
Time Crystal Phenomena

Doctoral dissertation by
Krzysztof Giergiel
PREPARED UNDER THE SUPERVISION OF
prof. dr hab. Krzysztof Sacha



ATOMIC OPTICS DEPARTMENT
INSTITUTE OF THEORETICAL PHYSICS
Jagiellonian University in Cracow
2021

Wydział Fizyki, Astronomii i Informatyki Stosowanej
Uniwersytet Jagielloński

Oświadczenie

Ja niżej podpisany Krzysztof Giergiel (nr indeksu: 1061900) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „Time Crystal Phenomena” jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr hab. Krzysztofa Sachy. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia

.....
(podpis doktoranta)

Kochanej Magdzie

Strzeszczenie

W tej rozprawie doktorskiej zebrane są prace przedstawiające analogii zjawisk znanych z fizyki ciała stałego odnalezione w domenie czasowej wraz z analizą ich eksperymentalnej realizacji. W załączonym zbiorze prac opisane są dwa układy doświadczalne - atomy Rydbergowskie w periodycznym polu elektrycznym i kondensat Bosego-Einsteina atomów grawitacyjnie opadających na wibrujące lustro atomowe. W kwantowych układach, które napędzane są periodycznie, w wyniku oddziaływań między materią dochodzi do realizacji faz materii znanych z fizyki układów periodycznych przestrzennie. W tych układach rozpoznaliśmy spontaniczne łamanie symetrii translacyjnej w czasie, lokalizację Andersona objawiającą się jako prawdopodobieństwo pomiaru cząstki eksponencjalnie zlokalizowane w czasie, układy dla których topologicznie chronione są struktury w czasie oraz czasowe analogi kwazikryształów. Cztery prace opisują analogii fizyki ciała stałego w czasie, a dwie z prac poświęcone są dogłębnej analizie powstającego układu doświadczalnego.

Abstract

This doctoral dissertation contains works presenting analogies of phenomena known from solid state physics found in the temporal domain, together with the analysis of their experimental implementation. The attached collection of works describes two experimental systems - Rydberg atoms in a periodic electric field and Bose-Einstein condensate of atoms gravitationally falling on a vibrating atomic mirror. In quantum systems that are driven periodically as a result of interactions between particles, the phases of matter known from the physics of spatially periodic systems are realized in the time domain. In these systems, we recognized the spontaneous breaking of time translational symmetry, Anderson's localization, appearing as an exponential localization of the probability of measurement of particles in time, systems where temporal structures are topologically protected, and temporal analogs of quasicrystals. Four of the works describe analogues of solid state phases in the time domain, the other two are devoted to an in-depth analysis of the experimental system, which is being built right now.

Lista publikacji zebranych w rozprawie doktorskiej

- [1] K. Giergiel and K. Sacha, "Anderson localization of a rydberg electron along a classical orbit," *Physical review A*, vol. 95, no. 6, p. 063402, 2017.
- [2] K. Giergiel, A. Miroszewski, and K. Sacha, "Time crystal platform: From quasicrystal structures in time to systems with exotic interactions," *Physical review letters*, vol. 120, no. 14, p. 140401, 2018.
- [3] K. Giergiel, A. Kosior, P. Hannaford, and K. Sacha, "Time crystals: Analysis of experimental conditions," *Physical Review A*, vol. 98, no. 1, p. 013613, 2018.
- [4] K. Giergiel, A. Dauphin, M. Lewenstein, J. Zakrzewski, and K. Sacha, "Topological time crystals," *New Journal of Physics*, vol. 21, no. 5, p. 052003, 2019.
- [5] K. Giergiel, A. Kuroś, and K. Sacha, "Discrete time quasicrystals," *Physical Review B*, vol. 99, no. 22, p. 220303, 2019.
- [6] K. Giergiel, T. Tran, A. Zaheer, A. Singh, A. Sidorov, K. Sacha, and P. Hannaford, "Creating big time crystals with ultracold atoms," *New Journal of Physics*, vol. 22, no. 8, p. 085004, 2020.

Pozostałe publikacje powstałe w trakcie doktoratu

- [1] M. Mierzejewski, K. Giergiel, and K. Sacha, "Many-body localization caused by temporal disorder," *Physical Review B*, vol. 96, no. 14, p. 140201, 2017.
- [2] K. Amini, J. Biegert, F. Calegari, A. Chacón, M. F. Ciappina, A. Dauphin, D. K. Efimov, C. F. de Morisson Faria, K. Giergiel, P. Gniewek, *et al.*, "Symphony on strong field approximation," *Reports on Progress in Physics*, vol. 82, no. 11, p. 116001, 2019.
- [3] A. Kuroś, R. Mukherjee, W. Golletz, F. Sauvage, K. Giergiel, F. Mintert, and K. Sacha, "Phase diagram and optimal control for n-tupling discrete time crystal," *New Journal of Physics*, vol. 22, no. 9, p. 095001, 2020.
- [4] P. Matus, K. Giergiel, and K. Sacha, "Anderson complexes: Bound states of atoms due to anderson localization," *Physical Review A*, vol. 103, no. 2, p. 023320, 2021.

I. WSTĘP

W tej rozprawie doktorskiej zebrane są prace [1–6] przedstawiające analogii zjawisk znanych z fizyki ciała stałego odnalezione w domenie czasowej wraz z analizą ich eksperymentalnej realizacji. W ostatniej dekadzie fizyka w wymiarze czasowym doczekała się równie intensywnych badań, co fizyka układów przestrzennych. W pracy Franka Wilczka z 2012 roku[7] stawia on pytanie, co do możliwości spontanicznego łamania symetrii translacji w czasie. Czy tak jak to się dzieje w przypadku powstawania struktur krystalicznych w przestrzeni, w naturze mogą w stanie podstawowym pojawiać się układy periodycznie ewoluujące w czasie? Próba odpowiedzi na to pytanie przez Wilczka okazała się niepoprawna - generalnie układy kwantowe w swoim stanie podstawowym, i dość ogólnie w stanach termicznych, nie mogą krystalizować w czasie[8, 9]. Niemniej praca ta zainspirowała naukowców do postawienia wielu innych pytań dotyczących obserwacji związanych z czasem. Dyskretna wersja krystalizacji w czasie została opisana w roku 2014 przez promotora tej pracy doktorskiej[10]. Opisane tu zjawiska związane są właśnie z takimi problemami.

W kwantowych układach, które napędzane są periodycznie, w wyniku oddziaływań między materią dochodzi do realizacji faz materii znanych z fizyki układów periodycznych przestrzennie. Eksperymentalnie zaobserwowano podwajanie i potrajanie częstości ewolucji przez szereg układów: łańcuchy spinów oparte na jonach [11], centrach NV^- w diamentach[12], spinach jądrowych [13–15], czy kondensacie Bosego-Einsteina [16]. W temacie dostępne jest coraz więcej artykułów przeglądowych[17–20] i książka[21].

W załączonym zbiorze prac opisane są dwa układy doświadczalne - atomy Rydbergowskie w periodycznym polu elektrycznym i kondensat Bosego-Einsteina atomów grawitacyjnie opadających na wibrujące lustro atomowe. W tych układach rozpoznaliśmy spontaniczne łamanie dyskretnej symetrii translacyjnej w czasie, lokalizację Andersona objawiającą się jako prawdopodobieństwo pomiaru cząstki eksponencjalnie zlokalizowane w czasie, układy dla których topologicznie chronione są struktury w czasie oraz czasowe analogi kwazikryształów. Dwie z prac poświęcone są dogłębnej analizie układu doświadczalnego i są częścią przygotowań do realizacji zaproponowanych doświadczeń w grupie prof. Petera Hannaforda, który jest współautorem tych artykułów.

W następującej części tego wstępu opiszę krótko czym są dyskretne kryształy czasowe. W kolejnej części opiszę, nieco bardziej technicznie, metody budowy modeli fizyki ciała stałego w czasie. Będzie to nieco odmienny punkt widzenia niż ten, który prezentuję w publikacjach będących właściwą częścią rozprawy doktorskiej. Uzbrojeni w tę wiedzę możemy przejść w ostatnim rozdziale do opisu proponowanego eksperymentu - kondensatu Bosego-Einsteina odbijającego się na wibrującym lustrze atomowym.

II. DYSKRETNE KRYSZTAŁY CZASOWE

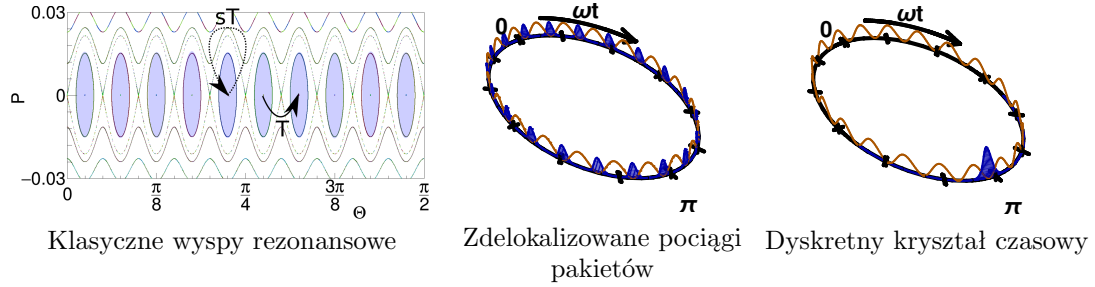
Termin dyskretne kryształy czasowe, lub równoważny - kryształy czasowe Floquet, opisuje układy, w których łamana jest dyskretna symetria translacyjna w czasie. Oznacza to, że rozważamy układy gdzie obecna jest siła periodycznie zależna od czasu. Odpowiedź układu, w wyniku oddziaływań między cząstkami, spontanicznie łamie narzucaną przez wymuszenie symetrię. Obserwowana struktura porusza się periodycznie z wielokrotnością okresu wymuszenia - co nazywamy dyskretnym kryształem czasowym.

Rozważmy teorię Floquet[22] - główne wyniki analizy periodycznych równań różniczkowych, zostały odkryte wielokrotnie i pod różnymi nazwami przez Georga Williama Hilla[23], Alexandra Lyapunowa[24] i wreszcie Felixa Blocha[25]. Centralny wynik w języku fizycznym jest nam doskonale znany. Stany własne periodycznego Hamiltonianu są periodyczne z tym samym okresem co Hamiltonian. Ważnym założeniem tego twierdzenia jest rozmiar przestrzeni rozwiązań. Jest

ono prawdziwe, jeśli rozmiar ten jest skończony. W granicy termodynamicznej (t.j. nieskończonej liczby oddziałujących cząstek) periodycznie zaburzony układ może spontanicznie wybrać inny, dłuższy okres ruchu.

Oryginalny pomysł stojący za dyskretnymi kryształami jest bardzo sprytny. Patrzymy na klasyczny periodycznie zaburzony układ jednocząstkowy. Rysujemy mapę wielu trajektorii w przestrzeni fazowej stroboskopowo, z okresem wymuszenia. Identyfikujemy w nim obszary przestrzeni fazowej, które są odwzorowane w przeciągu jednego okresu ewolucji na siebie. Możemy wśród nich znaleźć stany składające się z wybranej liczby s rozłącznych wysp rezonansowych. W takim przypadku dana wyspa odwzorowywana jest sama na siebie po s okresach ruchu. W procesie półklasycznej kwantyzacji, gdy objętość takich wysp rezonansowych jest wystarczająco duża, każda z nich zawiera zlokalizowany pakiet kwantowy poruszający się po klasycznej trajektorii. W wyniku przekrywania się eksponencjalnych ogonów takich pakietów stany własne układu są zdelokalizowane po wszystkich wyspach i tworzą superpozycję zlokalizowanych pakietów. Jeżeli przejdziemy do układu wielu bozonów wystarczy niewielkie oddziaływanie przyciągające, aby spontanicznie złamana została symetria i taka makroskopowa superpozycja została zniesiona.

Łamiący symetrię stan wielocząstkowy jest zlokalizowany w jednej wyspie rezonansowej i porusza się po klasycznej trajektorii, której okres jest s -krotnie dłuższy od okresu wymuszenia:



W ogólności wszystkie dotychczasowe kryształy czasowe Floquet działają na tej samej zasadzie. Stany o dłuższym okresie ewolucji - zlokalizowanej gęstości ładunku, bądź spinu w wyniku lokalnych oddziaływań stają się w granicy termodynamicznej zdegenerowanymi stanami własnymi problemu. Dla układu przygotowanego w takim stanie w wyniku spontanicznego łamania symetrii tylko jeden z takich stanów jest obsadzony i tym samym łamana jest symetria translacji w czasie o okres wymuszenia.

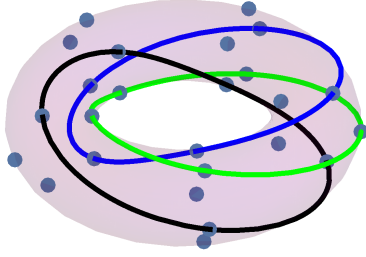
III. FIZYKA CIAŁA STAŁEGO W WYMIARZE CZASU

Klasyczne Hamiltoniany układów całkowalnych można zawsze zapisać jako:

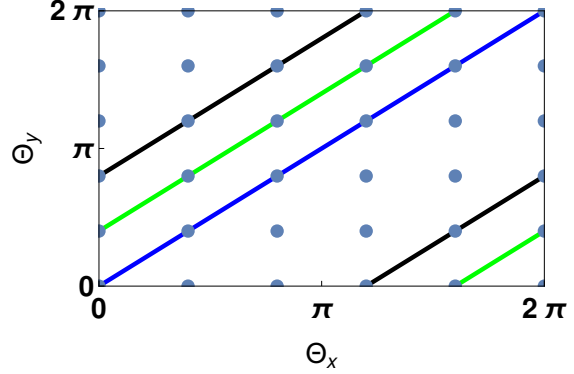
$$H_0 = f(J_1, J_2, J_3, \dots, J_d), \quad (\text{III.1})$$

gdzie (J_i, θ_i) są parami zmiennych kanonicznych, nazywanymi działaniem, kąt. Każdy zbiór całek ruchu $J_1, J_2, \dots, J_d$ wyznacza d wymiarowy torus $\mathbf{T}^d = S^1 \times \dots \times S^1$, na którym ruch cząstki opisany jest zmiennymi kątowymi $\theta_1, \theta_2, \dots, \theta_d$. Po powierzchni tego torusa nawijają się równoważne trajektorie, o różnych warunkach początkowych. Na przykład, gdy dla $d = 2$ zaburzymy periodyczny ruch cząstki na torusie wymuszeniem o s -krotnie wyższej częstotliwości będą na nim

wyróżnione punkty równoważne o translacje o $1/s$ -tą okresu niezaburzonego ruchu w kierunkach kątowych Θ_x i Θ_y . Takie warunki mogą prowadzić do dyskretnej sieci krystalicznej $s \times s$:



Schemat torusa, wraz z trajektoriami o identycznej szybkości w obu kierunkach kątowych.



Różne kolory opisują trajektorie, o różnych punktach początkowych. Punkty są tożsame z dokładnością o translacje czasowe o $1/6$ okresu ruchu cząstki.

Rozważmy teraz jednowymiarowy układ opisujący cząstkę uwięzioną w potencjale w zmiennych kąt działanie (1-wymiarowy torus to okrąg) i przeprowadzimy jego kwantyzację. Wówczas zamiast ciągłej funkcji działania otrzymujemy szereg stanów numerowany liczbą kwantową n :

$$\hat{H}_0 = \sum_n E_n |n\rangle \langle n|. \quad (\text{III.2})$$

Kolejne energie odpowiadają w przybliżeniu wartościom funkcji $f(n \cdot \hbar)$ z równania III.1. Energie są zawsze rosnące, gdyż funkcja $H_0(J) = f(J)$, jest rosnąca w miarę osłabiania energii wiązania w układzie. Procedura półklasycznej kwantyzacji polega na pokryciu przestrzeni fazowej (kąt, działanie) przez obszary o objętościach, różniących się od siebie o wielokrotność $\hbar$. Różnica energii między kolejnymi stanami to:

$$\Delta E_n = E_{n+1} - E_n \equiv \omega \approx \hbar \cdot H'_0(n \cdot \hbar) = \hbar \cdot \dot{\theta} \Big|_{J=n\hbar}. \quad (\text{III.3})$$

ΔE_n jest lokalną skalą energii, wprowadzoną przez kwantyzację, i równocześnie szybkością ruchu kołowego klasycznej zmiennej kąt, dla trajektorii o działaniu $J = n\hbar$. Można spodziewać się, że gdy będziemy układ zaburzać w rezonansie z powiązaną z tą energią skalą czasową to zaobserwować możemy ciekawe zjawiska.

Jak budować Hamiltoniany zależne od czasu? Wymaga to klasycznego zależnego od czasu pola (i od przestrzeni to jest o niezerowej pochodnej), z którym związana będzie siła wymuszająca. Dla silnych pól wiadomo, że przy dużej intensywności i dla wysoko wzbudzonych układów można bez problemu używać teorii półklasycznych zamiast ich kwantowych odpowiedników [26, 27]. Zapiszmy więc przykładowe wymuszenie:

$$V(x, t) = g(x) \cdot a(t), \quad (\text{III.4})$$

gdzie zależność od czasu jest periodyczna $a(t + T) = a(t)$. Przez okres zaburzenia T , wprowadziliśmy teraz nową skalę czasową, a przez funkcję $g(x)$, wprowadziliśmy przestrzenną strukturę oddziaływania. Główny wpływ na dynamikę układu jest tutaj zależny od wartości potencjału

uśrednionego wzdłuż klasycznej trajektorii. Takie zachowanie jest jasne, jak zaraz zobaczymy, gdy przyglądamy się układowi w bazie rotującej.

Gdy rozważymy niezaburzony Hamiltonian rozwinięty wokół pewnej energii E_{n_0} otrzymamy:

$$\hat{H}_0 = E_{n_0} + \omega(\hat{n} - n_0) + \frac{1}{2m_{eff}}(\hat{n} - n_0)^2 + O((\hat{n} - n_0)^3). \quad (\text{III.5})$$

Stosując zależną od czasu transformację unitarną $\hat{U} = \exp(-i\hat{n}\omega t)$ przechodzimy do tak zwanej bazy rotującej. W konsekwencji znika część liniowa Hamiltonianu i pozostaje tylko część kwadratowa (i wyższe rzędy rozwinięcia). Otrzymaliśmy więc (lokalnie) kwadratową relację dyspersji cząstki swobodnej o masie m_{eff} poruszającej się z pędem $\hat{n} - n_0$ po okręgu. Klasyczny odpowiednik takiej transformacji prowadzi do nowej zmiennej kątowej $\theta \rightarrow \theta - \omega t$, która rotuje z szybkością kołową ω , w porównaniu do oryginalnej bazy, skąd pochodzi nazwa transformacji. Taka zamiana zmiennych wymaga zależnej od czasu transformacji i sprawia, że wartości średnie Hamiltonianu w bazie rotującej nie są identyfikowane z energią, tylko kwazi-energią.

Zapisanie elementu macierzowego zaburzenia zależnego od czasu daje wówczas:

$$\text{klasycznie: } V(x, t) = g[x(\theta(t))] \cdot a(t) \quad (\text{III.6})$$

$$\text{kwantowo: } \langle n(t) | \hat{V}(t) | m(t) \rangle = {}_0 \langle n | g(\hat{x}) | m \rangle_0 a(t) \exp(i(n - m)\omega t), \quad (\text{III.7})$$

gdzie stany $|n\rangle_0$ pochodzą z bazy przed rotacją i są stanami własnymi H_0 . Następnie uśredniamy po czasie:

$$\text{klasycznie: } \frac{1}{T} \int dt V(t) = \frac{1}{T} \int_{t=0}^T dt g(x(\theta(t))) \cdot a(t) \quad (\text{III.8})$$

$$\text{kwantowo: } \frac{1}{T} \int dt \langle n(t) | \hat{V}(t) | m(t) \rangle = \frac{1}{T} {}_0 \langle n | g(\hat{x}) | m \rangle_0 \mathcal{F}_a[\omega(n - m)]. \quad (\text{III.9})$$

Górna linia jest jak zapowiadałem, uśrednieniem po trajektorii. Natomiast kwantowo wynikiem jest $\mathcal{F}_a$ - transformata Fouriera funkcji opisującej wymuszenie $a(t)$. Na przykład dla siły wymuszającej o zależności od czasu $a(t) = \cos(\Omega t)$ jej transformata nie znika tylko wtedy gdy $n - m = \frac{\Omega}{\omega}$. Oznacza to, że zaburzenia harmoniczne rezonansowe do trajektorii wprowadzają efektywne pole periodyczne $V(\theta)$ na okręgu(torusie) o liczbie powtórzeń $\frac{\Omega}{\omega}$.

Dodatkowo, gdy rozważymy układ wielu cząstek i uwzględnimy oddziaływania kontaktowe między nimi, które również może być zależne od czasu dostajemy doskonałe narzędzie programowania poszczególnych składowych bardzo różnych Hamiltonianów fizyki fazy skondensowanej, których dynamikę będzie realizować rozważany układ. Daje to bloki budulcowe do tworzenia różnych interesujących układów periodycznych - analogów doświadczeń z wykorzystaniem fizyki kryształów przestrzennych - dających potencjalnie ogromną kontrolę nad strukturą periodyczną i oddziaływaniami przez prostą zmianę zależności od czasu siły wymuszającej.

Powyżej przedstawiłem zarys strategii jaką stosowałem do opisu zjawisk krystalicznych w czasie. Więcej szczegółów czytelnik znajdzie w publikacjach załączonych do niniejszej rozprawy, w szczególności w materiałach uzupełniających opublikowanych wraz z nimi. Szczegóły dodatkowych wymagań, które muszą spełniać H_0 i V zawarte są w najnowszej prezentowanej pracy: [6], na przykładzie konstruowanego układu doświadczalnego.

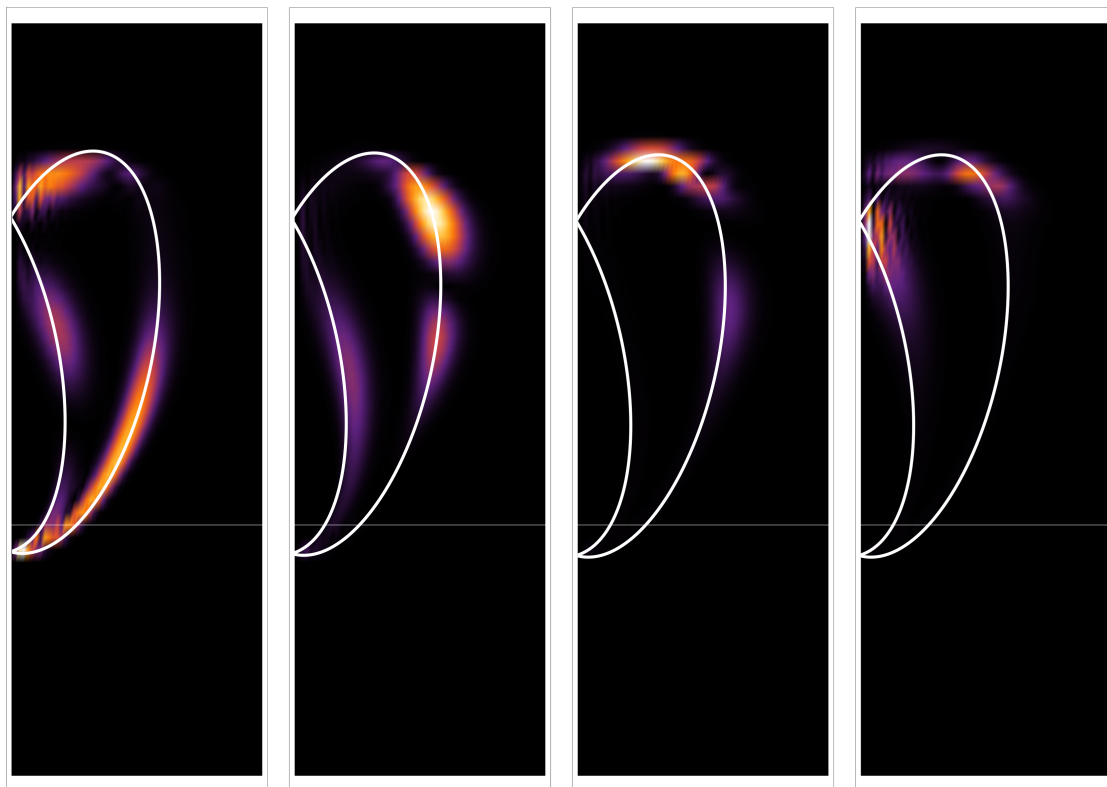
IV. REALIZACJE EKSPERYMENTALNE

Proste jednowymiarowe modele opisujące układy całkowalne to:

- oscylator harmoniczny - $H_0 = p^2/2 + x^2$,
- atom wodoru - $H_0 = p_r^2/2 + \frac{1}{r}$,
- cząstka w polu grawitacyjnym odbijająca się od sztywnego podłoża - $H_0 = p^2/2 + z, z \geq 0$.

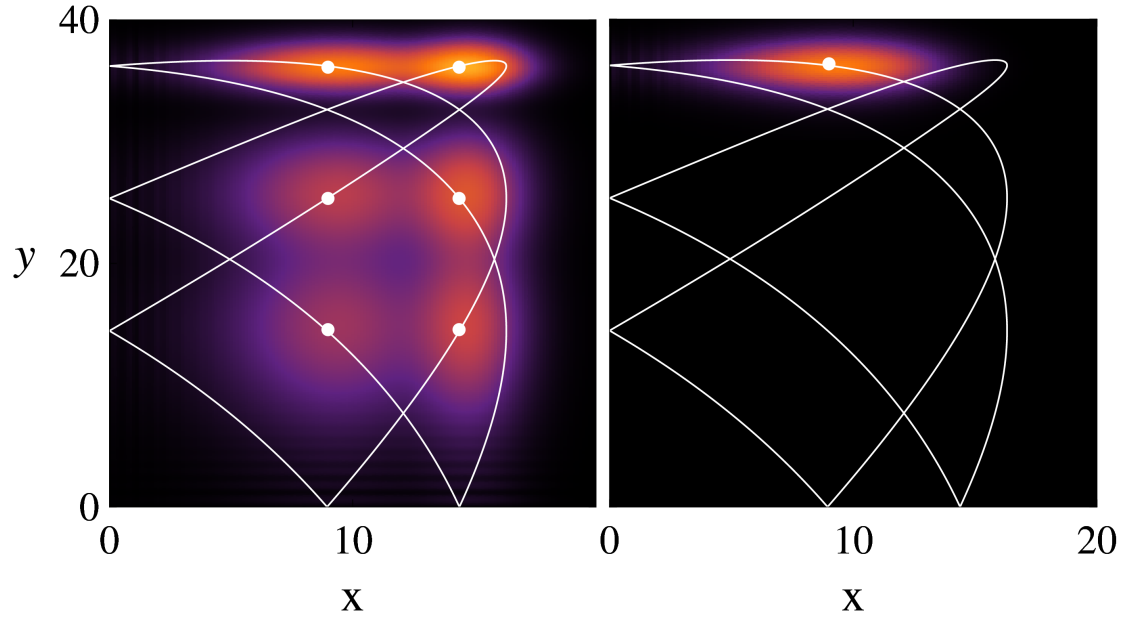
Energia niezaburzonego układu dla oscylatora harmonicznego nie posiada drugiego członu rozwinięcia Taylora, co odpowiada nieskończonej masie efektywnej w równaniu III.5. Nie możemy odseparować szybkich i wolnych stopni swobody. Taki układ nie nadaje się do realizacji zachowań krystalicznych w czasie.

Najprostszy układ związany z potencjałem centralnym to elektron atomu wodoru. Pojedynczy elektron pozwala na realizację w domenie czasu jedynie jedno-cząstkowych problemów fizyki fazy skondensowanej. W takim układzie opisaliśmy zjawisko lokalizacji Andersona w czasie. Numerycznie pokazywałem istnienie nierozpływających stanów elektronowych podróżujących wzdłuż orbity klasycznej stabilizowanych przez lokalizację Andersona w czasie (Rys. 1). Realizacja eksperymentalna tego układu wymaga niestety przygotowania wysoko wzbudzonych stanów Rydbergowskich.

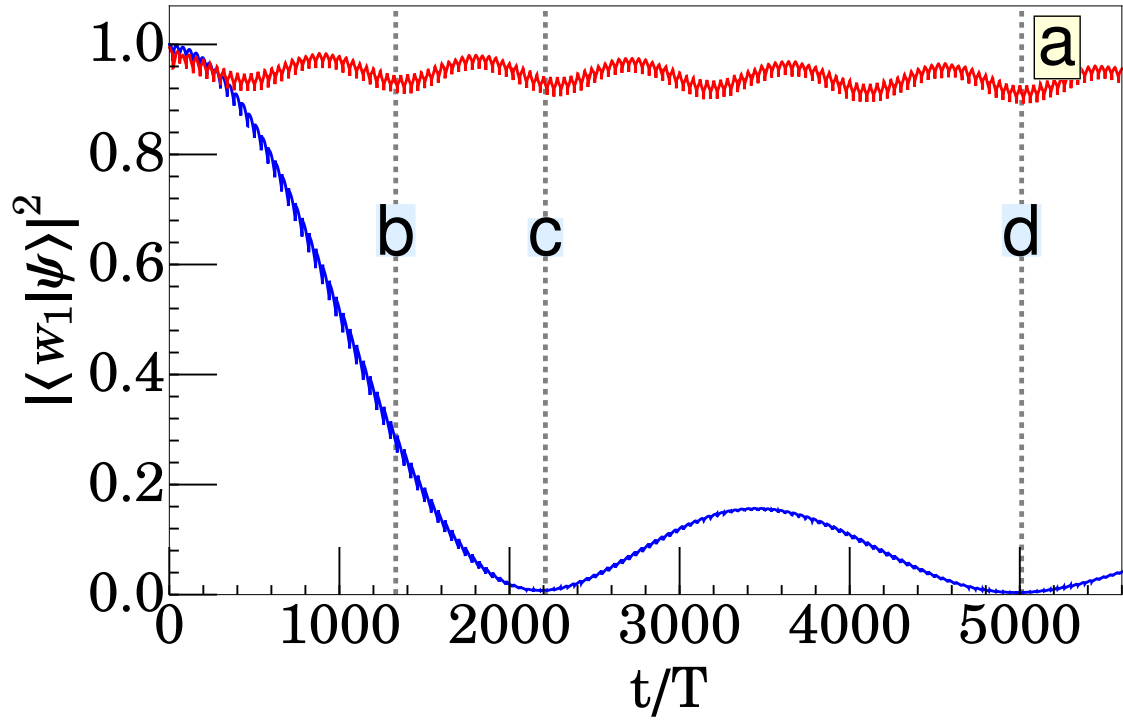


Rysunek 1: Schematyczny wykres czterech kolejnych równo odległych w czasie profili gęstości prawdopodobieństwa stanu Floquet atomu wodoru w fluktuującym w czasie polu elektrycznym skierowanym wzdłuż pionowej osi rysunku. Stan ten wykazuje eksponencjalną lokalizację i porusza się wzdłuż klasycznej trajektorii. Na wykresie oś pozioma jest osią radialną, stąd narysowana na białą klasyczna elipsa nakłada się z swoim odbiciem lustrzanym. Zjawisko to nazywane jest lokalizacją Andersona w czasie, gdyż odczyt profilu prawdopodobieństwa w zależności od czasu na ustalonym punkcie elipsy ujawnia lokalizację Andersona wokół pewnej chwili czasowej. Zachowanie takie powtarza się periodycznie w czasie, ponieważ pole elektryczne również powtarza się okresowo. Zaadaptowane z [1].

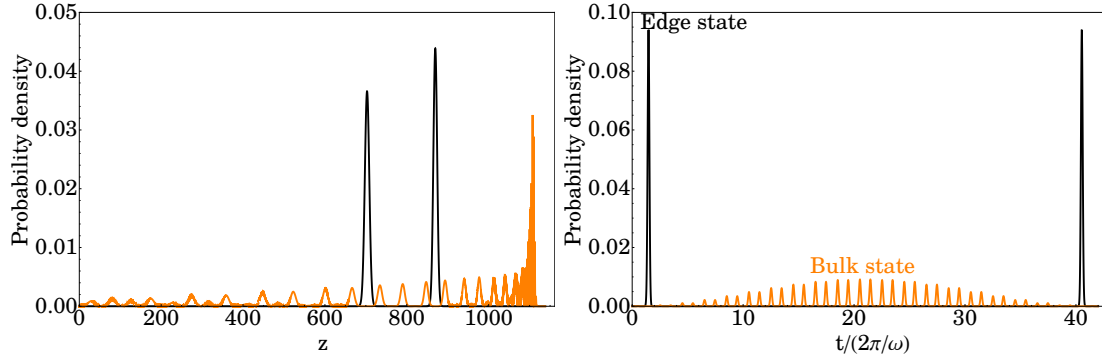
Pozostaje trzeci z “oczywistych” kandydatów: układ neutralnych elektrycznie, oddziaływujących atomów opadających w polu grawitacyjnym. Siłą wymuszającą są vibracje lustra atomowego, od którego chmura atomów okresowo się odbija. Jest to wybór dający nadzieję na eksperymentalną konstrukcję platformy do doświadczeń fizyki czasu. Odbijanie opadających w jednorodnym polu grawitacyjnym atomów to układ doświadczalny, któremu poświęcone są przykłady we wszystkich, z wyjątkiem jednej, załączonych publikacji. Pokazujemy w nim krystalizację czasową, w tym powstawanie kwazikryształów (Rys. 2), dzięki przyciągającym oddziaływaniom między atomami. Dogłębnie analizowaliśmy możliwości realizacji tego eksperymentu (Rys. 3), który jest w tej chwili budowany przez grupę prof. Petera Hannaforda w Australii. Dzięki bardziej skomplikowanym realizacjom drgań lustra możliwe powinno być tworzenie układów topologicznych takich jak na przykład realizacja modelu Su-Schrieffer-Heeger’a (tzw. SSH) (Rys. 4). Załączony zestaw prac potwierdza zasadność tytułu drugiej pracy z prezentowanego cyklu zatytułowanej platforma do doświadczeń z kryształami czasowymi[2].



Rysunek 2: Po lewej stan rozmyty równomiernie po punktach równoważnych przez translacje o $1/6$ okresu ruchu na orbicie klasycznej. Po prawej stan zlokalizowany opisujący chmurę atomów poruszających się wzdłuż klasycznej orbity. Stan jest stabilny gdy atomy oddziałują przyciągająco, łamiąc symetrię translacyjną w czasie. Zjawisko to związane jest z krystalizacją czasową w dwóch wymiarach. Sekwencja odbić chmury atomowej od luster położonych w $x = 0$ i $y = 0$ tworzy fragment ciągu Fibonacciego. Wybierając coraz dłuższą orbitę klasyczną jesteśmy w stanie zrealizować dowolnie długi ciąg Fibonacciego w czasie, tj. dowolnie "duży" kwazi-kryształ czasu. Zaadaptowane z [3].



Rysunek 3: Prawdopodobieństwo obserwacji dyskretnego kryształu czasowego w kondensacie Bosego-Einsteina odbijanym przez oscylujące lustro atomowe. Wyniki odpowiadają realistycznej sytuacji eksperymentalnej. Na wykresie widać, że uwzględnione eksperymentalne niedokładności nie zmieniają kluczowych wniosków z analiz przybliżonych. Na niebiesko widać rozwiązanie rozprzyskające się po wszystkich równoważnych przez translacje w czasie pakietach, a na czerwono rozwiązanie krystaliczne zlokalizowane na konkretnej trajektorii przez bardzo długi czas. T oznacza okres ruchu lustra. Litery na liniach przerywanych odpowiadają chwilom czasowym opisanym w [4].



Rysunek 4: Po lewej gęstość prawdopodobieństwa obserwacji chmury atomów w funkcji odległości od oscylującego lustra atomowego. Po prawej gęstość prawdopodobieństwa detekcji atomów w pobliżu lustra w funkcji czasu. Na czarno zaprezentowano chroniony topologicznie stan brzegowy obecny w tym układzie przez analogię do przestrzennego modelu SSH. Zaproponowanie eksperymentalnej realizacji stanów topologicznych było częścią pracy [5].

V. PODSUMOWANIE

Zaprezentowany zbiór prac jest prezentacją teoretycznie oczekiwanych zjawisk i możliwości ich eksperymentalnej realizacji w układach ultra-zimnych atomów. Doświadczalna realizacja wciąż leży w przyszłości i dopiero ona będzie przypieczętowaniem tych prac. Niemniej niezwykle ciekawe było badanie układów, które przez swoje kwantowe własności zdają się udawać układy klasyczne poruszając się jako pakiety cząstek wzdłuż klasycznych trajektorii. Te wyniki opisujące stabilizację obiektów kwantowych poruszających się po klasycznych orbitach mogą potencjalnie znaleźć w przyszłości zastosowania w technologii. Mój mały wkład w nadanie wymiarowi czasowemu równie hojnego traktowania, jakie zostało poświęcone fazom materii rozróżnialnymi przez ich własności przestrzenne jest wkładem tej rozprawy w naukę.

-
- [1] K. Giergiel and K. Sacha, Physical Review A **95**, 063402 (2017), ISSN 24699934, 1703.08339, URL <https://link.aps.org/doi/10.1103/PhysRevA.95.063402>.
 - [2] K. Giergiel, A. Miroszewski, and K. Sacha, Physical Review Letters **120**, 140401 (2018), ISSN 10797114, 1710.10087, URL <https://doi.org/10.1103/PhysRevLett.120.140401>.
 - [3] K. Giergiel, A. Kuroś, and K. Sacha, Physical Review B **99**, 220303 (2019), ISSN 24699969, 1807.02105, URL <https://link.aps.org/doi/10.1103/PhysRevB.99.220303>.
 - [4] K. Giergiel, A. Kosior, P. Hannaford, and K. Sacha, Physical Review A **98**, 013613 (2018), ISSN 24699934, 1805.05718, URL <https://link.aps.org/doi/10.1103/PhysRevA.98.013613>.
 - [5] K. Giergiel, A. Dauphin, M. Lewenstein, J. Zakrzewski, and K. Sacha, New Journal of Physics **21**, 052003 (2019), ISSN 1367-2630, 1806.10536, URL <https://iopscience.iop.org/article/10.1088/1367-2630/ab1e5f>.
 - [6] K. Giergiel, T. Tran, A. Zaheer, A. Singh, A. Sidorov, K. Sacha, and P. Hannaford, New Journal of Physics **22**, 085004 (2020), ISSN 1367-2630, URL <https://iopscience.iop.org/article/10.1088/1367-2630/aba3e6>.
 - [7] F. Wilczek, Physical Review Letters **109**, 160401 (2012), ISSN 00319007, 1202.2539, URL <http://link.aps.org/doi/10.1103/PhysRevLett.109.160401>.
 - [8] H. Watanabe and M. Oshikawa, Physical Review Letters **114**, 251603 (2015), ISSN 10797114, 1410.2143, URL <http://link.aps.org/doi/10.1103/PhysRevLett.114.251603>.
 - [9] H. Watanabe, M. Oshikawa, and T. Koma, Journal of Statistical Physics **178**, 926 (2020), ISSN 0022-4715, 1911.12939, URL <http://link.springer.com/10.1007/s10955-019-02471-5>.
 - [10] K. Sacha, Physical Review A - Atomic, Molecular, and Optical Physics **91**, 33617 (2015), ISSN 10941622, 1410.3638, URL <http://link.aps.org/doi/10.1103/PhysRevA.91.033617>.
 - [11] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter, A. Vishwanath, et al., Nature **543**, 217 (2017), ISSN 14764687, 1609.08684, URL <http://dx.doi.org/10.1038/nature21413>.
 - [12] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, et al., Nature **543**, 221 (2017), ISSN 14764687, 1610.08057, URL <http://dx.doi.org/10.1038/nature21426>.
 - [13] J. Rovny, R. L. Blum, and S. E. Barrett, Physical Review Letters **120**, 180603 (2018), ISSN 0031-9007, 1802.00126, URL <https://link.aps.org/doi/10.1103/PhysRevLett.120.180603>.
 - [14] J. Rovny, R. L. Blum, and S. E. Barrett, Physical Review B **97**, 184301 (2018), ISSN 24699969, 1802.00457, URL <https://link.aps.org/doi/10.1103/PhysRevB.97.184301>.
 - [15] S. Pal, N. Nishad, T. S. Mahesh, and G. J. Sreejith, Physical Review Letters **120**, 180602 (2018), ISSN 10797114, URL <https://link.aps.org/doi/10.1103/PhysRevLett.120.180602>.
 - [16] J. Smits, L. Liao, H. T. C. Stoof, and P. van der Straten, Physical Review Letters **121**, 185301 (2018), ISSN 0031-9007, 1807.05904, URL <https://link.aps.org/doi/10.1103/PhysRevLett.121.185301>.
 - [17] K. Sacha and J. Zakrzewski, Reports on Progress in Physics **81**, 016401 (2018), ISSN 0034-4885, 1704.03735, URL <https://iopscience.iop.org/article/10.1088/1361-6633/aa8b38>.
 - [18] V. Khemani, R. Moessner, and S. L. Sondhi, *A Brief History of Time Crystals* (2019), 1910.10745.
 - [19] D. V. Else, C. Monroe, C. Nayak, and N. Y. Yao, Annual Review of Condensed Matter Physics **11**, 467 (2020), ISSN 1947-5454, 1905.13232, URL <https://www.annualreviews.org/doi/10.1146/annurev-conmatphys-031119-050658>.
 - [20] L. Guo and P. Liang, New Journal of Physics **22**, 075003 (2020), ISSN 1367-2630, 2005.03138, URL <https://iopscience.iop.org/article/10.1088/1367-2630/ab9d54>.
 - [21] K. Sacha, *Time Crystals*, vol. 114 of *Springer Series on Atomic, Optical, and Plasma Physics* (Springer International Publishing, Cham, 2020), ISBN 978-3-030-52522-4, URL <http://link.springer.com/10.1007/978-3-030-52523-1>.
 - [22] G. Floquet, Annales scientifiques de l'École normale supérieure **12**, 47 (1883), ISSN 0012-9593.
 - [23] G. W. Hill, Acta Mathematica **8**, 1 (1886), ISSN 00015962.
 - [24] A. M. Lyapunov, International Journal of Control **55**, 531 (1982), ISSN 13665820.
 - [25] F. Bloch, Zeitschrift für Physik **52**, 555 (1929), ISSN 14346001.
 - [26] J. H. Shirley, Physical Review **138** (1965), ISSN 0031899X.

- [27] Y. Zel'Dovich, Soviet Journal of Experimental and Theoretical Physics' **24**, 1006 (1967), ISSN 1063-7761.

Anderson localization of a Rydberg electron along a classical orbit

Krzysztof Giergiel¹ and Krzysztof Sacha^{1,2}

¹*Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński,
ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

²*Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński,
ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

(Received 29 March 2017; published 1 June 2017)

Anderson localization is related to exponential localization of a particle in the configuration space in the presence of a disorder potential. Anderson localization can be also observed in the momentum space and corresponds to quantum suppression of classical diffusion in systems that are classically chaotic. Another kind of Anderson localization has been recently proposed, i.e., localization in the time domain due to the presence of *disorder* in time. That is, the probability density for the detection of a system at a fixed position in the configuration space is localized exponentially around a certain moment of time if a system is driven by a force that fluctuates in time. We show that an electron in a Rydberg atom, perturbed by a fluctuating microwave field, Anderson localizes along a classical periodic orbit. In other words the probability density for the detection of an electron at a fixed position on an orbit is exponentially localized around a certain time moment. This phenomenon can be experimentally observed.

DOI: [10.1103/PhysRevA.95.063402](https://doi.org/10.1103/PhysRevA.95.063402)

I. INTRODUCTION

Transport of a particle in the presence of a disorder potential can stop totally due destructive interference effects. This phenomenon, which is accompanied by exponential localization of eigenstates of a system in the configuration space, is the famous Anderson localization [1]. It can occur in a variety of different disordered systems ranging from acoustic waves to matter waves of ultra-cold atomic gases [2–4].

Classical particles can perform diffusive motion in the phase space if the dynamics of a system is chaotic. In the quantum description one can observe Anderson localization in momentum space that is induced not by external disorder but by underlying chaotic classical dynamics. Such a dynamical Anderson localization, predicted in the kicked rotor system, is extensively investigated both theoretically and experimentally [5–9].

Recently it has been shown that Anderson localization can also occur in the time domain in systems that are perturbed by a force fluctuating in time [10–12]. This phenomenon is related to time crystals, i.e., systems which can spontaneously switch to periodic motion [13,14]. While it is not easy to find a system that spontaneously moves when it is prepared in the ground state, models of time crystals with the help of periodically driven systems are interesting and can reveal new phenomena [15–32]. If a periodic perturbation starts fluctuating in time, then, in analogy to space crystals with disorder, Anderson localization of a system in the time domain can be observed. In other words, a detector situated at a certain position clicks with a probability that is localized exponentially around a certain moment in time. Anderson localization in the time domain is a general phenomenon that can be observed in many different systems where periodic driving starts behaving randomly [11]. Despite the fact that the time degree of freedom forms a one-dimensional *space*, it is possible to realize Anderson localization in the time domain with properties of multidimensional disorder systems where phase transition between localized and delocalized states can be observed [12].

In the present paper we investigate Anderson localization in the time domain in a Rydberg atom perturbed by a fluctuating microwave field. Rydberg atoms driven by a monochromatic microwave field can reveal nonspreading wave-packet motion where a resonantly driven electron is represented by a localized wave packet that moves periodically along a classical orbit and does not spread. This phenomenon, discovered some time ago [33–42] and observed in experiments [43–46], can be realized in different dynamical systems if nonlinear classical resonances can form [47]. Switching from a monochromatic microwave field to a randomly fluctuating field we show that in the frame moving with an electron along a classical orbit, Anderson localization takes place. In the laboratory frame, probability for detection of an electron at a given point on the orbit is localized exponentially around a certain moment of time. Such a behavior is repeated periodically in analogy to Anderson localization of a particle in a space crystal with disorder and with periodic boundary conditions where by traveling periodically around the ring, one observes periodically an exponentially localized density profile.

The paper is organized as follows. In Sec. II we explain the phenomenon of Anderson localization in the time domain in a one-dimensional (1D) model of a hydrogen atom. Then, in Sec. III we switch to the 3D case and analyze a realistic description of the system. Section IV is devoted to conclusions.

II. 1D MODEL

Let us consider a hydrogen atom perturbed by a linearly polarized microwave field in the 1D model [47]. The classical Hamiltonian of the system, in the dipole approximation and in atomic units, reads

$$H = H_0 + H_1, \quad (1)$$

$$H_0 = \frac{p^2}{2} - \frac{1}{z}, \quad (2)$$

$$H_1 = Fzf(t), \quad (3)$$

where $z \geq 0$. Such a 1D model can describe classical elliptical orbits degenerated to a line along the field polarization axis. F is a parameter that allows one to change the amplitude of the perturbation and $f(t)$ is a periodic function which describes the time dependence of the electric field. The electric field is assumed to oscillate with a frequency ω , but between $t = 0$ and $t = 2\pi/\omega$ it performs random fluctuations, i.e.,

$$f(t) = f(t + 2\pi/\omega) = \sum_{k \neq 0} f_k e^{ik\omega t}, \quad (4)$$

where Fourier components $f_k = f_{-k}^*$ are random numbers.

We are interested in a resonant driving of the atom when the frequency ω matches the frequency of electron motion on an unperturbed orbit. The description of the system close to the resonant trajectory can be significantly simplified if we employ the secular perturbation theory [47,48]. First, we switch to the action-angle variables, J and θ , of the unperturbed H atom that results in the following form of the Hamiltonian:

$$H_0 = -\frac{1}{2J^2}, \quad (5)$$

$$H_1 = F J^2 \left(\frac{3}{2} - \sum_{m \neq 0} \frac{\mathcal{J}'_m(m)}{m} e^{im\theta} \right) f(t), \quad (6)$$

where $\mathcal{J}'_m$'s are the derivatives of the ordinary Bessel functions. The resonant condition means that

$$\omega = \frac{\partial H_0(J_0)}{\partial J_0} = \frac{1}{J_0^3}. \quad (7)$$

In the moving frame, the position of an electron $\Theta = \theta - \omega t$ is a slow variable if we choose the conjugate momentum close to the resonant value $P = J - J_0 \approx 0$. Then, averaging the Hamiltonian over time leads to the effective time-independent Hamiltonian

$$H_{\text{eff}} = -\frac{3}{2J_0^4} P^2 + F J_0^2 \sum_{k \neq 0} \frac{\mathcal{J}'_k(k)}{k} f_{-k} e^{ik\Theta}, \quad (8)$$

where the constant term $-\frac{3}{2J_0^4}$ has been omitted. The first-order secular approximation (8) is the accurate description of the system provided the second-order contribution (that scales like $F^2 J_0^6$) is small as compared to the first-order term [48]. It implies we may restrict ourselves to the first-order effective Hamiltonian if $F J_0^4 \ll 1$; for discussion of the validity of the secular approximation in the context of the Anderson localization in time see the Appendix in [11].

Before we move on, we would like to discuss the secular Hamiltonian (8) and the conditions necessary to observe the Anderson localization we are interested in. If $f(t)$ were a simple single harmonic function like $\cos(k\omega t)$, then we would deal with a monochromatic resonant driving, and well-known nonspreading wave packets could be realized [47]. That is, for $k = 1$ we would deal with a single localized wave packet moving along a Kepler orbit with the period $2\pi/\omega$; classically, in the laboratory frame, an electron would move along a Kepler orbit with small oscillations around it if the initial momentum $P \approx 0$. For $k > 1$, k wave packets would move along a Kepler orbit and each wave packet would evolve with a period $k2\pi/\omega$, but each of them would be delayed with respect to

its neighbor by $2\pi/\omega$ [47]. We will see that in order to observe the Anderson localization we need $f(t)$ to consist of many harmonics at the same time. Then, the secular Hamiltonian (8) is the result of the coherent addition of resonant terms between the spatial harmonics of an unperturbed electronic motion and the corresponding temporal harmonics of the disordered driving amplitude. It also indicates that the electric dipole moment z expressed in terms of the action-angle variables must possess many spatial harmonics, cf. Eq. (6). It is possible provided an electron moves along an elliptical orbit and approaches closely to the nucleus. Thus, circular Kepler orbits are not suitable for realization of the Anderson localization in the time domain.

Harmonics of an unperturbed motion of an electron decrease with k like $\mathcal{J}'_k(k)/k$, cf. Eq. (6). If the fluctuations of the microwave field are engineered so that the components f_{-k} fulfill

$$\left| \frac{\mathcal{J}'_k(k)}{k} f_{-k} \right| = \frac{1}{\sqrt{2k_0}} \quad (9)$$

for $|k| \leq k_0$ and zero otherwise, and $\varphi_{-k} = -\varphi_k = \text{Arg}(f_{-k})$ are random numbers chosen uniformly from the interval $[0, 2\pi)$, we end up with the effective Hamiltonian

$$H_{\text{eff}} = \frac{P^2}{2m_{\text{eff}}} + F J_0^2 V(\Theta), \quad (10)$$

$$V(\Theta) = \frac{1}{\sqrt{2k_0}} \sum_{\substack{|k| \leq k_0 \\ k \neq 0}} e^{i(k\Theta + \varphi_k)}, \quad (11)$$

$$m_{\text{eff}} = -\frac{J_0^4}{3}, \quad (12)$$

which describes a particle with the negative effective mass m_{eff} in the disorder potential characterized by the variance $F^2 J_0^4$ and the finite correlation length $\zeta = \sqrt{2}/k_0$ [11]. Note that F is not the amplitude of the electric field because the microwave field is not monochromatic and $\frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt f^2(t) \neq \frac{1}{2}$. The intensity of the microwave field reads

$$I = \frac{F^2}{2} \sum_{k \neq 0} |f_k|^2 = \frac{F^2}{4k_0} \sum_{\substack{|k| \leq k_0 \\ k \neq 0}} \left(\frac{k}{\mathcal{J}'_k(k)} \right)^2. \quad (13)$$

In the present paper we have chosen the fluctuations of the microwave field that lead to the disorder potential (11) as an example. By a proper choice of $f(t)$ one can realize many different effective disorder potentials.

In the case of the quantum version of the Hamiltonian (10), i.e., quantizing the action angle variables (where J_0 becomes the principal quantum number n_0 of a H atom), we can expect Anderson localization. In 1D systems the presence of a disorder results in Anderson localization regardless how weak the disorder is. In the present case, the configuration space extends from $\Theta = 0$ to 2π due to the periodic boundary conditions. Therefore, Anderson localization can be observed if the localization length $\xi_{\text{loc}} \ll 2\pi$. In the weak disorder limit, i.e., when $F^2 n_0^4 \ll E E_\zeta$ where E is an energy eigenvalue and $E_\zeta = k_0^2/2m_{\text{eff}}$ is the so-called correlation energy, ξ_{loc} can be calculated by means of the Born approximation [4,49] and it

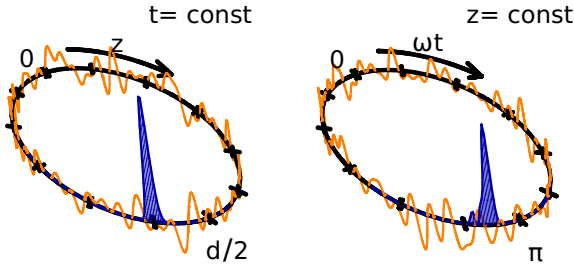


FIG. 1. Left panel: schematic illustration of the standard Anderson localization in the configuration space, i.e., localization of a particle on a ring of length d in the presence of a time-independent disordered potential with periodic boundary conditions. Right panel: Anderson localization in the time domain. For a fixed position in the configuration space, the probability density for the detection of a particle is exponentially localized around a certain moment of time. Such a behavior is repeated with a period $T = 2\pi/\omega$, similar to that in the left panel where if one travels periodically around the ring, one observes periodically an exponentially localized density profile.

reads

$$\frac{\xi_{\text{loc}}}{\zeta} = \frac{2\sqrt{2}}{\pi} \frac{EE_\zeta}{F^2 n_0^4} \quad (14)$$

for $E \geq E_\zeta/4$. Note that the effective mass m_{eff} is negative, thus, the Hamiltonian (10) is bounded from above not from below as in a usual case. Consequently, the greatest eigenvalue of Eq. (10) corresponds to the strongest localization. The localization length increases with a decrease of E . The validity of the Born approximation requires the localization length ξ_{loc} to be much greater than the correlation length ζ , which, together with the requirement $\xi_{\text{loc}} \ll 2\pi$, can be easily fulfilled for sufficiently large n_0 and with an appropriate choice of k_0 .

Anderson localization in the moving frame that is predicted with the help of the Hamiltonian (10) means an exponential localization in the time domain when we switch to the laboratory frame [10–12]. Indeed, in the laboratory frame a localized wave packet will move along an unperturbed classical orbit. If we ask how the probability for the detection of an electron, at a fixed position on an orbit, changes in time, it will turn out that the exponential localization in the Θ space translates into an exponential localization in time around a certain t_0 because for a fixed position in the laboratory frame the relation between Θ and t is linear, i.e., $\Theta = \theta - \omega t$. However, if we fix t and ask whether an electron is also exponentially localized in the configuration space, the answer is no because the transformation from the Θ space to the Cartesian space is nonlinear, compare Eqs. (3) and (6). In Fig. 1 we illustrate schematically the idea of Anderson localization in the time domain and compare it to the standard Anderson localization in the configuration space in systems with spatially disordered potentials and periodic boundary conditions.

We have obtained the prediction for Anderson localization in the time domain by means of the effective classical Hamiltonian and subsequent quantization of the classical action angle variables. However, the same results can be obtained starting with the full quantum Hamiltonian and applying a quantum version of the secular approach [33]. The Hamiltonian (1) is

time periodic which means that although there are no energy eigenstates, one can find a kind of *stationary* states $\psi_m(z, t)$ that are time periodic eigenfunctions of the so-called Floquet Hamiltonian $H_F(t)$ [47],

$$H_F \psi_m = (H - i\partial_t) \psi_m = E_m \psi_m, \quad (15)$$

where E_m are real eigenvalues which are called quasienergies of a system. The Floquet theorem [50] that is used here is in full analogy to the Bloch theorem known in condensed-matter physics but we deal with a system periodic in time not in space.

Performing time-dependent unitary transformation $\hat{U} = e^{i\hat{n}\omega t}$, where $\hat{n} = (-2\hat{H}_0)^{-2}$ is the operator of the principal quantum number of a hydrogen atom, we switch to the moving frame as in the classical approach. Then, the matrix elements of the resulting Floquet Hamiltonian in the hydrogenic eigenbasis read

$$\begin{aligned} \langle n' | \hat{H}'_F(t) | n \rangle &= \langle n' | \hat{U} \hat{H}_F(t) \hat{U}^\dagger | n \rangle \\ &= \left(-\frac{1}{2n^2} - n\omega - i\partial_t \right) \delta_{nn'} \\ &\quad + F \langle n' | z | n \rangle f(t) e^{-i(n-n')\omega t}. \end{aligned} \quad (16)$$

Averaging the last term of Eq. (16) over time we obtain the quantum version of the effective Hamiltonian [33],

$$\langle n' | \hat{H}_{\text{eff}} | n \rangle = \left(-\frac{1}{2n^2} - n\omega \right) \delta_{nn'} + F \langle n' | z | n \rangle f_{n-n'}, \quad (17)$$

where the operator $i\partial_t$ has been omitted because for the time-independent effective Hamiltonian it would only introduce a shift of energy eigenvalues. The effective Hamiltonian (17) corresponds to a single block of the matrix of the Floquet Hamiltonian $\langle n', k' | \hat{H}'_F | n, k \rangle$ in the basis spanned by the hydrogenic eigenvectors $|n\rangle$ and the Fourier functions $\langle t | k \rangle = \sqrt{\frac{\omega}{2\pi}} e^{ik\omega t}$, i.e., to the block with $k = k' = 0$. The omission of couplings between the chosen block and other blocks is valid if we want to describe the Hilbert subspace of states with principal quantum numbers n close to the resonant value n_0 , where $n_0 = \omega^{-1/3}$, and if $F n_0^4 \ll 1$.

Numerical diagonalization of Eq. (17) allows us to obtain Floquet eigenstates $\psi'_m(z)$ of the system in the moving frame within the secular approximation. Switching to the laboratory frame, $\psi_m(z, t) = e^{-i\hat{n}\omega t} \psi'_m(z)$, we expect that time evolution of the probability density for the detection of an electron at a fixed position will reveal exponential localization in time if Anderson localization takes place.

In Figs. 2 and 3 we show how the probability for the detection of an electron close to the position of the nucleus of a H atom changes in time for two different sets of the parameters. Figure 2 presents the results for the case of $n_0 = 10^6$ where the Anderson localization can be described with the help of the Born approximation, while in Fig. 3 we have used values of the parameters that are attainable in present-day laboratories. In order to afford small correlation length of the effective disordered potential, $\zeta = \sqrt{2}/k_0$, a big value of k_0 is needed. Then, however, the value of F must be decreased to stay in the validity range of the effective Hamiltonian because the intensity (13) increases with k_0 . As the result, high excitation of a hydrogen atom is necessary in order to observe the Anderson localization in time. In Fig. 3 we have chosen an

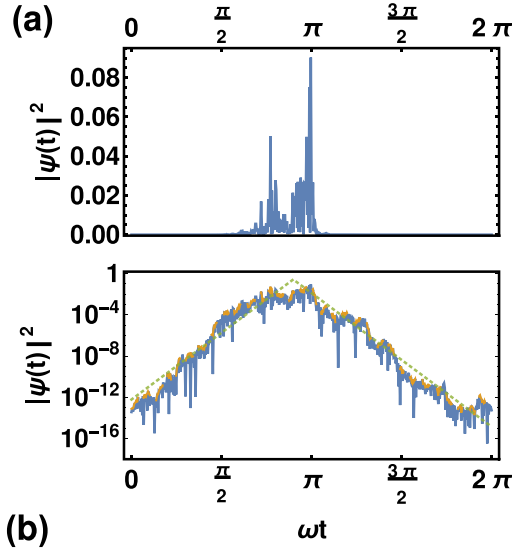


FIG. 2. Hydrogen atom in a fluctuating microwave field in the 1D model. Panels show probability density for the detection of an electron close to the nucleus versus time (solid blue lines) in (a) linear and (b) logarithmic scale corresponding to a Floquet eigenstate with a quasienergy $n_0^2 E = -\frac{3}{2} - 1.53 \times 10^{-8}$ of the Hamiltonian (17). Microwave field is resonant with electron motion, i.e., $\omega = n_0^{-3}$ where $n_0 = 10^6$, and consists of $k_0 = 500$ harmonics with randomly chosen relative phases; the amplitude parameter $F n_0^4 = 1.6 \times 10^{-8}$. An additional smooth envelope [orange dashed line in (b)] corresponds to the norm $||\psi(t)||^2 = |\psi(t)|^2 + \alpha |\partial_t \psi(t)|^2$ which better visualizes an exponential profile; small positive α is introduced to avoid zeros of $||\psi(t)||$ [51]. For such system parameters, the localization length in the time domain can be predicted with the help of the Born approximation (14) which results in $0.18/\omega$ which agrees quite well with the value of the localization length, $0.21/\omega$, obtained by a numerical fit of an exponentially localized function [dotted green line in (b)].

experimentally attainable value, $n_0 = 300$. Then for $k_0 \approx 5$ and suitable choice of F the effective Hamiltonian is valid and we can see signatures of Anderson localization in the time domain.

We would like to note that the spectra obtained in diagonalization of the quantum secular Hamiltonian (17) and the quantized version of the classical secular Hamiltonian (10) are identical and the same is true for the time evolution of the probabilities for the detection of an electron close to the nucleus that are shown in Figs. 2 and 3.

III. 3D DESCRIPTION

The 1D model allowed us to present the idea of Anderson localization in the time domain in a Rydberg atom and to explain the classical and quantum secular approximation methods. In this section we show that the Anderson localization predicted in Sec. II survives when we switch to the 3D description. Moreover, the 3D case allows also the observation of Anderson localization of an electron not only on an elongated one-dimensional orbit but also on an elliptical trajectory.

We consider a H atom perturbed by a linearly polarized microwave field, as in Sec. II, but now we add a static electric

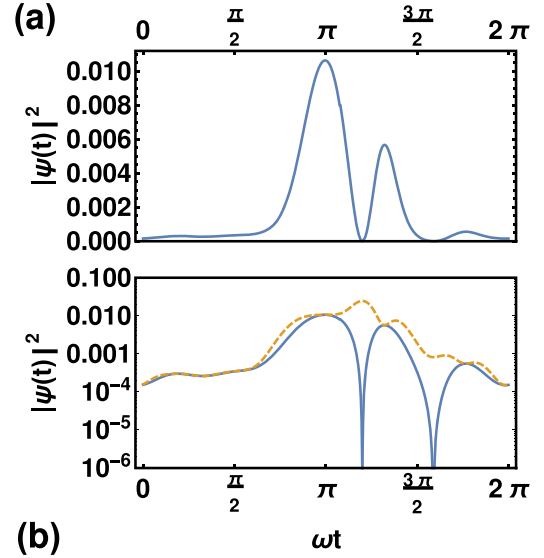


FIG. 3. Same as in Fig. 2 but for parameters attainable in a laboratory, i.e., $\omega = n_0^{-3}$ where $n_0 = 300$, $k_0 = 5$, and $F n_0^4 = 0.00016$ [i.e., the intensity $I = 0.72 \frac{\text{W}}{\text{cm}^2}$, Eq. (13)]. The Floquet eigenstate corresponding to the third (from above) quasienergy of the Hamiltonian (17) is chosen, i.e., $n_0^2 E = -\frac{3}{2} - 2.45 \times 10^{-5}$.

field along the polarization axis of the microwave field, i.e., the Hamiltonian of the system reads $H = H_0 + H_1$ with

$$H_0 = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad H_1 = F z f(t) - F_s z, \quad (18)$$

where $f(t)$ fulfills Eq. (4) and F_s stands for the static electric field amplitude. Projection of the angular momentum of an electron on the z axis is conserved and in the following we assume it is zero.

Let us begin with the classical description and perform the canonical transformation to the action angle variables (J, θ, L, Ψ) . The canonically conjugate variables J and θ are the same as in the 1D model, i.e., they describe the energy of an unperturbed electron and its position on a Kepler ellipse, respectively. The total angular momentum L is conjugate to Ψ and the latter is the angle between the major axis of a Kepler ellipse and the z axis [41]. In the frame moving with an electron, i.e., $\Theta = \theta - \omega t$ and $P = J - J_0$ where $J_0 = \omega^{-1/3}$, Θ and P vary slowly in the vicinity of a resonant orbit, i.e., when $P \approx 0$. The total angular momentum L and the angle Ψ are also slowly varying variables if $F J_0^4 \ll 1$ and $F_s J_0^4 \ll 1$ [42]. Thus, as in the 1D model, we may apply the secular approximation and obtain the classical effective Hamiltonian in the rotating frame by averaging the original Hamiltonian over time, which yields

$$H_{\text{eff}} = \frac{P^2}{2m_{\text{eff}}} + F J_0^2 \sum_{k \neq 0} U_k(L, \Psi) f_{-k} e^{ik\Theta} + F_s J_0^2 \frac{3\tilde{e}}{2} \cos \Psi, \quad (19)$$

where

$$U_k(L, \Psi) = \frac{J'_k(k\tilde{e})}{k} \cos \Psi + i \frac{\sqrt{1 - \tilde{e}^2}}{k\tilde{e}} J_k(k\tilde{e}) \sin \Psi, \quad (20)$$

and $\tilde{e} = \sqrt{1 - L^2/J_0^2}$ is the eccentricity of a Kepler ellipse. As in Eq. (8) the constant term $-\frac{3}{2J_0^2}$ has been omitted.

The Hamiltonian (19) describes electronic motion on a resonant Kepler orbit and a slow precession of the orbit itself, i.e., slow changes of the variables L and Ψ . The motion in the (P, Θ) space effectively decouples from the slow motion in the (L, Ψ) space and the semiclassical quantization of the system can be performed in the spirit of the Born-Oppenheimer approximation [41,42]. That is, first quantize the P and Θ variables, for frozen L and Ψ , and then use the results to construct an effective Hamiltonian for L and Ψ which are subsequently quantized. This procedure is particularly simple if we are interested in eigenstates which are concentrated, in the semiclassical picture, around a stable fixed point (L_0, Ψ_0) in the (L, Ψ) phase space. Then, we may treat Eq. (19) as a 1D Hamiltonian where $L = L_0$ and $\Psi = \Psi_0$ are fixed parameters.

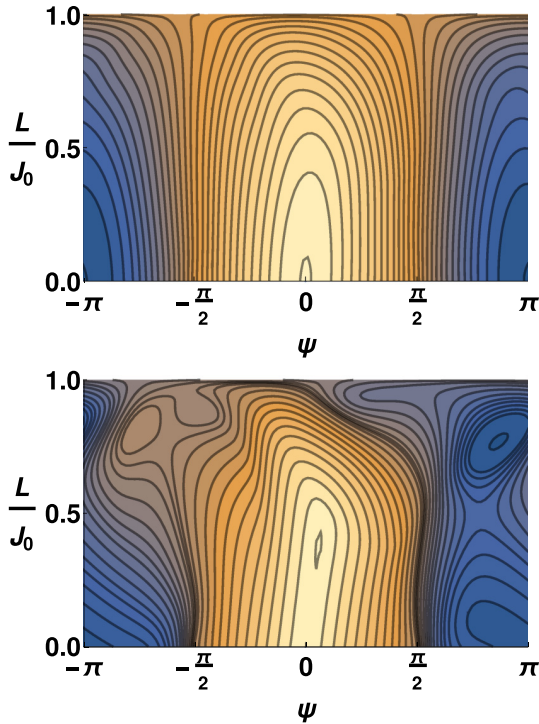


FIG. 4. Structure of the (L, Ψ) phase space related to the effective Hamiltonian (19) after the first stage of the Born-Oppenheimer approximation, i.e., when the motion of J and Θ was quantized and, e.g., the third eigenenergy (from above) was chosen. Then, the total energy is a function L and Ψ only, and its isovalue contours are presented for $F_s/F = 1.5$ (top panel) and $F_s/F = 0.33$ (bottom panel)—the lighter the area, the greater the energy. Position $(L_0 = 0, \Psi_0 = 0)$ of the stable fixed point visible in the top panel depends neither on an eigenenergy chosen in the first stage of the Born-Oppenheimer approach nor on a realization of the fluctuating microwave field. Position $(L_0/J_0 = 0.4, \Psi_0 = 0.06\pi)$ of the fixed point visible in the bottom panel changes weakly in different realizations of a randomly fluctuating field; this fixed point corresponds to an elliptical Kepler orbit whose major axis is not precisely oriented along the z axis. The other parameters are the following: $\omega = J_0^{-3}$ where $J_0 = 300$, $J_0^4 F = 0.00016$, and f_k as in Eq. (9) with $k_0 = 5$.

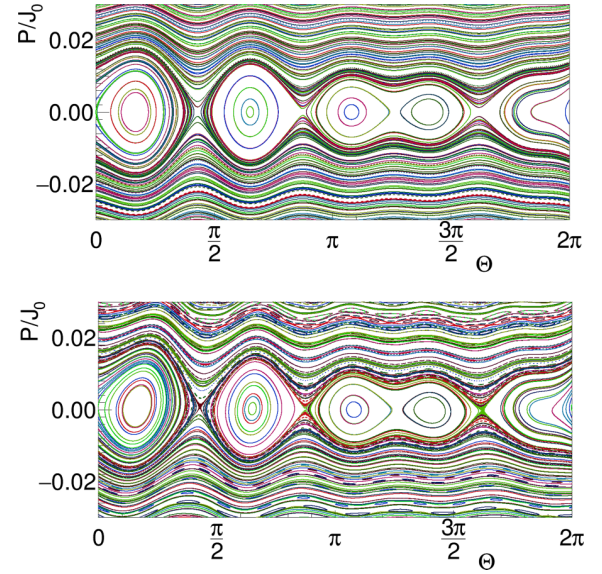


FIG. 5. Top panel: portrait of the (P, Θ) phase space generated by the 1D effective Hamiltonian (10). Bottom panel: stroboscopic picture of the (P, Θ) phase space obtained in numerical integration of the full classical equations of motion with initial values $L/J_0 = 0.1$ and $\Psi = 0$. The values of the parameters are the following: $FJ_0^4 = 0.00016$, $F_s J_0^4 = 0.00024$, and f_k as in Eq. (9) with $k_0 = 5$.

In Sec. II we have analyzed Anderson localization of an electron on an orbit degenerated into a line within the 1D model. However, such an orbit is not stable in the 3D description if only a linearly polarized microwave field is applied. In order to change the structure of the classical (L, Ψ) phase space and make $(L_0 = 0, \Psi_0 = 0)$ a stable fixed point we have included the static electric field in Eq. (18). If F_s is sufficiently big the orbit with $L_0 = 0$ oriented along the

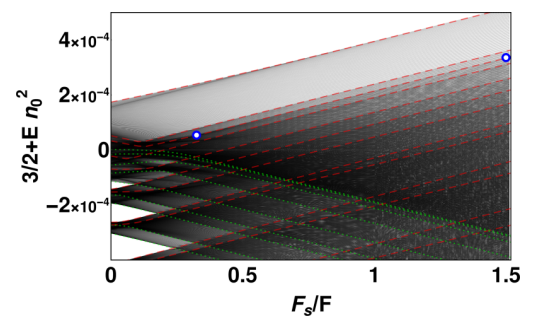


FIG. 6. Black solid lines: quasi-energy levels versus F_s obtained in diagonalization of the 3D quantum effective Hamiltonian (22). Red dashed lines: eigenvalues obtained semiclassically and within the Born-Oppenheimer approximation that correspond to the stable fixed point of the highest energy visible in Fig. 4; different dashed lines are related to different quantum numbers of the quantized (P, Θ) motion. Green dotted lines: similar semiclassical eigenvalues as indicated with red dashed lines but related to a fixed point in the (L, Ψ) space of the lowest energy. Open circles: quasienergy levels that correspond to two Floquet eigenstates presented in Figs. 7 and 8. The other parameters are the following: $\omega = n_0^{-3}$ where $n_0 = 300$, $F n_0^4 = 0.00016$, and f_k as in Eq. (9) with $k_0 = 5$.

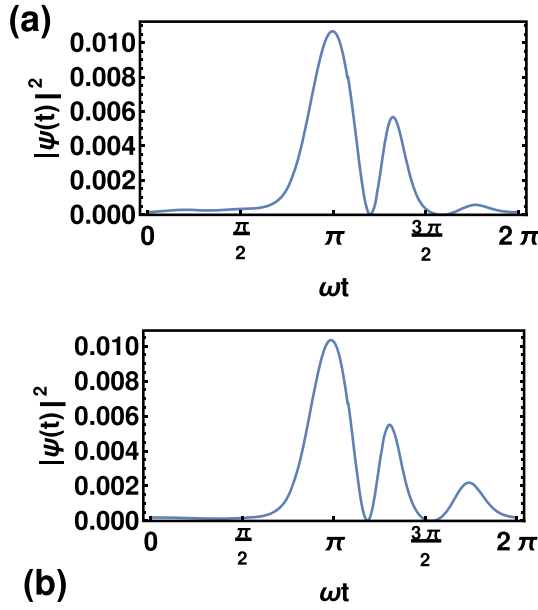


FIG. 7. Hydrogen atom in a fluctuating microwave field in the 3D model. Panels show probability density for the detection of an electron at the nucleus versus time for (a) $F_s/F = 1.5$ and (b) $F_s/F = 0.33$. The Floquet eigenstate presented in (a) corresponds to the eigenstate obtained in the 1D model and shown in Fig. 3. Panel (b) presents a Floquet eigenstate related to Anderson localization of an electron on an elliptical trajectory. The corresponding quasienergy levels are indicated in Fig. 6 with open circles. All parameters are the same as in Fig. 6.

microwave field polarization axis does not precess and it keeps its shape in time [41,42], see Fig. 4. Then, the coefficients (20) become

$$U_k(0,0) = \frac{\mathcal{J}'_k(k)}{k}, \quad (21)$$

and the Hamiltonian (19) reduces to the 1D effective Hamiltonian (8). With f_{-k} like in Eq. (9) we reproduce the previous 1D predictions. This is illustrated in Fig. 5 where we present the phase-space portrait generated by the 1D secular Hamiltonian (10) and the stroboscopic picture of the (P, Θ) phase space obtained in the full 3D classical evolution.

Quantum results that we will present in the following are achieved by means of the quantum version of the 3D secular Hamiltonian,

$$\begin{aligned} \langle n', l' | \hat{H}_{\text{eff}} | n, l \rangle &= \left(-\frac{1}{2n^2} - n\omega \right) \delta_{nn'} \delta_{ll'} \\ &+ \langle n', l' | z | n, l \rangle (F f_{n-n'} - F_s \delta_{nn'}), \end{aligned} \quad (22)$$

where $|n, l\rangle$ is a hydrogenic eigenstate with the principal quantum number n , total angular momentum l , and the projection of the angular momentum on the z axis equal to zero, cf. the 1D counterpart in Eq. (17). Diagonalization of Eq. (22) results in a bunch of quasienergy levels as presented in Fig. 6 versus F_s . In order to identify desired eigenstates it is very helpful to perform semiclassical quantization of the Hamiltonian (19) within the Born-Oppenheimer approximation; the obtained quasienergy

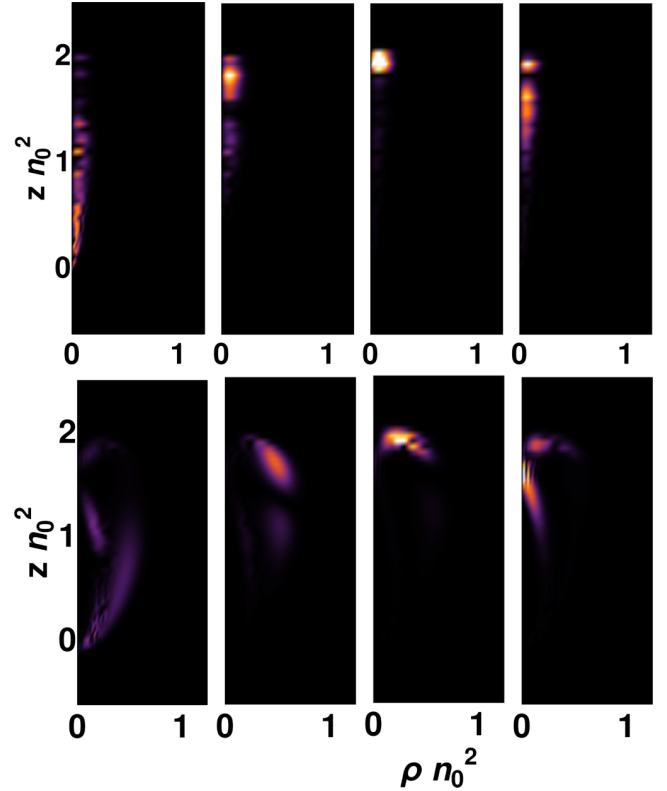


FIG. 8. Probability densities in the configuration space for different moments in time, i.e., $\omega t = \pi, \frac{3}{2}\pi, 2\pi, \frac{3}{2}\pi$ from left to right, for two Floquet eigenstates. Top panels are related to the eigenstate shown in Fig. 7(a) while bottom panels to the eigenstate presented in Fig. 7(b). Panels show cuts along an arbitrary plane containing the z axis multiplied by ρ to simulate the density in cylindrical coordinates. The ρ on the horizontal axis is either x or y or any other direction in the xy plane. The axis scaling factors are identical. Note that in the bottom panels a localized electron evolves along an elliptical Kepler orbit whose major axis is not precisely oriented along the z axis. Consequently the axial symmetry of the system results in two parts of the orbits visible in the plots.

levels follow very closely the desired levels of the Hamiltonian (22), see Fig. 6.

In Fig. 7 we present time evolution of the probability density for the detection of an electron close to the position of a nucleus of a H atom for two different Floquet eigenstates related to two different values of the static electric field amplitude F_s . Figure 7(a) corresponds to the parameters for which an electron Anderson localizes along an orbit degenerated into a line which is the 3D counterpart of the case described in Sec. II in the 1D model, cf. Fig. 3. Figure 7(b) is related to the case where the classical fixed point is located at $(L/n_0 \approx 0.4, \Psi = 0.06\pi)$, see Fig. 4. Then, an electron localizes along an elliptical trajectory. Evolution of the probability densities in the configuration space is illustrated in Fig. 8 where one can see different shapes of the orbits on which a localized electron moves. The major axis of an elliptical orbit visible in the bottom panels of Fig. 8 is not exactly oriented along the z axis. Therefore, due to the axial symmetry, one can see two parts of an ellipse, i.e., a part corresponding to, e.g., $x > 0$ and a part related to $x < 0$ but reflected with respect to $x = 0$ plane.

IV. CONCLUSIONS

Anderson localization in the time domain is a localization phenomenon that can be observed in time evolution of a system exposed to a fluctuating force. The probability density for detection of a system at a fixed position in the configuration space becomes exponentially localized around a certain moment of time due to the presence of *disorder* in time [10–12].

In the present paper we show that a Rydberg atom perturbed by a fluctuating microwave field constitutes a suitable system for realization of Anderson localization in the time domain. Driving of a Rydberg electron by a superposition of a monochromatic microwave field and its few harmonics with random relative phases leads to Anderson localization of an electron along a classical Kepler orbit. That is, the probability for measurement of the electron, e.g., close to the nucleus, is exponentially localized around a certain moment of time.

We analyze a Rydberg atom and present a range of parameters for which Anderson localization in the time domain can be observed experimentally. It seems that the phenomenon can be realized in a laboratory very soon. There are two groups which succeeded in creation of the so-called nonspreading wave packets in Rydberg atoms driven by microwave fields [43–46]. Realization of the Anderson localization in time requires modification of the microwave fields only, i.e., switching from the monochromatic fields to fields that fluctuate in time. It would be the first experimental demonstration of Anderson localization in the time domain.

ACKNOWLEDGMENT

Support of the National Science Centre, Poland, via Project No. 2016/20/W/ST4/00314 is acknowledged.

-
- [1] P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
 - [2] B. A. van Tiggelen, in *Diffuse Waves in Complex Media*, edited by J.-P. Fouque, NATO Advanced Study Institutes, Series C (Kluwer, Dordrecht, 1999), Vol. 531.
 - [3] A. Lagendijk, B. A. van Tiggelen, and D. S. Wiersma, *Phys. Today* **62**, 24 (2009).
 - [4] C. A. Müller and D. Delande, in *Lecture Notes of the Les Houches Summer School in Singapore: Ultracold Gases and Quantum Information*, edited by C. Miniatura *et al.* (Oxford University Press, Oxford, 2011), Vol. 91, Chap. 9.
 - [5] F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, *Phys. Rev. Lett.* **75**, 4598 (1995).
 - [6] S. Fishman, D. R. Grempel, and R. E. Prange, *Phys. Rev. Lett.* **49**, 509 (1982).
 - [7] G. Casati, I. Guarneri, and D. L. Shepelyansky, *Phys. Rev. Lett.* **62**, 345 (1989).
 - [8] G. Lemarié, J. Chabé, P. Szriftgiser, J. C. Garreau, B. Grémaud, and D. Delande, *Phys. Rev. A* **80**, 043626 (2009).
 - [9] I. Manai, J.-F. Clément, R. Chicireanu, C. Hainaut, J. C. Garreau, P. Szriftgiser, and D. Delande, *Phys. Rev. Lett.* **115**, 240603 (2015).
 - [10] K. Sacha, *Sci. Rep.* **5**, 10787 (2015).
 - [11] K. Sacha and D. Delande, *Phys. Rev. A* **94**, 023633 (2016).
 - [12] D. Delande, L. Morales-Molina, and K. Sacha, *arXiv:1702.03591*.
 - [13] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [14] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *Phys. Rev. Lett.* **109**, 163001 (2012).
 - [15] M. N. Chernodub, *Phys. Rev. D* **87**, 025021 (2013).
 - [16] F. Wilczek, *Phys. Rev. Lett.* **111**, 250402 (2013).
 - [17] P. Bruno, *Phys. Rev. Lett.* **110**, 118901 (2013).
 - [18] F. Wilczek, *Phys. Rev. Lett.* **110**, 118902 (2013).
 - [19] P. Bruno, *Phys. Rev. Lett.* **111**, 029301 (2013).
 - [20] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *arXiv:1212.6959*.
 - [21] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [22] L. Guo, M. Marthaler, and G. Schön, *Phys. Rev. Lett.* **111**, 205303 (2013).
 - [23] H. Watanabe and M. Oshikawa, *Phys. Rev. Lett.* **114**, 251603 (2015).
 - [24] K. Sacha, *Phys. Rev. A* **91**, 033617 (2015).
 - [25] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).
 - [26] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
 - [27] C. W. von Keyserlingk, V. Khemani, and S. L. Sondhi, *Phys. Rev. B* **94**, 085112 (2016).
 - [28] N. Y. Yao, A. C. Potter, I. D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
 - [29] S. A. Weidinger and M. Knap, *Sci. Rep.* **7**, 45382 (2017).
 - [30] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, *Nature* **543**, 217 (2017).
 - [31] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, *Nature* **543**, 221 (2017).
 - [32] A. Syrwid, J. Zakrzewski, and K. Sacha, *arXiv:1702.05006*.
 - [33] G. P. Berman and G. M. Zaslavsky, *Phys. Lett. A* **61**, 295 (1977).
 - [34] J. Henkel and M. Holthaus, *Phys. Rev. A* **45**, 1978 (1992).
 - [35] M. Holthaus, *Chaos Solitons Fractals* **5**, 1143 (1995).
 - [36] D. Delande and A. Buchleitner, *Adv. At. Mol. Opt. Phys.* **34**, 85 (1994).
 - [37] I. Bialynicki-Birula, M. Kaliński, and J. H. Eberly, *Phys. Rev. Lett.* **73**, 1777 (1994).
 - [38] A. Buchleitner and D. Delande, *Phys. Rev. Lett.* **75**, 1487 (1995).
 - [39] D. Delande, J. Zakrzewski, and A. Buchleitner, *Europhys. Lett.* **32**, 107 (1995).
 - [40] J. Zakrzewski, D. Delande, and A. Buchleitner, *Phys. Rev. Lett.* **75**, 4015 (1995).
 - [41] K. Sacha, J. Zakrzewski, and D. Delande, *Eur. Phys. J. D* **1**, 231 (1998).
 - [42] D. Delande, K. Sacha, and J. Zakrzewski, *Acta Phys. Pol. B* **33**, 2097 (2002), *arXiv:quant-ph/0206033*.
 - [43] H. Maeda and T. F. Gallagher, *Phys. Rev. Lett.* **92**, 133004 (2004).

- [44] H. Maeda and T. F. Gallagher, *Phys. Rev. A* **75**, 033410 (2007).
- [45] H. Maeda, J. H. Gurian, and T. F. Gallagher, *Phys. Rev. Lett.* **102**, 103001 (2009).
- [46] B. Wyker, S. Ye, F. B. Dunning, S. Yoshida, C. O. Reinhold, and J. Burgdörfer, *Phys. Rev. Lett.* **108**, 043001 (2012).
- [47] A. Buchleitner, D. Delande, and J. Zakrzewski, *Phys. Rep.* **368**, 409 (2002).
- [48] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, edited by F. John *et al.*, Applied Mathematical Sciences Vol. 38 (Springer, Berlin, 1983).
- [49] R. C. Kuhn, O. Sigwarth, C. Miniatura, D. Delande, and C. A. Müller, *New J. Phys.* **9**, 161 (2007).
- [50] J. H. Shirley, *Phys. Rev.* **138**, B979 (1965).
- [51] P. Lugan, Ph.D. thesis, Ecole Polytechnique, 2010, <https://tel.archives-ouvertes.fr/tel-00468888/document>

Time Crystal Platform: From Quasicrystal Structures in Time to Systems with Exotic Interactions

Krzysztof Giergiel,¹ Artur Miroszewski,^{1,2} and Krzysztof Sacha^{1,3}

¹*Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

²*National Centre for Nuclear Research, ul. Hoża 69, PL-00-681 Warsaw, Poland*

³*Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*



(Received 21 November 2017; published 2 April 2018)

Time crystals are quantum many-body systems that, due to interactions between particles, are able to spontaneously self-organize their motion in a periodic way in time by analogy with the formation of crystalline structures in space in condensed matter physics. In solid state physics properties of space crystals are often investigated with the help of external potentials that are spatially periodic and reflect various crystalline structures. A similar approach can be applied for time crystals, as periodically driven systems constitute counterparts of spatially periodic systems, but in the time domain. Here we show that condensed matter problems ranging from single particles in potentials of quasicrystal structure to many-body systems with exotic long-range interactions can be realized in the time domain with an appropriate periodic driving. Moreover, it is possible to create molecules where atoms are bound together due to destructive interference if the atomic scattering length is modulated in time.

DOI: [10.1103/PhysRevLett.120.140401](https://doi.org/10.1103/PhysRevLett.120.140401)

Although crystals have been known for years, time crystals sound more like science fiction than a serious scientific concept. In 2012 Frank Wilczek initiated a new research area by suggesting that periodic structures in time can be formed spontaneously by a quantum many-body system [1]. While the original Wilczek idea could not be realized because it was based on a system in the ground state [2–5], it turned out that spontaneous breaking of discrete time translation symmetry and self-reorganization of motion of a periodically driven quantum many-body system was possible [6]. This phenomenon was dubbed “discrete time crystals” [7–9] and it was already realized experimentally [10,11]; for a review see [12].

Apart from the realization of spontaneous breaking of discrete time translation symmetry, periodically driven quantum systems can also be employed to model crystalline structures in time in a similar way as external time-independent spatially periodic potentials allow one to model space crystals [13,14]. It should be stressed that driven systems with crystalline properties in time do not require external spatially periodic potentials. Crystal structures in time emerge due to periodic driving provided it is resonant with the unperturbed motion of a system. It is possible to investigate Anderson localization [15] in the time domain [14,16–18] or many-body localization caused by temporal disorder [19]. In the following we show that proper manipulation of higher temporal harmonics of a periodic perturbation is a perfect tool to engineer a wide class of condensed matter systems in the time domain

including many-body systems with exotic interactions. Moreover, it is possible to create molecules where atoms are bound together via disordered potentials.

Let us begin with a classical single particle system in one dimension described by the Hamiltonian $H_0(x, p)$. If the motion of a particle is bounded it is convenient to perform a canonical transformation to the so-called action-angle variables [20]. Then, $H_0 = H_0(I)$ where the momentum (action) I is a constant of motion and the canonically conjugate angle θ changes linearly with time, i.e., $\theta(t) = \Omega t + \theta(0)$ where $\Omega(I) = (dH_0(I)/dI)$ is a frequency of periodic evolution of a particle. Assume we turn on a periodic driving of the form $H_1 = \lambda h(x)f(t)$ where $f(t + 2\pi/\omega) = f(t) = \sum_k f_k e^{ik\omega t}$ and λ determines the strength of the driving. The spatial part of H_1 can be expanded in a Fourier series $h(x) = \sum_n h_n(I) e^{in\theta}$. A particle is resonantly driven if the period of its unperturbed motion is equal to an integer multiple of the driving period, i.e., $\omega = s\Omega(I_0)$ where s is an integer number and I_0 is a resonant value of the action. In order to analyze motion of a particle in the vicinity of a resonant trajectory; i.e., for $I \approx I_0$, it is convenient to switch to the moving frame, $\Theta = \theta - (\omega/s)t$, and apply the secular approximation [20]. It results in the effective time-independent Hamiltonian, $H_{\text{eff}} = (P^2/2m_{\text{eff}}) + \lambda V_{\text{eff}}(\Theta)$, where $P = I - I_0$, the effective mass $(1/m_{\text{eff}}) = (d^2 H_0(I_0)/dI_0^2)$ and the potential $V_{\text{eff}}(\Theta) = \sum_n h_{ns}(I_0) f_{-n} e^{ins\Theta}$. If the second order corrections are negligible, which can be easily monitored [20,21], H_{eff} provides an exact

description of particle motion in the vicinity of a resonant trajectory.

The effective Hamiltonian, H_{eff} , indicates that a resonantly driven system behaves like a particle on a ring, i.e., $0 < \Theta \leq 2\pi$, with a certain effective mass and in the presence of a time-independent effective potential $V_{\text{eff}}(\Theta)$. If there are many nonzero Fourier components of $h(x)$, a proper choice of temporal Fourier components of $f(t)$ allows one to create a practically arbitrary effective potential. Indeed, any potential on a ring can be expanded in a series $V_{\text{eff}}(\Theta) = \sum_n d_n e^{in\Theta}$ and in order to realize it we can choose the fundamental 1:1 resonance ($s = 1$) and periodic driving with the Fourier components $f_{-n} = (d_n/h_n(I_0))$. If $s > 1$, a potential energy structure is duplicated s times. For $s \gg 1$, $V_{\text{eff}}(\Theta)$ allows one to reproduce condensed matter problems where a particle can move in a potential with s identical wells of arbitrary shape and with periodic boundary conditions.

Before we illustrate our idea with an example we have to address two issues. First, so far our approach was classical but we deal with quantum systems. In order to obtain a quantum description one can either perform quantization of the effective Hamiltonian, i.e., $(P, \Theta) \rightarrow (\hat{P}, \hat{\Theta})$, or apply a quantum version of the secular approximation from the very beginning [22]. Both approaches lead to the same results. Eigenstates of the effective Hamiltonian in the moving frame correspond to time-periodic Floquet eigenstates of the original Floquet Hamiltonian, $H_F = H_0 + H_1 - i\hbar\partial_t$, in the laboratory frame [21,23]. The second issue is the following: What is the relation of the class of problems we consider with time crystals? Space crystals are related to periodic arrangement of particles in space. If we take a snapshot of a space crystal at some moment in time ($t = \text{const}$), then we can observe a crystalline structure in space. Switching to time crystals the role of time and space is exchanged. We fix position in configuration space ($x = \text{const}$); i.e., we choose the location for the detector, and ask if the probability of clicking of a detector behaves periodically in time. We have shown that in the frame moving along a classical resonant orbit, $\Theta = \theta - (\omega t/s)$, we obtain an effective Hamiltonian that can describe a solid state problem. Such a crystalline structure in Θ is reproduced in the time domain if we return to the laboratory frame, as the relation between Θ and t is linear. Thus, if we locate a detector close to a classical resonant trajectory, the probability of detection of a particle as a function of time reproduces crystalline structure described by means of H_{eff} in the moving frame.

In Refs. [24,25] it was proven that stable orbits of classical dissipative systems can reveal quasicrystal tiling in time. We show that quantum properties of quasicrystals in time can be investigated; see also [26,27]. In condensed matter physics quasicrystals are systems that do not have any minimal part that appears periodically in space. Nevertheless, two or more unit cells are not placed

randomly because a d -dimensional quasicrystal can be constructed as a slice through a $(2d)$ -dimensional periodic crystal [28–30]. We focus on the $d = 1$ case when one-dimensional quasicrystal structure can be constructed as a cut through a two-dimensional square lattice. The cut with the line whose gradient is the golden ratio generates the Fibonacci quasicrystal that can also be constructed with the help of the so-called inflation rule [24]: $B \rightarrow BS$ and $S \rightarrow B$ where B and S denote, e.g., big and small wells, respectively, of a potential energy of a single particle. Successive application of the inflation rule shows the process of growing of the quasicrystal, i.e., $B \rightarrow BS \rightarrow BSB \rightarrow BSBBS \rightarrow BSBBSBSB \rightarrow \dots$

In order to illustrate how to realize the Fibonacci quasicrystal in the time domain experimentally, let us consider, e.g., a particle that bounces on a vibrating mirror in the presence of a gravitational field [23,31–33] in a one-dimensional model. In the coordinate frame vibrating with the mirror, the mirror is fixed but the gravitation strength oscillates in time. Then the Hamiltonian of the system, in gravitational units [23], reads $H = (p^2/2) + x + \lambda x f(t)$ where $f(t) = \sum_k f_k e^{ik\omega t}$ and (λ/ω^2) is related to the amplitude of the mirror vibration. The secular approximation leads to the previously derived effective Hamiltonian with $m_{\text{eff}} = -(\pi^2/\omega^4)$ and $h_n = -(-1)^n/(n^2\omega^2)$ if the 1:1 resonance condition ($s = 1$) is fulfilled. A proper choice of $f(t)$ allows one to realize the effective potential $V_{\text{eff}}(\Theta)$ that reproduces any finite Fibonacci quasicrystal. In Fig. 1 we show what kind of driving leads to a quasicrystal with the total number of big and small potential wells given by the seventh Fibonacci number. Transport properties in the quasicrystal that can be analyzed with the help of the effective Hamiltonian in the frame moving along the 1:1 resonant orbit are observed in the time domain in the laboratory frame.

Now we demonstrate that periodically driven many-body systems allow for realization of solid state problems with exotic interactions. Let us illustrate this idea with ultracold atoms bouncing on a mirror that oscillates harmonically

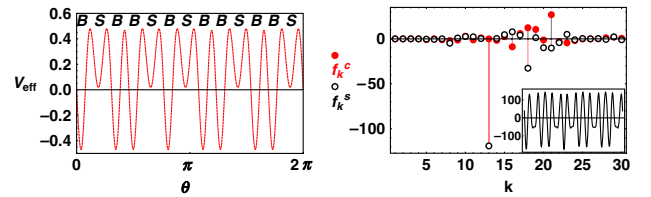


FIG. 1. Time quasicrystal: a particle bouncing on a vibrating mirror in a gravitational field in a one-dimensional model. The 1:1 resonance condition is assumed. Left: the effective potential $V_{\text{eff}}(\Theta)$ with the quasicrystal structure corresponding to the seventh Fibonacci number. Right: Fourier components of the periodic vibration of the mirror, $(f(t)/\omega^2) = \sum_k (f_k^c \cos k\omega t + f_k^s \sin k\omega t)$, that result in $V_{\text{eff}}(\Theta)$ presented in the left panel. Full symbols are related to f_k^c , open symbols to f_k^s . The inset of this panel shows $(f(t)/\omega^2)$ over one period.

with frequency ω . If the $s:1$ resonance condition is fulfilled, the single-particle effective Hamiltonian in the moving frame reads $H_{\text{eff}} = (P^2/2m_{\text{eff}}) + V_0 \cos(s\Theta)$ where $m_{\text{eff}} = -(\pi^2 s^4/\omega^4)$ and $V_0 = -(\lambda(-1)^s/\omega^2)$. Let us assume $s \gg 1$ and V_0 sufficiently big so that in the quantum description eigenvalues of H_{eff} form well-separated energy bands and eigenstates are Bloch waves $e^{ik\Theta} v_k(\Theta)$ where $v_k(\Theta + (2\pi/s)) = v_k(\Theta)$. Note that for a fixed position in the laboratory frame, $\theta = \text{const}$, the periodic Bloch waves character emerges in time, $e^{ik(\theta - \omega t/s)} v_k(\theta - (\omega t/s))$, with the period $(2\pi/\omega)$. Because of the negative effective mass m_{eff} , the effective Hamiltonian H_{eff} is bounded from above, not from below. Therefore, the first energy band possesses the highest energy. For simplicity, let us restrict ourselves to the first band and choose as the basis in the corresponding Hilbert subspace the Wannier states $w_j = w(\Theta - j(2\pi/s))$ where j denotes at which site of the effective potential a Wannier function is localized [34]. In the laboratory frame the Wannier states $w_j(x, t)$ describe localized wave packets moving along the resonant trajectory. We assume the normalization $\int_0^{s2\pi/\omega} dt \langle w_j | w_j \rangle = 1$. Thus, s sites of the effective potential in the moving frame correspond to s Wannier wave packets evolving in the laboratory frame. The width of the first energy band of H_{eff} is determined by $J = -2 \int_0^{s2\pi/\omega} dt \langle w_{i+1} | H_{\text{eff}} | w_i \rangle$, which is an amplitude of nearest neighbor tunnelings.

In ultracold atomic gases interactions are described by the contact Dirac-delta potential, $g_0 \delta(x)$, where g_0 is determined by the atomic scattering length, which can be modulated in time by means of a Feshbach resonance [35]. We are going to show that these contact interactions between atoms can result in exotic long-range interactions in the effective description of the resonantly driven many-body system (effective long-range interactions in the phase space crystals [13,36] have been considered in [37,38]; see also [39]). For example in the case of bosonic particles, when we restrict ourselves to the Hilbert subspace spanned by Fock states $|\dots, n_j, \dots\rangle$, where n_j is the number of atoms occupying a mode w_j , we obtain a many-body effective Hamiltonian of the Bose-Hubbard form,

$$\hat{H}_{\text{eff}} = -\frac{J}{2} \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{i,j} U_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i, \quad (1)$$

where the bosonic operators $\hat{a}_j$ annihilate particles in modes w_j 's and $U_{ij} = \int_0^{s2\pi/\omega} dt g_0(t) u_{ij}(t)$ with $u_{ij}(t) = 2 \int_0^\infty dx |w_i|^2 |w_j|^2$ for $i \neq j$ and $u_{ii} = \int_0^\infty dx |w_i|^4$ [14], where we assume that the atomic scattering length $g_0(t)$ can be modulated in time. The Hamiltonian (1) is valid provided the interaction energy per particle is always smaller than the energy gap between the lowest and first excited energy bands of the single-particle system. A given interaction coefficient U_{ij} is determined mostly by $g_0(t)$ at the moment when the corresponding wave packets overlap.

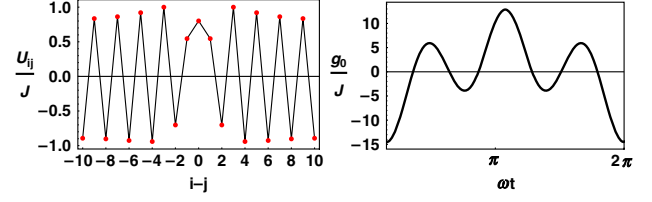


FIG. 2. System with exotic interactions: ultracold atoms bouncing on a harmonically oscillating mirror in a one-dimensional model. The 20:1 resonance condition is fulfilled and the many-body system is described by the Hamiltonian (1). The left panel shows the interaction coefficients U_{ij} corresponding to the scattering length $g_0(t)$ that is presented in the right panel. The frequency $\omega = 2.8$ of the mirror oscillations and $\lambda = 0.1$ which result in $J = 3.7 \times 10^{-5}$ and the gap of $302J$ between the lowest and first excited energy bands. Temporary interaction coefficients $(s2\pi/\omega)|g_0(t)|u_{ij}(t) \leq 85J$.

Suitable modulation of the scattering length $g_0(t)$ allows us to shape the interactions in (1). In order to perform a systematic analysis one can apply the singular value decomposition of the matrix $u_{ij}(t)$ where (i, j) and t are treated as indices of rows and columns, respectively. Left singular vectors tell us which sets of interaction coefficients U_{ij} can be realized, while the corresponding right singular vectors give the recipes for $g_0(t)$. In Fig. 2 we present an example of the interaction coefficients and the corresponding function $g_0(t)$. In this example the magnitude of the interactions of a particle located at a given site with other particles located at the same or distant sites is nearly the same, but their repulsive or attractive character changes in an oscillatory way.

Time is a single degree of freedom and it is hard to imagine multidimensional time crystals. However, we are going to show that resonantly driven systems can reveal properties of two-dimensional or three-dimensional space crystals in the time domain. Let us begin with a single particle bouncing between two mirrors that oscillate harmonically in two orthogonal directions with frequency ω ; see Fig. 3—generalization to the three-dimensional case is straightforward. The single-particle Hamiltonian reads $H = [(p_x^2 + p_y^2)/2] + x + y + \lambda x \cos \omega t + \lambda y \cos(\omega t + \varphi)$ where φ is the relative phase of the mirror oscillations. Assuming that for each of the two independent degrees of freedom the $s:1$ resonance condition is fulfilled we obtain (in terms of the action-angle variables and in the moving frame $\Theta_j = x, y = \theta_j - (\omega t/s)$) the effective Hamiltonian, $H_{\text{eff}} = [(P_x^2 + P_y^2)/2m_{\text{eff}}] + V_0[\cos(s\Theta_x) + \cos(s\Theta_y)]$, which describes a particle in a two-dimensional square lattice. For $s \gg 1$, eigenstates of H_{eff} are Bloch waves $e^{i(k_x \Theta_x + k_y \Theta_y)} v_{k_x}(\Theta_x) v_{k_y}(\Theta_y)$. When we fix a position in the laboratory frame, i.e., we fix θ_x and θ_y , periodic character of Bloch waves emerges in time, $e^{i[k_x \theta_x + k_y \theta_y - (k_x + k_y) \omega t/s]} v_{k_x}(\theta_x - (\omega t/s)) v_{k_y}(\theta_y - (\omega t/s))$. Different fixed values of θ_x and θ_y allow us to observe in

the time domain different cuts of the square lattice described by H_{eff} . We restrict ourselves to the first energy band of H_{eff} and define the Wannier state basis $W_{\mathbf{j}} = w_x(\Theta_x - j_x(2\pi/s))w_y(\Theta_y - j_y(2\pi/s))$ where $\mathbf{j} = (j_x, j_y)$ denotes at which site of the effective potential a Wannier function is localized. In the laboratory frame the Wannier states $W_{\mathbf{j}}(x, y, t)$ describe localized wave packets moving along resonant trajectories. The shape of the trajectories depends on the relative phase φ of the mirror oscillations. For $\varphi \neq (\pi/2)$ there are s different trajectories in the configuration space and s wave packets $W_{\mathbf{j}}(x, y, t)$ moving along each of them; see Fig. 3. Thus, s^2 sites of the effective potential in the moving frame correspond to s^2 Wannier wave packets evolving in the laboratory frame. Switching to the many-body case we obtain, for ultracold bosons, a two-dimensional version of the Hamiltonian (1). If the scattering length is not modulated in time, i.e., $g_0(t) = \text{const}$, the on-site interactions are dominant and the system reproduces, in the moving frame, a two-dimensional squared lattice problem with on-site interactions [34]. If we locate detectors at different positions in the laboratory frame, the time dependence of the probabilities of detection reflects cuts of the two-dimensional square lattice; see Fig. 3.

Finally, let us show that periodic driving allows one to create a molecule where Anderson localization is

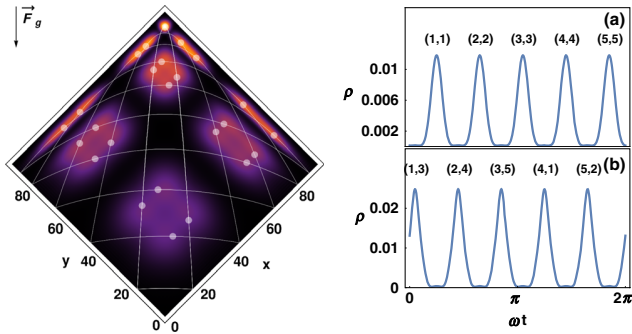


FIG. 3. Time crystal with properties of a two-dimensional space crystal: ultracold atoms bouncing between two perpendicular and harmonically oscillating mirrors with $\omega = 1.05$, $\lambda = 0.02$, and $\varphi = (\pi/2)$. The 5:1 resonance conditions are fulfilled and the many-body system is described by a two-dimensional version of the Bose-Hubbard Hamiltonian (1). For $g_0(t) = \text{const}$, the on-site interactions are dominant, i.e., $U_{ii}/g_0 \in [0.2, 0.3]J$, where different values correspond to different classical trajectories, while $U_{ij \neq i}/g_0 < 0.07J$. The left panel shows 25 Wannier wave packets, i.e., $\rho(x, y, t) = \sum_{\mathbf{j}} |W_{\mathbf{j}}(x, y, t)|^2$, at $t = (4/\omega)$ and trajectories along which they propagate—dots indicate positions of the centers of the wave packets. Two perpendicular mirrors are located at $x = 0$ and $y = 0$ and they form a $(\pi/4)$ angle with respect to the gravitational force $\vec{F}_g$. Right panels present $\rho(x, y, t)$ at $(x, y) = (90, 90)$ (a) and $(82, 82)$ (b) versus t : these plots reflect cuts of a square lattice described by the Bose-Hubbard model in the moving frame. Numbers in parentheses indicate which lattice sites $\mathbf{j} = (j_x, j_y)$ are located along the cuts.

responsible for the binding of two atoms. Assume that two atoms move on a ring and their scattering length is modulated in time employing a Feshbach resonance so that the Hamiltonian of the system reads $H = [(p_1^2 + p_2^2)/2] + 2\pi\lambda\delta(\theta_1 - \theta_2)f(t)$ where λ is a constant, $f(t) = \sum_{k \neq 0} f_k e^{ik\omega t}$, and $\theta_{1,2}$ denote positions of the atoms on the ring (see Fig. 4). If the first atom is moving in the clockwise direction with momentum $p_1 \approx \omega$, and the other in the anticlockwise direction with $p_2 \approx -\omega$, then the secular approximation results in $H_{\text{eff}} = [(P_1^2 + P_2^2)/2] + \lambda V_{\text{eff}}(\Theta_1 - \Theta_2)$ in the moving frame, i.e., $\Theta_1 = \theta_1 - \omega t$ and $\Theta_2 = \theta_2 + \omega t$. Interactions between atoms are described by the effective potential $V_{\text{eff}} = \sum_n f_{-2n} e^{in(\Theta_1 - \Theta_2)}$ whose shape can be engineered at will by a suitable choice of the Fourier components f_k of the periodic driving. For example if $f_k = (1/\sqrt{k_0})e^{i\varphi_k}$ for $|k| \leq (k_0/2)$ and 0 otherwise, where $\varphi_k = -\varphi_{-k}$ are random variables chosen from a uniform distribution, the atoms interact via the effective disordered potential characterized by the correlation length $(\sqrt{2}/k_0)$ and the standard deviation λ . Then, eigenstates $\psi(\Theta_1 - \Theta_2)$ of H_{eff} are Anderson localized around different values θ_0 of the relative coordinate [15], i.e., $|\psi|^2 \propto e^{-|\Theta_1 - \Theta_2 - \theta_0|/l_0}$, provided the localization length $l_0 \ll 2\pi$: within the Born approximation $l_0 = (Ek_0^2/\pi\lambda^2)$, which is valid when $(\lambda^2/k_0^2) \ll E \ll (k_0^2/4)$, where E is energy of the system in the moving frame [17,40]. Hence, we are dealing with a situation where two atoms are bound together not by attractive interactions but due to destructive interference, i.e., due to the Anderson localization phenomenon induced by disordered mutual interactions [21]. If atoms are identical bosons (fermions), an eigenstate must be symmetric (antisymmetric) under their exchange. This symmetry is easily restored because we can exchange the role of the atoms. That is, the first atom can move in the anticlockwise direction, $p_1 \approx -\omega$, while the other one can

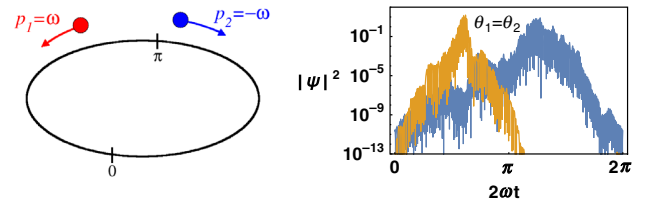


FIG. 4. Two atoms bound together due to destructive interference. The left panel shows a schematic plot of an experiment where two distinguishable atoms move on a ring in opposite directions with momenta $\pm\omega$. Resonant modulation of atomic scattering length allows one to create a molecule where the atoms are bound together due to destructive interference; i.e., in the moving frame eigenstates of the system are Anderson localized. The right panel presents probability densities for detection of the atoms at $\theta_1 = \theta_2$ in the laboratory frame versus t for two eigenstates related to energies $E = 6020$ (orange line) and 10460 (blue line) for $\lambda = 5660$ and $k_0 = 500$.

move in the clockwise direction, $p_2 \approx \omega$. Consequently, proper Floquet eigenstates for bosons or fermions, in the laboratory frame, read $\psi(\theta_1 - \theta_2 - 2\omega t) \pm \psi(\theta_2 - \theta_1 - 2\omega t)$. Experimental demonstration of two atoms bound due to destructive interference seems straightforward if atoms are prepared in a toroidal trap [41–43].

In summary, we have shown that a wide class of condensed matter problems can be realized in the time domain if single-particle or many-body systems are resonantly driven. It opens up unexplored territory for investigation of condensed matter physics in time and for the invention of novel time devices because time is our new ally. As an example we have demonstrated that periodic driving allows one to realize molecules where atoms are bound together not due to attractive mutual interactions but due to destructive interference.

Support of the National Science Centre, Poland, via Grants No. 2016/20/W/ST4/00314 (K. G.) and No. 2016/21/B/ST2/01095 (K. S.) is acknowledged. This work was performed with the support of the EU via Horizon2020 FET project QUIC (Grant No. 641122).

-
- [1] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [2] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [3] H. Watanabe and M. Oshikawa, *Phys. Rev. Lett.* **114**, 251603 (2015).
 - [4] A. Syrwid, J. Zakrzewski, and K. Sacha, *Phys. Rev. Lett.* **119**, 250602 (2017).
 - [5] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, *arXiv:1708.05014*.
 - [6] K. Sacha, *Phys. Rev. A* **91**, 033617 (2015).
 - [7] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).
 - [8] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
 - [9] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
 - [10] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath *et al.*, *Nature (London)* **543**, 217 (2017).
 - [11] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani *et al.*, *Nature (London)* **543**, 221 (2017).
 - [12] K. Sacha and J. Zakrzewski, *Rep. Prog. Phys.* **81**, 016401 (2018).
 - [13] L. Guo, M. Marthaler, and G. Schön, *Phys. Rev. Lett.* **111**, 205303 (2013).
 - [14] K. Sacha, *Sci. Rep.* **5**, 10787 (2015).
 - [15] C. A. Müller and D. Delande, *Disorder and Interference: Localization Phenomena* (Oxford Scholarship, Oxford, 2011), Vol. 91, Chap. 9.
 - [16] K. Sacha and D. Delande, *Phys. Rev. A* **94**, 023633 (2016).
 - [17] K. Giergiel and K. Sacha, *Phys. Rev. A* **95**, 063402 (2017).
 - [18] D. Delande, L. Morales-Molina, and K. Sacha, *Phys. Rev. Lett.* **119**, 230404 (2017).
 - [19] M. Mierzejewski, K. Giergiel, and K. Sacha, *Phys. Rev. B* **96**, 140201 (2017).
 - [20] A. Lichtenberg and M. Lieberman, *Regular and Chaotic Dynamics*, Applied Mathematical Sciences (Springer-Verlag, Berlin, 1992).
 - [21] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.120.140401> for the analysis of the validity of effective Hamiltonians and for a short introduction to the description of periodically driven systems.
 - [22] G. Berman and G. Zaslavsky, *Phys. Lett.* **61A**, 295 (1977).
 - [23] A. Buchleitner, D. Delande, and J. Zakrzewski, *Phys. Rep.* **368**, 409 (2002).
 - [24] F. Flicker, *arXiv:1707.09371* [SciPost (to be published)].
 - [25] F. Flicker, *arXiv:1707.09333*.
 - [26] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *Phys. Rev. Lett.* **109**, 163001 (2012).
 - [27] Y. Huang, T. Li, and Z. Q. Yin, *Phys. Rev. A* **97**, 012115 (2018).
 - [28] J. E. S. Socolar and P. J. Steinhardt, *Phys. Rev. B* **34**, 617 (1986).
 - [29] C. Janot, *Quasicrystals: A Primer* (Oxford University Press, Oxford, 1994).
 - [30] M. Senechal, *Quasicrystals and Geometry* (Cambridge University Press, Cambridge, 1995).
 - [31] A. Steane, P. Szriftgiser, P. Desbiolles, and J. Dalibard, *Phys. Rev. Lett.* **74**, 4972 (1995).
 - [32] D. C. Lau, A. I. Sidorov, G. I. Opat, R. J. McLean, W. J. Rowlands, and P. Hannaford, *Eur. Phys. J. D* **5**, 193 (1999).
 - [33] K. Bongs, S. Burger, G. Birkl, K. Sengstock, W. Ertmer, K. Rzażewski, A. Sanpera, and M. Lewenstein, *Phys. Rev. Lett.* **83**, 3577 (1999).
 - [34] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Lühmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, *Rep. Prog. Phys.* **78**, 066001 (2015).
 - [35] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, *Rev. Mod. Phys.* **82**, 1225 (2010).
 - [36] L. Guo and M. Marthaler, *New J. Phys.* **18**, 023006 (2016).
 - [37] L. Guo, M. Liu, and M. Marthaler, *Phys. Rev. A* **93**, 053616 (2016).
 - [38] P. Liang, M. Marthaler, and L. Guo, *New J. Phys.* **20**, 023043 (2018).
 - [39] E. Anisimovas, G. Žlabys, B. M. Anderson, G. Juzeliūnas, and A. Eckardt, *Phys. Rev. B* **91**, 245135 (2015).
 - [40] R. C. Kuhn, O. Sigwarth, C. Miniatura, D. Delande, and C. A. Müller, *New J. Phys.* **9**, 161 (2007).
 - [41] M. Clifford, J. Arlt, J. Courtial, and K. Dholakia, *Opt. Commun.* **156**, 300 (1998).
 - [42] E. M. Wright, J. Arlt, and K. Dholakia, *Phys. Rev. A* **63**, 013608 (2000).
 - [43] Z. Y. Wang, Z. Zhang, and Q. Lin, *Eur. Phys. J. D* **53**, 127 (2009).

Supplemental material: Time crystal platform: from quasi-crystal structures in time to systems with exotic interactions

Krzysztof Giergiel,¹ Artur Miroszewski,^{1,2} and Krzysztof Sacha^{1,3}

¹ *Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

² *National Centre for Nuclear Research, ul. Hoża 69, PL-00-681 Warsaw, Poland*

³ *Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

In this Supplemental Material we present an analysis of the validity of the effective Hamiltonian approach for two examples: a particle bouncing on a vibrating mirror and formation of a molecule where two atoms are bound together due to destructive interference. However, we begin with a short introduction to the classical and quantum description of periodically driven systems.

If the Hamiltonian of, e.g., a single-particle system in one-dimensional space depends explicitly on time, $H(x, p, t)$, the energy is not conserved. In the classical description one can extend the phase space of the system by defining t as an additional *dimension* of the configuration space and $p_t = -H$ as the conjugate momentum [1, 2]. Then, the new Hamiltonian in such an extended phase space reads $\mathcal{H} = H(x, p, t) + p_t$ and the motion of a particle is parametrized by some fictitious time τ . Because $\frac{dt}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_t} = 1$, t and τ are essentially identical. The new Hamiltonian does not depend explicitly on τ and thus it is conserved as τ evolves.

In the quantum description there are no energy eigenstates because the energy is not conserved. However, if a particle is periodically driven, $H(x, p, t + \frac{2\pi}{\omega}) = H(x, p, t)$, we can look for a kind of stationary states of the form of $\psi_n(x, t) = e^{-iE_n t} v_n(x, t)$ where $v_n(x, t + \frac{2\pi}{\omega}) = v_n(x, t)$. Indeed, substituting $\psi_n(x, t)$ in the time-dependent Schrödinger equation we obtain the eigenvalue problem,

$$(H(x, p, t) - i\partial_t) v_n(x, t) = E_n v_n(x, t), \quad (1)$$

where $v_n(x, t)$ fulfills periodic boundary conditions in time [2, 3]. Thus, one may define a new Hamiltonian (so-called Floquet Hamiltonian) $H_F = H - i\partial_t$ whose eigenstates (called Floquet states) are time-periodic and form a complete basis in the Hilbert space of the system at any time t [3]. The Floquet Hamiltonian can be considered as the quantized version of the classical Hamiltonian $\mathcal{H}$ in the extended phase space where $p_t \rightarrow -i\partial_t$.

In the Letter we show different periodically driven systems whose Floquet states possess properties of condensed matter systems in the time domain. In order to identify systems with such properties and find suitable parameters we begin with the classical approach, obtain effective Hamiltonians and then switch to the quantum description.

PARTICLE BOUNCING ON A VIBRATING MIRROR

Let us consider a particle which bounces on a periodically vibrating mirror in the presence of the gravitational force [2]. In the frame moving with the mirror, the mirror does not vibrate but the gravitational potential changes periodically in time. The Hamiltonian of the system, in gravitational units, reads $H = H_0 + H_1$ with

$$H_0(x, p) = \frac{p^2}{2} + x, \quad H_1(x, t) = \lambda x f(t), \quad (2)$$

where $f(t + 2\pi/\omega) = f(t)$ and λ determines the strength of the perturbation.

For $\lambda = 0$, the classical description of the system becomes very simple if we apply the canonical transformation to the so-called action-angle variables (I, θ) [1]. Then, $H_0 = H_0(I) = \frac{1}{2} (3\pi I)^{2/3}$ where the action I (new momentum) is a constant of motion and the conjugate angle θ (new position variable) changes linearly in time. Now, $\theta(t) = \Omega t + \theta(0)$ where $\Omega(I) = \frac{dH_0(I)}{dI}$ is the frequency of a periodic motion of the unperturbed particle [2]. The portrait of the (θ, I) phase space is very simple because it consists of straight lines corresponding to different values of $I = \text{constant}$.

When $\lambda \neq 0$, the phase space structure changes around resonant values of I . Let us focus on $f(t) = \sum_k f_k e^{ik\omega t}$ where f_k 's correspond to the time quasi-crystal structure presented in Fig. 1 of the Letter. In the present Supplemental Material in Fig. 1(a)-1(b) we show a stroboscopic picture of the (Θ, P) space obtained by integration of the classical equations of motion generated by the Hamiltonian (2) where $\Theta = \theta - \omega t$ and $P = I - I_0$ with I_0 corresponding to the 1:1 resonance condition between the driving force and the unperturbed particle motion, i.e. $\omega = \Omega(I_0)$. We have chosen $I_0 = 1$ and two different values of λ . If λ is sufficiently small the phase space portrait around elliptical islands does not reveal chaotic motion and we may expect that the first order secular approximation [1] is able to perfectly describe motion of a particle close to the resonant value of I . The Hamiltonian (2) in the frame moving along the resonant orbit, i.e. in the $\Theta = \theta - \omega t$ and I variables, reads

$$H = H_0(I) - \omega I + \lambda f(t) \sum_n h_n(I) e^{in(\Theta + \omega t)}, \quad (3)$$

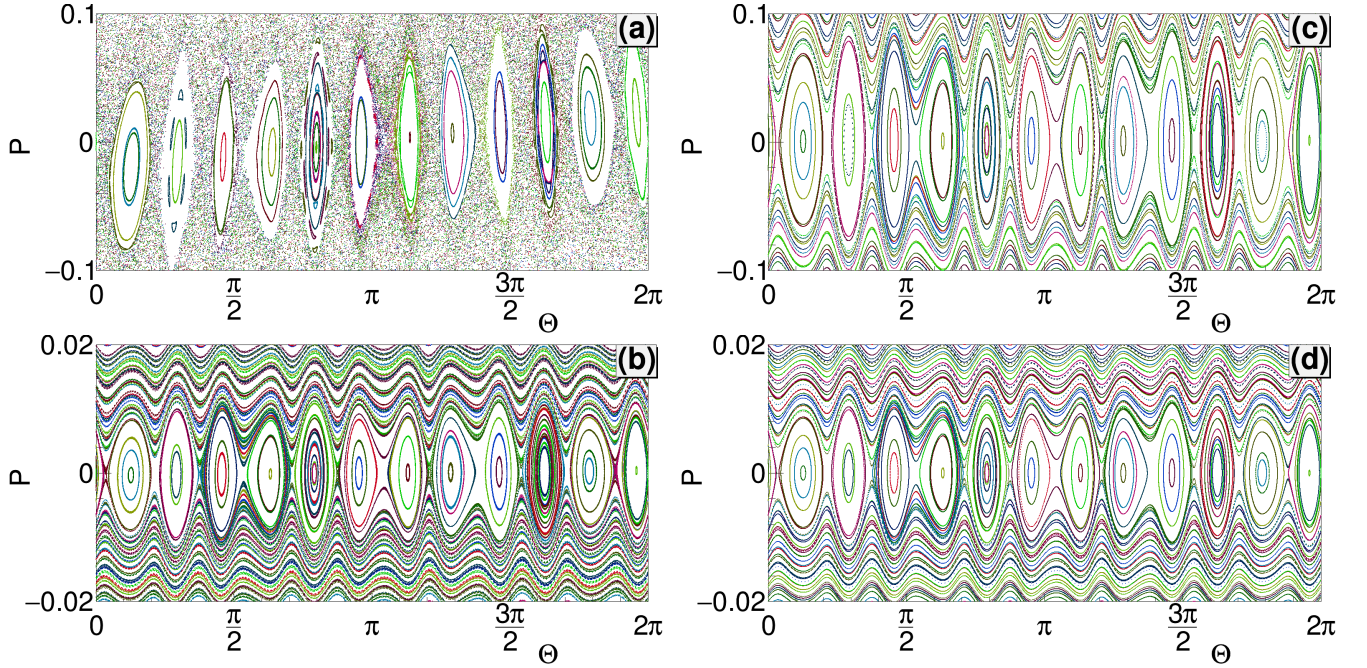


FIG. 1: Panels (a) and (b): stroboscopic picture of the phase space of a particle bouncing on a vibrating mirror. Equations of motion generated by the full Hamiltonian (2) have been integrated and after every period $\frac{2\pi}{\omega}$ of the mirror oscillations position of a particle in the (Θ, P) space has been plotted. The variable $\Theta = \theta - \omega t$ denotes position of a particle in the frame moving along the 1:1 resonant orbit. The canonically conjugate momentum $P = I - I_0$ where I_0 fulfills the 1:1 resonant condition $\Omega(I_0) = \omega$ (we have chosen such ω that $I_0 = 1$). The strength of the perturbation $\lambda = 0.6$ (a) and $\lambda = 0.01$ (b). In panels (c) and (d) the phase space portraits generated by the effective Hamiltonian (4) are presented which correspond to the same parameters of the system as in the panels (a) and (b), respectively.

where $h_0 = \left(\frac{\pi I}{\sqrt{3}}\right)^{2/3}$ and $h_n = \frac{(-1)^{n+1}}{n^2} \left(\frac{3I}{\pi^2}\right)^{2/3}$ for $n \neq 0$. In the moving frame, Θ and I are slow variables if $P = I - I_0 \approx 0$. Then, averaging (3) over fast time variable and performing Taylor expansion around the resonant value of I lead to the effective secular Hamiltonian,

$$H_{\text{eff}} = \frac{P^2}{2m_{\text{eff}}} + \lambda \sum_n h_n(I_0) f_{-n} e^{in\Theta}, \quad (4)$$

where $m_{\text{eff}} = -\frac{\pi^2}{\omega^4}$. The phase space portraits generated by (4) are shown in Fig. 1(c)-1(d) for the values of λ corresponding to the exact portraits presented in Fig. 1(a)-1(b). In the case when $\lambda = 0.01$, the effective Hamiltonian results and the exact data are identical. Thus, the motion of a particle in the quasi-crystal potential predicted by the effective Hamiltonian is reproduced by the full classical dynamics provided λ is sufficiently small.

In order to switch to quantum effective description we can either perform quantization of the classical effective Hamiltonian (4) or apply quantum version of secular approximation for the Hamiltonian (2) [4]. Let us first discuss the former approach. Classical equations of motion possess the scaling symmetry which implies that by a proper rescaling of the parameters and dynamical variables of the system we obtain the same behavior as presented in Fig. 1 but around arbitrary value of $I_0 \neq 1$

[2]. That is, when we redefine $\omega' = I_0^{-1/3}\omega$ and $\lambda' = \lambda$ we can use the results presented in Fig. 2 if we rescale $p' = I_0^{1/3}p$, $x' = I_0^{2/3}x$ and $t' = I_0^{1/3}t$. In the quantum description the scaling symmetry is broken because the Planck constant sets a scale in the phase space,

$$[x, p] = i \quad \Rightarrow \quad [x', p'] = \frac{i}{I_0}. \quad (5)$$

For $I_0 \gg 1$, the quantized version of the effective Hamiltonian (4), i.e. when $P \rightarrow -i\frac{\partial}{\partial\Theta}$, provides perfect quantum description of the resonant behavior of the system. The same quantum results can be obtained by applying the quantum secular approach [4] which yields

$$\langle n' | H_{\text{eff}} | n \rangle = (E_n - n\omega) \delta_{nn'} + \lambda \langle n' | x | n \rangle f_{n-n'}, \quad (6)$$

where $|n\rangle$'s are eigenstates of the unperturbed ($\lambda = 0$) system and $2^{1/3}E_n$ are zeros of the Airy function [2]. Equation (6) has been obtained by switching to the moving frame, with the help of the unitary transformation $\hat{U} = e^{i\hat{n}\omega t}$, and by averaging the Hamiltonian over the short time scale $\frac{2\pi}{\omega}$ [4].

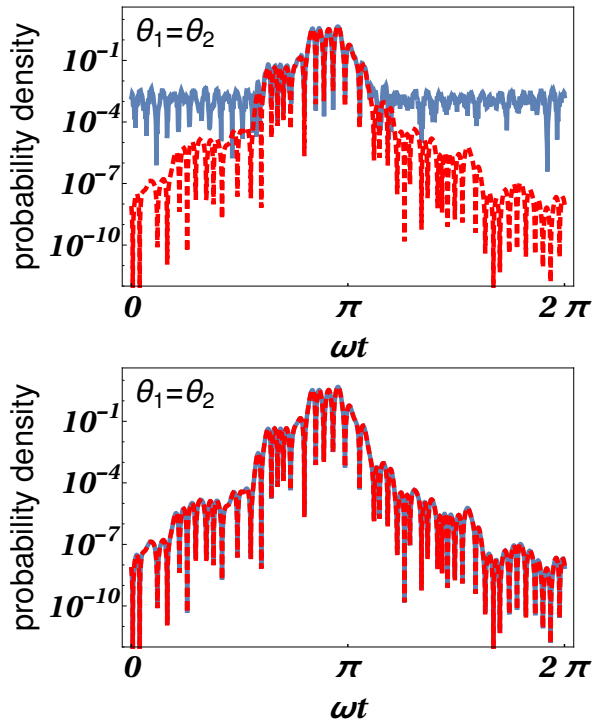


FIG. 2: Red dashed lines show probability densities for detection of two atoms at $\theta_1 = \theta_2$ in the laboratory frame versus t for an eigenstate of the effective Hamiltonian (9) related to the energy $E = 616$, $\lambda = 600$ and $k_0 = 100$. Blue solid lines depict the probability density of the corresponding Floquet eigenstate of the full Hamiltonian (7) for $\omega = 10^4$ (top panel) and $\omega = 3 \times 10^5$ (bottom panel).

MOLECULE FORMATION DUE TO DESTRUCTIVE INTERFERENCE

Let us consider two atoms which move on a ring and interact via a contact potential,

$$H = \frac{p_1^2 + p_2^2}{2} + 2\pi\lambda\delta(\theta_1 - \theta_2)f(t), \quad (7)$$

where $f(t) = \sum_{k \neq 0} f_k e^{i\omega t}$ with $f_k = \frac{1}{\sqrt{k_0}} e^{i\varphi_k}$ for $|k| \leq \frac{k_0}{2}$ and zero otherwise. $\varphi_k = -\varphi_{-k}$ are random variables chosen from the uniform distribution.

If the momenta of atoms fulfill $p_1 \approx \omega$ and $p_2 \approx -\omega$, then in the moving frame,

$$\begin{aligned} \Theta_1 &= \theta_1 - \omega t, & P_1 &= p_1 - \omega, \\ \Theta_2 &= \theta_2 + \omega t, & P_2 &= p_2 + \omega, \end{aligned} \quad (8)$$

the secular approximation results in the effective Hamiltonian

$$H_{\text{eff}} = \frac{P_1^2 + P_2^2}{2} + \lambda \sum_n f_{-2n} e^{in(\Theta_1 - \Theta_2)}, \quad (9)$$

which describes two particles interacting via a disorder potential. For a suitable choice of the system parameters, two atoms can be bound together due to Anderson localization.

The effective Hamiltonian does not depend on ω but its validity does. For a given λ , the effective Hamiltonian (9) is always valid if ω is sufficiently large as the second order corrections are proportional to $\frac{\lambda^2}{\omega^2}$. This is illustrated in Fig. 2 where we compare an eigenstate of (9) with the corresponding Floquet eigenstate of the full Hamiltonian (7) for two different values of ω . Thus, for sufficiently large ω , higher order terms neglected in the first order secular approximation do not modify the Anderson localization phenomena predicted by the effective Hamiltonian (9).

-
- [1] A. Lichtenberg and M. Lieberman, *Regular and chaotic dynamics*, Applied mathematical sciences (Springer-Verlag, 1992), ISBN 9783540977452, URL <https://books.google.pl/books?id=2ssPAQAAMAAJ>.
 - [2] A. Buchleitner, D. Delande, and J. Zakrzewski, *Physics reports* **368**, 409 (2002), URL <http://www.sciencedirect.com/science/article/pii/S0370157302002703>.
 - [3] J. H. Shirley, *Phys. Rev.* **138**, B979 (1965), URL <https://link.aps.org/doi/10.1103/PhysRev.138.B979>.
 - [4] G. Berman and G. Zaslavsky, *Physics Letters A* **61**, 295 (1977), ISSN 0375-9601, URL <http://www.sciencedirect.com/science/article/pii/0375960177906181>.

Time crystals: Analysis of experimental conditionsKrzysztof Giergiel,¹ Arkadiusz Kosior,¹ Peter Hannaford,² and Krzysztof Sacha^{1,3}¹*Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*²*Centre for Quantum and Optical Science, Swinburne University of Technology, Hawthorn, Victoria 3122, Australia*³*Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

(Received 21 May 2018; published 12 July 2018)

Time crystals are quantum many-body systems which are able to self-organize their motion in a periodic way in time. Discrete time crystals have been experimentally demonstrated in spin systems. However, the first idea of spontaneous breaking of discrete time translation symmetry, in ultracold atoms bouncing on an oscillating mirror, still awaits experimental demonstration. Here, we perform a detailed analysis of the experimental conditions needed for the realization of such a discrete time crystal. Importantly, the considered system allows for the realization of dramatic breaking of discrete time translation symmetry where a symmetry broken state evolves with a period tens of times longer than the driving period. Moreover, atoms bouncing on an oscillating mirror constitute a suitable system for the realization of dynamical quantum phase transitions in discrete time crystals and for the demonstration of various nontrivial condensed-matter phenomena in the time domain. We show that Anderson localization effects, which are typically associated with spatial disorder and exponential localization of eigenstates of a particle in configuration space, can be observed in the time domain when ultracold atoms are bouncing on a randomly moving mirror.

DOI: [10.1103/PhysRevA.98.013613](https://doi.org/10.1103/PhysRevA.98.013613)**I. INTRODUCTION**

Time crystals are quantum many-body systems which due to interactions between the particles are able to self-organize in a periodic way in time which is in full analogy to the formation of space crystals due to mutual interactions between atoms in condensed-matter physics [1]. Wilczek initiated time crystal research but his original idea concerning the formation of a crystalline structure in time turned out to be impossible to realize because he considered a time-independent system in the ground state [2–8]. However, soon after, another version of time crystals was proposed in which a periodically driven quantum many-body system spontaneously breaks discrete time translation symmetry of the Hamiltonian and starts evolving with a period twice as long as the period of the external driving [9–21]. This kind of quantum self-reorganization of the motion of quantum many-body systems, called discrete time crystals, has already been realized experimentally in spin systems [22–27]. It should be mentioned that in the classical regime breaking of discrete time translation symmetry in an atomic system has also been demonstrated in the laboratory [28,29].

In the present paper we return to the experimental proposal of a discrete time crystal published in 2015 and analyze in detail the conditions for its experimental realization. In Ref. [9] it was shown that an ultracold atomic cloud bouncing on an oscillating atom mirror is able to spontaneously self-reorganize its motion and to move with a period twice as long as the mirror oscillation period if the interactions between atoms are sufficiently strong. The system itself is suitable for realization not only of sponta-

neous breaking of discrete time translation symmetry of the Hamiltonian into a motion with twice as long a period but also into a motion with any multiple period of the driving. We focus here on the self-reorganization of the motion of the system with a period 40 times longer than the period of the driving. Such a dramatic breaking of discrete time translation symmetry of the Hamiltonian is not possible to observe in spin systems unless one deals with a very large spin quantum number.

The formation of a discrete time crystal is related to a quantum phase transition [9,30]. In order to form a time crystal, the strength of the interactions between atoms bouncing on a mirror must be greater than a critical value. It turns out that when one suddenly changes the interaction strength from the time crystal regime to the noninteracting regime, a dynamical quantum phase transition can be observed in the time evolution of the system [31]. Dynamical quantum phase transitions are recently discovered analogs of equilibrium phase transitions where the nonanalytical behavior of a system is observed not as a function of a control parameter but versus time [32–35]. We analyze the experimental conditions which allow one to observe that the return probability of the system to the initial degenerate manifold reveals a cusp at a critical moment of time after the sudden change from the time crystal phase to the noninteracting regime.

Ultracold atoms bouncing on an oscillating atom mirror [36] (for stationary mirror experiments see [37–44]) constitute a promising system for experimental realization of various condensed-matter phases in the time domain [1]. If a single- or many-body system is periodically and resonantly driven and the resonance is of a high order, such as 40 times longer than

the driving period, the system behaves like electrons in a space crystal [45,46]. Importantly, such a crystalline behavior of a resonantly driven system is observed in the time domain when the detection is carried out in the laboratory frame. A proper manipulation of the periodic driving allows one to realize various solid-state phenomena in the time domain [47]. In the present paper we focus on Anderson localization in time [46,48–50] and show that the introduction of randomness in the motion of the atom mirror leads to the Anderson localization phenomenon which is observed in the laboratory frame as an exponential localization versus time of the probability for the detection of atoms at a fixed position.

The paper is organized as follows. In Sec. II we introduce the system and provide the theoretical description needed to analyze the phenomena we are interested in. In Sec. III we perform a detailed analysis of the experimental conditions required for the realization of discrete time crystals. The results of Sec. III also serve as a base for an analysis of the experimental conditions for realization of dynamical quantum phase transitions in time crystals, Sec. IV, and Anderson localization in time, Sec. V. We conclude in Sec. VI.

II. DESCRIPTION OF THE SYSTEM

In this section we introduce the system and its theoretical description. We begin with a single-particle problem which is followed by the many-body generalization.

A. Single-particle problem

In the present paper we use the gravitational units where the length, energy, and time are given by

$$l_0 = \left(\frac{\hbar^2}{m^2 g} \right)^{1/3}, \quad E_0 = mgl_0, \quad t_0 = \frac{\hbar}{mgl_0}, \quad (1)$$

respectively, with m the atom mass and g the gravitational acceleration. In the laboratory frame a single atom bouncing on an oscillating atom mirror is described by the following Hamiltonian:

$$H = \frac{p^2}{2} + z + F[z - f(t)], \quad (2)$$

where $F(z)$ describes the mirror, i.e., the profile of the reflecting potential, and $f(t) = f(t + T)$, with

$$T = \frac{2\pi}{\omega}, \quad (3)$$

determines the periodic oscillations of frequency ω of the mirror position. A theoretical description of the system is much more convenient when we switch from the laboratory frame to the frame oscillating with the mirror. Then, the mirror is fixed but the effective gravitational acceleration oscillates in time,

$$H = \frac{p^2}{2} + z + zf''(t) + F(z). \quad (4)$$

Except in Sec. V, we focus on the case where

$$f(t) = -\gamma \cos \omega t, \quad \gamma = \frac{\lambda}{\omega^2}. \quad (5)$$

The ratio λ/ω^2 determines the amplitude of the harmonic oscillations of the mirror in the laboratory frame. Assuming the mirror can be modeled by a hard wall potential located

at $z = 0$ in the oscillating frame, we may drop the $F(z)$ in Eq. (4) which leads to the final form of the single-particle Hamiltonian [51]

$$H(t) = \frac{p^2}{2} + z + \lambda z \cos \omega t, \quad z \geq 0. \quad (6)$$

The energy of the considered system is not conserved because the Hamiltonian (6) depends explicitly on time. However, due to the time periodicity, there are eigenstates of the so-called Floquet Hamiltonian [51,52],

$$\mathcal{H} = H(t) - i\partial_t, \quad (7)$$

which evolve periodically with the period of the driving. The corresponding eigenvalues are called quasienergies of the system. The Floquet formalism is in full analogy to the Bloch theorem approach known in condensed-matter physics. The aim of the present paper is to analyze experimental conditions of the Floquet eigenstates of the single-particle system (6) and its many-body version.

The description of a particle bouncing resonantly on an oscillating mirror, which we are interested in, can be simplified by employing the quantum secular approximation [53]. However, starting with the classical description, which at the end is quantized, we not only arrive at the same results but also gain an intuitive picture of the system dynamics.

Let us begin with the classical version of the Hamiltonian (6) and apply a canonical transformation from the Cartesian position and momentum to the so-called action-angle variables I and θ of the unperturbed Hamiltonian $H_0 = p^2/2 + z$ [51,54]. In terms of this new pair of canonically conjugate variables, the unperturbed Hamiltonian depends on the momentum (action) only, $H_0(I) = (3\pi I)^{2/3}/2$, and the periodic particle trajectory can be found immediately. Indeed, the action $I = \text{const}$ and the position of the particle on a periodic orbit is given by an angle which changes linearly in time, $\theta(t) = \Omega(I)t + \theta(0)$, where $\Omega(I) = dH_0(I)/dI$ is the frequency of the periodic motion of an unperturbed particle. The distance of the classical turning point of a particle from the mirror is

$$h = \frac{\pi^2}{2\Omega^2}. \quad (8)$$

In the presence of the mirror oscillations, we are interested in the motion of a particle in the vicinity of a periodic orbit which is resonant with the time-dependent driving, i.e., when $I \approx I_s$, where $\omega = s\Omega(I_s)$ with integer s . The $s : 1$ resonant motion can be described by a simple effective Hamiltonian when we switch to the frame moving along a resonant orbit, defined by

$$\Theta = \theta - \frac{\omega}{s}t, \quad (9)$$

and perform averaging over time. In the moving frame, Θ and $P = I - I_s$ are slow variables provided we are close to a resonant orbit, i.e., provided $P \approx 0$. Then, averaging the Hamiltonian (6) over time yields [51,54]

$$H_{\text{eff}} \approx \frac{P^2}{2m_{\text{eff}}} + V_0 \cos(s\Theta), \quad (10)$$

where a constant term has been omitted and

$$m_{\text{eff}} = -\frac{\pi^2 s^4}{\omega^4}, \quad V_0 = \frac{\lambda}{\omega^2} (-1)^{s+1}. \quad (11)$$

The Hamiltonian (10) indicates that in the moving frame a resonantly bouncing atom behaves effectively like a particle in a periodic lattice, i.e., like an electron in a space crystal [45,46,55–57]. Importantly, the crystal behavior in the moving frame will be reproduced in the time domain when we return to the laboratory frame [46–48,58]. Indeed, the linear relation (9) between the position Θ in the moving frame and the time t in the laboratory frame ensures that in the quantum description when we fix the position in the laboratory frame $\theta = \text{const}$, the crystalline behavior in Θ , described by the effective Hamiltonian (10), will be observed in the evolution of the probability of detection of an atom versus t . This simple argumentation shows that it is not the presence of any external space periodic potential but the resonant dynamics itself that is responsible for the emergence of a crystalline behavior. This also shows that we have a platform for realization of a broad range of condensed-matter phenomena in the time domain—as an example we analyze Anderson localization in time in Sec. V. However, in Secs. III and IV we consider a different aspect of the system. We show that ultracold atoms bouncing resonantly on a harmonically oscillating mirror can reveal spontaneous breaking of the discrete time translation symmetry of the Hamiltonian if the interaction between atoms is sufficiently strong and a discrete time crystal phenomenon forms [9,31].

Before we switch to the description of time crystals we need to analyze the validity of the approach we use. Classical equations of motion generated by the Hamiltonian (6) possess scaling symmetry [51]. It means that when we appropriately rescale the parameters and the position and momentum, the equations of motion do not change. This allows us to set one of the parameters, perform the analysis of the system, and use the results for other values of the chosen parameter by applying the scaling transformation. Let us set the resonant action $I_s = 1$ which implies the resonant driving frequency $\omega = s(\pi^2/3)^{1/3}$ and perform an analysis of the validity of the effective Hamiltonian (10) for different values of the perturbation amplitude λ . The obtained results will allow us to predict the resonant behavior for any value of I_s provided they are rescaled according to $\omega' = I_s^{-1/3}\omega$, $\lambda' = \lambda$, $p' = I_s^{1/3}p$, $x' = I_s^{2/3}x$, and $t' = I_s^{1/3}t$. In Fig. 1 phase space portraits corresponding to the 40 : 1 resonance ($s = 40$), $\lambda = 0.2$ and $\lambda = 0.4$ are presented. In the latter case the phase space possesses a significant chaotic area but in the former case the phase space is mostly regular and resembles the behavior predicted by the effective Hamiltonian (10).

In the quantum description the scaling symmetry is broken because the Planck constant sets a scale in phase space which can be easily seen from the commutation relation

$$[x', p'] = i \Rightarrow [x, p] = \frac{i}{I_s}, \quad (12)$$

which also indicates that I_s^{-1} can be treated as an effective Planck constant. For $I_s \gg 1$, the quantized version of the classical effective Hamiltonian (10), i.e., when $P \rightarrow -i\frac{\partial}{\partial\Theta}$, provides a perfect quantum description of the resonant behavior of the system [51]. As already mentioned the same quantum results can be obtained by applying the quantum secular approximation [53].

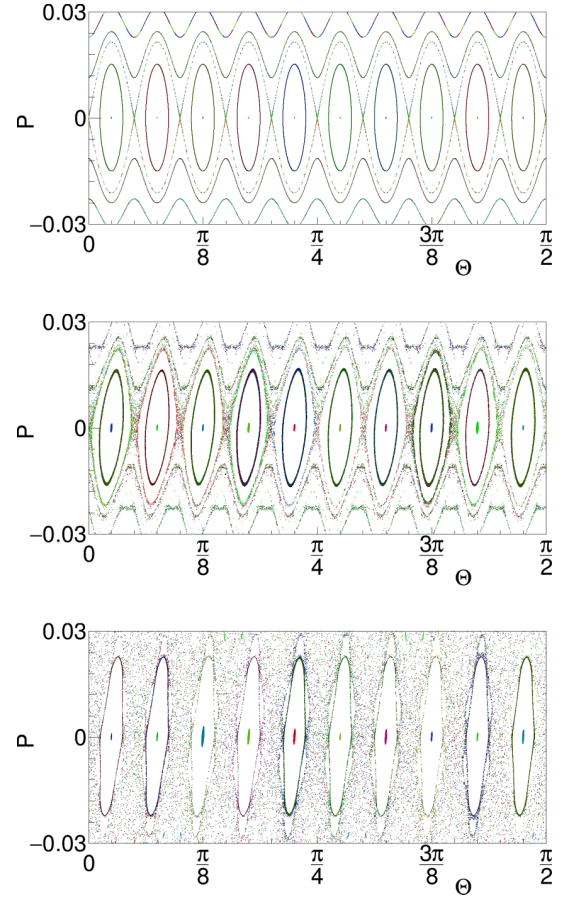


FIG. 1. Top panel: phase space portrait generated by the effective Hamiltonian (10) for $\lambda = 0.2$. Middle and bottom panels show the stroboscopic phase space portraits generated by the exact Hamiltonian (6) for $\lambda = 0.2$ and $\lambda = 0.4$, respectively. The frequency of the mirror oscillations, $\omega = 40(\pi^2/3)^{1/3}$, corresponds to the 40 : 1 resonance ($s = 40$) for the resonant value of the action $I_{40} = 1$. Note that for the sake of clarity we show only a quarter of the full 2π range of Θ .

Now we can define a simple strategy on how to choose suitable parameters of the system. If we are interested in a certain $s : 1$ resonance behavior, we set $I_s = 1$ and $\omega = s(\pi^2/3)^{1/3}$ and perform an analysis of which values of λ are allowed in order to deal with the behavior predicted by the effective Hamiltonian (10). Then, in order to choose which value of I_s is suitable in the quantum case, we have to keep in mind that the effective Planck constant I_s^{-1} must be smaller or at least comparable to the area of a single regular elliptic island (cf. Fig. 1), otherwise no quantum state is localized (in a semiclassical sense) in an island and the picture of a particle moving like an electron in a space crystal is lost. Having determined I_s and employing the scaling transformation we obtain the desired $\omega' = I_s^{-1/3}s(\pi^2/3)^{1/3}$ and $\lambda' = \lambda$.

Eigenstates of the effective Hamiltonian (10) obtained in the moving frame correspond to the Floquet states of the original Floquet Hamiltonian (7). For $s \gg 1$ the eigenstates of (10) are Bloch waves [47]. If we are interested in the first energy band of (10) only, the effective description can be further simplified. That is, expanding the wave function of the quantum version of (10) in the basis of Wannier states $w_i(\Theta)$ localized in each

site of the effective periodic potential in (10) and belonging to the first energy band, $\psi(\Theta) = \sum_{i=1}^s a_i w_i(\Theta)$, we obtain an expression for the particle energy in the tight-binding form [59]

$$E \approx -\frac{J}{2} \sum_{i=1}^s (a_{i+1}^* a_i + \text{c.c.}), \quad (13)$$

where $J = -2\langle w_{i+1} | H_{\text{eff}} | w_i \rangle$ is the amplitude of tunneling of an atom between neighboring Wannier states. Note that the Wannier states are represented in the laboratory frame by localized wave packets evolving periodically along the $s : 1$ resonant orbit with a period s times longer than the mirror oscillation period T .

B. Many-body problem

The interaction between atoms couples different degrees of freedom and we may not consider different degrees of freedom independently without justification.

We assume that a cloud of ultracold atoms is bouncing on the oscillating atom mirror in the presence of a transverse harmonic potential characterized by the frequency $\omega_{\perp}$. The many-body Hamiltonian of our system, in the second quantization formalism and in the gravitational units (1), reads [60]

$$\hat{H} = \int d^3r \hat{\Psi}^{\dagger} \left[\frac{\mathbf{p}^2}{2} + z + \lambda z \cos \omega t + \frac{\omega_{\perp}^2 (x^2 + y^2)}{2} \right. \\ \left. + \frac{g_0}{2} \hat{\Psi}^{\dagger} \hat{\Psi} \right] \hat{\Psi}, \quad (14)$$

where $\hat{\Psi}(\mathbf{r}, t)$ is the bosonic field operator and the contact interaction between atoms is described by the coefficient $g_0 = 4\pi a_s$, with a_s being the atomic s -wave scattering length given in the gravitational units (1). If the transverse confinement is sufficiently strong (see the analysis in Sec. III A), ultracold atoms do not populate excited states along the transverse directions and we may restrict to the ground state $\phi_0(x, y)$ of the harmonic oscillator trap of frequency $\omega_{\perp}$. That is, substituting $\hat{\Psi}(\mathbf{r}, t) = \phi_0(x, y) \hat{\psi}(z, t)$ in (14) and integrating over x and y we arrive at the one-dimensional version of the many-body Hamiltonian [60]

$$\hat{H} = \int_0^{\infty} dz \hat{\psi}^{\dagger} \left[\frac{p^2}{2} + z + \lambda z \cos \omega t + \frac{g_{1D}}{2} \hat{\psi}^{\dagger} \hat{\psi} \right] \hat{\psi}, \quad (15)$$

where a constant term has been omitted and

$$g_{1D} = g_0 \frac{\omega_{\perp}}{2\pi} = 2\omega_{\perp} a_s, \quad (16)$$

with the scattering length a_s given in the gravitational units (1).

The Hamiltonian (15) is the many-body counterpart of the single-particle Hamiltonian (6). Similarly, as in the single-particle case, the energy is not conserved but there exist many-body Floquet eigenstates which evolve periodically with the period of the mirror oscillations, T . We will focus on quantum states related to the classical $s : 1$ resonant dynamics. For sufficiently strong interaction, the corresponding many-body Floquet eigenstates become extremely vulnerable to any perturbation because they form macroscopic superpositions (Schrödinger cat-like states) and even an infinitesimally weak perturbation, for example, measurement of the position of

a single atom, is sufficient to cause the collapse of the Schrödinger cat states into one of the states which forms the macroscopic superposition. In the $s : 1$ resonant case this means that the discrete time translation symmetry of the Hamiltonian is spontaneously broken into another discrete time translation symmetry [9]. In other words, the Floquet eigenstates, which must obey the symmetry of the Hamiltonian, evolve with the period of the mirror oscillations T but the system spontaneously chooses symmetry broken states which evolve with a period s times longer. This phenomenon is dubbed discrete time crystal formation because a quantum many-body system, due to interaction between the particles, spontaneously self-reorganizes its motion and starts evolving with a period different from the period expected from the symmetry of the Hamiltonian [9–11]. It is in analogy to space crystal formation where atoms, due to mutual interactions, spontaneously form a periodic arrangement in space which breaks the continuous space translation symmetry of the solid state Hamiltonians. In the $2 : 1$ resonant case ($s = 2$), a full many-body analysis was possible and even simulations of the spontaneous symmetry breaking process in the course of the measurements of the position of the particles could be performed [9]. Here, we are interested in $s \gg 1$ and the full many-body calculations are not attainable. Therefore, in order to describe experimentally relevant conditions we are going to apply the mean-field approach where the spontaneous time translation symmetry breaking is indicated by the loss of stability of the mean-field solutions which evolve with period T and the emergence of stable solutions evolving with period sT . The mean-field approach is valid provided that the number of atoms depleted from a Bose-Einstein condensate is much smaller than the total number of particles. It is true in the regime where no spontaneous breaking of time translation symmetry occurs and deep in the discrete time crystal regime where symmetry broken states are Bose-Einstein condensates [9,30].

The simplest way to switch from the full many-body description to the mean-field approach is to exchange the bosonic field operator $\hat{\psi}(z, t)$ with a classical field $\psi(z, t)$. Then, the Hamiltonian (15) becomes the energy functional of a Bose-Einstein condensate with $\psi(z, t)$ being the condensate wave function. We are looking for periodic solutions of the periodically driven system within the mean-field approach, and therefore we introduce the energy functional related to the Floquet Hamiltonian [46]

$$E = \frac{1}{sT} \int_0^{sT} dt \int_0^{\infty} dz \psi^* \left[\frac{p^2}{2} + z + \lambda z \cos \omega t \right. \\ \left. + \frac{g_{1D}N}{2} |\psi|^2 - i \partial_t \right] \psi, \quad (17)$$

where N stands for the total number of atoms and we assume that $\int_0^{\infty} dz |\psi(z, t)|^2 = 1$. The energy (17) can be considered as the energy of the system per particle averaged over time. Anticipating the emergence of stable solutions with period sT we have introduced averaging over a time period s times longer than the driving period T . Solving the Gross-Pitaevskii equation (GPE) [60] corresponding to (17),

$$\left[\frac{p^2}{2} + z + \lambda z \cos \omega t + g_{1D}N |\psi(z, t)|^2 - i \partial_t \right] \psi = \mu \psi, \quad (18)$$

we can obtain condensate wave functions which evolve periodically in time.

In the absence of particle interactions, the single-particle Floquet eigenstates, related to the classical $s : 1$ resonant dynamics, form an s -dimensional Hilbert subspace and are superpositions of s localized wave packets $w_i(z, t)$ which evolve along the classical $s : 1$ resonant orbit. Despite the fact that the wave packets $w_i(z, t)$ propagate with period sT , the Floquet eigenstates are periodic with period T because after each period of the mirror oscillations the $w_i(z, t)$'s exchange their positions. The wave packets $w_i(z, t)$ are Wannier states corresponding to the first energy band of the effective Hamiltonian (10). Restricting to the s -dimensional Hilbert subspace we can expand solutions of the GPE in the Wannier-like basis, $\psi(z, t) = \sum_{i=1}^s a_i w_i(z, t)$, and obtain the energy functional (17) within the tight-binding approximation [cf. Eq. (13)],

$$E \approx -\frac{J}{2} \sum_{i=1}^s (a_{i+1}^* a_i + \text{c.c.}) + \frac{1}{2} \sum_{i,j=1}^s U_{ij} |a_i|^2 |a_j|^2, \quad (19)$$

where

$$J = -\frac{2}{sT} \int_0^{sT} dt \int_0^\infty dz w_{i+1}^* \left[\frac{p^2}{2} + z + \lambda z \cos \omega t - i \partial_t \right] w_i, \quad (20)$$

$$U_{ii} = \frac{g_{1D} N}{sT} \int_0^{sT} dt \int_0^\infty dz |w_i|^4, \quad (21)$$

$$U_{ij} = \frac{2g_{1D} N}{sT} \int_0^{sT} dt \int_0^\infty dz |w_i|^2 |w_j|^2, \quad \text{for } i \neq j. \quad (22)$$

Note that the amplitude J is related to the tunneling of atoms between the wave packets that are neighbors on the classical trajectory. The tight-binding approximation (19) is valid provided the interaction energy per particle is much smaller than the energy gap between the first and the second energy bands of the effective Hamiltonian (10) [59].

We have completed all the necessary theoretical tools to analyze the conditions for experimental realization of various time crystal phenomena.

III. DISCRETE TIME CRYSTALS

The formation of space crystals is related to the spontaneous breaking of continuous space translation symmetry. Hamiltonians of solid-state systems do not change if all particles are translated by an arbitrary vector. This symmetry implies that if the systems are prepared in eigenstates, the probability for detection of a single particle must be uniform in space and no crystalline structures are visible. However, exact symmetric ground states of condensed-matter systems are strongly vulnerable to a perturbation and even measurement of the position of one particle is sufficient to uncover a crystalline structure in space due to localization of the center of mass of the system. Such a space crystal lives infinitely long if we deal with a macroscopic system because quantum spreading of the center of mass lasts so long that it is not measurable [1].

The Wilczek idea that the same phenomenon can be observed in the time domain if certain quantum many-body systems are prepared in ground states turned out to be impossible [1]. However, spontaneous breaking of time translation sym-

metry is possible if one relaxes the requirement and considers excited eigenstates of quantum many-body systems. The so-called discrete time crystals are related to periodically driven systems which spontaneously break discrete time translation symmetries of Hamiltonians and self-reorganize their motion and start evolving with a period different from the period of the driving. The first idea of a discrete time crystal was based on ultracold atoms bouncing on an oscillating atom mirror [9]. In the present section we are going to analyze the experimental conditions for realization of this system.

In the original version of the discrete time crystal the $2 : 1$ resonant dynamics of ultracold atoms bouncing on an oscillating mirror was considered [9]. From an experimental point of view, in order to reduce atomic losses, it is better to work with a higher resonance. We have chosen the $40 : 1$ resonance. Discrete time crystal formation in this case means that the system spontaneously self-reorganizes its motion and starts evolving with a period 40 times longer than the period expected from the symmetry of the Hamiltonian. We will first focus on the theoretical considerations and then switch to analysis of the experimental conditions.

A. Theoretical analysis

A quantum many-body description shows that for sufficiently strong attractive interactions between the ultracold atoms, low-lying energy states in the Hilbert subspace related to the classical $2 : 1$ resonant dynamics possess Schrödinger cat-like structures and the measurement of the position of a particle in the system prepared in one of them breaks discrete time translation symmetry and a discrete time crystal forms [9]. This phenomenon can also be described by means of the mean-field theory which indicates a loss of the stability of the condensate wave function propagating with the period of the mirror oscillation T and the emergence of new stable solutions evolving with the period $2T$ [9]. We apply the latter approach to the $40 : 1$ resonance case.

The simplest way to identify the critical strength of the interactions between atoms that leads to the breaking of discrete time translation symmetry is to analyze the lowest energy solution of the energy functional in the tight-binding approximation (19). From the classical analysis presented in Fig. 1 we know that for $s = 40$ the exact phase portrait is reproduced by the prediction of the effective Hamiltonian (10) for $\lambda = 0.2$. Having determined λ we need to choose the frequency of the mirror oscillation ω or equivalently the resonant action $I_{40} \equiv I_{s=40}$. The latter determines an effective Planck constant I_{40}^{-1} which has to be smaller than the area of one of the elliptical resonant islands visible in Fig. 1. From an experimental point of view we need to ensure that the tunneling time of atoms between the elliptical islands, $2.4/J$, is much shorter than the lifetime of a Bose-Einstein condensate in the laboratory. On the other hand, we cannot afford J to be too large because then the energy gap between the first and the second energy bands of the effective single-particle Hamiltonian (10) is very small and when we turn on the particle interactions the simple picture of the time crystal formation, described by the single band tight-binding model (19), is lost. A compromise is to choose ω such that the energy gap is at least of the order of $10J$. Simple diagonalization of the single-particle effective

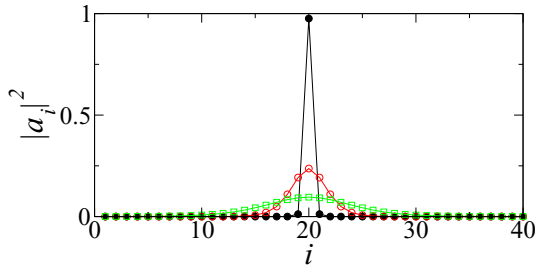


FIG. 2. Lowest energy solutions corresponding to the mean-field energy functional within the tight-binding approximation (19) for interaction strengths $g_{1D}N = -0.12$ (black filled circles), -0.01 (red open circles), and -0.003 (green open squares). For $g_{1D}N \gtrsim -1.64 \times 10^{-3}$ the symmetry is not broken and the lowest energy solutions are uniform.

Hamiltonian (10) shows that for $\omega = 4.9$ (i.e., $I_{40} = 1790$) the energy gap equals $9.5J$, where $J = 8.6 \times 10^{-4}$.

Having chosen the parameters of the single-particle problem we can now analyze what strength of the attractive interactions is needed to break the symmetry and which value is suitable for an experiment. For $g_{1D} = 0$, the energy (19) is minimized by the uniform solution, $a_i = 1/\sqrt{40}$, which corresponds to a wave function $\psi(z, t)$ that evolves with the period T . For $g_{1D}N \lesssim -1.64 \times 10^{-3}$ the uniform solution becomes unstable and new degenerate stable solutions are born which evolve with the period $40T$. For $|g_{1D}N|$ greater than the critical value but close to it, the lowest energy solution is slightly nonuniform and it could be difficult to prepare and detect it experimentally. Therefore, it is much better to increase $|g_{1D}N|$ so that the symmetry broken solutions are localized in single sites of the tight-binding problem (19). In Fig. 2 we present the solutions for different interaction strengths. The value of about $g_{1D}N = -0.12$ results in the symmetry broken solutions being nearly entirely localized in single sites which correspond to the interaction energy per particle $|U_{ii}|/2 = 3.8J$. The latter is 2.5 times smaller than the energy gap between the first and second energy bands which guarantees the validity of the tight-binding approximation.

In three-dimensional space a Bose gas can collapse if the attractive contact interactions are sufficiently strong [61–65]. In order to prevent such a bosenova effect a transverse trap has to be present and the number of atoms cannot be too large. We have already assumed that in the transverse directions the condensate wave function $\phi_0(x, y)$ corresponds to the ground state of the harmonic trap of frequency $\omega_\perp$. We will see that a discrete time crystal is represented by a wave packet $\psi(z, t)$ evolving along a classical trajectory whose probability distribution reveals the smallest standard deviation at the classical turning point, which we denote by σ . The system will be stable against collapse if the resulting interaction energy per particle at the turning point, $g_0N \int d^3r |\phi_0\psi|^4$, is smaller than the excitation energy along the transverse directions, $\omega_\perp$, which leads to the following condition [61]:

$$\sigma \gtrsim |a_s|N. \quad (23)$$

The present subsection has allowed us to determine the optimal parameters for the formation of a discrete time crystal on the

basis of the 40 : 1 resonant dynamics. In the next subsection we analyze the experimental conditions and perform time-dependent numerical simulations of the realization of the discrete time crystal by integrating the full GPE (18), i.e., without referring to the tight-binding approximation.

B. Experimental conditions

A suitable atomic system for performing a discrete time crystal experiment based on ultracold atoms bouncing on an oscillating mirror is the ^{85}Rb $|F = 2, m_F = -2\rangle$ state, which has a broad Feshbach resonance that allows precise tuning of the s -wave scattering length and hence the inter-particle interaction [66]. Using gravitational units for ^{85}Rb , $l_0 = 0.385 \mu\text{m}$, and $t_0 = 0.198 \text{ ms}$, we obtain the following values for the parameters determined in the previous subsection: mirror oscillation frequency $\omega/(2\pi t_0) = 3.94 \text{ kHz}$ and amplitude $\gamma l_0 = 3.2 \text{ nm}$, distance of the classical turning point from the mirror $hl_0 = 127 \mu\text{m}$, and standard deviation of the atomic density along the longitudinal direction at the classical turning point $\sigma l_0 = 0.77 \mu\text{m}$. The maximal allowed number of atoms is $N \leq \sigma/|a_s|$, which for $a_s l_0 \approx -0.1 \text{ nm}$ ($-2a_0$) yields $N \leq 8000$. The desired interaction strength, $g_{1D}N = 2\omega_\perp a_s N = -0.12$, can be achieved by a proper choice of $\omega_\perp$, a_s , and N ; e.g., $a_s l_0 = -0.1 \text{ nm}$, $\omega_\perp/(2\pi t_0) = 37 \text{ Hz}$, and $N = 5000$. These parameters are considered to be realistic for an experiment.

In the original proposal of a discrete time crystal based on ultracold atoms bouncing on an oscillating mirror [9], we considered the simple case of a 2 : 1 resonance in which the initial state was a single Floquet state consisting of a superposition of $s = 2$ wave packets. In order to keep the required number of bounces to a reasonable number to reduce possible atom losses, it is better to work with a higher resonance, such as the 40 : 1 resonance. However, it is difficult to prepare a single Floquet state for the 40 : 1 resonance because it is a superposition of 40 localized wave packets moving with different velocities and with certain mutual phase relations. On the other hand, it is relatively easy to prepare a single localized wave packet that moves periodically along the 40 : 1 resonant orbit and to monitor the evolution of the wave packet with and without the attractive interactions.

The experiment starts with the ground state of a BEC of ^{85}Rb atoms in a three-dimensional harmonic optical dipole trap whose center is located at the classical turning point. The frequencies of the (pancake shape) trap are $\omega_\perp/(2\pi t_0) = 37 \text{ Hz}$ and $\omega_\parallel/(2\pi t_0) = 100 \text{ Hz}$ along the transverse and longitudinal directions, respectively. The value of the longitudinal trap frequency ensures that the standard deviation of the ground-state distribution along the longitudinal direction at the classical turning point is $\sigma l_0 = 0.77 \mu\text{m}$.

Once the initial state of the system is prepared, we turn off the trapping potential along the longitudinal direction and the atom cloud starts falling on the oscillating atom mirror under the influence of gravity and the tightly confining one-dimensional red-detuned optical waveguide. The frequency of the mirror oscillations is adjusted to the frequency of the classical 40 : 1 resonant orbit which depends on the initial distance of the cloud from the mirror. The initial phase of the mirror oscillation corresponds to $t = 0$ in Eq. (5) and hence

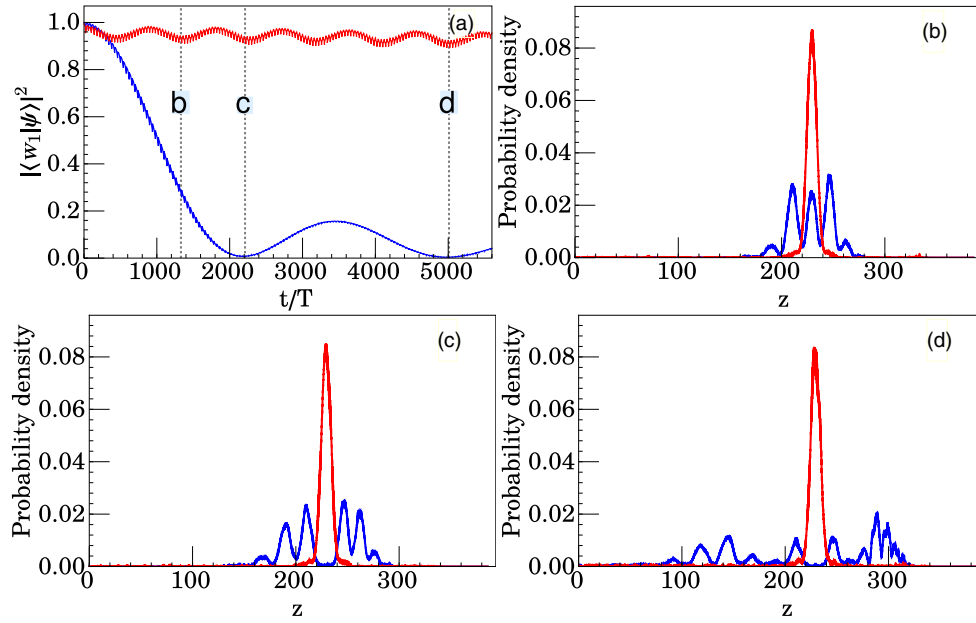


FIG. 3. Demonstration of the discrete time crystal formation; see also Fig. 4. Initially an atomic cloud is prepared in a trap above a periodically oscillating mirror. The parameters of the trap are chosen so that the longitudinal width of the atomic distribution $|\psi(z,0)|^2$ fits the width of the Wannier wave packet $|w_1(z,0)|^2$ located at the classical turning point. When the longitudinal trapping potential is turned off, atoms fall on the mirror and their time evolution is described by the full GPE (18). In the absence of interactions, atoms tunnel slowly to neighboring Wannier wave packets which is indicated by a decrease of the overlap between the time evolving wave function and the initially chosen Wannier state $|\langle w_1(t)|\psi(t)\rangle|^2$ [see blue curve in (a)]—at $t/T = 2210$ all atoms tunnel out from the initially chosen Wannier wave packet and $|\langle w_1(t)|\psi(t)\rangle|^2 = 0$. However, when the attractive interactions are turned on ($g_{1D}N = -0.12$), the system chooses a periodic solution evolving with a period 40 times longer than the period expected from the symmetry of the Hamiltonian and the discrete time translation symmetry is broken. This situation is presented in (a) by the red curve where the interaction strength $g_{1D}N = -0.12$. In panels (b)–(d) the densities of interacting (noninteracting) atoms are presented with the red (blue) curves at different moments of time—we have chosen time moments when the atomic clouds are midway between the mirror location ($z = 0$) and the classical turning point ($z = 329$). Panel (b) corresponds to $t/T = 1330$ when in the noninteracting case the initial and neighboring Wannier wave packets are equally populated. Panel (c) is related to the time moment $t/T = 2210$ when all noninteracting atoms tunnel out of the initial Wannier wave packet while panel (d) corresponds to $t/T = 5010$. During the entire evolution the wave function of the interacting system does not decay, demonstrating the stability of the discrete time crystal—see red curves in (b)–(d).

the mirror needs to be in the lowest position with zero velocity. Both adjustments can be carried out by tuning the driving of the mirror.

We now analyze the above optimal parameters for the case of the 40 : 1 resonance. In Figs. 3 and 4 we present the evolution of the atomic cloud, both in the presence of the desired attractive interactions ($g_{1D}N = -0.12$) and in the absence of the interactions. Without the interactions, atoms prepared in the localized wave packet tunnel slowly to neighboring localized wave packets which move on the 40 : 1 resonant classical trajectory, as indicated by the overlap between the time evolving wave function and the initially chosen Wannier state $|\langle w_1(t)|\psi(t)\rangle|^2$ [Fig. 3(a), blue line]. After $t/T = 2210$ mirror periods (i.e., after 55 bounces of the atom cloud on the mirror) no atoms remain in the initial wave packet [Figs. 3(c), blue line, and Fig. 4(b), green line] because all the atoms have tunneled to neighboring wave packets. In the presence of sufficiently strong attractive interactions the initial Gaussian wave packet does not decay—it evolves freely along the classical trajectory with a period 40 times longer than the mirror oscillation period for long times without tunneling losses to other wave packets [Fig. 3, red lines and Fig. 4(a)].

The ^{85}Rb BEC is prepared by setting the s -wave scattering length to $a_s l_0 \approx 300a_0$ by means of the broad Feshbach resonance at the magnetic field 155.041(18) G [66] and sympathetically cooling with a ^{87}Rb BEC in an optical dipole trap [67]. Once the BEC is reached, the evaporative cooling can be continued which allows one to reduce the thermal cloud and to obtain the desired number of atoms in the trap. The atom mirror is a blue-detuned repulsive light sheet created by focusing 532 nm light, e.g., from a 10-W frequency-doubled Nd : YVO₄ laser, with a cylindrical lens [41]. The bouncing atoms are confined in a single transverse mode of the tightly confining red-detuned optical waveguide. Such an atom mirror can be modulated with the required frequency of ~ 4 kHz and amplitude of ~ 3 nm by vibrating a beam-guiding optical mirror with a piezocrystal, by modulating the beam with an AOM or by modulating the optical potential. A similar light-sheet atom mirror has previously been used to bounce a BEC of ^{87}Rb atoms dropped from heights of $\sim 100 \mu\text{m}$ by Bongs *et al.* [41], who demonstrated the coherent evolution of the bouncing BEC. For the 40 : 1 resonance and the above parameters, about 55 bounces, or about 0.6 s, are required without significant loss of atoms or loss of phase coherence to

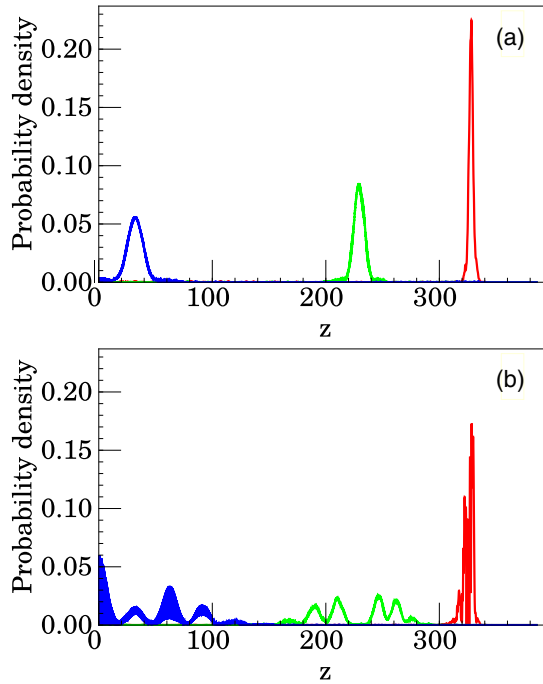


FIG. 4. Experimental signatures of the formation of the discrete time crystal. A periodically driven quantum many-body system, due to the interactions between particles, does not follow the driving force but spontaneously chooses motion with a period 40 times longer than the period of the driving. This is illustrated in panel (a) where we plot the densities of the atomic cloud around $t/T = 2210$, i.e., at $t/T = 2200$ (red), 2210 (green), and 2220 (blue) for $g_{1D}N = -0.12$. The time crystal behavior is in contrast to the evolution of the system without interactions. Panel (b) shows that without interactions the evolution of the initially localized wave packet is not stable because atoms have tunneled to other localized wave packets moving along the classical resonant orbit. The filled-in blue regions between $z = 0$ and $z = 100$ are interference fringes related to the reflection of the wave packets from the mirror. In both panels the same initial state, described in Fig. 3, has been chosen.

allow time for the atoms to tunnel to neighboring wave packets in the absence of interactions.

The evolution of the atom density of the bouncing ^{85}Rb atoms is monitored when the s -wave scattering length is adiabatically changed from $\approx 300a_0$ to the following:

(i) zero, to turn off the interactions to allow the atoms to start to tunnel to other wave packets in times of order $1/J$. To control the scattering length to $\pm 0.1a_0$ requires the magnetic field to be stable to about ± 2 mG;

(ii) $a_s l_0 = -2a_0$, corresponding to an attractive interaction $g_{1D}N = -0.12$ for $N = 5000$. This is sufficiently strong to break the time-translation symmetry to form a discrete time crystal which evolves freely with period $40T$ for long times without tunneling to other wave packets. The stability of the discrete time crystal can be tested by introducing controlled fluctuations of the mirror amplitude.

We now consider the influence of possible experimental imperfections:

(1) Precise control of the total number of atoms N , the s -wave scattering length a_s , and the transverse confinement

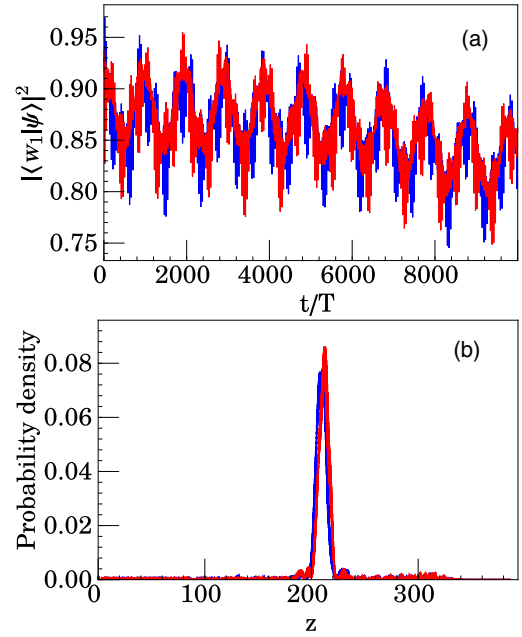


FIG. 5. Similar data to that presented in Fig. 3 in the interacting case ($g_{1D}N = -0.12$); however, the initial atomic distribution $|\psi(z, 0)|^2$ is displaced along the longitudinal direction with respect to the classical turning point by $+0.5\sigma$ (red curves) and -0.5σ (blue curves), where σ is the standard deviation of the probability density $|\psi(z, 0)|^2$. Panel (a) shows the corresponding $|\langle w_1(t) | \psi(t) \rangle|^2$ while panel (b) presents $|\psi(z, t)|^2$ at $t/T = 9990$.

frequency $\omega_{\perp}$ is not important because all of these parameters influence the value of $g_{1D}N$ only. For example, when $g_{1D}N$ is changed by 10%, the results presented in Fig. 3(a) do not change.

(2) Precise control of the frequency $\omega_{\parallel}$ of the initial longitudinal trapping potential is also not essential. When the width of the initial atomic distribution along the longitudinal direction is changed by 10% with respect to the optimal value, the results shown in Fig. 3(a) do not change.

(3) Precise location of the initial atomic distribution at the desired classical turning point is not important because deviations can be corrected by an appropriate adjustment of the frequency of the mirror oscillations. The important factor is the stability of the location of the atom distribution in different realizations of the experiment. In Fig. 5 we show the results when the initial distribution is displaced with respect to the optimal position by 0.5 times its standard deviation, i.e., by $0.5\sigma l_0 = 0.38 \mu\text{m}$. There is a drop of the squared overlap by only a few percent. Thus, in different realizations of the experiment the allowed deviations of the location of the initial distribution with respect to the mirror position are of the order of 0.5σ .

IV. DYNAMICAL QUANTUM PHASE TRANSITION

In equilibrium statistical physics phase transitions are indicated by a nonanalytical behavior of the macroscopic quantities as a function of a control parameter [68,69]. It turns out that a similar nonanalytical behavior can also be observed as a function of time in the nonequilibrium dynamics of many-body

systems which is induced by a quantum quench, i.e., a sudden change of a control parameter across a critical value [32,35]. Such a dynamical quantum phase transition is indicated by a nonanalyticity of the return probability of a system to the initial state at a certain critical moment of time if the thermodynamic limit is considered. This phenomenon can be interpreted as the partial loss of information on the system evolution—when we observe a system from the point of view of the initial state there is a breakdown of the short-time expansion at a critical moment of time. While dynamical quantum phase transitions have been analyzed mostly in time-independent systems, recently a similar behavior has been predicted in a periodically driven system which reveals discrete time crystal formation [31]. Here, we analyze the experimental signatures of a dynamical quantum phase transition for ultracold atoms bouncing on a periodically oscillating mirror after a quantum quench from the time crystal phase to a weakly interacting regime.

A quantum many-body description of ultracold bosons bouncing resonantly on an oscillating mirror can be reduced to a Bose-Hubbard Hamiltonian. That is, when we restrict to the Hilbert subspace spanned by Fock states $|n_1, \dots, n_s\rangle$, where n_i denotes the number of bosons occupying a Wannier-like wave packet $w_i(z, t)$ evolving along the $s : 1$ resonant orbit, we end up with a many-body Bose-Hubbard Hamiltonian similar to Eq. (19) but with the complex numbers a_i replaced by the standard bosonic annihilation operators $\hat{a}_i$. If the attractive interactions between bosons are sufficiently strong, the low-lying energy eigenstates of the Bose-Hubbard Hamiltonian are Schrödinger cat-like states. For example, the ground state $|\Psi\rangle$ can be approximated by a macroscopic superposition $(|N, 0, \dots, 0\rangle + |0, N, 0, \dots, 0\rangle + \dots + |0, \dots, 0, N\rangle)/\sqrt{s}$. For $s = 2$ it was shown that starting with $|\Psi\rangle$ and performing a quench to a weakly interacting regime, the so-called Loschmidt echo $\mathcal{L}(t) = |\langle\Psi(t)|\tilde{\Psi}(t)\rangle|^2$, where $|\Psi(t)\rangle$ and $|\tilde{\Psi}(t)\rangle$ evolve according to the Hamiltonians before and after the quench, respectively, reveals a cusp at a critical moment of time t_c [31]. In an experiment it is extremely difficult to prepare a Schrödinger cat state because the system spontaneously chooses a symmetry broken state and a discrete time crystal forms, which has been analyzed in Sec. III. However, the signature of the dynamical quantum phase transition can still be observed experimentally even if we prepare initially a discrete time crystal state, e.g., $|\Psi\rangle = |\Phi_1\rangle = |N, 0, \dots, 0\rangle$, where all bosons occupy a single Wannier-like wave packet $w_1(z, t)$, and after the quench we perform measurements of the return probability of the system to the lowest energy manifold. Indeed, in the limit when $N \rightarrow \infty$, the return probability to the degenerate ground-state manifold of the Bose-Hubbard Hamiltonian reproduces the Loschmidt echo [31,35], i.e.,

$$\mathcal{L}(t) = \sum_{i=1}^s p_i(t) = \sum_{i=1}^s e^{-N\lambda_i(t)}, \quad (24)$$

where $p_i(t) = |\langle\Phi_i(t)|\tilde{\Psi}(t)\rangle|^2$ is the probability of finding the evolving state $|\tilde{\Psi}(t)\rangle$ in a time crystal state $|\Phi_i(t)\rangle$ where N bosons occupy a Wannier-like wave packet $w_i(z, t)$. In (24) we have introduced the so-called rates $\lambda_i(t)$ which are intensive

quantities and read

$$\lambda_i(t) = -\frac{1}{N} \ln p_i(t). \quad (25)$$

In the thermodynamic limit, the Loschmidt echo is determined by the smallest rate at a given moment of time t ,

$$\mathcal{L}(t) \propto e^{-N\lambda_{\min}(t)}, \quad (26)$$

where $\lambda_{\min}(t) = \min\{\lambda_1(t), \dots, \lambda_s(t)\}$ [35]. A nonanalytical behavior of $\mathcal{L}(t)$ corresponds to a time moment when initially the smallest rate $\lambda_i(t)$ has increased so much that it becomes greater than another rate $\lambda_j(t)$. Then, the minimal rate $\lambda_{\min}(t)$ reveals a cusp which corresponds to a cusp of the Loschmidt echo $\mathcal{L}(t)$; see Fig. 6(a). Experimentally, it is very difficult to measure directly the Loschmidt echo because $\mathcal{L}(t)$ drops exponentially with N . However, the rates themselves can be measured and the dynamical quantum phase transition can be observed in the laboratory [33,34].

The experiment demonstrating a dynamical quantum phase transition in a discrete time crystal can be performed with the parameters determined in Sec. III. Initially, for the interaction strength $g_{1D}N = -0.12$, one prepares a discrete time crystal state $|\Psi(t)\rangle \approx |\Phi_1(t)\rangle$ which evolves with a period $s = 40$ times longer than the mirror oscillation period T ; see Figs. 3 and 4. At a certain moment of time t , the interactions are instantly turned off ($g_{1D} = 0$) and the system starts evolving according to a new Hamiltonian. We are interested in the rates (25) where $p_i(t)$ are the probabilities of finding the evolving state $|\tilde{\Psi}(t)\rangle$ in the different time crystal states $|\Phi_i(t)\rangle$. Due to the fact that the initial state and the time crystal states $|\Phi_i(t)\rangle$ are Bose-Einstein condensates where macroscopic numbers of atoms occupy Wannier-like wave packets $w_i(z, t)$, the entire experiment can be described by the GPE (18). The projections of the solution $\tilde{\psi}(z, t)$ of the GPE on the Wannier-like wave packets allows us to calculate the probabilities $p_i(t) = |\langle w_i(t)|\tilde{\psi}(t)\rangle|^2$ and consequently the rates $\lambda_i(t) = -\ln |\langle w_i(t)|\tilde{\psi}(t)\rangle|^2$. In an experiment, the rates can be obtained by measuring atomic densities. Indeed, by determining the fractions of the total number of atoms which form a given localized wave packet we obtain estimates for $|\langle w_i(t)|\tilde{\psi}(t)\rangle|^2$. This is illustrated in Fig. 6 where we present the rates $\lambda_i(t)$ obtained by fitting a sum of Gaussian distributions to the atomic densities. The crossing point of the smallest rates corresponds to the critical time t_c when the return probability of the system (24) reveals a cusp indicating the dynamical quantum phase transition.

V. ANDERSON LOCALIZATION IN TIME

In this section we still analyze ultracold atoms bouncing on an oscillating mirror but we do not consider spontaneous breaking of time translation symmetry and formation of discrete time crystals.

In Sec. II we showed that atoms bouncing resonantly on the mirror behave like electrons in a space crystal and such a crystalline behavior is observed in the time domain. In this section we show that when we introduce disorder in the driving, Anderson localization phenomena known in condensed-matter physics can be realized in time.

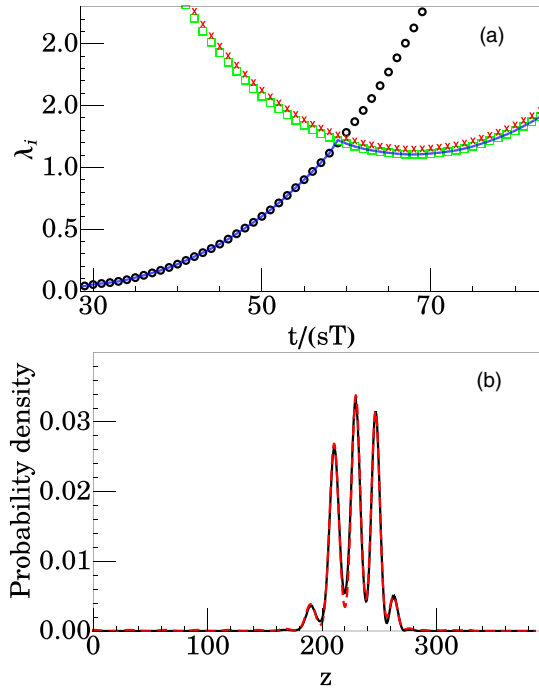


FIG. 6. Dynamical quantum phase transition in a discrete time crystal. Initially, ultracold atoms are prepared in a harmonic trap above an oscillating mirror and at $t = 0$ the longitudinal trapping potential is turned off and atoms fall on the mirror and form a discrete time crystal, i.e., they break the time translation symmetry and evolve with a period $s = 40$ longer than T . At time $t/(sT) = 28$ the interactions between atoms are turned off and the system evolves according to a different Hamiltonian and the time crystal decays. Panel (a) shows the evolution of the smallest three rates $\lambda_i(t)$, sampled with the period $sT = 40T$, obtained by fitting a sum of Gaussian distributions to the atomic densities—each Gaussian represents a Wannier-like wave packet $|w_i(z, t)|^2$. Black symbols correspond to the overlap of $\tilde{\psi}(z, t)$ with the initial Wannier-like wave packet $w_1(z, t)$, while red crosses and green squares correspond to the overlap with the two neighboring Wannier functions. Blue solid line is related to the minimal rate $\lambda_{\min}(t)$. The crosses and squares are not identical because the state prepared initially in the trap is not a perfect Wannier function $w_1(z, t)$. An example of the result of the fitting is presented in panel (b), where we have chosen t close to the critical time $t_c/(sT) = 59$, i.e., at the moment of time when initially the smallest rate approaches two other rates corresponding to the projection of $\tilde{\psi}(z, t)$ on two neighboring Wannier-like wave packets. In panel (b) the solid black line corresponds to $|\tilde{\psi}(z, t)|^2$ and the dashed red line to the fitted sum of the Gaussian distributions.

Anderson localization is a well-known phenomenon which takes place in configuration space and relies on an exponential localization of eigenstates of a particle [70,71]. When a time-independent spatially periodic potential is contaminated by a spatially disordered contribution, extended Bloch waves turn into exponentially localized eigenstates due to destructive interference between different multiple-scattering paths. The localization of eigenstates is accompanied by the inhibition of transport in a disordered system. Anderson localization has also been studied in disordered systems with fast periodic time modulations [72–74]. Actually, the presence of a spatially

periodic potential is not necessary. Indeed, even without a crystalline structure in space, the presence of a disordered potential characterized by a finite correlation length results in the localization of a particle in configuration space. Anderson localization can also be observed in momentum space and it is related to the quantum suppression of classical diffusion of classically chaotic systems [75–77]. Yet another kind of Anderson localization has recently been proposed: localization in the time domain due to the presence of disorder in time [46,48–50].

In this section we show that noninteracting ultracold atoms bouncing on an atom mirror can reveal Anderson localization in time if the mirror performs random motion. Experimental signatures of the localization are related to an exponential localization, around a certain moment of time, of the probability for the detection of atoms at a fixed position on a resonant periodic orbit [46].

We begin with an atom bouncing on a harmonically oscillating mirror where we assume that the $s : 1$ resonance condition is fulfilled. For $s \gg 1$, the eigenstates of the effective single-particle Hamiltonian (10) are Bloch waves which in the laboratory frame appear as trains of localized Wannier-like wave packets, $w_i(z, t)$, moving periodically along a classical resonant orbit. When we restrict to the first energy band of the effective Hamiltonian (10), the energy of an atom can be described by the tight-binding model (13).

Let us assume that the motion of the mirror is not perfectly harmonic because we turn on a fluctuating perturbation. That is, the mirror motion is no longer described by Eq. (5) but by

$$f(t) = f(t + sT) = -\gamma \cos \omega t - \frac{V_d}{\sqrt{2s}} \sum_{\substack{k=-s \\ k \neq 0}}^s e^{i(k\omega t/s + \varphi_k)}, \quad (27)$$

where $T = 2\pi/\omega$ and the phases $\varphi_k = -\varphi_{-k}$ are chosen randomly from the uniform distribution in the interval $[0, 2\pi]$. The first part of $f(t)$ is responsible for the crystalline structure described by (10) while the other part is the fluctuating perturbation whose strength is characterized by V_d . For the parameters chosen in Sec. III, the first part corresponds to the main driving with frequency $\omega/(2\pi t_0) = 3.94$ kHz and the perturbation part consists of its $s = 40$ subharmonics. The presence of the perturbation results in additional terms in the effective Hamiltonian (10) which currently reads

$$H_{\text{eff}} \approx \frac{P^2}{2m_{\text{eff}}} + V_0 \cos(s\Theta) + \frac{V_d}{\sqrt{2s}} \sum_{\substack{k=-s \\ k \neq 0}}^s e^{i(k\Theta + \varphi_k)}, \quad (28)$$

with $m_{\text{eff}} = -\pi^2 s^4/\omega^4$ and $V_0 = \gamma(-1)^{s+1}$. If we restrict to the first energy band we obtain the tight-binding model

$$E \approx -\frac{J}{2} \sum_{i=1}^s (a_{i+1}^* a_i + \text{c.c.}) + \sum_{i=1}^s \epsilon_i |a_i|^2, \quad (29)$$

where $\epsilon_i = (sT)^{-1} \int_0^{sT} dt \int_0^\infty dz z f''(t) |w_i|^2$. Employing the central limit theorem it can be shown that the randomness of the phases φ_k implies that the ϵ_i are random numbers corresponding to a normal distribution with standard deviation V_d . The tight-binding model (29) is actually a one-dimensional

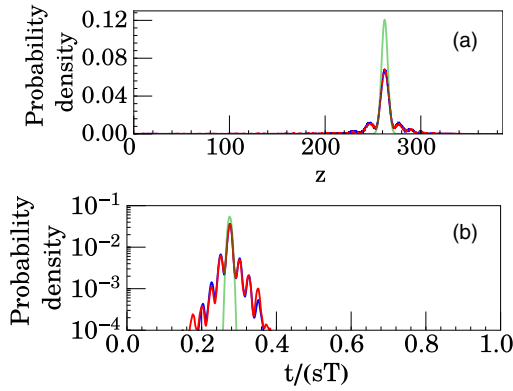


FIG. 7. Anderson localization in the time domain in the presence of crystalline structure in time. Panel (a) shows the initial atomic density (green curve) and the densities averaged over 50 realizations of the random motion of the mirror, see Eq. (27), at long evolution times, i.e., $t = 2210T$ (blue curve) and $t = 5010T$ (red curve). Panel (b) presents the Anderson localization in the time domain, i.e., the average probability densities for the detection of atoms at $z = 1$ vs time. Green, blue, and red curves correspond to the results presented in panel (a), i.e., to the probability of the detection of atoms around $t = 0$, $t = 2210T$, and $t = 5010T$, respectively. Once the Anderson localization freezes the spreading of the wave packet, the probability density for the detection of atoms at $z = 1$ does not change its shape and it is repeated with period sT due to the periodic boundary conditions in time—the presented results correspond to $s = 40$.

Anderson model and for $s \rightarrow \infty$ Anderson localization takes place regardless of how small V_d is. In the case when $s = 40$, which we are interested in, localization can be observed provided eigenstates localize on a number of neighboring Wannier-like wave packets smaller than 40. For the parameters described in Sec. III, the amplitude of the harmonic oscillations of the mirror is $\gamma l_0 = 3.2$ nm, which results in $J = 8.6 \times 10^{-4}$, and in order to analyze the Anderson localization we choose $V_d = 5.1J$, which corresponds to $V_d l_0 = 1.7$ nm.

The experimental realization of Anderson localization in time can start with precisely the same initial state as considered in Secs. III and IV, i.e., with ultracold atoms trapped above the mirror. However, in the present case the interactions between atoms have to be turned off and the mirror performs the motion described by (27). The initial state is a superposition of the exponentially localized eigenstates described by (28) and in the course of time evolution it does not spread over the entire resonant periodic orbit but tends to a localized periodically evolving probability distribution [78–80]. We have performed numerical integration of the full Schrödinger equation of an atom bouncing on a harmonically oscillating mirror and in the presence of random fluctuations; see (27). In Fig. 7(a) we show the density of atoms in configuration space averaged over 50 different realizations of the disordered motion of the mirror at $t = 2210T$ and $t = 5010T$. The results indicate that after sufficiently long evolution the atomic density freezes its shape—it essentially stops spreading already at time t of the order of $1000T$. This behavior is in contrast to the evolution without the mirror fluctuations ($V_d = 0$) presented in Figs. 3 and 4(b) where tunneling of atoms does not stop

and atoms tend to spread along the entire classical resonant orbit.

Anderson localization in the time domain is illustrated in Fig. 7(b) where the average probability density for the detection of atoms at $z = 1$ is shown as a function of time. Once the atomic distribution stops spreading, the localization of the probability density around a certain moment of time is observed and it is repeated with the period sT due to the periodic boundary conditions in time.

The experimental setup considered in this subsection can also be used for realization of many-body localization where interactions between particles and strong disorder result in the absence of thermalization of a system, vanishing of dc transport, and logarithmic growth of the entanglement entropy [81–88]. This phenomenon can be observed in the time domain when together with the presence of the fluctuating atom mirror, repulsive interactions between atoms are turned on [58].

VI. SUMMARY AND CONCLUSIONS

In the present paper we have considered various aspects of crystalline behavior in time of ultracold atoms bouncing on an oscillating atom mirror and performed a detailed analysis of the realistic experimental conditions needed for their realization. Experiments with cold and ultracold atoms bouncing on a mirror have already been carried out [36–44]. Such a system turns out to be a promising platform for realization of time crystals.

We began with considerations of the formation of discrete time crystals [9–21]. Discrete time crystals have already been demonstrated in the laboratory with the help of spin systems [22–27]. Atoms bouncing on a mirror are able to reveal a dramatic breaking of discrete time translation symmetry of the system Hamiltonian where the symmetry broken states evolve with periods tens of times longer than the driving period. Such discrete time crystals cannot be realized in spin systems unless the spin quantum numbers are very large. We identified experimentally realistic conditions and performed numerical simulations of the formation of a discrete time crystal. We also performed an analysis of the influence of experimental imperfections on the realization of the phenomenon.

Quantum many-body systems which form discrete time crystals can reveal a dynamical quantum phase transition after suddenly turning off the interactions between particles [31]. We demonstrated that ultracold atoms bouncing on an oscillating mirror are also a suitable system for realization of such a nonanalytical behavior in time. Indeed, we showed that measurements of atomic densities after a quantum quench (i.e., after a sudden turn-off of the interactions) allow one to obtain the return probability of the system to the initial time crystal manifold which reveals nonanalytical behavior around a critical moment of time.

Atoms resonantly bouncing on a mirror are also a promising system for realization of various condensed-matter phenomena in the time domain [1,47]. We focused on Anderson localization and showed that when the mirror fluctuates in time, the randomness in the driving of the atoms

results in Anderson localization of the system in the time domain [46,48–50]. The latter phenomenon corresponds to an exponential localization in time of the probability for the detection of atoms at a fixed position in configuration space. Analysis of the considered system showed that the observation of Anderson localization in time is attainable experimentally.

ACKNOWLEDGMENTS

We thank J. Wang, A. Sidorov, N. Robins, and C. Kuhn for fruitful discussions. Support of the National Science Centre, Poland via Projects No. 2016/20/W/ST4/00314 (K.G.) and No. 2016/21/B/ST2/01086 (A.K.), and under QuantERA Program No. 2017/25/Z/ST2/03027 (K.S.) is acknowledged.

-
- [1] K. Sacha and J. Zakrzewski, *Rep. Prog. Phys.* **81**, 016401 (2018).
 - [2] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [3] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [4] H. Watanabe and M. Oshikawa, *Phys. Rev. Lett.* **114**, 251603 (2015).
 - [5] A. Syrwid, J. Zakrzewski, and K. Sacha, *Phys. Rev. Lett.* **119**, 250602 (2017).
 - [6] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, *arXiv:1708.05014*.
 - [7] Y. Huang, T. Li, and Z.-q. Yin, *Phys. Rev. A* **97**, 012115 (2018).
 - [8] N. V. Prokof'ev and B. V. Svistunov, *arXiv:1710.00721*.
 - [9] K. Sacha, *Phys. Rev. A* **91**, 033617 (2015).
 - [10] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).
 - [11] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
 - [12] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
 - [13] A. Lazarides and R. Moessner, *Phys. Rev. B* **95**, 195135 (2017).
 - [14] A. Russomanno, F. Iemini, M. Dalmonte, and R. Fazio, *Phys. Rev. B* **95**, 214307 (2017).
 - [15] T.-S. Zeng and D. N. Sheng, *Phys. Rev. B* **96**, 094202 (2017).
 - [16] K. Nakatsugawa, T. Fujii, and S. Tanda, *Phys. Rev. B* **96**, 094308 (2017).
 - [17] W. W. Ho, S. Choi, M. D. Lukin, and D. A. Abanin, *Phys. Rev. Lett.* **119**, 010602 (2017).
 - [18] B. Huang, Y.-H. Wu, and W. V. Liu, *Phys. Rev. Lett.* **120**, 110603 (2018).
 - [19] Z. Gong, R. Hamazaki, and M. Ueda, *Phys. Rev. Lett.* **120**, 040404 (2018).
 - [20] R. R. W. Wang, B. Xing, G. G. Carlo, and D. Poletti, *Phys. Rev. E* **97**, 020202 (2018).
 - [21] K. Mizuta, K. Takasan, M. Nakagawa, and N. Kawakami, *arXiv:1804.01291*.
 - [22] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath *et al.*, *Nature (London)* **543**, 217 (2017).
 - [23] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani *et al.*, *Nature (London)* **543**, 221 (2017).
 - [24] S. Pal, N. Nishad, T. S. Mahesh, and G. J. Sreejith, *Phys. Rev. Lett.* **120**, 180602 (2018).
 - [25] J. Rovny, R. L. Blum, and S. E. Barrett, *Phys. Rev. Lett.* **120**, 180603 (2018).
 - [26] J. Rovny, R. L. Blum, and S. E. Barrett, *Phys. Rev. B* **97**, 184301 (2018).
 - [27] S. Autti, V. B. Eltsov, and G. E. Volovik, *Phys. Rev. Lett.* **120**, 215301 (2018).
 - [28] K. Kim, M.-S. Heo, K.-H. Lee, K. Jang, H.-R. Noh, D. Kim, and W. Jhe, *Phys. Rev. Lett.* **96**, 150601 (2006).
 - [29] M.-S. Heo, Y. Kim, K. Kim, G. Moon, J. Lee, H.-R. Noh, M. I. Dykman, and W. Jhe, *Phys. Rev. E* **82**, 031134 (2010).
 - [30] P. Ziń, J. Chwedeńczuk, B. Oleś, K. Sacha, and M. Trippenbach, *Europhys. Lett.* **83**, 64007 (2008).
 - [31] A. Kosior and K. Sacha, *Phys. Rev. A* **97**, 053621 (2018).
 - [32] M. Heyl, A. Polkovnikov, and S. Kehrein, *Phys. Rev. Lett.* **110**, 135704 (2013).
 - [33] P. Jurcevic, H. Shen, P. Hauke, C. Maier, T. Brydges, C. Hempel, B. P. Lanyon, M. Heyl, R. Blatt, and C. F. Roos, *Phys. Rev. Lett.* **119**, 080501 (2017).
 - [34] N. Fläschner, D. Vogel, M. Tarnowski, B. S. Rem, D.-S. Lühmann, M. Heyl, J. C. Budich, L. Mathey, K. Sengstock, and C. Weitenberg, *Nat. Phys.* **14**, 265 (2018).
 - [35] M. Heyl, *Rep. Prog. Phys.* **81**, 054001 (2018).
 - [36] A. Steane, P. Szriftgiser, P. Desbiolles, and J. Dalibard, *Phys. Rev. Lett.* **74**, 4972 (1995).
 - [37] T. M. Roach, H. Abele, M. G. Boshier, H. L. Grossman, K. P. Zetie, and E. A. Hinds, *Phys. Rev. Lett.* **75**, 629 (1995).
 - [38] A. I. Sidorov, R. J. McLean, W. J. Rowlands, D. C. Lau, J. E. Murphy, M. Walkiewicz, G. I. Opat, and P. Hannaford, *Quantum Semiclassical Opt.: J. Eur. Opt. Soc., Part B* **8**, 713 (1996).
 - [39] N. Westbrook, C. I. Westbrook, A. Landragin, G. Labeyrie, L. Cognet, V. Savalli, G. Horvath, A. Aspect, C. Hendel, K. Moelmer *et al.*, *Phys. Scr.* **T78**, 7 (1998).
 - [40] D. C. Lau, A. I. Sidorov, G. I. Opat, R. J. McLean, W. J. Rowlands, and P. Hannaford, *Eur. Phys. J. D* **5**, 193 (1999).
 - [41] K. Bongs, S. Burger, G. Birkel, K. Sengstock, W. Ertmer, K. Rzążewski, A. Sanpera, and M. Lewenstein, *Phys. Rev. Lett.* **83**, 3577 (1999).
 - [42] A. I. Sidorov, R. J. McLean, F. Scharnberg, D. S. Gough, T. J. Davis, B. J. Sexton, G. I. Opat, and P. Hannaford, *Acta Phys. Pol. B* **33**, 2137 (2002).
 - [43] J. Fiutowski, D. Bartoszek-Bober, T. Dohnalik, and T. Kawalec, *Opt. Commun.* **297**, 59 (2013).
 - [44] T. Kawalec, D. Bartoszek-Bober, R. Panaś, J. Fiutowski, A. Pławecka, and H.-G. Rubahn, *Opt. Lett.* **39**, 2932 (2014).
 - [45] L. Guo, M. Marthaler, and G. Schön, *Phys. Rev. Lett.* **111**, 205303 (2013).
 - [46] K. Sacha, *Sci. Rep.* **5**, 10787 (2015).
 - [47] K. Giergiel, A. Miroszewski, and K. Sacha, *Phys. Rev. Lett.* **120**, 140401 (2018).
 - [48] K. Sacha and D. Delande, *Phys. Rev. A* **94**, 023633 (2016).
 - [49] K. Giergiel and K. Sacha, *Phys. Rev. A* **95**, 063402 (2017).
 - [50] D. Delande, L. Morales-Molina, and K. Sacha, *Phys. Rev. Lett.* **119**, 230404 (2017).
 - [51] A. Buchleitner, D. Delande, and J. Zakrzewski, *Phys. Rep.* **368**, 409 (2002).

- [52] J. H. Shirley, *Phys. Rev.* **138**, B979 (1965).
- [53] G. Berman and G. Zaslavsky, *Phys. Lett. A* **61**, 295 (1977).
- [54] A. Lichtenberg and M. Lieberman, *Regular and Chaotic Dynamics*, Applied Mathematical Sciences (Springer-Verlag, New York, 1992).
- [55] L. Guo and M. Marthaler, *New J. Phys.* **18**, 023006 (2016).
- [56] L. Guo, M. Liu, and M. Marthaler, *Phys. Rev. A* **93**, 053616 (2016).
- [57] P. Liang, M. Marthaler, and G. Lingzhen, *New J. Phys.* **20**, 023043 (2018).
- [58] M. Mierzejewski, K. Giergiel, and K. Sacha, *Phys. Rev. B* **96**, 140201 (2017).
- [59] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Lühmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, *Rep. Prog. Phys.* **78**, 066001 (2015).
- [60] C. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2002).
- [61] Y. Castin, in *Coherent Atomic Matter Waves*, edited by R. Kaiser, C. Westbrook, and F. David (Springer, Berlin, New York, 2001), pp. 1–136.
- [62] H. Saito and M. Ueda, *Phys. Rev. Lett.* **86**, 1406 (2001).
- [63] H. Saito and M. Ueda, *Phys. Rev. A* **63**, 043601 (2001).
- [64] E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wieman, *Nature (London)* **412**, 295 (2001).
- [65] P. A. Altin, G. R. Dennis, G. D. McDonald, D. Döring, J. E. Debs, J. D. Close, C. M. Savage, and N. P. Robins, *Phys. Rev. A* **84**, 033632 (2011).
- [66] N. R. Claussen, S. J. J. M. F. Kokkelmans, S. T. Thompson, E. A. Donley, E. Hodby, and C. E. Wieman, *Phys. Rev. A* **67**, 060701 (2003).
- [67] C. C. N. Kuhn, G. D. McDonald, K. S. Hardman, S. Bennetts, P. J. Everitt, P. A. Altin, J. E. Debs, J. D. Close, and N. P. Robins, *New J. Phys.* **16**, 073035 (2014).
- [68] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England 2011).
- [69] J. Dziarmaga, *Adv. Phys.* **59**, 1063 (2010).
- [70] P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
- [71] C. A. Müller and D. Delande, *Disorder and Interference: Localization Phenomena* (Oxford University Press, New York, 2011), Vol. 91, Chap. 9.
- [72] A. Kosior, J. Major, M. Płodzień, and J. Zakrzewski, *Phys. Rev. A* **92**, 023606 (2015).
- [73] J. Major, *Phys. Rev. A* **94**, 053613 (2016).
- [74] J. Major, M. Płodzień, O. Dutta, and J. Zakrzewski, *Phys. Rev. A* **96**, 033620 (2017).
- [75] S. Fishman, D. R. Grempel, and R. E. Prange, *Phys. Rev. Lett.* **49**, 509 (1982).
- [76] G. Casati, I. Guarneri, and D. L. Shepelyansky, *Phys. Rev. Lett.* **62**, 345 (1989).
- [77] G. Lemarié, J. Chabé, P. Szriftgiser, J. C. Garreau, B. Grémaud, and D. Delande, *Phys. Rev. A* **80**, 043626 (2009).
- [78] L. Sanchez-Palencia, D. Clément, P. Lugan, P. Bouyer, G. V. Shlyapnikov, and A. Aspect, *Phys. Rev. Lett.* **98**, 210401 (2007).
- [79] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, *Nature (London)* **453**, 891 (2008).
- [80] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, *Nature (London)* **453**, 895 (2008).
- [81] D. Basko, I. Aleiner, and B. Altshuler, *Ann. Phys. (N.Y.)* **321**, 1126 (2006).
- [82] V. Oganesyan and D. A. Huse, *Phys. Rev. B* **75**, 155111 (2007).
- [83] M. Žnidarič, T. Prosen, and P. Prelovšek, *Phys. Rev. B* **77**, 064426 (2008).
- [84] D. A. Huse, R. Nandkishore, and V. Oganesyan, *Phys. Rev. B* **90**, 174202 (2014).
- [85] R. Nandkishore and D. A. Huse, *Ann. Rev. Condens. Mater. Phys.* **6**, 15 (2015).
- [86] M. Kozarzewski, P. Prelovšek, and M. Mierzejewski, *Phys. Rev. B* **93**, 235151 (2016).
- [87] M. Mierzejewski, J. Herbrych, and P. Prelovšek, *Phys. Rev. B* **94**, 224207 (2016).
- [88] P. Sierant and J. Zakrzewski, *New J. Phys.* **20**, 043032 (2018).



FAST TRACK COMMUNICATION • OPEN ACCESS

Topological time crystals

To cite this article: Krzysztof Giergiel *et al* 2019 *New J. Phys.* **21** 052003

View the [article online](#) for updates and enhancements.

Recent citations

- [Broadband frequency translation through time refraction in an epsilon-near-zero material](#)
Yiyu Zhou *et al*
- [Planar optomagnonic cavities driven by surface spin waves](#)
Petros Andreas Pantazopoulos and Nikolaos Stefanou
- [Discrete time crystal in a finite chain of Rydberg atoms without disorder](#)
Chu-hui Fan *et al*



FAST TRACK COMMUNICATION

Topological time crystals

OPEN ACCESS

RECEIVED
20 February 2019REVISED
24 April 2019ACCEPTED FOR PUBLICATION
1 May 2019PUBLISHED
29 May 2019

Original content from this
work may be used under
the terms of the [Creative
Commons Attribution 3.0
licence](#).

Any further distribution of
this work must maintain
attribution to the
author(s) and the title of
the work, journal citation
and DOI.

Krzysztof Giergiel¹, Alexandre Dauphin² , Maciej Lewenstein^{2,3}, Jakub Zakrzewski^{1,4} and
Krzysztof Sacha^{1,4}¹ Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348
Kraków, Poland² ICFO—Institut de Ciències Fotoniques, The Barcelona Institute of Science and Technology, Av. Carl Friedrich Gauss 3, E-08860
Castelldefels (Barcelona), Spain³ ICREA, Pg. Lluís Companys 23, E-08010 Barcelona, Spain⁴ Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków,
PolandE-mail: krzysztof.sacha@uj.edu.pl**Keywords:** time crystals, topological phases, ultra-cold atoms**Abstract**

By analogy with the formation of space crystals, crystalline structures can also appear in the time domain. While in the case of space crystals we often ask about periodic arrangements of atoms in space at a moment of a detection, in time crystals the role of space and time is exchanged. That is, we fix a space point and ask if the probability density for detection of a system at this point behaves periodically in time. Here, we show that in periodically driven systems it is possible to realize topological insulators, which can be observed in time. The bulk-edge correspondence is related to the edge in time, where edge states localize. We focus on two examples: Su-Schrieffer-Heeger model in time and Bose Haldane insulator which emerges in the dynamics of a periodically driven many-body system.

1. Introduction

Time crystals are quantum systems where a crystalline structure emerges in time with no initial crystalline structure in space [1–5] (for the classical version of time crystals, including topologically protected systems, see [6–11]). A crystalline structure in time is related to the periodic dynamics of a system. It has been shown that periodically driven quantum many-body systems can spontaneously self-reorganize their motion and start moving with a period different from the period of the driving [12–14] (see also [15–29]). This kind of spontaneous formation of a new crystalline structure in time is dubbed discrete or Floquet time crystals and has been already realized experimentally [30–34]. However, periodically driven systems can also reveal a whole variety of condensed matter phases in the time domain even if no spontaneous process is involved in the emergence of such crystalline structures in time [28, 35–40] (see [41–44] for phase space crystals). Indeed, if a single- or many-body system is driven resonantly, its resonant dynamics can be reduced to solid state-like behavior and importantly such condensed matter physics emerges not in the configuration space but in the time domain—e.g. Anderson or many-body localization in time, superfluid-Mott insulator (MI) transition or quasi-crystals in the time domain have been demonstrated [35–40].

In this paper we show that it is possible to drive resonantly a system so that the emerging crystalline structure in time is a symmetry protected topological (SPT) phase [45]. The topological time crystals we consider should not be confused with the so-called Floquet topological systems. In the latter, a crystalline structure (usually an optical lattice) is present in space and it is periodically driven so that its effective parameters can be changed and the system can reveal topological properties in space but no crystalline structure can be observed in time [46–49]. Our systems are also different from Floquet–Bloch systems where time periodicity is considered as an additional synthetic dimensional combined with a crystalline structure in space [50–55]. We consider systems where external forces do not reveal any periodic behavior in space. They drive systems periodically in time and due to the resonant driving condensed matter phenomena emerge in dynamics of systems.

In the models considered, resonant dynamics leads to an emergence of a time crystalline structure that may possess a SPT phase. SPT phases constitute a new paradigm: they are characterized by a global topological invariant and, therefore, can not be described by the Landau theory of phase transitions. Remarkable examples of these phases are, among others, the Haldane phase [56, 57] and the topological insulators [58]. The study of the latter with and without interactions has attracted much interest in condensed matter and in quantum simulators [58–60]. Quantum simulators constitute very versatile platforms with an unprecedented degree of control of the parameters of the system such as the hopping or the interactions [61]. Quantum simulators have successfully simulated and detected topological insulators in one-dimensional (1D) [48, 62–67], 2D [68–70] and 4D [71, 72]. All these realizations of the topological insulators rely on an initial underlying lattice. In the present paper we do not assume any underlining spatially periodic structure with a non-trivial topology [24]. We show that time crystals with topological properties can emerge due to an appropriate resonant driving and discuss possible implementation and detection schemes in quantum simulators.

2. The model

We focus on ultra-cold atoms bouncing on an oscillating atom mirror in the presence of the gravitational field [73] (for the stationary mirror experiments see [74–81]) but the phenomena we investigate can be realized in any periodically driven system which can reveal nonlinear resonances in the classical description [82]. The single-particle Hamiltonian, in the gravitational units and in the frame oscillating with the mirror [83], reads $H = H_0 + H_1$ where $H_0 = p^2/2 + x$ and

$$H_1 = x[\lambda \cos(s\omega t) + \lambda_1 \cos(s\omega t/2) + f(t)], \quad (1)$$

with $f(t) = f(t + 2\pi/\omega) = \sum_k f_k e^{ik\omega t}$ and integer-valued $s \gg 1$. Let us start with classical mech. description. In order to describe a resonant driving of the system it is convenient to perform canonical transformation to the so-called *action angle variables*, where the unperturbed Hamiltonian depends only on the new momentum (action), $H_0(I) = (3\pi I)^{2/3}/2$ [82, 83]. In the absence of the perturbation ($H_1 = 0$) the action I is a constant of motion, and the conjugate position variable (angle) changes linearly in time $\theta(t) = \Omega t + \theta(0)$, where $\Omega(I) = dH_0(I)/dI$ is the frequency of periodic evolution of a particle. We assume the resonant driving $\Omega(I_0) = \omega$, where I_0 is the resonant value of the action. Then, by means of the secular approximation [82, 83] (see the [appendix](#)) in the frame moving along the classical resonant orbit $\Theta = \theta - \omega t$, we obtain the effective Hamiltonian

$$H_{\text{eff}} = \frac{p^2}{2m_{\text{eff}}} - V_0 \cos(s\Theta) - V_1 \cos(s\Theta/2) - V_2(\Theta), \quad (2)$$

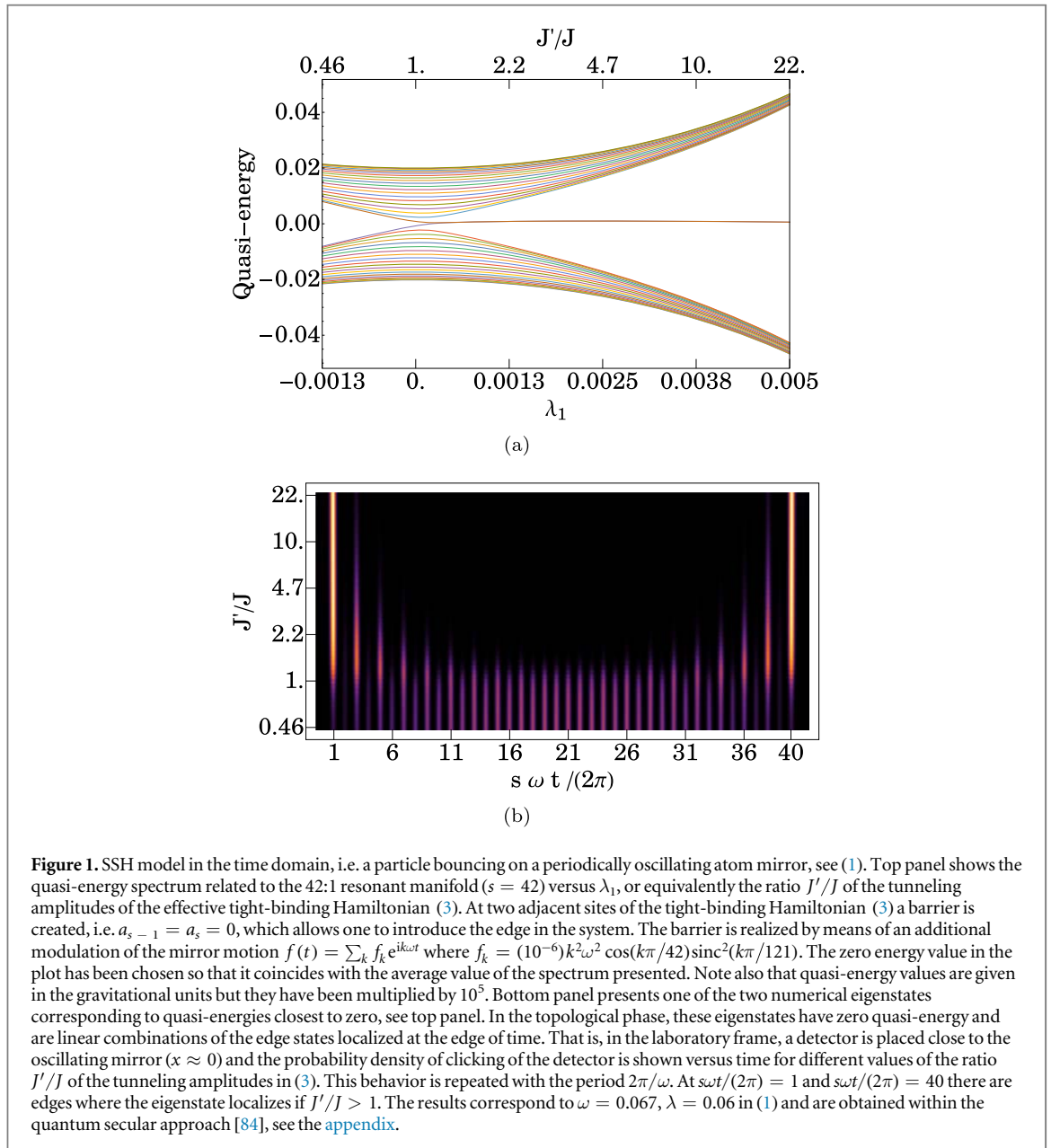
with $m_{\text{eff}} = -\pi^2/\omega^4$, $V_0 = \lambda(-1)^s/(s^2\omega^2)$, $V_1 = 4\lambda_1(-1)^{s/2}/(s^2\omega^2)$ and $V_2(\Theta) = \sum_n e^{in\Theta}(-1)^n f_{-n}/(n^2\omega^2)$. This result is valid in the regime close to resonance, i.e. when $P = I - I_0 \approx 0$. Equation (2) indicates that for $s \gg 1$ and $V_2(\Theta) = 0$, a resonantly driven particle behaves like an electron in a one-dimensional space crystal. In the laboratory frame, the crystalline behavior described by equation (2) is reproduced in the time domain due to the linearity of the transformation $\Theta = \theta - \omega t$ [35, 36]. In other words, when we switch to the quantum description, the clicking probability of a detector located in the laboratory frame close to the classical resonant orbit, reflects periodic behavior of the Bloch waves $\psi(\Theta)$ of the Hamiltonian (2) in the moving frame. In order to introduce an edge in the system and investigate the edge-bulk correspondence in a topological regime, we introduce a barrier $V_2(\Theta)$ localized on the sites $s-1$ and s of the periodic potential in (2). It is done by means of an additional modulation of the mirror motion $f(t)$ in (1) whose Fourier components are $f_k = (10^{-6})k^2\omega^2 \cos(k\pi/42) \text{sinc}^2(k\pi/121)$ for the case of $s = 42$ that is considered in figures 1–2.

3. A time crystal topological insulator

Let us focus on the first energy band of the quantum version of the effective Hamiltonian of equation (2) with $V_2(\Theta) = 0$. The Wannier states $w_i(\Theta)$ of the first energy band [85], which are localized in single sites of the periodic potential in equation (2), correspond to localized wavepackets $w_i(x, t)$ moving along the classical resonant orbit with the period $2\pi/\omega$.⁵ For both non-vanishing V_0 and V_1 , the effective periodic potential describes a Bravais lattice with a two-point basis. Restricting ourselves to the first energy band, i.e. the wavefunction of a particle is expanded as $\psi = \sum_{i=1}^s a_i w_i(\Theta)$, we obtain a tight-binding Hamiltonian

$$H_{\text{eff}} \approx -\frac{1}{2} \sum_{i=1}^s J_i (a_{i+1}^* a_i + \text{c.c.}), \quad (3)$$

⁵ Note that due to the negative effective mass m_{eff} , the first energy band is the highest in energy.



where $J_{2i} = J'$ and $J_{2i-1} = J$, which is identical to the SSH model [86]. The latter describes spinless fermions hopping on a 1D-lattice with staggered hopping amplitudes. Changing the ratio λ_1/λ in (1), allows one to control the ratio J'/J . This effective Hamiltonian belongs to the BDI class of the periodic table of the topological insulators and superconductors [87] and is characterized by a $\mathbb{Z}$ topological invariant, the winding number ν . For an infinite system with $J' > J$ ($J' < J$), the system is in a topological (trivial) phase with winding number $\nu = 1$ ($\nu = 0$). For a finite system, the topological phase exhibits zero energy edge states protected by the topology of the bulk. The SSH model has been experimentally realized in quantum simulators and both the presence of edge states and the winding number have been measured [63–65]. Let us emphasize that the Wannier states $w_i(x, t)$ of equation (2) are localized wavepackets of the effective SSH Hamiltonian of equation (3). We then discuss how such states allow one to detect the topology of the SSH Hamiltonian.

4. Detection of the topology

The experimental detection of the edge states and the corresponding bulk winding number ν can be realized in ultra-cold atoms which form a Bose–Einstein condensate (BEC) bouncing on an oscillating atom mirror. The Hamiltonian of equation (3) fulfills periodic boundary conditions, i.e. $a_{s+1} = a_1$. However, it is possible to introduce an edge in our system by means of a proper modulation of the mirror motion. Indeed, if f_k 's in the definition of $f(t)$ are suitably chosen, the resulting effective potential $V_2(\theta)$ can have a shape of a barrier localized

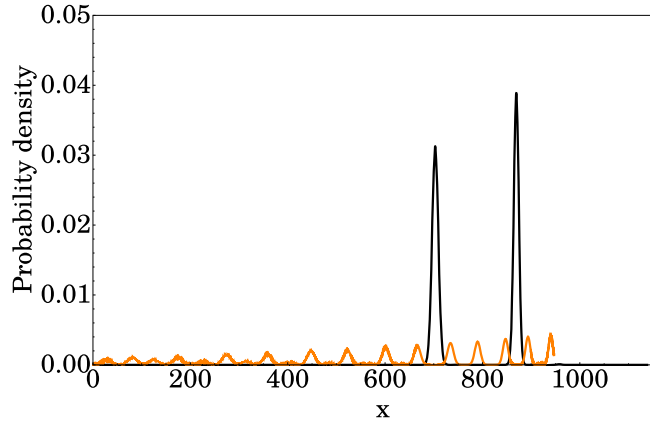


Figure 2. SSH model in the time domain. Black line: probability density of the eigenstate localized at the edge of time as shown in the bottom panel of figure 1 for $J'/J = 22$ but plotted in the configuration space for $s\omega t = 39\pi/2$. Orange line presents an example of a bulk eigenstate. The mirror is located at $x \approx 0$ while the classical turning point corresponds to $x = 1110$. Note that the crystalline behavior described by equation (3) is not observed in the configuration space shown in this plot but in the time domain, see bottom panel of figure 1.

on two adjacent sites of the periodic potential in equation (2). Then, the edge is created and two eigenstates can localize exponentially close to it in the topological phase. The corresponding quasi-energy spectrum of the full Floquet Hamiltonian $\mathcal{H}(t) = H - i\partial_t$ is shown in figure 1(a). For $J'/J > 1$ two degenerate zero-energy levels form which correspond to two eigenstates localized close to the barrier created by the potential $V_2(\Theta)$. In the laboratory frame these two eigenstates are related to Floquet states that evolve periodically along the classical resonant orbit. For a detector located close to the orbit, the probability of a detector clicking reveals an edge in time and these two Floquet states localize close to it, as shown in figure 1(b) for increasing values of J'/J . In [28] it was proposed how a BEC of non-interacting atoms can be loaded from a trap on a classical resonant orbit: a localized atomic cloud has to be released from a trap at the position of the turning point of the classical resonant orbit above the oscillating mirror. Then, the initial state of the system corresponds to all atoms occupying a single localized state $w_i(x, t)$ which evolves along the orbit. An atomic cloud loaded when the edge state is passing close to the turning point remains close to the edge as the edge states do not penetrate much the bulk [63]. How one of the zero-energy eigenstates with the localization close to the edge in time looks like in the configuration space is depicted in figure 2 for $s\omega t = 39\pi/2$. Measurements of atomic density at different times therefore allow one to confirm the localization properties of the state loaded on the edge.

On the other hand, if an atomic cloud is released in the bulk, then the subsequent time evolution leads to non-zero populations $|a_i(t)|^2$ of many Wannier wavepackets. Measurement of the atomic density allows one to obtain $|a_i(t)|^2$ and consequently the winding number which is determined by the mean chiral displacement, i.e. $\nu \approx 2\sum_i (i - i_0)(|a_{2i+1}(t)|^2 - |a_{2i}(t)|^2)$ where i_0 is a number of a cell of the Bravais lattice where the atomic cloud is initially loaded [48, 49, 66]. The relation between ν and the mean chiral displacement is valid after a long-time evolution when time averaged [66].

5. A time crystal Haldane phase

We now switch from single-particle to many-body systems which are resonantly driven and which can be characterized by non-trivial topology. It is known that a Bose gas in a time-independent spatially periodic lattice with repulsive on-site and nearest-neighbor interactions, described by the following Bose–Hubbard Hamiltonian

$$\hat{H} = -\frac{J}{2} \sum_i (\hat{a}_{i+1}^\dagger \hat{a}_i + \text{h.c.}) + \frac{U}{2} \sum_i \hat{n}_i(\hat{n}_i - 1) + V \sum_i \hat{n}_{i+1} \hat{n}_i, \quad (4)$$

can reveal a topological behavior. For large J the Hamiltonian (4) describes superfluid phase of bosons, for large U the MI emerges and for large V the density wave (DW) phase is present where translation symmetry of the Hamiltonian (4) is spontaneously broken. However, between the MI phase and the DW phase there is the topological Haldane insulator (HI) phase [88–91]—or more precisely a bosonic analog of HI in spin-1 chain [56, 57]. As discussed in [88] this phase breaks a hidden Z_2 symmetry related to a highly nonlocal string order parameter [92].

To realize a resonantly driven system that, in the time-periodic Wannier-like modes $w_i(x, t)$ of equation (2), is effectively described by the Hamiltonian of (4), we again consider ultra-cold bosons bouncing on an

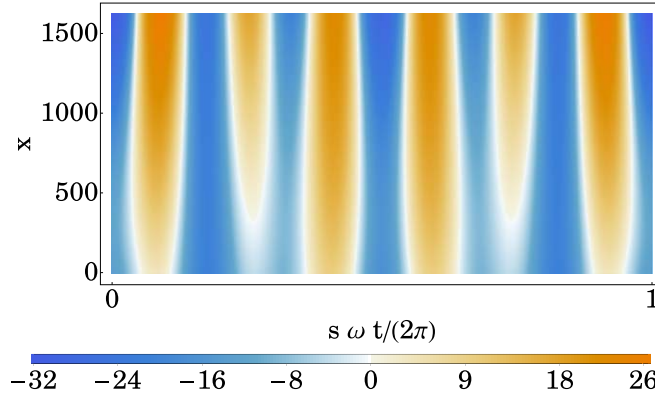


Figure 3. Ultra-cold atoms bouncing on a harmonically oscillating mirror, i.e. $\lambda_1 = 0$ and $f(t) = 0$ in (1). Figure shows an example of the dependence of the atomic s-wave scattering length $g_0(x, t) \times 10^{-4}$ on a position in space and on time which leads to the effective interactions between bosons described by the on-site and nearest neighbor interaction coefficients $U/J = 3$ and $V/J = 2.5$. If unit boson filling factor in the effective Bose–Hubbard model is assumed, then the results are related to the HI phase [90]. To realize MI or DW phases, the appropriate $g_0(x, t)$ is similar with the amplitude of its oscillations around zero being twice smaller or greater by at least factor 1.5, respectively. The other parameters of the system are the following: $\omega = 0.056$, $s = 64$ (i.e. 64:1 resonance), $\lambda = 0.14$ and $M = 2$ in the expansion $g_0 = \sum_{m=0}^M \alpha(t, m)x^m$. The mirror is located at $x = 0$ and the turning point of the classical resonant orbit at $x \approx 1600$. The maximal temporal interaction energy per particle is about $1.6 \times 10^4 J$ and thus much smaller than the gap between the first and second quasi-energy bands of the corresponding single-particle problem which is $1.1 \times 10^5 J$ where $J = 5.5 \times 10^{-8}$. The results are obtained within the quantum secular approach [84], see the [appendix](#).

oscillating mirror. We assume that the single-particle Hamiltonian corresponds to equation (1) with $\lambda_1 = 0$ and $f(t) = 0$. For the resonant driving described above, we may restrict to the first energy band of the effective Hamiltonian (2) that leads to the tight-binding model (3). In the many-body case when we restrict to the Hilbert subspace spanned by Fock states $|n_1, \dots, n_s\rangle$, where n_i 's denote numbers of bosons occupying Wannier wavepackets $w_i(x, t)$, we obtain the effective many-body Hamiltonian which resembles (4) but with the interaction terms $1/2 \sum_{i,j=1}^s U_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_i \hat{a}_j$ where $U_{ij} = 2 \int_0^{2\pi/\omega} dt \int_0^\infty dx g_0 |w_i|^2 |w_j|^2$ for $i \neq j$ and similar U_{ii} but twice smaller [35] (see the [appendix](#)). The effective interaction coefficients U_{ij} depend on the atomic s-wave scattering length g_0 , shapes of $|w_i(x, t)|^2$ and how densities of different Wannier wavepackets overlap in the course of time evolution on the classical resonant orbit. Despite the fact that the original interactions between ultra-cold atoms are contact, the effective interactions can be long-range [35, 42, 93]. Moreover, they can be controlled by changing the s-wave scattering length in space and periodically in time by means of a Feshbach resonance, i.e. $g_0 = g_0(x, t)$. Indeed, if the applied magnetic field results in appropriate oscillations of the scattering length around zero, nearly arbitrary effective long-range interactions can be created. In order to perform a systematic analysis of the control of U_{ij} , we assume that $g_0(x, t) = \sum_{m=0}^M \alpha(t, m)x^m$ and write the interaction coefficients in the form $U_{ij} = \sum_{m=0}^M \int_0^{2\pi/\omega} dt \alpha(t, m) u_{ij}(t, m)$. To find a suitable $\alpha(t, m)$ one can apply the singular value decomposition of the matrix $u_{ij}(t, m)$ where (i, j) and (t, m) are treated as indices of rows and columns, respectively [40]. Left singular vectors tell us which sets of interaction coefficients U_{ij} can be realized, while the corresponding right singular vectors give the recipes for $\alpha(t, m)$ and consequently for $g_0(x, t)$. In figure 3 we show an example of $g_0(x, t)$ corresponding to $U_{ii} = U$, $U_{ij} = V$ for $|i - j| = 1$ and $U_{ij} = 0$ when $|i - j| > 1$ where $U/J = 3$ and $V/J = 2.5$. These values are related to the HI phase if the unit mean boson filling factor is assumed [90]. In order to realize topologically trivial MI or DW phases, the corresponding $g_0(x, t)/J$ may take a similar form with the amplitude of its oscillations around zero being about twice smaller for MI or at least a factor 1.5 greater for the DW phase. For example, for the parameters of [28], if one wants to modulate the scattering length around zero value with the amplitude of two Bohr radius, the magnetic field has to change with the speed 0.5 G ms^{-1} —well within the experimental reach [94].

It is quite straightforward to prepare initially all bosons in a single Wannier state $w_i(x, t)$ by releasing atomic cloud from a trap at the classical turning point above the mirror [28]. However, the preparation of the system in lowest quasi-energy state within the resonant subspace, i.e. in the ground state of (4) is not easy. An open question remains whether the time-evolution after a quench of a localized initial state can reveal the topology of the Haldane Insulating phase. This problem is beyond the scope of the present paper.

Finally let us analyze whether the resonant driving does not heat the system considerably. Consider first a single particle problem. While a harmonic oscillator can absorb continuously energy when resonantly driven, we consider a model with nonlinear resonances—a particle bouncing resonantly on an oscillating mirror [83]. For such a system its period of motion depends on energy, so when a particle absorbs energy, its period changes and a particle gets out of the resonance. The resonant transfer of energy stops [82]. The nonlinear resonances are

related to the presence of elliptical islands in the classical phase space [82], see figure A1 in the appendix. Floquet states located inside the islands are strongly localized [83]. Their coupling to other states localized outside the islands, for any type of weak additional coupling is exponentially small precisely due to the semiclassical localization. Such a coupling may be provided by e.g. weak interactions among the particles that require a consideration of necessarily enlarged $2N$ -dimensional phase space. Such an additional coupling of a subspace of Floquet states localized in the islands to the complementary Hilbert subspace can be tested within the mean field approach as an exact many-body analysis is obviously beyond the current possibilities. A careful mean-field analysis performed in [28] indicates that, on the relevant experimentally time scale, such a coupling (and the resulting heating) is negligible also if typical experimental imperfections are taken into account. Thus the relevant physics takes place in the resonant subspace of Floquet states localized in the phase space and are immune to heating, at least for a time scale much longer than the experimentally relevant one [28], without any disorder present. If interactions are sufficiently weak then the physics may be described within the lowest band of the Bose–Hubbard model constructed in the paper.

6. Discussion

We have shown how one can construct topological insulators in the time domain. To this end we considered bosons bouncing on the periodically oscillating atom mirror, topological phases are obtained by an appropriate tuning the shape of the oscillations. We presented explicit schemes for the realization of the effective SSH and the extended Bose–Hubbard Hamiltonians which possess topological behavior in the time domain. In the case of the topological SSH, we proposed a detection method of the topological properties in time with state-of-the-art techniques [28]. In the case of the HI, the experimental detection of the topological phase remains a challenge.

Acknowledgments

AD and ML acknowledge fundings from the Spanish Ministry MINECO (National Plan 15 Grant: FISICATEAMO No. FIS2016-79508-P, SEVERO OCHOA No. SEV-2015-0522, FPI), European Social Fund, Fundació Cellex, Generalitat de Catalunya (AGAUR Grant No. 2017 SGR 1341 and CERCA/Program), ERC AdG OSYRIS and EU FETPRO QUIC. AD is financed by a Cellex-ICFO-MPQ fellowship. Support of the National Science Centre, Poland via Projects No. 2016/20/W/ST4/00314 (KG, ML), 2016/21/B/ST2/01086 (JZ), and No. 2016/21/B/ST2/01095 (KS) is acknowledged.

Note added

After the submission of the present manuscript we learned about the work on a similar topic but in photonic materials [95].

Appendix

In this appendix we present the derivation of the effective Hamiltonians used in the paper. We also provide a detailed explanation of the emergence of crystalline structures in the time domain in the course of dynamical evolution of resonantly driven systems.

A1. Classical effective Hamiltonian

We consider a single atom bouncing on an oscillating atom mirror in the presence of the gravitational force. The Hamiltonian of the system, in the frame oscillating with the mirror [83] (then the mirror is fixed) and in the gravitational units, reads

$$H = H_0 + H_1, \quad (\text{A.1})$$

$$H_0 = \frac{p^2}{2} + x, \quad (\text{A.2})$$

$$H_1 = x[\lambda \cos(s\omega t) + \lambda_1 \cos(s\omega t/2) + f(t)], \quad (\text{A.3})$$

where H_1 describes the oscillation of the mirror composed of two harmonics. s is an integer number and

$$f(t) = f(t + 2\pi/\omega) = \sum_k f_k e^{ik\omega t}. \quad (\text{A.4})$$

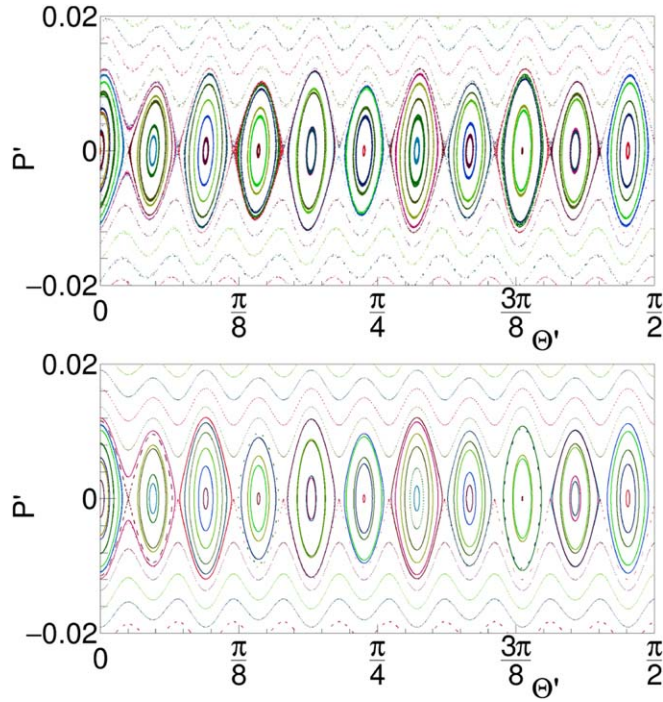


Figure A1. Comparison of the exact stroboscopic picture (top panel) of the system phase space with the phase space portrait (bottom panel) obtained with the help of the effective Hamiltonian (A.10). The parameters of the system are the same as those used in the paper in figures 1–2, i.e. $s = 42$, $\omega = 0.067$, $\lambda = 0.06$, $\lambda_1 = 0.005$ and $f_k = (10^{-6})k^2\omega^2 \cos(k\pi/42)\text{sinc}^2(k\pi/121)$. Scaled variables have been used in the plot, i.e. $P' = P/I_0$ and $\Theta' = \Theta$. Note that for the sake of clarity we show only a quarter of the full 2π range of Θ' .

We assume $x \geq 0$ with the mirror located at $x = 0$. At first glance the system does not have anything in common with a solid state-like behavior. In order to demonstrate that for a resonant driving of the particle motion, the system behaves like a condensed matter system (and in particular may possess topological properties) let us begin with the classical description and apply the secular approximation [82, 83].

First we apply the canonical transformation from the Cartesian position x and momentum p to the so-called action-angle variables. Then, the unperturbed Hamiltonian depends on the new momentum (action) only [83]

$$H_0 = \frac{1}{2}(3\pi I)^{2/3}, \quad (\text{A.5})$$

and the solution of the unperturbed particle motion is straightforward. That is, the action is a constant of motion, $I = \text{const}$, and the conjugate position variable (the angle) evolves linearly in time, $\theta(t) = \Omega(I)t + \theta(0)$, where $\Omega(I) = dH_0(I)/dI$ is the frequency of periodic evolution of a particle. The perturbation part of the Hamiltonian, in the new canonically conjugate variables, reads

$$H_1 = \left(\sum_n h_n(I) e^{in\theta} \right) \left(\sum_m F_m e^{im\omega t} \right), \quad (\text{A.6})$$

where $h_0 = \left(\frac{\pi I}{\sqrt{3}} \right)^{2/3}$ and $h_n = \frac{(-1)^{n+1}}{n^2} \left(\frac{3I}{\pi^2} \right)^{2/3}$ for $n \neq 0$, and F_m are Fourier components of the time-periodic function in (A.3).

Next, we assume that a particle is resonantly driven, that is the frequency ω is equal to the frequency of the unperturbed motion, i.e. $\omega = \Omega(I_0)$ where I_0 is the resonant value of the action. We are interested in initial conditions of a particle where the action $I \approx I_0$. Then, it is convenient to perform another canonical transformation to the moving frame, associated with the motion along the resonant orbit [82]

$$\Theta = \theta - \omega t, \quad (\text{A.7})$$

$$\mathcal{I} = I. \quad (\text{A.8})$$

In that frame, the Hamiltonian of the system takes the form

$$H = \frac{1}{2}(3\pi \mathcal{I})^{2/3} - \omega \mathcal{I} + \sum_{n,m} h_n(\mathcal{I}) F_m e^{in\Theta} e^{i(n+m)\omega t}. \quad (\text{A.9})$$

Now, if we are interested in the motion close to the resonant orbit, i.e. when $\mathcal{I} \approx I_0$, both $\mathcal{I}$ and Θ are changing slowly if the perturbation is weak. The only fast variable is time and we can significantly simplify the description by averaging the Hamiltonian (A.9) over time [82] that results in

$$H_{\text{eff}} = \frac{P^2}{2m_{\text{eff}}} - V_0 \cos(s\Theta) - V_1 \cos(s\Theta/2) - V_2(\Theta), \quad (\text{A.10})$$

which is the effective Hamiltonian (2) in the paper. In (A.10)

$$m_{\text{eff}} = -\frac{\pi^2}{\omega^4}, \quad (\text{A.11})$$

$$V_0 = \lambda \frac{(-1)^s}{s^2 \omega^2}, \quad (\text{A.12})$$

$$V_1 = \lambda_1 \frac{4(-1)^{s/2}}{s^2 \omega^2}, \quad (\text{A.13})$$

$$V_2(\Theta) = \sum_n \frac{(-1)^n}{n^2 \omega^2} f_{-n} e^{in\Theta}, \quad (\text{A.14})$$

and we have performed Taylor expansion around $P = \mathcal{I} - I_0 \approx 0$. The validity of the effective Hamiltonian (A.10) can be easily tested by comparing the exact stroboscopic picture of the system phase space with the phase space portrait generated by the Hamiltonian (A.10). It is done in figure A1 which demonstrates that the secular approach captures quantitatively the dynamics of the system.

A2. Quantum effective Hamiltonian

So far we have performed the classical analysis. In order to switch to the effective quantum description we can either perform a quantization of the classical effective Hamiltonian (A.10) or apply a quantum version of the secular approximation for the Hamiltonian (A.1) [84]. Let us first discuss the former approach. Classical equations of motion possess the scaling symmetry which implies that by a proper rescaling of the parameters and the dynamical variables of the system we obtain the same behavior as presented in figure A1 but around arbitrary value of I_0 [83]. That is, when we redefine $\omega' = I_0^{-1/3} \omega$ and $\lambda' = \lambda$ we can use the results presented in figure A1 if we rescale $p' = I_0^{1/3} p$, $x' = I_0^{2/3} x$ and $t' = I_0^{1/3} t$. In the quantum description the scaling symmetry is broken because the Planck constant sets a scale in the phase space

$$[x, p] = i \quad \Rightarrow \quad [x', p'] = \frac{i}{I_0}. \quad (\text{A.15})$$

From (A.15) we see that $1/I_0$ plays a role of the effective Planck constant. Thus, for $I_0 \gg 1$, quantization of the effective Hamiltonian (A.10), obtained in the action-angle variables, i.e. when $P \rightarrow -i \frac{\partial}{\partial \Theta}$, provides perfect quantum description of the resonant behavior of the system. The same quantum results can be obtained by applying the quantum secular approach [84] which yields

$$\langle n' | H_{\text{eff}} | n \rangle = (E_n - n\omega) \delta_{nn'} + \langle n' | x | n \rangle F_{n-n'}, \quad (\text{A.16})$$

where $|n\rangle$'s are eigenstates of the unperturbed system and $2^{1/3} E_n$ are zeros of the Airy function [83] and F_n are Fourier components of the time periodic function in (A.3). Equation (A.16) has been obtained by switching to the moving frame, with the help of the unitary transformation $\hat{U} = e^{i\hat{n}\omega t}$, and by averaging the resulting Hamiltonian over time [84].

A3. Crystalline structure in time

For $f(t) = 0$, and consequently for $V_2(\Theta) = 0$ in (A.10), and for $s \gg 1$, corresponding classically to the $s:1$ resonance between the motion of the particle in the gravitational field and mirror oscillations, the effective Hamiltonian becomes identical to a solid state system with an electron moving in a periodic potential in space with periodic boundary conditions. In the quantum description, eigenstates of (A.10) are Bloch waves $e^{ik\Theta} u_n(\Theta)$ where k is a quasi-momentum, n denotes a number of an energy band and $u_n(\Theta + 2\pi/s) = u_n(\Theta)$. Such a crystalline structure that we observe in the frame moving along the resonant orbit is reproduced in the time domain when we return to the laboratory frame [35, 36]. Indeed, when we locate a detector in the laboratory frame close to the resonant orbit (i.e. at a space point where $I \approx I_0$ and $\theta = \text{const}$), probability for clicking of the detector as a function of time reproduces periodic behavior versus Θ that we obtain in the moving frame with the help of the effective Hamiltonian (A.10). This is the crucial property of the system we consider here and it is related to the fact that the transformation between the laboratory frame and the moving frame is linear in time, see (A.7). Thus, for $\theta = \text{const}$ we obtain the same behavior in time as versus Θ in the moving frame, $e^{ik(\theta - \omega t)} u_n(\theta - \omega t) = e^{ik\Theta} u_n(\Theta)$. We would like to stress that such a crystalline structure in time is not a result of the presence of any potential periodic in space. It emerges in the dynamics of the system due to the resonant driving and a high resonance order s .

A4. Tight-binding approximation

Assume that $V_2(\Theta) = 0$ in (A.10). If we are interested in the first energy band of the effective Hamiltonian (A.10), the description of the system can still be simplified. Suppose that $w_i(\Theta)$ are Wannier functions corresponding to the first energy band of (A.10). Then, expanding the wavefunction of a particle in the Wannier function basis, $\psi(\Theta) = \sum_{i=1}^s a_i w_i(\Theta)$, we obtain the tight-binding Hamiltonian, see (3) in the paper

$$H_{\text{eff}} \approx -\frac{1}{2} \sum_{i=1}^s J_i (a_{i+1}^* a_i + \text{c.c.}), \quad (\text{A.17})$$

where tunneling amplitudes $J_i = -2 \int_0^{2\pi} d\Theta w_{i+1}^* H_{\text{eff}} w_i$ describe tunneling of a particle between neighboring sites of the potential in (A.10). There are also longer range tunnelings but they are orders of magnitude weaker for the parameters we consider in the paper. For $\lambda_1 \neq 0$ when $J = J_{2i-1}$, $J' = J_{2i}$ and $J \neq J'$, the Hamiltonian (A.17) describes a particle in a Bravais lattice with a two-point basis. Actually it is the SSH model [86] which can possess topological properties.

The tight-binding Hamiltonian (A.17) describes a particle in the Wannier function basis. In the laboratory frame this is the basis of localized wavepackets which evolve periodically along the resonant orbit, i.e. in the Cartesian coordinate frame $w_i(x, t + 2\pi/\omega) = w_i(x, t)$. It shows that condensed matter physics in the time domain we consider here can be described by a solid state model but in the time-periodic basis.

A5. Many-body problem

Having introduced a tight-binding model for a single particle problem, we can easily switch to the many-body case. Indeed, assume that we have N bosons which interact via Dirac-delta potential $g_0 \delta(x)$ and we restrict to the resonant Hilbert subspace of the system. That is, we restrict to the Hilbert subspace spanned by Fock states $|n_1, \dots, n_s\rangle$ where n_i denotes a number of particles that occupy a Wannier state $w_i(x, t)$. Then, the effective many-body Hamiltonian takes the form of the Bose–Hubbard model [35]

$$\hat{H} = -\frac{1}{2} \sum_i (J_i \hat{a}_{i+1}^\dagger \hat{a}_i + \text{h.c.}) + \frac{1}{2} \sum_{i,j=1}^s U_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i, \quad (\text{A.18})$$

where $\hat{a}_i$ are standard bosonic annihilation operators and

$$\begin{aligned} U_{ij} &= 2 \int_0^{2\pi/\omega} dt \int_0^\infty dx g_0 |w_i(x, t)|^2 |w_j(x, t)|^2, \quad \text{for } i \neq j, \\ U_{ii} &= \int_0^{2\pi/\omega} dt \int_0^\infty dx g_0 |w_i(x, t)|^4, \end{aligned} \quad (\text{A.19})$$

are coefficients which describe interactions between particles. If n_i bosons occupy a given localized Wannier-like wavepacket $w_i(x, t)$ they interact that is related to the on-site interactions in (A.18). However, bosons occupying different localized wavepackets $w_i(x, t)$ and $w_j(x, t)$ also interact because in the course of time evolution along the resonant orbit the wavepackets pass each other. That leads to effective long-range interactions in the Bose–Hubbard Hamiltonian (A.18). Moreover, the long-range interaction can be engineered if the s -wave scattering length of atoms is periodically modulated in time and depends on a position in space, i.e. $g_0 = g_0(x, t)$.

The effective Bose–Hubbard Hamiltonian is valid if the maximal interaction energy per particle is much smaller than the energy gap ΔE between the first and second energy bands of the single-particle Hamiltonian (A.10). In the paper we assume that the interaction energy is larger than the width of the first energy band (which is of the order of J) but much smaller than ΔE . There is no problem to fulfill this condition because $\Delta E \gg J$.

ORCID iDs

Alexandre Dauphin  <https://orcid.org/0000-0003-4996-2561>

Jakub Zakrzewski  <https://orcid.org/0000-0003-0998-9460>

References

- [1] Wilczek F 2012 *Phys. Rev. Lett.* **109** 160401
- [2] Bruno P 2013 *Phys. Rev. Lett.* **111** 070402
- [3] Watanabe H and Oshikawa M 2015 *Phys. Rev. Lett.* **114** 251603
- [4] Syrwid A, Zakrzewski J and Sacha K 2017 *Phys. Rev. Lett.* **119** 250602
- [5] Sacha K and Zakrzewski J 2018 *Rep. Prog. Phys.* **81** 016401
- [6] Shapere A and Wilczek F 2012 *Phys. Rev. Lett.* **109** 160402
- [7] Ghosh S 2014 *Physica A* **407** 245–51
- [8] Yao N Y, Nayak C, Balents L and Zaletel M P 2018 arXiv:1801.02628
- [9] Das P, Pan S, Ghosh S and Pal P 2018 *Phys. Rev. D* **98** 024004
- [10] Alvarez P, Canfora F, Dimakis N and Paliathanasis A 2017 *Phys. Lett. B* **773** 401–7

- [11] Avilés L, Canfora F, Dimakis N and Hidalgo D 2017 *Phys. Rev. D* **96** 125005
- [12] Sacha K 2015 *Phys. Rev. A* **91** 033617
- [13] Khemani V, Lazarides A, Moessner R and Sondhi S L 2016 *Phys. Rev. Lett.* **116** 250401
- [14] Else D V, Bauer B and Nayak C 2016 *Phys. Rev. Lett.* **117** 090402
- [15] Yao N Y, Potter A C, Potirniche I D and Vishwanath A 2017 *Phys. Rev. Lett.* **118** 030401
- [16] Lazarides A and Moessner R 2017 *Phys. Rev. B* **95** 195135
- [17] Russomanno A, Iemini F, Dalmonte M and Fazio R 2017 *Phys. Rev. B* **95** 214307
- [18] Zeng T S and Sheng D N 2017 *Phys. Rev. B* **96** 094202
- [19] Nakatsugawa K, Fujii T and Tanda S 2017 *Phys. Rev. B* **96** 094308
- [20] Ho W W, Choi S, Lukin M D and Abanin D A 2017 *Phys. Rev. Lett.* **119** 010602
- [21] Huang B, Wu Y H and Liu W V 2018 *Phys. Rev. Lett.* **120** 110603
- [22] Gong Z, Hamazaki R and Ueda M 2018 *Phys. Rev. Lett.* **120** 040404
- [23] Wang R R W, Xing B, Carlo G G and Poletti D 2018 *Phys. Rev. E* **97** 020202
- [24] Bomantara R W and Gong J 2018 *Phys. Rev. Lett.* **120** 230405
- [25] Autti S, Eltsov V B and Volovik G E 2018 *Phys. Rev. Lett.* **120** 215301
- [26] Kosior A and Sacha K 2018 *Phys. Rev. A* **97** 053621
- [27] Mizuta K, Takasan K, Nakagawa M and Kawakami N 2018 *Phys. Rev. Lett.* **121** 093001
- [28] Giergiel K, Kosior A, Hannaford P and Sacha K 2018 *Phys. Rev. A* **98** 013613
- [29] Kosior A, Syrwid A and Sacha K 2018 *Phys. Rev. A* **98** 023612
- [30] Zhang J *et al* 2017 *Nature* **543** 217–20
- [31] Choi S *et al* 2017 *Nature* **543** 221–5
- [32] Pal S, Nishad N, Mahesh T S and Sreejith G J 2018 *Phys. Rev. Lett.* **120** 180602
- [33] Rovny J, Blum R L and Barrett S E 2018 *Phys. Rev. Lett.* **120** 180603
- [34] Rovny J, Blum R L and Barrett S E 2018 *Phys. Rev. B* **97** 184301
- [35] Sacha K 2015 *Sci. Rep.* **5** 10787
- [36] Sacha K and Delande D 2016 *Phys. Rev. A* **94** 023633
- [37] Giergiel K and Sacha K 2017 *Phys. Rev. A* **95** 063402
- [38] Mierzejewski M, Giergiel K and Sacha K 2017 *Phys. Rev. B* **96** 140201
- [39] Delande D, Morales-Molina L and Sacha K 2017 *Phys. Rev. Lett.* **119** 230404
- [40] Giergiel K, Miroszewski A and Sacha K 2018 *Phys. Rev. Lett.* **120** 140401
- [41] Guo L, Marthaler M and Schön G 2013 *Phys. Rev. Lett.* **111** 205303
- [42] Guo L and Marthaler M 2016 *New J. Phys.* **18** 023006
- [43] Guo L, Liu M and Marthaler M 2016 *Phys. Rev. A* **93** 053616
- [44] Liang P, Marthaler M and Guo L 2018 *New J. Phys.* **20** 023043
- [45] Senthil T 2015 *Annu. Rev. Condens. Matter Phys.* **6** 299
- [46] Przysiężna A, Dutta O and Zakrzewski J 2015 *New J. Phys.* **17** 013018
- [47] Biedroń K, Dutta O and Zakrzewski J 2016 *Phys. Rev. A* **93** 033631
- [48] Cardano F *et al* 2017 *Nat. Commun.* **8** 15516
- [49] Maffei M, Dauphin A, Cardano F, Lewenstein M and Massignan P 2018 *New J. Phys.* **20** 013023
- [50] Kitagawa T, Berg E, Rudner M and Demler E 2010 *Phys. Rev. B* **82** 235114
- [51] von Keyserlingk C W and Sondhi S L 2016 *Phys. Rev. B* **93** 245145
- [52] Else D V and Nayak C 2016 *Phys. Rev. B* **93** 201103
- [53] Potter A C, Morimoto T and Vishwanath A 2016 *Phys. Rev. X* **6** 041001
- [54] Roy R and Harper F 2016 *Phys. Rev. B* **94** 125105
- [55] Xu S and Wu C 2018 *Phys. Rev. Lett.* **120** 096401
- [56] Haldane F D M 1983 *Phys. Lett. A* **93** 464
- [57] Haldane F D M 1983 *Phys. Rev. Lett.* **50** 1153
- [58] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045–67
- [59] Goldman N, Juzeliunas G, Öhberg P and Spielman I B 2014 *Rep. Prog. Phys.* **77** 126401
- [60] Ozawa T *et al* 2019 *Rev. Mod. Phys.* **91** 015006
- [61] Lewenstein M, Sanpera A and Ahufinger V 2007 *Ultracold Atoms in Optical Lattices: Simulating Quantum Many Body Physics* (Oxford: Oxford University Press)
- [62] Kitagawa T, Broome M a, Fedrizzi A, Rudner M S, Berg E, Kassal I, Aspuru-Guzik A, Demler E and White A G 2012 *Nat. Commun.* **3** 882
- [63] Meier Eric J, An Fangzhao A and Gadway B 2016 *Nat. Commun.* **7** 13986
- [64] Atala M, Aidelburger M, Barreiro J T, Abanin D, Kitagawa T, Demler E and Bloch I 2013 *Nat. Phys.* **9** 79
- [65] St-Jean P, Goblot V, Galopin E, Lemaître A, Ozawa T, Gratiet L L, Sagnes I, Bloch J and Amo A 2017 *Nat. Photon.* **11** 651
- [66] Meier E J, An F A, Dauphin A, Maffei M, Massignan P, Hughes T L and Gadway B 2018 arXiv:1802.02109
- [67] de Léséleuc S, Lienhard V, Scholl P, Barredo D, Weber S, Lang N, Büchler H P, Lahaye T and Browaeys A 2018 arXiv:1810.13286
- [68] Aidelburger M, Lohse M, Schweizer C, Atala M, Barreiro J T, Nascimbène S, Cooper N R, Bloch I and Goldman N 2015 *Nat. Phys.* **11** 162
- [69] Stuhl B K, Lu H I, Ayccock L M, Genkina D and Spielman I B 2015 *Science* **349** 1514–8
- [70] Tarnowski M, Nur Ünal F, Fläschner N, Rem B S, Eckardt A, Sengstock K and Weitenberg C 2019 *Nat. Commun.* **10** 1728
- [71] Lohse M, Schweizer C, Price H M, Zilberberg O and Bloch I 2018 *Nature* **553** 55
- [72] Zilberberg O, Huang S, Guglielmon J, Wang M, Chen K P, Kraus Y E and Rechtsman M C 2018 *Nature* **553** 59
- [73] Steane A, Szriftgiser P, Desbiolles P and Dalibard J 1995 *Phys. Rev. Lett.* **74** 4972–5
- [74] Roach T M, Abele H, Boshier M G, Grossman H L, Zetie K P and Hinds E A 1995 *Phys. Rev. Lett.* **75** 629–32
- [75] Sidorov A I, McLean R J, Rowlands W J, Lau D C, Murphy J E, Walkiewicz M, Opat G I and Hannaford P 1996 *Quantum Semiclass. Opt.: J. Eur. Opt. Soc. B* **8** 713
- [76] Westbrook N *et al* 1998 *Phys. Scr.* **1998** 7
- [77] Lau D C, Sidorov A I, Opat G I, McLean R J, Rowlands W J and Hannaford P 1999 *Eur. Phys. J. D* **5** 193–9
- [78] Bongs K, Burger S, Birkel G, Sengstock K, Ertmer W, Rzążewski K, Sanpera A and Lewenstein M 1999 *Phys. Rev. Lett.* **83** 3577–80
- [79] Sidorov A, McLean R, Scharnberg F, Gough D, Davis T, Sexton B, Opat G and Hannaford P 2002 *Acta Phys. Pol. B* **33** 2137

- [80] Fiutowski J, Bartoszek-Bober D, Dohnalik T and Kawalec T 2013 *Opt. Commun.* **297** 59
- [81] Kawalec T, Bartoszek-Bober D, Panaś R, Fiutowski J, Pławecka A and Rubahn H G 2014 *Opt. Lett.* **39** 2932–5
- [82] Lichtenberg A and Lieberman M 1992 *Regular and Chaotic Dynamics Applied Mathematical Sciences* (Berlin: Springer)
- [83] Buchleitner A, Delande D and Zakrzewski J 2002 *Phys. Rep.* **368** 409–547
- [84] Berman G and Zaslavsky G 1977 *Phys. Lett. A* **61** 295–6
- [85] Dutta O, Gajda M, Hauke P, Lewenstein M, Lühmann D-S, Malomed B A, Sowiński T and Zakrzewski J 2015 *Rep. Prog. Phys.* **78** 066001
- [86] Su W P, Schrieffer J R and Heeger A J 1979 *Phys. Rev. Lett.* **42** 1698–701
- [87] Chiu C K, Teo J C Y, Schnyder A P and Ryu S 2016 *Rev. Mod. Phys.* **88** 035005
- [88] Dalla Torre E G, Berg E and Altman E 2006 *Phys. Rev. Lett.* **97** 260401
- [89] Rossini D and Fazio R 2012 *New J. Phys.* **14** 065012
- [90] Ejima S and Fehske H 2015 *J. Phys.: Conf. Ser.* **592** 012134
- [91] Biedroń K, Łacki M and Zakrzewski J 2018 *Phys. Rev. B* **97** 245102
- [92] Kennedy T and Tasaki H 1992 *Phys. Rev. B* **45** 304–7
- [93] Anisimovas E, Žlabys G, Anderson B M, Juzeliūnas G and Eckardt A 2015 *Phys. Rev. B* **91** 245135
- [94] Mark M J, Meinert F, Lauber K and Nägerl H C 2018 *SciPost Phys.* **5** 55
- [95] Lustig E, Sharabi Y and Segev M 2018 *Optica* **5** 1390

Discrete time quasicrystals

Krzysztof Giergiel,¹ Arkadiusz Kuroś,¹ and Krzysztof Sacha^{1,2}¹*Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*²*Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

(Received 11 July 2018; revised manuscript received 3 December 2018; published 11 June 2019)

Between space crystals and amorphous materials there exists a third class of aperiodic structures which lack translational symmetry but reveal long-range order. They are dubbed quasicrystals and their formation, similar to the formation of space crystals, is related to spontaneous breaking of translational symmetry of underlying Hamiltonians. Here, we investigate spontaneous emergence of quasicrystals in periodically driven systems. We consider a quantum many-body system which is driven by a harmonically oscillating force and show that interactions between particles result in spontaneous self-reorganization of the motion of a quantum many-body system and in the formation of a quasicrystal structure in time.

DOI: [10.1103/PhysRevB.99.220303](https://doi.org/10.1103/PhysRevB.99.220303)

Quasicrystals are related to spatial structures which cannot be reproduced by translation of an elementary cell but reveal long-range order [1–3]. Quasicrystals are a subject of research in solid state physics but also in optics [4–7] and ultracold atomic gases [8].

Recently research of crystalline structures has migrated to the time domain [9] (for phase-space crystals see Refs. [10–13]). Indeed, a quantum many-body system can spontaneously self-organize its motion and start moving periodically forming a crystalline structure in the time domain. While the first idea of such time crystals turned out to be impossible for the realization [14–17] another type of spontaneous formation of crystalline structures in time was proposed. These are the so-called discrete time crystals that are periodically driven quantum many-body systems which break spontaneously discrete time translation symmetry of Hamiltonians and start moving with a period different from the driving period [18–21]. Discrete time crystals have already been realized in laboratories [22–26] and they draw considerable attention in the literature [27–47] (see also Refs. [48–53] for a classical version of time crystals). In the field of time crystals, quasicrystal structures have been investigated in classical systems [54], quantum systems [40,55,56], and in an experiment on magnon condensation [57]. In Refs. [58,59] quasicrystal response of systems which are driven quasiperiodically in time was demonstrated. Quasiperiodic response of a periodically driven many-body system was analyzed in Ref. [60] but with no spontaneous time translational symmetry breaking process involved. In the present Rapid Communication we analyze how a quasicrystal structure forms due to spontaneous breaking of discrete time translation symmetry of a many-body time-periodic Hamiltonian.

One-dimensional (1D) quasicrystal sequence can be generated by a cut of a square lattice with the help of a line whose gradient is an irrational number [61–63]. For the Fibonacci quasicrystal the gradient is the golden ratio and the successive cuts of vertical and horizontal lines of the

square lattice produce a sequence $LRLRLRL\ldots$ of two elementary cells which we denote by L and R , see Fig. 1. The sequence corresponds to a quasicrystal structure where there is no translation symmetry but two elementary cells are not distributed randomly so that the sequence reveals long-range order [1]. A finite fragment of the Fibonacci quasicrystal sequence can be obtained by cutting the square lattice with a line whose gradient is a rational number that approximates the golden ratio, see Fig. 1. In the following we show how any finite fragment of the Fibonacci quasicrystal structure can spontaneously emerge in the time evolution of a periodically driven many-body system if interactions between particles are sufficiently strong.

We focus on ultracold atoms bouncing between two orthogonal harmonically oscillating mirrors in a 2D model. Such a system can be realized experimentally [64] (for the stationary mirror experiments see Refs. [65–72]). A single atom bouncing between the mirrors is described, in the frame oscillating with the mirrors [73,74], by the Hamiltonian [75]

$$H = \frac{p_x^2 + p_y^2}{2} + x + y + \lambda_{x,y} \cos(\omega t + \Delta\phi) + \lambda_y y \cos(\omega t), \quad (1)$$

where ω is the frequency of the mirrors' oscillations, $\Delta\phi$ the relative phase, and $\lambda_{x,y}$ the amplitudes of the oscillations. The mirrors are located at $x = 0$ and at $y = 0$ and the gravitational force $\vec{F}_g$ points in the $-(\mathbf{x} + \mathbf{y})$ direction, see inset of Fig. 1. We assume that in the many-body case, N bosons are bouncing between the mirrors and interact via Dirac-delta potential $g_0\delta(\mathbf{r})$ [76]. Such contact interactions are determined by the s-wave scattering length of atoms which is assumed to be negative $g_0 < 0$. The system is periodically driven, thus, we may look for a kind of stationary states which evolve periodically in time. They are eigenstates of the Floquet Hamiltonian $\hat{H}(t) = \hat{H} - i\partial_t$, where $\hat{H}$ is a many-body version of (1) with the contact interactions between

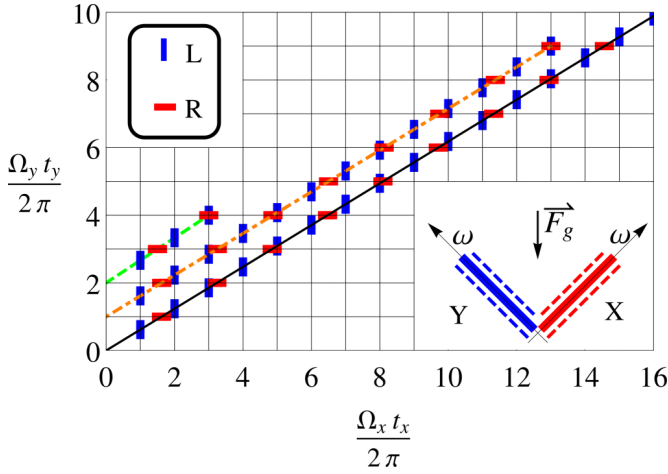


FIG. 1. Generation of the one-dimensional Fibonacci quasicrystal. The solid black line cuts the square lattice. The tangent of the angle that the line forms with the vertical axis is equal to the golden ratio. Consecutive cuts of the line with the vertical (L) and horizontal (R) lines of the lattice form the Fibonacci quasicrystal sequence, *LRLRLRL*... Dashed green and dotted-dashed orange lines correspond to rational approximation of the golden ratio, $3/2$ and $13/8$, respectively. In the case of ultracold atoms bouncing between two mirrors which oscillate with frequency ω (see the schematic plot in the inset where $\vec{F}_g$ denotes the gravitational force), the green and orange lines are related to spontaneous formation of finite fragments of the Fibonacci quasicrystal in the time domain with $\Omega_x/\Omega_y = s_y/s_x = 3/2$ and $13/8$, respectively. That is, bounces of atoms off the left *L* and right *R* mirrors form a sequence of events that reproduces a fragment of the Fibonacci quasicrystal of length $s_x + s_y$. Parameters Ω_x and Ω_y are frequencies of bouncing of atoms off the left and right mirrors, respectively, see text. The axes of the main figure can be considered as two independent time axes [54], $t_{x,y}$, related to periodic motion along the x and y directions.

particles included, see Ref. [73]. The corresponding eigenvalues are called quasienergies of the system [75,77]. The discrete time translation symmetry of the time-periodic Hamiltonian, $\hat{H}(t + 2\pi/\omega) = \hat{H}(t)$, implies that all Floquet eigenstates must evolve with the driving period $2\pi/\omega$. In the following we show that in the limit when the number of particles $N \rightarrow \infty$ but $g_0 N = \text{const.}$ [78], there are subspaces of the Hilbert space of the system where low-lying quasienergy eigenstates are fragile because they form macroscopic superposition. Consequently even an infinitesimally weak perturbation, e.g., a measurement of a position of one atom, is sufficient to induce collapse of the many-body state to one of the superimposed states. It results in breaking of the discrete time translation symmetry of the Hamiltonian [18]. Interestingly an evolving symmetry broken state can reveal a sequence of events (bounces of atoms off the left *L* and right *R* mirrors) which forms a finite fragment of the Fibonacci quasicrystal in time.

Let us start with the single-particle problem (1) which consists of the independent motion along x and y directions. We are interested in a resonant driving of the system, i.e., the frequencies Ω_x and Ω_y of the unperturbed particle motion along the x and y directions fulfill $s_x \Omega_x = \omega$ and $s_y \Omega_y = \omega$ with integer s_x and s_y . The description of a resonantly

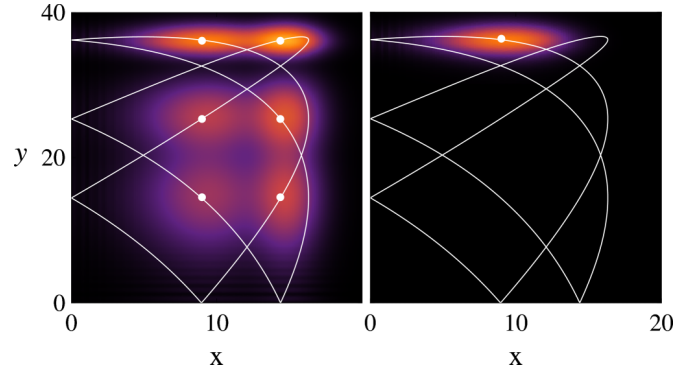


FIG. 2. Density of atoms bouncing between two orthogonal oscillating mirrors at $t = 2\pi/3\omega$. The left (*L*) mirror is located at $x = 0$ and the right (*R*) mirror at $y = 0$ and the gravitational force $\vec{F}_g$ points in the $-(x+y)$ direction, see inset of Fig. 1. Left panel is related to the symmetry preserving state which evolves periodically with the driving period $2\pi/\omega$ —the left and right mirrors are visited by atoms alternately: *LRLRLR*. The presented density consists of $s_x s_y$ localized Wannier-like wavepackets ($s_x = 2$, $s_y = 3$). The trajectory of the Wannier wavepackets are moving along is drawn in the panels. Right panel corresponds to a symmetry broken state where interactions between atoms result in spontaneous breaking of discrete time translation symmetry of the Hamiltonian and emergence of a quasicrystal structure in time. Atoms are visiting the left and right mirrors in an order that matches the sequence *LRLRL* i.e. a finite fragment of the Fibonacci quasicrystal is reproduced because the golden ratio gradient of the line in Fig. 1 is approximated by the rational number $\Omega_x/\Omega_y = s_y/s_x = 3/2$. The parameters of the system are: $\lambda_x = 0.094$, $\lambda_y = 0.030$, $\omega = 1.1$, $\Delta\phi = 2\pi/3$, $g_0 N = 0$ (left) and $g_0 N = -0.022$ (right). The latter results in $U_{ii} N/J = -81$, with $J = 4.8 \times 10^{-6}$, in the Hamiltonian (2) that describes an effective $s_x \times s_y$ lattice. The results are obtained within the quantum secular approach [79].

driven particle can be reduced to an effective tight-binding Hamiltonian [28,73,75,80,81]. When we switch from a single particle to many bosons resonantly driven, the single-particle tight-binding Hamiltonian becomes the Bose-Hubbard Hamiltonian,

$$\hat{\mathcal{H}}_{\text{eff}} = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_{i,j} U_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_i, \quad (2)$$

which is the many-body Floquet Hamiltonian written in a basis of time-periodic functions $W_{i=(i_x, i_y)}(\mathbf{r}, t) = w_{i_x}(x, t)w_{i_y}(y, t)$ which are localized wave packets propagating along the classical resonant orbit with the period $T = s_x s_y 2\pi/\omega$ and which play a role of Wannier functions known in condensed matter physics [82], see Fig. 2. There are $s_x s_y$ Wannier functions W_i which are products of localized wave packets $w_{i_x}(x, t)$ and $w_{i_y}(y, t)$ moving along the x and y directions with the periods $2\pi/\Omega_x$ and $2\pi/\Omega_y$, respectively. In (2), $\hat{a}_i$'s are the standard bosonic annihilation operators, the nearest neighbor tunneling amplitudes $J_{ij} = -(2/T) \int_0^T dt \int d^2 \mathbf{r} W_i^*(\mathbf{r}, t) [H - i\partial_t] W_j(\mathbf{r}, t)$, and the coefficients of the effective interactions $U_{ij} = (2/T) \int_0^T dt \int d^2 \mathbf{r} g_0 |W_i|^2 |W_j|^2$ for $i \neq j$ and similar U_{ii} but by factor two smaller [28,40,73]. In the present Rapid Communication we choose the amplitudes of the mirrors' oscillations,

λ_x and λ_y , so that the resulting amplitudes for nearest neighbor tunnelings along the x and y directions are the same, $J \equiv J_{ij}$. Typically, the coefficient for the on-site interactions U_{ii} is at least an order of magnitude larger than U_{ij} for long-range interactions ($i \neq j$).

To conclude this part, the description of the resonantly driven many-body system is reduced, in the time-periodic basis $W_i(\mathbf{r}, t)$, to the Bose-Hubbard Hamiltonian (2) [73,83]. The resonant driving is related to nonlinear classical resonances where a particle cannot absorb an unlimited amount of energy because transfer of the energy changes a period of motion of the system, a particle goes out of the resonance and the transfer stops [73].

For negligible interactions between particles the ground state of $\hat{H}_{\text{eff}}$ is a Bose-Einstein condensate $\Psi_0(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = \prod_{j=1}^N \psi(\mathbf{r}_j, t)$, i.e., all atoms occupy a condensate wave function $\psi(\mathbf{r}, t) \propto \sum_i W_i(\mathbf{r}, t)$ which evolves with the driving period $2\pi/\omega$. Indeed, despite the fact that each W_i evolves with the period $T = s_x s_y 2\pi/\omega$, after each period $2\pi/\omega$, the Wannier wave functions W_i exchange their positions so that the condensate wave function $\psi(\mathbf{r}, t)$ propagates with the driving period, see Fig. 2. When the interactions between atoms are attractive and sufficiently strong it is energetically favorable to group all atoms in a single localized wave packet $W_i(\mathbf{r}, t)$ [18]. Then, we expect the ground state of $\hat{H}_{\text{eff}}$ to be of the form $\Psi_0 = \prod_{j=1}^N W_i(\mathbf{r}_j, t)$ where i can be arbitrary. However, such a state cannot be a Floquet eigenstate of the system because it evolves with the period $T = s_x s_y 2\pi/\omega$ while the discrete time translation symmetry of the Hamiltonian requires that all Floquet eigenstates must evolve with the period of the driving $2\pi/\omega$. In order to reconcile the energy and symmetry requirements, the ground state of $\hat{H}_{\text{eff}}$ takes the form $\Psi_0 \propto \sum_i \prod_{j=1}^N W_i(\mathbf{r}_j, t)$ which is a macroscopic superposition of Bose-Einstein condensates [84,85]. However, such a macroscopic superposition is extremely fragile and it is sufficient, e.g., to measure the position of one atom and the ground state collapses to one of the Bose-Einstein condensates which form the macroscopic superposition, $\Psi_0 \rightarrow \Psi \approx \prod_{j=1}^N W_i(\mathbf{r}_j, t)$ [85,86]—which W_i is chosen depends on a result of the measurement. In the limit when $N \rightarrow \infty$ but $U_{ii}N = \text{const.}$, the latter state is robust and evolves with the period $T = s_x s_y 2\pi/\omega$ and thus breaks time translation symmetry of the many-body Hamiltonian [18]. The described scenario is an example of a process of spontaneous breaking of time translation symmetry in the quantum many-body system. Similar spontaneous symmetry breaking phenomenon is not present in [60] because Floquet states are related to single Fock states in the position representation.

In order to describe the system we apply the mean-field approach [18,44,73,87]. The mean-field approximation is valid because the ground state of (2) for negligible interactions and also symmetry broken states, $\Psi \approx \prod_{j=1}^N W_i(\mathbf{r}_j, t)$, in the regime of the quasicrystal formation are Bose-Einstein condensates. The mean-field energy of the system per particle reads $E = -(J/2) \sum_{(i,j)} a_i^* a_j + (N/2) \sum_{ij} U_{ij} |a_i|^2 |a_j|^2$ [44,73] and we are looking for a condensate wave function $\psi(\mathbf{r}, t) = \sum_i a_i W_i(\mathbf{r}, t)$ which minimizes E [78,87]. In the

left panel of Fig. 2 we show such a wave function $\psi(\mathbf{r}, t)$ obtained for negligible interactions and for $\Omega_x = \omega/2$ and $\Omega_y = \omega/3$ (i.e., $s_x = 2$, $s_y = 3$). The wave function $\psi(\mathbf{r}, t)$ is a uniform superposition of $s_x s_y = 6$ localized Wannier wave packets, it evolves with the period $2\pi/\omega$ and describes atoms bouncing alternately off the left (L) and right (R) mirrors. If we plot probabilities for the measurement of atoms close to the left mirror, $\rho_L(t) = \int dy |\psi(x \approx 0, y, t)|^2$, and close to the right mirror, $\rho_R(t) = \int dx |\psi(x, y \approx 0, t)|^2$, we can see that maximal values of $\rho_{L,R}(t)$ appear alternately and form a periodic sequence of events $LRLR \dots$, see Fig. 3. However, if the interactions are sufficiently strong, i.e., $U_{ii}N/J \lesssim -6.5$, the system chooses spontaneously motion with the period $T = s_x s_y 2\pi/\omega$. That is, the mean-field approach shows that the ground state energy is degenerate and the corresponding wave functions are not uniform superposition of W_i . The system prepared in a lowest energy mean-field state breaks discrete time translation symmetry of the many-body Hamiltonian because it evolves with the period different from the driving period. For $U_{ii}N/J \lesssim -25$ the symmetry broken degenerate ground states reduce to $\psi(\mathbf{r}, t) \approx W_i(\mathbf{r}, t)$ with accuracy better than 99%—which W_i is chosen by the system is determined in a spontaneous symmetry breaking process. In Fig. 2 we show an example of such a ground state wave function $\psi(\mathbf{r}, t)$ where a single localized wave packet bouncing between the mirrors is visible. The corresponding probabilities $\rho_{L,R}(t)$ form a sequence of events $LRLLR$, whose length is $s_x + s_y = 5$, which is repeated with the period T , see Fig. 3. The sequence is a fragment of the Fibonacci quasicrystal. The time quasicrystal states predicted by the mean-field approach lives forever. The predictions are valid in the limit when $N \rightarrow \infty$ but $g_0N = \text{const}$ because then the corresponding symmetry preserving eigenstates of the quantum many-body model (2) are degenerate and their superpositions, that form the symmetry-broken states, do not decay [18].

It now becomes clear how to realize conditions where any finite fragment of the Fibonacci quasicrystal emerges due to spontaneous breaking of discrete time translation symmetry of the Hamiltonian: (i) One has to choose a rational number s_y/s_x which approximates the golden ratio and reproduces a given fragment of the Fibonacci quasicrystal sequence when it is taken as the gradient of the line in Fig. 1. (ii) Then, we know which resonant subspace of the periodically driven many-body system is able to realize such a quasicrystal, i.e., the subspace corresponding to the frequencies of unperturbed single-particle motion $\Omega_x = \omega/s_x$ and $\Omega_y = \omega/s_y$. (iii) If the many-body system is prepared in a low-lying eigenstate within this subspace, then either atoms are bouncing off the left and right mirrors in the alternate way (if the interactions are negligible) or the bounces on the mirrors form a sequence of events that reproduces a finite fragment of the Fibonacci quasicrystal (if the interactions are sufficiently strong). In the right panels of Fig. 3 we illustrate these two situations for $s_x = 8$ and $s_y = 13$. In the symmetry preserving case, the probabilities for detection atoms close to the left and right mirrors, $\rho_{L,R}(t)$, show a periodic sequence of maxima $LRLR \dots$. However, when the attractive interactions are sufficiently strong, the discrete time translation symmetry is spontaneously broken and the Fibonacci quasicrystal $LRLRLRL \dots$ is formed [54]. We would like to stress that the quasicrystal structure formed

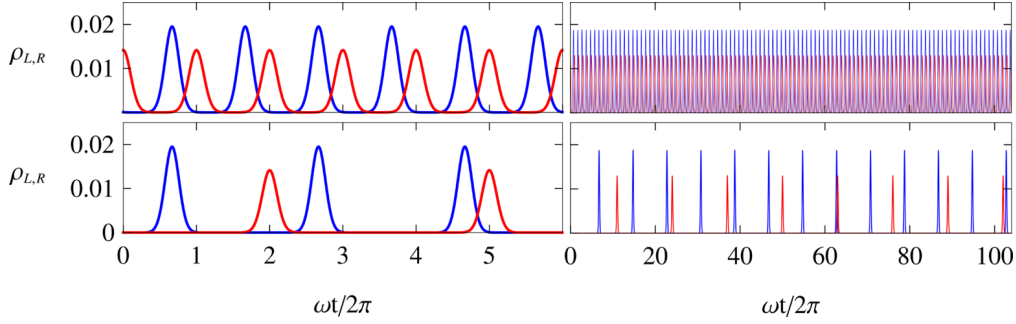


FIG. 3. Scaled probabilities for the detection of atoms close to the left mirror $\rho_L(t)$ (blue lines) and close to the right mirror $\rho_R(t)$ (red lines), where $\rho_L(t) = \int dy |\psi(x \approx 0, y, t)|^2$ and there is an analogous expression for $\rho_R(t)$. Top panels are related to symmetry preserving states while bottom panels to states where the discrete time translation symmetry of the Hamiltonian is spontaneously broken. Left panels correspond to $\Omega_x/\Omega_y = s_y/s_x = 3/2$ while in the right panels such ratios are equal 13/8. Symmetry preserving states form periodic sequences of the elementary cells L and R associated with the alternate appearance of maxima of $\rho_L(t)$ and $\rho_R(t)$. In the symmetry broken case, bounces of atoms off the left and right mirrors form a sequence of events that reproduces a finite fragment (of length $s_x + s_y$) of the Fibonacci quasicrystal which is repeated in the time evolution of the system with the period $T = s_x s_y 2\pi/\omega$. The results shown in the left panels correspond to the same parameters as in Fig. 2, while in the right panels: $\lambda_x = 0.087$, $\lambda_y = 0.026$, $\omega = 1.77$, $\Delta\phi = \pi/2$, $s_y = 13$, $s_x = 8$, $g_0 N = 0$ (top right panel) and $g_0 N = -0.029$ (bottom right panel). The latter results in $U_{\text{ff}} N/J = -80$ and $J = 2.3 \times 10^{-6}$ in the Bose-Hubbard Hamiltonian that describes an effective $s_x \times s_y$ lattice.

by the bouncing atoms is related to the sequence of bounces not to the sequence of time intervals between the bounces—the latter can be different, see Fig. 3. In the experiment, the time intervals which are very small can be disrupted due to imperfections of the motion of the mirrors which can result in defects in the Fibonacci quasicrystal. Long time stability of our phenomenon resulting from the coupling of the system to the subspace complementary to the resonant subspace requires further investigation but we expect that the considered quasicrystal is a prethermal state.

To conclude, quasicrystal structures can emerge in the time domain due to spontaneous breaking of discrete time translation symmetry of the time-periodic many-body Hamiltonian. They can be realized in ultracold atoms bouncing between oscillating atom mirrors if atoms are loaded to a resonant classical orbit. The latter can be done if an atomic Bose-Einstein condensate is prepared in a trap located at a classical turning point of a resonant trajectory and afterwards the trapping potential is turned off at a proper moment of

time [44]—the mirrors can be realized by two blue-detuned repulsive light sheets formed by focusing laser beams with cylindrical lenses. It results in a quantum state where all atoms occupy a single localized Wannier-like wave packet that evolves along a resonant orbit. For sufficiently strong attractive interactions between atoms, the localized atomic wave packet will perform evolution with a quasicrystal structure in time and will not decay. In contrast, for negligible interactions, atoms will tunnel to other localized wave packets evolving along the orbit which indicates decay of the quasicrystal.

We are grateful to Peter Hannaford for his valuable comments. This work was performed with the support of EU via Horizon 2020 FET project QUIC (No. 641122). Support of the National Science Centre, Poland via Projects Symfonia No. 2016/20/W/ST/00314 (K.G.), QuantERA Programme No. 2017/25/Z/ST2/03027 (A.K.) and No. 2016/21/B/ST2/01095 (K.S.) is acknowledged.

-
- [1] C. Janot, *Quasicrystals: A Primer* (Oxford University Press, Oxford, 1994).
 - [2] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, *Phys. Rev. Lett.* **53**, 1951 (1984).
 - [3] D. Levine and P. J. Steinhardt, *Phys. Rev. Lett.* **53**, 2477 (1984).
 - [4] M. Kohmoto, B. Sutherland, and K. Iguchi, *Phys. Rev. Lett.* **58**, 2436 (1987).
 - [5] W. Steurer and D. Sutter-Widmer, *J. Phys. D* **40**, R229 (2007).
 - [6] E. Albuquerque and M. Cottam, *Phys. Rep.* **376**, 225 (2003).
 - [7] V. Z. Valy, N. Ajay, and A. Amit, *Nat. Photon.* **7**, 177 (2013).
 - [8] K. Viebahn, M. Sbroscia, E. Carter, J.-C. Yu, and U. Schneider, *Phys. Rev. Lett.* **122**, 110404 (2019).
 - [9] K. Sacha and J. Zakrzewski, *Rep. Prog. Phys.* **81**, 016401 (2018).
 - [10] L. Guo, M. Marthaler, and G. Schön, *Phys. Rev. Lett.* **111**, 205303 (2013).
 - [11] L. Guo and M. Marthaler, *New J. Phys.* **18**, 023006 (2016).
 - [12] L. Guo, M. Liu, and M. Marthaler, *Phys. Rev. A* **93**, 053616 (2016).
 - [13] L. Pengfei, M. Michael, and L. Guo, *New J. Phys.* **20**, 023043 (2018).
 - [14] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
 - [15] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013).
 - [16] H. Watanabe and M. Oshikawa, *Phys. Rev. Lett.* **114**, 251603 (2015).
 - [17] A. Syrwid, J. Zakrzewski, and K. Sacha, *Phys. Rev. Lett.* **119**, 250602 (2017).
 - [18] K. Sacha, *Phys. Rev. A* **91**, 033617 (2015).
 - [19] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).

- [20] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016).
- [21] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
- [22] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath *et al.*, *Nature (London)* **543**, 217 (2017).
- [23] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani *et al.*, *Nature (London)* **543**, 221 (2017).
- [24] S. Pal, N. Nishad, T. S. Mahesh, and G. J. Sreejith, *Phys. Rev. Lett.* **120**, 180602 (2018).
- [25] J. Rovny, R. L. Blum, and S. E. Barrett, *Phys. Rev. Lett.* **120**, 180603 (2018).
- [26] J. Rovny, R. L. Blum, and S. E. Barrett, *Phys. Rev. B* **97**, 184301 (2018).
- [27] C. Nayak, *Nature (London)* **543**, 185 (2017).
- [28] K. Sacha, *Sci. Rep.* **5**, 10787 (2015).
- [29] D. Delande, L. Morales-Molina, and K. Sacha, *Phys. Rev. Lett.* **119**, 230404 (2017).
- [30] A. Lazarides and R. Moessner, *Phys. Rev. B* **95**, 195135 (2017).
- [31] A. Russomanno, F. Iemini, M. Dalmonte, and R. Fazio, *Phys. Rev. B* **95**, 214307 (2017).
- [32] T.-S. Zeng and D. N. Sheng, *Phys. Rev. B* **96**, 094202 (2017).
- [33] K. Nakatsugawa, T. Fujii, and S. Tanda, *Phys. Rev. B* **96**, 094308 (2017).
- [34] W. W. Ho, S. Choi, M. D. Lukin, and D. A. Abanin, *Phys. Rev. Lett.* **119**, 010602 (2017).
- [35] B. Huang, Y.-H. Wu, and W. V. Liu, *Phys. Rev. Lett.* **120**, 110603 (2018).
- [36] Z. Gong, R. Hamazaki, and M. Ueda, *Phys. Rev. Lett.* **120**, 040404 (2018).
- [37] R. R. W. Wang, B. Xing, G. G. Carlo, and D. Poletti, *Phys. Rev. E* **97**, 020202(R) (2018).
- [38] F. Iemini, A. Russomanno, J. Keeling, M. Schirò, M. Dalmonte, and R. Fazio, *Phys. Rev. Lett.* **121**, 035301 (2018).
- [39] M. Mierzejewski, K. Giergiel, and K. Sacha, *Phys. Rev. B* **96**, 140201(R) (2017).
- [40] K. Giergiel, A. Miroszewski, and K. Sacha, *Phys. Rev. Lett.* **120**, 140401 (2018).
- [41] R. W. Bomantara and J. Gong, *Phys. Rev. Lett.* **120**, 230405 (2018).
- [42] A. Kosior and K. Sacha, *Phys. Rev. A* **97**, 053621 (2018).
- [43] K. Mizuta, K. Takasan, M. Nakagawa, and N. Kawakami, *Phys. Rev. Lett.* **121**, 093001 (2018).
- [44] K. Giergiel, A. Kosior, P. Hannaford, and K. Sacha, *Phys. Rev. A* **98**, 013613 (2018).
- [45] K. Tucker, B. Zhu, R. J. Lewis-Swan, J. Marino, F. Jimenez, J. G. Restrepo, and A. M. Rey, *New J. Phys.* **20**, 123003 (2018).
- [46] A. Kosior, A. Syrwid, and K. Sacha, *Phys. Rev. A* **98**, 023612 (2018).
- [47] W. C. Yu, J. Tangpanitanon, A. W. Glaetzle, D. Jaksch, and D. G. Angelakis, *Phys. Rev. A* **99**, 033618 (2018).
- [48] A. Shapere and F. Wilczek, *Phys. Rev. Lett.* **109**, 160402 (2012).
- [49] S. Ghosh, *Physica A (Amsterdam, Neth.)* **407**, 245 (2014).
- [50] N. Y. Yao, C. Nayak, L. Balents, and M. P. Zaletel, *arXiv:1801.02628*.
- [51] P. Das, S. Pan, S. Ghosh, and P. Pal, *Phys. Rev. D* **98**, 024004 (2018).
- [52] P. Alvarez, F. Canfora, N. Dimakis, and A. Paliathanasis, *Phys. Lett. B* **773**, 401 (2017).
- [53] L. Avilés, F. Canfora, N. Dimakis, and D. Hidalgo, *Phys. Rev. D* **96**, 125005 (2017).
- [54] F. Flicker, *SciPost Phys.* **5**, 1 (2018).
- [55] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *Phys. Rev. Lett.* **109**, 163001 (2012).
- [56] Y. Huang, T. Li, and Z.-q. Yin, *Phys. Rev. A* **97**, 012115 (2018).
- [57] S. Autti, V. B. Eltsov, and G. E. Volovik, *Phys. Rev. Lett.* **120**, 215301 (2018).
- [58] P. T. Dumitrescu, R. Vasseur, and A. C. Potter, *Phys. Rev. Lett.* **120**, 070602 (2018).
- [59] Y. Peng and G. Refael, *Phys. Rev. B* **98**, 220509(R) (2018).
- [60] D. J. Luitz, A. Lazarides, and Y. Bar Lev, *Phys. Rev. B* **97**, 020303(R) (2018).
- [61] M. C. Valsakumar and V. Kumar, *Pramana* **26**, 215 (1986).
- [62] E. Bombieri and J. E. Taylor, *J. Phys. Colloques* **47**, C3-19 (1986).
- [63] Z. Lin, H. Kubo, and M. Goda, *Z. Phys. B* **98**, 111 (1995).
- [64] A. Steane, P. Szriftgiser, P. Desbiolles, and J. Dalibard, *Phys. Rev. Lett.* **74**, 4972 (1995).
- [65] T. M. Roach, H. Abele, M. G. Boshier, H. L. Grossman, K. P. Zetie, and E. A. Hinds, *Phys. Rev. Lett.* **75**, 629 (1995).
- [66] A. I. Sidorov, R. J. McLean, W. J. Rowlands, D. C. Lau, J. E. Murphy, M. Walkiewicz, G. I. Opat, and P. Hannaford, *Quantum Semiclassical Opt.* **8**, 713 (1996).
- [67] N. Westbrook, C. I. Westbrook, A. Landragin, G. Labeyrie, L. Cognet, V. Savalli, G. Horvath, A. Aspect, C. Hendel, K. Moelmer *et al.*, *Phys. Scr.* **T78**, 7 (1998).
- [68] D. C. Lau, A. I. Sidorov, G. I. Opat, R. J. McLean, W. J. Rowlands, and P. Hannaford, *Eur. Phys. J. D* **5**, 193 (1999).
- [69] K. Bongs, S. Burger, G. Birkel, K. Sengstock, W. Ertmer, K. Rzażewski, A. Sanpera, and M. Lewenstein, *Phys. Rev. Lett.* **83**, 3577 (1999).
- [70] A. Sidorov, R. McLean, F. Scharnberg, D. Gough, T. Davis, B. Sexton, G. Opat, and P. Hannaford, *Acta Phys. Pol. B* **33**, 2137 (2002).
- [71] J. Fiutowski, D. Bartoszek-Bober, T. Dohnalik, and T. Kawalec, *Opt. Commun.* **297**, 59 (2013).
- [72] T. Kawalec, D. Bartoszek-Bober, R. Panaś, J. Fiutowski, A. Pławecka, and H.-G. Rubahn, *Opt. Lett.* **39**, 2932 (2014).
- [73] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevB.99.220303> for the detailed description of the resonant motion of atoms bouncing on oscillating atom mirrors and for the description of the approach used in the analysis of the spontaneous emergence of quasicrystals in time. A short discussion of experimental aspects and stability of resonantly driven systems is also included.
- [74] In order to switch from the laboratory frame (where one mirror oscillates like $-\frac{\lambda_x}{\omega^2} \cos(\omega t + \Delta\phi)$ along the x direction and the other like $-\frac{\lambda_y}{\omega^2} \cos \omega t$ along the y direction) to the coordinate frame where the mirrors do not move, the unitary transformation $U_y = e^{iy \frac{\lambda_y}{\omega} \sin \omega t} e^{ip_y \frac{\lambda_y}{\omega^2} \cos \omega t}$, and a similar one for the motion along x , has been applied. We use the gravitational units but assume that the gravitational acceleration is given by $g/\sqrt{2}$.

- [75] A. Buchleitner, D. Delande, and J. Zakrzewski, [Phys. Rep. **368**, 409 \(2002\)](#).
- [76] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, [Rev. Mod. Phys. **82**, 1225 \(2010\)](#).
- [77] J. H. Shirley, [Phys. Rev. **138**, B979 \(1965\)](#).
- [78] E. H. Lieb, R. Seiringer, and J. Yngvason, [Phys. Rev. A **61**, 043602 \(2000\)](#).
- [79] G. Berman and G. Zaslavsky, [Phys. Lett. A **61**, 295 \(1977\)](#).
- [80] A. Lichtenberg and M. Lieberman, *Regular and Chaotic Dynamics*, Applied Mathematical Sciences (Springer-Verlag, New York, 1992).
- [81] K. Sacha and D. Delande, [Phys. Rev. A **94**, 023633 \(2016\)](#).
- [82] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D. S. Lühmann, B. A. Malomed, T. Sowiński, and J. Zakrzewski, [Rep. Prog. Phys. **78**, 066001 \(2015\)](#).
- [83] M. Łacki and J. Zakrzewski, [Phys. Rev. Lett. **110**, 065301 \(2013\)](#).
- [84] P. Ziń, J. Chwedeńczuk, B. Oleś, K. Sacha, and M. Trippenbach, [Europhys. Lett. **83**, 64007 \(2008\)](#).
- [85] B. Oleś, P. Ziń, J. Chwedeńczuk, K. Sacha, and M. Trippenbach, [Laser Physics **20**, 671 \(2010\)](#).
- [86] K. W. Mahmud, J. N. Kutz, and W. P. Reinhardt, [Phys. Rev. A **66**, 063607 \(2002\)](#).
- [87] C. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2002).

Supplemental material: Discrete Time Quasi-Crystals

Krzysztof Giergiel,¹ Arkadiusz Kuroś,¹ and Krzysztof Sacha^{1,2}

¹ *Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński,
ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

² *Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński,
ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland*

In this Supplemental Material we describe details of the description of the many-body system of atoms bouncing resonantly between two orthogonal oscillating atom mirrors in the presence of the gravitational force. We begin with a single-particle problem and then generalize the approach to the many-body case. We also discuss shortly experimental implementation and address the problem of stability of the quasi-crystal structures in time.

I. SINGLE-PARTICLE PROBLEM

We start with a short introduction to the Floquet formalism [1] and then switch to the description of a single-particle system which we are interested in. We show how to obtain an effective Floquet Hamiltonian that describes resonant dynamics of the system.

A. Floquet states

Consider the following time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(x, t) = H(t) \psi(x, t) \quad (1)$$

with Hamiltonian having a discrete time translation symmetry $H(t + T) = H(t)$. According to the Floquet theorem, solution of the equation (1) may be written as a linear combination of the functions of the form

$$\psi_k(x, t) = e^{-i\varepsilon_k t} \phi_k(x, t), \quad (2)$$

where $\phi_k(x, t + T) = \phi_k(x, t)$ are time-periodic with the same period T as the Hamiltonian [1]. Substituting the solution (2) into the Schrödinger equation (1) we obtain

$$\mathcal{H} \phi_k(x, t) = \varepsilon_k \phi_k(x, t), \quad (3)$$

where $\mathcal{H} = H(t) - i \frac{\partial}{\partial t}$ is termed the Floquet Hamiltonian and $\phi_k(x, t)$ are so-called Floquet eigenstates. Since the function $\phi_k(x, t) e^{i \frac{2\pi n}{T} t}$, where n is integer, is also a solution of the eigenequation (3) corresponding to the eigenvalue $\varepsilon_k + n \frac{2\pi}{T}$, the quasi-energy spectrum is periodic with the period $\frac{2\pi}{T}$ and in the description of a system it is actually sufficient to restrict to a fragment of the spectrum, i.e. to a single Floquet zone of the width of $\frac{2\pi}{T}$.

B. Single-particle Hamiltonian

Let us consider an atom bouncing between two orthogonal oscillating atom mirrors which form the angle $\pi/4$ with the gravitational force vector. The Hamiltonian, in the standard gravitational units [2] but with the gravitational acceleration $g \rightarrow g/\sqrt{2}$, reads

$$H = \frac{p_x^2 + p_y^2}{2} + x + y + F_x \left(x + \frac{\lambda_x}{\omega^2} \cos(\omega t + \Delta\phi) \right) + F_y \left(y + \frac{\lambda_y}{\omega^2} \cos(\omega t) \right), \quad (4)$$

where $F_x(x)$ and $F_y(y)$ describe the mirrors, i.e., the profile of the reflecting potentials along x and y directions, respectively. The mirrors oscillate harmonically with the frequency ω around $x = 0$ and $y = 0$ with the amplitudes λ_x/ω^2 and λ_y/ω^2 . Description of the system is more convenient if we switch from the laboratory frame to the frame oscillating with the mirrors. In the classical case it can be done by means of the canonical transformation,

$$\begin{aligned} x' &= x + \frac{\lambda_x}{\omega^2} \cos(\omega t + \Delta\phi), & y' &= y + \frac{\lambda_y}{\omega^2} \cos(\omega t), \\ p'_x &= p_x - \frac{\lambda_x}{\omega} \sin(\omega t + \Delta\phi), & p'_y &= p_y - \frac{\lambda_y}{\omega} \sin(\omega t), \end{aligned} \quad (5)$$

while in the quantum case by the corresponding unitary transformation, i.e. $U_y = e^{iy \frac{\lambda_y}{\omega} \sin \omega t} e^{ip_y \frac{\lambda_y}{\omega^2} \cos \omega t}$ and a similar one for the motion along x . The resulting Hamiltonian is the following

$$H = \frac{p_x'^2 + p_y'^2}{2} + x' + y' + \lambda_x x' \cos(\omega t + \Delta\phi) + \lambda_y y' \cos(\omega t) + F_x(x') + F_y(y'). \quad (6)$$

We assume that the mirrors can be modeled by hard wall potentials located at $x' = 0$ and $y' = 0$ and therefore we may drop the $F_x(x')$ and $F_y(y')$ in (6) keeping in mind that motion of a particle takes place for $x' \geq 0$ and $y' \geq 0$. In the following we also drop primes and the final single-particle Hamiltonian reads

$$H = \frac{p_x^2 + p_y^2}{2} + x + y + \lambda_x x \cos(\omega t + \Delta\phi) + \lambda_y y \cos(\omega t). \quad (7)$$

C. Description of resonant dynamics

Let us start with the classical mechanics. The single-particle problem described by the Hamiltonian (7) consists of independent motion along x and y directions. We are interested in a resonance driving and in order to describe resonant dynamics we perform canonical transformation to the so-called action-angle variables of the unperturbed problem (i.e. when $\lambda_x = \lambda_y = 0$) [3]. In these new canonically conjugate variables, the unperturbed Hamiltonian depends on the new momenta (the actions I_x and I_y) only,

$$\begin{aligned} H_0(I_x, I_y) &= \frac{p_x^2 + p_y^2}{2} + x + y \\ &= \frac{(3\pi)^{2/3}}{2} \left(I_x^{2/3} + I_y^{2/3} \right). \end{aligned} \quad (8)$$

If $\lambda_x = \lambda_y = 0$, the actions are constant of motion ($I_x, I_y = \text{const}$) and the corresponding position variables (the angles θ_x and θ_y) evolve linearly in time, i.e. $\theta_{x,y} = \Omega_{x,y}t + \theta_{x,y}(0)$ where

$$\Omega_x(I_x) = \frac{dH_0(I_x, I_y)}{dI_x}, \quad \Omega_y(I_y) = \frac{dH_0(I_x, I_y)}{dI_y}, \quad (9)$$

are frequencies of an unperturbed periodic bouncing of a particle on the static mirrors. The total Hamiltonian (7) in the action-angle variables takes the form [4]

$$\begin{aligned} H &= H_0(I_x, I_y) + \lambda_x \cos(\omega t + \Delta\phi) \sum_n h_n(I_x) e^{in\theta_x} \\ &\quad + \lambda_y \cos(\omega t) \sum_n h_n(I_y) e^{in\theta_y}, \end{aligned} \quad (10)$$

where $h_0(I_{x,y}) = \left(\frac{\pi I_{x,y}}{\sqrt{3}} \right)^{2/3}$ and $h_n(I_{x,y}) = \frac{(-1)^{n+1}}{n^2} \left(\frac{3I_{x,y}}{\pi^2} \right)^{2/3}$ if $n \neq 0$.

The resonant driving of a particle corresponds to the conditions

$$s_x \Omega_x(I_{x0}) = \omega, \quad s_y \Omega_y(I_{y0}) = \omega, \quad (11)$$

where s_x and s_y are integers and I_{x0} and I_{y0} are resonant values of the actions. In order to obtain an effective Hamiltonian that describes motion of a particle close to a resonant orbit we apply the classical secular approximation [3, 4]. First, we extend the phase space of the system by the time t variable and its canonically conjugate momentum $p_t = -H$ which play a role of additional coordinates. The Hamiltonian in the extended phase space $\mathcal{H} = H + p_t$ is the classical analogue of the quantum Floquet Hamiltonian, where $p_t \rightarrow -i\frac{\partial}{\partial t}$. Next, we perform a canonical transformation to the frame moving along a resonant orbit,

$$\Theta_x = \theta_x - \frac{\omega}{s_x} t, \quad (12)$$

$$\Theta_y = \theta_y - \frac{\omega}{s_y} t, \quad (13)$$

$$P_t = p_t + \frac{\omega I_x}{s_x} + \frac{\omega I_y}{s_y}, \quad (14)$$

which results in

$$\begin{aligned} \mathcal{H} &= H_0(I_x, I_y) - \frac{\omega I_x}{s_x} - \frac{\omega I_y}{s_y} + P_t \\ &\quad + \lambda_x \cos(\omega t + \Delta\phi) \sum_n h_n(I_x) e^{in\Theta_x} e^{in\omega t/s_x} \\ &\quad + \lambda_y \cos(\omega t) \sum_n h_n(I_y) e^{in\Theta_y} e^{in\omega t/s_y}, \end{aligned} \quad (15)$$

and carry out averaging over time keeping all dynamical variables fixed. The latter is allowed because in the moving frame (14) both the actions and the angles and P_t are slowly varying quantities if we are close to the resonant orbit (i.e. $P_x = I_x - I_{x0} \approx 0$ and $P_y = I_y - I_{y0} \approx 0$) and if the time-dependent perturbation is weak, i.e.

$$\begin{aligned} \frac{d\Theta_x}{dt} &= \Omega_x(I_x) - \frac{\omega}{s_x} + \mathcal{O}(\lambda_x) \approx 0, \quad \text{for } I_x \approx I_{x0}, \\ \frac{d\Theta_y}{dt} &= \Omega_y(I_y) - \frac{\omega}{s_y} + \mathcal{O}(\lambda_y) \approx 0, \quad \text{for } I_y \approx I_{y0}. \end{aligned} \quad (16)$$

The resulting effective Hamiltonian reads [3, 4]

$$\begin{aligned} \mathcal{H}_{\text{eff}} &= \frac{P_x^2}{2m_{\text{eff},x}} + V_x \cos(s_x \Theta_x + \Delta\phi) \\ &\quad + \frac{P_y^2}{2m_{\text{eff},y}} + V_y \cos(s_y \Theta_y) + P_t, \end{aligned} \quad (17)$$

where $V_x = \lambda_x h_{-s_x}(I_{x0})$, $m_{\text{eff},x}^{-1} = \frac{d^2 H_0(I_{x0}, I_{y0})}{dI_{x0}^2}$ and similar expressions for V_y and $m_{\text{eff},y}$.

The Hamiltonian (17) has been obtained within the classical approach. In order to switch to the quantum description one has to quantize (17), i.e. $P_{x,y} = -i\frac{\partial}{\partial \Theta_{x,y}}$ and $P_t = -i\frac{\partial}{\partial t}$. The other option is to apply the quantum version of the secular approximation from the very beginning [5] that leads to the same result if we choose $I_{x0} \gg 1$ and $I_{y0} \gg 1$ [2]. The secular Hamiltonian (17) is time-independent which implies that $P_t = \text{constant}$ and can be dropped. Actually in the quantum description, due to the time-periodicity of the system, eigenvalues $k\omega$ of P_t are quantized (i.e. k is integer [4]) that makes the quasi-energy spectrum to repeat itself with the period ω as expected, see Sec. IA. In the following we consider Floquet eigenstates corresponding to $k = 0$.

If we focus on a resonance where $s_{x,y} \gg 1$, the Hamiltonian (17) corresponds to a solid state problem of an electron moving in a two-dimensional space crystal. We will be considering the first energy band of the quantized version of (17) and therefore the description of a resonantly driven particle can still be simplified. Indeed, superposing the Bloch wave eigenfunctions of (17) corresponding to the first energy band we can construct Wannier functions $W_{\mathbf{i}=(i_x, i_y)}(\Theta_x, \Theta_y) = w_{i_x}(\Theta_x) w_{i_y}(\Theta_y)$ localized in different sites of the periodic potential in (17). These Wannier functions in the laboratory frame appear as localized wavepackets moving along a classical resonant orbit with the period $T = s_x s_y 2\pi/\omega$, i.e.

$W_{\mathbf{i}}(x, y, t) = w_{i_x}(x, t)w_{i_y}(y, t)$ where $w_{i_x}(x, t)$ is periodic with the period $s_x 2\pi/\omega$ and $w_{i_y}(y, t)$ is periodic with the period $s_y 2\pi/\omega$. In the basis of the Wannier functions, i.e. when we restrict to wavefunctions of the form $\psi = \sum_{\mathbf{i}} a_{\mathbf{i}} W_{\mathbf{i}}$, the Hamiltonian (17) reads [6, 7]

$$\mathcal{H}_{\text{eff}} \approx -\frac{1}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\mathbf{ij}} a_{\mathbf{i}}^* a_{\mathbf{j}}, \quad (18)$$

where the sum runs over nearest neighbour sites of the potential in (17) and

$$J_{\mathbf{ij}} = -2 \langle W_{\mathbf{i}} | \mathcal{H}_{\text{eff}} | W_{\mathbf{j}} \rangle, \quad (19)$$

stand for amplitudes of tunneling of a particle between neighbouring sites.

Equation (18) is a standard tight-binding model and it indicates that a resonantly driven particle is equivalent to a solid state problem if we use the basis of localized wavepackets $W_{\mathbf{i}}(x, y, t)$ which are evolving periodically along a resonant classical trajectory [7].

II. MANY-BODY PROBLEM

A. Many-body Hamiltonian

In the present section we would like to switch from the single-particle problem to many ultra-cold atoms which are bosons and which are bouncing between two oscillating orthogonal mirrors. We focus on a resonant driving and restrict to the Hilbert subspace which is spanned by the localized Wannier wavepackets $W_{\mathbf{i}}(x, y, t)$ introduced in the previous section. In other words we consider the subspace spanned by the Fock states $|n_{(1,1)}, \dots, n_{(s_x, s_y)}\rangle$, where $n_{(i_x, i_y)}$ denotes number of bosons occupying a Wannier wavepacket $W_{\mathbf{i}=(i_x, i_y)}$. Restricting to this subspace and expanding the bosonic field operator in the Wannier basis,

$$\hat{\psi}(x, y, t) \approx \sum_{\mathbf{i}} W_{\mathbf{i}}(x, y, t) \hat{a}_{\mathbf{i}}, \quad (20)$$

where $\hat{a}_{\mathbf{i}}$ are the standard bosonic annihilation operators, and substituting (20) to the many-body Floquet Hamiltonian we obtain a many-body version of the tight-binding model (18),

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{T} \int_0^T dt \int dx dy \hat{\psi}^\dagger \left[H(t) + \frac{g_0}{2} \hat{\psi}^\dagger \hat{\psi} - i\partial_t \right] \hat{\psi}, \\ &\approx -\frac{1}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\mathbf{ij}} \hat{a}_{\mathbf{i}}^\dagger \hat{a}_{\mathbf{j}} + \frac{1}{2} \sum_{\mathbf{i}, \mathbf{j}} U_{\mathbf{ij}} \hat{a}_{\mathbf{i}}^\dagger \hat{a}_{\mathbf{j}}^\dagger \hat{a}_{\mathbf{j}} \hat{a}_{\mathbf{i}}, \end{aligned} \quad (21)$$

where $H(t)$ is given in (7), $T = s_x s_y 2\pi/\omega$ and g_0 is the parameter (proportional to the atomic s-wave scattering

length) that characterizes the potential of contact interactions between atoms, $g_0 \delta(x) \delta(y)$. The effective interaction coefficients in (21) read

$$U_{\mathbf{ij}} = \frac{2g_0}{T} \int_0^T dt \int dx dy |W_{\mathbf{i}}|^2 |W_{\mathbf{j}}|^2, \quad (22)$$

for $\mathbf{i} \neq \mathbf{j}$ and similar $U_{\mathbf{ii}}$ but by factor two smaller [7, 8]. The Hamiltonian (21) is the standard Bose-Hubbard Hamiltonian written in the time-periodic basis. Application of a time-dependent basis requires corrections to the Hamiltonian coming from the time derivative of the basis vectors [9]. Here, it is included automatically when we perform the action of the Floquet operator, $H(t) - i\partial_t$, on the Wannier functions.

Typically the *on-site* interaction coefficients $U_{\mathbf{ii}}$ are at least an order of magnitude larger than $U_{\mathbf{ij}}$ for *long-range* interactions ($\mathbf{i} \neq \mathbf{j}$). In the Letter we choose the parameters of the system so that all nearest neighbour tunneling amplitudes are the same,

$$J \equiv J_{\mathbf{ij}}. \quad (23)$$

The Hamiltonian (21) is actually the Bose-Hubbard model which is a many-body counterpart of the single-particle tight-binding Hamiltonian (18). It is valid if the interaction energy per particle is much smaller than the gap E_{gap} between the first and the second energy bands of the single-particle problem (17) [7, 8]. For example for the parameters used in Figs. 2-3 in the Letter, the maximal *on-site* interaction energy per particle $|U_{\mathbf{ii}}|N/J \approx 80$, where N is the total number of atoms, that is much smaller than the energy gap $E_{\text{gap}}/J \approx 3000$.

B. Mean-field approximation

If the interaction between atoms are negligible (i.e. $g_0 \rightarrow 0$), the ground state of the Bose-Hubbard Hamiltonian (21) is a Bose-Einstein condensate where all atoms occupy a condensate wavefunction which is a uniform superposition of all Wannier functions

$$\psi(x, y, t) \propto \sum_{\mathbf{i}} W_{\mathbf{i}}(x, y, t). \quad (24)$$

That is, the many-body ground state reads

$$\Psi(x_1, y_1, \dots, x_N, y_N, t) = \prod_{j=1}^N \psi(x_j, y_j, t). \quad (25)$$

On the other hand, if interactions between atoms are attractive and sufficiently strong it is energetically favorable to group all atoms in a single localized Wannier wavepacket because it decreases the energy [10–12]. Then, we expect the ground state to be of the form $\Psi = \prod_{j=1}^N W_{\mathbf{i}}(x_j, y_j, t)$ where $\mathbf{i} = (i_x, i_y)$ is arbitrary. However, such a state cannot be a Floquet eigenstate of the system because it evolves with the period

$T = s_x s_y 2\pi/\omega$ while the discrete time translation symmetry of the Hamiltonian requires that all Floquet eigenstates must evolve with the period of the driving $2\pi/\omega$. In order to reconcile the energy and symmetry requirements, the ground state takes the form

$$\Psi(x_1, y_1, \dots, x_N, y_N, t) \propto \sum_{\mathbf{i}} \left(\prod_{j=1}^N W_{\mathbf{i}}(x_j, y_j, t) \right), \quad (26)$$

which in the Fock states basis reads

$$|\Psi\rangle = \frac{1}{\sqrt{s_x s_y}} (|N, 0, \dots, 0\rangle + |0, N, 0, \dots, 0\rangle + \dots + |0, \dots, 0, N\rangle). \quad (27)$$

Such a ground state is actually a Schrödinger cat-like state, i.e. a superposition of macroscopic states, and it is extremely fragile to any perturbation. For example it is sufficient to measure position of a single atom and the Schrödinger cat state collapses to one of the states which form the superposition,

$$\Psi \rightarrow \prod_{j=1}^N W_{\mathbf{i}}(x_j, y_j, t), \quad (28)$$

in the Fock states basis it corresponds to

$$|\Psi\rangle \rightarrow |0, \dots, 0, N, 0, 0, \dots, 0\rangle. \quad (29)$$

Which localized Wannier wavepacket $W_{\mathbf{i}}$ is chosen in (28) depends on the result of the measurement of the position of an atom. In other words this is an example of a spontaneous process which is responsible for spontaneous breaking of the discrete time translation symmetry of the Hamiltonian.

Note that both the ground state (25) of the weakly interacting system and a symmetry broken state (28) are Bose-Einstein condensates which can be described within the mean-field approximation. Thus, we may use the mean-field approach to describe the phenomenon we are after. In the mean-field description, the spontaneous time translation symmetry breaking will be indicated by appearance of a bifurcation where the symmetry-preserving ground state solution (25) loses its stability and new stable solutions are born which break the discrete time translation symmetry of the Hamiltonian and evolve with a period which is different from the driving period.

It is straightforward to obtain the mean-field equations. Indeed, the mean-field quasi-energy functional reads [2, 6]

$$E = \frac{1}{T} \int_0^T dt \int dxdy \psi^* \left[H + \frac{g_0 N}{2} |\psi|^2 - i\partial_t \right] \psi, \\ \approx -\frac{1}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} J_{\mathbf{ij}} a_{\mathbf{i}}^* a_{\mathbf{j}} + \frac{N}{2} \sum_{\mathbf{i}, \mathbf{j}} U_{\mathbf{ij}} |a_{\mathbf{i}}|^2 |a_{\mathbf{j}}|^2, \quad (30)$$

where we have restricted to the resonant subspace with a condensate wavefunction $\psi(x, y, t) = \sum_{\mathbf{i}} a_{\mathbf{i}} W_{\mathbf{i}}(x, y, t)$. The ground state of (30) can be found by solving the corresponding Gross-Pitaevskii equation [13]. For negligible interactions, the ground state is of the form (24) while for sufficiently strong attractive interactions there are $s_x s_y$ degenerate mean-field ground state solutions $\psi(x, y, t) \approx W_{\mathbf{i}}(x, y, t)$ corresponding to different values of $\mathbf{i} = (i_x, i_y)$.

The mean-field approach has been used in the Letter in order to obtain quantitative prediction for a range of the system parameters where time quasi-crystals form spontaneously. The time quasi-crystal states predicted by the mean-field approach lives forever. The predictions are valid in the limit when $N \rightarrow \infty$ but $g_0 N = \text{const}$ because then the corresponding symmetry preserving eigenstates of the quantum many-body Bose-Hubbard model (21) are degenerate and their superpositions that form the symmetry-broken states do not decay [6].

III. SCENARIO FOR THE EXPERIMENTAL REALIZATION

An oscillating atom mirror can be realized when an evanescent wave, created close to the surface of dielectric material, is modulated in time or by means of an oscillating blue-detuned repulsive light sheet formed by focusing a laser beam with cylindrical lens. The latter can be quite easily reconfigured into two orthogonal oscillating mirrors.

In order to demonstrate our finding in the laboratory one has to prepare a Bose-Einstein condensate of ultra-cold atoms in a trap at the position $(x, y) \approx (16, 37)$ for the parameters used in Fig. 2 of the Letter, i.e. close to the classical turning point. Then, at the moment of time which is synchronized with the oscillations of the mirrors, the trapping potential has to be turned off and the atomic cloud starts falling on the mirrors. If the orientation of the mirrors is adjusted so that the gravitational force points along a fragment of the trajectory that connects the points $(x, y) \approx (16, 37)$ and $(x, y) \approx (0, 15)$, cf. Fig. 2 of the Letter, the atomic cloud falling on the mirrors is already moving along the proper resonant orbit. It means that all atoms are prepared in a single Wannier-like wavepacket $W_{\mathbf{i}}(\mathbf{r}, t)$ of the lower band. If the interactions between atoms are weak, one will observe tunneling of atoms to other Wannier-like wavepackets. However, if the interactions are sufficiently strong, atoms do not tunnel and the stable time quasi-crystal structure emerges in the course of the time evolution. This strategy is similar to the strategy proposed for the experiments on discrete time crystals, see [2] for details.

IV. DISCUSSION OF STABILITY OF PERIODICALLY DRIVEN SYSTEMS WITH CLASSICAL NON-LINEAR RESONANCES

A periodically driven system can be heated by absorbing unlimited amount of energy like in the case of a resonantly driven harmonic oscillator. However, we deal with a system which, in the single-particle case, possesses non-linear classical resonances. The latter are well known in the field of dynamical systems or in quantum chaos [3, 4]. In such systems a period of unperturbed motion of a particle depends on its energy and therefore a particle is not able to absorb continuously the energy: when the energy of a particle increases, its period changes, the system goes out of the resonance and the transfer of the energy stops. Such single particle resonantly driven systems are effectively described by secular Hamiltonians which are time-independent in the frame moving along the classical resonant orbit and which describe resonant elliptical islands created in the phase space [3, 4], see Eq (17). When these effective Hamiltonians are valid, a particle cannot be heated. The validity of an effective Hamiltonian can easily be monitored by looking at the exact structure of the classical phase space: if chaotic layers

are absent, the effective Hamiltonian captures the exact dynamics of a particle [2]. The robustness of the resonantly driven systems against heating is related to stable resonant elliptical islands in the classical phase space.

In the many-body case, the interaction between particles introduces an additional perturbation. However, we consider the interaction energy per particle which is orders of magnitude smaller than the energy gap between the first and second resonant quasi-energy bands — lack of the coupling of these bands is the most critical condition for the validity of the effective many-body Hamiltonian Eq. (21). Such a situation is similar to the standard tight-binding approximation commonly used in solid state physics and in ultra-cold atoms. That is, a single-band approximation is valid if the interaction energy per particle is much smaller than the energy gap between the bands. The effective description of many-body resonant behavior of atoms bouncing on an oscillating mirror was tested in a different context in Refs. [2, 6]. There, it has been shown that the mean-field description of a Bose system within the full Gross-Pitaevskii equation is identical to the mean-field results obtained within the effective Hamiltonian approach.

-
- [1] J. H. Shirley, Phys. Rev. **138**, B979 (1965), URL <https://link.aps.org/doi/10.1103/PhysRev.138.B979>.
 - [2] K. Giergiel, A. Kosior, P. Hannaford, and K. Sacha, Phys. Rev. A **98**, 013613 (2018), URL <https://link.aps.org/doi/10.1103/PhysRevA.98.013613>.
 - [3] A. Lichtenberg and M. Lieberman, *Regular and chaotic dynamics*, Applied mathematical sciences (Springer-Verlag, 1992), ISBN 9783540977452, URL <https://books.google.pl/books?id=2ssPAQAAMAAJ>.
 - [4] A. Buchleitner, D. Delande, and J. Zakrzewski, Physics reports **368**, 409 (2002), URL <http://www.sciencedirect.com/science/article/pii/S0370157302002703>.
 - [5] G. Berman and G. Zaslavsky, Physics Letters A **61**, 295 (1977), ISSN 0375-9601, URL <http://www.sciencedirect.com/science/article/pii/0375960177906181>.
 - [6] K. Sacha, Phys. Rev. A **91**, 033617 (2015), URL <https://link.aps.org/doi/10.1103/PhysRevA.91.033617>.
 - [7] K. Sacha, Sci. Rep. **5**, 10787 (2015), URL <https://www.nature.com/articles/srep10787>.
 - [8] K. Giergiel, A. Miroszewski, and K. Sacha, Phys. Rev. Lett. **120**, 140401 (2018), URL <https://link.aps.org/doi/10.1103/PhysRevLett.120.140401>.
 - [9] M. Lacki and J. Zakrzewski, Phys. Rev. Lett. **110**, 065301 (2013), URL <https://link.aps.org/doi/10.1103/PhysRevLett.110.065301>.
 - [10] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005), URL <https://link.aps.org/doi/10.1103/PhysRevLett.95.010402>.
 - [11] P. Ziń, J. Chwedeńczuk, B. Oleś, K. Sacha, and M. Trippenbach, EPL (Europhysics Letters) **83**, 64007 (2008), URL <http://stacks.iop.org/0295-5075/83/i=6/a=64007>.
 - [12] B. Oleś, P. Ziń, J. Chwedeńczuk, K. Sacha, and M. Trippenbach, Laser Physics **20**, 671 (2010), ISSN 1555-6611, URL <http://dx.doi.org/10.1134/S1054660X10050130>.
 - [13] C. Pethick and H. Smith, *Bose-Einstein condensation in dilute gases* (Cambridge University Press, Cambridge, England, 2002).



PAPER

Creating big time crystals with ultracold atoms

Krzysztof Giergiel¹, Tien Tran², Ali Zaheer², Arpana Singh², Andrei Sidorov², Krzysztof Sacha¹ and Peter Hannaford²¹ Instytut Fizyki Teoretycznej, Uniwersytet Jagielloński, ulica Profesora Stanisława Łojasiewicza 11, PL-30-348 Krakow, Poland² Optical Sciences Centre, Swinburne University of Technology, Hawthorn, Victoria 3122, AustraliaE-mail: phannaford@swin.edu.au and krzysztof.sacha@uj.edu.pl

Keywords: time crystals, Bose–Einstein condensate, ultracold atoms

RECEIVED
1 April 2020REVISED
23 June 2020ACCEPTED FOR PUBLICATION
8 July 2020PUBLISHED
17 August 2020

Original content from
this work may be used
under the terms of the
[Creative Commons
Attribution 4.0 licence](#).

Any further distribution
of this work must
maintain attribution to
the author(s) and the
title of the work, journal
citation and DOI.



Abstract

We investigate the size of discrete time crystals s (ratio of response period to driving period) that can be created for a Bose–Einstein condensate (BEC) bouncing resonantly on an oscillating mirror. We find that time crystals can be created with sizes in the range $s \approx 20$ –100 and that such big time crystals are easier to realize experimentally than a period-doubling ($s=2$) time crystal because they require either a larger drop height or a smaller number of bounces on the mirror. We also investigate the effects of having a realistic soft Gaussian potential mirror for the bouncing BEC, such as that produced by a repulsive light-sheet, which is found to make the experiment easier to implement than a hard-wall potential mirror. Finally, we discuss the choice of atomic system for creating time crystals based on a bouncing BEC and present an experimental protocol for realizing big time crystals. Such big time crystals provide a flexible platform for investigating a broad range of non-trivial condensed matter phenomena in the time domain.

1. Introduction

In 2012 Frank Wilczek proposed that a quantum many-body system in the lowest state could spontaneously break time-translation symmetry to form a time crystal, in analogy with the formation of a crystal in space [1]. Although such time crystals cannot exist in the lowest state of a quantum system with two-body interactions [2, 3] (see [4] for possible time crystals with long-range multi-particle interactions), it was later demonstrated that a periodically driven many-body quantum system can spontaneously break discrete time-translation symmetry to form a discrete time crystal which evolves with a period two-times ($s = 2$) longer than the driving period [5]. Such a time crystal is predicted to be robust against external perturbations and to persist perpetually in the limit of a large number of particles [6, 7]. Similar ideas of discrete time crystals were later proposed for periodically driven *spin* systems [8–10], which in the case of a spin-1/2 system evolve with a period twice as long as the driving period. Experimental evidence of discrete time crystals has since been reported for a range of spin systems, including a spin chain of ions [11], nitrogen-vacancy spin impurities in diamond [12] and nuclear spins in organic molecules [13] and ordered ADP crystals [14, 15]. In addition, space-time crystals—with periodicity in both space and time—have been reported for a superfluid Bose–Einstein condensate (BEC) of atoms [16, 17]. Experiments demonstrating spontaneous emergence of periodic evolution that does not require periodic driving have also been performed in magnon BECs consisting of bosonic quasi-particles [18, 19]. In references [20, 21] it has been shown that periodically driven systems can also reveal crystalline structures in phase space. A number of comprehensive reviews on time crystals have recently been published [22–25].

In a recent paper [6] we presented mean-field calculations for a BEC of attractively interacting atoms bouncing resonantly on an oscillating mirror that exhibited dramatic breaking of time-translation symmetry to form a discrete time crystal. These time crystals can evolve with a period more than an order of magnitude longer ($s \gg 10$) than the driving period, thereby creating a large number of available ‘lattice sites’ in the time domain. Such a system provides a flexible platform for investigating a broad range of

nontrivial condensed matter phenomena in the time domain [6, 26–31]. Other time crystal systems with large values of s have also recently been proposed [32–34].

In this paper we investigate the range of sizes (s -values) of discrete time crystals that can be created for a BEC of ultracold atoms bouncing resonantly on an oscillating mirror. We find that time crystals can be created in the range $s \approx 20$ –100 and, furthermore, that such big time crystals are much easier to realize experimentally than a period-doubling time crystal ($s = 2$) because they require either a larger drop height or a smaller number of bounces on the mirror. We also investigate the effects of having a realistic soft Gaussian potential atom mirror (rather than a theoretical hard-wall mirror)—such as that produced by a repulsive light-sheet—which allows us to operate with a much larger mirror oscillation amplitude and in turn makes the experiment easier. Finally, we discuss the choice of atomic system for creating time crystals based on a bouncing BEC and present an experimental protocol for realizing big time crystals.

2. Theoretical

2.1. Single-particle case for a hard-wall mirror

We first consider a single atom bouncing in the vertical direction z on a harmonically oscillating hard-wall mirror in the presence of strong transverse harmonic confinement. Introducing gravitational units $l_0 = (\hbar^2 / (m^2 g))^{1/3}$, $E_0 = mgl_0$, $t_0 = \hbar / (mgl_0)$ and assuming a one-dimensional (1D) approximation, the Hamiltonian of the system in the laboratory frame can be written

$$H = p^2/2 + \tilde{V}[z + (\lambda/\omega^2)z \cos(\omega t)] + z, \quad (1)$$

where the potential $\tilde{V}(z \leq 0) = \infty$ and $\tilde{V}(z > 0) = 0$ for the hard-wall mirror, ω and λ/ω^2 are the frequency and the amplitude of the oscillating mirror in the laboratory frame, respectively, and m and g are the atom mass and gravitational acceleration. Use of gravitational units allows calculations to be performed with dimensionless parameters that are independent of the mass of the atom.

Transforming from the laboratory frame to the oscillating frame of the mirror, equation (1) becomes [5, 35]

$$H = p^2/2 + \tilde{V}(z) + z + \lambda z \cos(\omega t), \quad (2)$$

where λ is the amplitude of the time-periodic perturbation in the oscillating frame (hereafter referred to as the amplitude of the oscillating mirror). The case of a realistic soft Gaussian potential mirror is considered in section 2.6.

In the classical description, transforming to action-angle variables (I, θ)

$$z = 1/2(3\pi I)^{2/3} \left[1 - \left(\frac{\theta}{\pi} \right)^2 \right]; \quad p = - \left(\frac{3I}{\pi^2} \right)^{1/3} \theta, \quad (3)$$

where $0 \leq \theta < 2\pi$, and using the secular approximation [35, 36], equation (2) can be expressed as a single-particle time-independent Hamiltonian [6]

$$H_F \approx P^2/(2m_{\text{eff}}) + \lambda \langle z \rangle \cos(s\Theta), \quad (4)$$

where $\Theta = \theta - \omega s/t$, $P = I - I_s$, $I_s = \pi^2 s^3 / (3\omega^3)$ is the action of the $s : 1$ resonant orbit. The effective mass and classical average value of z are given by

$$m_{\text{eff}} = -\frac{\pi^2 s^4}{\omega^4} = -\left(\frac{9}{\pi} \right)^{2/3} I_s^{4/3}; \quad \langle z \rangle = \frac{1}{\omega^2} = \frac{1}{s^2} \left(\frac{3I_s}{\pi^2} \right)^{2/3}, \quad (5)$$

where $s = \omega/\Omega$ and Ω is the bounce frequency of the unperturbed particle (i.e., for a static mirror). The height of the classical turning point (drop height) of the bouncing atom is then $h_0 = \pi^2/(2\Omega^2)$.

Equation (4) is the effective Hamiltonian of a particle in the frame moving along an $s : 1$ resonant orbit and indicates that for $s \gg 1$ a single resonantly driven atom in the vicinity of a resonant trajectory behaves like an electron moving in a crystalline structure created by ions in a solid state system. That is, eigenvalues of the quantized version of the Hamiltonian (4) form energy bands (see figure 1) and the corresponding eigenstates are Bloch waves. These eigenvalues are actually quasi-energies of a periodically driven particle while the eigenstates are Floquet states obtained in the frame moving along the resonant orbit [5, 6, 26, 35]. In the quantum description we will apply the tight-binding approximation by restricting the analysis to the first energy band of the quantized version of the Hamiltonian (4). In the *many-body* case such an approximation is valid provided the interaction energy per particle is smaller than the energy gap ΔE between the first and second energy bands shown in figure 1.

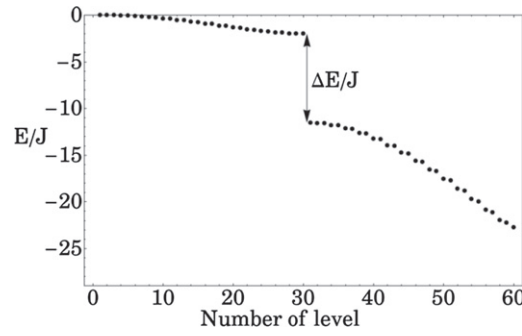


Figure 1. Eigenenergies E for the first and second energy bands of the quantized version of the Hamiltonian (4) for $s = 30$. Because the effective mass in (4) is negative, cf (5), the first energy band is the highest on the energy scale. The mirror oscillation frequency $\omega = 4.45$ and the oscillation amplitude $\lambda = 0.2$ are chosen so that the energy gap $\Delta E/J \approx 10$. The energies are given in units of the nearest-neighbour tunnelling amplitude J corresponding to the first energy band.

2.2. Optimal value of the driving strength λ for a hard-wall mirror

We estimate the largest λ for a hard-wall potential mirror that is allowed before the dynamics become chaotic, making use of the Chirikov criterion [37]. This criterion estimates the value of λ for which two neighbouring resonance islands, $s : 1$ and $(s + 1) : 1$, described by equation (4), overlap. The distance between the islands (for $s \gg 1$) and the half-width of the islands are

$$I_{s+1} - I_s \approx (\pi^2/\omega^3)s^2; \quad \frac{1}{2}(\Delta I)_s = 2\sqrt{|m_{\text{eff}}\lambda\langle z \rangle|} = \sqrt{\lambda} 2\pi s^2/\omega^3. \quad (6)$$

The resonance islands nearly overlap when

$$\frac{2}{3}(I_{s+1} - I_s) \approx \frac{1}{2}(\Delta I)_{s+1} + \frac{1}{2}(\Delta I)_s, \quad (7)$$

which leads to $\lambda \approx (\pi/6)^2 \approx 0.27$. The $2/3$ factor in (7) is an empirical correction that allows for the presence of higher-order resonances [35]. This means that for $s \gg 1$ the critical value of λ is a constant independent of s and ω . For $\lambda = 0.2$, we find that, although there is some chaos between the neighbouring $s : 1$ and $(s + 1) : 1$ islands, the islands themselves are still not perturbed [6]. For $\lambda < 0.2$, the resonance islands become smaller and less suitable for realization of a time crystal. Indeed, quantum states that describe time crystals are located inside the resonance islands and if the islands are too small we need to choose a large value of I_s in order to realize a time crystal. While this is in principle possible, the resulting evolution of ultracold atoms that demonstrates time crystal behaviour becomes very long, see section 2.4. We conclude that for a hard-wall potential mirror $\lambda = 0.2$ is universally good for any $s : 1$ resonance for which $s \gg 1$.

2.3. Scaling of parameters with $s = \omega/\Omega$

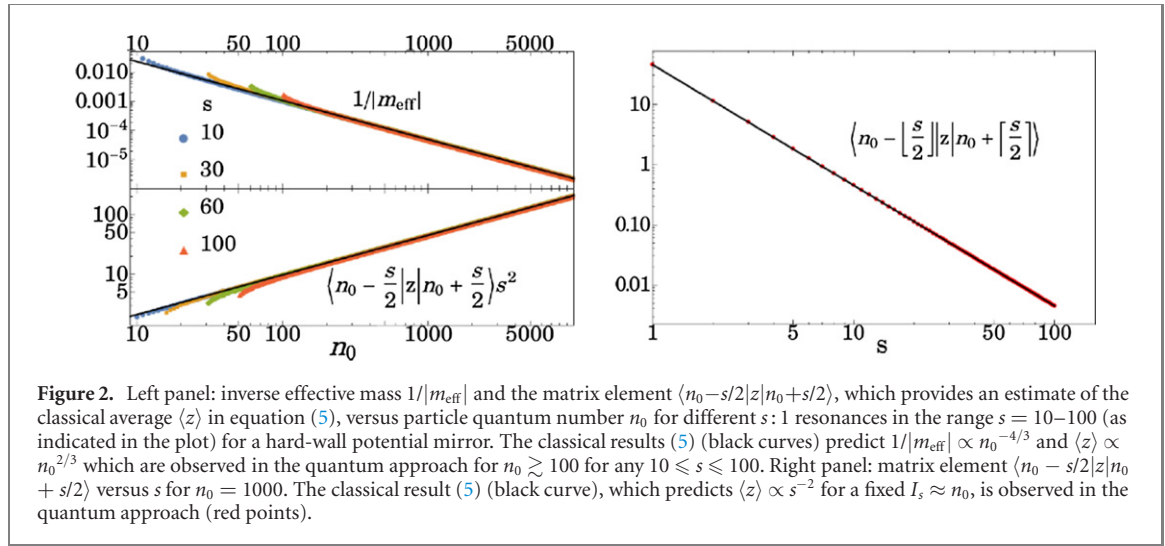
The quantum secular approximation allows us to obtain the quantum version of the classical Hamiltonian (4) [6]. That is, switching to the oscillating frame by means of the unitary transformation $U = e^{i\hat{n}\omega t/s}$, the matrix elements of the time-averaged quantum Hamiltonian describing the $s : 1$ resonance dynamics reads

$$\langle n' | H_F | n \rangle \approx \left(E_n - \frac{n\omega}{s} \right) \delta_{n',n} + \frac{\lambda}{2} \langle n' | z | n \rangle (\delta_{n',n+s} + \delta_{n',n-s}), \quad (8)$$

where the $|n\rangle$ are eigenstates of the unperturbed part of the Hamiltonian (2), i.e., H with $\lambda = 0$, with the corresponding eigenvalues E_n , and $\hat{n}|n\rangle = n|n\rangle$. The validity of the quantum secular approximation (8) can be checked for a given set of parameters by comparing results of the classical secular approach with the exact classical approach. That is, if the classical secular and exact treatments agree, the quantum secular approach is also valid [6]. Around the resonant value of the quantum number of the unperturbed particle, i.e., for $n \approx n_0 \approx I_s$, the first term on the right hand side of (8) can be approximated by

$$E_n - \frac{n\omega}{s} \approx E_{n_0} - \frac{n_0\omega}{s} + \frac{(n - n_0)^2}{2m_{\text{eff}}}. \quad (9)$$

We know how the parameters of equation (4) scale with I_s . We now investigate how the effective mass m_{eff} , obtained from the quantum approach (8), and the matrix element $\langle n_0 - s/2 | z | n_0 + s/2 \rangle$, which provides an estimate of the classical average value $\langle z \rangle$, scale with n_0 and s . The results, presented in figure 2, show that



for $s \leq 100$, the classical scaling is reproduced in the quantum approach if the particle quantum number $n_0 \gtrsim 100$. This allows us to use the classical analytical expressions to determine the optimal parameters for an experiment.

2.4. Number of bounces and energy gap

To demonstrate that a time crystal is created one needs to show that the ultracold atoms evolve periodically with a period s -times longer than the driving period $T = 2\pi/\omega$. If the interactions between atoms are too weak, the subharmonic periodic evolution will be destroyed because the atoms will start to tunnel between lattice sites of the potential in (4), or in other words between wave-packets propagating along the resonant orbit. Therefore, to demonstrate that a time crystal is created, the system needs to have evolved for at least the time period corresponding to the tunnelling time t_{tunnel} of a single atom between neighbouring lattice sites. The tunnelling time (for $s \gg 1$) and the bounce period T_{bounce} of the atom are given by [6]

$$t_{\text{tunnel}} \approx 2.4/J; \quad T_{\text{bounce}} = 2\pi/\Omega \quad (10)$$

where J is the tunnelling amplitude of the particle between neighbouring sites of the periodic potential in (4), $\Omega = [\pi^2/(3n_0)]^{1/3}$, and I_s is denoted here by the quantum number of the unperturbed particle, i.e., $I_s \approx n_0$. We require t_{tunnel} to be as short as possible because then the number of bounces needed to demonstrate that a time crystal is robust against single-particle tunnelling is relatively small—each bounce off the mirror is a potential source of loss of atoms from the BEC. The number of bounces required to observe quantum tunnelling for non-interacting particles is then

$$N_b = \frac{t_{\text{tunnel}}}{T_{\text{bounce}}} \approx 2.4\Omega/(2\pi J) \quad \text{for } s \gg 1. \quad (11)$$

Another important parameter that we wish to control is the energy gap ΔE between the first and second bands of the quantum version of the Hamiltonian (4), see figure 1. To realize a time crystal, sufficiently strong interactions between atoms need to be present. However, if the interactions are too strong, the single-band description within the Bose–Hubbard model (see section 2.5) is no longer valid because higher bands of the Hamiltonian (4) become involved. Thus, ΔE should be as large as possible but this requirement is in contradiction with the requirement of a short tunnelling time t_{tunnel} and we need to find a compromise.

Dividing the classical Hamiltonian (4) by $\lambda\langle z \rangle$ and rescaling $s\Theta \rightarrow \Theta$, gives

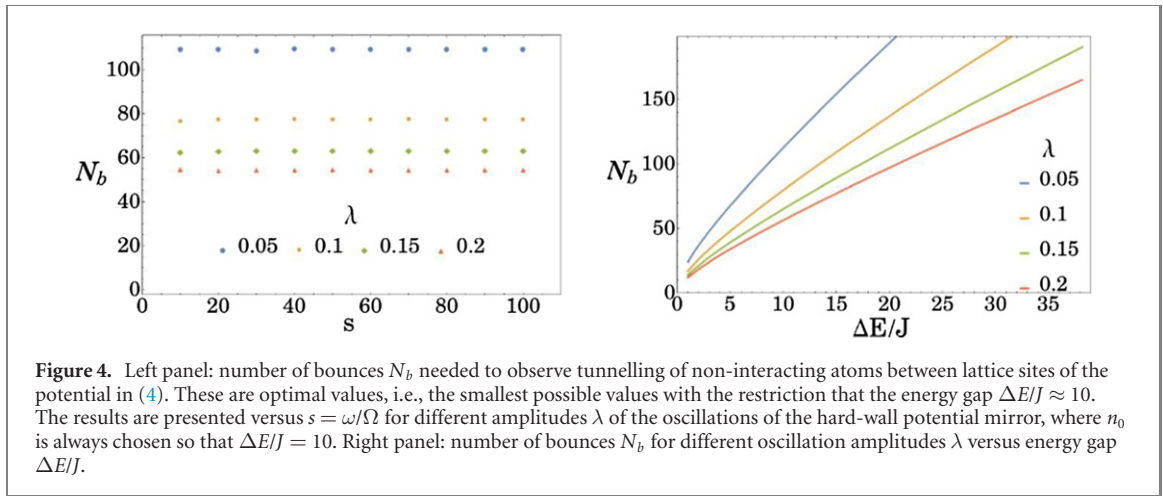
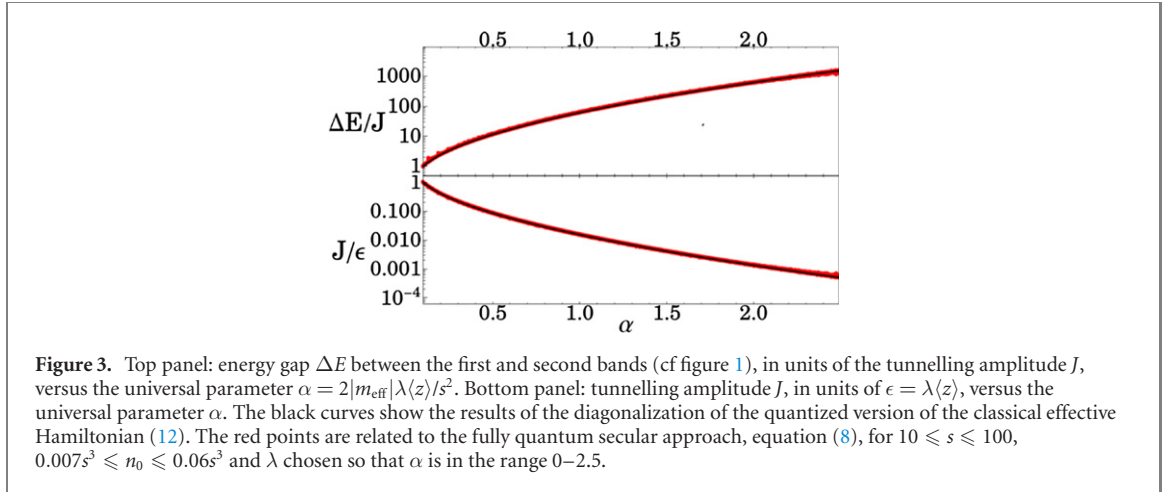
$$H_F \approx P^2/[2m_{\text{eff}}\lambda\langle z \rangle/s^2] + \cos \Theta = P^2/\alpha + \cos \Theta, \quad (12)$$

which depends only on the single universal parameter

$$\alpha = 2|m_{\text{eff}}|\lambda\langle z \rangle/s^2 = 2\pi^2 s^2 \lambda/\omega^6. \quad (13)$$

The division by $\lambda\langle z \rangle$ also means that in order to analyse the scaling properties of the system the quasi-energy gap ΔE and the tunnelling amplitude J should be expressed in units of $\epsilon = \lambda\langle z \rangle = \lambda/\omega^2$.

Figure 3 (top panel) presents calculations of the energy gap $\Delta E/J$, obtained from the quantum secular Hamiltonian (8), cf figure 1, versus the universal parameter α for a range of values of s and n_0 for a



hard-wall potential mirror. We see that for different values of s and n_0 all the quantum data lie on a single universal curve. Similarly, in figure 3 (bottom panel), the tunnelling amplitude J , in units of $\epsilon = \lambda\langle z\rangle$, obtained from diagonalization of the effective Hamiltonian (4) and from the fully quantum secular approach (8), versus α for different values of s and n_0 all lie on the same universal curve which we denote by $f(\alpha) = J(\alpha)/\epsilon$. These results demonstrate that the scaling deduced from our analysis is valid and can be used for further purposes.

With the help of $f(\alpha) = J(\alpha)/\epsilon$, we can now express the number of bounces as

$$N_b = \frac{t_{\text{tunnel}}}{T_{\text{bounce}}} = 2.4/[\sqrt{2\lambda\alpha}f(\alpha)], \quad (14)$$

which indicates that if we choose λ (e.g., the optimal value $\lambda = 0.2$ for a hard-wall potential mirror) and a value for the energy gap $\Delta E(\alpha)/J(\alpha)$ (which determines α) the number of bounces N_b is constant and independent of the values of s that we choose. In other words, for fixed λ and α we can choose any s : 1 resonance (then from $\alpha = \text{constant}$ we obtain n_0 and consequently ω) and the number of bounces is always the same. This is illustrated in the left panel of figure 4 where for different values of λ we always choose α so that the gap $\Delta E/J$ between the first and second energy bands of the quantum version of the Hamiltonian (4) is about 10 (which for $\lambda = 0.2$ corresponds to $\alpha = 0.456$).

The key results of the analysis are presented in the right panel of figure 4 where the number of bounces N_b needed for tunnelling of non-interacting atoms between lattice sites of the potential in (4) versus the band gap $\Delta E/J$ are presented. The smaller the band gap we can afford, the smaller the optimum number of bounces.

2.5. Many-body case

We now switch to the case of a BEC of interacting bosonic atoms bouncing on an oscillating mirror in the presence of strong transverse harmonic confinement $\omega_\perp$ and assume the 1D approximation. Note that we focus on a spatially finite 1D system where for sufficiently weak interactions, the BEC is not destroyed by

long-wavelength quantum fluctuations. We restrict the analysis to the Hilbert subspace corresponding to the first energy band of the quantum version of the Hamiltonian (4). This is the resonant subspace where atoms occupy s localized wavepackets $w_i(z, t)$ evolving along the $s : 1$ resonant orbit with period sT , where T is the driving period [6]. These wave-packets are the time-periodic version of the Wannier states in solid state physics. Within the mean-field approach and restricting to the s -dimensional resonant Hilbert subspace we can expand solutions of the Gross–Pitaevskii equation in terms of localized Wannier-like states $\psi(z, t) = \sum_{i=1}^s a_i w_i(z, t)$ and obtain the energy functional (actually the quasi-energy functional) in the form [6]

$$E \approx -\frac{1}{2} \sum_{i,j=1}^s J_{ij} (a_i^* a_j + \text{c.c.}) + \frac{1}{2} \sum_{i,j=1}^s U_{ij} |a_i|^2 |a_j|^2, \quad (15)$$

with

$$J_{jj} = -\frac{2}{sT} \int_0^{sT} dt \int_0^\infty dz w_j^* \left[\frac{p^2}{2} + z + \lambda z \cos \omega t - i\partial_t \right] w_j,$$

$$U_{ii} = \frac{g_{1D}N}{sT} \int_0^{sT} dt \int_0^\infty dz |w_i|^4,$$

$$U_{ij} = \frac{2g_{1D}N}{sT} \int_0^{sT} dt \int_0^\infty dz |w_i|^2 |w_j|^2, \quad \text{for } i \neq j,$$

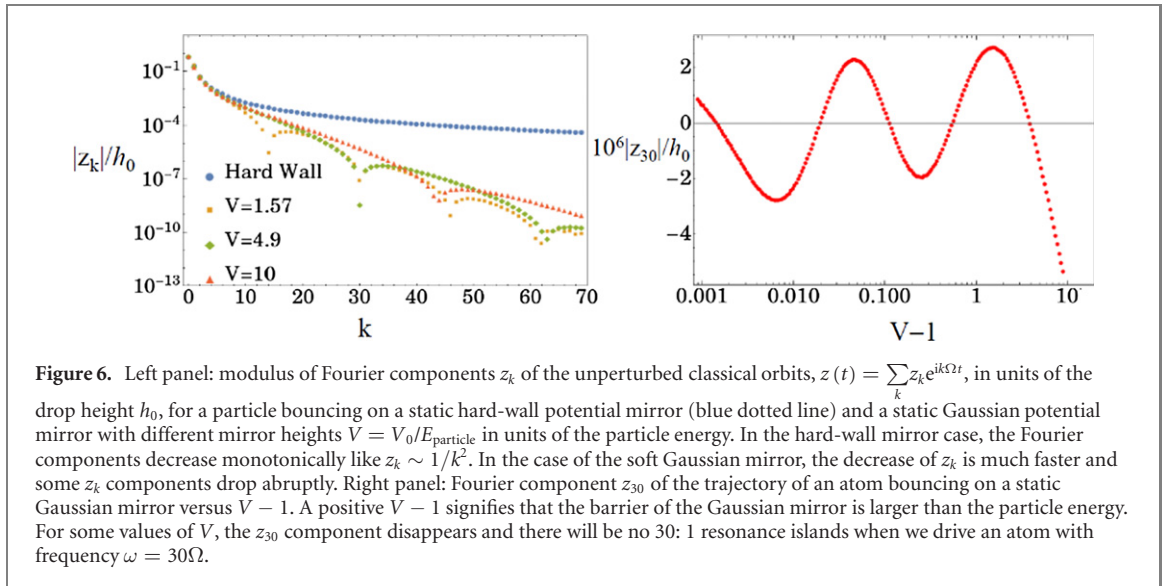
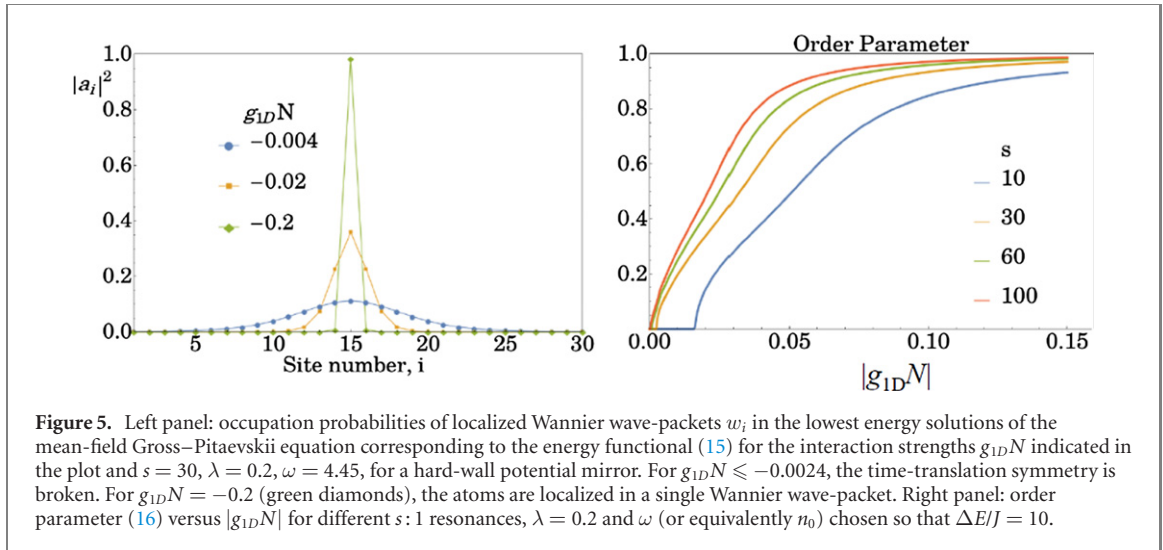
where N is the total number of atoms, $g_{1D} = 2\omega_\perp a_s$ describes the contact interaction between the atoms, a_s is the s -wave scattering length (in gravitational units) and the U_{ii} and U_{ij} describe the on-site and long-range interaction energies per particle in the Bose–Hubbard model (15). The leading tunnelling amplitudes correspond to the nearest-neighbour hopping $J = |J_{i,i+1}|$ and only this hopping can be kept in the model when we want to describe time crystal dynamics. Such a tight-binding approximation is valid provided the interaction energy per particle is much smaller than the energy gap ΔE between the first and second bands of the effective Hamiltonian (4), see figure 1.

In the left panel of figure 5 the lowest energy solutions of the mean-field energy (15) for $s = 30$, $\lambda = 0.2$, $\omega = 4.45$ and different values of the interaction strength $g_{1D}N$ are presented. We emphasize that these solutions are related to the lowest quasi-energy of the driven system within the resonant Hilbert subspace; they do not correspond to the ground state which does not exist in a periodically driven system. For $g_{1D}N > -0.0024$, the lowest energy solution is a uniform superposition of all s Wannier-like wave-packets, $\psi = \sum_{i=1}^s w_i / \sqrt{s}$. The wavefunction ψ evolves with the driving period T and no spontaneous breaking of discrete time-translation symmetry of the many-body Hamiltonian takes place. However, if the attractive interactions are sufficiently strong, it becomes energetically favourable for the atoms to localize, and the lowest energy mean-field solution reveals spontaneous breaking of time-translation symmetry; i.e., ψ evolves with a period s -times longer than T . The stronger the interactions, the better the localization of the atoms, and for $g_{1D}N \leq -0.07$, we obtain $\psi \approx w_i$ with the squared-overlap $|\langle \psi | w_i \rangle|^2 > 0.9$. The corresponding interaction energy per particle $|U_{ii}|/(2J) \approx 0.8$ (where $J = |J_{i,i+1}|$) is smaller than the energy gap $\Delta E/J = 10$ between the first and second bands of the quantum version of the Hamiltonian (4), and consequently the Bose–Hubbard model (15) is valid. In the right panel of figure 5, the order parameter

$$O = \frac{\text{Max}_i (|\langle \psi | w_i \rangle|^2) - 1/s}{1 - 1/s}, \quad (16)$$

which characterizes the overlap of the lowest energy solution ψ with a single Wannier-like wave-packet w_i versus the interaction strength $g_{1D}N$ is presented for different s . The critical values of the interaction strength can be identified in the plots. The figure also shows how strong the interactions need to be in order to be dealing with the lowest energy solution that practically reduces to a single Wannier-like wave-packet w_i . This is important information because time crystals in which $\psi(z, t) \approx w_i(z, t)$ can be easily prepared in an experiment, see section 4. The case $s = 40$ has been extensively analysed in [6], where the numerical integration of the full Gross–Pitaevskii equation confirmed the results based on the Bose–Hubbard model.

For different $s:1$ resonances, the critical interaction strength $g_{1D}N$ differs because the tunnelling amplitudes J_{ij} are slightly different and also because the same $g_{1D}N$ does not necessarily mean the same interaction coefficients U_{ij} in (15). The coefficients U_{ij} depend on the longitudinal width of the atom cloud which, for the optimal values of the parameters, varies from $\sigma_z = 1.3$ – 2.7 for $s = 10$ – 100 (table 2). Thus, a slightly different interaction parameter $g_{1D}N$ and transverse confinement frequency $\omega_\perp$ is required for different s .



2.6. Case of a soft Gaussian potential mirror

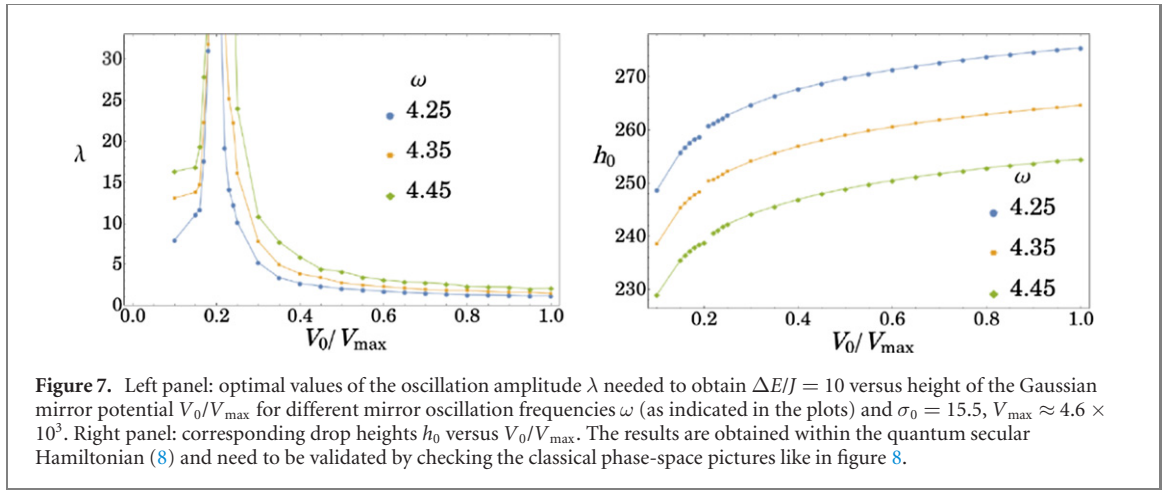
The calculations in previous sections were based on a simple hard-wall mirror potential. We now consider the case of a realistic soft Gaussian mirror potential, such as that produced by a repulsive light-sheet,

$$V(z) = V_0 \exp(-z^2/2\sigma_0^2), \quad (17)$$

where V_0 and σ_0 are the height and width of the Gaussian mirror potential.

For a hard-wall potential mirror, the trajectories of the bouncing atoms reverse their direction abruptly at the reflection point (i.e., $p \rightarrow -p$), so that the Fourier transform of the unperturbed periodic trajectories $z(t) = \sum_k z_k e^{ik\Omega t}$ (where Ω is the frequency of the bouncing atom in the absence of mirror oscillations, see (10)) results in amplitudes of the harmonics that decrease with k like $z_k \sim 1/k^2$. In the case of a soft Gaussian potential mirror, the trajectories of the atoms are smoothly reflected, so that the harmonics decrease much faster with k , see left panel of figure 6. Because for an $s : 1$ resonance the amplitude of the potential in (4) is determined by the amplitude of the s th harmonic, i.e., $\langle z \rangle \approx z_s$, the Gaussian potential mirror needs to oscillate with an amplitude λ much larger than the oscillation amplitude of a hard-wall potential mirror in order to have the same effect on the bouncing atoms.

When a particle bounces off a soft Gaussian potential mirror some harmonics of the classical unperturbed orbits are not created at all. The question of which harmonics disappear for a given $V = V_0/E_{\text{particle}} \geq 1$ (where E_{particle} is the particle's energy) is a complex problem. In the left panel of figure 6 we see dramatic drops of certain Fourier components which have a uniform spacing increasing from $\Delta k = 16$ – 33 for $V \approx 1.6$ – 5 . We focus here on the 30:1 resonance. The right panel of figure 6 shows the amplitude z_{30} of the 30th harmonic of an unperturbed periodic orbit as a function of $V = V_0/E_{\text{particle}}$.



When V approaches one, i.e., $V_0 = E_{\text{particle}}$, from above there is a series of zeros of z_{30} . However, for $V \gtrsim 10$, there is no dramatic drop of $|z_{30}|$ and an experiment demonstrating a discrete time crystal can be carried out in this regime.

For a given choice of V_0 and σ_0 , our primary restriction is to obtain an energy gap of $\Delta E/J \approx 10$. First, we choose a mirror oscillation frequency ω and by applying the quantum secular approximation (8) we determine the optimal oscillation amplitude λ that leads to $\Delta E/J \approx 10$. We then need to check if the secular approximation is still valid for these parameters by examining the classical phase-space pictures of the action I versus angle Θ to see if the dynamics is still regular or if it is already chaotic.

We assume here a soft Gaussian mirror potential which corresponds approximately to the repulsive light-sheet mirror used in [38] for the reflection of a ^{87}Rb BEC dropped from heights up to $300 \mu\text{m}$. The mirror is formed by a $\leq 3\text{W}$, 532 nm laser beam with a waist $\sigma_0 l_0 = 10 \mu\text{m}$ and horizontal extension $200 \mu\text{m}$, which for ^{39}K atoms corresponds to $\sigma_0 = 15.5$ and $V_{\max} \approx 4.6 \times 10^3$ in gravitational units. The hardness of the mirror can be varied by varying the beam waist σ_0 . For a given Gaussian width $\sigma_0 = 15.5$, different mirror potential heights $V_0/V_{\max}$, and oscillation frequencies around the optimal hard-wall mirror value $\omega = 4.45$, we have used the quantum secular approach (8) to determine the amplitude of the mirror oscillations λ required to obtain $\Delta E/J \approx 10$. The results are shown in the left panel of figure 7. Now that we have predictions for all parameters, the corresponding classical phase-space pictures have been obtained, examples of which are shown in figure 8 for different mirror heights $V_0/V_{\max}$. For $\omega = 4.45$, the phase space around the 30:1 resonance islands is regular in all cases except $V_0/V_{\max} \approx 0.2$, for which the λ needed to obtain $\Delta E/J \approx 10$ is extremely high (>100) and the classical motion is no longer regular, and consequently the 30:1 resonance for $V_0/V_{\max} \approx 0.2$ is not suitable for realization of a time crystal. This is because for $V_0/V_{\max} \approx 0.2$ (and for a particle energy that fulfills $V = V_0/E_{\text{particle}} \approx 5$), the 30th harmonic of the resonant periodic orbit disappears (figure 6). For $V_0/V_{\max} > 0.2$, the drop height h_0 does not change too much as a function of $V_0/V_{\max}$ (figure 7, right panel) and the optimal number of bounces required for tunnelling to neighbouring wave-packets and the optimal tunnelling amplitude remain nearly constant at $N_b \approx 54$ and $J \approx 0.0010$ for $\omega = 4.45$. All of these parameters are very close to the optimal values for the hard-wall mirror.

For $\omega > 4.45$ and λ corresponding to $\Delta E/J = 10$, the classical motion becomes more chaotic than for $\omega = 4.45$ and the secular approach is not fully valid, while for $\omega < 4.45$ the number of bounces N_b required for tunnelling to neighbouring wave-packets increases. Thus, the optimal value of ω for the soft Gaussian potential mirror is close to the optimal value $\omega = 4.45$ for the hard-wall potential mirror for $s = 30$.

The relatively large optimal values of λ in the case of the soft Gaussian potential mirror, e.g., $\lambda = 2.3$ (corresponding to $\lambda l_0/\omega^2 = 74 \text{ nm}$ for ^{39}K in the laboratory frame) at $V_0/V_{\max} = 0.8$, $\omega = 4.45$, required to obtain a band gap $\Delta E/J = 10$ are more readily accessible experimentally than the optimal hard-wall mirror value $\lambda = 0.2$ (6.5 nm for ^{39}K). If still larger mirror oscillation amplitudes λ are required in the laboratory, one could operate closer to the peak of the λ versus mirror potential height curve in figure 7, left panel, provided the corresponding classical resonance islands are not destroyed by chaos.

2.7. Constraints on the maximum number of atoms

To operate in the quasi-1D regime, the transverse standard deviation $\sigma_{\perp}$ of the atomic density needs to be less than or equal to the standard deviation σ_z along the longitudinal direction and the interaction energy per particle should not exceed the excitation energy in the transverse directions ($\sigma_{\perp}$ and σ_z are defined as the Gaussian fits to the density of the atomic cloud). The transverse width is determined by the frequency

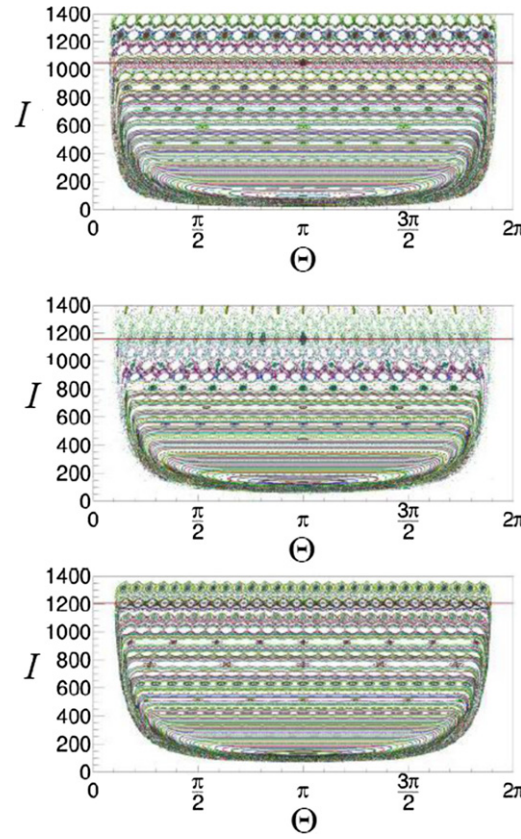


Figure 8. Classical phase-space pictures of the action I versus angle Θ for the reflection of atoms from a soft Gaussian mirror potential with different mirror potential heights $V_0/V_{\max} = 0.1, 0.2, 0.8$ (top to bottom), where $V_{\max} \approx 4.6 \times 10^3$ and $\sigma_0 = 15.5$, $\omega = 4.45$. The mirror oscillation amplitude λ is chosen so that the quantum secular approximation (8) predicts a band gap $\Delta E/J = 10$. The phase space around the $s = 30$ resonance islands, which are located at $I \approx 1000$ – 1200 (red horizontal lines), is regular in all cases except $V_0 \approx 0.2V_{\max}$, for which the λ needed to obtain $\Delta E/J = 10$ is extremely high (cf figure 7, left panel) and the classical motion is no longer regular. Not all islands are visible because some are hidden in the lower, narrow and elongated part of the phase space. For Θ close to 0 and 2π , i.e., close to the Gaussian mirror, the action I drops to zero because we have used action-angle variables suitable for the hard-wall mirror (which are given analytically) rather than for the Gaussian mirror.

$\omega_{\perp}$ of the harmonic potential in the transverse directions, i.e., $\sigma_{\perp} = \sqrt{\hbar/(2m\omega_{\perp})}$. These requirements imply that we need $\sigma_z \geq \sigma_{\perp}$ and $\sigma_z > |a_s|N$, where a_s is the atomic s -wave scattering length. When these criteria are fulfilled, a BEC of ultracold atoms is also stable against ‘bosonova’ collapse that can occur for attractive interactions in three-dimensional space. For atoms bouncing resonantly on an oscillating mirror, the longitudinal width is smallest at the classical turning point and corresponds to $\sigma_z = 1.81$ for $s = 30$. For ^{39}K atoms, $s = 30$, $\omega = 4.45$ and a Gaussian mirror with $V_0/V_{\max} = 0.8$, $\sigma_0 l_0 = 10 \mu\text{m}$, the minimum longitudinal width is $\sigma_z l_0 = 1.17 \mu\text{m}$; for example, for $a_s l_0 = -1.7a_0$, the maximum number of atoms is $N_{\max} \approx 10\,000$.

To minimize losses due to three-body recombination, the mean-square atom density $\langle n^2 \rangle$ at the classical turning point needs to be less than $1/(K_3 \tau_{\text{BEC}})$, where K_3 is the three-body recombination constant and τ_{BEC} is the lifetime of the BEC. For ^{39}K $|1, +1\rangle$ atoms and taking $K_3 = 1.3(5) \times 10^{-29} \text{ cm}^6 \text{ s}^{-1}$ near the zero-crossing point [39] and $\tau_{\text{BEC}} \approx 1 \text{ s}$, we obtain $\langle n^2 \rangle_{\max} \approx 10^{29} \text{ cm}^{-6}$; for example, for $\sigma_z l_0 = \sigma_{\perp} l_0 = 1.17 \mu\text{m}$ at the classical turning point, the maximum number of atoms is $N_{\max} \approx 16\,000$.

3. Choice of atomic system

In our earlier paper [6], we focussed on the ^{85}Rb $|F = 2, m_F = -2\rangle$ system, which has a broad Feshbach resonance at 155 G with a zero crossing point at 166 G. In table 1, we compare the Feshbach resonance parameters and time crystal parameters for a hard-wall potential mirror and $s = 30$ for four bosonic alkali systems that have broad Feshbach resonances: ^{85}Rb $|F = 2, m_F = -2\rangle$, ^{39}K $|1, +1\rangle$, ^{39}K $|1, -1\rangle$ and ^7Li $|1, 0\rangle$.

In a time crystal experiment, the s -wave scattering length a_s needs to be adjusted to zero for the non-time crystal phase and to small negative (attractive) values (e.g., $a_s l_0 = -1.6a_0$) for the time crystal phase. The sensitivity of a_s to magnetic fields in a Feshbach resonance is determined by the ratio of the

Table 1. Feshbach resonance and time crystal parameters for ^{85}Rb , ^{39}K and ^7Li atoms for a hard-wall potential mirror and $s = 30$.

	^{85}Rb	^{39}K		^7Li
	$ 2, -2\rangle$	$ 1, +1\rangle$	$ 1, -1\rangle$	$ 1, 0\rangle$
Feshbach Resonances				
Centre magnetic field, B_∞ (G)	155.04 [40]	402.5 [42]	560.7 [43]	736.8 [44]
Width of resonance, Δ (G)	10.7	−52	−56	−192.3
Zero crossing field, B_0 (G)	165.7	350.5	504.7	849.9
Background scattering length, a_{bg} [a_0]	−443	−29	−29	−25
Background to width ratio $ a_{bg}/\Delta $ [a_0/G]	−41.4	0.56	0.52	0.130
δB to give $\delta a_0 = \pm 0.1 a_0$ (mG)	2.4	179	193	764
3-body recombination at B_0 : K_3 ($\text{cm}^6 \text{s}^{-1}$)	8.3×10^{-28} [41]	$1.3(5) \times 10^{-29}$ [39]	$1.5(6) \times 10^{-29}$ [43]	1×10^{-27} [45]
Gravitational units				
Length, l_0 (μm)	0.385		0.647	2.034
Time, t_0 (ms)	0.198		0.256	0.455
Energy, E_0/k_B (nK)	38.6		29.7	16.8
Time crystal parameters ($s = 30$)				
Mirror frequency, $\omega/(2\pi t_0)$ (kHz)	3.6		2.8	1.56
Hard-wall mirror amplitude, $\lambda l_0/\omega^2$ (nm)	3.9		6.5	20.5
Atom bounce period, $T_{\text{bounce}} t_0$ (ms)	8.4		10.9	19.3
Drop height, $h_0 l_0$ (μm)	86		145	460
Tunnel amplitude, J/t_0 (s^{-1})	5.3		4.1	2.3
Tunnel time, $t_{\text{tunnel}} t_0$ (s)	0.45		0.59	1.04
Number of bounces during t_{tunnel} , N_b	54		54	54
Longitudinal width, $\sigma_z l_0$ (μm)	0.70		1.17	3.7
v_{rms} at 100 nK (mm s^{-1})	5.4		8.0	19

background scattering length to the width of the resonance a_{bg}/Δ . For ^7Li and ^{39}K , a_{bg}/Δ is 320 and 75 times smaller, respectively, than for ^{85}Rb and therefore much less sensitive to stray magnetic fields. ^{39}K also has the flexibility of having two broad Feshbach resonances, one involving a high-field seeking $|1, +1\rangle$ state at $B_\infty = 402.5$ G and the other a low-field seeking $|1, -1\rangle$ state at $B_\infty = 560.7$ G, which may be useful if the atoms need to be trapped in a magnetic trap prior to producing a BEC in an optical dipole trap.

The gravitational unit of length l_0 (which scales as $\text{m}^{-2/3}$) is 5.3 and 1.7 times larger for ^7Li and ^{39}K atoms than for ^{85}Rb , and hence the mirror oscillation amplitude and drop height are larger by these factors. A larger mirror oscillation amplitude is more accessible in an experiment while a larger drop height allows the atom density to be probed with higher spatial resolution during a bounce cycle. On the other hand, the gravitational unit of time t_0 (which scales as $\text{m}^{-1/3}$) is 2.3 and 1.3 times larger for ^7Li and ^{39}K atoms than for ^{85}Rb , and hence the bounce period and the time for tunnelling to neighbouring wave-packets are longer by these amounts, which makes the experiment longer compared to the lifetime of the bouncing BEC. In addition, the rms velocity (which scales as $\text{m}^{-1/2}$) for a thermal cloud is 3.5 and 1.5 larger for ^7Li and ^{39}K atoms than for ^{85}Rb , which means an atom cloud needs to be 12 and 2.2 times colder than for ^{85}Rb to have the same velocity spread.

Potassium-39 $|1, +1\rangle$ and $|1, -1\rangle$ atoms have much smaller three-body collision loss rates than ^{85}Rb $|2, -2\rangle$ and ^7Li $|1, 0\rangle$ atoms (table 1), which allows higher atom densities to be used. ^{39}K also has certain technical advantages compared with ^7Li : high-power tapered amplifier laser systems at 767 nm (D2) and 770 nm (D1) are readily available commercially, ^{39}K does not require a Zeeman slower, and it is easier to access the zero crossing point ($B_0 = 350$ G) than for a ^7Li resonance ($B_0 = 545$ G) and to quickly switch or ramp the Feshbach magnetic field.

From the above considerations, we focus here on ^{39}K as an optimal atomic system. Table 2 summarizes the parameters for ^{39}K atoms for a hard-wall potential mirror for the range $s = 10$ –100 and $\lambda = 0.2$, $\Delta E/J = 10$, $N_b = 54$. Ideally, we require a large mirror oscillation amplitude ($\lambda l_0/\omega^2 \gg 10$ nm) which is more readily accessible in an experiment, a large drop height ($h_0 l_0 \geq 100 \mu\text{m}$) to allow high spatial resolution probing of the atom density during a bounce cycle, and a short tunnelling time (< 1 s) to allow the experiment to be performed in times shorter than the lifetime of a bouncing BEC. The results of table 2 suggest that concerning the values of the drop height and tunnelling time, it should be feasible to create time crystals with sizes in the range $s \approx 20$ –100.

Table 2. Calculated parameters for ^{39}K atom for a hard-wall potential mirror for different values of s and $\lambda = 0.2$, $\Delta E/J = 10$, $N_b = 54$.

s	ω	$\omega/(2\pi t_0)$ (kHz)	$\lambda l_0/\omega^2$ (nm)	$h_0 l_0$ (μm)	$\sigma_z l_0$ (μm)	n_0	J/t_0 (s^{-1})	$t_{\text{tunnel}} t_0$ (s)
10	3.08	1.92	13.6	33.5	0.83	112	8.5	0.28
20	3.89	2.42	8.56	84.5	1.02	448	5.3	0.45
30	4.45	2.77	6.53	145	1.17	1007	4.1	0.59
40	4.90	3.05	5.38	213	1.29	1790	3.4	0.71
50	5.28	3.28	4.64	281	1.38	2797	2.9	0.83
60	5.61	3.49	4.11	365	1.47	4027	2.6	0.93
70	5.90	3.67	3.71	449	1.55	5481	2.3	1.03
80	6.17	3.84	3.40	536	1.62	7159	2.1	1.13
90	6.42	3.99	3.14	627	1.68	9060	2.0	1.22
100	6.65	4.13	2.93	722	1.74	11186	1.8	1.31

For $s = 10$ the drop height becomes significantly smaller than $100 \mu\text{m}$ if we want to have an energy gap $\Delta E/J \approx 10$ and a number of bounces not greater than $N_b = 54$. In the extreme case of $s = 2$, even if we allow for a drop height as small as $10 \mu\text{m}$ but keep $\Delta E/J \approx 10$ (i.e., we choose $\omega = 1.1$ and $\lambda = 0.0145$), we obtain the number of bounces required for tunnelling of non-interacting atoms $N_b = 235$. Clearly, time crystals of small size are much more demanding experimentally than the big time crystals.

For the soft Gaussian potential mirror and $s = 30$, $\omega = 4.45$, we obtain $N_b \approx 54$, $h_0 \approx 253$ and thus similar parameters to the corresponding hard-wall potential mirror case. However, the mirror oscillation amplitude for the soft Gaussian potential mirror is much more accessible experimentally, i.e., ($\sim 75 \text{ nm}$ for ^{39}K and a Gaussian width $\sigma_0 l_0 = 10 \mu\text{m}$ and height $V_0/V_{\text{max}} = 0.8$, where $V_{\text{max}} \approx 4.6 \times 10^3$). Still larger mirror oscillation amplitudes could be accessed, if required, by choosing a mirror potential height V_0/V_{max} closer to the peak in figure 7 (left panel), provided the corresponding classical resonance islands remain stable.

4. Experimental protocol

We present here an experimental protocol to realize a discrete time crystal based on a ^{39}K BEC bouncing resonantly on an oscillating mirror. As an example, we focus on an $s = 30$ system and a 532 nm Gaussian potential light-sheet mirror with $\sigma_0 l_0 = 10 \mu\text{m}$, $V_0/V_{\text{max}} = 0.8$, $V_{\text{max}} \approx 4.6 \times 10^3$. Other time crystals in the range $s \approx 20$ – 100 can be accessed using estimates of the optimal parameters given in table 2.

- Preparation of initial atom cloud.* We start with a BEC of $N \approx 5000$ ^{39}K atoms in a large 1064 nm crossed optical dipole trap (CODT) [46] located at a drop height $h_0 l_0 \approx 145 \mu\text{m}$ above the atom mirror. The longitudinal and transverse trap frequencies are adjusted to about 95 Hz to produce a spherical CODT, so that the standard deviation of the Gaussian atomic distribution $\sigma_z l_0 \approx 1.17 \mu\text{m}$ matches the width of the Wannier wave-packet $|w_i(z, 0)|^2$ at the classical turning point.
- Release of BEC from optical dipole trap.* The longitudinal trapping potential is then switched off to release the BEC from the CODT to fall on to the repulsive light-sheet mirror in the presence of a vertical 1064 nm optical waveguide with confinement frequency 95 Hz .
- Frequency and amplitude of oscillating mirror.* Next, the frequency of the oscillating mirror is tuned to the $s : 1$ resonance: $\omega = \frac{1}{2}s[g/(2h_0)]^{1/2}$ (e.g., $\omega/(2\pi t_0) = 2.8 \text{ kHz}$ for $s = 30$ and $h_0 l_0 \approx 145 \mu\text{m}$), and the amplitude of the mirror oscillation is set to $\lambda l_0/\omega^2 \approx 75 \text{ nm}$ (for a Gaussian potential mirror with $\sigma_0 l_0 = 10 \mu\text{m}$, $V_0/V_{\text{max}} \approx 0.8$) to create an energy gap $\Delta E/J \approx 10$. The light-sheet atom mirror is oscillated, e.g., by reflecting the beam off a piezo-driven optical mirror or possibly by modulating the intensity of the light-sheet beam.
- Detection of time crystal.* The atom density is measured at fixed positions between the classical turning point and the atom mirror and at different moments in time out to typically $t/T = 2000$ mirror oscillations (or about 0.6 s). Two kinds of measurements are performed: (i) With the particle interaction set to zero ($g_{1D}N = 0$), so that all atoms will have tunnelled out of the initially populated wave-packet into neighbouring wave-packets at $t/T = 2000$ to form a bunch of spatially separated wave-packets that spread with evolving time, as indicated in figures 3 and 4 of reference [6]. (ii) With the particle interaction sufficiently large to break the time-translation symmetry (e.g., $g_{1D}N = -0.2$), so that atoms will not tunnel from the initially populated wave-packet and the system evolves with period sT and without decay out to at least $t/T = 2000$, as indicated in figures 3 and 4 of reference [6]. The first measurement (i) is needed in order to demonstrate that without a sufficiently strong particle interaction there is no (sT) -periodic time evolution that breaks the time translation symmetry.

5. Experimental feasibility

Previous experiments have been successfully performed on a bouncing BEC dropped from heights 150–300 μm on to a static 532 nm light-sheet atom mirror [38], similar to the atom mirror proposed here, and the phase coherence of the BEC was found to be preserved following multiple bounces. Similar results have been found for a thermal atom cloud bouncing from an evanescent-wave light mirror [47].

5.1. Lifetime of bouncing BEC

The lifetime of a bouncing BEC may be limited by losses due to three-body recombination collisions, two-body collisions with background atoms and molecules, photon scattering caused by stray near-resonant light, and atoms missing the mirror due to spreading of the wave-packet in the transverse directions or walk-off.

The lifetime due to three-body recombination is given by $\tau_{3b} = [\langle n^2 \rangle K_3]^{-1}$, where $K_3 = 1.3 (5) \times 10^{-29} \text{ cm}^6 \text{ s}^{-1}$ for $^{39}\text{K} |1, +1\rangle$ atoms near the zero-crossing point [40]; for example, for $\sigma_z l_0 = \sigma_\perp l_0 = 1.17 \mu\text{m}$ (at the classical turning point) and $N = 5000$ atoms, we obtain $\langle n^2 \rangle \approx 8 \times 10^{27} \text{ cm}^{-6}$ and hence $\tau_{3b} \approx 10 (4) \text{ s}$. Losses due to collisions with background atoms can be kept to a negligible level by maintaining a high-quality vacuum ($< 10^{-11}$ millibar) and losses due to photon scattering should be negligible for light from the far-detuned 532 nm light-sheet mirror. Spreading of the atom cloud along the transverse directions will be suppressed due to the presence of the vertical optical waveguide. For atoms bouncing resonantly on an oscillating atom mirror, there will be no spreading in the longitudinal direction because each bounce from the oscillating mirror refocuses the atoms back [35]. In the calculations presented here, we assume about 50 bounces (corresponding to 1500 mirror oscillations for $s = 30$ and lasting about 0.6 s) can be achieved without significant atom losses for drop heights of around 200 μm .

5.2. Spatial resolution

For $s = 30$, $\omega = 4.45$ and $\omega_z/(2\pi t_0) = \omega_\perp/(2\pi t_0) \approx 95 \text{ Hz}$, the drop height $hl_0 \approx 145 \mu\text{m}$ is much larger than the longitudinal extension of the atom cloud ($2\sigma_z l_0 \approx 2\text{--}10 \mu\text{m}$ between the turning point and the atom mirror) and thus it should be possible to probe the position of the atom cloud with reasonable spatial resolution. Higher resolution could be achieved, if required, by choosing a higher s resonance, for which the drop height (in SI units) scales as $s^{4/3}$ (table 2).

5.3. Amplitude of mirror oscillation

For a soft Gaussian potential mirror with $\sigma_0 l_0 = 10 \mu\text{m}$, $V_0/V_{\text{max}} = 0.8$, $\omega = 4.45$, the optimal oscillation amplitude is $\lambda = 2.3$, or $\lambda l_0/\omega^2 = 74 \text{ nm}$ at a mirror oscillation frequency of 2.8 kHz for ^{39}K in the laboratory frame. Still larger mirror oscillation amplitudes could be used, if required, by operating closer to the peak in figure 7(a) provided the corresponding classical resonance islands remain stable. Our estimates show that perturbations due to mechanical vibrations transmitted by a typical optical table are negligible when we want to control the motion of the atom mirror to a few nanometers at oscillation frequencies of a few kilohertz.

5.4. Stray magnetic fields

In the present experiment, the s -wave scattering length a_s needs to be adjusted precisely to small negative values (e.g., $a_s l_0 = -1.6a_0$ for $g_{1D}N = -0.2$ with $N = 5000$) for the time crystal phase or to zero for the non-time crystal phase. For the broad $^{39}\text{K} |1, +1\rangle$ Feshbach resonance, tuning the scattering length to a precision of $\pm 0.1a_0$ requires the magnetic field to be stable to $\pm 0.18 \text{ G}$ which is much larger than the stray AC and DC magnetic fields achievable in the laboratory.

6. Discussion and conclusions

We have investigated the range of sizes s of discrete time crystals that can be created for a BEC of attractively interacting atoms bouncing resonantly on an oscillating mirror. We have also considered the effects of having a realistic soft Gaussian potential mirror, such as that produced by a repulsive light-sheet, and suitable atomic systems for performing a time crystal experiment.

We find that for reflection from a soft Gaussian potential mirror the optimal amplitude of the mirror oscillations ($\sim 75 \text{ nm}$ for ^{39}K atoms with $s = 30$, $\omega/(2\pi t_0) = 2.8 \text{ kHz}$) is about an order of magnitude larger, and hence more experimentally accessible, than the optimal oscillation amplitude for a hard-wall mirror. For reflection from a Gaussian potential mirror the trajectories of the atoms are smooth at the position of reflection, so that smaller amplitude harmonics are created and, as a result, larger mirror oscillation

amplitudes are needed to create a sufficiently large band gap ($\Delta E/J \approx 10$) and stable resonance islands, compared with a hard-wall potential mirror.

We find that using a ^{39}K BEC and realistic experimental parameters, it should be possible to create discrete time crystals with sizes in the range $s \approx 20\text{--}100$. For $s < 20$, the drop height starts to become small ($< 80\text{ }\mu\text{m}$ for the parameters considered), which makes it difficult to probe the atom density at different fixed positions with high spatial resolution, while for $s > 100$, the optimal mirror oscillation amplitude becomes small and the tunnelling times in the absence of interactions start to become long ($> 1.3\text{ s}$), which increases the time needed to perform an experiment compared with the lifetime of a bouncing BEC.

The robustness and stability of these time crystals against small perturbations are summarised in the form of a phase diagram of the detuning parameter Δh_0 against interaction strength $g_{1D}N$ in an accompanying paper in this issue [7].

Time crystals involving ultracold atoms bouncing resonantly on an atom mirror provide a platform for investigating a broad range of non-trivial condensed matter phenomena in the time domain. These include Mott insulator-like phases in the time-domain [26]; Anderson localization [6, 26] and many-body localization [27] due to temporal disorder; dynamical quantum phase transitions in time crystals [6, 28]; many-body systems with exotic long-range interactions [29]; time quasi-crystals—which are ordered but not periodic in time [29, 30]; and topological time crystals [31].

Acknowledgments

We thank Zoran Hadzibabic for suggesting the use of ^{39}K atoms and Shannon Whitlock for fruitful discussions. Support of the Australian Research Council (DP190100815) and the National Science Centre Poland via Projects No 2016/20/W/ST4/00314 and 2019/32/T/ST2/00413 (KG) and 2018/31/B/ST2/00349 (KS) is gratefully acknowledged. KG acknowledges the support of the Foundation for Polish Science (FNP).

References

- [1] Wilczek F 2012 Quantum time crystals *Phys. Rev. Lett.* **109** 160401
- [2] Bruno P 2013 Impossibility of spontaneously rotating time crystals: a No-Go *Phys. Rev. Lett.* **111** 070402
- [3] Watanabe H and Oshikawa M 2015 Absence of quantum time crystals *Phys. Rev. Lett.* **11** 251603
- [4] Kozin V K and Kyriienko O 2019 Quantum time crystals from Hamiltonians with long-range interactions *Phys. Rev. Lett.* **123** 210602
- [5] Sacha K 2015 Modeling spontaneous breaking of time-translation symmetry *Phys. Rev. A* **91** 033617
- [6] Giergiel K, Kosior A, Hannaford P and Sacha K 2018 Time crystals: analysis of experimental conditions *Phys. Rev. A* **98** 013613
- [7] Kuroś A, Mukherjee R, Golletz W, Sauvage F, Giergiel K, Mintert F and Sacha K 2020 Phase diagram and optimal control for n-tupling discrete time crystal (arXiv:2004.14982)
- [8] Khemani V, Lazarides A, Moessner R and Sondhi S L 2016 Phase structure of driven quantum systems *Phys. Rev. Lett.* **116** 250401
- [9] Else D V, Bauer B and Nayak C 2016 Floquet time crystals *Phys. Rev. Lett.* **117** 090402
- [10] Yao N Y, Potter A C, Potirniche I D and Vishwanath A 2017 Discrete time crystals. Rigidity, criticality, and realizations *Phys. Rev. Lett.* **118** 030401
- [11] Zhang J et al 2017 Observation of a discrete time crystal *Nature* **543** 217
- [12] Choi S et al 2017 Observation of discrete time-crystalline order in a disordered dipolar many body system *Nature* **543** 221
- [13] Pal S, Nishad N, Mahesh T S and Sreejith G J 2018 Temporal order in periodically driven spins in star-shaped clusters *Phys. Rev. Lett.* **120** 180602
- [14] Rovny J, Blum R L and Barrett S E 2018 Observation of discrete-time-crystal signatures in an ordered dipolar many-body system *Phys. Rev. Lett.* **120** 180603
- [15] Rovny J, Blum R L and Barrett S E 2018 ^{31}P NMR study of discrete time-crystalline signatures in an ordered crystal of ammonium dihydrogen phosphate *Phys. Rev. B* **97** 184301
- [16] Smits J, Liao L, Stoof H T C and van der Straten P 2018 Observation of a space-time crystal in a superfluid quantum gas *Phys. Rev. Lett.* **121** 185301
- [17] Liao L, Smits J, van der Straten P and Stoof H T C 2018 Dynamics of a space-time crystal in an atomic Bose-Einstein condensate *Phys. Rev. A* **99** 013625
- [18] Autti S, Eltsov V B and Volovick G E 2018 Observation of a time quasicrystal and its transition to a superfluid time crystal *Phys. Rev. Lett.* **120** 215301
- [19] Kreil A J E, Musienko-Shmarova H Y, Eggert S, Serga A A, Hillebrands B, Bozhko D A, Pomyalov A and L'vov V S 2019 Tunable space-time crystal in room temperature magnetodielectrics *Phys. Rev. B* **100** 020406
- [20] Guo L, Marthaler M and Schön G 2013 Phase space crystals: a new way to create a quasienergy band structure *Phys. Rev. Lett.* **111** 205303
- [21] Liang P, Marthaler M and Guo L 2018 Floquet many-body engineering: topology and many-body physics in phase space lattices *New J. Phys.* **20** 023043
- [22] Sacha K and Zakrzewski J 2018 Time crystals: a review *Rep. Prog. Phys.* **81** 016401
- [23] Else D V, Monroe C, Nayak C and Yao N Y 2020 Discrete time crystals *Annu. Rev. Condens. Matter Phys.* **11** 467
- [24] Khemani V, Moessner R and Sondhi S L 2019 A brief history of time crystals arXiv:1910.10745v1
- [25] Guo L and Liang P 2020 Condensed matter physics in time crystals arXiv:2005.03138

- [26] Sacha K 2015 Anderson localization and Mott insulator phase in the time domain *Sci. Rep.* **5** 10787
- [27] Mierzejewski M, Giergiel K and Sacha K 2017 Many-body localization caused by temporal disorder *Phys. Rev. B* **96** 140201
- [28] Kosior A and Sacha K 2018 Dynamic quantum phase transitions in discrete time crystals *Phys. Rev. A* **97** 053621
- [29] Giergiel K, Miroszewski A and Sacha K 2018 Time crystal platform: from quasicrystal structures in time to systems with exotic interactions *Phys. Rev. Lett.* **120** 140401
- [30] Giergiel K, Kuroś A and Sacha K 2019 Discrete time quasicrystals *Phys. Rev. B* **99** 220303
- [31] Giergiel K, Dauphin A, Lewenstein M, Zakrzewski J and Sacha K 2019 Topological time crystals *New J. Phys.* **21** 052003
- [32] Pizzi A, Knolle J and Nunnenkamp A 2020 Period-n discrete time crystals and quasicrystals with ultracold bosons *Phys. Rev. Lett.* **123** 150601
- [33] Pizzi A, Knolle J and Nunnenkamp A 2020 Higher-order and fractional discrete time crystals in clean long-range interacting systems arXiv:1910.07539
- [34] Surace F M, Russomanno A, Dalmonte M, Silva A, Fazio R and Iemini F 2019 Floquet time crystals in clock models *Phys. Rev. B* **99** 104303
- [35] Buchleitner A, Delande D and Zakrzewski J 2002 Non-dispersive wave packets in periodically driven quantum systems *Phys. Rep.* **368** 409
- [36] Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics (Applied Mathematical Sciences)* (Berlin: Springer)
- [37] Chirikov B V 1979 A universal instability of many-dimensional oscillator systems *Phys. Rep.* **52** 263
- [38] Bongs K, Burger S, Birkel G, Sengstock K, Ertmer W, Rzazewski K, Sanpera A and Lewenstein M 1999 Coherent evolution of bouncing Bose-Einstein condensates *Phys. Rev. Lett.* **83** 3577
- [39] Fattori M, D'Errica C, Roati G, Zaccanti M, Jona-Lasinio M, Modugno M, Inguscio M and Modugno G 2008 Atom interferometry with a weakly interacting Bose-Einstein condensate *Phys. Rev. Lett.* **100** 080405
- [40] Claussen N R, Kokkelmans S J J M F, Thompson S T, Donley E A, Hodby E and Wieman C E 2003 Very-high-precision bound-state spectroscopy near a ^{85}Rb Feshbach resonance *Phys. Rev. A* **67** 060701(R)
- [41] Marchant A L, Händel S, Hopkins S A, Wiles T P and Cornish S L 2012 Bose-Einstein condensation of ^{85}Rb by direct evaporation in an optical dipole trap *Phys. Rev. A* **85** 053647
- [42] D'Errico C, Zaccanti M, Fattori M, Roati G, Inguscio M, Modugno G and Simoni A 2007 Feshbach resonances in ultracold ^{39}K *New J. Phys.* **9** 223
- [43] Lepoutre S, Fouché L, Boissé A, Berthet G, Salomon G, Aspect A and Bourdel T 2016 Production of strongly bound K 39 bright solitons *Phys. Rev. A* **94** 053626
- [44] Pollack S E, Dries D, Junker M, Chen Y P, Corcovilos T A and Hulet R G 2009 Extreme tunability of interactions in a Li 7 Bose-Einstein condensate *Phys. Rev. Lett.* **102** 090402
- [45] Shotan Z, Machtay O, Kokkelmans S and Khaykovich L 2014 Three-body recombination at vanishing scattering lengths in an ultracold Bose gas *Phys. Rev. Lett.* **113** 053202
- [46] Salomon G, Fouché L, Lepoutre S, Aspect A and Bourdel T 2014 All-optical cooling of K 39 to Bose-Einstein condensation *Phys. Rev. A* **90** 033405
- [47] Steane A, Szriftgiser P, Delbailles P and Dalibard J 1995 Phase modulation of atomic de Broglie waves *Phys. Rev. Lett.* **74** 4972

