, k$. Interestingly, the reverse statement is also true. Indeed, take the group automorphisms of a graph, and suppose it is edge transitive. Without loss of generality suppose that vertices $1, \dots, k$ are connected in an edge e_0 . Observe that any other edge e is a result of some automorphism h , i.e., $e = h(e_0)$ (all terms in an excitation state might be obtained from $|1 \dots 10 \dots 0\rangle$ by an action of automorphisms). Moreover, none of the edges is distinguished; hence each of them is achieved the same number of times, as the coefficients in front of the edges are equal. ■

Let us briefly compare the concept of constructing genuinely entangled quantum states presented above with the notion of graph states. First, notice that the excitation state is evidently different than the graph state associated with the same graph. The latter is strictly connected to the quantum circuits, in which edges of a graph are replaced by controlled-NOT (CNOT) gates. Therefore, the graph state is closely related to its deterministic realization by the aforementioned two-qubit gates. At the same time, encoding the closed formula of a graph state (written in computational basis) and its entanglement properties is rather a challenging task. For an

excitation state the situation is different. The exact form of a state and its entanglement properties are straightforward, contrary to its physical realization under quantum circuits. In fact, the quantum circuit for excitation states refers to the graph construction as well. Nevertheless, it is significantly more complex than for the graph states (see Sec. VII for further discussion).

IV. ENTANGLEMENT PROPERTIES

An excitation state $|G\rangle$ forms in general a genuinely entangled state. As we will see, its entanglement properties reflect the structure of the (hyper)graph. We analyze reductions of the N -partite pure state $\rho^G = |G\rangle\langle G|$ onto a bipartite system S and describe its entanglement in terms of the concurrence C [66]. This quantity is a faithful entanglement measure, easy to compute for any two-qubit mixed state and for any bipartition of a multipartite pure state.

For any two-qubit mixed state ρ its concurrence reads [66]

$$C(\rho) := \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (11)$$

in which $\lambda_1, \dots, \lambda_4$ denote square roots of the eigenvalues of a Hermitian matrix $\sqrt{\rho}\tilde{\rho}\sqrt{\rho}$ ordered decreasingly, where

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$$

is the so-called spin-flipped form of ρ . Concurrence forms an entanglement monotone [67], and it is faithful, as it admits strictly positive values for entangled states and it vanishes for separable states.

The generalized concurrence $C_{v|\text{rest}}$ measures the entanglement between the subsystem v and the rest of the system. For pure states, this quantity is determined by the relevant reduced density matrix [66], $C_{v|\text{rest}} = 2 \sqrt{\det \rho_v}$, so its values belong to the range $[0, 1]$.

The distribution of bipartite quantum entanglement, measured by the concurrence among N qubits, satisfies a monogamy inequality [68,69]:

$$C_{v_1 v_2, \dots, v_N}^2 \geq C_{v_1 v_2}^2 + \dots + C_{v_1 v_N}^2, \quad (12)$$

where $v_2, \dots, v_N$ are vertices relevant to subsystems. In that sense, the squared concurrence, also called a *tangle* [66], correctly quantifies bipartite entanglement in multipartite systems, inasmuch as the entanglement between subsystem v_1 and the rest of the system does not exceed the sum of entanglement between v_1 and any other qubit.

For any subsystem S of the studied system we define its complementary subsystem $\bar{S}$. For any vertex $v \in S$ we introduce the set $S_v := S \setminus v$. First, we show that, for any subsystem S and any distinguished particle $v \in S$ which is far enough from the rest of the subsystem S , the reduced state $\rho_S^G := \text{Tr}_{\bar{S}} \rho^G$ separates across the partition $v|S_v$. Moreover, we show that this is a necessary but not sufficient condition for separability in any subsystem. Second, we provide exact formulas relating the two-party concurrence C_{vw} between any two particles, v and w , with the generalized concurrence $C_{v|\text{rest}}$ between particle v and the rest of the system. We are going to use this result for quantifying entanglement of various examples in Sec. V.

As we will see, inequality (12) is generally not saturated, which implies the presence of multipartite entanglement.

Finally, we discuss separability of an excitation state $|G\rangle$ and provide necessary and sufficient conditions for separability in terms of edge structure of the relevant graph. Our results show that an excitation state $|G\rangle$ is typically strongly entangled.

We restrict the analysis to the k -uniform hypergraphs, and assume our graphs to be connected. Not-connected graphs are relevant for the tensor product of two excitation states, and hence might be analyzed separately. By distance d between vertices v_0 and w , we understand the minimal number of vertices $v_1, \dots, v_d$, such that there exist (hyper)edges $e_1, \dots, e_d$:

$$v_{i-1}, v_i \in e_i$$

and $v_d = w$.

Consider a subsystem S with one distinguished particle v and suppose that the distance between corresponding nodes in the graph is $d(v, w) > 2$ for any $w \in S_v = S \setminus v$. The reduced density matrix ρ_S^G is separable across the partition $v|S_v$. Indeed, one can show that

$$\begin{aligned} \rho_S^G &\propto d_v |1\rangle_v \langle 1| \otimes |0 \dots 0\rangle_{v^c} \langle 0 \dots 0| \\ &+ |0\rangle_v \langle 0| \otimes \underbrace{\left(\rho_{S_v}^G - d_v |0 \dots 0\rangle_{v^c} \langle 0 \dots 0| \right)}_{\text{all terms are non-negative}}, \end{aligned}$$

where d_v denotes the degree of the relevant vertex. Obviously, the presented form of the reduced density matrix is completely separable. We conclude it in the following lemma.

Lemma 1. The reduced density matrix ρ_G^S of a subsystem $S = v \cup S'$ is completely separable with respect to the partition $v|S'$ for any particle v such that $d(v, S') > 2$.

The *degree* d_v of the vertex v is the number of edges on which v is incident. The *joint neighborhood* n_{vw} of two vertices is the number of sets W such that both $W \cup v$ and $W \cup w$ constitute an edge. The *section* s_{vw} is the number of edges on which v and w are incident. Note that for graphs, the joint neighborhood is simply the number of vertices adjacent to both v and w , while section $s_{vw} = 1$ or $s_{vw} = 0$ depending whether vertices v and w are connected.

Consider now the reduction of a pure state, $\rho^G = |G\rangle\langle G|$, to the subsystem $S = \{v, w\}$ consisting of two parties. Careful analysis of a (hyper)graph structure yields the following form of reduced density matrix $\rho_{vw}^G = \text{Tr}_{\bar{S}} \rho^G$:

$$\rho_{vw}^G = \frac{1}{|E|} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & d_v - s_{vw} & n_{vw} & 0 \\ 0 & n_{vw} & d_w - s_{vw} & 0 \\ 0 & 0 & 0 & s_{vw} \end{pmatrix}, \quad (13)$$

where the first element $\lambda = |E| - d_v - d_w + s_{vw}$. By the positive partial transpose (PPT) test the reduced state ρ_{vw}^G is entangled iff $n_{vw}^2 > \lambda s_{vw}$, while the exact amount of entanglement can be characterized by the concurrence.

To simplify the notation, for a given excitation state $|G\rangle$ and two selected subsystems v and w , the concurrence of the reduced state ρ_{vw}^G will be denoted as

$$C_{vw} := C(\rho_{vw}^G). \quad (14)$$

We compute the two-party concurrence C_{vw} according to Eq. (11):

$$C_{vw} = \max \left\{ 0, \frac{2}{|E|} (n_{vw} - \sqrt{s_{vw}\lambda}) \right\}, \quad (15)$$

where $\lambda = |E| - d_v - d_w + s_{vw}$.

Since concurrence is a faithful entanglement measure for two-party systems [67,70], the positivity of the right-hand side of the above equation is necessary and sufficient for entanglement between subsystems v and w . Therefore, we first analyze the case for which Eq. (15) reduces to zero. Notice that the concurrence trivially vanishes if $n_{vw} = 0$, which means that v and w have no common neighborhood. This condition is fulfilled precisely in two situations, either $d(v, w) = 1$ or $d(v, w) > 2$. The above observation matches the previous discussion that the system $\rho_{vS'}^G$ always separates if $d(v, S') > 2$. Consider now the case in which Eq. (15) takes strictly positive values. Indeed, for v and w of distance $d(v, w) = 2$, the neighborhood $n_{vw} \neq 0$ and the section $s_{vw} = 0$. We summarize the above discussion in the following.

Lemma 2. The reduced two-party state ρ_{vw}^G is entangled iff $d(v, w) = 2$ or $d(v, w) = 1$ and

$$n_{vw}^2 > s_{vw}(|E| - d_v - d_w + s_{vw}).$$

Roughly speaking, we observe the entanglement between parties v and w if they share common neighbors. On the other hand, entanglement can disappear if both nodes become the closest neighbors. The inequality above is usually satisfied in almost complete (hyper)graphs. For graphs with a local-like structure, it will be violated. In such a case the bipartite entanglement will be present only for two distant parties. Some statements in this spirit are presented below.

Recall that a hypergraph might be identified with a pair (V, E) . We define the *product* of two disjoint hypergraphs $(V_1, E_1) \sqcup (V_2, E_2)$ as a hypergraph $(V_1 \cup V_2, E_1 \sqcup E_2)$ with vertices being the union $V_1 \cup V_2$ of vertex sets and with edges of the following form:

$$E_1 \sqcup E_2 := \sum_{e_1 \in E_1, e_2 \in E_2} e_1 \cup e_2.$$

In short, we say that such a hypergraph is a *product hypergraph* with respect to the division $V_1|V_2$. We derive the following criterion for separability of the excitation state $|G\rangle$ corresponding to a k -hypergraph in terms of separation of its edges.

Proposition 2. The excitation state $|G\rangle$ corresponding to a k -hypergraph is separable, $|G\rangle = |G\rangle_{V_1} \otimes |G\rangle_{V_2}$, iff it is a product hypergraph with respect to the division $V_1|V_2$.

Proof. On one hand, it is a straightforward observation that for a product hypergraph $(V_1 \cup V_2, E_1 \sqcup E_2)$ the relevant excitation state $|G\rangle$ is separable across the division $V_1|V_2$, i.e., $|G\rangle = |G\rangle_{V_1} \otimes |G\rangle_{V_2}$.

On the other hand, the following two conditions are necessary for the excitation state $|G\rangle$ to be separable:

(1) The number of excitations in V_1 and V_2 is the same; i.e., there are numbers c_1, c_2 such that $c_1 + c_2 = k$ and for each edge $e \in E$, $e \cap V_i = c_i$.

(2) The set of edges factors across the partition $V_1|V_2$; i.e., there are sets $E_1 \in P(V_1)$, $E_2 \in P(V_2)$, such that $E = \sum_{e_1 \in E_1, e_2 \in E_2} e_1 \cup e_2$.

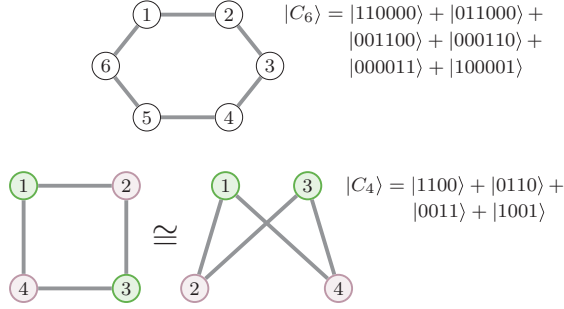
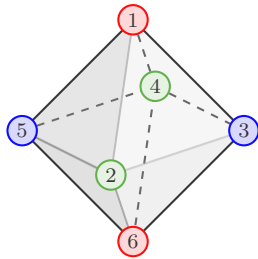


FIG. 4. Two cycle graphs: C_4 and C_6 with the corresponding states $|C_4\rangle$ and $|C_6\rangle$ listed on the right. Observe that the cycle C_4 forms a complete bipartite graph, $C_4 = K_{2,2}$. According to Corollary 1, $|C_6\rangle$ is genuinely entangled while $|C_4\rangle$ is separable, as $|C_4\rangle = (|01\rangle + |10\rangle)_{13} \otimes (|01\rangle + |10\rangle)_{24}$.

One may observe that those two conditions are equivalent to the fact that G is the product hypergraph. ■

Some exemplary separable excitation states are presented in Figs. 4 and 5. Verification of the above criterion is straightforward. Observe that for excitation states $|G\rangle$ the separability criterion is fulfilled by the complete bipartite graph only. Indeed, the first condition simplifies due to the fact that there might be edges between $V_1 \cup V_2 = V$ only. The second condition comes to completeness of the bipartite graph under the assumption of connectivity of the graph. Following the usual notation in graph theory, let $K_{V_1 V_2}$ be a *complete bipartite graph*, while $G = K_{V_1, \dots, V_k}$ is a *complete multipartite k -hypergraph*. Recall that a multipartite (k -partite) hypergraph G is a hypergraph whose vertices can be partitioned into k different independent sets. Equivalently, it is a hypergraph that can be colored with k colors. As a consequence, we obtain the following three corollaries.

Corollary 1 (Only for graphs). The excitation state $|G\rangle$ is separable, $|G\rangle = |G\rangle_{V_1} \otimes |G\rangle_{V_2}$, iff the relevant graph G is a complete bipartite graph, $G = K_{V_1 V_2}$. In fact, such a state forms



$$(|10\rangle + |01\rangle)_{16} \otimes (|10\rangle + |01\rangle)_{24} \otimes (|10\rangle + |01\rangle)_{35}$$

FIG. 5. 3-regular hypergraph on six vertices. Each hyperedge is represented by a triangle. Observe that the presented hypergraph is complete tripartite hypergraph $K_{2,2,2}$. Indeed, vertices 1,6 and 2,4 and 3,5 are colored differently according to the partition. All hyperedges containing vertices of different colors are present. According to Corollary 3 the state becomes separable across partition 16|24|35, as it is indicated in the bottom.

a tensor product of two $|W\rangle$ -like states:

$$|G\rangle = |W\rangle_{V_1} \otimes |W\rangle_{V_2}.$$

We reformulate these separability criteria in terms of symmetries of the Dicke-like states.

Corollary 2. A Dicke-like state $|D_N^2\rangle_H$ with two excitations is separable across the bipartition $N_1|N_2$ iff its group symmetry is equal to $\mathcal{S}_{N_1} \times \mathcal{S}_{N_2}$.

We can derive a similar conclusion to Corollary 1 in the case of k -hypergraphs and multiseparability.

Corollary 3. The k -regular excitation state $|G\rangle$ separates $|G\rangle = |G\rangle_{V_1} \otimes \dots \otimes |G\rangle_{V_k}$ iff the relevant hypergraph is a complete multipartite hypergraph $G = K_{V_1, \dots, V_k}$. Such a state is a tensor product of k states from the class $|W\rangle$,

$$|G\rangle = |W\rangle_{V_1} \otimes \dots \otimes |W\rangle_{V_k}.$$

We discuss now how tight are the monogamy relations (12) for parties in an excitation state. Recall that if the aforementioned bound is saturated, the entanglement of a given part has the bipartite form with other parties; otherwise multipartite entanglement can be observed.

For further discussion, we assume *regularity* of a graph, which means that the degree d_v is constant for any vertex v . Moreover, we assume an additional condition: If vertices v and w are connected, they are connected with the same number of hyperedges. In other words, the section s_{vw} takes the same values depending on the distance between vertices:

$$s_{vw} = \begin{cases} s & \text{for } d(v, w) = 1 \\ 0 & \text{for } d(v, w) > 1. \end{cases}$$

We call such a graph a *distance-1 regular graph*. In particular, for standard graphs this value reads 1 for neighboring nodes and zero otherwise. We mention here a particular class of graphs widely discussed in the graph theory. A *distance-regular graph* [71] is a regular graph such that for any two vertices v and w , the number of vertices at distance j from v and at distance ℓ from w depends only upon j, ℓ , and the distance $d(v, w)$. Our assumptions on (hyper)graphs correspond to distance-regular graphs with distances $d(v, w) = 1$ and $d(v, w) = 2$. We conjecture that regularity concerning the vertices of larger distances, $d(v, w) > 2$, yields regularity in the amount of entanglement in subsystems consisting of three or more parties. Unfortunately, for larger systems we have no satisfactory entanglement quantification [39] required to formalize this statement. From Eq. (15), we derive an expression for the concurrence C_{vw} between subsystems corresponding to vertices of a distance-1 regular graph G .

Proposition 3. For a connected and regular graph G the concurrence C_{vw} between two nodes v and w reads

$$C_{vw}(G) = \frac{2}{|E|} \begin{cases} \max\{0, C\} & \text{for } d(v, w) = 1 \\ n_{vw} & \text{for } d(v, w) = 2 \\ 0 & \text{for } d(v, w) > 2, \end{cases} \quad (16)$$

where $C = n_{vw} - \sqrt{s(|E| - 2d + s)}$, d is the degree of each node, and s is a section for each adjacent vertex.

Elementary calculations lead to the following result.

Proposition 4. The square of the generalized concurrence, $C_{v|\text{rest}}^2$, between the particle v and the rest of the system,

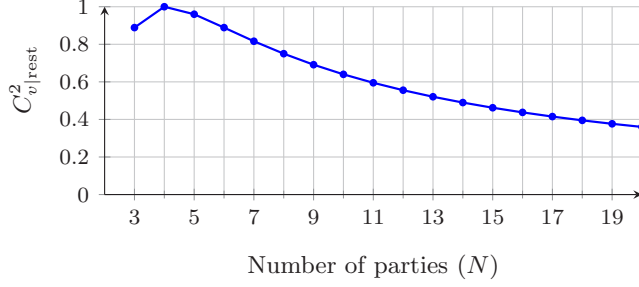


FIG. 6. Squared concurrence (2-tangle) $C_{v|rest}^2$ between particle v and the rest of the system depends on the number of vertices, N , and uniformity k only, so it takes the same value for an arbitrary party of a regular k -hypergraph. In general, the tangle $C_{v|rest}^2$ takes the maximal value for $k = N/2$. For instance, in the case of graphs, it takes the maximal permissible value for concurrence for regular graphs on $N = 4$ vertices. Asymptotically, $C_{v|rest}^2 \sim 8/N$.

expressed as a function of the number of edges $|E|$ and the number of vertices N , reads

$$C_{v|rest}^2 = 4 \frac{d_v(|E| - d_v)}{|E|^2} = 4 \frac{k(N - k)}{N^2} \quad (17)$$

(see Fig. 6). The second equation is valid under the assumption of the regularity of a k -hypergraph; i.e., the degrees of vertices are the same.

V. EXAMPLES OF DICKE-LIKE STATES

In previous sections we presented two similar, but different, constructions of genuinely entangled states: excitation states $|G\rangle$, related to a graph G , and Dicke-like states, determined by a subgroup, which restricts the summation in the classical expression of Dicke. The (hyper)graph construction is by far more general, as it leads to quantum circuits corresponding to excitation states $|G\rangle$. A Dicke-like state can be considered a special case of the excitation states, which exhibits a certain symmetry structure.

We combine both representations and investigate highly symmetric objects, such as regular polygons, Platonic solids, and regular plane tilings. It is worth mentioning that such symmetric objects, especially Platonic solids, were already used in various contexts concerning multipartite entanglement, including quantification of entanglement of permutation-symmetric states [47,56], identification of quantumness of a state [72], the search for the maximally entangled symmetric state [46], or general geometrical quantification of entanglement [48,73] especially among states with imposed symmetries on the roots of Majorana polynomials [55,56].

We present particular examples of quantum states shared among N parties positioned in a highly symmetric way. By using results from Sec. IV, we discuss their entanglement properties. First, we computed the concurrence in bipartite subsystems for all presented examples. Recall, that for a given number of excitations k , the entanglement shared between a particular node v and the rest of the system in an excitation state depends only on the number of parties, N ,

$$C_{v|rest}^2 = 4 \frac{k(N - k)}{N^2}, \quad (18)$$

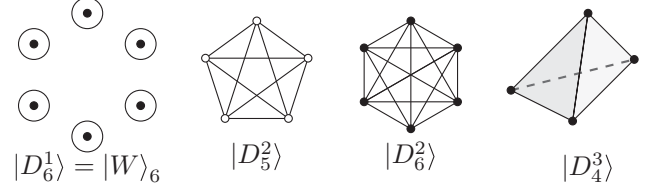


FIG. 7. Dicke state $|D_N^k\rangle$ is associated with a complete k -regular hypergraph, while a 1-uniform complete hypergraph leads to the state $|W_N\rangle$. The graph and the corresponding state are completely symmetric, so the labels of the nodes can be omitted. Dicke states $|D_N^k\rangle$ with two excitations are related to complete graphs on N vertices. A tetrahedron represents the Dicke states $|D_4^1\rangle$, $|D_4^2\rangle$ and $|D_4^3\rangle$, depending on whether we consider the vertices, the edges, or the faces of the tetrahedron. Each face represents the hyperedge of a 3-uniform complete hypergraph on four vertices, relevant to $|D_4^3\rangle$. The graph formed by edges of a tetrahedron forms the complete graph on four vertices, relevant to $|D_4^2\rangle$. Four vertices of the tetrahedron provide a 1-uniform graph, related to $|D_4^1\rangle$. This generalizes to the fact that the simplex of dimension $N - 1$ represents the full family $\{|D_N^k\rangle\}_{k=1}^{N-1}$ of Dicke states of N vertices.

as it was shown in Eq. (17). We define the *entanglement ratio* Γ_v for the node v as

$$\Gamma_v := \frac{\sum_{i \neq v} C_{v|i}^2}{C_{v|rest}^2} \in [0, 1], \quad (19)$$

which measures the ratio of entanglement shared between particular parties in a bipartite way in comparison to the amount of entanglement shared in the multipartite way. Since concurrence satisfies monogamy inequality (12), the parameter Γ_v takes values in the range $[0, 1]$. Figure 13 compares the entanglement ratio of various examples in the context of the application of provided states.

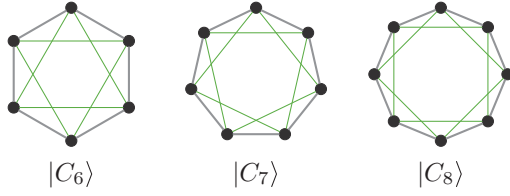
A. Dicke states

The Dicke states $|D_N^k\rangle$ are excitation states for complete k -regular hypergraphs on N vertices (see Fig. 7). Their group symmetry is the full permutation group $\mathcal{S}_N$. In a particular case, if the number of excitations is equal to 1, the Dicke state coincides with the state $|W_N\rangle$ and the bound in Eq. (12) is tight [69].

Consider any vertex v and its neighbor w of a complete k -regular hypergraph. The parameters are as follows: the section is $s = \binom{N-2}{k-2}$, the degree $d = \binom{N-1}{k-1}$, neighborhood $n_{vw} = \binom{N-2}{k-1}$, and the number of all hyperedges $|E| = \binom{N}{k}$. By eqs. 16 and 17, and elementary calculation, the concurrence in bipartite subsystems takes the following value:

$$C_{v|w} = 2 \binom{N}{k}^{-1} \left(\binom{N-2}{k-1} - \sqrt{\binom{N-2}{k} \binom{N-2}{k-2}} \right).$$

Observe that the formula above is invariant with respect to the change $k \leftrightarrow N - k$. Indeed, Dicke states $|D_N^k\rangle$ and $|D_N^{N-k}\rangle$ are equivalent up to local change of basis $|0\rangle \leftrightarrow |1\rangle$; hence they share the same entanglement properties. The entanglement



$$|C_7\rangle = |1100000\rangle + |0110000\rangle + |0011000\rangle + |0001100\rangle + |0000110\rangle + |0000011\rangle + |1000001\rangle$$

FIG. 8. Three cycle graphs: C_6 , C_7 , and C_8 indicated by gray edges. One of the corresponding states, $|C_7\rangle$, is listed below. Entanglement on a two-party subsystem is vanishing except for parties of distance 2, $d(v, w) = 2$, for which the concurrence C_{vw} reads $2/6$, $2/7$, and $2/8$ for states $|C_6\rangle$, $|C_7\rangle$, and $|C_8\rangle$, respectively. Nonvanishing concurrence is indicated by green lines connecting relevant parties, so that the graph of entanglement splits into two for an even number of parties, $N = 6$ or 8 .

ratio Γ_v for a given node v is equal to

$$\Gamma_v(D_N^k) = \frac{N-1}{\binom{N-1}{k} \binom{N-1}{k-1}} \left(\binom{N-2}{k-1} - \sqrt{\binom{N-2}{k} \binom{N-2}{k-2}} \right)^2.$$

In a particular case, for $k = 1$, $N = 1$, the entanglement ratio saturates its range, $\Gamma_v = 1$. In these two cases the Dicke states belong to the family $|W\rangle$ and saturate the monogamy bound (12) derived in [68,69].

By Propositions 3 and 4, we get the following.

Corollary 4. For the Dicke states $|D_N^k\rangle$ the entanglement ratio at infinite dimension is nonzero:

$$\lim_{N \rightarrow \infty} \Gamma_v(D_N^k) = 2k - 1 - \sqrt{k(k-1)}.$$

In particular, for states related to graphs, $k = 1$, we find

$$\lim_{N \rightarrow \infty} \Gamma_v(D_N^1) = 3 - 2\sqrt{2}.$$

B. Cyclic states

The simplest nontrivial subgroup of the permutation group S_N is a cyclic group C_N . In general states $|D_N^k\rangle_{C_N}$ are *translationally invariant*, i.e., invariant under a cyclic permutation of qubits [74,75]. The family of such states is widely considered in several one-dimensional (1D) models applied in condensed-matter physics, like the XY model or the Heisenberg model. For further considerations, assume that the number of excitations is equal to $k = 2$. According to Definition 2, we can construct the Dicke-like state $|D_N^2\rangle_{C_N}$ by taking the superposition of all elements:

$$|0 \dots 0110 \dots 0\rangle,$$

where excitations are always on adjacent positions (see Fig. 8).

On the other hand, the state $|D_N^2\rangle_{C_N}$ can be constructed as an excitation state. To this end consider the cyclic graph on N vertices. The relevant excitation state, denoted by $|C_N\rangle$, matches perfectly $|D_N^2\rangle_{C_N}$. The group of automorphisms of a cyclic graph C_N forms a *dihedral* group $\mathcal{D}_{2N}$, where the lower index stands for the number of elements in the group,

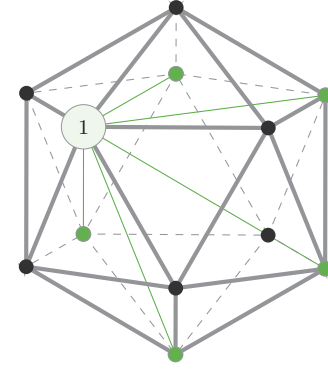


FIG. 9. Icosahedron, one of five Platonic solids. Its edges (indicated with gray) determine the excitation state $|P^e\rangle_{20}$ on 20 qubits. For a chosen node 1, the concurrence of the two-party subsystem C_{1v} is vanishing except for subsystems corresponding to distance-2 nodes indicated by green lines and green nodes.

$|\mathcal{D}_{2N}| = 2N$. It describes the group of symmetries of a regular N -polygon, which consists of rotations and reflections of the figure. Each reflection is an idempotent element; the order of rotations may vary. Using those properties of a cyclic graph C_N , we conclude that the group of symmetries of the quantum state $|D_N^2\rangle_{C_N} = |C_N\rangle$ is a dihedral group $\mathcal{D}_{2N}$. Notice that the systems with dihedral $\mathcal{D}_{2N}$ symmetries were considered in a context of correlation theory of the chemical bond [35]. Molecules invariant under a rotation and inversion were investigated [35]. Figure 10 presents the stellar representation of the six-qubit state $|C_6\rangle$, which serves as an example of the relation between the Dicke-like states and the construction of excitation states.

In order to analyze entanglement properties of $|C_N\rangle$ we calculate the parameters of the cyclic graph C_N . The number of edges $|E| = N$, while for any vertex v its degree reads $d_v = 2$. Any two vertices v and w might be related in three following ways: either they are neighbors and then $s = 1$, $n_{vw} = 0$; or they are of distance $d(v, w) = 2$ and then $n_{vw} = 2$, or the distance $d(v, w) > 0$. The concurrence in bipartite subsystems

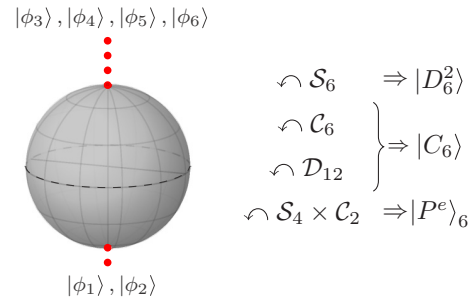


FIG. 10. Stellar representation of the Dicke-like states $|D_6^2\rangle_H$ relevant to the action of various groups $H < S_6$. Acting by the full symmetric group S_6 on a Bloch sphere we obtain the Dicke state $|D_6^2\rangle$. By action of a cyclic group C_6 or the dihedral group $\mathcal{D}_{12}$, we obtain the cyclic state $|C_6\rangle$. Choosing the action of $S_4 \times C_2$, as described in Sec. VC, we obtain the Platonic state $|P^e\rangle_6$.

takes the following value:

$$C_{v|w}(C_N) = \begin{cases} 2/N & \text{for } d(v, w) = 2 \\ 0 & \text{otherwise,} \end{cases}$$

with a small correction for $N = 4$, where $C_{v|w}^2 = 1$ for vertices of distance 2. The entanglement ratio Γ_v for a given node reads

$$\Gamma_v(C_N) = \frac{1}{N-2},$$

with the same correction in the case $N = 4$, for which $\Gamma_v = 1$.

We emphasize that the cyclic and dihedral groups were already successfully used for constructing fully symmetric states with additional (cyclic and/or dihedral) symmetry imposed on roots of related Majorana polynomials [56]. In our work, instead of imposing additional symmetries on roots, we restrict the symmetrization procedure in Eq. (9) to the subgroup $H < \mathcal{S}_N$ of the symmetric group $\mathcal{S}_N$.

C. Platonic states

Platonic solids—highly symmetric geometric objects in $\mathbb{R}^3$ —were used to construct quantum states in various contexts [48,72,73]. With any platonic solid we associate in this work two quantum excitation states. First, we can construct an excitation state simply by looking at the edges of a solid, see Fig. 9. Second, we may associate faces of a solid with hyperedges of a hypergraph. For instance, the tetrahedron is linked to the Dicke state $|D_4^2\rangle$ (by reading edges) and $|D_4^3\rangle$ (by reading faces) (see Fig. 7).

We use the notation $|P^e\rangle_N$ for Platonic states constructed by looking at the edges of a solid, while $|P^f\rangle_N$ refers to the construction involving faces (N denotes the number of vertices in a related Platonic solid).

Due to the extraordinary symmetry of Platonic solids, it is not surprising that both constructions of quantum states lead to Dicke-like states. In order to show it, we need to find appropriate groups, which action yields Platonic states. Consider the automorphism group of relevant graphs or hypergraphs. An elementary argument from the representation theory shows that it is determined by the group of symmetries of the Platonic solid.

Fact 1. Consider a Platonic solid. Each permutation $\sigma \in \mathcal{S}_{|V|}$ of vertices which preserves the structure of faces or edges represents a symmetry of the solid.

Since groups of symmetries of Platonic solids are transitive on edges and faces, any such state can be realized by an action of the aforementioned group.

There exist five Platonic solids: a self-dual tetrahedron and two dual pairs of the same group of symmetries: (a) cube and octahedron and (b) dodecahedron and icosahedron. Any symmetry of a solid determines a permutation $\sigma \in \mathcal{S}_N$ on vertices. For a tetrahedron, the reverse statement holds true. For the first dual pair, consider a cube. Symmetries of the cube coincide with permutations of its four diagonals plus one nonrotation symmetry, the point reflection with respect to the center. For the second dual pair, consider a dodecahedron. One can embed five cubes in the dodecahedron, forming the compound of five cubes. Any rotation of dodecahedron determines an even permutation of the cubes. As in the previous case, there exists

TABLE I. Concurrence C_{12} and entanglement ratio Γ_v for two-qubit systems obtained by partial trace of $N - 2$ subsystems of distance 2 of Platonic states determined by the edges of the solid.

	$ P^e\rangle_4$	$ P^e\rangle_6$	$ P^e\rangle_8$	$ P^e\rangle_{12}$	$ P^e\rangle_{20}$
Concurrence, C_{12}	0.333	0.667	0.333	0.133	0.067
Entanglement ratio, Γ_v	0.333	0.500	0.444	0.160	0.074

additionally a reflection symmetry with respect to the center. This classical analysis implies the following statement.

Observation 3. The symmetry group of Platonic states $|P^e\rangle_4$ and $|P^f\rangle_4$ related to the tetrahedron is the full symmetric group $\mathcal{S}_4$. Indeed,

$$|P^e\rangle_4 = |D_4^2\rangle, \quad |P^f\rangle_4 = |D_4^3\rangle.$$

The symmetry group of $|P^e\rangle_6$ and $|P^f\rangle_6$ related to the octahedron, and $|P^e\rangle_8$ and $|P^f\rangle_8$ related to the cube, is

$$\mathcal{S}_4 \times \mathcal{S}_2,$$

nontrivially embedded into $\mathcal{S}_6$ and $\mathcal{S}_8$, respectively, see Fig. 10. The symmetry group of states $|P^e\rangle_{12}$ and $|P^f\rangle_{12}$ related to the dodecahedron, and $|P^e\rangle_{20}$, $|P^f\rangle_{20}$ related to the icosahedron, is

$$\mathcal{A}_5 \times \mathcal{S}_2,$$

nontrivially embedded into $\mathcal{S}_{12}$ and $\mathcal{S}_{20}$, respectively.

Note that the Platonic states listed above form nontrivial examples of symmetric states. In particular, in the case of the dodecahedron the alternating symmetry $\mathcal{A}_5$ is observed, which is not easy to construct.

For Platonic states determined by the edges of the solid, the concurrence in bipartite subsystems takes a positive value only for nodes of distance 2. For such nodes 1 and 2, the concurrence C_{12} and the entanglement ratio Γ_v are presented in Table I.

D. Regular m -polytope families

The line of thinking applied to the Platonic solids can be extended to the regular polytopes in higher dimensions. Such families include the self-dual m -simplices and m -hypercubes with dual m -orthoplexes. Each of these polytopes provides us with a set of k -uniform hypergraphs for $1 \leq k \leq m - 1$ defined by the set of their k -dimensional hyperedges. For the sake of this section we will denote the states corresponding to the k -dimensional hyperedges of an m -simplex as $|S_m^k\rangle$ and analogously $|B_{2m}^k\rangle$ for m -hypercubes and $|O_{2m}^k\rangle$ for m -orthoplexes, where the lower index stands for the number of subsystems, N , in the state ($N = m, 2^m$, and $2m$ for m -simplex, m -hypercube, and m -orthoplex, respectively).

First, we may state a simple observation concerning m -simplices.

Observation 4. The states related to the m -simplices are equivalent to the Dicke states,

$$|S_m^k\rangle = |D_m^k\rangle,$$

and the symmetry group is trivially $\mathcal{S}_m$.

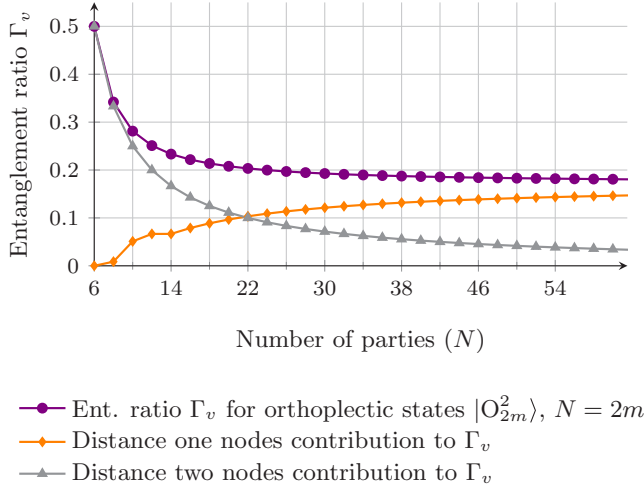


FIG. 11. The entanglement ratio Γ_v quantifies the ratio of the bipartite and multipartite entanglement of a particular subsystem v . The distribution of Γ_v for the family $|O_{2m}^2\rangle$ of states with $N = 2m$ parties and $k = 2$ excitations, related to the m -orthoplexes, is presented. The contribution from distance-2 vertices is dominating for small systems. For larger systems the contribution from distance-2 nodes dominates Γ_v , while the related graph becomes almost complete. The entanglement ratio at infinity goes to a nonzero value, $3 - 2\sqrt{2} \approx 0.17$.

The symmetry group of $|B_{2m}^k\rangle$ and $|O_{2m}^k\rangle$ is given by the nontrivial hyperoctahedral group B_m . We may first formulate a simple observation about the orthoplectic states.

Observation 5. The states $|O_{2m}^{m-1}\rangle$ are separable with respect to the partition $12|34|\dots|(2m-1)2m$, with each pair of vertices being distance 2 or, in other words, lying on a common diagonal of the related m -orthoplex.

Furthermore, one can easily calculate the concurrence C_{vw} for the states $|O_{2m}^2\rangle$ connected to the 2-edges of the m -orthoplex,

$$C_{vw}(O_{2m}^2) = \begin{cases} \max\{0, \frac{\sqrt{2m^2-4m+3}-2m+4}{2m-2m^2}\} & \text{for } d(v, w) = 1 \\ 2/m & \text{for } d(v, w) = 2 \\ 0 & \text{for } d(v, w) > 2, \end{cases} \quad (20)$$

and similarly the entanglement ratio

$$\Gamma_v(O_{2m}^2) = \begin{cases} 1/(m-1) & \text{for } m \leq 3 \\ \frac{6m^2-4m(\sqrt{2m^2-6m+5}+5)+8\sqrt{2m^2-6m+5}+19}{2(m-1)^2}, & \text{for } m > 3, \end{cases} \quad (21)$$

where the first case holds for $m \leq 3$ and the second one for $m > 3$. By Propositions 3 and 4, we get the following.

Corollary 5. Similarly to Dicke states, the entanglement ratio for the $|O_{2m}^2\rangle$ states converges to a nonzero value:

$$\lim_{n \rightarrow \infty} \Gamma_v(O_{2m}^2) = 3 - 2\sqrt{2} \approx 0.1716. \quad (22)$$

Interestingly, for small values m the main contribution to this value comes from distance-2 nodes, while for large values of m from distance-1 nodes, see Fig. 11.

The situation is much simpler for the hypercubic states $|B_{2m}^2\rangle$, where the concurrence occurs only between the

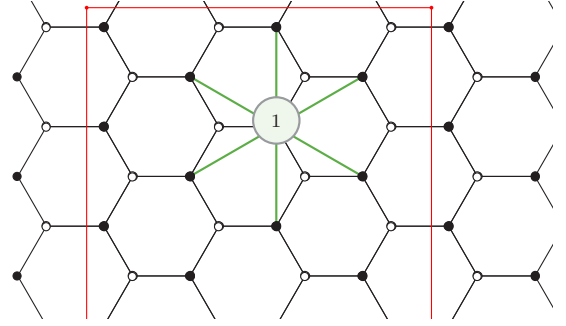


FIG. 12. Hexagonal tiling of a plane. One may choose a rectangular region (indicated by the red line) and glue the boundary left-right and up-down to form a torus. In this specific choice of the rectangular cut, there are four nodes in each row and six in each column. For chosen node v , the concurrence on the two-party subsystem C_{1v} is vanishing except for distance-2 parties (indicated by green lines). In such a case $C_{1v} = 1/12$, and the entanglement ratio $\Gamma_v \approx 0.06$. For the similar tiling by triangles (restriction to 25 nodes), the entanglement ratio takes the value $\Gamma_v \approx 0.072$. Even though the triangular network is twice as dense as the hexagonal, for such a network much more entanglement is concentrated in the bipartite subsystems of distance 2.

distance-2 vertices, $C_{12}(B_{2m}^2) = 2^{3-m}/m$, while the entanglement ratio reads

$$\Gamma_v(B_{2m}^2) = \frac{4(m-1)}{(2^m-2)m},$$

which asymptotically tends to zero.

E. Plane regular tilings

Other highly symmetric objects, including regular and semiregular tessellation of the plane, can also be used to generate symmetric quantum states. In the regular tiling, the plane is covered by regular polygons, all of the same shape. In semiregular tiling, tiles of more than one shape are allowed but every corner is identically arranged. Regular tiling provides an edge-transitive graph, while semiregular tiling leads to a vertex-transitive graph. Even though the local properties of such tiling might be various, the global pattern itself can be categorized into 17 wallpaper groups.

Each such a group consists of two types of generators: shifts and reflections. In general, tessellations take the form of infinite patterns, but one may restrict to the arbitrary regular region by appropriate gluing on the boundary. In such a way, we obtain a finite tiling, which might be related to excitation states under construction provided. By the property of edge transitivity the regular tilings correspond to Dicke-like states. The symmetry group is given by a relevant wallpaper group, restricted to the chosen size of the tiling. Since semiregular tilings do not exhibit edge transitivity, they in general do not correspond to Dicke-like states.

Figure 12 presents the hexagonal tiling of the plane and relevant excitation state $|H_N\rangle$. For hexagonal tiling, the concurrence in bipartite subsystems takes the following value:

$$C_{v|w}(H_N) = \begin{cases} 2/N & \text{for } d(v, w) = 2 \\ 0 & \text{otherwise,} \end{cases}$$

for the tiling restricted to N nodes (minimal size of a cut is 3×3). The entanglement ratio Γ_v for a given node v reads

$$\Gamma_v(H_N) = \frac{4}{3} \frac{1}{N-2}.$$

Even though the value of the bipartite concurrence $C_{v|w}$ is the same as in the cyclic case, the parameter Γ_v takes a larger value. Indeed, each node in the hexagonal case has six distance-2 vertices, which causes the increase of the entanglement ratio Γ_v .

F. General local network

The approach presented allows one to construct multipartite quantum states associated to local networks. We assume that certain properties of the network are the same at each node. In particular, all vertices have the same degree d . Our analysis is precise for a sufficiently large number of nodes, N . By locality, we understand that the number of edges $|E|$ scales as the number of vertices, $|E| = O(N)$. This assumption leads to vanishing of the concurrence for all bipartite reduced systems corresponding to two vertices at distance 1. For simplicity we restrict our attention to the case of graph networks; i.e., edges connect always two vertices.

For a given node labeled as v , define the following function:

$$\gamma_v := \sum_{v': d(v, v')=2} n_{vv'}^2, \quad (23)$$

where the sum runs over all distance-2 vertices, while the joint neighborhood $n_{vv'}$ (defined in Sec. IV) simplifies here to the number of paths of length 2 from v to v' . Observe that by regularity, the number of paths connecting distance-2 vertices takes always the same value:

$$\sum_{v': d(v, v')=2} n_{vv'} = d(d-1).$$

The possible displacement of distance-2 nodes responds for the parameter Γ_v for the given network. Indeed, from Propositions 3 and 4 follows that

$$\Gamma_v(\text{network}) = \frac{2}{N-2} \frac{1}{d^2} \gamma_v,$$

and the only nonglobal parameter above is γ_v . Therefore, optimization of the entanglement ratio Γ_v is equivalent to optimization of γ_v . In order to maximize Eq. (23), distance-2 paths from node v should lead to the minimal number of nodes. Conversely, in order to minimize (23), the distance-2 nodes should be connected by a single path only. For instance, Fig. 9 presents the network in which all vertices of distance 2 are connected by two paths, while in the network presented in Fig. 12 all such vertices are connected by a unique path.

Observation 6. For a regular local network, i.e., $|E| = O(N)$, the entanglement ratio Γ_v of relevant excitation states has the following lower bound:

$$\Gamma_v(\text{local network}) \geq \frac{2}{N-2} \frac{d-1}{d}.$$

Moreover, it has the following asymptotic complexity: $\Gamma_v = O(1/N)$.

Observe that the bound above is tight for cyclic states $|C_N\rangle$ and for states $|H_N\rangle$ related to the hexagonal tiling. Hypercubic states $|B_{2m}^2\rangle$ achieve exactly twice this bound.

Remarkably, the entanglement ratio Γ_v is strongly related to the curvature of the network. For a flatlike network one can expect that the number of distance-2 nodes will be twice larger than the number of distance-1 nodes. This is the case of the hexagonal tiling (see Fig. 12) and all the regular tessellations but not all semiregular tilings. By assumption of flatness (understood as above), and that the same number of paths connects all distance-2 vertices, equal to $(d-1)/2$, the entanglement ratio Γ_v can be estimated as follows:

$$\Gamma_v(\text{flat local network}) \simeq \frac{1}{N-2} \frac{(d-1)^2}{d}.$$

Notice that the above estimation is not precise if the number of paths connecting distance-2 vertices is not uniformly distributed. Observe that hypercubic states $|B_{2m}^2\rangle$ are not relevant to the flat networks, and indeed, their entanglement ratio scales differently,

$$\Gamma_v(B_{2m}^2) = \frac{4}{2^m-2} \frac{(m-1)}{m} = \frac{4}{N-2} \frac{(d-1)}{d},$$

with respect to the local degree d parameter.

VI. PHASE TRANSITIONS

As discussed before, the entanglement ratio Γ_v measures the amount of entanglement not present in the bipartite entanglement. Therefore, it indicates the robustness of an entangled state. We investigate the asymptotic behavior of the parameter Γ_v of an excitation state $|G\rangle$ with the change of the local degree d of the relevant hypergraph.

For simplicity consider Dicke states with two excitations, $|D_N^2\rangle$. According to exact calculations, the asymptotic entanglement ratio Γ_v is

$$\Gamma_v(D_N^2) \sim (3 - 2\sqrt{2}).$$

Recall that for local networks one has $\Gamma_v \gtrsim d/N$. Therefore, the complexity of entanglement ratio Γ_v for Dicke states and a general local network is not the same, $O(N)$ and $O(1/N)$, respectively.

This phenomenon is related to the fact that for regular networks, the concurrence $C_{v|w}$ between two parties may take two distinct nonvanishing values according to the distance $d(v, w)$ (see Propositions 3). While nonvanishing concurrence is always present for distance-2 nodes, for neighboring nodes it takes a positive value, iff $n_{vw}^2 \geq (d(N-4) + 2)/2$. Since the number of common neighbors, n_{vw} , is bounded by the degree d , it might occur only if $d \gtrsim N/2$. Notice that the regular graph satisfying this inequality is highly connected since at least half of all possible edges are present. In such a highly connected graph, the vertices are of distance 1 or 2. Therefore, the joint neighborhood might be estimated by $n_{vw} \sim d^2/N$ (under the assumption of random distribution of edges). Hence, by Propositions 3, for a graph with vertices of degree

$$d = \frac{1}{\sqrt[3]{2}} N \simeq 0.794N,$$

we observe a first phase transition, in which the entanglement between distance-1 parties starts to appear. At this moment, however, the entanglement between such parties is negligible in comparison to the entanglement between distance-2 vertices.

The second phase transitions refer to the threshold in which entanglement between distance-1 vertices is protruding over the entanglement present between vertices of distance 2. We quantify this intuition by comparing the contribution of those two distinct types of bipartite concurrence on the right-hand side of Eq. (12). Regarding the previous discussion, this phenomenon takes place for a graph with vertices of degree

$$d(n_{vw} - \sqrt{dN/2})^2 \approx (N - d)n_{vw}^2.$$

Taking into account that $n_{vw} \sim d^2/N$ for such strongly connected graphs, the second phase transition takes place for

$$d \approx 0.973N,$$

which was computed numerically. We summarize the discussion in the following statement.

Observation 7. Consider the asymptotic behavior of an excitation state $|G\rangle$ corresponding to a graph of N vertices with respect to the average degree of each vertex equal to d . Two phase transitions occur: The first one, for the average vertex degree $d = 0.794N$, refers to the threshold in which the entanglement between distance-1 parties starts to appear. The second one, for the average vertex degree $d = 0.973N$, refers to the threshold in which entanglement between distance-1 vertices is protruding over the entanglement present between vertices of distance 2.

Note that there is no substantial difference between local networks with average degree d of each vertex being a finite number and global networks with the degree d depending on N , until the limit $d \simeq 0.794N$ is achieved. Indeed, in both cases, the entanglement ratio $\Gamma_v \sim d/N$, and the entanglement is present only between distance-2 nodes. Once this limit is exceeded, the bipartite entanglement between distance-1 nodes starts to appear. Its amount is, however, negligible in comparison to the entanglement between distance-2 vertices.

Entanglement between distance-1 nodes prevails once the threshold $d \approx 0.973N$ is exceeded. Since the entanglement between distance-1 nodes is significantly weaker than the entanglement corresponding to nodes at distance 2, this leads to a global consequence in entanglement distribution. Indeed, the entanglement ratio Γ_v (adequate to amount of entanglement in bipartite systems in comparison to the amount of entanglement shared in a multipartite way) changes its scaling from $\Gamma_1 \sim d/N$ to $\Gamma_1 \sim (3 - 2\sqrt{2})$. Therefore, the resulting states share much less bipartite entanglement. Hence the global amount of entanglement remains on the same level; states become more robust and persistent, which is perfectly observed at the limit case of the Dicke states.

The aforementioned transitions are clearly visible for the $|O_{2m}^2\rangle$ family of states, related to the m -orthoplexes (see Fig. 11). Indeed, for a small number of particles, $N = 2m$, the entanglement is mainly present between distance-2 vertices. Already for an $N = 8$ particle system $|O_8^2\rangle$, modest entanglement between distance-1 nodes can be detected. The contribution from distance-1 vertices becomes dominating for larger

systems, starting from systems with $N = 22$ particles. This reflects the fact that for a large number N , the related graph becomes almost complete. Figure 13 compares the asymptotic behavior of different families of Dicke-like states.

VII. QUANTUM CIRCUITS

We present below a quantum circuit efficiently transforming a fully separable tensor product state into the excitation state proposed in this work. Construction of this circuit was inspired by similar circuits for Dicke states developed in [36,37]. However, it refers explicitly to the graph structure and is more intuitive in that sense. As we will see, our scheme uses between $\sim 4|V|$ and $\sim 10|E|$ CNOT gates, depending on the structure of a graph. For clarity of demonstration, we present the construction of a unitary circuit transforming the excitation state into the separable state. The provided construction can be extended in a simple manner to the case of hypergraphs.

We construct a quantum circuit iteratively relative to the set of vertices. For a vertex $v \in V$, we apply $d_v - 1$ three-qubit unitary operations and one two-qubit operation. This can be represented graphically by deleting edges adjacent to the consecutive vertices. Although any three-qubit unitary operation can be simulated by at most 21 CNOT gates [76], considered operations can be simulated by six or ten two-qubit gates only, according to [77]. We present the exact form of unitary operations below and, in parallel, its realizations by accessible two-qubit gates, where we assume standard implementation of Toffoli gate with 6 CNOT gates. The exact form of quantum circuit depends on the order of deleted vertices and adjacent edges; the procedure itself, however, might be performed for any such order.

First step. Choose a vertex v , and take all adjacent vertices $v_1, \dots, v_d$, where d is the degree of v . Without loss of generality, suppose that each v_i is related to the i th particle, while v is related to the $(d + 1)$ th particle. We consecutively apply the following three-qubit gates $U_{\{i,d,d+1\}}^{(1)}$ on parties i , d , and $d + 1$:

$$\begin{aligned} |101\rangle + \sqrt{i-1}|011\rangle &\mapsto \sqrt{i}|011\rangle, \\ |110\rangle &\mapsto |110\rangle, \\ |100\rangle &\mapsto |100\rangle, \\ |010\rangle &\mapsto |010\rangle, \\ |001\rangle &\mapsto |001\rangle, \\ |000\rangle &\mapsto |000\rangle, \end{aligned} \quad (24)$$

for $i = 1, \dots, d - 2$. Applying $U_{\{i,d,d+1\}}^{(1)}$ the operation is relevant to the graphical operation of deleting an edge $e = \{i, d + 1\}$. It may be simulated by ten CNOT gates, as indicated in Fig. 14.

Second, we apply the following three-qubit operator, $U_{\{d-1,d,d+1\}}^{(2)}$, on parties $d - 1$, d , and $d + 1$,

$$\begin{aligned} |101\rangle + \sqrt{d-2}|001\rangle &\mapsto \sqrt{d-1}|001\rangle, \\ |011\rangle &\mapsto |011\rangle, \\ |100\rangle &\mapsto |100\rangle, \\ |010\rangle &\mapsto |010\rangle, \\ |000\rangle &\mapsto |000\rangle, \end{aligned} \quad (25)$$

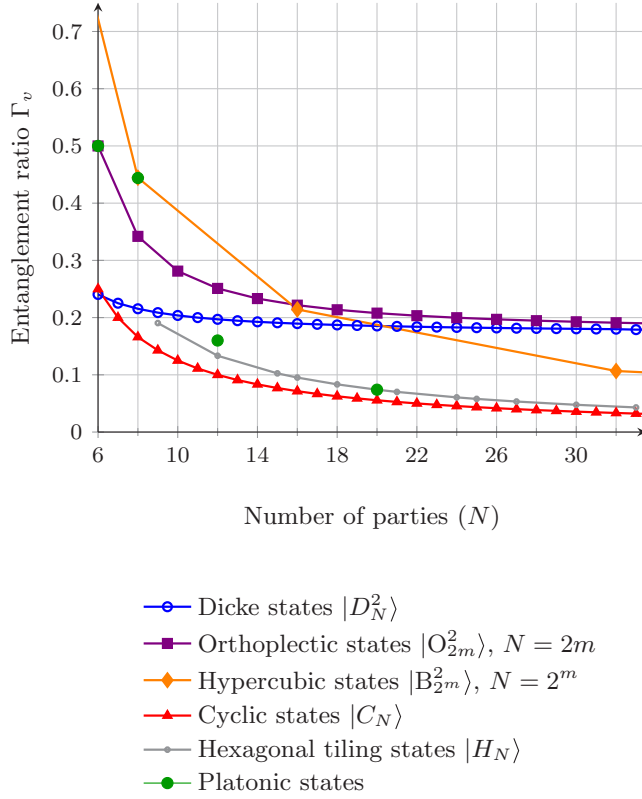


FIG. 13. Distribution of entanglement for Dicke-like states with $k = 2$ excitations is compared. The entanglement ratio Γ_v quantifies the ratio of the bipartite and multipartite entanglement of a particular subsystem v . The distribution of entanglement varies for states relevant to local and global networks, i.e., graphs with asymptotically different densities. For states related to dense graphs, $|D_N^2\rangle$ and $|O_{2m}^2\rangle$, the ratio Γ_v tends to a nonzero value $3 - 2\sqrt{2} \approx 0.17$. For families of states in which the local degree does not scale with the size of a graph ($|B_{2m}^2\rangle$, $|C_N\rangle$, $|H_N\rangle$), the complexity of Γ_v is $O(1/N)$. The exact scaling factor for local networks relies on a local degree d in a way depending on the curvature of the network. For states related to flat ($|C_N\rangle$, $|H_N\rangle$) and spherical (Platonic states) networks one has $\Gamma_v \sim d/N$, while for the hyperbolic case ($|B_{2m}^2\rangle$), the ratio scales as $\Gamma_v \sim 1/N$. For flat networks, the exact scaling parameter does not exceed unity ($\Gamma_v \sim d/2N$ for cyclic states $|C_N\rangle$), and $\Gamma_v \sim 4d/9N$ for states related to a hexagonal tiling), and exceeds unity for a spherical networks (Platonic states).

which is relevant to the graphical operation of deleting an edge $e = \{d - 1, d + 1\}$. Observe that Eq. (25) is defined only on five-dimensional subspace, and is arbitrary on the remaining part of the Hilbert space. This leaves room for optimization of simulation of $U_{\{d-1, d, d+1\}}^{(2)}$. Indeed, it may be simulated only by six CNOT gates or a single Toffoli gate (see Fig. 14). For comparison, for simulation of Eq. (24) already ten CNOT gates were needed.

Finally we apply the following two-qubit operator, $U_{\{d, d+1\}}^{(3)}$, on parties d and $d + 1$:

$$\begin{aligned} |11\rangle + \sqrt{d-1}|01\rangle &\mapsto \sqrt{d}|01\rangle, \\ |00\rangle &\mapsto |00\rangle, \quad |10\rangle \mapsto |10\rangle, \end{aligned} \quad (26)$$

which is relevant to deleting the only remaining edge: $e = \{d, d + 1\}$. Those three operations are presented in Fig. 14. There is a simple logic behind those operations. Consecutively, we combine all terms having excitations on position $d + 1$ into a single term with an excitation on this position. Observe that after applying these operations, the state takes the form of the following superposition:

$$\frac{1}{\sqrt{|E|}} \left(\sqrt{d}|1\rangle_v \otimes |0 \cdots 0\rangle + |0\rangle_v \otimes \sum_{e \in E \setminus v} |\psi_e\rangle \right),$$

where $E \setminus v$ denotes a set of edges which do not contain vertex v .

Second and next steps. Consider now the graph $G' = (V \setminus v, E \setminus v)$. Notice that the degrees of vertices initially adjacent to the vertex v have changed. We faithfully repeat the procedure from the first step for an arbitrary vertex w from the graph G' . Observe that after applying the aforementioned gates, the state takes the following form:

$$\begin{aligned} &\frac{1}{\sqrt{|E|}} \left(\sqrt{d_v}|1\rangle_v \otimes |0 \cdots 0\rangle_{v^c} \right. \\ &\quad \left. + \sqrt{d'_w}|1\rangle_w \otimes |0 \cdots 0\rangle_{w^c} + |00\rangle_{vw} \otimes \sum_{e \in E \setminus v, w} |\psi_e\rangle \right), \end{aligned}$$

where d_v is the degree of the vertex v , while d'_w is the degree of the vertex w of the reduced graph, after deleting the vertex v (the degree of w can decrease by 1). We repeat this procedure further, for $G'' = (V \setminus v, w; E \setminus v, w)$, until we delete $N - 1$ vertices, which fully separates the initial graph.

Final step. After applying the presented procedure iteratively $N - 1$ times, the state takes the following form:

$$\frac{1}{\sqrt{|E|}} \sum_{v \in V} \sqrt{d'_v} |1\rangle_v \otimes |0 \cdots 0\rangle_{v^c},$$

where d'_v denotes the degree of the vertex v of the graph reduced according to the presented procedure. Note that the form of the above state is similar to the state $|W_N\rangle$, as it is equivalent to it with respect to the transformations from the stochastic local operations and classical communication (SLOCC) class. The separation of this state is a well-known procedure [78], and might be obtained by performing two-qubit gates $U_{li}^{(4)}$:

$$\begin{aligned} \sqrt{d'_{v_1} + \cdots + d'_{v_{i-1}}} |10\rangle + \sqrt{d'_{v_i}} |01\rangle &\mapsto \sqrt{d'_{v_1} + \cdots + d'_{v_i}} |10\rangle, \\ |00\rangle &\mapsto |00\rangle, \end{aligned}$$

on particles 1 and i . This gate may be simulated only by three CNOT gates (see Fig. 14).

Cost estimation. We start with a connected graph with $|V|$ vertices and $|E|$ edges, where the degrees of each vertex are $d_1, d_2, \dots < |V|$. In order to delete a given vertex of degree d we need $1 + 6\theta(d - 1) + 10\theta(d - 2)(d - 2)$, where $\theta(\cdot)$ is the Heaviside theta function. By considering reverse engineering, the spanning tree of the graph costs $|V| - 1$ CNOT operations. In the best case the spanning tree will be a star graph with degrees $\{|V| - 1, 1, \dots, 1\}$. In the next step we pair up the degree-1 vertices and connect them with $\frac{|V|-1}{2}$ edges. Depending on oddness or evenness we are left with

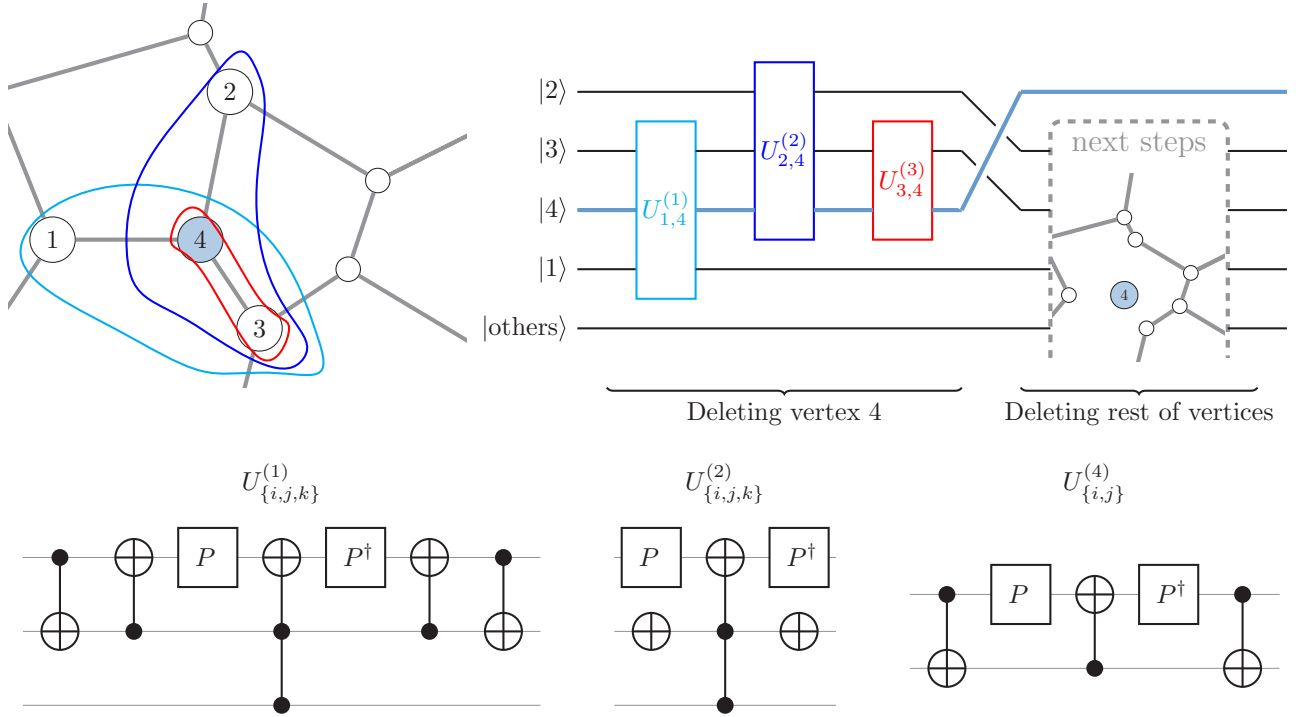


FIG. 14. Procedure sketched for vertex 4 of degree $d = 3$. First, we apply unitary three-qubit operations $U_{\{1,3,4\}}^{(1)}$ on parties 1, 3, 4; second, $U_{\{2,3,4\}}^{(2)}$ on parties 2, 3, and 4; and finally the two-qubit gate $U_{\{3,4\}}^{(3)}$ on parties 3 and 4. As a result, we obtain the state separable with respect to the fourth particle. In the next step, we work with the graph with all edges adjacent to the fourth vertex deleted. Presented unitary operations $U_{\{1,3,4\}}^{(1)}$, $U_{\{2,3,4\}}^{(2)}$, and $U_{\{3,4\}}^{(3)}$ might be simulated by the following quantum gates. All three constructions rely on an auxiliary unitary gate P , which satisfies $P^\dagger \sigma_x P = \sqrt{a - b\sigma_z} + \sqrt{b}\sigma_x$ with a and b set according to the corresponding operation.

an additional edge, which takes up the circuit cost to $6\lceil \frac{|V|-1}{2} \rceil$ CNOT operations and leaves all the vertices of degree at least 2. The remaining edges cost in total $10(|E| - \lceil \frac{|V|-1}{2} \rceil - |V| + 1) \approx 10|E| - 5(|V| - 1)$ CNOT operations. The final additions are the $U_{ij}^{(4)}$ operations which cost at most $3(|V| - 1)$ CNOT operations. Thus we arrive at three different regimes:

(1) $|E| \sim |V|$, for which the main cost comes from the spanning tree and can be estimated as $4(|V| - 1)$ CNOT operations.

(2) $|E| \sim \frac{3}{2}|V|$, for which the cost is based on vertices of degree 1 and 2, and is given as $7(|V| - 1)$.

(3) $|E| \gg \frac{3}{2}|V|$, for which the cost is dominated by the degree greater than 2 and is estimated as $10|E| + 2|V| - 2$ CNOT operations.

VIII. FIVE-QUBIT CYCLIC STATE $|C_5\rangle$ IMPLEMENTATION

In order to demonstrate the viability of the provided construction we have explicitly evaluated the construction for the simplest nontrivial cyclic state $|C_5\rangle$. Initialization of the state is done basically in two steps. The first step involves initialization of excitations and distributing them in order to arrive at the state of the form

$$\frac{1}{\sqrt{|E|}} \sum_{v \in V} \sqrt{d'_v} |1\rangle_v \otimes |0 \cdots 0\rangle_{v^c}.$$

In the next step, the distribution of edges between the vertices is effected by applying the unitaries (25) and (26) in

reversed order to finally reconstruct the graph corresponding to $|C_5\rangle$. The overall circuit requires a single operator $U^{(2)}$ and remaining operations are done by using $U^{(3)}$ and $U^{(4)}$, which totals to an estimated 22 CNOT operations, which does not take into account the topology of the quantum computer to be used.

Such a circuit can be realized on the state-of-the-art five-qubit quantum computers provided by IBM: linear-topology Santiago and Athens with quantum volumes (QVs) of 32 and Vigo with QV of 16 with T topology. In total we used more than 740 000 samples over all three computers, which gives distribution with significant values for all expected computational states proceeding from cyclic permutations of the $|00011\rangle$ state with the probability 0.487 of finding the system in one of them (see Fig. 15).

We decided to compare the results to a model of noise composed from the exponential probability of decay to the base state and a constant probability of flip from zero to one. In order to perform the noise analysis we first divide the readouts into signal,

$$\mathcal{S} = \left\{ (i_1, \dots, i_5) : \sum_{j=1}^5 i_j = 2, \exists j i_j = i_{j+1} = 1 \right\},$$

and noise, $\mathcal{N} = \{0, 1\}^{\otimes 5} \setminus \mathcal{S}$. Furthermore, the noise has been divided into the subsets with constant numbers of excitations, $\mathcal{N}_k = \{(i_1, \dots, i_5) \in \mathcal{N}, i_1 + \dots + i_5 = k\}$, hence $\mathcal{N} = \mathcal{N}_0 \cup \dots \cup \mathcal{N}_5$.

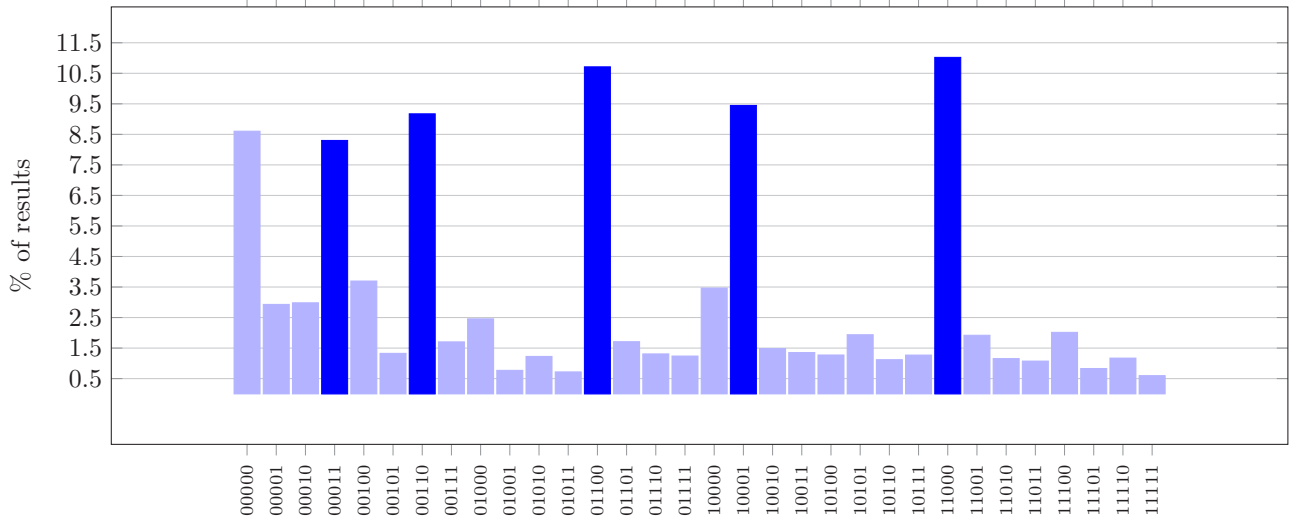


FIG. 15. Mean results acquired over 740 000 samples acquired on IBM Vigo, Athens, and Santiago computers. Prominent peaks at all cyclic permutations of $|00011\rangle$ are plotted in dark, with an additional significant peak at $|00000\rangle$. Due to effects of decoherence we observe a background of all remaining states represented by light bars.

This division allows us to introduce the mean probability of noise with n excitations,

$$\bar{P}_k = \frac{1}{|\mathcal{N}_k|} \sum_{i_1, \dots, i_5 \in \mathcal{N}_k} p_{i_1, \dots, i_5}, \quad (27)$$

where the probabilities are found from empirical data by dividing the number hits for a specific state $N_{i_1, \dots, i_5}$ by the total number of hits:

$$p_{i_1, \dots, i_5} = \frac{N_{i_1, \dots, i_5}}{N_{tot}}.$$

The second step was to fit the model with exponential and flip noises superposed, which resulted in a satisfactory fit with total squared distances equal to $\sum_{k=0}^4 (\bar{P}_k - \bar{P}_{k,fit})^2 \approx 0.075$ (see Fig. 16).

In conclusion, the experiment shows that implementation of the introduced preparation procedure and subsequently executing it on a real-world quantum machine is not only viable, but follows simple predictions about the nature of errors present in such computers.

IX. HAMILTONIANS OF EXCITATION STATES

The Dicke states are ground states of the Dicke Hamiltonian with a well-defined number of excitations. The excitation states introduced above can be considered as ground states of analogous Hamiltonians, with the same property of a well-defined excitation number.

The single-mode Dicke Hamiltonian (known also as the Tavis-Cummings model or generalized Jaynes-Cummings model) can be written as [31,32]

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}} \underbrace{(a^\dagger + a)(J_- + J_+)}_{\text{interaction}},$$

where J_α are the collective operators:

$$J_z \equiv \sum_{i=1}^N \sigma_z^i, \quad J_\pm \equiv \sum_{i=1}^N \sigma_\pm^i.$$

At zero interaction, $\lambda = 0$, the coupling term in the Dicke Hamiltonian vanishes. Thus its eigenstates have the tensor product form with a factor representing the Fock states of the field and the other factor as an eigenstate of the collective angular momentum operator J_z . The operator J_z has the

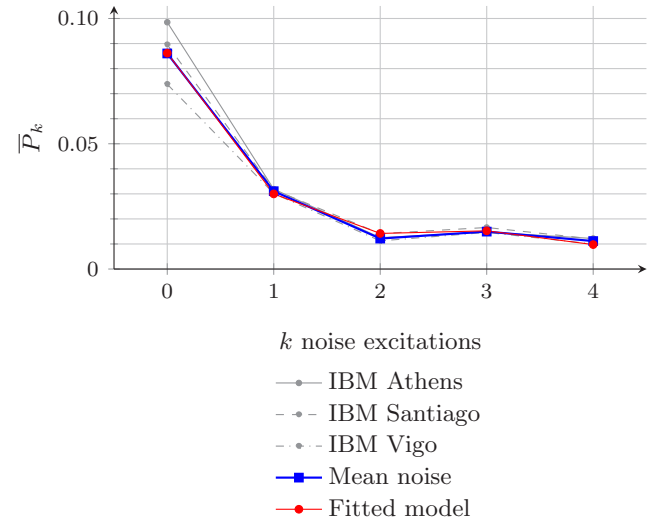


FIG. 16. Mean probability of noise (27) for each machine collected according to the number of excitations. Averaged noise (blue) has been taken as a basis for fitting of the noise model, composed of exponential decay to the ground state of the entire system superposed with a random 0-1 flip with equal probability for all qubits. The resulting fit, plotted in red, is satisfactory, with a total deviation of $\sum_{k=0}^4 (\bar{P}_k - \bar{P}_{k,fit})^2 \approx 0.075$.

eigenvalues $-N/2, -N/2 + 1, \dots, N/2$, which are strongly degenerated. The further partition of its eigenspaces is done by the effective total angular momentum operator J , which might be written as

$$J \equiv J_+ J_- + J_z(J_z + \mathbb{I}),$$

in terms of the raising and lowering operators. The Dicke states $|j, m\rangle_N$ are eigenstates of both operators J_z and J :

$$\begin{aligned} J_z |j, m\rangle_N &= m |j, m\rangle_N, \\ J |j, m\rangle_N &= j(j+1) |j, m\rangle_N, \end{aligned}$$

with $m = -j, -j+1, \dots, j$ representing the number of excitations, while $j = 1/2, 3/2, \dots, N/2$ (for N odd) and $j = 0, 1, \dots, N/2$ (for N even) correspond to the cooperation number. In general, atomic configurations for $N > 2$ contain entanglement [79] and are degenerated for $j < N/2$ [32]. Taking the maximal value $j = N/2$ results in the aforementioned fully symmetric Dicke states $|D_N^k\rangle$,

$$|D_N^k\rangle := |N/2, k + N/2\rangle_N.$$

In several physical situations, it is convenient to consider the eigenvectors spanned by the fully symmetric Dicke states [79]. Indeed, this significantly reduces the dimension of space from 2^N to $N+1$.

Independently of the excitation number k , the Dicke states $|D_N^k\rangle$ are eigenstates of the operator $J_+ J_-$. The latter operator can be rewritten as a fully symmetric, two-body Hamiltonian,

$$H := J_+ J_- = \sum_{i=1}^N \sigma_-^{(i)} \sigma_+^{(i)} + \sum_{i \neq j}^N \sigma_-^{(i)} \sigma_+^{(j)},$$

in which excitations might be exchanged between any two parties. The Dicke states $|D_N^k\rangle$ are eigenvectors corresponding to the nondegenerated maximal eigenvalue $k(N+1-k)$. In some physical problems the particles are not distributed symmetrically as some interactions are more likely to occur. The related two-body Hamiltonian of the system of N particles with spin-1/2 considered in [80] generalizes the Heisenberg model.

We show that excitation states, and hence Dicke-like states, are ground states for Hamiltonians with an interaction term analogous to $J_+ J_-$. As in the Dicke case we discuss the class of states with a well-defined excitation number and analyze a single subspace of the J_z operator. First, we restrict ourselves to the subspace of the operator J_z relevant to two excitations; i.e., we set $k = 2$. For any graph G , consider the following four-body Hamiltonian, where excitations might be exchanged between any pair of edges:

$$H_G := J_+^G J_-^G, \quad (28)$$

where

$$J_\pm^G \equiv \sum_{vv' \in E} \sigma_\pm^{(v)} \sigma_\pm^{(v')}.$$

Notice that operators J_+^G, J_-^G are constructed analogously to the operators J_+, J_- . The maximal eigenvalue of the Hamiltonian H_G is equal to the squared number of edges, $|E|^2$. This energy level is nondegenerated and the corresponding eigenstate is given by the excitation state $|G\rangle$. Indeed, the

Hamiltonian H_G is a sum of $|E|^2$ projections, which justifies the bound of the energy. The state $|G\rangle$ saturates this bound, contrary to any other state with specified number of excitations, $k = 2$.

The excitation state $|G\rangle$ can be seen as a ground state of the three-body interaction Hamiltonian,

$$H_G = \sum_{v \in V} \sum_{\substack{vv' \in E \\ v'v'' \in E}} \sigma_-^{(v)} (\sigma_- \sigma_+)^{(v')} \sigma_+^{(v'')}. \quad (29)$$

The above Hamiltonian describes the case in which the pairs of excitations might be exchanged only between neighboring edges. The maximal eigenvalue is equal to $\binom{d}{2}|V|$, where d is the degree of a vertex. In analogy to the Hamiltonian (28) the maximal eigenvalue is not degenerated and the state $|G\rangle$ forms the relevant eigenvector. Notice that the Dicke state $|D_N^2\rangle$ is a ground state of the Hamiltonian associated with the full graph G on N vertices.

It is worth noting that $\sigma_- \sigma_+ = (\mathbb{I} - \sigma_z)/2 = |1\rangle\langle 1|$ and the Hamiltonian can be written in an alternative way:

$$H_G = \sum_{v \in V} \sum_{\substack{vv' \in E \\ v'v'' \in E}} \sigma_-^{(v)} |1\rangle\langle 1|^{(v')} \sigma_+^{(v'')}. \quad (30)$$

In this form the term $\sigma_- \sigma_+$ can be interpreted as a conditional hopping interaction—if the site v' is occupied, the hopping from v to v'' is effected, otherwise no interaction happens. The behavior is very much akin to what is called the quantum transistor. One of the simplest models of such a quantum transistor [81] involves left and right qubits and a two-qubit gate, an interaction very similar to that observed here.

Similarly, excitation states related to the k -uniform hypergraphs G might be described as ground states of the Hamiltonian with $2k$ -body interaction, $H = J_+^G J_-^G$. The excitation state $|G\rangle$ relevant to the k -hypergraph G is a ground state of the Hamiltonian (28), where the operators $J_\pm^G$ are defined as

$$J_\pm^G \equiv \sum_{\{v^1, \dots, v^k\} \in E} \sigma_\pm^{(v^1)} \dots \sigma_\pm^{(v^k)}.$$

In summary, the Dicke states $|D_N^m\rangle$ are uniquely determined as eigenvectors of operators J_z and the two-body interaction Hamiltonian, $H = J_+ J_-$. Excitation states are also eigenvectors of an operator J_z ; nevertheless, the further division of its subspaces differs from the one corresponding to the Dicke states. Indeed, for any number k of excitations and an excitation state $|G\rangle$ one may construct a $(k+1)$ -body interaction Hamiltonian with the state $|G\rangle$ being its nondegenerated ground state. Note that the construction of the Hamiltonian presented above depends on the number of excitations.

X. CLASSIFICATION PROBLEM

From a theoretical point of view symmetric states of N qubits possess remarkably simple classification with respect to SLOCC [82] as they can be classified by the *Young diagrams* of a size N [83]. This is in a strong contrast to the classification of a general N -qubit state, for which the full classification is known for systems consisting of four qubits only [84–86]. Furthermore, the admissible symmetries of stabilizers of symmetric states were also successfully classified

[87,88]. Classifying states with limited symmetries might be an accessible task and a step toward the goal to extend the current limits of understanding of multipartite entanglement in the case of a large number of subsystems.

XI. CONCLUDING REMARKS

In this work we advocate an original approach to construct genuinely entangled multipartite states by using a group action or, more generally, graphs and hypergraphs. Obtained states resemble the Dicke states, whereas the excitations are chosen only between specific subsystems related to the structure of the graph. We propose a quantum circuit generating the states constructed, with the same complexity as similar circuits for the Dicke states. We successfully simulated considered states on available quantum computers: IBM Santiago, Vigo, and Athens. Moreover, discussed states are shown to form ground states of three-body Hamiltonians, analogous to the original Hamiltonian of Dicke. The presented interaction, in turn, resembles quantum transistor behavior.

In various realistic scenarios the full symmetry between collaborating systems is not required or not desirable. Excitation states appear as good candidates for such less-symmetric tasks. For this reason, we investigate the possible symmetries of such states and demonstrate that not all kinds of symmetry are possible. This reflects the impossibility of arranging specific tasks of a given symmetry. Therefore, special attention is dedicated to the family of excitation states, which exhibits a highly symmetric structure. We refer to such states as Dicke-like, since they are a superposition of terms obtained by permutations of $|1 \cdots 10 \cdots 0\rangle$, whereas only permutations from a specified subgroup H of the permutation group, S_N , are taken. We introduce specific families of quantum states related to highly symmetric objects, such as regular polygons, Platonic solids, and regular plane tilings.

We investigate the entanglement properties of introduced families of states. First, we provide the general separability conditions for excitation states and present them in a twofold way: as specific conditions satisfied by a relevant graph or specific conditions imposed on the symmetry group. Second, we specifically quantify the entanglement presence in introduced families of states. In order to compare the entanglement distribution, we compute the concurrence in bipartite systems

C_{vw} and the generalized concurrence $C_{v|\text{rest}}$ between particle v and the rest of the system (Propositions 3 and 4). Analytic formulas for these quantities are provided and illustrated on several examples. For regular hypergraphs, with equal degrees of each node, the following general observations hold:

(1) The concurrence $C_{v|\text{rest}}$ describing entanglement between a given subsystem v and the rest of the system depends only on the total number of nodes and uniformity of the graph and does not depend on its local structure.

(2) The concurrence C_{vw} between nodes v and w is positive only for distance-2 nodes, and in states related to almost complete hypergraphs, i.e., $|E| \sim \binom{N}{k}$, also for distance-1 nodes. For all local networks, the second possibility does not occur.

(3) The concurrence C_{vw} between nodes of distance 2 is proportional to the number of shared neighbors.

Moreover, we introduce the notion of entanglement ratio Γ_v , which measures the ratio of entanglement shared between particular subsystems in a bipartite way in comparison to the amount of entanglement shared in the multipartite way. We investigate the asymptotic behavior of the parameter Γ_v of an excitation state $|G\rangle$ with the change of the local degree d of the relevant hypergraph. Two phase transitions with respect to the bipartite entanglement in the excitation-state network are reported. For small average degree d of vertices, the entanglement ratio scales as $\Gamma_v \sim d/N$, and the entanglement is present only between distance-2 nodes. Once the limit $d \simeq 0.794N$ is exceeded, the bipartite entanglement between distance-1 nodes starts to appear. Its amount is, however, negligible in comparison to the entanglement between distance-2 vertices. Entanglement between distance-1 nodes prevails once the threshold $d \approx 0.973N$ is exceeded.

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**D Thirty-six entangled officers of Euler: Quantum
solution to a classically impossible problem**

Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem

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The negative solution to the famous problem of 36 officers of Euler implies that there are no two orthogonal Latin squares of order six. We show that the problem has a solution, provided the officers are entangled, and construct orthogonal quantum Latin squares of this size. As a consequence, we find an example of the long elusive Absolutely Maximally Entangled state AME(4,6) of four subsystems with six levels each, equivalently a 2-unitary matrix of size 36, which maximizes the entangling power among all bipartite unitary gates of this dimension, or a perfect tensor with four indices, each running from one to six. This special state deserves the appellation *golden AME state* as the golden ratio appears prominently in its elements. This result allows us to construct a pure nonadditive quhex quantum error detection code $((3,6,2))_6$, which saturates the Singleton bound and allows one to encode a 6-level state into a triplet of such states.

Introduction: It is well-known that quantum entanglement leads to peculiar consequences, and enable technical tasks such as teleportation, secret sharing, secure key cryptography, and error correcting codes [1]. Special entangled states such as the cluster states are a resource for measurement based quantum computing [2, 3]. Multipartite entanglement is also implicated in quantum many-body states and leads to thermalization in isolated systems [4, 5]. While the understanding of multipartite entanglement is still evolving, an extreme class of states: the absolutely maximally entangled (AME) ones are clearly singled out and easily defined. AME states find applications in several quantum protocols including quantum secret sharing and parallel teleportation [2], holographic quantum error correcting codes [4, 8], and quantum repeaters [9]. However, they are not easy to construct and often their very existence is unknown.

A pure quantum state, $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties, each of a local dimension d , is *absolutely maximally entangled* [1, 2], written as AME(N, d), if it is maximally entangled for every bipartition. For $N = 2$ and $N = 3$ parties, generalizations of the Bell and GHZ states, $\sum_{k=1}^d |kk\rangle/\sqrt{d}$ and $\sum_{k=1}^d |kkk\rangle/\sqrt{d}$ are AME states. The smallest number of parties of interest is therefore $N = 4$ where it is known that AME(4,2) does not exist, four qubits cannot be absolutely maximally entangled [15, 20]. However for all $d > 2$, except for $d = 6$, AME states of 4 parties are known to exist. Thus arguably the most interesting open case was the existence of AME(4,6). It has featured on open-problem lists of quantum information [13, 14] and in this Letter we settle this positively by explicitly constructing an example, leaving only four qubits as exceptional.

To shed light on the peculiar properties of such states, consider a collection of four dice. Alice selects *any* two dice and rolls them, obtaining one of 36 equally likely outcomes, as Bob rolls the remaining ones. If the entire state is AME(4,6), Alice can always deduce the result obtained in Bob's part of

the 4-party system.



FIG. 1: Four dice in the golden absolutely maximally entangled state of four-quhex, AME(4,6), corresponding to 36 entangled officers of Euler. Any pair of dice is unbiased, although their outcome determines the state of the other two. This is not possible if the four dice are replaced by four coins (qubits).

Furthermore, such a state allows one to teleport any unknown, two-dice quantum state, from any two owners of two subsystems to the lab possessing the two other dice of the entangled state of the four-party system. These are not possible if the dice is replaced by two-sided coins as AME(4,2) states do not exist [20].

The existence of AME(4, d) for all $d > 2$ and $d \neq 6$ has been known from the existence of particular combinatorial designs, namely orthogonal Latin squares (OLS), also known as Graeco-Latin squares. A simple example of a combinatorial design [15] is given by a single *Latin square*: a $d \times d$ array filled with d copies of d different symbols, such that each occurs once in each row and each column. For the design of experiments, one uses Graeco-Latin squares: two Latin squares arranged in such a way that the ordered pairs of entries in all cells of the square are distinct. The name of the design refers to a popular way to represent such a pair by one Greek character and one Latin.

It is easy to see that there are no Graeco-Latin squares of size two. Furthermore, it is not difficult to construct such a combinatorial design for $d = 3$ – see Fig. 2. Analogous con-

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βC	γA	αB		$K\spadesuit$	$Q\spadesuit$	$A\clubsuit$		2,3	3,1	1,2
γB	αC	βA	=	$Q\clubsuit$	$A\spadesuit$	$K\spadesuit$	=	3,2	1,3	2,1
αA	βB	γC		$A\spadesuit$	$K\clubsuit$	$Q\spadesuit$		1,1	2,2	3,3

FIG. 2: An example of Graeco-Latin square of order $d = 3$. In the middle, Greek and Latin letters are replaced by ranks and suits of cards, on the right by pairs of numbers.

structions also work for $d = 4, 5$, and in general, for odd numbers d and multiples of four [16]. The case of $d = 6$ is thus special. Leonhard Euler examined the now-famous problem [17]: “Six different regiments have six officers, each one belonging to different ranks. Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one of each regiment?” As Euler observed, such an arrangement, equivalent to a Graeco-Latin square of order 6, written OLS(6), does not exist, which was proven much later by Gaston Tarry [18]. Two OLS of size $d = 10$ were constructed in the 20th century and it is known that such designs exist [19] for any natural number $d \neq 2$, or $\neq 6$. This enables the construction of AME(4, d) for any d other than $d = 2$ and $d = 6$ [19].

While classical designs are built of discrete objects, their quantum analogs, introduced in the seminal thesis of Zauner [21], are composed of pure quantum states: normalized vectors from a complex Hilbert space $\mathcal{H}_d$ of dimension d . Such distinguished configurations of states play a significant role in quantum information, as they describe generalized quantum measurements with special properties. In particular, quantum analogs of Latin squares were introduced in [22], while various approaches to orthogonal quantum LS were advocated in [9–11].

In this work, we formulate a quantum analog of Euler’s problem of 36 officers [9] and present its complete analytic solution. As AME(4, 2) does not exist and standard OLS exist for any natural d , different from 2 and 6, the latter case corresponding to the quantum version of the Euler problem of 36 officers, was the only open problem for 4-partite systems [3, 27]. Our construction of an AME(4, 6) state, shows that for 4 parties, $d = 6$ is the only dimension in which there are no classical OLS, but there exists a quantum one. Even though OLS of order six do not exist, we present a coarse-grained OLS of that order, whose structure stands behind its constructed quantum analog.

Multipartite entangled states from Graeco-Latin squares: A pure quantum state $|\psi\rangle \in \mathcal{H}_d^A \otimes \mathcal{H}_d^B$ of a bipartite $d \times d$ system can be expanded in a product basis, $|\psi\rangle = \sum_{i,j=1}^d C_{ij} |i\rangle |j\rangle$, where the matrix C of coefficients satisfies the normalization condition, $\|C\|^2 = \text{Tr} CC^\dagger = 1$. A pure state is maximally entangled if its partial trace is maximally mixed, so that the matrix of coefficients is unitary up to rescaling, $CC^\dagger = I/d$. A pure quantum state, $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties, each of a local dimension d is AME(N, d) if the partial trace $\text{Tr}_S |\psi\rangle \langle \psi| \propto I$, for any subsystem S of $|S| = \lfloor N/2 \rfloor$ parties.

Any Graeco-Latin square of order d determines a 4-party quantum state $|\Psi\rangle \in \mathcal{H}_d^{\otimes 4}$,

$$|\Psi\rangle = \frac{1}{d} \sum_{i,j,k,\ell=1}^d T_{ijkl} |i\rangle |j\rangle |k\rangle |\ell\rangle, \quad (1)$$

where coefficients $T_{ijkl} = 1$ if the pair (k, ℓ) is an entry in i -th row and j -th column, while $T_{ijkl} = 0$ otherwise. Note that the matrix of order d^2 , indexed by doubled indices, $U_{ij,kl} := T_{ijkl}$, forms a permutation. Orthogonality conditions imposed on Latin squares implies that the matrices corresponding to the other two bipartitions of the four indices into pairs: $ik|j\ell$ and $i\ell|jk$, also form permutations. This follows as every relation between any two pairs of features (such as the column number and the card suit versus the row number and the card rank shown in Fig. 2) is a bijection.

It is convenient to recall reorderings of entries of a matrix of size d^2 used in quantum theory [17]. Representing the matrix in a product basis as $U_{ij,kl} = \langle i, j | U | k, \ell \rangle$, one defines the partially transposed matrix, U^Γ and the reshuffled matrix U^R ,

$$U_{ij,kl}^\Gamma = U_{i\ell,kj}, \quad U_{ij,kl}^R = U_{ik,j\ell}. \quad (2)$$

Making use of this notation, one can say that a permutation matrix U of order d^2 yields an OLS(d) if the reordered matrices, U^R and U^Γ are also permutations. In this case, the 4-partite state of Eq. 1 has a particular property: It is maximally entangled with respect to any bipartition of four indices i, j, k, ℓ into two pairs, and is hence an AME(4, d) state.

Orthogonal quantum Latin squares: To obtain other AME states of this class, we retain the condition that the matrices U, U^R and U^Γ be unitary while relaxing the condition that they are permutations. Such a matrix U is called 2-unitary [5], see supplemental material section I for more details. In fact, any 2-unitary matrix $U \in \mathbb{U}(d^2)$ yields an AME(4, d) state

$$|\text{AME}(4, d)\rangle = \frac{1}{d} \sum_{i,j=1}^d |i\rangle |j\rangle |\psi_{ij}\rangle, \quad (3)$$

where $|\psi_{ij}\rangle = U|i\rangle|j\rangle$. This allows one to say that the corresponding quantum design of d^2 bipartite quantum states $|\psi_{ij}\rangle$, $i, j = 1, \dots, d$, forms an orthogonal quantum Latin square (OQLS) – for a formal definition see supplemental material (SM) section I.

It may be noted that a 2-unitary matrix U of size d^2 , treated as a 4-index tensor, $T_{ijkl} := U_{(ij),(kl)}$ has been called a *perfect tensor* [8] and used in constructing quantum error correcting codes. Any of its 2-index flattenings $T_{ij}^{k\ell}, T_{ik}^{j\ell}, T_{i\ell}^{jk}$ is unitary and provides an *isometry* between any pair of its indices. The partial trace of the 4-party state $|\Psi\rangle$ related to a perfect tensor as in Eq. 3 is maximally mixed for any symmetric bipartition of the system. Thus existence of such a perfect tensor with 4 indices running from 1 to d is equivalent to the existence of an AME state of four qudits and OQLS of size d .

Searching for 36 entangled officers: To tackle the quan-

tum Euler problem of AME(4,6), equivalent to finding a 2-unitary matrix of order 36, we used an iterative numerical technique based on nonlinear maps in the space of unitary matrices $\mathbb{U}(d^2)$ introduced recently in [7]. A closely related Sinkhorn-like algorithm to generate unitary matrices such that only their partial transpose is unitary was presented earlier in [12]. One map that produces 2-unitaries $\mathcal{M}_{\Gamma R} : U_0 \mapsto U_1$, consists of two parts (i) R and Γ operations in that order $U_0 \mapsto U_0^R \mapsto (U_0^R)^\Gamma := U_0^{\Gamma R}$, and (ii) projection onto the nearest unitary matrix $U_0^{\Gamma R} \mapsto U_1$ using the polar decomposition, $U_0^{\Gamma R} = U_1 H$, where H is a positive semi-definite matrix. It is straightforward to see that 2-unitary matrices are fixed points of the map $\mathcal{M}_{\Gamma R}^3$, or period-3 orbits of $\mathcal{M}_{\Gamma R}$. After n iterations of the map, $\mathcal{M}_{\Gamma R}^n[U_0] = U_n$, and as $n \rightarrow \infty$ this converges to 2-unitary matrices with high probabilities for $d = 3$ and $d = 4$, even using seeds U_0 sampled randomly according to the Haar measure on the unitary group [7].

The key to generating a 2-unitary matrix for $d > 4$ is to choose an appropriate seed matrix. The neighborhood of permutation matrices approximating OLS is a natural choice. In the most interesting case of $d = 6$, for which there are no OLS, seeds in the vicinity of the permutation matrix P_{36} [19] that is closest to an OLS, surprisingly, do not lead to a 2-unitary. However, there do exist other suitable permutation matrices whose vicinity contains seeds that under the map $\mathcal{M}_{\Gamma R}$ converge to 2-unitaries. An example of a seed that leads to the 2-unitary solution displayed in this work is provided in section III of SM.

A 2-unitary matrix remains 2-unitary on multiplication by local unitary operators. Using this freedom, we applied a search algorithm over the group $\mathbb{U}(6) \otimes \mathbb{U}(6)$ of local unitary operations, to orthogonalize certain rows and columns in a given numerical 2-unitary matrix U and its rearrangements $U^R, (U^R)^\Gamma$. We searched for block structures in all these three matrices abstracting from the exact form of the matrices. The particular choice of the orthogonality relations corresponds to the block structure of the eventually obtained analytical solution. These tools can be generalized to construct multi-unitary operators and corresponding AME states in other local dimensions and number of parties. While the solution presented is the smallest one of arguably greatest interest, namely AME(4,6), the methods can potentially yield maximally entangled states that are not created by presently known techniques.

Solution found: To present the solution to the problem of the 36 entangled officers of Euler, we display the coefficients T_{ijkl} of the AME(4,6) quantum state $\in \mathcal{H}_6^{\otimes 4}$ (four quhex state), see Eq. 3. Non-vanishing coefficients T_{ijkl} might be conveniently written in form of a table, see Fig. 3. The provided construction is based on the root of unity of order 20, denoted by $\omega = \exp(i\pi/10)$. There exist only three non-zero ampli-

tudes:

$$\begin{aligned} a &= (\sqrt{2}(\omega + \bar{\omega}))^{-1} = (5 + \sqrt{5})^{-1/2}, \\ b &= (\sqrt{2}(\omega^3 + \bar{\omega}^3))^{-1} = ((5 + \sqrt{5})/20)^{1/2}, \\ c &= 1/\sqrt{2}, \end{aligned} \quad (4)$$

whose numerical values are $\simeq (0.3717, 0.6015, 0.7071)$, respectively. The bar over the symbol indicates the complex conjugate. The relations $a^2 + b^2 = c^2 = 1/2$ and $b/a = \varphi = (1 + \sqrt{5})/2$ the golden ratio, determine all amplitudes appearing in the solution and explain why the constructed AME state deserves to be called the *golden AME state*. Checking the property of being an AME state reduces to verification of several equations involving roots of unity of order 20, which we elaborate in a detailed way in SM section II. The key result of this work consists, therefore, of the following assertion proved in SM:

Theorem 1. *There exists an AME(4,6) state of 4 parties with 6 levels each.*

This statement is equivalent to the existence of a 2-unitary matrix $\mathcal{U}_{36} \in \mathbb{U}(36)$, and a perfect tensor $T_{ijkl} := \mathcal{U}_{36}(ij)(kl)$, and a solution to the quantum analog of the 36-officers problem of Euler.

An explicit solution of the generalized Euler problem can be conveniently written in terms of a *quantum* OLS in the form of 36 states $|\psi_{ij}\rangle = \mathcal{U}_{36}|i\rangle|j\rangle \in \mathcal{H}_6^{\otimes 2}$ representing the entangled officers. Following Euler's notation used for Graeco-Latin squares we label the rank of each officer by a rank of the card A, K, Q, J, 10, 9, and (extended set of) suits ♠, ♣, ♦, ♥, ♠*, ♣* are used to label her regiment:

$$\begin{aligned} |\psi_{11}\rangle &= c|A\spadesuit\rangle + c|K\spadesuit\rangle, \\ |\psi_{12}\rangle &= c\omega^{17}|A\diamondsuit\rangle + c\omega^{19}|K\heartsuit\rangle, \\ &\vdots \\ |\psi_{66}\rangle &= b\omega^9|10\spadesuit\rangle + b\omega^{13}|9\clubsuit\rangle + a\omega^{16}|10\clubsuit\rangle + a\omega^{16}|9\spadesuit\rangle, \end{aligned} \quad (5)$$

while expressions for the remaining thirty-three states can be directly read out from the 2-unitary matrix presented in Fig. 3. Notice that the related state in Eq. 3 is an AME state. Remarkably, all the 36 states $|\psi_{ij}\rangle$ are maximally entangled as two-qubit states. While this is evident for states such as $|\psi_{11}\rangle$, even $|\psi_{66}\rangle$ and others with support on 4 states are maximally entangled, thanks to the special value of the phases.

Structure of the AME(4,6) state: Recall that a classical OLS corresponds to a 2-unitary permutation matrix. Since there is no solution to the original problem of Euler, the 2-unitary permutation matrix of size 36 does not exist. Nonetheless, we can present the AME(4,6) state obtained by us in a form similar to the classical solution of AME(4,3) generated from classical OLS in Fig. 2. Let us consider every row of a 2-unitary matrix as a place to put an “officer” in, then we can express the entanglement in our solution by showing which two (or four) officers are entangled, thus producing Fig. 4.

The 2-unitary matrix $\mathcal{U}_{36}$ described here has, up to permutations, the structure of nine unitary blocks of size 4. Moreover, the block structure is also characteristic for the reshuffled

(1,1)	(2,2)	(1,2)	(2,1)		(1,3)	(2,4)	(1,4)	(2,3)		(1,5)	(2,6)	(1,6)	(2,5)	
$a \omega^{10}$	a	$b \omega^{15}$	$b \omega^5$	(6,3)	$a \omega^2$	$a \omega^{14}$	$b \omega$	$b \omega^5$	(2,5)	$a \omega$	$a \omega^{19}$	$b \omega^{14}$	$b \omega^{16}$	(4,1)
		c	c	(1,1)			$c \omega^5$	$c \omega^{19}$	(3,3)	$a \omega$	$a \omega^3$	$b \omega^{10}$	$b \omega^4$	(3,4)
c	c			(5,6)	$c \omega^{17}$	$c \omega^{19}$			(1,2)	$b \omega^4$	$b \omega^{18}$	$a \omega^3$	$a \omega^9$	(2,6)
$b \omega^{10}$	b	$a \omega^5$	$a \omega^{15}$	(4,2)	$b \omega^{14}$	$b \omega^6$	$a \omega^3$	$a \omega^7$	(6,4)	$b \omega^2$	$b \omega^8$	$a \omega^5$	$a \omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)		(3,3)	(4,4)	(3,4)	(4,3)		(3,5)	(4,6)	(3,6)	(4,5)	
$a \omega^4$	$a \omega^{10}$	$b \omega^{17}$	$b \omega^7$	(4,5)	a	a	$b \omega^{15}$	$b \omega^{15}$	(4,6)	$a \omega^2$	a	$b \omega^{19}$	$b \omega^{13}$	(2,3)
		$c \omega^2$	$c \omega^2$	(3,2)			c	$c \omega^{10}$	(6,1)			$c \omega^{16}$	c	(6,2)
$c \omega^{10}$	$c \omega^6$			(2,4)	c	$c \omega^{10}$			(5,4)	$c \omega^8$	$c \omega^{16}$			(3,1)
$b \omega^7$	$b \omega^{13}$	$a \omega^{10}$	a	(5,3)	b	b	$a \omega^5$	$a \omega^5$	(1,5)	$b \omega^{14}$	$b \omega^{12}$	$a \omega$	$a \omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)		(5,3)	(6,4)	(5,4)	(6,3)		(5,5)	(6,6)	(5,6)	(6,5)	
$a \omega^3$	$a \omega^7$	b	b	(1,4)	$a \omega^{12}$	$a \omega^{14}$	$b \omega^{15}$	$b \omega$	(3,6)	$a \omega^{18}$	$a \omega^{18}$	$b \omega^3$	$b \omega^3$	(1,3)
		c	$c \omega^{10}$	(2,1)			$c \omega^{14}$	$c \omega^{10}$	(5,1)			c	$c \omega^{10}$	(5,2)
$c \omega^{13}$	$c \omega^7$			(3,5)	$c \omega^7$	$c \omega^{19}$			(2,2)	$c \omega$	$c \omega^{11}$			(6,5)
$b \omega^9$	$b \omega^{13}$	$a \omega^{16}$	$a \omega^{16}$	(6,6)	$b \omega^{14}$	$b \omega^{16}$	$a \omega^7$	$a \omega^{13}$	(4,3)	$b \omega^{10}$	$b \omega^{10}$	$a \omega^5$	$a \omega^5$	(4,4)

FIG. 3: Non-vanishing coefficients T_{ijkl} of the AME(4,6) state $|\Psi\rangle$. Indices (i, j) are indicated in rows, (k, l) in columns, respectively. Treating each coordinates (i, j) as a position (row and column) of Euler's officer, its rank and regiment are in superposition of two or four canonical ranks and regiments. Equivalently, the picture shows non-zero entries of 2-unitary matrix $\mathcal{U}_{36}$ of size 36 where $j + 6(i - 1)$ labels the relevant row while $\ell + 6(k - 1)$ determines the column. Note that all 36 officers, each represented by a single row of the matrix, are maximally entangled qubit states, 16 of them have minimal support being bipartite Bell states (red rows), while the remaining 20 have maximal support (yellow/green rows) – see Eq. 5 and SM. The order of the blocks reflects an additional structure as the numbering of columns increases by six as we proceed to a block below. Depicted matrix preserves its structure of nine unitary blocks of size four with respect to transformations of reshuffling $j \longleftrightarrow k$, and partial transpose $j \longleftrightarrow \ell$.

matrix $\mathcal{U}_{36}^R$ and the partially transposed matrix $\mathcal{U}_{36}^\Gamma$. In other words, in the original matrix $\mathcal{U}_{36}$ we found the block structure *invariant* under reshuffling and partial transpose. The problem of finding 36 entangled officers of Euler splits into two sub-problems: to identify first a block-invariant structure and then to select adequate non-zero elements within them. A particular combinatorial design underlies the invariant structure. Grouping symbols of indices k and ℓ in the presented perfect tensor T_{ijkl} in pairs: $1, 2 \rightarrow A/\alpha$, and $3, 4 \rightarrow B/\beta$, and $5, 6 \rightarrow C/\gamma$ for k/ℓ respectively, results in a coarse-grained OLS, which reveals the described block structure, see Fig. 5.

Quantum codes: Several examples of quantum error correction codes discussed in the literature belong to the class of *additive (stabilizer) codes*, quantum analogs of classical additive codes [9, 32]. In particular all hitherto known AME states, are either stabilizer states [9], or might be derived from the stabilizer construction [33]. Stabilizer codes have the structure of an eigenspace of an abelian group generated by multilocal generalized Pauli operators. The stabilizer approach is especially effective for codes with local prime power dimensions d . Stabilizers of an additive code might be presented in their standard form [9]. We examined all stabilizer sets of four quhex in their standard form and did not find an AME(4,6) state. Therefore, the presented AME(4,6) state is a nonadditive one.

Nonadditive quantum codes are in general more difficult to construct, however, in many cases, they outperform the stabi-

lizer codes [34]. Thus far, the stabilizer approach practically contained the combinatorial approach to constructing AME and k -uniform states. As we demonstrated, the consideration of coarse-grained combinatorial structures might be successful in constructing genuinely entangled states and have advantages over the stabilizer approach.

In order to successfully use a pure $((4, 1, 3))_6$ code in an error correction scheme, one may apply the shortening procedure [35, 36] and obtain $((3, 6, 2))_6$ code. In such way, a single quhex $|i\rangle \in \mathcal{H}_6$ is encoded into a three quhex state $|\tilde{i}\rangle \in \mathcal{H}_6^{\otimes 3}$ defined by

$$|i\rangle \rightarrow |\tilde{i}\rangle := \frac{1}{\sqrt{6}} \sum_{j,k,\ell=1}^6 T_{ijkl} |j, k, \ell\rangle.$$

Both codes, the initial and shortened ones, are optimal, meeting the quantum Singleton bound [36]. Thus, the presented construction of AME states of four subsystems with 6 levels each sheds some light on how to construct nonadditive quantum error correction codes, in a case for which the stabilizer approach fails.

Summary and Outlook: The famous combinatorial problem of 36 officers was posed by Euler, who claimed in 1779 that no solution exists. The first paper with proof of this statement, by Tarry [18], came only 121 years later, in 1900. After another 121 years, we have presented a solution to the quan-

$ K\spadesuit\rangle$	$ A\clubsuit\rangle$	$ A\diamond\rangle$	$ 10\spadesuit\rangle$ $ 10\star\rangle$	$ 10\spadesuit\rangle$ $ 10\clubsuit\rangle$	$ Q\diamond\rangle$ $ Q\heartsuit\rangle$	$ Q\clubsuit\rangle$ $ Q\star\rangle$
		$ K\heartsuit\rangle$	$ 9\spadesuit\rangle$ $ 9\star\rangle$	$ 9\spadesuit\rangle$ $ 9\clubsuit\rangle$	$ J\diamond\rangle$ $ J\heartsuit\rangle$	$ J\clubsuit\rangle$ $ J\star\rangle$
$ 9\spadesuit\rangle$	$ 10\clubsuit\rangle$	$ 10\diamond\rangle$	$ Q\spadesuit\rangle$ $ Q\star\rangle$	$ Q\spadesuit\rangle$	$ A\diamond\rangle$ $ A\heartsuit\rangle$	$ A\clubsuit\rangle$ $ A\star\rangle$
		$ 9\heartsuit\rangle$	$ J\spadesuit\rangle$ $ J\star\rangle$	$ J\spadesuit\rangle$ $ J\clubsuit\rangle$	$ K\diamond\rangle$ $ K\heartsuit\rangle$	$ K\clubsuit\rangle$ $ K\star\rangle$
$ Q\clubsuit\rangle$	$ J\star\rangle$	$ J\spadesuit\rangle$	$ Q\clubsuit\rangle$	$ A\spadesuit\rangle$ $ A\star\rangle$	$ 10\spadesuit\rangle$	$ 10\diamond\rangle$ $ 10\heartsuit\rangle$
			$ K\diamond\rangle$ $ A\heartsuit\rangle$	$ K\clubsuit\rangle$ $ K\star\rangle$	$ 9\clubsuit\rangle$	$ 9\diamond\rangle$ $ 9\heartsuit\rangle$
$ A\spadesuit\rangle$ $ A\star\rangle$	$ A\spadesuit\rangle$ $ A\clubsuit\rangle$	$ 10\diamond\rangle$ $ 10\heartsuit\rangle$	$ 10\spadesuit\rangle$ $ 10\star\rangle$	$ Q\spadesuit\rangle$ $ Q\clubsuit\rangle$	$ Q\diamond\rangle$ $ Q\heartsuit\rangle$	
$ K\spadesuit\rangle$ $ K\star\rangle$	$ K\spadesuit\rangle$ $ K\clubsuit\rangle$	$ 9\diamond\rangle$ $ 9\heartsuit\rangle$	$ 9\spadesuit\rangle$ $ 9\star\rangle$	$ J\spadesuit\rangle$ $ J\clubsuit\rangle$	$ J\diamond\rangle$ $ J\heartsuit\rangle$	
$ 10\heartsuit\rangle$	$ 10\star\rangle$	$ Q\spadesuit\rangle$ $ Q\clubsuit\rangle$	$ Q\diamond\rangle$ $ J\heartsuit\rangle$	$ A\clubsuit\rangle$ $ A\star\rangle$	$ A\spadesuit\rangle$	$ K\clubsuit\rangle$
$ 9\diamond\rangle$	$ 9\clubsuit\rangle$	$ J\spadesuit\rangle$ $ J\clubsuit\rangle$		$ K\spadesuit\rangle$ $ K\star\rangle$		
$ Q\heartsuit\rangle$	$ Q\star\rangle$	$ A\spadesuit\rangle$ $ A\clubsuit\rangle$	$ A\diamond\rangle$ $ A\heartsuit\rangle$	$ 10\spadesuit\rangle$	$ 10\spadesuit\rangle$ $ 10\clubsuit\rangle$	
$ J\diamond\rangle$	$ J\clubsuit\rangle$	$ K\spadesuit\rangle$ $ K\clubsuit\rangle$	$ K\diamond\rangle$ $ K\heartsuit\rangle$	$ 10\spadesuit\rangle$	$ 9\spadesuit\rangle$ $ 9\clubsuit\rangle$	

FIG. 4: Visualization of entanglement between Euler's officers. Position of each officer: i -th row and j -th column, is presented in appropriate row and column on the array. Ranks and regiments of officers (relevant to the indices k and ℓ in T_{ijkl}) are represented in form of cards with corresponding ranks and suits (in particular, $1 \rightarrow A/\spadesuit$, $2 \rightarrow K/\clubsuit$, $3 \rightarrow Q/\diamond$, $4 \rightarrow J/\heartsuit$, $5 \rightarrow 10/\spadesuit$, $6 \rightarrow 9/\star$, for k/ℓ respectively). The grade of each officer is in a superposition of four basic grades: two ranks, and two regiments. The font size of each ket corresponds to the amplitude of the related tensor element. Notice that a classical solution to Euler's problem would correspond to the array with only one card in each entry. Note the structures arising in the array, e.g. aces are entangled only with kings, queens with jacks, and 10s with 9s. Moreover, the *colors* of the cards are not entangled with each other. Hence, officers are grouped into nine sets with four elements, each sharing the same pairs of figures and colors of the card. For example, officers on positions $(1, 1)$, $(4, 2)$, $(5, 6)$, $(6, 3)$ share two black figures A, K of a black suits $\spadesuit, \clubsuit$, which corresponds to the top-left clique on Fig. 3.

$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$	$B\beta$	$B\gamma$
$C\alpha$	$C\beta$	$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$
$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$	$C\alpha$	$C\beta$
$A\gamma$	$A\alpha$	$C\beta$	$C\gamma$	$B\alpha$	$B\beta$
$C\beta$	$C\gamma$	$B\alpha$	$B\beta$	$A\gamma$	$A\alpha$
$B\beta$	$B\gamma$	$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$

FIG. 5: A coarse-grained OLS of order 6, which reveals the block structure of a perfect tensor T_{ijkl} . Indices of non-vanishing elements of the tensor T_{ijkl} are presented: i in row, j in column, while a pair of coarse-grained indices k, ℓ in the entry. Each pair of symbols repeats exactly four times in the array. Moreover, each symbol on each position repeats exactly twice in each row and column. Notice that the array above corresponds to Fig. 4 by coarse-graining figures $A/K \rightarrow A$; $Q/J \rightarrow B$; $10/9 \rightarrow C$, and suits $\spadesuit/\clubsuit \rightarrow \alpha$; $\diamond/\heartsuit \rightarrow \beta$; $\spadesuit/\star \rightarrow \gamma$ of cards in entries.

tum version wherein the officers can be entangled. This unexpected result implies constructive solutions to the related problems of the existence of absolutely maximally entangled states of four subsystems with six levels each, a 2-unitary ma-

trix $\mathcal{U}_{36}$ of size 36 with maximal entangling power and a perfect tensor T_{ijkl} with four indices, each running from one to six. Our results allowed us to construct original quantum error correction codes: a pure code $((4, 1, 3))_6$, and a shortened code $((3, 6, 2))_6$, which allows encoding a 6-level state into a set of three such subsystems. It is tempting to believe that the quantum design presented here will trigger further research on quantum combinatorics.

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Supplemental Material for “Thirty-six entangled officers of Euler: Quantum solution to a classically impossible problem”

I. ORTHOGONAL QUANTUM LATIN SQUARES AND 4-PARTITE ENTANGLED STATES.

In this section, we collect the definitions introduced in the literature in various contexts and demonstrate the equivalence between different notions used in the main body of the work.

Definition 1. AME state [S1, S2]. A pure quantum state, $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties, each of a local dimension d , is called *absolutely maximally entangled (AME)*, written as $|\text{AME}(N, d)\rangle$, if it is maximally entangled for every bipartition, i.e. the partial trace $\text{Tr}_S |\psi\rangle\langle\psi| \propto \mathbb{I}$, for any subsystem S of $|S| = \lfloor N/2 \rfloor$ parties.

In this work, we analyze the case of a fourpartite state, $N = 4$. A list of known AME states is available at [S3]. In general, an AME state consisting of an even number N of subsystems with d levels each leads to a pure quantum error correction code $((N, 1, N/2 + 1))_d$, which saturates the Singleton bound [S4].

Definition 2. Multiunitary matrix [S5]. A unitary matrix U of order d^2 is called *2-unitary* if the reshuffled matrix U^R and partially transposed matrix U^Γ are also unitary.

Matrices with the last condition relaxed also play a role in studies on many-body quantum dynamics: A unitary matrix U of size d^2 , for which U^R is unitary, is called *dual-unitary* [S6, S7]. In general, a matrix U of order d^k is called *k-unitary* (or multiunitary), if it remains unitary for all $(2k-1)(k-1)$ reorderings of $2k$ indices which define the matrix. Any 2-unitary matrix U of size d^2 describes a bipartite quantum gate, which maximizes the entangling power, $e_p(U) = 1$, see Supplementary Material Section 3.

Definition 3. Perfect tensor [S8]. A tensor $T_{i_1 \dots i_{2k}}$ with $2k$ indices, each running from 1 to d is called *perfect*, if any of its flattening into a matrix of order d^k is unitary.

The notion of orthogonal quantum Latin squares (OQLS) was introduced in [S9, S10]. Here we follow an alternative approach of Rico [S11] to assure full consistency with 2-unitarity. Let $|\chi^+\rangle = \sum_{i=1}^d |i, i\rangle$ denote the (non-normalized) maximally entangled Bell state of a 2-qudit system. Any bipartite quantum state, $|\psi\rangle = \sum_{k,\ell=1}^d C_{k,\ell} |k, \ell\rangle$, can be also written as $|\psi\rangle = (C \otimes \mathbb{I}) |\chi^+\rangle$.

Definition 4. OQLS. Consider a set of d^2 bipartite states in $\mathcal{H}_d \otimes \mathcal{H}_d$, which can be written in a product basis, $|\psi_{ij}\rangle = \sum_{k,\ell=1}^d C_{k\ell}^{i,j} |k, \ell\rangle$ for $i, j = 1, \dots, d$. Such a set forms an OQLS if (a) the states satisfy orthogonality relations, $\langle \psi_{ij} | \psi_{k\ell} \rangle = \delta_{ik} \delta_{j\ell}$; and the block matrix $\tilde{C}$ of size d^2 written $\tilde{C} = (C^{1,1}, \dots, C^{1,d}; \dots; C^{d,1}, \dots, C^{d,d})$ is block unitary, so that the conditions (b) $\sum_{i=1}^d C_{i,j}^{i,\ell} (C^{i,\ell})^\dagger = \delta_{j,\ell} \mathbb{I}$, and (c) $\sum_{j=1}^d C_{i,j}^{i,j} (C^{k,j})^\dagger = \delta_{i,k} \mathbb{I}$, are satisfied.

Observe that the orthogonality relations, equivalent to $\text{Tr } C^{i,j} (C^{k,\ell})^\dagger = \delta_{ik} \delta_{j\ell}$, correspond to the ‘different location’ conditions for strong sudoku satisfied by the 2-unitary permutations presented in Eq. S3 and S4. Furthermore, note that the block unitarity of $\tilde{C}$ implies that the related block matrix, $\tilde{B}$ with blocks $B^{i,j} = C^{i,j} (C^{i,j})^\dagger$ is block bistochastic, $\sum_{i=1}^d B^{i,j} = \mathbb{I}$, and $\sum_{j=1}^d B^{i,j} = \mathbb{I}$. Hence any OQLS described by the block matrix $\tilde{C}$ implies a block bistochastic matrix $\tilde{B}$ introduced in [S12], and recently studied in [S13] under the name of *quantum magic square*.

To show these notions in action, we shall analyze the case of a Graeco-Latin square (OLS) of size three, shown in Fig. 2. To get a set of 9 bipartite vectors $|\psi_{ij}\rangle$ in a given cell (A, α) we replace the classical symbols by a bipartite quantum state $|A, \alpha\rangle$. Thus each state has a product form, $|\psi_{i,j}\rangle = |i \oplus j\rangle \otimes |i \oplus 2j\rangle$, which is a consequence of the fact that OLS are classical. Notice that the operations inside kets are performed modulo 3.

The same information is encoded in the tensor determined by

$$T_{ijkl} = \delta_{k, i \oplus j} \delta_{\ell, i \oplus 2j}, \quad (\text{S1})$$

with addition operations performed modulo 3. It is easy to check that this tensor is perfect. Furthermore, the corresponding state

$$|\text{AME}(4, 3)\rangle = \frac{1}{3} \sum_{i,j=1,2,3} |i\rangle |j\rangle |i \oplus j\rangle |i \oplus 2j\rangle \quad (\text{S2})$$

belongs to AME states, as all reduced density matrices to two qutrits are equal to $\mathbb{I}_9/9$. One may present the tensor T_{ijkl} in a form of 9×9 permutation matrix P_9 , where non-vanishing element of T_{ijkl} corresponds to a non-zero entry of P_9 on the intersection

of $j + 3(i - 1)$ -th row with $\ell + 3(k - 1)$ -th column. In other words, the corresponding permutation matrix

$$P_9 = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad (S3)$$

is 2-unitary, since the reshuffled and partially transposed matrices

$$P_9^R = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad \text{and} \quad P_9^\Gamma = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \quad (S4)$$

also form permutation matrices. Observe that positions of non-zero entries obey the rules of a strong Sudoku: in each row, column and block there is a single entry equal to 1. Furthermore, all the locations of these entries in each block are different.

Let us return now to the general case of an arbitrary dimension d and relax the assumption that the 2-unitary matrix of order d^2 is a permutation. To demonstrate relations between different notions introduced independently in different communities we recall the equivalence relations.

Proposition 1. *The following statements are equivalent:*

1. *There exist orthogonal quantum Latin squares of size d ,*
2. *There exists an AME state of four subsystems with d levels each,*
3. *There exists a 2-unitary matrix U of size d^2 ,*
4. *There exists a perfect tensor with four indices, each running from 1 to d .*

To show that this is the case, note first that definition 3 of a perfect tensor is equivalent to definition 2 of a multiunitary matrix. Furthermore, these properties imply that the state given by Eq. 1 has all reductions of size two maximally mixed, so according to definition 1 it forms an AME state $|\Psi_{ABCD}\rangle$ of four parties. The three constraints, (a), (b) and (c) in definition 4 are equivalent to the fact that partial traces over selected subsystems, CD , BD and AD , respectively are maximally mixed, so existence of OQLS and an AME state of the corresponding system, is equivalent.

Observe that a matrix U is 2-unitary if and only if the corresponding matrix U^R , or equivalently U^Γ , is 2-unitary. Indeed, all three corresponding states are related by the permutation of subsystems. Note that conditions imposed on AME states are invariant under the permutation of subsystems.

Furthermore, a 2-unitary matrix U remains 2-unitary after any local operation, i.e. $U \mapsto \tilde{U}(U_A \otimes U_B)U(U_C \otimes U_D)$ where $U_A, U_B, U_C, U_D \in \mathbb{U}(6)$ are unitary matrices of size 6. Thus, by this mean one can entangle officers from a classical OLS, creating *apparently* quantum solution of OQLS. By apparent, we express the fact that it can be disentangled to a classical OLS by applying reverse local operation $(U_A^\dagger \otimes U_B^\dagger)\tilde{U}(U_C^\dagger \otimes U_D^\dagger)$. Observe that a similar disentangling procedure is not feasible in the case of any AME(4, 6) state, since it could not have been produced using classical designs. Consequently, we call our result a *genuinely* quantum solution to OQLS of size $d = 6$.

Note that the state AME(4, 3) corresponds to a classical OLS, as the 2-unitary matrix is a permutation P_9 , so the entries of the design are product states. We are not aware whether for $d = 3$ there exist genuinely quantum orthogonal Latin squares. However, such a design exists for $d = 6$, as will be demonstrated below.

II. PROOF OF THEOREM 1 ON THE EXISTENCE OF AME(4, 6)

In this section we prove Theorem 1 by construction. We introduce a matrix $\mathcal{U}$ of size 36 and show that it is 2-unitary. Its non vanishing entries, determined by the tensor, $\mathcal{U}_{p,s} = T_{ijkl}$, with $p = j + 6(i - 1)$ and $s = \ell(k - 1)$, are shown in Fig 3. To show 2-unitarity of the matrix $\mathcal{U}$ (also denoted $\mathcal{U}_{36}$ in the main text) we verify that three related matrices: $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ are unitary. By definition 2, this implies that the matrix $\mathcal{U}^R$ is 2-unitary, which is equivalent to the fact that the matrix $\mathcal{U}$ is 2-unitary. Each of those matrices has the structure of nine 4×4 blocks. Hence our task simplifies to verification that constituent blocks are unitary matrices.

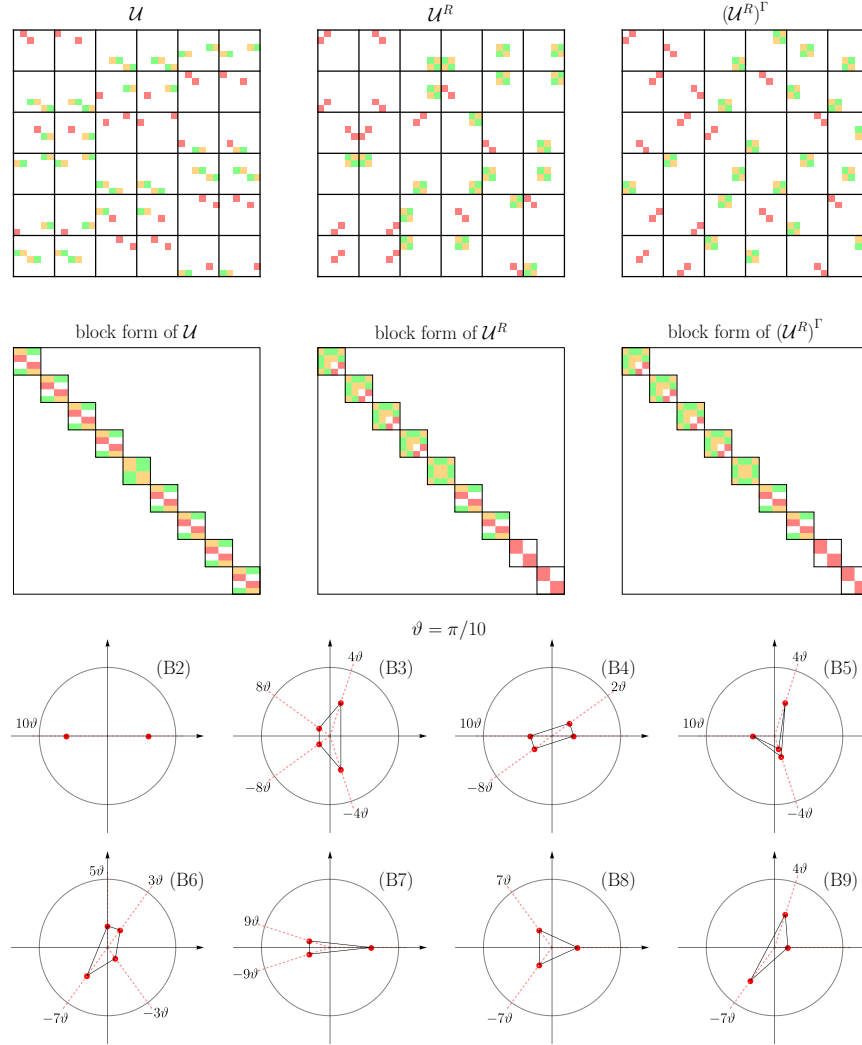


FIG. S1: Structure of three related matrices $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ of order 36 are presented on top. The modulus of a non-vanishing element is represented by the intensity of the background color. Note that similarly to rows and columns, blocks also satisfy orthogonality conditions b) and c) from definition 4 of OQLS. Each matrix has the structure of nine 4×4 blocks, the structure within the blocks, however, is different for matrix $\mathcal{U}$ and matrices $\mathcal{U}^R$, $(\mathcal{U}^R)^\Gamma$. Orthogonality between pairs of rows in block matrices might be presented as a constellation of two, three, or four points on the complex plane which sum up to zero. Constellations related to Eqs. S5-S12 are indicated. In most cases, constellations are simply pairs or double pairs of antipodal points, which clearly sum up to zero. There are six non-trivial constellations, all of them are depicted. Note that all phases are multiples of $\vartheta = \pi/10$.

Consider the 2-unitary matrix $\mathcal{U}$. Interestingly, except for one block component in the matrix $\mathcal{U}$, all eight block components

are equivalent (up to a multiplication of rows and columns by adequate phases) to the following 4×4 unitary matrix:

$$V = \begin{bmatrix} a & a & b & b \\ 0 & 0 & c & -c \\ c & -c & 0 & 0 \\ b & b & -a & -a \end{bmatrix}.$$

Orthogonality relations between rows in the matrix above might be presented as pairs of antipodal points on the complex plane, for example, orthogonality between the first two rows reads

$$bc(1 - 1) = 0. \quad (\text{S5})$$

The exceptional block of matrix $\mathcal{U}$ is presented on the right top corner in Fig. 3. Six orthogonality relations between rows read

$$a^2(\omega^8 + \omega^{-8}) + b^2(\omega^4 + \omega^{-4}) = 0, \quad (\text{S6})$$

$$ab(1 + \omega^2 + \omega^{-8} - 1) = 0, \quad (\text{S7})$$

$$ab(\omega^{-2} + \omega^2 + \omega^{-8} + \omega^8) = 0,$$

up to a phase factor, with $\omega = \exp(i\pi/10)$. Each equation might be presented as a unitarity rectangle - a constellation of four points in the complex plane which sum up to zero, as it is shown on Fig. S1. Observe that the second and third equations above are relevant to two pairs of antipodal points on the complex plane. Geometric interpretation of the numbers a and b is shown in Fig. S2.

Although the matrices $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ enjoy the structure of nine 4×4 blocks, similar to $\mathcal{U}$, the particular arrangement inside their blocks is significantly different from the $\mathcal{U}$ matrix. Blocks in $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$ matrices are of four distinct types up to multiplication of their rows and columns by phase factors, see Fig. S1. Orthogonality relations between rows of both matrices reflect their diversified structure. In particular, we distinguished five additional orthogonality relations given by the following equations:

$$a^2\omega^4 + ab(\omega^{10} + \omega^{-4}) + b^2\omega^{-4} = 0, \quad (\text{S8})$$

$$a^2\omega^{-3} + ab(\omega^5 + \omega^3) + b^2\omega^{-7} = 0, \quad (\text{S9})$$

$$ab(\omega^{-4} + \omega^{-6}) + bc\omega^5 = 0, \quad (\text{S10})$$

$$ab(\omega^{-8} + \omega^{-2}) + ac\omega^5 = 0, \quad (\text{S11})$$

$$a^2 + b^2\omega^4 + bc\omega^{-7} = 0. \quad (\text{S12})$$

Related constellations are presented on Fig. S1. The above-listed equalities provide orthogonality between rows in the three matrices $\mathcal{U}$, $\mathcal{U}^R$ and $(\mathcal{U}^R)^\Gamma$.

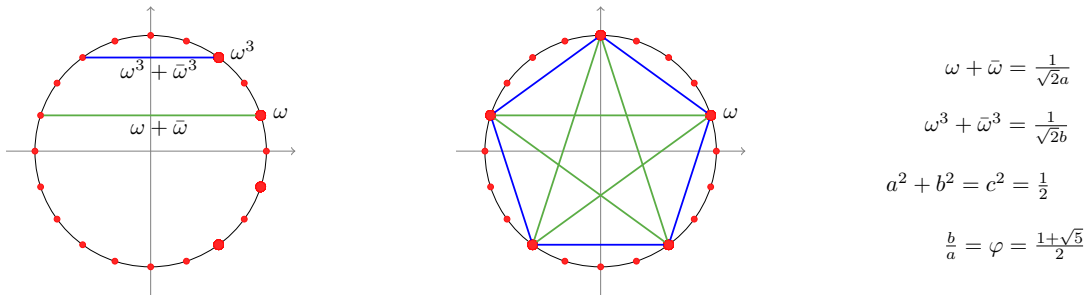


FIG. S2: Geometric interpretation of the constants a, b and c appearing in Eq. 4 with use of a regular pentagon. First two numbers contain the phase $\omega = \exp(i\pi/10)$ and are related to the golden ratio ϕ .

Observe that in the presented solution each officer is entangled with no more than 3 officers out of remaining 35. This implies the 2-unitary matrix $\mathcal{U}$ is sparse. An explicit form of the matrix $\mathcal{U}$ is available on line in several formats [S14], together with an explicit form of the corresponding AME(4, 6) state determined by $\mathcal{U}$ and Eq. 3. Further numerical results suggest that any 2-unitary matrix $\mathcal{U}_{36}$ has complex entries. Therefore, it is tempting to conjecture that there exists no solution in the set $\mathbb{O}(36)$ of orthogonal matrices of this size.

Three amplitudes a, b, c which appear in the presented construction might be defined as the unique solution of the following algebraic equations: $a^2 + b^2 = c^2 = 1/2$ and $b/a = \phi = (1 + \sqrt{5})/2$, see Fig. S2. Notice the similarities between algebraic equations which lead to values a, b, c , and the algebraic equations which lead to the amplitudes in a heterogenous AME state in $2 \times 3 \times 3 \times 3$ system presented in [S15].

The phases of the coefficients shown in Fig. 3 of the main body of the paper, being multiples of $\omega = \exp(i\pi/10)$, are chosen in such a way that all 36 quantum states $|\psi_{ij}\rangle$, each represented by a single row of the 2-unitary matrix $\mathcal{U}$, are equivalent to the standard, two-qubit Bell state. This is fact easy to see for any state formed by two coefficients of moduli, $|c| = 1/\sqrt{2}$, as states $|\psi_{11}\rangle$ or $|\psi_{56}\rangle$ represented in the second and the third line in the upper left block in Fig. 3 respectively. To show that this property holds also for other states, note that the state $|\psi_{63}\rangle$ corresponding to the first line of the aforementioned block can be written in the product basis as

$$|\psi_{63}\rangle = a\omega^{10}|11\rangle + b\omega^{15}|12\rangle + b\omega^5|21\rangle + a|22\rangle.$$

Thus the partial trace of the projector reads, $\text{Tr}_B |\psi_{63}\rangle\langle\psi_{63}| = \text{diag}(a^2 + b^2, a^2 + b^2) = \mathbb{I}/2$. This proves that $|\psi_{63}\rangle$ is locally equivalent to the maximally entangled Bell state. A similar reasoning works for all other states consisting of four terms and represented in Fig. 3 by green and yellow elements. Hence all 36 states, corresponding to 36 entangled officers of Euler, can be considered as maximally entangled, two qubit states.

III. GENERATING 2-UNITARY MATRIX OF ORDER 36 USING A DYNAMICAL MAP

Entangling power of a bipartite unitary gate U is defined as the mean entanglement produced by the gate, $e_p(U) = C_d \overline{\mathcal{E}(|\psi_{AB}\rangle)}$ with $|\psi_{AB}\rangle = U(|\phi_A\rangle \otimes |\phi_B\rangle)$. The average, indicated by the overline, is taken over the Haar measure of random states in each subsystem [S16] and we choose the normalization $C_d = (d+1)/(d-1)$ so that the maximum value of the entangling power is 1. As a measure of entanglement it is convenient to choose the linear entropy of the reduced density matrix, $\mathcal{E}(|\psi\rangle_{AB}) = 1 - \text{Tr}(\rho_A^2)$, where $\rho_A = \text{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$. An alternative approach to measure entanglement in a bipartite gate U of order d^2 is to use its operator Schmidt decomposition [S17] $U = \sum_{j=1}^{d^2} \sqrt{\lambda_j} A_j \otimes B_j$, where the matrices A_j and B_j form an orthonormal basis in the space of operators, while the Schmidt coefficients λ_j are given by squared singular values of the reshuffled matrix U^R – see Eq. 2. To quantify non-locality of the gate one uses the *operator entanglement*, defined by the linear entropy of the Schmidt vector, $E(U) = 1 - (\sum_{j=1}^{d^2} \lambda_j^2)/d^4$. It is convenient to introduce the SWAP operator S , defined by the relation $S(|\phi_A\rangle \otimes |\phi_B\rangle) = |\phi_B\rangle \otimes |\phi_A\rangle$. Note that S^R is unitary, which implies that $E(S) = 1 - 1/d^2$. It allows one to show a direct link between both quantities and express the entangling power of U in terms of the operator entanglement [S16],

$$e_p(U) = \frac{1}{E(S)} (E(U) + E(US) - E(S)), \quad (\text{S13})$$

which implies the normalization, $0 \leq e_p(U) \leq 1$. Lower bound, $e_p(U) = 0$ is saturated by any local gate of a product form, $U = u_A \otimes u_B$, and the SWAP gate S .

$\tilde{P} =$	11	22	33	44	55	66
	23	14	45	36	61	52
	32	41	64	53	16	25
	46	35	51	62	24	13
	54	63	26	15	42	31
	65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10♣	9*
K♦	A♥	J♣	Q*	9♠	10♣
Q♣	J♠	9♥	10♦	A*	K♣
J*	Q♣	10♠	9♣	K♥	A♦
10♥	9♦	K*	A♣	J♣	Q♠
9♣	10*	A♣	K♠	Q♦	J♥

FIG. S3: The design in $d = 6$ that is the closest possible to being an OLS [S19]. The two pairs of marked entries are repeated and not all possible 36 pairs are found.

Entangling power does not distinguish between locally inequivalent gates like U and US , as $e_p(U) = e_p(US)$. To distinguish

them one can use a complementary quantity called *gate typicality* [S18],

$$g_t(U) = \frac{1}{2E(S)} \left(E(U) - E(US) + E(S) \right), \quad (\text{S14})$$

satisfying $0 \leq g_t(U) \leq 1$. Lower bound, $g_t(U) = 0$, is attained by gates of a product form, while the upper bound $g_t(U) = 1$ is achieved by the SWAP gate and locally equivalent gates. Any 2-unitary matrix $\mathcal{U}$ is maximally non-local, also if composed with SWAP, and one has $E(\mathcal{U}) = E(\mathcal{U}S) = E(S)$, so that $g_t(\mathcal{U}) = 1/2$ and $e_p(\mathcal{U}) = 1$. Note that there is no 2-unitary matrix of size $d^2 = 4$, for which the maximal value $e_p = 1$ of the entangling power is achieved [S16, S19]. This is equivalent to the fact that there are no AME states for a four-qubit system [S20].

To look for 2-unitary matrices of size d^2 using the dynamical map, $U_{n+1} = \mathcal{M}_{\Gamma R}[U_n]$, presented in the main body of the text one needs to find an appropriate initial matrix. While for $d = 3$ a random unitary matrix of order nine with a significant probability generates a 2-unitary matrix, finding a suitable seed for $d = 6$ is considerably more demanding. It is natural to consider the design which gives the best approximation to a Graeco-Latin square [S19] and this is shown in Fig. S3.

$\tilde{P}_s =$	11 22 33 44 55 66	$=$	$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$	$J\heartsuit$	$10\clubsuit$	$9\spadesuit$
	23 14 45 36 61 52		$K\diamondsuit$	$A\heartsuit$	$J\clubsuit$	$Q\spadesuit$	$9\spadesuit$	$10\clubsuit$
	32 41 64 53 16 25		$Q\clubsuit$	$J\spadesuit$	$9\heartsuit$	$10\diamondsuit$	$A\spadesuit$	$K\clubsuit$
	46 35 51 62 24 13		$J\spadesuit$	$Q\clubsuit$	$10\spadesuit$	$9\clubsuit$	$K\heartsuit$	$A\diamondsuit$
	64 56 26 15 43 31		$9\heartsuit$	$10\spadesuit$	$K\spadesuit$	$A\clubsuit$	$J\diamondsuit$	$Q\spadesuit$
	55 63 12 21 42 34		$10\clubsuit$	$9\diamondsuit$	$A\clubsuit$	$K\spadesuit$	$J\clubsuit$	$Q\heartsuit$

FIG. S4: The design in $\tilde{P}_s$ does not form an OLS of dimension 6 since two pair of ranks are in the same columns. Its small perturbation used as a seed to the algorithm gives the 2-unitary matrix $\mathcal{U}_{36}$ and the desired state $|\text{AME}(4, 6)\rangle$.

A permutation matrix of order 36, denoted as P_{36} , can be obtained from $\tilde{P}$: if P_{36} is partitioned into 6×6 blocks, the only nonzero ($= 1$) entry of the ij -th block is given by the corresponding entry in $\tilde{P}$. For example, the $(5, 1)$ block is such that its 4th column and 5th row is 1. This has the maximum entangling power over all permutations of order 36 and is given by $e_p(P_{36}) = 314/315 \approx 0.996825$. If one starts with P_{36} as the seed, then $e_p(U_n) \rightarrow e_p(A) = 419/420 \approx 0.9976$ as $n \rightarrow \infty$ where $U_n = \mathcal{M}_{\Gamma R}^n[P_{36}]$. Interestingly, A is an orthogonal matrix such that A^Γ is unitary but A^R fails to be one.

This is an encouraging result and gives a way to explore unitary operators which have entangling powers larger than that of P_{36} . Similarly, if we start in the neighborhood of P_{36} , *i.e.* we take seeds of the form $P_{36} \exp(i\varepsilon H)$ where $\varepsilon \ll 1$ and H is Hermitian; $H = (M + M^T)/2$ with entries of M sampled from the normal distribution, then the map converges to unitaries with larger values of entangling power. However, $e_p(U_n) \approx 0.9991$ is the largest value obtained by us while starting in the vicinity of P_{36} (and for sufficiently large n).

In order to generate a 2-unitary matrix of size 36, one can search in the vicinity of other permutation matrices. Since the search space is very large ($36! \approx 10^{41}$), we restrict our attention to permutations that are in the vicinity of P_{36} . One of the permutations that works, denoted below as P_s , can be constructed from the design in Fig. S4 as described above.

The matrix $\tilde{P}_s$ differs from $\tilde{P}$ in the last two rows, see Fig. S4. The entangling power of the permutation matrix P_s is $e_p(P_s) = 104/105 \approx 0.9905$. Much better results can be obtained if one starts with a matrix from the neighborhood of P_s , as indicated above. The dynamical map with a finite probability ($p \approx 6\%$) converges to a 2-unitary matrix such that $e_p(U_n) = 1$ up to a machine precision for $n \sim 10^3$. Trajectories of some initial seed unitaries in the neighborhood of P_s are shown in Fig. S5. Every third iteration is marked. Most of these seed unitaries converge to strongly attracting fixed points of the map, such as A or AS (local extrema) while a few converge to 2-unitaries (global extrema). Nearby initial conditions with almost the same values of entangling power and gate typicality converge to different fixed points and explain the complex dynamics induced by the map on the (e_p, g_t) plane. The iteration procedure and convergence of the trajectory U_n to 2-unitary matrix on the (e_p, g_t) plane is visualised in Fig. S5. Convergence speed and the accuracy obtained is quantitatively described in Fig. S6. and implies that after a sufficiently large number of iterations, matrix U_n becomes 2-unitary. The output matrix, $\mathcal{U} = U_n$, can be put in a block diagonal form consisting of three blocks of size 12. There are also other ways of getting appropriate seeds to generate 2-unitary operators, such as enphasing of the permutation P_s (equivalent to multiplication by a diagonal unitary).

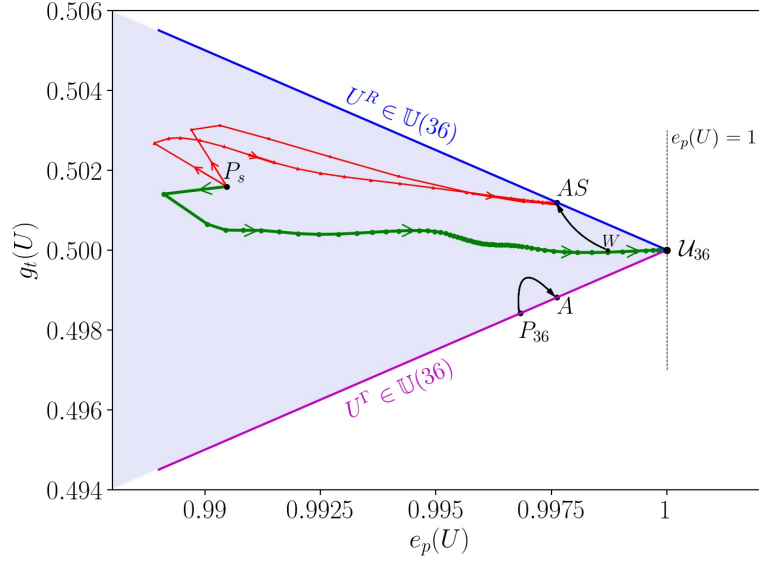


FIG. S5: A sample of trajectories initiated in the neighborhood of P_s and generated by the map $\mathcal{M}_{\Gamma R}$ are shown, where every third iteration is marked. Many of the trajectories end up at a strongly attracting fixed point labelled A or its partner that is multiplied by the SWAP gate S , AS while a small fraction are able to reach the 2-unitary fixed point, $\mathcal{U}_{36}$. If the map is initiated from the best approximation to an OLS, P_{36} , or from its neighborhood the trajectory ends up mostly at the point corresponding to the matrix A , which has a larger entangling power than P_{36} . If one starts from W which is an orthogonal matrix with an even larger entangling power than A , it ends up at the point corresponding to AS .

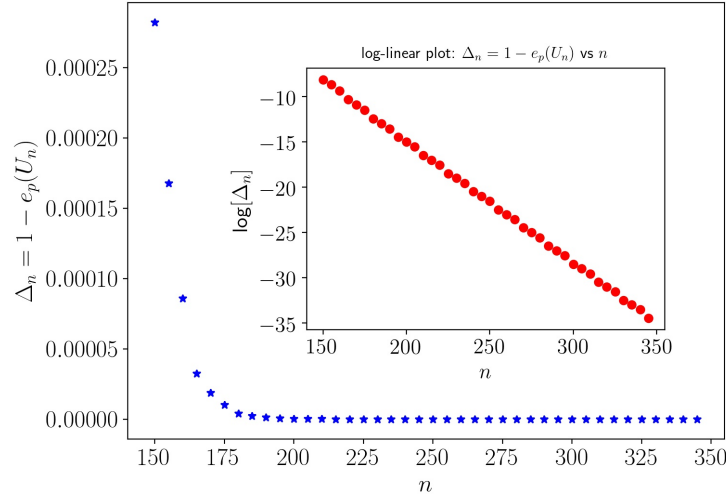


FIG. S6: Convergence of a trajectory initiated in the neighborhood of P_s to a 2-unitary matrix quantified by the deviation $\Delta_n = 1 - e_p(U_n)$ is plotted as a function of the number n of iterations. For $n > 150$ the deviation Δ_n decays exponentially as shown in the inset.

IV. AN APPARENT SOLUTION OF THE EULER'S PROBLEM FOR $d = 6$

An interesting practical application of the standard Euler's problem of 36 officers is worth to be mentioned here. A puzzle called *36cuBe* designed in 2008 by D. C. Niederman is directly linked to this mathematical question: the player obtains 36 pieces of six colors and of six different heights and is supposed to place them in the square to obey all the rules of OLS - see Fig. S7.

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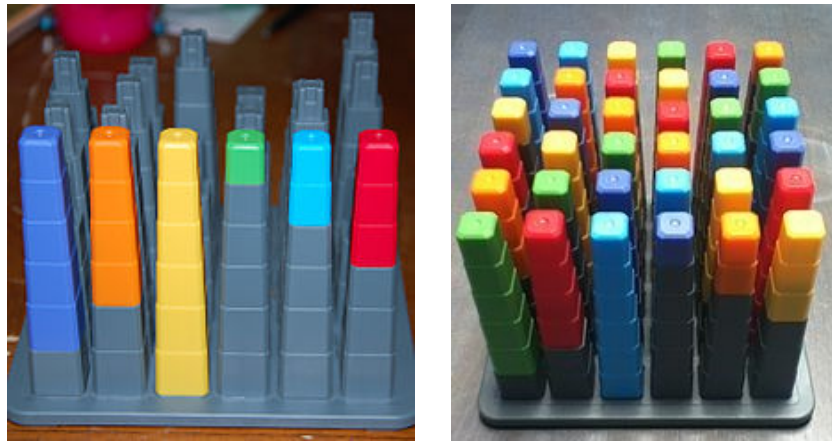


FIG. S7: Left panel: Puzzle 36cuBe advertised as *the World's Most Challenging Puzzle* consisting of 36 pieces of six colors and six heights and (right panel) its apparent solution. Closer inspection of all the pieces reveals that it corresponds to the pattern presented in Fig. S4, as two columns contain two pieces of the same height, in agreement with the conjecture of Euler and the theorem of Tarry.

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E Entanglement Classification via Single Entanglement Measure

Entanglement Classification via Single Entanglement Measure

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We provide necessary and sufficient conditions for generic n -qubit states to be equivalent under Stochastic Local Operations with Classical Communication (SLOCC) using a single polynomial entanglement measure. SLOCC operations may be represented geometrically by Möbius transformations on the roots of the entanglement measure on the Bloch sphere. Moreover, we show how the roots of the 3-tangle measure classify 4-qubit generic states, and propose a method to obtain the normal form of a 4-qubit state which bypasses the possibly infinite iterative procedure.

Quantum entanglement is one of the key manifestations of quantum mechanics and the main resource for technologies founded on quantum information science. In particular, quantum states with non-equivalent entanglement represent distinct resources which may be useful for different protocols. The idea of clustering states into classes exhibiting different qualities under quantum information processing tasks resulted in their classification under stochastic local operations assisted by classical communication (SLOCC). Such a classification was successfully presented for two, three and four qubits [1–4]. However, the full classification of larger systems is completely unknown. Even the much simpler problem of detecting if two n -qubit states ($n > 4$) are SLOCC-equivalent is, in general, quite demanding [5–8].

Among several approaches to the entanglement quantification and classification problem, a particularly useful one is via *SL-invariant polynomial* (SLIP) measures. Well-known examples are concurrence and 3-tangle, which measure the 2-body and 3-body quantum correlations of the system [9, 10]. SLIP measures provide not only a convenient method for entanglement classification but also its practical detection. Indeed, it was shown that almost all SLOCC equivalence classes can be distinguished by ratios of such measures [3]. Any given two n -qubit states are then SLOCC-equivalent if a *complete set of SLIP measures* has the same values for both of them [5]. For more than four qubits, however, the size of such a set grows exponentially, making it intractable to use this approach to discriminate SLOCC-equivalent states with more than four qubits [11].

In this Letter, we show that a single SLIP measure is enough to provide necessary and sufficient conditions for any two generic pure n -qubit states to be SLOCC-equivalent, by looking at how the roots of the SLIP measure for those states behave under SLOCC. In essence, if the states are SLOCC-equivalent, then the roots of the

SLIP measure for each state must be related by a Möbius transformation, which is straightforward to verify. We use this procedure to show how the 3-tangle measure is enough to discriminate between generic 4-qubit states. Finally, we show how one may use the roots of a SLIP entanglement measure to obtain the normal form of a 4-qubit state which bypasses the possibly infinite iterative standard procedure.

Polynomial Invariant Measures—An *entanglement measure* is a function $E(|\psi\rangle)$ defined for pure states of n qubits which vanishes on the set of separable states. One of the desired features of entanglement measures is invariance under SLOCC operations. Mathematically, a SLOCC operation might be uniquely determined by the action of local invertible operators $L \in \text{SL}(2, \mathbb{C})^{\otimes n}$ [1]. An entanglement measure $E(|\psi\rangle)$ defined on the system of n qubits is called a *SL-invariant polynomial of homogeneous degree h* if it is polynomial in the coefficients of a pure state $|\psi\rangle$ and satisfies

$$E(\kappa L |\psi\rangle) = \kappa^h E(|\psi\rangle) \quad (1)$$

for each real constant $\kappa > 0$ and invertible linear operator L [1, 3, 12]. We denote such a measure as SLIP_n^h , where the upper index indicates the degree of the polynomial and the lower index is related to the number of qubits.

Any SLIP measure E can also be extended to mixed states by determining the largest convex function on the set of mixed states which coincides with E on the set of pure states [13]. Despite its simple definition, the evaluation of a convex roof extension requires non-linear minimization procedure, and for a general density matrix is a challenging task [14–17]. An attempt to address this challenging task was carried out by introducing the so-called *zero-polytope*, the convex hull of pure states with vanishing E measure [18–21]. In the particular case of rank-2 density matrices ρ , the zero-polytope can be represented inside a Bloch sphere, spanned by the roots of

E [19, 22]. We adapt this approach focusing only on the roots of polynomial invariants, equivalently the vertices of the zero-polytope.

System of roots—Consider a $(n+1)$ -partite qubit state $|\psi\rangle$. The state $|\psi\rangle$ can be uniquely written as

$$|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle, \quad (2)$$

providing the canonical decomposition of the reduced density matrix $\rho = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$ obtained by tracing out the first qubit. Note that the states $|\psi_0\rangle$ and $|\psi_1\rangle$ are in general neither normalized nor orthogonal. Consider now the family of states

$$|\psi_z\rangle = z |\psi_0\rangle + |\psi_1\rangle, \quad (3)$$

where z is taken from the extended complex plane $\hat{\mathbb{C}}$, i.e., complex numbers plus infinity. We denote this the *extended plane representation*. In addition, consider any SLIP_n^h measure E defined on the set of n -partite pure qubit states. Since E is polynomial in the coefficients of $|\psi_z\rangle$, it is also polynomial in the complex variable z [18]. Therefore, the polynomial $E(z |\psi_0\rangle + |\psi_1\rangle)$ has exactly h roots: $\zeta_1, \dots, \zeta_h$ (which may be degenerated and/or at infinity), related to the degree of E . By using the complex number z , the states $|\psi_z\rangle$ can be mapped to the surface of a sphere via the standard *stereographic projection* $(\theta, \phi) := (\arctan 1/|z|, -\arg z)$ written in spherical coordinates. This way, a point on the unit 2-sphere (θ, ϕ) can be associated with the quantum state

$$|\tilde{\psi}_z\rangle := \cos \frac{\theta}{2} |\psi_0\rangle + \sin \frac{\theta}{2} e^{i\phi} |\psi_1\rangle \quad (4)$$

with $z = \text{ctg}(\theta/2) e^{-i\phi}$, such that $|\psi_0\rangle$ lies in the North pole and $|\psi_1\rangle$ lies in the South pole, see Figure 1. We denote this the *Bloch sphere representation*. Note that $|\tilde{\psi}_z\rangle \propto |\psi_z\rangle$ and that neither of these states is normalized, since $|\psi_0\rangle$ and $|\psi_1\rangle$ are not normalized in general either.

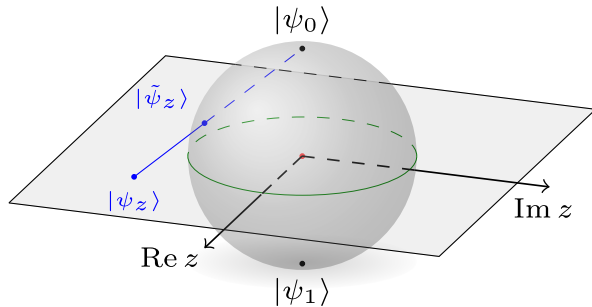


Figure 1. The stereographic projection relating the family of states $|\psi_z\rangle$ on the extended complex plane with the associated family of states $|\tilde{\psi}_z\rangle$ on the Bloch sphere. The spherical coordinates (θ, ϕ) and the complex coordinate z are related via the stereographic projection $z = \text{ctg}(\theta/2) e^{-i\phi}$.

Local operations on the system of roots—To each linear invertible operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one may associate a Möbius transformation $z \mapsto z' := \frac{az+b}{cz+d}$, mapping the extended complex plane $\hat{\mathbb{C}}$ into itself [23, 24]. The composition of such transformations represents the multiplication of the associated operators. In particular, $z \mapsto z' := \frac{dz-b}{-cz+a}$ is an inverse Möbius transformation related with $\mathcal{O}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Note that although Möbius transformations are typically represented on the extended complex plane, one may represent them as transformations on the Bloch sphere via the stereographic projection. The correspondence between invertible operators and Möbius transformations represented on the Bloch sphere was already successfully used for SLOCC classification of permutation-symmetric states [25–27].

To study the effect of SLOCC operations on the system of roots we begin by acting on the first qubit of a state $|\psi\rangle$ written in the form of Eq. (2) with an invertible linear operator $\mathcal{O}$. In terms of the family of states $|\psi_z\rangle$ in Eq. (3), this operation induces the map

$$|\psi_z\rangle \mapsto |\psi_{z'}\rangle = \frac{az+b}{cz+d} |\psi_0\rangle + |\psi_1\rangle, \quad (5)$$

i.e. the index is mapped via the Möbius transformation $z \mapsto z' := \frac{az+b}{cz+d}$ (see Supplementary Material Section 1). In addition, since $|\tilde{\psi}_z\rangle \propto |\psi_z\rangle$, we have that the family of states $|\tilde{\psi}_z\rangle$ also transforms according to Eq. (5). This reflects the fact that the states $|\psi_z\rangle$ and $|\tilde{\psi}_z\rangle$ associated to the extended complex plane and the Bloch sphere are related by a stereographic projection of the variable z . Using Eq. (5) and the defining equation $E(\zeta_i |\psi_0\rangle + |\psi_1\rangle) = 0$ for the roots ζ_i of the polynomial E , one concludes that the roots transform according to the inverse Möbius transformation associated to the operator $\mathcal{O}$, i.e. $\zeta_i \mapsto \frac{d\zeta_i-b}{-c\zeta_i+a}$. Finally, although the system of roots changes with local operations acting on the qubit that is being traced out in Eq. (2), it is invariant under local operations acting on any other qubit since the polynomial E is $\text{SL}(2, \mathbb{C})^{\otimes n}$ invariant. We summarize these results in the theorem below (see Supplementary Material Section 1 for a detailed proof).

Theorem 1. Consider an $(n+1)$ -partite pure quantum state $|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle$. The roots ζ_i of any SLIP_n^h entanglement measure associated to the partial trace of the first qubit:

1. are invariant under invertible operators, i.e. invariant under $\mathbf{1} \otimes \mathcal{O}_{\vec{n}} \in \text{SL}(2, \mathbb{C})^{\otimes n}$ operators;
2. transform via an inverse Möbius transformation $\zeta'_i = \frac{d\zeta_i-b}{-c\zeta_i+a}$ w.r.t the $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \mathbf{1}^n$ operator.

It is crucial to emphasize that normalizing the states $|\psi_0\rangle$ and $|\psi_1\rangle$ after the action of the operator $\mathcal{O}$, as is the case in existing related works [18, 19, 21, 22, 28], would spoil

the mapping of Eq. (5). As a consequence, the action of SLOCC operators on the states $|\psi_z\rangle$ would no longer be given by the corresponding Möbius transformation, and the statements in Theorem 1 would no longer hold.

State discrimination—The decomposition (2) can be performed with respect to any other subsystem, each with its own system of roots. This is particularly important for entanglement classification schemes where the presence of entanglement after partial trace is an aspect of focus, such as in [29, 30]. Any local operator $\mathcal{O}_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the k -th qubit will influence independently the corresponding k -th system of roots and its associated zero-polytope via the Möbius transformation $\zeta_i \mapsto \frac{d\zeta_i - b}{-c\zeta_i + a}$. On the other hand, if acting globally with a local operator $\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_{n+1}$, all roots (and thus all zero-polytopes) will be affected. Since a Möbius transformation is a bijective mapping on the Bloch sphere, the total number of roots will always be preserved [28]. Moreover, the existence of a local transformation between two given states becomes straightforward to verify since Möbius transformations are fully classified.

This way, Theorem 1 provides a solution for the problem of discriminating $(n+1)$ -qubit states up to SLOCC-equivalence [5–8]. Indeed, Möbius transformation associated with the SLOCC operation transforms a system of roots of polynomial invariants of a given state onto a system of roots of an equivalent state. To verify if two pure states are SLOCC-equivalent, one can thus use the following procedure:

- 1) Choose any SLIP_n^h entanglement measure of degree $h \geq 3$ and calculate its roots for each subsystem for both states (note that a generic state will always have h roots for each subsystem);
- 2) Focus on one subsystem i , $1 \leq i \leq n+1$ and choose 3 of the h roots from each state;
- 3) Write the unique Möbius transformations between the two triplets of roots and derive the local operator $\mathcal{O}_i$ associated to it;
- 4) Choose a different set of 3 out of the h roots of the second state and repeat step 3). Do this for all $3! \binom{h}{3}$ possibilities;
- 5) Repeat steps 3) and 4) for all other subsystems and then consider the tensor products of all the local operators obtained, resulting in a finite set of operators of the form $\mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_{n+1}$;
- 6) If the two given $(n+1)$ -qubit states are SLOCC-equivalent, one of these operators must transform one state into the other. Otherwise, they are not SLOCC-equivalent.

The above procedure has two main important features. Firstly, it factorizes the problem of finding SLOCC-equivalence, i.e. local operations are determined separately for each subsystem. Secondly, it discretizes the initial discrimination task since there are at most $(3! \binom{h}{3})^{n+1}$ local operators which might provide SLOCC equivalence between initial states.

State classification—Any three distinct points on the sphere can be transformed onto any other three distinct points via a unique Möbius transformation. While this is not the case for four points, it is possible to take any four complex points z_1, z_2, z_3, z_4 and associate a so-called *cross-ratio*

$$\lambda(z_1, z_2, z_3, z_4) := \frac{z_3 - z_1}{z_3 - z_2} \frac{z_4 - z_2}{z_4 - z_1}, \quad (6)$$

which is preserved under Möbius transformations [23, 26]. Systems of four distinct points are related via Möbius transformations if their cross-ratios are related in the same way. The cross-ratio is not invariant under permutations of points, however, and depending on the ordering taken for the four points, it takes six values: $\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}$ [26]. A particular interesting set of four points is one of the form $z, 1/z, -z, -1/z$, which we call a *normal system*. Any set of four points may be mapped into a normal system, for which $z, 1/z, -z, -1/z$ will be the solutions of the fourth degree equation $\lambda = 4z^2/(1+z^2)^2$, where λ is the corresponding cross-ratio from Eq. (6). Such a map is unique up to symmetries of the cube, i.e the group of 24 rotations generated by $\pi/2$ rotations along the X, Y, Z axis, denoted by $\mathcal{G}_{24}$.

It is particularly straightforward how one may use the normal system of roots for SLOCC-classification of small dimensional systems. Focusing first on the three-qubit case, genuinely entangled pure states are SLOCC-equivalent to either $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ or $|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ [31]. Using the 2-tangle $\tau^{(2)}$ [9] as the entanglement measure, one may use the roots to distinguish between the two classes. Indeed, all rank-2 reduced density matrices of the $|\text{W}\rangle$ state have a single root, while there are always two distinct roots for the $|\text{GHZ}\rangle$ state [22].

Contrary to the three qubit case, there are infinitely many SLOCC classes of four qubit states [31]. Although four qubit states were divided into nine families [2, 32, 33], we will focus on generic 4-qubit states, i.e. 4-qubit states with random coefficients belonging to the so called G_{abcd} family - the 4-qubit SLOCC family with the most degrees of freedom. The representative state is of the form $|G_{abcd}\rangle = \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle)$, where $a^2 \neq b^2 \neq c^2 \neq d^2$ are pairwise different. Choosing the 3-tangle $\tau^{(3)}$ [34] as the entanglement measure, the states $|G_{abcd}\rangle$ have four non-degenerate roots already in the normal form (see Supplementary Material Section 3).

Since the normal form of roots is unique up to the group $\mathcal{G}_{24}$, the problem of SLOCC-equivalence of states $|G_{abcd}\rangle$ becomes solvable, with a discrete amount of solutions. Indeed, it can quickly be confirmed if two states in the G_{abcd} class are SLOCC equivalent by checking if one can be obtained from the other by the action of an element of the finite class of operators $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. We thus find that exactly 192 states of the form $|G_{abcd}\rangle$ are SLOCC-equivalent.

Proposition 1. Two states $|G_{abcd}\rangle$ and $|G_{a'b'c'd'}\rangle$ are SLOCC-equivalent iff their coefficients are related by the following three operations: multiplication by a phase factor $(a', b', c', d') = e^{i\phi}(a, b, c, d)$, and permutation of coefficients $(a', b', c', d') = \sigma(a, b, c, d)$, and change of sign in front of two coefficients from a, b, c, d .

See Supplementary Material Section 3 for a more detailed analysis. The symmetry in Proposition 1 is given by the Weyl group of Cartan type D_4 and it has already been observed that the generators of four-qubit polynomial invariants exhibit this type of symmetry [32, 35]. As a consequence, this result constitutes a new relation between 4-qubit invariants and the convex roof extension of 3-tangle $\tau^{(3)}$, which may shed some light on the problem of generalizing the CKW inequality [10] for four qubit states [12, 16, 17, 36, 37], and beyond [36–38].

Finally, we remark a link between normal systems of roots and the normal form of pure states of four qubits belonging to the G_{abcd} class. A state is in its normal form if its reductions are maximally mixed [39]. Although the process to determine the normal form of a state (if it exists) is straightforward, it may also turn out to be an infinite iterative process [39]. However, the results of Theorem 1 applied to the four qubits states of the G_{abcd} class show that this difficulty can be avoided. Indeed, the representative state of the G_{abcd} family is in normal form [39] and its corresponding roots associated with the $\tau^{(3)}$ measure also form a normal system. Thus, one may calculate the roots associated with any state in the G_{abcd} class, quickly transform them into a normal system of roots using a Möbius transformation and use the associated SLOCC operator to convert the initial state into a state in the normal form. We illustrate this procedure by transforming the widely discussed four-partite state $\frac{1}{\sqrt{2}}(|0\rangle|GHZ\rangle + |1\rangle|W\rangle)$ [19, 21] into its normal form, see Figure 2. Without this technique, the standard way of obtaining the normal form would indeed result in an infinite iterative procedure.

Conclusions—In this letter, we showed how a single SLIP_n^h entanglement measure can be used to derive necessary and sufficient conditions for any two $(n+1)$ -qubit states to be SLOCC equivalent. This was possible by showing that the roots of any SLIP_n^h measure transform via Möbius transformations under the SLOCC operations performed on the subsystems. In that way, SLOCC

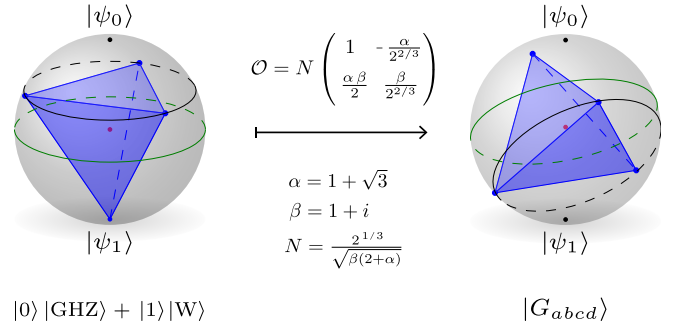


Figure 2. The system of four roots (represented as blue dots) related to the 3-tangle polynomial measure $\tau^{(3)}$ evaluated on the first subsystem of the state $1/\sqrt{2}(|0\rangle|GHZ\rangle + |1\rangle|W\rangle)$. This system of four points can be mapped into a normal system (i.e. symmetrically related points $z, -z, 1/z, -1/z$) by a Möbius transformation. Similar local transformations can be performed with respect to other subsystems, transforming the states into a state in the normal form.

equivalence between two states is implied by the easily verifiable existence of a Möbius transformation relating aforementioned roots for each subsystem. We demonstrated our approach on 4-qubit states, and showed that the roots of the 3-tangle measure $\tau^{(3)}$ are enough to classify 4-qubit states from the most generic G_{abcd} family. Lastly, a procedure was presented to determine the normal form of states in the G_{abcd} family that circumvents the possibility of an infinite iterative process of the standard procedure.

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Supplemental Material for “Entanglement Classification via Single Entanglement Measure”

SECTION 1. PROOF OF THEOREM 1

We present a proof for Theorem 1. Any $n + 1$ partite qubit state $|\psi\rangle \in \mathcal{H}_2^{\otimes(n+1)}$ might be written as

$$|\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle. \quad (\text{S1})$$

Such a form provides the canonical decomposition of the reduced density matrix $\rho = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$ over the non-normalized states $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}_2^{\otimes N}$, obtained by tracing out the first qubit. Consider now a reversible operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ acting on the first qubit. Under the action of this operator, the state $|\psi\rangle$ is transformed into

$$|\psi'\rangle := \mathcal{O}|\psi\rangle = |0\rangle (a|\psi_0\rangle + b|\psi_1\rangle) + |1\rangle (c|\psi_0\rangle + d|\psi_1\rangle) = |0\rangle |\psi'_0\rangle + |1\rangle |\psi'_1\rangle, \quad (\text{S2})$$

where

$$|\psi'_0\rangle := a|\psi_0\rangle + b|\psi_1\rangle, \quad (\text{S3})$$

$$|\psi'_1\rangle := c|\psi_0\rangle + d|\psi_1\rangle. \quad (\text{S4})$$

Consider now any superposition of states $|\psi'_0\rangle$ and $|\psi'_1\rangle$. Observe that

$$\begin{aligned} |\psi'_z\rangle &:= z|\psi'_0\rangle + |\psi'_1\rangle = z(a|\psi_0\rangle + b|\psi_1\rangle) + c|\psi_0\rangle + d|\psi_1\rangle \\ &= (az + b)|\psi_0\rangle + (cz + d)|\psi_1\rangle \\ &\propto \frac{az + b}{cz + d} |\psi_0\rangle + |\psi_1\rangle, \end{aligned}$$

where the complex number $cz + d$ was factored out in order for the transformation to map states from the extended plane representation to the extended plane representation. In other words, we have

$$\mathcal{O}|\psi_z\rangle = |\psi_{z'}\rangle, \quad z' = \frac{az + b}{cz + d}, \quad (\text{S5})$$

i.e. the operator $\mathcal{O}$ transforms states in the extended plane representation by applying a Möbius transformation on the index z . Suppose now that ζ_i is a zero of a h -degree polynomial function E , i.e. $E(\zeta_i|\psi_0\rangle + |\psi_1\rangle) = 0$. Acting on the first qubit with $\mathcal{O}$, the density matrix after tracing out the first qubit becomes $|\psi'_0\rangle\langle\psi'_0| + |\psi'_1\rangle\langle\psi'_1|$, so the entanglement measure E will be zero for some new roots ζ'_i , such that $E(\zeta'_i|\psi'_0\rangle + |\psi'_1\rangle) = 0$. Using Eqs. (S3)-(S4), the later equation can be transformed into

$$E\left((c\zeta'_i + d)\left(\frac{a\zeta'_i + b}{c\zeta'_i + d}|\psi_0\rangle + |\psi_1\rangle\right)\right) = 0 \quad (\text{S6})$$

where the factor $(c\zeta'_i + d)$ is irrelevant since any root multiplied by it will still be a valid root. Comparing with the equation for the roots before the action of $\mathcal{O}$, we reach the conclusion that the roots transform according to the inverse Möbius transformation as

$$\zeta'_i = \frac{d\zeta_i - b}{-c\zeta_i + a}, \quad (\text{S7})$$

under the action of the operator $\mathcal{O}$. As a consequence, the roots of the zero-polytope transform with respect to the inverse Möbius transformation associated to the operator $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Analyze now the case when $\mathcal{O}$ is a unitary operator $\mathcal{O} = \mathcal{U}$. Since any unitary operator $\mathcal{U}$ can be represented as a rotation $\mathcal{R} = \begin{pmatrix} \cos\alpha & \sin\alpha e^{i\phi} \\ -\sin\alpha e^{-i\phi} & \cos\alpha \end{pmatrix}$ (up to an irrelevant global phase), it will simply rotate the Bloch ball, together with the zero-polytope.

Consider now multilocal operators $\mathcal{O}_{\vec{n}} = \mathcal{O}_1 \otimes \dots \otimes \mathcal{O}_n$ acting on the remaining qubits of the state $|\psi\rangle$ from Eq. (S1). The state $|\psi\rangle$ will transform accordingly as

$$|\psi'\rangle := \mathcal{O}_{\vec{n}}|\psi\rangle = |0\rangle \underbrace{\mathcal{O}_{\vec{n}}|\psi_0\rangle}_{:=|\psi'_0\rangle} + |1\rangle \underbrace{\mathcal{O}_{\vec{n}}|\psi_1\rangle}_{:=|\psi'_1\rangle}. \quad (\text{S8})$$

After the action of $\mathcal{O}_{\vec{n}}$, a state in the extended plane representation will have a value of entanglement measure E equal to

$$E\left(z|\psi'_0\rangle + |\psi'_1\rangle\right) = E\left(\mathcal{O}_{\vec{n}}(z|\psi_0\rangle + |\psi_1\rangle)\right).$$

However, since E is $\text{SL}(2, \mathbb{C})^{\otimes n}$ invariant, we have that $E(z|\psi_0\rangle + |\psi_1\rangle) = 0$ iff $E(z|\psi'_0\rangle + |\psi'_1\rangle) = 0$, and so the roots of both polynomial equations are the same. As a consequence, the roots of the zero-polytope will remain unchanged under the action of $\mathcal{O}_{\vec{n}}$. This concludes the proof of Theorem 1.

SECTION 2. NORMAL FORM

Consider the set of four symmetrically related points $\Phi = \{z, \frac{1}{z}, -z, -\frac{1}{z}\}$. It is very convenient to associate with them the cuboid spanned by eight points:

$$\Phi \cup \bar{\Phi} = \left\{z, \frac{1}{z}, -z, -\frac{1}{z}, \bar{z}, \frac{1}{\bar{z}}, -\bar{z}, -\frac{1}{\bar{z}}\right\},$$

as it is presented on Figure S1. Observe, that all six faces of the cuboid are parallel to one of the planes: XZ, XY , or YZ . In fact, this property is equivalent to the initial assumption that the set of points Φ is in normal form. Clearly, all rotations of the Bloch ball preserve the form of the cuboid. Nevertheless, only a special subgroup of all rotations preserve faces of the cuboid being parallel to XZ, XY , or YZ . This special subgroup $\mathcal{G}_{24}$ contains 24 elements spanned by three rotations of $\pi/2$ around X, Y , and Z axis, given by:

$$\mathbf{R}_x(\pi/2) = \begin{pmatrix} \cos \pi/4 & -i \sin \pi/4 \\ -i \sin \pi/4 & \cos \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad (\text{S9})$$

$$\mathbf{R}_y(\pi/2) = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (\text{S10})$$

$$\mathbf{R}_z(\pi/2) = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} \quad (\text{S11})$$

In fact, this is a group of rotations preserving the regular cube (the group of orientable cube symmetries). Clearly, all rotations in the $\mathcal{G}_{24}$ group preserve the normal-form structure of Φ . On the other hand, the normal form is uniquely determined up to 24 rotations in the $\mathcal{G}_{24}$ group.

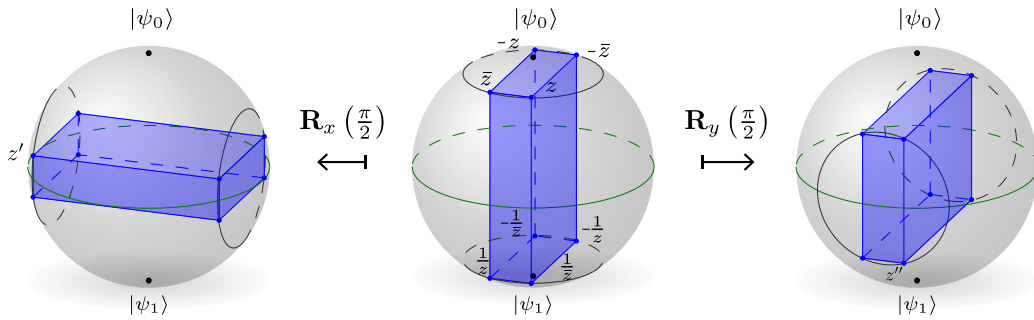


Figure S1. A normal system of roots $z, 1/z, -z, -1/z$ together with the conjugate points $\bar{z}, 1/\bar{z}, -\bar{z}, -1/\bar{z}$ span the cuboid whose faces are parallel to the XZ, XY and YZ planes. There are 24 rotations of the Bloch sphere which preserve this property, composing the elements of the group $\mathcal{G}_{24}$. Two of them, namely the rotation by a $\pi/2$ angle around X and Y axes are presented. The system of roots transforms according to Eqs. S9-S11, giving $z \mapsto z' := \frac{z-i}{-iz+1}$ and $z \mapsto z'' := \frac{z-1}{z+1}$ for the two rotations.

Proposition 2. Each non-degenerated four points z_1, z_2, z_3, z_4 on the Bloch sphere can be transformed onto the normal form $z, \frac{1}{z}, -z, -\frac{1}{z}$ via a Möbius transformation T . The latter is uniquely defined up to 24 rotations in the group $\mathcal{G}_{24}$.

Proof. For each complex number λ there exists another complex number z , such that the cross-ratio of the four points is equal to λ , i.e.

$$\left(z, \frac{1}{z}; -z, -\frac{1}{z}\right) = \lambda. \quad (\text{S12})$$

Indeed, the cross-ratio on the left side equals $4z^2/(1+z^2)^2$, and the equation $4z^2/(1+z^2)^2 = \lambda$ has exactly four solutions

$$z_0 = \frac{4 - 2\lambda + \sqrt{1 - \lambda}}{2\lambda}, \frac{1}{z_0}, -z_0, -\frac{1}{z_0}. \quad (\text{S13})$$

Therefore, for a given value λ there exists a unique λ -normal system, such that the cross-ratio of its vertices is given by $(z_0, \frac{1}{z_0}; -z_0, -\frac{1}{z_0}) = \lambda$. Replacing the vertex z_0 by any other vertex $z_0, 1/z_0, -z_0$, or $-1/z_0$ does not change the value of the cross-ratio $(z_0, \frac{1}{z_0}; -z_0, -\frac{1}{z_0}) = \lambda$. Note that there exists a unique Möbius transformation T which maps z_1, z_2, z_3 onto $z_0, 1/z_0, -z_0$, with the remaining z_4 mapped onto $-1/z_0$. Observe as well that the value of z_0 is unique up to its inverse, opposite and opposite inverse elements, according to Eq. (S13), with the corresponding Möbius transformations associated to the matrices $T, \sigma_x T, \sigma_y T$ and $\sigma_z T$. Each of those transformations maps the set of points $\{z_1, z_2, z_3, z_4\}$ onto the same set of points $\{z_0, 1/z_0, -z_0, -1/z_0\}$, although the exact bijection between those two sets of roots is different for each transformation.

Depending on the order of four points $\{z_1, z_2, z_3, z_4\}$, the corresponding cross-ratio takes six values: $\lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}$ and $\frac{\lambda}{\lambda-1}$. For each of these, there is a corresponding set of solutions of the form $\{z_0, 1/z_0, -z_0, -1/z_0\}$ via Eq. (S13) with four related Möbius transformations. Therefore, there are in total 24 Möbius transformations that map any four non-degenerated points onto a normal system, each of them related by an element of the group $\mathcal{G}_{24}$ which has exactly 24 elements. $\square$

SECTION 3. PROOF OF PROPOSITION 1

Consider the state $|G_{abcd}\rangle$ and its decomposition with respect to the first subsystem $|G_{abcd}\rangle = |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle$, where

$$\begin{aligned} |\psi_0\rangle &= \frac{a+d}{2}|000\rangle + \frac{a-d}{2}|011\rangle + \frac{b+c}{2}|101\rangle + \frac{b-c}{2}|110\rangle, \\ |\psi_1\rangle &= \frac{a+d}{2}|111\rangle + \frac{a-d}{2}|100\rangle + \frac{b+c}{2}|010\rangle + \frac{b-c}{2}|001\rangle. \end{aligned}$$

Suppose that $\tau^{(3)}(\zeta|\psi_0\rangle + |\psi_1\rangle) = 0$. Since $\tau^{(3)}$ is a $\text{SL}(2, \mathbb{C})^{\otimes 3}$ invariant, for any local operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ we have

$$\tau^{(3)}\left((\mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3)(\zeta|\psi_0\rangle + |\psi_1\rangle)\right) = 0.$$

Observe that

$$\begin{aligned} |\psi_0\rangle &= (\sigma_x \otimes \sigma_x \otimes \sigma_x) |\psi_1\rangle, \\ |\psi_1\rangle &= (\sigma_x \otimes \sigma_x \otimes \sigma_x) |\psi_0\rangle, \end{aligned}$$

where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices. Therefore by taking all local operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ equal to σ_x , we may conclude that

$$0 = \tau^{(3)}\left((\sigma_x \otimes \sigma_x \otimes \sigma_x)(\zeta|\psi_0\rangle + |\psi_1\rangle)\right) = \zeta|\psi_1\rangle + |\psi_0\rangle \propto \frac{1}{\zeta}|\psi_0\rangle + |\psi_1\rangle, \quad (\text{S14})$$

hence $1/\zeta$ is another root of $\tau^{(3)}$. Similarly, by considering $(\sigma_y \otimes \sigma_y \otimes \sigma_y)$ and $(\sigma_z \otimes \sigma_z \otimes \sigma_z)$, one may find another two roots $-\zeta, -1/\zeta$ of $\tau^{(3)}$. This shows that the roots of $\tau^{(3)}$ evaluated on any state from the G_{abcd} family are symmetrical with respect to rotations around X, Y, Z axes by the angle π . Writting $\tau^{(3)}(z|\psi_0\rangle + |\psi_1\rangle) = 0$ explicitly, we obtain the equation

$$\tau^{(3)}(z|\psi_0\rangle + |\psi_1\rangle) = Az^4 - 2(2B + A)z^2 + A = 0, \quad (\text{S15})$$

where $A = (b^2 - c^2)(a^2 - d^2)$ and $B = (c^2 - d^2)(a^2 - b^2)$. The above equation is non-degenerated iff $A, B, A + 2B \neq 0$, which happens iff $a^2 \neq b^2 \neq c^2 \neq d^2$ are pairwise different.

Lemma 1. Any local operator $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \mathcal{O}_3 \otimes \mathcal{O}_4 \in \text{SL}(2, \mathbb{C})^{\otimes 4}$ which transforms states $|G_{a'b'c'd'}\rangle \propto \mathcal{O} |G_{abcd}\rangle$ with $a^2 \neq b^2 \neq c^2 \neq d^2$, is of the form $\mathcal{O}_i \in \mathcal{G}_{24}$.

Proof. A local operator $\mathcal{O}_1$ acting on the first qubit and transforming the state $|G_{abcd}\rangle$ onto $|G_{a'b'c'd'}\rangle$, also transforms their systems of roots denoted as Λ and Λ' , respectively, via the action of the corresponding Möbius transformation. Note that both systems Λ and Λ' are in the normal form, therefore, according to Proposition 2, we have that $\mathcal{O}_i \in \mathcal{G}_{24}$. A similar analysis with respect to all other qubits shows that $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4 \in \mathcal{G}_{24}$. $\square$

This way, searching for SLOCC-equivalence between the states $|G_{abcd}\rangle$ and $|G_{a'b'c'd'}\rangle$ becomes restricted to the search within the finite class of operators $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. Since the group $\mathcal{G}_{24}$ has only 24 elements, one may numerically verify that there are exactly $8 \times 24 = 192$ states in the G_{abcd} family which are SLOCC-equivalent to $|G_{abcd}\rangle$ by $\mathcal{O} \in \mathcal{G}_{24}^{\otimes 4}$. For example, the following operation

$$\mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_x\left(\frac{\pi}{2}\right) \quad (\text{S16})$$

transforms state $|G_{abcd}\rangle$ into $|G_{-b-acd}\rangle$. This might be simply written as a transformation of a tuples of indices: the tuple (a, b, c, d) is transformed into the tuple $(-b, -a, c, b)$. Similarly, the operators showed on the following right hand sides provide the corresponding transformations of the tuple (a, b, c, d) on the left side:

$$\begin{aligned} \mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_y\left(\frac{\pi}{2}\right) &\longleftrightarrow (a, \textcolor{blue}{d}, c, \textcolor{blue}{b}), \\ \mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) \otimes \mathbf{R}_z\left(\frac{\pi}{2}\right) &\longleftrightarrow (-\textcolor{blue}{d}, b, c, -a), \\ \mathbf{R}_y(\pi) \otimes \mathbf{R}_y(\pi) \otimes \mathbf{1} \otimes \mathbf{1} &\longleftrightarrow (a, -\textcolor{blue}{b}, -\textcolor{blue}{c}, d), \\ \mathbf{R}_x(\pi) \otimes \mathbf{R}_x(\pi) \otimes \mathbf{1} \otimes \mathbf{1} &\longleftrightarrow (a, b, -\textcolor{blue}{c}, -\textcolor{blue}{d}), \\ \mathbf{R}_y(\pi) \otimes \mathbf{1} \otimes \mathbf{R}_y(\pi) \otimes \mathbf{1} &\longleftrightarrow (\textcolor{blue}{d}, c, \textcolor{blue}{b}, a), \\ \mathbf{R}_x(\pi) \otimes \mathbf{1} \otimes \mathbf{R}_x(\pi) \otimes \mathbf{1} &\longleftrightarrow (b, a, \textcolor{blue}{d}, c). \end{aligned}$$

Additionally, the tuples (a, b, c, d) and $(-a, -b, -c, -d)$ represent the same state. Note that any composition of the above operations also provides SLOCC equivalences between $|G_{abcd}\rangle$ states. The eight aforementioned transformations of tuples generate all permutations of the a, b, c, d indices, together with the change of a sign of any two or all four indices. There are exactly 24 permutations and for each permutation the signs can be matched in exactly $1 + \binom{4}{2} + 1 = 8$ ways. This gives in total 192 tuples representing SLOCC equivalent states, which perfectly matches the numerical result.

Finally, another trivial manipulation with indices a, b, c, d comes from multiplying by a global phase, which is an irrelevant operation due the fact that quantum states are elements of a projective space. This operation transforms the indices as

$$(e^{i\theta}a, e^{i\theta}b, e^{i\theta}c, e^{i\theta}d) \sim (a, b, c, d),$$

resulting in the same quantum state for any real number $\theta \in [0, 2\pi)$. In particular, for $\theta = \pi$, we observe that system of opposite indices determines the same state as the initial one, i.e. $(-a, -b, -c, -d) \sim (a, b, c, d)$.