

Negative radiation pressure in soliton dynamics



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Abstract

This thesis aims to present the research on the so-called negative radiation pressure, i.e. a phenomenon when a travelling wave exerts a pulling force on a soliton. The effect was studied for three (1+1)-dimensional models: two-component Bose–Einstein condensates, abelian Higgs model and Q-balls. Additionally, the process of relaxation of a squashed Q-ball was analyzed. In order to explain the origin of the negative radiation pressure, we found the linearized spectrum of the solitons and derived the effective force and mass. The results show that the phenomenon is ubiquitous in physical systems, with several diverse underlying mechanisms. The thesis consists of three papers and an introduction.

Streszczenie

Celem tej rozprawy jest przedstawienie badań dotyczących tzw. ujemnego ciśnienia promieniowania – zjawiska polegającego na wywieraniu siły przyciągającej soliton przez poruszającą się falę. Efekt został zbadany w trzech (1+1)-wymiarowych modelach: dwuskładnikowym kondensacie Bosego–Einsteina, abelowym modelu Higgsa oraz Q-ballach. Dodatkowo została przeprowadzona analiza procesu relaksacji ściśniętego Q-balla. Aby wyjaśnić genezę ujemnego ciśnienia promieniowania, znaleźliśmy zlinearyzowane spektrum solitonów i obliczyliśmy ich efektywną siłę oraz masę. Wyniki pokazują, że zjawisko to jest powszechne w układach fizycznych, a stoi za nim kilka różnych mechanizmów. Rozprawa składa się z trzech publikacji oraz wstępu.

Contents

Abstract	1
Content of the thesis	3
Acknowledgements	4
1 Negative radiation pressure	5
1.1 Introduction to NRP	5
1.2 Linearization	7
1.3 Newtonian motion: effective force and mass	8
2 Summary of the results	9
3 Numerical methods	11
3.1 Initial remarks and software	11
3.2 RK4 method for partial differential equations	11
3.3 Split-step method	16
3.4 Gradient flow	17
3.5 Boundary value problems	18
4 Other work	21
4.1 Other published research	21
4.2 Heavy kinks	22
References	26
Articles	28

Content of the thesis

The thesis consists of the following articles:

- [A1] D. Ciurla, P. Forgács, Á. Lukács, and T. Romańczukiewicz, “Negative radiation pressure in Bose-Einstein condensates”, *Phys. Rev. E* **109**, 014228 (2024).
- [A2] D. Ciurla, “Negative radiation pressure in the abelian Higgs model”, *Physics Letters B* **856**, 138954 (2024).
- [A3] D. Ciurla, P. Dorey, T. Romańczukiewicz, and Y. Shnir, “Perturbations of Q-balls: from spectral structure to radiation pressure”, *Journal of High Energy Physics* **2024**, 196 (2024).

The appropriate statements from the coauthors of [A1] and [A3], confirming my contribution, were attached during the submission of the thesis. My contribution to the papers is listed below.

- I did the majority of the work in [A1], including the most of analytical calculations, all numerical simulations and a bulk of writing during preparation of the manuscript. I estimate my contribution to be 85% of the total work.
- I am the sole author of the letter [A2].
- In [A3], I did important numerical simulations and also contributed to analytical calculations and preparation of the manuscript. I estimate my contribution to be 55% of the total work.

The thesis is organized as follows. The introduction consist of four sections. In Section 1 the concept of the negative radiation pressure is introduced, together with discussion of the methods used to study this effect, such as linearization in Section 1.2 and computation of the effective force in Section 1.3. The main results of the articles [A1–A3] are summarized in Section 2. Then, the numerical methods used in the research are presented in Section 3. In Section 4 other work done by me during my PhD studies is discussed, including the brief summary of published research in Section 4.1, as well as the description of unpublished studies on the so-called heavy kinks in 4.2. Finally, after the introduction, the articles [A1–A3] are reprinted, being the main part of this thesis.

Acknowledgements

I would like to thank my supervisor, Tomasz Romańczukiewicz, for his guidance, and for introducing me to the fascinating topic of soliton dynamics. I also express my gratitude towards all of my collaborators and colleagues, especially the co-authors of the papers I published. Finally, I thank my partner, friends, and family for their support.

1 Negative radiation pressure

1.1 Introduction to NRP

A single, universally adopted definition of a soliton does not exist. For the purpose of this thesis, let us loosely define a soliton as a localized solution to nonlinear differential equations, which maintains its shape when propagating with a constant velocity. The first description of a soliton is attributed to John Scott Russell, who in 1834 observed such a wave in a body of water, and later reproduced the phenomenon experimentally [1]. Additional experiments were made [2], but a theoretical description was lacking until the publication of works by Boussinesq in 1871 [3], John William Strutt (better known as Lord Rayleigh) in 1876 [4], and Diederik Korteweg and Gustav de Vries in 1895 [5], with their famous Korteweg-de Vries (KdV) equation. Many years after these initial discoveries, the subject was reborn in 1958 with the theoretical studies of solitons in plasma physics by J. H. Adlam and J. E. Allen [6, 7], followed by their second paper in 1960 [8]. In 1965, Norman Zabusky and Martin Kruskal, showed, using numerical simulations, that solitons in the KdV equation retain their shapes and velocities after collision. In that paper, they coined the term *soliton*, which we can mark as a symbolic beginning of a modern era of soliton studies. Since then, solitons have been found and studied, both theoretically and experimentally, in many areas of physics, such as condensed matter physics [9, 10], optics [11, 12], nuclear physics [13], Bose–Einstein condensates [14], cosmology [15], particle physics [16], DNA physics [17–19], solid mechanics, fluid mechanics, biomechanics [20] and more.

One of the key features of solitons is their particle-like behavior. Nevertheless, the analogy to physical particles breaks down in some scenarios, leading to seemingly unexpected dynamics. One of such cases can occur when a soliton is interacting with a travelling wave (radiation). Intuitively, the soliton should be pushed in the direction of propagation of the incoming wave. However, sometimes the pulling force is exerted on the soliton, and we call this phenomenon a negative radiation pressure (NRP). It was observed for the first time for kinks in ϕ^4 model [21–23], and then for domain walls in ϕ^6 model [24, 25], graphene simulations [26] and other models [27, 28]. The articles contained in this thesis expand this list by dark-bright solitons in Bose–Einstein condensates [A1], domain walls in (1+1)-dimensional abelian Higgs model [A2] and (1+1)-dimensional Q-balls [A3].

The definition of NRP in terms of soliton acceleration had been sufficient for most cases, until the need for a more accurate definition arose during the research of the dark and dark-bright solitons in Bose–Einstein condensates. For such solitons the effective mass is negative, and they behave more like holes than particles. In that case, the force exerted in the direction of the incoming wave pulls the soliton, and we want to differentiate that from our usual notion of NRP. Therefore, following the conventions introduced in [A1], we define that a negative radiation pressure (NRP) is present when the effective force, exerted on a soliton by an incoming wave, has the opposite direction as the direction of propagation of the wave. Similarly, a positive radiation pressure (PRP) occurs when the force has the same direction as the wave. This definition, which uses force instead of acceleration, can be unambiguously applied to all the papers this thesis contains.

In general, NRP occurs due to a transfer of energy from the incoming wave to scattering states with different momenta. However, the details of the underlying cause can vary con-

siderably. To investigate the possible mechanisms, the incoming wave is treated as a small perturbation of the soliton, assuming its amplitude is sufficiently small. For example, for a real scalar field the full solution of the nonlinear equation of motion is approximated as

$$\phi = \phi_s + \xi = \phi_s + a\xi^{(1)} + a^2\xi^{(2)} + \dots, \quad (1)$$

where ϕ_s is the soliton and a is the amplitude of the incoming wave ξ . Henceforth, whenever we refer to a first, second, linear or nonlinear order etc., we mean the order in the amplitude of the incoming wave. So far, the following mechanisms responsible for NRP have been observed:

- I Nonlinear coupling together with a reflectionless linearized potential.
- II Two coupled fields with different dispersion relations in scattering modes.
- III Complex field rotating in phase, having two possible scattering channels for each incoming wave.
- IV Asymmetric solitons.

Type I NRP was observed for kinks in the ϕ^4 model and its small perturbations [21–23], and although historically the earliest, this is believed to be a very rare mechanism. It is worth noting that the reflectionlessness of a scattering potential in the linear order does not necessarily imply the existence of NRP. For example, in the well-known sine-Gordon model there is no force exerted on kinks by a radiation, and therefore no NRP. This is due to the fact that the potential is reflectionless in all orders, not only the first one, and as a result any momentum cannot be transferred to the soliton. Mechanisms of the types II, III and IV can already be visible in the linear approximation and are believed to be more widespread. Type II was discovered for a toy model in [28], but we used it for a more physical setting of two-component Bose–Einstein condensates in [A1]. We found type III NRP for Q-balls in [A3]. Type IV was observed in ϕ^6 model [24, 25], and we found it in the abelian Higgs model [A2], in both cases due to the presence of asymmetric domain walls between different vacua.

Most of the research on NRP, this thesis included, focuses on 1-dimensional models, although 2D systems were considered in [24, 27] and a 3D model was briefly studied in [24]. It is worth emphasizing that this is simply due to simplicity of 1D models, not the lack of NRP in higher dimensions. On the contrary, we expect NRP in 2D and 3D systems to be an interesting field for further studies.

At the time of writing this thesis, no experimental realization of NRP is known. However, our findings for Bose–Einstein condensates in [A1] are especially promising from the phenomenological point of view, and we hope that they will be confirmed in experiments. It is also worth noting that NRP was observed in simulations of buckled graphene nanoribbons [26] and in some other promising physical models [27]. Moreover, we anticipate that the other results contained in this thesis will help to better understand the phenomenon of negative radiation pressure and to find it in real physical systems.

1.2 Linearization

NRP in the articles contained in this thesis was explained with a reasonable accuracy by a linear order approximation. This method is useful not only for explaining the negative radiation pressure, but it is also a standard procedure in many other problems, e.g. soliton stability analysis [29–31]. Although the details of a linearization depend on the model, let us present a simplified broad introduction to this approach for (1+1) dimensions. The specific equations are described in the papers [A1–A3].

We start with a perturbative expansion, such as (1), and substitute it into the full partial differential equations (PDEs) of the model. In this way another system of time-dependent PDEs is obtained, which by separation of time t and space x variables can be reduced to Schrödinger-like ODEs of the form

$$-\eta''(x) + V(x)\eta(x) = E\eta(x), \quad (2)$$

where η can be a single function or a vector of functions; then $V(x)$ and E are numbers or matrices respectively. In the latter case, E is assumed to be diagonal, but not necessarily proportional to the identity. The number of components in this system of equations can be greater than the number of components in the original PDE. In particular, if the soliton is rotating in phase as in [A1, A3], each perturbation splits into two solutions with two eigenvalues, therefore doubling the number of components.

Different types of solutions to (2) can be found, including:

- Bound modes, i.e. localized solutions corresponding to the discrete spectrum. They include zero modes, i.e. those with zero eigenvalue.
- Scattering modes, related to the continuous spectrum.
- Quasinormal modes, i.e. states with complex eigenvalues: they oscillate similarly as the scattering modes, but decay exponentially. They are also sometimes called resonance modes, but we follow the terminology used in [32].

Sometimes more terminology is introduced, e.g. in [A3] we used the name *half-propagating modes* to describe solutions which propagates in only one of the two components. For the analysis of NRP at the linear level, it is sufficient to know the scattering modes. However, in higher orders, couplings with bound modes may be visible as resonances in the soliton acceleration. This was observed in [23] for kinks in the ϕ^4 model, although the detailed description requires further research. Quasinormal modes may also play a similar role, but it remains to be studied.

Analytical solutions to (2) can be found in some models (e.g. [A2]), but in many cases we have to implement numerical methods, described in Section 3.5. However, usually asymptotic form of scattering modes can be easily obtained, from which we get the range of the continuous spectrum and appropriate dispersion relations. This allows us to extract the usual scattering data, i.e. transmission and reflection coefficients (or S -matrix elements).

1.3 Newtonian motion: effective force and mass

In the linear order, we expect that solitons behave approximately as Newtonian particles. This approach is based on an assumption that the soliton interacting with small-amplitude waves approximately does not change its shape, which allows us to treat the position of its center x_0 as a function of time $x_0(t)$, obeying the Newton's law

$$M\ddot{x}_0 = F \quad (3)$$

with an effective mass M and an effective force F . These effective quantities are obtained from the total energy and momentum, and the conservation laws. First, we compute the energy-momentum tensor T^μ_ν for a given model. Then, the total energy and momentum are

$$E = \int_{-\infty}^{\infty} T^0_0 dx, \quad P = - \int_{-\infty}^{\infty} T^0_1 dx, \quad (4)$$

where 0 is the temporal, and 1 the spatial coordinate.

Let us denote the total momentum of the soliton moving with velocity v as $P_s(v)$. Then, the effective mass can be defined as

$$M = \left. \frac{dP_s}{dv} \right|_{v=0}, \quad (5)$$

or, alternatively, using the total energy $E_s(v)$:

$$M = \left. \frac{d^2 E_s}{dv^2} \right|_{v=0}. \quad (6)$$

These two definitions should give the same result, provided the Newton's law approximation indeed holds for the considered soliton motion. In most cases, computing the total momentum and the total energy of the moving soliton is straightforward using (4). In some models there are, however, some nuances, which when neglected can lead to the incorrect definition of P_s and E_s , and thus the wrong mass. Dark solitons, and the dark components of the dark-bright solitons, in coupled nonlinear Schrödinger equations have a form of a dip in a constant background. In [A1] we discuss how P_s and E_s can be properly renormalized to get rid of the contribution from the background. In that case, the resulting mass is negative, which lead us to the more precise definition of NRP discussed earlier. In relativistic models (with $c = 1$), such as [A2, A3], we expect that M coincides with the rest energy of the soliton.

Let P be the total momentum of the soliton and the waves, then the effective force is defined as

$$F = \langle \partial_t P \rangle_T, \quad (7)$$

where $\langle \cdot \rangle_T$ is the average over a period of the radiation. To compute the rate of change of the momentum, we use the conservation law

$$\partial_t P = T^1_1 \Big|_{-\infty}^{\infty}, \quad (8)$$

derived from the Noether's theorem. Since the formula requires only the values of the energy-momentum tensor at infinities, we can use the asymptotic form of the waves with appropriate

transmission and reflection coefficients. Let us assume that the linearized equations (2) for the considered model have n components. Usually, the obtained formula for a force exerted by a plane wave of the amplitude a incoming from the i -th scattering component has a form

$$F \sim a^2 \sum_{j=1}^n [k_j^2 (\delta_{ij} + R_j) - p_j^2 T_j] , \quad (9)$$

where R_j and T_j are the reflection and transmission components in the j -th component, and k_j and p_j are wavenumbers at $-\infty$ and $+\infty$ respectively. It is clear from the above equation, that the wavenumbers play a crucial role in changing the sign of the force, and therefore in the mechanisms behind the negative radiation pressure. For NRP of the types II and III the wavenumbers k_j and p_j are equal for given j , but they differ between components. There is always more than one scattering component in models with NRP of these types: for the type III NRP the number of components is doubled due to the rotation in phase of the soliton (as discussed in the Section 1.2), and for the type II there are multiple scattering channels simply due to the presence of multiple fields. For the type IV we usually have one component, but with different k_1 and p_1 .

Let us finish this discussion with a remark about the classification of NRP in the two-component Bose–Einstein components, described in [A1]. In this model, there are in fact four scattering components, because the solitons are both rotating in phase and spanned across two-component wavefunctions. However, since there is no radiation pressure in the one-component condensates, we decided that the presence of the multiple fields is crucial, thus we ultimately classified NRP in this model as the type II.

2 Summary of the results

The articles [A1–A3] are an integral part of this thesis. However, since they are reprinted without any rewriting, we briefly summarize their main results below, for the convenience of the reader, and for completeness of this introduction.

In the first article [A1], we studied one-dimensional, two-component Bose–Einstein condensates in the mean-field regime, in which they are described by two coupled nonlinear Schrödinger (Gross–Pitaevskii) equations:

$$\begin{aligned} i\hbar \frac{\partial \psi_1}{\partial t} &= -\frac{1}{2m_1} \frac{\partial^2 \psi_1}{\partial x^2} + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2) \psi_1 + V_1(x)\psi_1 , \\ i\hbar \frac{\partial \psi_2}{\partial t} &= -\frac{1}{2m_2} \frac{\partial^2 \psi_2}{\partial x^2} + (g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2) \psi_2 + V_2(x)\psi_2 . \end{aligned} \quad (10)$$

Physically, such equations describe a condensate in a trap elongated in one of the directions, with two distinguishable components. We focused on a case when the condensate is made of two different spin states of the same atomic species, therefore the atomic masses m_1 and m_2 are equal. Moreover, we assumed $g_{11} = g_{22} = 1$ and kept the coupling g_{12} as a free parameter. In the main part of the paper, we omitted the trapping potentials V_1 and V_2 , and later confirmed that our results still qualitatively apply to the equations with a harmonic trap.

First, we analyzed stationary solutions to (10) called dark solitons, which have a form of a dip in the probability density of one of the components, and their wavefunction rotates in phase. Their interaction with a wave in the same component is trivial: the system reduces to a single-component condensate, which is an integrable system, and there is no radiation pressure in this case. However, for the wave incoming in the other component, we observed a positive radiation pressure (and no pressure for $g_{12} = 0, 1$), although the acceleration was in the opposite direction due to the negative effective mass of the solitons. The accelerations obtained from the Newtonian approximation in the linear regime matched the numerical simulations with a very high accuracy.

Second, the dark-bright solitons were studied. They consist of a dark soliton in one component, and a bright soliton (i.e. a peak) in the second component, and, similarly as the dark soliton, they have a negative effective mass. We found that if the incoming wave is in the dark or bright component, the solitons experience NRP (type II) or PRP respectively. Our approximations matched the full PDEs simulations for most cases. We noticed discrepancy for larger amplitudes of the waves, which is expected, since we used only the linear order. However, we also encountered an inaccuracy for high frequency waves in the dark component, which is not well understood yet, and requires further research.

The second article [A2] contains the study of the well-known abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*D^\mu\phi - U(|\phi|^2), \quad (11)$$

but with a sextic potential

$$U(|\phi|^2) = |\phi|^2(|\phi|^2 - 1)^2. \quad (12)$$

We reduced this (3+1)-dimensional theory to a restricted (1+1)-dimensional model, assuming that the vector potential A_μ has only one nonvanishing component, orthogonal to x , and that the phase of the scalar field ϕ is fixed. This simplified model has an asymmetric domain wall solution, interpolating between the two different vacua. We studied the interactions of the domain wall with small-amplitude waves in the gauge field. We found that the soliton moves in the same direction, regardless of the direction of the incoming wave. Therefore, depending on which side the radiation comes from, both PRP and NRP are exerted. Since the effect is present due to the asymmetry of the domain wall, the observed NRP is of the type IV. For sufficiently small amplitudes and charges, the nonlinear effects on the accelerations were negligible, and we successfully described the soliton motion with the derived effective force.

Finally, in [A3] we researched Q-balls in a (1+1)-dimensional complex scalar field theory:

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - U(|\phi|^2), \quad (13)$$

with a potential

$$U(|\phi|^2) = |\phi|^2 - |\phi|^4 + \beta|\phi|^6. \quad (14)$$

Q-balls, similarly as the dark and dark-bright solitons mentioned above, are stationary solutions rotating in phase. Due to this fact, the number of scattering channels doubles (see Section 1.2) and allows for NRP of the type III to be present in this model. We found that for waves incoming in one of the scattering components, the solitons always experience PRP, while for waves in the second component both PRP and NRP can be present, depending on

the Q-ball and wave frequencies. Again, the linearized spectrum and the effective Newtonian motion were sufficient to explain the phenomenon.

Additionally, another type of perturbation was studied in the paper. We squashed a Q-ball and analyzed its relaxation, finding that the process is dominated by a decay of a bound mode into a quasinormal mode and scattering modes. We repeated the process for metastable Q-balls with $\beta = 0$, and compared our results with [33]. In [33] the authors concluded that the oscillations during the relaxation are some form of a breather, however our analysis have shown that this claim can be overstated.

In conclusion, we found negative radiation pressure of the types II, III and IV in three different (1+1)-dimensional models. It is now clear that the effect, initially thought to be a curiosity in some specific field theories, is ubiquitous in various physical systems. We also point out that there are many opportunities for further research in the field, such as understanding the role of the bound modes in NRP, searching for NRP in other models, analyzing e.g. dark-dark or bright-bright solitons in Bose–Einstein condensates, etc. Our hope is that the research contained in this thesis will contribute to better theoretical understanding of the phenomenon of negative radiation pressure, and will bring it closer to the experimental realization.

3 Numerical methods

3.1 Initial remarks and software

Most of the problems in soliton dynamics require the use of numerical approximations, and the articles contained in this thesis are no exception. Below, we present the most important methods we used. Since we were dealing exclusively with (1+1)-dimensional systems, in most cases we present 1-dimensional versions of the methods, with some exceptions in which the generalization to higher methods does not affect the clarity of the text (e.g. gradient flow method). All numerical calculations in [A1, A2], and these done by the author in [A3], were implemented using custom Python code with Numpy and Scipy libraries.

3.2 RK4 method for partial differential equations

Let us start the discussion of numerical methods with the methods used to solve original, non-simplified partial differential equations (PDEs) governing the dynamics of the models. We used two methods for that: the fourth order Runge–Kutta method (RK4) in [A2, A3] and the second order split-step method in [A1]. In this thesis, we explain both of them, starting with the former in this section. For a more comprehensive introduction to the RK4 method, see e.g. [34–36].

The RK4 method is the most well-known of the Runge–Kutta methods family. In its original form, it is used to solve initial value problems of the type

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y^0, \quad (15)$$

where y is an unknown function of a variable t , f is some function, and we look for a solution for $t > t_0$, where t_0 is some initial parameter for which y has a given value y^0 (0 is an index

here, not an exponent). To solve this problem, we pick some step size $\Delta t > 0$ and discretize the variable t :

$$t_n = t_0 + n \Delta t \quad n = 0, 1, \dots, N, \quad (16)$$

where N is some arbitrary number of points. We denote values of the function y at the discretized points as

$$y^n = y(t_n). \quad (17)$$

Then, we can iteratively compute a value of y at $(n+1)$ th step, knowing its value at n th step (or more previous steps in some other methods). To illustrate the principle, let us present an example of a simpler method, known as the explicit Euler method. We can approximate the derivative $\frac{dy}{dt}$ at the point t_n as the finite difference

$$\left. \frac{dy}{dt} \right|_{t=t_n} \approx \frac{y^{n+1} - y^n}{\Delta t}, \quad (18)$$

and after inserting that to the equation (15) and rearranging we obtain the expression for the value of the function y at the $(n+1)$ th step:

$$y^{n+1} = y^n + \Delta t f(t_n, y^n). \quad (19)$$

The above formula for Euler method can be also read from the Taylor expansion of $y^{n+1} \equiv y(t_n + \Delta t)$, which after dropping $\mathcal{O}(\Delta t^2)$ term gives the same equation as above. The Euler method, although simple, is not very accurate. One of the ways to improve accuracy is to evaluate function f at some points between t_n and t_{n+1} . The Runge–Kutta methods use this trick, and the expression for RK4 method is

$$y^{n+1} = y^n + \Delta t \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}, \quad (20)$$

where

$$\begin{aligned} k_1 &= f(t_n, y^n), \\ k_2 &= f\left(t_n + \frac{\Delta t}{2}, y^n + \frac{\Delta t}{2} k_1\right), \\ k_3 &= f\left(t_n + \frac{\Delta t}{2}, y^n + \frac{\Delta t}{2} k_2\right), \\ k_4 &= f(t_n + \Delta t, y^n + \Delta t k_3). \end{aligned} \quad (21)$$

We omit the exact proof of the above formula (for the detailed derivation see e.g. [35]), but let us provide some intuition. In the equation (20) the next function value y^{n+1} is computed as the present value y^n plus the steps multiplied by the weighted average of four parameters: k_1, k_2, k_3, k_4 . These parameters represent estimated slopes computed from the function f : k_1 is the slope at the beginning of the interval $[t_n, t_{n+1}]$ (i.e. slope from the Euler method), k_4 at the end and k_2, k_3 at the midpoint, but k_3 is calculated using k_2 slope instead of k_1 . Note, that they have to be computed in the order from k_1 to k_4 , since each slope reference the previous one. For more details on the RK4 slopes and comparison with the Euler method, see figure 1.

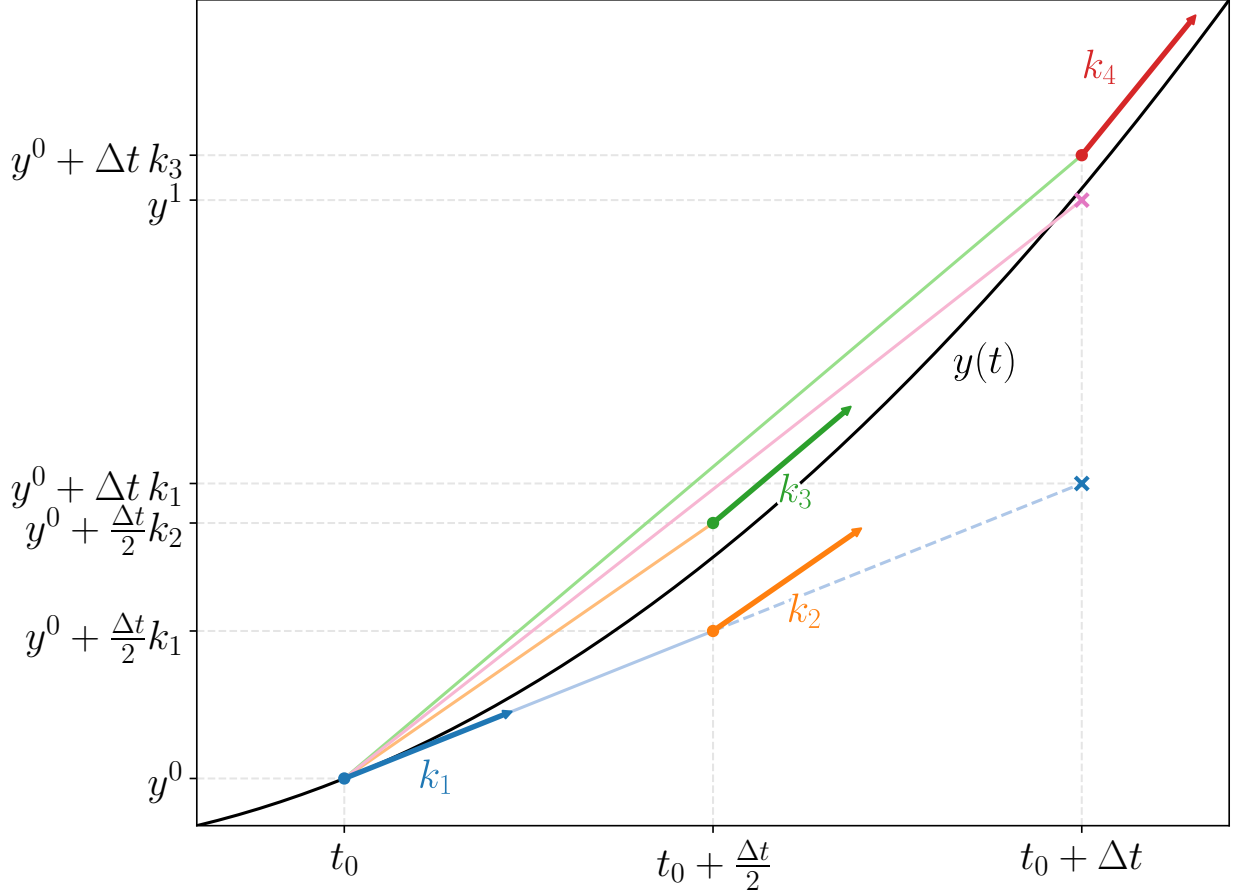


Figure 1: Illustration of the first step in RK4 and Euler methods. Both algorithms start at the point (t_0, y^0) (blue round point). In the Euler method, we progress by a step Δt with the slope k_1 , and we get to the final point (blue cross). In the RK4 method, we first use the slope k_1 , using only the half of the step size, and get to the orange point, at which k_2 is computed. Then we get back to the initial (blue round) point and repeat the process using the slope k_2 . This gets us to the green point, at which we compute k_3 . Then, we go back to the initial point again and use k_3 with the whole step to get to the red point, at which k_4 is calculated. Finally, the weighted average of all four slopes is computed, and it is used to get from the initial to the final point (purple cross), with y^1 given by the equation (20). Note that the final point of the step for RK4 method (purple cross) is much closer to the exact function value (black line) than the final point of the Euler method (blue cross).

Without any modifications, RK4 (and other similar explicit methods) can be applied to systems of multiple equations by performing each step simultaneously for all unknowns, but we have to remember to compute k_1 for all the equations before we proceed to k_2 etc. This also allows us to solve initial value problems for higher order ordinary differential equations (ODEs), since they always can be reduced to a system of first order ODEs. In particular, consider the second order initial value problem of the following form:

$$\frac{d^2y}{dt^2} = f(t, y), \quad y(t_0) = y^0, \quad \left. \frac{dy}{dt} \right|_{t=t_0} = v^0. \quad (22)$$

After introducing a new function $v(t)$, this can be reduced to a system of two equations:

$$\begin{aligned} \frac{dy}{dt} &= v(t), \quad y(t_0) = y^0, \\ \frac{dv}{dt} &= f(t, y), \quad v(t_0) = v^0, \end{aligned} \quad (23)$$

which can be solved using the simultaneous RK4 approach mentioned above.

Finally, let us apply this knowledge to partial differential equations (PDEs). In [A2, A3], we are dealing with (1+1)-dimensional nonlinear Klein–Gordon-type equations of the form

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = F(y, t, x), \quad y(0, x) = y^0, \quad \left. \frac{\partial y}{\partial x} \right|_{t=0} = v^0 \quad (24)$$

on the interval $x \in [x_0, x_M]$, with Dirichlet boundary conditions:

$$y(t, x_0) = a, \quad y(t, x_M) = b. \quad (25)$$

In the context of this thesis, the variable t in the above equation is interpreted as the time, and x as the spatial coordinate. To apply the RK4 method to such a PDE, we discretize x in the same way as the time coordinate in equation (16), with the step Δx . For some arbitrary number M , we denote

$$\begin{aligned} x_m &= x_0 + m \Delta x \quad m = 0, 1, \dots, M, \\ y_i(t) &= y(t, x_i). \end{aligned} \quad (26)$$

Then, the second derivative with respect to x can be approximated as a finite difference, similarly as the first derivative w.r.t. t was approximated in (18). We call such an approach a finite difference method, the alternative being e.g. spectral methods, such as the split-step method discussed later. There are infinitely many possible finite difference approximations for a second derivative, we choose a fourth-order central formula, also known as a five-point stencil:

$$\left. \frac{\partial^2 y}{\partial x^2} \right|_{x=x_i} \approx \frac{-y_{i-2} + 16y_{i-1} - 30y_i + 16y_{i+1} - y_{i+2}}{12\Delta x^2}. \quad (27)$$

This formula can be obtained by computing the Taylor series of $y_{i\pm 1}(t) \equiv y(t, x_i \pm \Delta x)$ and $y_{i\pm 2}(t) \equiv y(t, x_i \pm 2\Delta x)$ up to the fourth order in Δx , and finding the coefficients by which these four expansions and the fifth term $y_i(t) \equiv y(t, x_i)$ have to be multiplied in order to

add up to the second derivative. Combining this discretization with the introduction of $v_i(t)$ analogously as in (23), we reduce the PDE (24) to a system of $2M$ equations:

$$\begin{aligned}\frac{dy_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= \frac{-y_{i-2} + 16y_{i-1} - 30y_i + 16y_{i+1} - y_{i+2}}{12\Delta x^2} + F(y_i, t, x_i) \equiv f_i(t, y_i)\end{aligned}\quad (28)$$

with conditions

$$\begin{aligned}y_i(t_0) &= y_i^0, \quad v_i(t_0) = v_i^0, \\ y_0(t) &= a, \quad y_M(t) = b.\end{aligned}\quad (29)$$

Note, that since values y_0 and y_M are fixed, there is no need to compute derivatives at x_0 and x_M . However, we have to compute derivatives at x_1 and x_{M-1} using different formulas than (27), therefore we assume that equations (28) are appropriately modified for $i = 1, M - 1$: for the precise formulas see e.g. [37]. We can now discretize the time t as in (16), denoting

$$y_i^n = y(t_n, x_i), \quad v_i^n = v(t_n, x_i), \quad (30)$$

and apply the RK4 method to the system of equations (28).

Below, for completeness, we present the RK4 scheme applied to the equations (28). For the sake of brevity, we introduce a vector notation:

$$\begin{aligned}\vec{y}^n &= (y_0^n, \dots, y_M^n), \quad \vec{v}^n = (v_0^n, \dots, v_M^n), \\ \vec{k}_m &= ((k_m)_0, \dots, (k_m)_M), \quad \vec{p}_m = ((p_m)_0, \dots, (p_m)_M), \quad m = 1, 2, 3, 4, \\ \vec{f}(t, \vec{y}) &= (f_0(t, y_0), \dots, f_M(t, y_M)),\end{aligned}\quad (31)$$

with f_i defined in (28). Then, the RK4 step consists of the following successive substeps:

$$\begin{aligned}1) \quad \vec{k}_1 &= \vec{v}^n & \vec{p}_1 &= \vec{f}(t_n, \vec{y}^n) \\ 2) \quad \vec{k}_2 &= \vec{v}^n + \frac{\Delta t}{2} \vec{p}_1 & \vec{p}_2 &= \vec{f}(t_n + \frac{\Delta t}{2}, \vec{y}^n + \frac{\Delta t}{2} \vec{k}_1) \\ 3) \quad \vec{k}_3 &= \vec{v}^n + \frac{\Delta t}{2} \vec{p}_2 & \vec{p}_3 &= \vec{f}(t_n + \frac{\Delta t}{2}, \vec{y}^n + \frac{\Delta t}{2} \vec{k}_2) \\ 4) \quad \vec{k}_4 &= \vec{v}^n + \Delta t \vec{p}_3 & \vec{p}_4 &= \vec{f}(t_n + \Delta t, \vec{y}^n + \Delta t \vec{k}_3) \\ 5) \quad \vec{y}^{n+1} &= \vec{y}^n + \Delta t \frac{\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4}{6} & \vec{v}^{n+1} &= \vec{v}^n + \Delta t \frac{\vec{p}_1 + 2\vec{p}_2 + 2\vec{p}_3 + \vec{p}_4}{6}\end{aligned}\quad (32)$$

The above pattern can also be easily applied to systems of multiple equations of the type (24), as it was done in [A2]. To do that, each element y_i^n , v_i^n , $(k_m)_i$ and $(p_m)_i$ should be treated as a vector, with each component of the vector corresponding to one of the unknown functions.

The total accumulated error of the RK4 method is of the order $O(\Delta t^4)$. To control the errors, one can use a modified RK4 method with an adaptive step size. However, we used fixed step size and tested the accuracy by changing Δt and looking at the convergence of the results, and we performed similar tests for Δx .

3.3 Split-step method

To describe two-component Bose–Einstein condensates in [A1], we used two coupled nonlinear Schrödinger equations, which were solved using the so-called split-step method [38–40], described in this section. For simplicity, let us start with a single-component 1-dimensional nonlinear Schrödinger equation of the form

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + V(x)\psi, \quad (33)$$

where ψ is a function of x and t , and g is some real parameter. We can introduce a linear operator $\mathcal{L}$ and a nonlinear operator $\mathcal{N}$:

$$\mathcal{L} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}, \quad \mathcal{N} = g|\psi|^2 + V(x), \quad (34)$$

and then the equation (33) can be written as

$$i\frac{\partial\psi}{\partial t} = (\mathcal{L} + \mathcal{N})\psi. \quad (35)$$

Similarly as for the RK4 method, we discretize the time domain with the step Δt , assuming some initial time t_0 and denoting

$$\psi^n(x) = \psi(t_n, x). \quad (36)$$

Then, we can use the approximate formula

$$\psi^{n+1} = e^{-i\Delta t(\mathcal{L}+\mathcal{N})}\psi^n. \quad (37)$$

Note, that this formula would be exact if $\mathcal{L}$ and $\mathcal{N}$ were both independent of t . In order to be able to treat the derivative term $\mathcal{L}$ separately, we have to split the exponential functions into a product of two. In general, the operators $\mathcal{L}$ and $\mathcal{N}$ do not commute, but we can use only the first terms of the Baker–Campbell–Hausdorff formula and approximate:

$$\psi^{n+1} = e^{-i\Delta t\mathcal{L}}e^{-i\Delta t\mathcal{N}}\psi^n. \quad (38)$$

If we use the Fourier transform and go from x in the position space to p in the momentum space, the derivative $\frac{\partial}{\partial x}$ is substituted by ip , and consequently the operator $\mathcal{L}$ becomes $\frac{p^2}{2}$. Therefore, denoting the Fourier transform as $\mathcal{F}$, the formula (38) in the momentum space is

$$\mathcal{F}[\psi^{n+1}(x)] = e^{-\Delta t\frac{p^2}{2}}\mathcal{F}[e^{-i\Delta t\mathcal{N}}\psi^n(x)]. \quad (39)$$

Using the inverse transform $\mathcal{F}^{-1}$, we obtain the formula

$$\psi^{n+1} = \mathcal{F}^{-1}\left[e^{-\Delta t\frac{p^2}{2}}\mathcal{F}[e^{-i\Delta t\mathcal{N}}\psi^n]\right]. \quad (40)$$

Knowing this formula, we discretize the spatial coordinate x with a step Δx , similarly as in the RK4 method. Then, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are replaced with the discrete Fourier transform

(DFT) and its inverse. In practice, we use the fast Fourier transform (FFT) algorithm to compute DFT, and we may use the coordinate $\xi = \frac{p}{2\pi}$ instead of p , depending on the specific implementation. More about DFT and FFT can be found, e.g. in [36].

The above method is called the first order split-step method, and it can be generalized to higher accuracy schemes. In [A1], we used the second order split-step method, which involves taking only half of the time step with $\mathcal{N}$, then the full step with $\mathcal{L}$, and then the remaining half step with $\mathcal{N}$:

$$\psi^{n+1} = e^{-i\frac{\Delta t}{2}\mathcal{N}} \mathcal{F}^{-1} \left[e^{-\Delta t \frac{p^2}{2}} \mathcal{F} \left[e^{-i\frac{\Delta t}{2}\mathcal{N}} \psi^n \right] \right]. \quad (41)$$

Note, that it is not much more computationally expensive than the first-order method, since the Fourier transforms are evaluated the same number of times. This is not the case for methods of order higher than 2: such methods can be found e.g. in [38, 39].

Generalization to systems of equations is straightforward, by promoting ψ to be a vector of component functions, and rewriting the system in the form of equation (35). For example, consider the coupled equations from [A1]:

$$\begin{aligned} i \frac{\partial \psi_1}{\partial t} &= -\frac{1}{2} \frac{\partial^2 \psi_1}{\partial x^2} + (|\psi_1|^2 + g_{12}|\psi_2|^2) \psi_1 + V_1(x) \psi_1, \\ i \frac{\partial \psi_2}{\partial t} &= -\frac{1}{2} \frac{\partial^2 \psi_2}{\partial x^2} + (|\psi_2|^2 + g_{12}|\psi_1|^2) \psi_2 + V_2(x) \psi_2. \end{aligned} \quad (42)$$

They can be rewritten as (35) with ψ , $\mathcal{L}$ and $\mathcal{N}$ defined as:

$$\begin{aligned} \psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \mathcal{L} = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2}, \\ \mathcal{N} &= \begin{pmatrix} |\psi_1|^2 + g_{12}|\psi_2|^2 + V_1(x) & 0 \\ 0 & |\psi_2|^2 + g_{12}|\psi_1|^2 + V_2(x) \end{pmatrix}, \end{aligned} \quad (43)$$

and the method can be applied in the same way as for the single component. In our approach, we used the second order split-step method with periodic boundary conditions, although other methods with more sophisticated boundary conditions are possible to implement for such equations (e.g. [41]).

3.4 Gradient flow

Although full PDEs are usually possible to solve only using numerical methods, simple special stationary solutions, such as solitons, are quite often given analytically. However, in general this is not the case, e.g. most of dark-bright solitons in [A1] had to be computed numerically. To do that, we used a method called gradient flow (or gradient descent), which is very convenient due to its simplicity, but it has some drawbacks such as a slow convergence.

The basic version of gradient descent is used to find a local minimum of a function f of variables $\vec{x}$ [42]. We choose some small step $\gamma > 0$ and make the initial guess for the minimum, denote it $\vec{x}_0$. Then, we iterate:

$$\vec{x}^{n+1} = \vec{x}^n - \gamma \nabla f(\vec{x}^n). \quad (44)$$

Since the gradient points to the direction of fastest increase, subtracting it should get us to a local minimum, after a sufficiently large number of iterations. We can generalize this method (see e.g. [43]) to find static field configurations $\vec{\phi}$ which minimize the energy functional

$$E = \int \mathcal{E}(\vec{\phi}) d^D x, \quad (45)$$

where $\mathcal{E}$ is the energy density and D is the spatial dimension. Again, we guess some initial configuration $\vec{\phi}_0$, but instead of a gradient, we use a variation of the energy functional

$$\frac{\delta E}{\delta \vec{\phi}} = \frac{\partial \mathcal{E}}{\partial \vec{\phi}} - \partial_i \frac{\partial \mathcal{E}}{\partial (\partial_i \vec{\phi})}, \quad (46)$$

where ∂_i is the partial derivative with respect to the i -th spatial coordinate, and the Einstein summation convention is used. Then, the generalized gradient flow method has a form

$$\vec{\phi}^{n+1} = \vec{\phi}^n - \gamma \left. \frac{\delta E}{\delta \vec{\phi}} \right|_{\vec{\phi}=\vec{\phi}^n}. \quad (47)$$

As an example, let us consider the well-known ϕ^4 model in (1+1) dimensions. In this case there is only one scalar field ϕ , therefore the field configuration is $\vec{\phi} = \phi$. The energy functional for static fields has a form

$$E = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} (1 - \phi^2)^2 \right] dx. \quad (48)$$

The variation is

$$\frac{\delta E}{\delta \phi} = -\frac{\partial^2 \phi}{\partial x^2} - 2\phi(1 - \phi^2). \quad (49)$$

We can now discretize the x coordinate with a step Δx , approximate the second derivative in the variation with the formula (27) (with appropriate modifications at the boundary) and choose some gradient flow step γ . For the sake of demonstration, let us consider the kink solution

$$\phi_K(x) = \tanh(x) \quad (50)$$

on the interval $x \in [-10, 10]$. As an initial guess for the method, we can take a linear function with values -1 and 1 at the boundaries, respectively. As we can see in figure 2, the gradient flow method converges to the known kink solution.

3.5 Boundary value problems

When linearizing around the soliton, we reduce the problem to systems of ODEs of the form (2). For such equations, we are usually dealing with two types of boundary value problems:

- bound modes, vanishing at the boundaries, for which we are looking for η and E ;
- scattering modes, behaving as plane waves at the boundaries, for which E is given, and the only unknown is η .

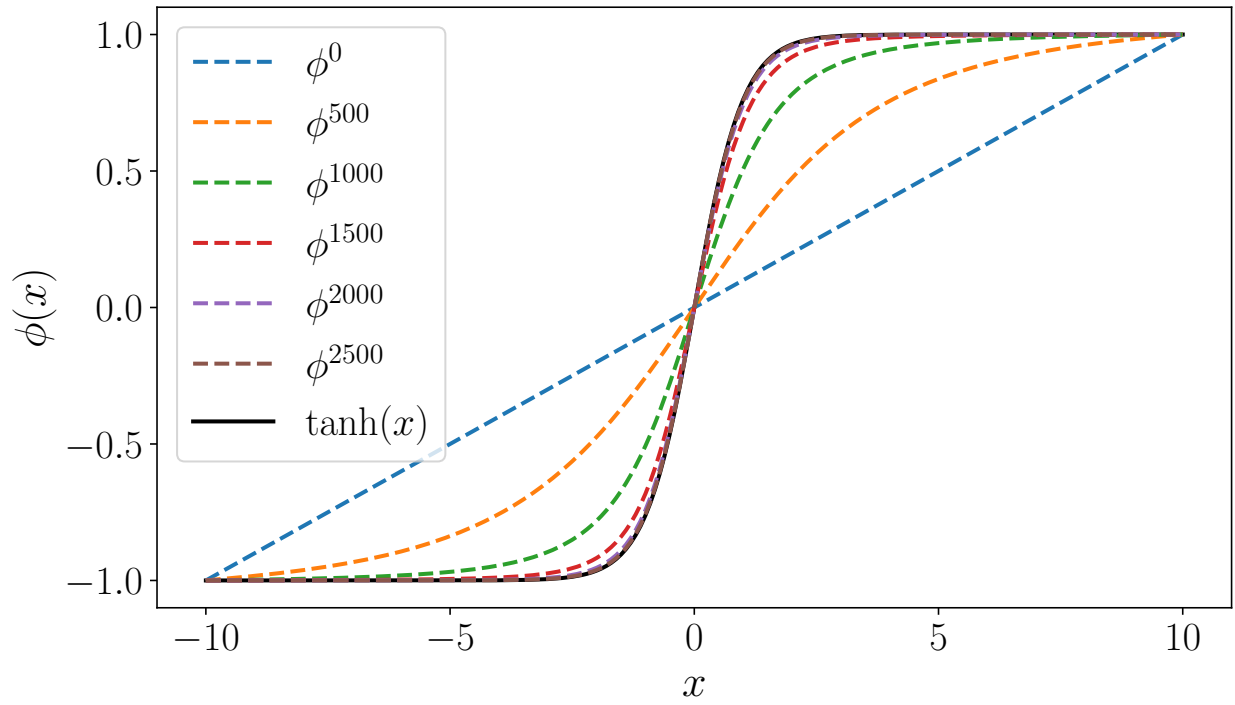


Figure 2: Gradient flow method example: kink in the ϕ^4 model, with $\Delta x = 0.1$ and $\gamma = 0.001$. Solid black line represent the analytical solution, while dashed lines are approximate solutions after different number of steps.

The range of E for which the scattering states are present can be deduced by analyzing asymptotic solutions, which are usually easily obtained analytically. In both cases, we can discretize the space with some step Δx and represent η as a column

$$\eta = \begin{pmatrix} \eta(x_0) \\ \eta(x_1) \\ \dots \\ \eta(x_N) \end{pmatrix} \quad \text{where } x_n = x_0 + n \Delta x, \quad n = 0, 1, \dots, N. \quad (51)$$

Then, we can use a finite difference to approximate the second derivative, obtaining its matrix D (for a more detailed discussion of a similar approach see e.g. [34, 36]). For example, for the five-point stencil (27) D is a block-diagonal matrix of the form

$$D = \frac{1}{12\Delta x^2} \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & -1 & 16 & -30 & 16 & -1 & \\ & & -1 & 16 & -30 & 16 & -1 \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (52)$$

with appropriate modifications for the points near and at the boundaries. V and E are discretized similarly, reducing (2) to a matrix equation

$$(-D + V)\eta = E\eta. \quad (53)$$

Let us describe how to find bound modes first. To impose zero Dirichlet boundary conditions, we can simply put the appropriate elements of D to zero. If η was initially a vector of functions, E could be in general not proportional to identity, but we can always multiply the equation (53) by some matrix B , such that $BE = \epsilon \mathbb{1}$, with ϵ being a number. Then we can solve the eigenproblem

$$B(-D + V)\eta = \epsilon\eta. \quad (54)$$

with any algorithm (we used Scipy library with sparse matrices).

To obtain scattering states, we also need to impose appropriate boundary conditions. For example, let η be a single function, with an asymptotic form

$$\begin{aligned} \eta(x \rightarrow -\infty) &= a (e^{ikx} + r e^{-ikx}) , \\ \eta(x \rightarrow +\infty) &= a t e^{ikx} . \end{aligned} \quad (55)$$

With sufficiently large grid, we can approximate $-\infty$ by x_0 and ∞ by x_N . Then, the boundary conditions can be written as

$$\begin{aligned} \eta'(x_0) + ik\eta(x_0) &= 2a i k e^{ikx_0} , \\ \eta'(x_N) - ik\eta(x_N) &= 0 . \end{aligned} \quad (56)$$

The conditions can be imposed by including l.h.s. in the matrix D , and modifying the equation (53):

$$(-D + V - E)\eta = C, \quad (57)$$

where C is a zero vector, except boundary terms, which are the r.h.s. of (56). Then, we obtain a large system of linear equations, which can be solved using any method (again, we used Scipy library and sparse matrices).

While solving boundary value problems, it is sometimes useful to use a simpler technique, called the shooting method. This approach reduces the system to an initial value problem, with some of the initial conditions given by some of the boundary conditions, while the rest of the initial conditions are varied, until the solution "hits" the remaining boundary conditions. The varying can be done in a more systematic way, using root-finding algorithms [34, 36].

4 Other work

4.1 Other published research

In this section, the other work done by me during my PhD studies, and published in academic journals, is briefly presented. The following papers, contrary to [A1–A3], are not an integral part of this thesis.

- [B1] C. Adam, D. Ciurla, K. Oleś, T. Romańczukiewicz, and A. Wereszczyński, “Relativistic moduli space and critical velocity in kink collisions”, *Phys. Rev. E* **108**, 024221 (2023).
- [B2] C. Adam, D. Ciurla, K. Oleś, T. Romańczukiewicz, and A. Wereszczyński, “Sphalerons and resonance phenomenon in kink-antikink collisions”, *Phys. Rev. D* **104**, 105022 (2021).
- [B3] D. Ciurla, L. Hadasz, and T. Williams, “A superspace Dirac operator in NCG and the "factorization" of the ordinary Dirac operator”, *Acta Physica Polonica B* **52**, 1449 (2021).

The kink collisions described in [B1, B2] are not directly related to the topic of NRP, but nevertheless they also consider soliton dynamics, and require similar computational methods. In both papers, my contribution involved numerical computations. I estimate that I did 30% of the total work in [B1] and 5% of the total work in [B2]. In [B1] we used the *perturbative relativistic moduli space* to reduce the field theory to a finite-dimensional system. This approach is based on promoting the amplitudes of the so-called Derrick modes to collective coordinates. In particular, we measured the critical velocity, at which the multibounce/annihilation becomes a single bounce collision. We have shown that for ϕ^4 , sine-Gordon and Christ–Lee models the critical velocity from the moduli space approaches the real value when the number of Derrick modes is increased. My work contained in this article was supported by the Polish National Science Centre (Grant No. NCN 2021/43/D/ST2/01122). In [B2], we investigated a family of models in which an unstable static solution, called sphaleron, exists. We found that such sphalerons can play a significant role in kink-antikink collisions. This result proves that to study soliton dynamics, the study of free solitons and their excitations is not always sufficient to understand the whole process.

The article [B3] contains research started before my PhD studies, not connected to the topic of this thesis. In our attempt to reconcile noncommutative geometry and supersymmetry, inspired by the procedure of taking a “square root” of the Dirac operator [44], we

performed similar factorization in the Euclidean spacetime. We proposed an application of this factorization to Dirac operators in the so-called almost-commutative geometry. My contribution was 30% of the total work.

4.2 Heavy kinks

Apart from the published research, summarized in the previous section, I did some preliminary studies on the so-called heavy kinks, in collaboration with P. Forgács, Á. Lukács and T. Romańczukiewicz. The defining property of these solitons is that their effective mass (see Section 1.3) is infinite. The properties of the heavy kinks make them a good simplified analogue of the global vortices in the Goldstone model, which are interesting also for the study of NRP. However, the dynamics of heavy kinks is complex enough to be worth investigating, leaving aside this additional motivation.

We studied (1+1)-dimensional scalar field theories with several potentials. To mimic the global vortices in the Goldstone model, we required the kink energy to diverge logarithmically, and the simplest potential we found with such a property was

$$U(\phi) = \frac{2}{\phi^2} \quad (58)$$

with a heavy kink solution

$$\phi_K(x) = 2\sqrt{|x|}. \quad (59)$$

Derivative of this kink, however, has a singularity at $x \rightarrow 0$, therefore we considered other potentials:

$$\begin{aligned} 1) \quad U(\phi) &= \frac{2}{\phi^2 + 1}, & x(\phi_K) &= \frac{1}{4} \left(\phi_K \sqrt{\phi_K^2 + 1} + \operatorname{arsinh}(\phi_K) \right), \\ 2) \quad U(\phi) &= \frac{4\sqrt{2}(64 + \phi^4)}{\left(32 + \phi^4 + \phi^2 \sqrt{64 + \phi^4} \right)^{3/2}}, & \phi_K(x) &= 2x(1 + x^2)^{-1/4}, \\ 3) \quad U(\phi) &= \frac{2}{(|\phi| + 2\sqrt{a})^2}, & \phi_K(x) &= 2 \left(\sqrt{|x| + a} - \sqrt{a} \right) \operatorname{sgn}(x), \end{aligned} \quad (60)$$

where in the right column the heavy kink solutions are presented. Each of these models has its benefits and drawbacks. The first one has a simple potential, but the kink profile is given in the implicit form that has to be inverted numerically, while the second one has a simple kink profile, but a complicated potential. The third model has analytical solutions in the form of Bessel functions for the linearized spectrum, but is difficult to simulate numerically due to discontinuities in derivatives, therefore we present simulations only of the first two models.

We studied the movement of the heavy kinks after a short external force was applied, and compared it with the sine-Gordon kink. If the applied impulse is symmetric (i.e. given by an even function of the spatial coordinate), the kinks are pushed in one direction. After the force is turned off, the sine-Gordon kink moves with a constant velocity, while the heavy kinks slow

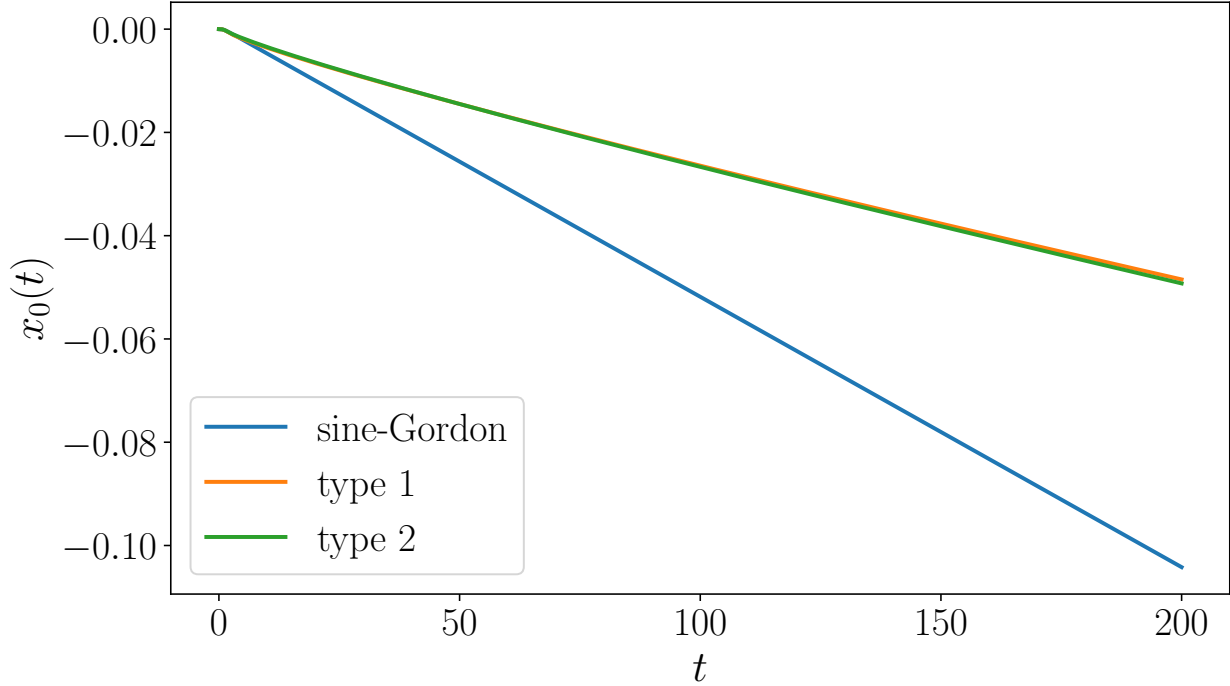


Figure 3: Position of the heavy kinks (models 1 and 2 from equation (4.2)) and the sine-Gordon kink as a function of time, after a short symmetric external impulse is applied.

down, as presented in figures 3 and 4. The trajectory of the sine-Gordon kink is predicted with reasonable accuracy by the Newtonian approximation: we can compute the effective force and mass in a straightforward manner using the collective coordinate approach. This is obviously not the case for the heavy kinks, since their mass is infinite. However, we tried a more general approach, which involved using Green's function and the linearized spectrum of the kink.

We want to investigate the evolution of the kink under the perturbation (external impulse) $\chi(t, x)$, which is described by the equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + U'(\phi) = \chi. \quad (61)$$

We can assume the perturbative expansion around the kink, similar as (1):

$$\phi(t, x) = \phi_K(x) + \xi(t, x) \quad (62)$$

and in the linear order we obtain

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} + V(x)\xi = \chi, \quad (63)$$

where $V(x) = U''(\phi_K(x))$. The standard way of solving such inhomogeneous equations is via the Green's function (propagator) $G(t, x, t', x')$, which we can obtain from the linearized

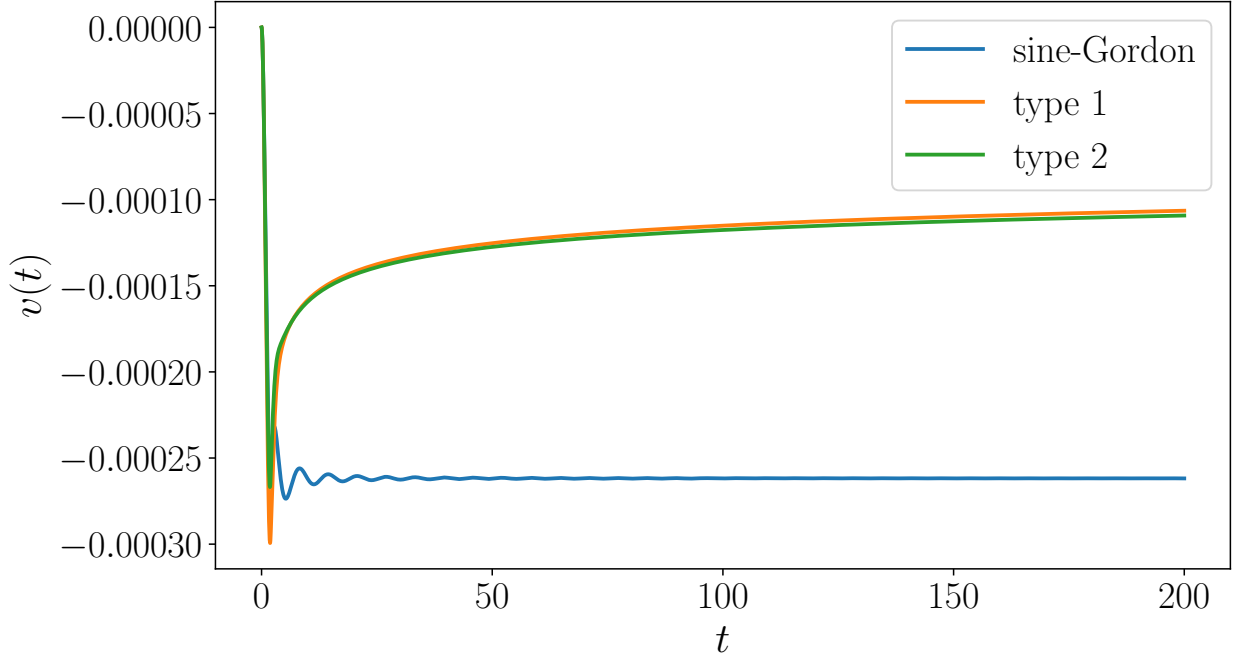


Figure 4: Velocity of the heavy kinks and the sine-Gordon kink after a short symmetric external impulse.

modes (see Section 1.2) of the kink. Then the solution to (63) with the initially static kink is

$$\xi(t, x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t, x, t', x') \chi(t', x') dx' dt'. \quad (64)$$

This formula can be used to compute the position of the center of the kink $x_0(t)$. Up to linear order, kink at x_0 has a form

$$\phi(t, x) = \phi_K(x - x_0) = \phi_K(x) - x_0 \phi'_K(x). \quad (65)$$

Substituting $x = 0$ we get

$$x_0 = -\frac{\phi(0)}{\phi'_K(0)}, \quad (66)$$

and combining this with (62) we obtain

$$x_0(t) = -\frac{\xi(t, 0)}{\phi'_K(0)}. \quad (67)$$

With this approach we were able to accurately compute the details of motion of the sine-Gordon kink, not captured by the Newtonian approximation, i.e. the decaying oscillations visible in figure 4. However, we were unable to successfully apply this method to the heavy kinks. One of the main reasons was the fact, that the Green's functions contain integrals over all possible scattering modes, and we have to either make an approximation for small wavenumbers, or integrate numerically over some finite truncated range. In both cases, the

large wavenumbers are omitted, and since they govern the dynamics for short times, the simulation times have to be longer in order to get to the regime well approximated by the small wavenumbers. Unfortunately, we suspect that the dynamics of the heavy kinks is dominated by the larger wavenumbers for longer times than in the sine-Gordon model, which makes the computations difficult, but we hope that the more accurate calculations would allow us to progress the research in the future.

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Negative radiation pressure in Bose-Einstein condensates

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In two-component nonlinear Schrödinger equations, the force exerted by incident monochromatic plane waves on an embedded dark soliton and on dark-bright-type solitons is investigated, both perturbatively and by numerical simulations. When the incoming wave is nonvanishing only in the orthogonal component to that of the embedded dark soliton, its acceleration is in the opposite direction to that of the incoming wave. This somewhat surprising phenomenon can be attributed to the well-known negative effective mass of the dark soliton. When a dark-bright soliton, whose effective mass is also negative, is hit by an incoming wave nonvanishing in the component corresponding to the dark soliton, the direction of its acceleration coincides with that of the incoming wave. This implies that the net force acting on it is in the opposite direction to that of the incoming wave. This rather counterintuitive effect is a yet another manifestation of negative radiation pressure exerted by the incident wave, observed in other systems. When a dark-bright soliton interacts with an incoming wave in the component of the bright soliton, it accelerates in the opposite direction; hence the force is pushing it now. We expect that these remarkable effects, in particular the negative radiation pressure, can be experimentally verified in Bose-Einstein condensates.

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I. INTRODUCTION

In this paper we consider the interaction of solitons and the sound waves in a two-component, nonlinear Schrödinger equation (NLSE) in one dimension. The NLSE is a widely used model, for example, in nonlinear optics [1], and in particular it serves to describe Bose-Einstein condensates (BECs) of neutral atoms. The motivation of our enterprise is to point out some simple but somewhat surprising physical phenomena, which are hopefully experimentally observable in BECs.

Bose-Einstein condensates were realized experimentally for the first time in [2] and have been produced in numerous experiments ever since. Many BECs can be described in a mean-field approximation, leading to the NLSE, for which classical field-theoretic methods are appropriate. Moreover, in a number of situations, the dynamics of solitons in a BEC can be well approximated by restricting the dynamics to one spatial dimension. Often, the trap used in experiments can be approximated by a harmonic potential, and choosing its frequencies in the two chosen dimensions to be much larger than in the remaining third dimension, an effective, quasi-1D cigar-shaped condensate is achieved [3]. Bose-Einstein condensates with two distinguishable components are described in a mean-field approximation by coupled nonlinear Schrödinger equations (CNLSEs) [4,5]. Experimentally, such two-component BECs can be achieved either by mixing two

different atomic species, e.g., ^{41}K and ^{87}Rb [6] ($m_1/m_2 \approx 0.47$), or by using two different spin states of the same species [7]. Although different kinds of solitons can be found in this system (see, e.g., [8,9]), we focus on the so-called dark-bright (DB) and dark solitons.

Dark solitons have been studied theoretically in the context of BEC [10] and subsequently in the context of optical fibers [11]. In particular, their dynamics was researched extensively, including multisoliton interactions [12–14], collisions [15,16], and interactions with perturbations [17–22]. Similarly, dark-bright solitons were a subject of many theoretical articles [4,23–27], together with their generalizations to vortex-bright solitons [28,29], discrete equations [30], and spinor condensates [31]. Dark and dark-bright solitons were realized experimentally both in BECs [32–40] and in nonlinear optics [41–43]. Many reviews of the topic can be found, e.g., [3,44–46].

In the present work we show that in the two-component CNLSE, the interaction of dark and dark-bright solitons with incoming small-amplitude plane waves can be reasonably well described by standard scattering theory. We derive the force acting on the solitons in terms of scattering data. When the amplitude of the plane waves is sufficiently small, linearization about the soliton provides a tractable approximation with good precision. In this case, the waveform is obtained as a solution of the Bogoliubov–de Gennes equations (BdGEs) [47]. The force acting on the soliton can be easily found from momentum conservation. For the case of main interest for us, when an incoming plane wave of amplitude a and wave number k_1 is nonzero only in one component, say, 1 (dark),

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then the induced force can be written as

$$F = a^2 [k_1^2(1 + R_1 - T_1) + (k_2^+)^2(R_2^+ - T_2^+) + (k_2^-)^2(R_2^- - T_2^-)], \quad (1)$$

where R_i and T_i denote the transmission and reflection coefficients, respectively, for an incoming wave into channel i and $k_2^\pm$ are the two possible wave numbers in the bright component. In realistic scenarios, for a equal to about 10% of a DB soliton's dark component amplitude, the accelerations due to this force are up to the order of 10^{-2} m/s² (see Sec. VD).

We note that the dynamics due to a force acting on dark and dark-bright solitons is somewhat counterintuitive, since the direction of the force and that of the resulting acceleration point in opposite directions because of their effective negative mass.

The value of the coupling g_{12} between the two components of the CNLSE plays an important role, since for $g_{12} = 1$ (in suitable units) the system is integrable [48], and in this special case the net force exerted by incoming waves on solitons is zero. In fact, for the integrable case, exact solutions corresponding to nonlinear superposition of cnoidal waves and solitons have been constructed [49]. We find that quite generally, for $g_{12} \neq 1$ in two-component CNLS systems, an incoming plane wave can exert a pulling force on certain solitons, referred to as the tractor beam effect or negative radiation pressure. As it has been already demonstrated for a number of cases in one and two dimensions, in the presence of two scattering channels with different dispersion relations, an incoming plane wave can exert a pulling force on the scatterer [50].

The paper is organized as follows. We review briefly some of the main properties of two-component CNLSEs in Sec. IIA, exhibit the expressions for the energy and field momentum in Sec. IIB, and present the linearized equations of motion around a soliton in Sec. IIC. In Sec. III we introduce the general notion of the Newtonian approximation using the effective mass and force. We apply these ideas to the specific cases of dark and dark-bright solitons, with a small wave in each component, in Secs. IV and V, respectively. Most importantly, we derive the acceleration of the solitons using an effective model and compare it with numerical simulations of the full CNLSE. In Sec. VD we verify how well the results derived from the homogeneous system apply to a dark-bright soliton in a harmonic trap. We summarize in Sec. VI.

II. MODEL

A. Coupled nonlinear Schrödinger (Gross-Pitaevskii) equation

In the mean-field regime, a one-dimensional two-component Bose–Einstein condensate can be described by two coupled nonlinear Schrödinger equations, also called Gross-Pitaevskii equations, of the form [4,5,35,51–53]

$$\begin{aligned} i\hbar\partial_t\psi_1 &= -\frac{1}{2m_1}\partial_{xx}\psi_1 + (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1 + V_1(x)\psi_1, \\ i\hbar\partial_t\psi_2 &= -\frac{1}{2m_2}\partial_{xx}\psi_2 + (g_{22}|\psi_2|^2 + g_{12}|\psi_1|^2)\psi_2 + V_2(x)\psi_2, \end{aligned} \quad (2)$$

where ψ_i ($i = 1, 2$) denote the (complex) wave functions of the two components of the condensate, m_i are their masses, and V_i are the trapping potentials experienced by the i th component. If $m_1 = m_2$, the couplings can be written as $g_{ij} = 2\hbar\omega_\perp a_{ij}$ [35,51–53], where a_{ij} denote the s -wave scattering lengths between the two components (or within one component in the case of a_{ii}) and $\omega_\perp$ is the transverse trapping frequency. The system can be described by the above one-dimensional equations if the frequencies of trapping potentials in the x direction, i.e., $V_i(x)$, are much smaller than $\omega_\perp$. Positive values of a_{ij} (and therefore also of g_{ij}) correspond to repulsive interaction between the components i and j , whereas a negative value corresponds to the interaction being attractive.

In order to simplify the problem, we reduce the number of parameters used. First, we consider a condensate made of two different spin states of the same atomic species; therefore, $m_1 = m_2$, and we can rescale $m_i = 1$. Second, we set $g_{ii} = 1$ and keep g_{12} as a free parameter. The former is justified, because the ratio of scattering lengths in experiments is often close to one, e.g., in the mixture of the $|2, 1\rangle$ and $|1, -1\rangle$ states of ⁸⁷Rb without additional tuning it is $a_{11}/a_{12}/a_{22} = 1.03/1/0.97$ [7], or for $|1, -1\rangle$ and $|2, -2\rangle$ states $a_{11}/a_{12}/a_{22} = 1.01/1/1$ [33]. Scattering lengths can be manipulated (both their magnitudes and their signs) using Feshbach resonances [54]. In particular, a_{12} can be tuned independently [55–57] and therefore different values of the g_{12} coefficient are achievable in experiments. Finally, we assume that $V_i = 0$, which should be a valid first approximation; we will verify that later in Sec. VD. Therefore, we assume $a_{11} = a_{22}$ and take $m = m_1 = m_2$ as the unit of mass, $\hbar/2a_{11}m\omega_\perp$ as the length unit, and $\hbar/4a_{11}^2m\omega_\perp^2$ as the unit of time. Then the set of equations (2) takes the form

$$\begin{aligned} i\partial_t\psi_1 &= -\frac{1}{2}\partial_{xx}\psi_1 + (|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1, \\ i\partial_t\psi_2 &= -\frac{1}{2}\partial_{xx}\psi_2 + (|\psi_2|^2 + g_{12}|\psi_1|^2)\psi_2. \end{aligned} \quad (3)$$

The system of equations (3) has various types of solitonic solutions (see the review in [44]). The simplest solutions correspond to the embedding of a scalar soliton into one of the components. We consider embedded dark solitons, for which the probability density has a dip, and dark-bright (DB) solitons, having a dip and a peak of the probability density in the components ψ_1 and ψ_2 , respectively. When $g_{12} = 1$, Eqs. (3) correspond to the Manakov system [48], which is known to be integrable (see also Ref. [58]). For the integrable case, the DB solitons are known analytically, while for values of $g_{12} \neq 1$ we have solved Eqs. (3) numerically. For such values of $g_{12} \neq 1$, we have shown analytically and also confirmed by numerical simulations that incoming sound waves (referred to as radiation) do exert a force on the solitons.

Usually, particlelike objects such as solitons are pushed in the direction of propagation of an incoming wave. However, in some cases the wave pulls a soliton in the opposite direction. We refer to such a situation as negative radiation pressure. In the case of dark and dark-bright solitons, this definition needs an additional clarification. Such solitons have negative effective mass; therefore, their acceleration has an opposite sign to the effective force. Thus, we define a positive (negative) radiation pressure [PRP (NRP)] as a setup in which the

force has the same (opposite) sign as (to) the direction of the incoming wave. This means that, in terms of an acceleration of dark and DB solitons, PRP and NRP correspond to a wave pulling and pushing a soliton, respectively. We remark that in the literature on NRP [50,59–63], in the systems considered up to now only positive masses occurred; therefore, the NRP exerted by incident plane waves has manifested by a pulling effect.

B. Integrals of motion

The Lagrangian density corresponding to the CNLSEs (3) is [64]

$$\mathcal{L} = \frac{1}{2} \sum_{i=1,2} [i(\psi_i^* \partial_t \psi_i - \psi_i \partial_t \psi_i^*) - |\partial_x \psi_i|^2 - |\psi_i|^4 - g_{12} |\psi_1|^2 |\psi_2|^2]. \quad (4)$$

Using the symmetries of the Lagrangian (4) and the Noether theorem (details in Appendix A), the total energy and momentum are derived as

$$E = \frac{1}{2} \sum_{i=1,2} \int_{-\infty}^{\infty} (|\partial_x \psi_i|^2 + |\psi_i|^4 + g_{12} |\psi_1|^2 |\psi_2|^2) dx, \quad (5)$$

$$P = \frac{i}{2} \sum_{i=1,2} \int_{-\infty}^{\infty} (\psi_i \partial_x \psi_i^* - \psi_i^* \partial_x \psi_i) dx, \quad (6)$$

and it is shown that they obey the equations

$$\partial_t E = \frac{1}{2} \sum_{i=1,2} (\partial_x \psi_i^* \partial_t \psi_i + \partial_x \psi_i \partial_t \psi_i^*)|_{-\infty}^{\infty}, \quad (7)$$

$$\partial_t P = \frac{1}{2} \sum_{i=1,2} [-i(\psi_i^* \partial_t \psi_i - \psi_i \partial_t \psi_i^*) - |\partial_x \psi_i|^2 + |\psi_i|^4 + g_{12} |\psi_1|^2 |\psi_2|^2]|_{-\infty}^{\infty}. \quad (8)$$

It is also worth noting that, due to the $U(1) \times U(1)$ symmetry, the solutions of Eqs. (3) obey the continuity equations (even if we include the trapping potential)

$$\partial_t |\psi_i|^2 + \partial_x J_i = 0, \quad (9)$$

where

$$J_i = \frac{i}{2} (\psi_i \partial_x \psi_i^* - \psi_i^* \partial_x \psi_i). \quad (10)$$

Note that $P = \sum_{i=1,2} \int_{-\infty}^{\infty} J_i dx$. Integrating Eq. (9) over the whole space and using the Newton-Leibniz theorem, we obtain

$$\partial_t \int_{-\infty}^{\infty} |\psi_i|^2 dx = -J_i|_{-\infty}^{\infty}. \quad (11)$$

We can choose the normalization as $\int |\psi_i(x, t)|^2 dx = N_i$, where N_i is the number of atoms in the i th component. These particle numbers are conserved.

C. Linearization around a soliton

We consider stationary solitons of Eqs. (3) of the form $\psi_i(x, t) = e^{-i\mu_i t} \Phi_i(x)$, where $\Phi_i(x)$ are real functions satisfying the following equations:

$$\begin{aligned} \mu_1 \Phi_1 &= -\frac{1}{2} \partial_{xx} \Phi_1 + (|\Phi_1|^2 + g_{12} |\Phi_2|^2) \Phi_1, \\ \mu_2 \Phi_2 &= -\frac{1}{2} \partial_{xx} \Phi_2 + (|\Phi_2|^2 + g_{12} |\Phi_1|^2) \Phi_2. \end{aligned} \quad (12)$$

If the wave function is normalized to the number of atoms in each component, then the μ_i are determined from these normalization conditions and they are interpreted as chemical potentials [65].

Let us consider a small perturbation of a soliton solution of Eqs. (3), of the form

$$\psi_i(x, t) = e^{-i\mu_i t} [\Phi_i(x) + a \xi_i(x, t)], \quad (13)$$

where the parameter of the perturbation $a \ll 1$. Moreover, let us make an ansatz

$$\xi_i(x, t) = e^{i\tilde{\omega} t} \xi_i^+(x) + e^{-i\tilde{\omega} t} \xi_i^-(x). \quad (14)$$

After inserting this ansatz into Eqs. (3) and keeping only terms linear in a , we obtain that ξ_i^- and ξ_i^+ satisfy

$$\begin{aligned} \left(-\frac{1}{2} \partial_{xx} + \mathbf{M}\right) \Xi &= \text{diag}(\mu_1 + \tilde{\omega}, \mu_1 - \tilde{\omega}, \mu_2 \\ &\quad + \tilde{\omega}, \mu_2 - \tilde{\omega}) \Xi, \end{aligned} \quad (15)$$

where diag means diagonal matrix, Ξ stands for the vector

$$\Xi = (\xi_1^-, \xi_1^{+*}, \xi_2^-, \xi_2^{+*})^T, \quad (16)$$

and

$$\mathbf{M} = \begin{pmatrix} 2\Phi_1^2 + g_{12}\Phi_2^2 & \Phi_1^2 & g_{12}\Phi_1\Phi_2 & g_{12}\Phi_1\Phi_2 \\ \Phi_1^2 & 2\Phi_1^2 + g_{12}\Phi_2^2 & g_{12}\Phi_1\Phi_2 & g_{12}\Phi_1\Phi_2 \\ g_{12}\Phi_1\Phi_2 & g_{12}\Phi_1\Phi_2 & g_{12}\Phi_1^2 + 2\Phi_2^2 & \Phi_2^2 \\ g_{12}\Phi_1\Phi_2 & g_{12}\Phi_1\Phi_2 & \Phi_2^2 & g_{12}\Phi_1^2 + 2\Phi_2^2 \end{pmatrix}. \quad (17)$$

Equation (15) is the generalization of the BdGE [47] to the two-component CNLSE. Here we consider plane-wave solutions of Eq. (15), but also note that the existence of bound-state solutions with complex $\tilde{\omega}$ would signal instability of the soliton. Such stability analyses have been performed, e.g., in Refs. [4,45] in similar coupled models, and in Refs. [66–68] for a single-component NLSE. In our case,

the numerical technique used for finding the soliton (gradient descent) already ensures that unstable solitons are not found.

We consider a setup consisting of a soliton and a wave. The wave is incoming from $-\infty$ in one of the two components of the BEC and is moving to the right. It corresponds to Eq. (13) with a interpreted as the amplitude (and the

appropriate boundary conditions discussed later). Using the linearization described above, the wave can be written as

$$ae^{-i\mu_i t} \xi_i(x, t) = ae^{-i(\mu_i - \tilde{\omega})t} \xi_i^+(x) + ae^{-i(\mu_i + \tilde{\omega})t} \xi_i^-(x) \quad (18)$$

and we can define its frequencies as

$$\omega_i^\pm = \mu_i \mp \tilde{\omega}, \quad (19)$$

which correspond to those on the right-hand side of Eq. (15). Sometimes we loosely refer to $\tilde{\omega}$ as the frequency, but the true frequencies of the wave are given by $\omega_i^\pm$. We assume that asymptotically these waves have a form of monochromatic plane waves. We define transmission and reflection coefficients, separately for each of the examples, as coefficients of the asymptotic plane-wave modes in a solution.

III. NEWTONIAN MOTION AND THE EFFECTIVE MASS OF SOLITONS

We expect both the dark and the dark-bright solitons of the NLSE to behave as Newtonian particles in the first approximation, albeit with unusual dynamics, due to their negative effective masses. In this present context, we refer the reader to Ref. [69] for the motion of dark solitons, and for a recent review of the dynamics of solitons in the vector NLSE see Ref. [45].

More precisely, we assume that in the presence of perturbations, the solitons do not change their shape and that we can treat the center of the soliton x_0 solely as a function of time, reducing the problem to one-dimensional classical dynamics. In this description, $x_0(t)$ is expected to obey the equation

$$M\ddot{x}_0(t) = F, \quad (20)$$

where M and F are the effective mass and force, respectively. These quantities are obtained from the integrals of motion discussed in Sec. II B. The description of the dynamics of the dark soliton is complicated by the fact that the wave function describes the soliton on top of a constant background [69].

The energy E and the momentum P of the dark soliton have to be defined carefully; they have to be renormalized in order to subtract the contribution from the background [17, 44]. The renormalized quantities P_s and E_s will be given separately for the dark and dark-bright soliton (see Secs. IV and V).

A useful definition of the effective mass M from the renormalized total energy E_s and momentum P_s for a soliton moving with velocity v is given as

$$M = \left. \frac{d^2 E_s}{dv^2} \right|_{v=0} = \left. \frac{dP_s}{dv} \right|_{v=0}. \quad (21)$$

If the soliton is indeed moving according to Newton's law, M computed from E_s should match that derived from P_s .

The effective force exerted by the sound waves on the soliton can be obtained as the time derivative of the total momentum $\partial_t P$. Since the renormalization corrections are time independent, $\partial_t P = \partial_t P_s$. We are interested in the force averaged over a period of the incoming wave; thus we are led to define the effective force as

$$F = \langle \partial_t P \rangle_T, \quad (22)$$

where P is the total momentum including the radiation and $\langle \cdot \rangle_T$ is the average over a period. We note that to evaluate

Eq. (8) it is sufficient to know the asymptotic form of the radiation in order to compute the effective force. In the computation of F , we can only keep the contributions of order a^2 since we have the solution up to linear order in the amplitude. In this linearized approximation (which turns out to be quite efficient), one can easily obtain the results for any incoming waveform.

IV. DARK SOLITON

A. Dark soliton solution

A particular solution of Eq. (3) with arbitrary g_{12} is a (scalar) dark soliton centered at x_0 [4], embedded into the vector NLSE

$$\psi_1 = e^{-i\mu t} \sqrt{\mu} \tanh[\sqrt{\mu}(x - x_0)], \quad \psi_2 = 0, \quad (23)$$

with chemical potential $\mu_1 = \mu > 0$. Such a soliton can be understood as a dip in the probability density obtained from the collective wave function of atoms in the condensate. We consider two scenarios: In the first case the additional atoms will be present in the second component in the form of a plane wave (a system still described by the two-component model) and in the second case the second component will be completely absent (a problem reduced to the one-component NLSE).

In order to examine the effective force exerted by an incoming plane wave on a dark soliton, we need to analyze the linearized equations. The relation between $\tilde{\omega}$ (see Sec. II C) and the wave numbers k_i is obtained from the $x \rightarrow \pm\infty$ asymptotic form of Eq. (15). Knowing that $\Phi_1(x) \rightarrow \pm\sqrt{\mu}$ and $\Phi'_1(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, the asymptotic form of the matrix from Eq. (17) is

$$\mathbf{M}_{x \rightarrow \pm\infty} = \mu \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & g_{12} & 0 \\ 0 & 0 & 0 & g_{12} \end{pmatrix}. \quad (24)$$

The linearized equations (15) with this $\mathbf{M}$ can easily be diagonalized and solved, obtaining

$$\begin{aligned} \xi_1^-(x) &= Ae^{ik_1 x} + Be^{-ik_1 x}, \\ \xi_1^+(x) &= \left(-\frac{\frac{k_1^2}{2} - \tilde{\omega}}{\mu} - 1 \right) (A^* e^{-ik_1 x} + B^* e^{ik_1 x}), \end{aligned} \quad (25)$$

where A and B are arbitrary constants and

$$k_1 = \sqrt{2\sqrt{\mu^2 + \tilde{\omega}^2} - \mu}. \quad (26)$$

In the above solution we omitted the solutions with imaginary wave number, since they describe nonpropagating solutions and therefore do not carry a momentum. Note that for $\tilde{\omega} \neq 0$ the wave number k_1 is real; thus Eqs. (25) describe propagating (traveling) waves.

As mentioned earlier, we consider waves coming from the left and moving to the right. The direction of the traveling wave is determined by a relative sign of the wave frequency and its wave vector. Assuming an incoming wave with $e^{ik_1 x}$ for ω_1^+ and (to be consistent with the above solution) $e^{-ik_1 x}$ for ω_1^- , the condition for moving to the right is that the frequency

$\omega_1^\pm = \mu \mp \tilde{\omega}$ is positive or negative, respectively [cf. Eqs. (18) and (19)]. This can be transformed into the following conditions for $\tilde{\omega}$: $\tilde{\omega} < \mu$ for ω_1^+ and $\tilde{\omega} < -\mu$ for ω_1^- . Therefore, for $\tilde{\omega} < -\mu$ both waves propagate to the right.

In the second component, the equations are already diagonal. Note that in this case μ_2 is arbitrary and choosing it can be interpreted as fixing a reference point for $\tilde{\omega}$. For simplicity, let us set $\mu_2 = 0$; then we can interpret $\mp\tilde{\omega} = \omega_2^\pm$ simply as the frequency of the incoming wave in the second sector. Then the solutions are

$$\begin{aligned}\xi_2^-(x) &= C e^{ik_2^- x} + D e^{-ik_2^- x}, \\ \xi_2^+(x) &= E e^{ik_2^+ x} + F e^{-ik_2^+ x},\end{aligned}\quad (27)$$

where

$$k_2^\pm = \sqrt{2\sqrt{\mp\tilde{\omega}} - \mu g_{12}} \quad (28)$$

and C, D, E , and F are arbitrary constants. These waves propagate when $k_2^\pm$ is real (and nonzero), that is, when $\mp\tilde{\omega} > \mu g_{12}$. This means that for $g_{12} < 0$ there exists a range of $\tilde{\omega}$ in which both waves can propagate with the same $\tilde{\omega}$ and for $g_{12} \geq 0$ with fixed $\tilde{\omega}$ only one (or neither) of the waves can propagate.

Let us analyze how the parameter $\tilde{\omega}$ affects the direction of propagation of the waves in the second component. Using the same logic as for the first component (assuming an incoming wave with $e^{ik_2^\pm x}$), we get the conditions for the range of $\tilde{\omega}$ in which the waves in the second component are moving to the right, namely, we obtain $\tilde{\omega} < 0$ for ω_2^+ and $\tilde{\omega} > 0$ for ω_2^- . This means that only one of them moves to the right for a given $\tilde{\omega}$.

In order to find the effective mass of the soliton, we repeat the derivation of the renormalized momentum done in [17] (for additional details see also [70]) and obtain the effective mass from it following [44]. First, we consider the total momentum P_s of a scalar dark soliton moving with a constant velocity v (often called the gray soliton):

$$\begin{aligned}\psi_1 &= e^{-i\mu t} \{ i v + \sqrt{\mu - v^2} \tanh[\sqrt{\mu - v^2}(x - x_0 - vt)] \}, \\ \psi_2 &= 0.\end{aligned}\quad (29)$$

Note that the moving soliton becomes shallower and shallower as the velocity increases, finally vanishing when $v^2 = \mu$, which defines its maximal velocity. However, the above wave function describes a dark soliton on top of a background, and we are interested in the total momentum of the soliton. Let us note that the solution with constant probability density, corresponding to the asymptotics of (29), i.e., $|\psi_1|^2 = \mu$ and $|\psi_2|^2 = 0$, is of the form

$$\begin{aligned}\psi_1 &= \sqrt{\mu} e^{-i(\mu + q^2/2)t} e^{iqx}, \\ \psi_2 &= 0,\end{aligned}\quad (30)$$

with some real q . Since we are interested in a nonmoving background, we choose $q = 0$, yielding the background part of the dark soliton. However, we have to take into account the phase change induced by the presence of the soliton (see [17]). Therefore, we assume that the background (in the first component) has the form

$$\psi_b = \sqrt{\mu} e^{-i\mu t} e^{ik(x)x}, \quad (31)$$

where k is some real function of x , reflecting the phase change induced by a soliton. It will turn out that the explicit form

of $k(x)$ is not needed. Although the probability density of the background is constant (equal to μ), the total momentum contribution also depends on the phase. Inserting Eq. (31) into Eq. (6) we obtain that the contribution of the background can be written as

$$\mu \Delta\phi \equiv \mu \int_{-\infty}^{\infty} [k(x) + x k'(x)] dx = \mu x k(x) \Big|_{-\infty}^{\infty}. \quad (32)$$

Comparing with Eq. (31), $\Delta\phi$ is readily identified with the induced phase change of the background. Therefore, the total momentum of the pure soliton can be expressed as (cf. [17,44])

$$P_s = \frac{i}{2} \int_{-\infty}^{\infty} (\psi_1 \partial_x \psi_1^* - \psi_1^* \partial_x \psi_1) dx - \mu \Delta\phi. \quad (33)$$

The phase change $\Delta\phi$ can be easily computed from the asymptotics of Eq. (29):

$$\Delta\phi = -2 \arctan \left(\frac{\sqrt{\mu - v^2}}{v} \right). \quad (34)$$

Using this, we obtain from Eq. (33) the momentum corresponding to the soliton

$$P_s = -2v\sqrt{\mu - v^2} + 2\mu \arctan \left(\frac{\sqrt{\mu - v^2}}{v} \right), \quad (35)$$

which allows us to compute its effective mass as (cf. [44])

$$M = \left. \frac{dP_s}{dv} \right|_{v \rightarrow 0} = -4\sqrt{\mu}. \quad (36)$$

Intuitively, the negative sign is not a surprise, because a dark soliton is, as mentioned before, a dip in the collective probability density of atoms. The same mass is obtained using the renormalized energy [17] (see Sec. III)

$$E_s = \frac{1}{2} \int_{-\infty}^{\infty} [|\partial_x \psi_1|^2 + (|\psi_1|^2 - \mu)^2] dx. \quad (37)$$

B. Wave in the second component

Let us now consider the interaction of an embedded dark soliton in the first component and an incoming wave in the second one. In this case one can obtain the analytic solutions of the linearized equations for the waveform in the second component. Since in the case of an embedded dark soliton [Eq. (23)] the linearized equations (15) for ξ_1 and ξ_2 are decoupled from each other, we may simply set $\xi_1 = 0$. Then Eq. (15) is reduced to

$$\begin{aligned}-\frac{1}{2} \partial_{xx} \xi_2^-(x) + g_{12} \mu \tanh^2(\sqrt{\mu} x) \xi_2^-(x) &= \tilde{\omega} \xi_2^-(x), \\ -\frac{1}{2} \partial_{xx} \xi_2^+(x) + g_{12} \mu \tanh^2(\sqrt{\mu} x) \xi_2^+(x) &= -\tilde{\omega} \xi_2^+(x).\end{aligned}\quad (38)$$

Regular solutions of Eq. (38) can be expressed in terms of associated Legendre functions of the first kind

$$\xi_2^\pm(x) = A P_\lambda^{ik_2^\pm/\sqrt{\mu}}[\tanh(\sqrt{\mu} x)], \quad (39)$$

where $\lambda = \frac{1}{2}(\sqrt{1 + 8g_{12}} - 1)$ and A is a normalization factor.

Since our boundary conditions correspond to a wave coming from $x = -\infty$, we impose the following asymptotic

behavior on ξ_2 :

$$\begin{aligned}\xi_2^\pm(x) &\xrightarrow{x \rightarrow -\infty} e^{ik_2^\pm x} + r_2^\pm e^{-ik_2^\pm x}, \\ \xi_2^\pm(x) &\xrightarrow{x \rightarrow +\infty} t_2^\pm e^{ik_2^\pm x}.\end{aligned}\quad (40)$$

This asymptotic behavior can be ensured by choosing the normalization constant A in Eq. (39) appropriately. The reflection and transmission coefficients are defined as $R_2^\pm = |r_2^\pm|^2$ and $T_2^\pm = |t_2^\pm|^2$, respectively, and can be written as

$$\begin{aligned}R_2^\pm &= \frac{2 \sin^2(\pi \lambda)}{\cosh\left(\frac{2\pi k_2^\pm}{\sqrt{\mu}}\right) - \cos(2\pi \lambda)}, \\ T_2^\pm &= \frac{2 \sinh^2\left(\frac{\pi k_2^\pm}{\sqrt{\mu}}\right)}{\cosh\left(\frac{2\pi k_2^\pm}{\sqrt{\mu}}\right) - \cos(2\pi \lambda)}.\end{aligned}\quad (41)$$

One can check that the following relation is satisfied:

$$R_2^\pm + T_2^\pm = 1. \quad (42)$$

Note that the reflection coefficient is zero not only for $g_{12} = 1$, but for any value of g_{12} such that λ is an integer.

We now investigate the dynamics of a dark soliton embedded in the first component under the influence of an incident plane wave coming from $x = -\infty$ embedded in the second component. We stick to the linearized approximation and we assume the amplitude of the wave a to be sufficiently small. Let us consider the setup in which $\tilde{\omega}$ is such that only one of the waves $\xi_2^\pm$ has a real wave number. Then we can omit the other one (ξ_2^+ or ξ_2^-), since it describes a nonpropagating solution and does not carry a momentum. In terms of full wave functions ψ_i , this setup has the following asymptotics:

$$\begin{aligned}\psi_1(x, t) &\xrightarrow{x \rightarrow -\infty} -\sqrt{\mu} e^{-i\mu t}, \\ \psi_1(x, t) &\xrightarrow{x \rightarrow +\infty} \sqrt{\mu} e^{-i\mu t}, \\ \psi_2(x, t) &\xrightarrow{x \rightarrow -\infty} a e^{-i\omega_2^\pm t} (e^{ik_2^\pm x} + r_2^\pm e^{-ik_2^\pm x}), \\ \psi_2(x, t) &\xrightarrow{x \rightarrow +\infty} a e^{-i\omega_2^\pm t} t_2^\pm e^{ik_2^\pm x}.\end{aligned}\quad (43)$$

In order to approximate the acceleration of the soliton, we assume Newtonian motion, with the force stemming from the radiation pressure, averaged over a period of the incoming wave.

The force is derived from Eq. (8). First, we substitute the above asymptotic form into this equation. Then we average it over time for the period $T = 2\pi/\tilde{\omega}$ (which in this case does not change anything) and omit the terms of the order higher than a^2 (since a is small). Finally, we replace $\tilde{\omega}$ with the appropriate dispersion relation with $k_2^\pm$, derived in the preceding section, obtaining the force

$$F = \langle \partial_t P \rangle_T = a^2 (k_2^\pm)^2 (1 + R_2^\pm - T_2^\pm), \quad (44)$$

where P is the total momentum and $\langle \cdot \rangle_T$ means the average over the period. Using the relation (42), the force can be simplified to $F = 2a^2 (k_2^\pm)^2 R_2^\pm$; therefore, in this case reflectionlessness implies no force (of the assumed order a^2).

Let us briefly consider the range of $\tilde{\omega}$ for which ξ_2^+ and ξ_2^- are both traveling waves. Then the initial wave $\xi_2^\pm$ can scatter into both ξ_2^+ and ξ_2^- and the analogous derivation leads to the

force

$$F = a^2 [(k_2^\pm)^2 + (k_2^\pm)^2 (\tilde{R}_2^\pm - \tilde{T}_2^\pm) + (k_2^\mp)^2 (\tilde{R}_2^\mp - \tilde{T}_2^\mp)], \quad (45)$$

where the incoming wave has a wave number $k_2^\pm$. However, after considering the boundary conditions, $\tilde{R}_2^\pm$ and $\tilde{T}_2^\pm$ are equal to $R_2^\pm$ and $T_2^\pm$ from Eq. (41) only for the incoming wave and for the other one are equal to zero. Therefore, the above expression ultimately reduces to Eq. (44). If both are the incoming waves, only one of them is moving to the right for given $\tilde{\omega}$ (see the discussion in Sec. IV A) and also the effective force is simply a sum of individual forces, so it is not interesting at this point.

Finally, using the reflection coefficient given by Eq. (41), the mass derived in the preceding section, and the above F , we can derive the explicit form of acceleration exerted on the scalar dark soliton in the first component by a wave with frequency $\tilde{\omega}$ in the second component. For the incoming wave with a wave number $k_2^\pm$ it is

$$\ddot{x}_0 = -\frac{a^2}{\sqrt{\mu}} \frac{(k_2^\pm)^2 \sin^2(\pi \lambda)}{\cosh\left(\frac{2\pi k_2^\pm}{\sqrt{\mu}}\right) - \cos(2\pi \lambda)}, \quad (46)$$

where x_0 is interpreted as the position of the soliton. Note that although the force (44) is always non-negative, the acceleration is always nonpositive due to the negative effective mass (36); therefore, we observe either the positive radiation pressure, i.e., positive force, or no pressure at all.

To verify the above results, we performed numerical simulations. The initial condition was a dark soliton in the first sector and a wave propagating from the left end of the interval with a given frequency and an appropriate wave number, where its amplitude was kept sufficiently small. However, the initial wave was multiplied by a superposition of hyperbolic tangents to cut it smoothly, in order to have the initial wave beginning slightly after the left boundary and ending slightly before the center of the soliton. This deviation from a plane-wave shape introduces a short kick exerted on the soliton and this results in a constant velocity, on top of which we observe the acceleration compared with Eq. (46). More precisely, the wave in the second component had the form

$$\psi_2(x, t = 0) = a e^{ik_2^\pm x} \Phi_{\text{cut}}(x), \quad (47)$$

with parameters such that a wave with $k_2^\pm$ propagates to the right, i.e., $\tilde{\omega} < 0$ and $\tilde{\omega} < -\mu g_{12}$. Here Φ_{cut} is the cutting function mentioned before,

$$\Phi_{\text{cut}}(x) = \frac{1}{2} [\tanh(x - x_{\min} - 10) - \tanh(x + 10)], \quad (48)$$

where $x_{\min}$ is the left boundary in space. The center of the soliton was computed as the minimum of the probability density in the first component. Then the acceleration was computed by fitting the quadratic function to the position of the center and compared with Eq. (46) (see Figs. 1–4). It turned out that our effective linearized model explains the observed accelerations quite well for a wide range of parameters.

C. Dark soliton and a wave in the same component

Since the linearized equations (15) for ξ_1 and ξ_2 , in the case of the dark soliton, are independent, we start with a similar ansatz as before, namely, $\xi_2 = 0$. Now we examine the case where the wave with small amplitude a comes from $-\infty$ in

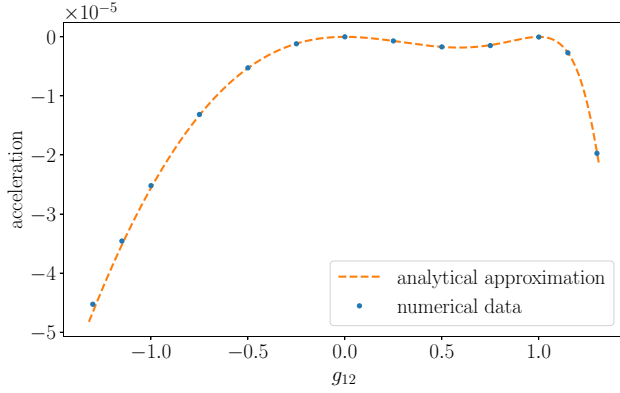


FIG. 1. Acceleration of a dark soliton with $\mu = 1$ under the influence of the wave in the second component coming from the left with a frequency $\omega_2^+ = -\tilde{\omega} = 1.4$ and amplitude $a = 0.05$ for different values of g_{12} .

the first component, which [taking into account the solution (25)] has the asymptotics

$$\begin{aligned} \psi_1(x, t) &\xrightarrow{x \rightarrow -\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) e^{-i\omega_1^+ t} (e^{ik_1 x} + r_1 e^{-ik_1 x}) \\ &\quad + a\beta e^{-i\omega_1^- t} (e^{-ik_1 x} + r_1^* e^{ik_1 x}) - \sqrt{\mu} e^{-i\mu t}, \\ \psi_1(x, t) &\xrightarrow{x \rightarrow +\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) e^{-i\omega_1^+ t} t_1 e^{ik_1 x} \\ &\quad + a\beta e^{-i\omega_1^- t} t_1^* e^{-ik_1 x} + \sqrt{\mu} e^{-i\mu t}, \\ \psi_2(x, t) &\xrightarrow{x \rightarrow \pm\infty} 0, \end{aligned} \quad (49)$$

where

$$\beta = \frac{\sqrt{2}\mu}{\sqrt{(k_1^2 + 2\mu)[k_1(\sqrt{k_1^2 + 4\mu} + k_1) + 2\mu]}} \quad (50)$$

is a normalization constant. The reflection and transmission coefficients are defined as $R_1 = |r_1|^2$ and $T_1 = |t_1|^2$, respectively. The coefficient β was chosen in such a way as to have

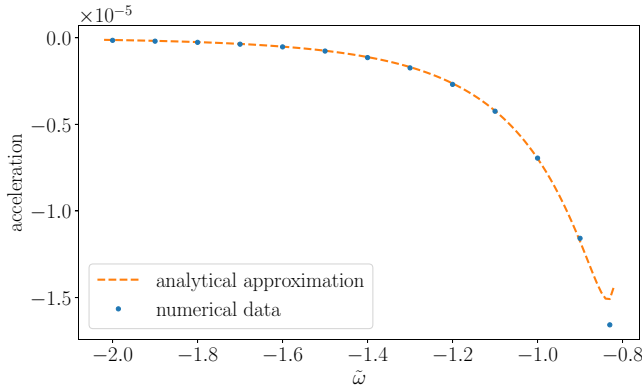


FIG. 2. Acceleration of a dark soliton with $\mu = 1$ under the influence of the wave in the second component coming from the left with different frequencies $\omega_2^+ = -\tilde{\omega}$, $g_{12} = 0.8$, and amplitude $a = 0.05$.

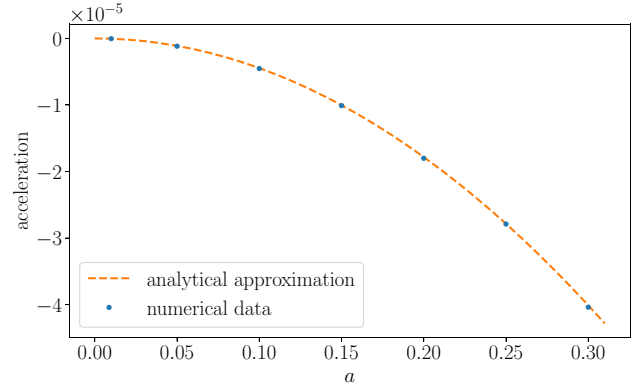


FIG. 3. Acceleration of a dark soliton with $\mu = 1$ under the influence of the wave in the second component coming from the left with a frequency $\omega_2^+ = -\tilde{\omega} = 1.4$, $g_{12} = 0.8$, and different amplitudes.

the simplest form of the force. From the above asymptotics, the effective force acting on a soliton can be derived analogously as in the previous case, obtaining

$$F = a^2 k_1^2 (1 + R_1 - T_1). \quad (51)$$

The solutions to the linearized equations (15) in this case with $\mu = 1$ can be found in [71–73]. Assuming the asymptotics (49) and transforming them for arbitrary μ , we obtain

$$\begin{aligned} \xi_1^{\pm}(x) &= \frac{2\beta k_1}{(k_1 \mp 2i\sqrt{\mu})(k_1^2 + 2\tilde{\omega})} \left[\left(\frac{k_1^2}{2} \mp \tilde{\omega} \right) \right. \\ &\quad \times \left(1 \pm \frac{2i\sqrt{\mu}}{k_1} \tanh(x\sqrt{\mu}) \right) + \frac{\mu}{\cosh^2(x\sqrt{\mu})} \left. \right] e^{\pm ik_1 x}, \end{aligned} \quad (52)$$

where β is given in (50). The solutions imply $R_1 = 0$ and $T_1 = 1$ and thus no force. However, we also solved the equations numerically, in order to show the more general method used later in this article. The infinities were approximated by sufficiently large L ; then $x \in [-L, L]$ [in general, the grid is different from the one used in the full partial differential equation (PDE) simulations of CNLSEs]. We changed the basis from (ξ_1^-, ξ_1^{+*}) to the solutions for which the asymptotic form of the equations (15) is diagonal. Let us denote the solutions

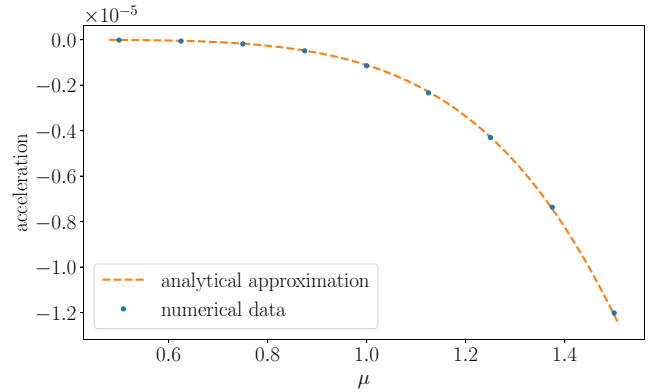


FIG. 4. Acceleration of a dark soliton with different values of μ under the influence of the wave in the second component coming from the left with a frequency $\omega_2^+ = -\tilde{\omega} = 1.4$, $g_{12} = 0.8$, and amplitude $a = 0.05$.

in the new basis by $(\tilde{\xi}_1^-, (\tilde{\xi}_1^+)^*)$. Then the boundary conditions were imposed as

$$(\tilde{\xi}_1^{-'} - ik_1 \tilde{\xi}_1^-)|_{x=-L} = -2ik_1 e^{ik_1 L} \quad (53)$$

to have $e^{-ik_1 x}$ at $x = -L$ and

$$(\tilde{\xi}_1^{-'} + ik_1 \tilde{\xi}_1^-)|_{x=L} = 0 \quad (54)$$

to ensure that there is no $e^{ik_1 x}$ at $x = L$. This corresponds to the incoming wave with $e^{ik_1 x}$ for ω_1^+ moving from left to right. Here $(\tilde{\xi}_1^+)^*$ describes the nonpropagating waves and we imposed analogous boundary conditions on it,

$$\{[(\tilde{\xi}_1^+)^*]' - ik_{\text{im}}(\tilde{\xi}_1^+)^*\}|_{x=-L} = \{[(\tilde{\xi}_1^+)^*]' + ik_{\text{im}}(\tilde{\xi}_1^+)^*\}|_{x=L} = 0, \quad (55)$$

with imaginary $k_{\text{im}} = \pm\sqrt{2}i\sqrt{\mu^2 + \tilde{\omega}^2 + \mu}$ instead of k_1 . Then R_1 and T_1 were computed from the numerical solutions, using the equations

$$R_1 = |\tilde{\xi}_1^-(-L) - e^{-ik_1 L}|^2, \\ T_1 = |\tilde{\xi}_1^-(L)|^2. \quad (56)$$

The acceleration, computed from the force (51) with numerically obtained R_1 and T_1 and the mass derived in the previous section, turned out to be close to zero, as expected.

The above result can be compared with the full PDE simulations of CNLSEs. Before we do that, let us discuss a particular difficulty present here. In the case of a scalar dark soliton with a wave in the second component, determining the center of the soliton from numerical data is relatively easy since the wave and the soliton are in completely different components. Then the center is simply given by a minimum of the probability density in the soliton component. In general, this is not the case because in many scenarios the waves scatter into both components of the condensate. Therefore, developing the strategy of extracting the position of a soliton from numerical data will be important not only for the waves initially in the first component, but for most other setups as well.

When a wave is present in the same component as the soliton, one can observe oscillations of the minimum (in the case of a bright soliton, the maximum) of the probability

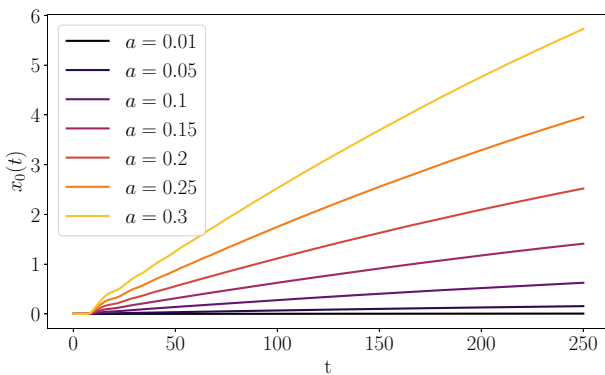


FIG. 5. Position of a dark soliton with $\mu = 1$ under the influence of the wave in the second component coming from the left with frequency $\omega_2^+ = -\tilde{\omega} = 1.4$, $g_{12} = 0.8$, and different amplitudes.

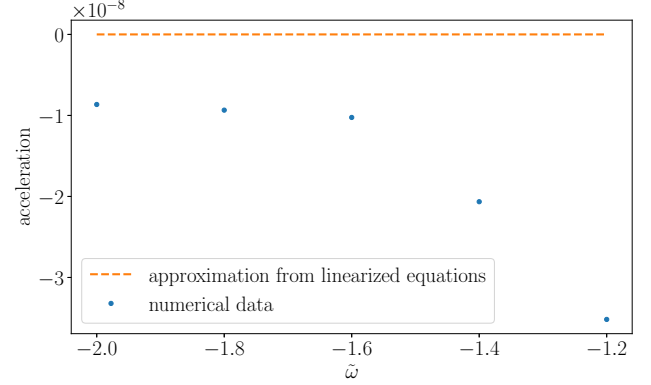


FIG. 6. Acceleration of a dark soliton with $\mu = 1$ under the influence of the wave in the first component coming from the left with different frequencies $\omega_1^\pm = \mu \mp \tilde{\omega}$ and amplitude $a = 0.05$.

density. These oscillations are due to the effect of the incident wave on the soliton (cf. Figs. 5 and 16). This interesting effect is the analog described by the Quist effect [74] for vortices. In the present case it complicates the extraction of the position of the soliton from numerical data. To improve the determination of the soliton positions, we used filtering of high frequencies from $|\psi(x, t)|^2$ at each instant t before computing the minimum. Since the amplitude of these oscillations becomes smaller than the observed trajectories during the time evolution, it pays off to make the time evolution as long as feasible.

The initial condition in the numerical simulations was $\psi_2(x, t = 0) = 0$ and

$$\psi_1(x, t = 0) = \Phi(x) + a\beta \left[\left(-\frac{k_1^2 - \tilde{\omega}}{\mu} - 1 \right) e^{ik_1 x} + e^{-ik_1 x} \right] \times \Phi_{\text{cut}}(x), \quad (57)$$

where Φ is the dark soliton for $t = 0$, Φ_{cut} is the same cutting function as in (48), and β is the normalization given by (50). We used $\tilde{\omega} < -\mu$ in order to have both $e^{\pm ik_1 x}$ starting a wave

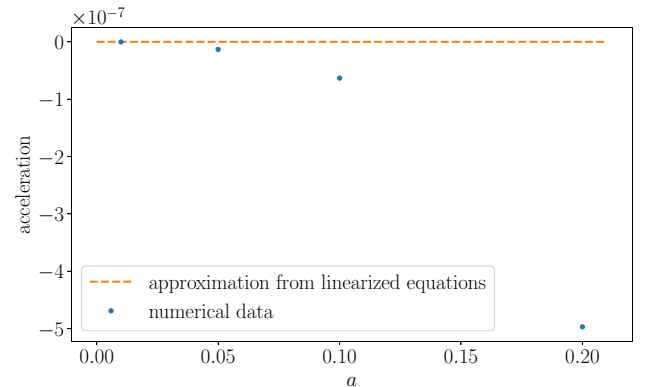


FIG. 7. Acceleration of a dark soliton with $\mu = 1$ under the influence of the wave in the first component coming from the left with frequencies $\omega_1^+ = 2.4$ and $\omega_1^- = -0.4$ (i.e., $\tilde{\omega} = -1.4$), $g_{12} = 0.8$, and different amplitudes.

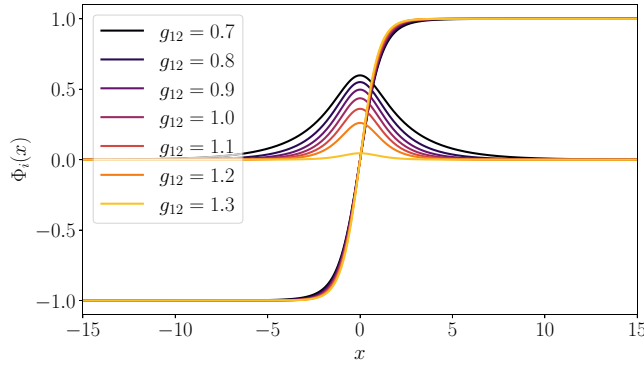


FIG. 8. Dark-bright solitons for different values of the parameter g_{12} , calculated using the gradient flow method. The parameters are $\mu = 1$, $\kappa = 0.9$, and μ_i as given in Eq. (59).

moving to the right. Acceleration computed from Eq. (51) divided by the appropriate effective mass with numerically obtained reflection and transmission coefficients has values close to zero. It was compared with the acceleration from the full PDE simulations (Figs. 6 and 7), which is also small. This seems to indicate that the force is indeed approximately zero.

V. DARK-BRIGHT SOLITONS

A. Dark-bright solutions

The CNLSE (3) with $g_{12} = 1$ possesses a particular solution [23]

$$\begin{aligned}\psi_1 &= e^{-i\mu t} \sqrt{\mu} \tanh[\kappa(x - x_0)], \\ \psi_2 &= e^{-i(\mu - \kappa^2/2)t} \sqrt{\mu - \kappa^2} \text{sech}[\kappa(x - x_0)],\end{aligned}\quad (58)$$

which is an example of a dark-bright soliton with

$$\mu_1 = \mu, \quad \mu_2 = \mu - \kappa^2/2. \quad (59)$$

Obviously, $0 < \kappa^2 < \mu$. Assuming $\psi_i(x, t) = e^{-i\mu_i t} \Phi_i(x)$, we can find DB solitons for other values of the parameter g_{12} . Their profiles Φ_i are presented in Fig. 8. They can be intuitively understood as a dip in the probability density of atoms

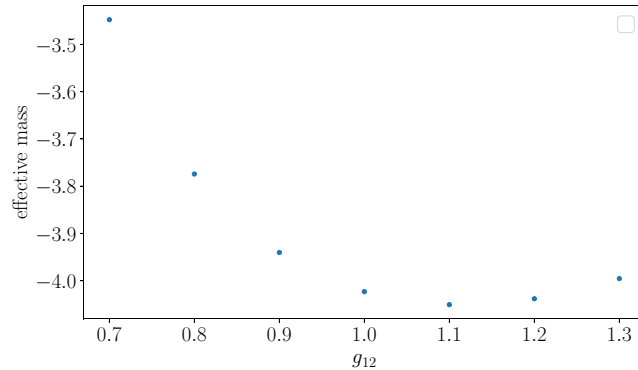


FIG. 9. Effective mass of DB solitons for different values of g_{12} , obtained using solitons at time $t = 60$ after pushing it using the external impulse (70) with the parameters $T = 1$ and $V_0 \in [0.005, 0.08]$. Velocities were obtained using the linear fit to the position of the center of the soliton.

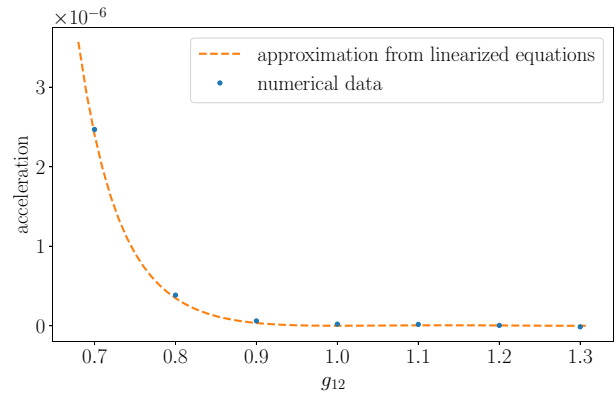


FIG. 10. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the dark component coming from the left with frequencies $\omega_1^+ = 2.4$ and $\omega_1^- = -0.4$ (i.e., $\tilde{\omega} = -1.4$), $a = 0.05$, and different values of g_{12} .

of one kind (species or spin state) and a relatively small peak in the probability density of atoms of the other kind.

Consider scattering on such solitons. It can be easily deduced from Eq. (12) that for $x \rightarrow \pm\infty$ DB solitons profiles $\Phi_1(x) \rightarrow \pm\sqrt{\mu}$, $\Phi_2(x) \rightarrow 0$, and $\Phi_i'(x) \rightarrow 0$ regardless of the value of g_{12} . Therefore, the matrix $\mathbf{M}$ in Eq. (17) becomes asymptotically

$$\mathbf{M}_{x \rightarrow \pm\infty} = \mu \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & g_{12} & 0 \\ 0 & 0 & 0 & g_{12} \end{pmatrix}. \quad (60)$$

Note that this asymptotic form is exactly the same as in the case of scalar dark soliton; therefore, the solutions are the same as in Sec. IV A, in particular the wave number

$$k_1 = \sqrt{2} \sqrt{\sqrt{\mu^2 + \tilde{\omega}^2} - \mu}, \quad (61)$$

and again the waves in the first component propagate for any $\tilde{\omega} \neq 0$. The only difference is that μ_2 is no longer arbitrary

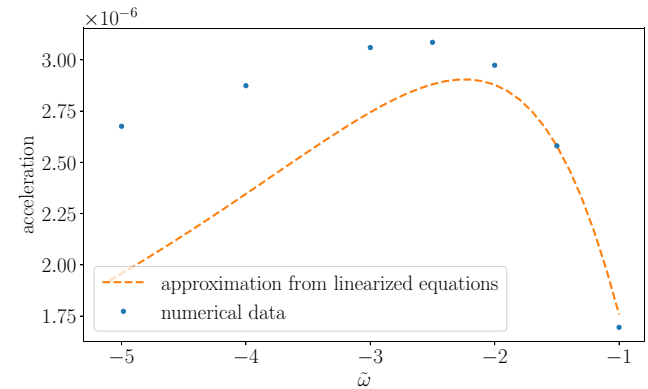


FIG. 11. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the dark component coming from the left with an amplitude $a = 0.05$, $g_{12} = 0.7$, and different frequencies $\omega_1^\pm = \mu \mp \tilde{\omega}$.

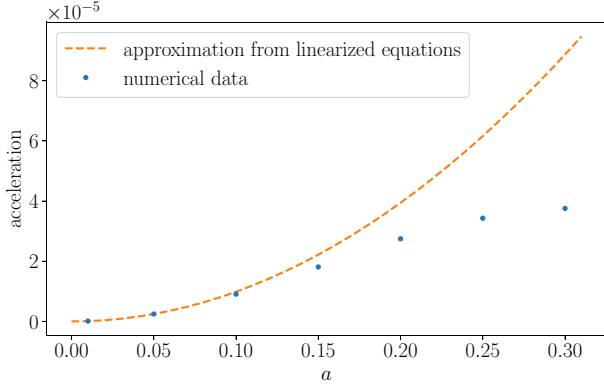


FIG. 12. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the dark component coming from the left with frequencies $\omega_1^+ = 2.4$ and $\omega_1^- = -0.4$ (i.e., $\tilde{\omega} = -1.4$), $g_{12} = 0.7$, and different amplitudes.

and therefore

$$k_2^\pm = \sqrt{2\sqrt{\mp\tilde{\omega} - \mu g_{12} + \mu - \kappa^2/2}}. \quad (62)$$

Thus, waves in the second component with wave number $k_2^\pm$ propagate when $\mp\tilde{\omega} > \mu g_{12} - \mu + \kappa^2/2$. This means that for $\kappa^2 < 2\mu(1 - g_{12})$ there exists a range of $\tilde{\omega}$ in which both waves can propagate with the same $\tilde{\omega}$; however, we were unable to find the stable solitons in this range. For $\kappa^2 \geq 2\mu(1 - g_{12})$ with fixed $\tilde{\omega}$ only one (or neither) of the waves can propagate. Note that this excludes the possibility of propagating both types of waves in the second component for $g_{12} \geq 1$.

Conditions for moving to the right in the first component are the same as for the scalar dark case. In the second component, they are $\tilde{\omega} < \mu - \kappa^2/2$ for ω_2^+ and $\tilde{\omega} > -\mu + \kappa^2/2$ for ω_2^- . Therefore, for $\tilde{\omega} \in (-\mu + \kappa^2/2, \mu - \kappa^2/2)$ both waves propagate to the right (it is always true because $\mu - \kappa^2/2$ is positive).

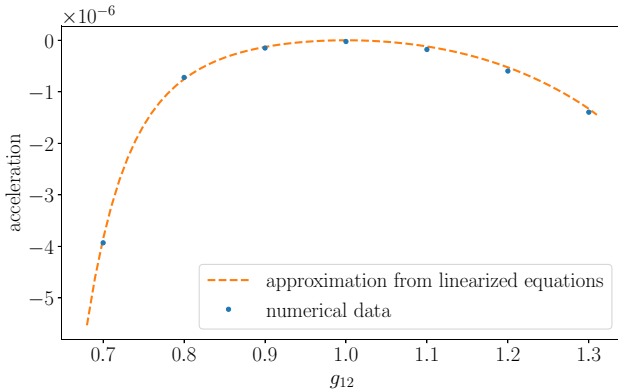


FIG. 13. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the bright component coming from the left with frequencies $\omega_2^+ = 1.995$ and $\omega_2^- = -0.805$ (i.e., $\tilde{\omega} = -1.4$), $a = 0.05$, and different values of g_{12} .

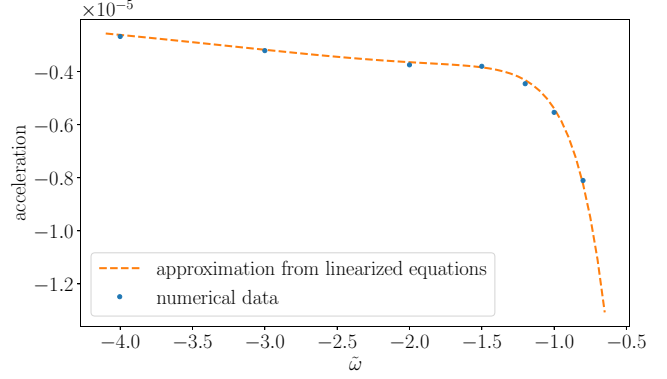


FIG. 14. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the bright component coming from the left with an amplitude $a = 0.05$, $g_{12} = 0.7$, and different frequencies $\omega_2^+ = \mu - \kappa^2/2 - \tilde{\omega}$.

The moving dark-bright soliton in the Manakov ($g_{12} = 1$) case with velocity v is [23]

$$\begin{aligned} \psi_1 &= e^{-i\mu t} \sqrt{\mu} \{ \cos \alpha \tanh[\tilde{\kappa}(x - x_0 - vt)] + i \sin \alpha \}, \\ \psi_2 &= e^{-i[\mu - \tilde{\kappa}^2(1 - \tan^2 \alpha)/2]t} e^{i v x} \sqrt{(\mu - \kappa^2) \frac{\tilde{\kappa}}{\kappa}} \operatorname{sech} \\ &\quad \times [\tilde{\kappa}(x - x_0 - vt)], \end{aligned} \quad (63)$$

where

$$\tilde{\kappa} = \frac{\kappa^2 - \mu + \sqrt{2\kappa^2 \mu \cos(2\alpha) + \kappa^4 + \mu^2}}{2\kappa}, \quad (64)$$

$$v = \tilde{\kappa} \tan \alpha. \quad (65)$$

We compute the renormalized total energy (analogously to the renormalized energy of a dark soliton)

$$\begin{aligned} E_s &= \frac{1}{2} \int_{-\infty}^{\infty} [|\partial_x \psi_1|^2 + |\partial_x \psi_2|^2 + (|\psi_1|^2 - \mu)^2 + |\psi_2|^4 \\ &\quad + 2|\psi_1|^2 |\psi_2|^2] dx, \end{aligned} \quad (66)$$

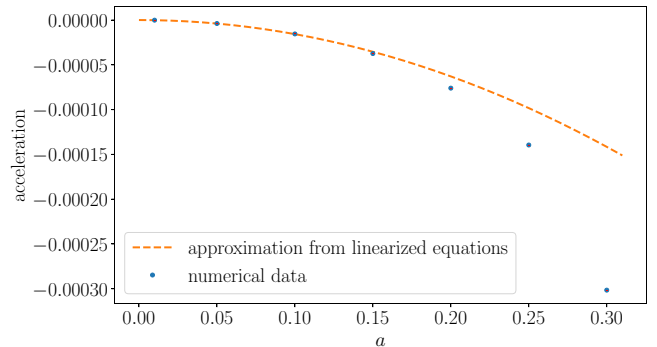


FIG. 15. Acceleration of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the bright component coming from the left with frequency $\omega_2^+ = 1.995$ (i.e., $\tilde{\omega} = -1.4$), $g_{12} = 0.7$, and different amplitudes.

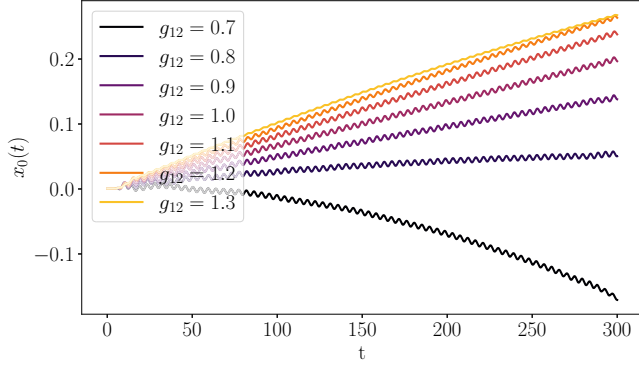


FIG. 16. Position of a dark-bright soliton with $\mu = 1$ and $\kappa = 0.9$ under the influence of the wave in the bright component coming from the left with frequency $\omega_2^+ = 1.995$ (i.e., $\tilde{\omega} = -1.4$), $a = 0.05$, and different values of g_{12} .

and from that we get the effective mass

$$M = \left. \frac{d^2 E_s}{dv^2} \right|_{v=0} = \left(\frac{dv}{d\alpha} \right)^{-2} \left(\frac{d^2 E_s}{d\alpha^2} - \frac{d^2 v}{d\alpha^2} \frac{dv}{d\alpha} \frac{dE_s}{d\alpha} \right) \Big|_{\alpha=0} = -\frac{2(\kappa^2 + \mu)}{\kappa}. \quad (67)$$

For $\kappa \rightarrow \sqrt{\mu}$ we reproduce the result for the scalar dark soliton $M = -4\sqrt{\mu}$. Using the renormalized total momentum

$$P_s = \frac{i}{2} \int_{-\infty}^{\infty} (\psi_1 \partial_x \psi_1^* - \psi_1^* \partial_x \psi_1 + \psi_2 \partial_x \psi_2^* - \psi_2^* \partial_x \psi_2) dx - \mu \Delta \phi, \quad (68)$$

where $\Delta \phi = 2\alpha - \pi$ is the phase change between $-\infty$ and $+\infty$ in the dark (first) component, we obtain the same effective mass

$$M = \left. \frac{dP_s}{dv} \right|_{v=0} = \left(\frac{dv}{d\alpha} \right)^{-1} \frac{dP_s}{d\alpha} \Big|_{\alpha=0} = -\frac{2(\kappa^2 + \mu)}{\kappa}. \quad (69)$$

This indicates that the motion of the dark-bright solitons in the Manakov case is indeed Newtonian and that we used the correct renormalization.

In the non-Manakov ($g_{12} \neq 1$) case we push the soliton using the short and localized external impulse

$$V_1(x) = V_0 t(T - t) \theta(t) \theta(T - t) \frac{\tanh(x)}{\cosh(x)}, \quad V_2(x) = 0 \quad (70)$$

and obtain the velocity and moving soliton profile after a sufficiently long time, from which we calculate E_s and compute $M = \left. \frac{d^2 E_s}{dv^2} \right|_{v=0}$ and $M = \left. \frac{dP_s}{dv} \right|_{v=0}$ by fitting a quadratic function

C. Wave in the bright component

If we consider a DB soliton with a wave in the second component with a wave number $k_2^\pm$ (and a small amplitude a) coming from the left, the asymptotics are

$$\psi_1(x, t) \xrightarrow{x \rightarrow -\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) \frac{1}{\mu} e^{-i\omega_1^+ t} r_1 e^{-ik_1 x} + a\beta e^{-i\omega_1^- t} r_1^* e^{ik_1 x} - \sqrt{\mu} e^{-i\mu t},$$

to $E_s(v)$ and a linear function to $P_s(v)$, respectively (Fig. 9). Henceforth, we will use the mass obtained from the momentum, because it is more accurate.

B. Wave in the dark component

Let us consider a DB soliton with a wave in the first component with a wave number k_1 and a small amplitude a coming from the left (provided adequate conditions, discussed in the preceding section, are met). The asymptotics are then

$$\begin{aligned} \psi_1(x, t) &\xrightarrow{x \rightarrow -\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) \frac{1}{\mu} e^{-i\omega_1^+ t} (e^{ik_1 x} + r_1 e^{-ik_1 x}) \\ &\quad + a\beta e^{-i\omega_1^- t} (e^{-ik_1 x} + r_1^* e^{ik_1 x}) - \sqrt{\mu} e^{-i\mu t}, \\ \psi_1(x, t) &\xrightarrow{x \rightarrow +\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) \frac{1}{\mu} e^{-i\omega_1^+ t} t_1 e^{ik_1 x} \\ &\quad + a\beta e^{-i\omega_1^- t} t_1^* e^{-ik_1 x} + \sqrt{\mu} e^{-i\mu t}, \\ \psi_2(x, t) &\xrightarrow{x \rightarrow -\infty} a e^{-i\omega_2^+ t} r_2^+ e^{-ik_2^+ x} + a e^{-i\omega_2^- t} r_2^- e^{-ik_2^- x}, \\ \psi_2(x, t) &\xrightarrow{x \rightarrow +\infty} a e^{-i\omega_2^+ t} t_2^+ e^{ik_2^+ x} + a e^{-i\omega_2^- t} t_2^- e^{ik_2^- x}, \end{aligned} \quad (71)$$

where β is the same as in Eq. (50). Using an analogous approach as before, we derive that the force exerted on the soliton by such a wave is

$$F = a^2 [k_1^2 (1 + R_1 - T_1) + (k_2^+)^2 (R_2^+ - T_2^+) + (k_2^-)^2 (R_2^- - T_2^-)], \quad (72)$$

where $R_1 = |r_1|^2$, $T_1 = |t_1|^2$, $R_2^\pm = |r_2^\pm|^2$, and $T_2^\pm = |t_2^\pm|^2$.

The setup (71) corresponds to the eigenwave (25) with $A = 0$; let us call it the first eigenwave. The other eigenwave ($B = 0$), i.e., Eq. (71) with $k_1 \rightarrow -k_1$, gives the same expression for the effective force, but with different values of reflection and transmission coefficients. Let us focus on the first eigenwave (note that the conditions for propagation to the right are derived above for the first eigenwave). Using an approach similar to that for the scalar soliton, we can compute values of these coefficients and (using the effective mass computed above) compare the resulting acceleration with full PDE simulations (with initial field configurations constructed analogously as for the scalar dark soliton). It turns out that we always get the negative radiation pressure, described well by our linear model for relatively small amplitudes and frequencies (Figs. 10–12). The nonlinear behavior for larger amplitudes is expected, since linear approach relies on the fact that the amplitude is small. However, the discrepancy for larger frequencies is surprising and requires further study.

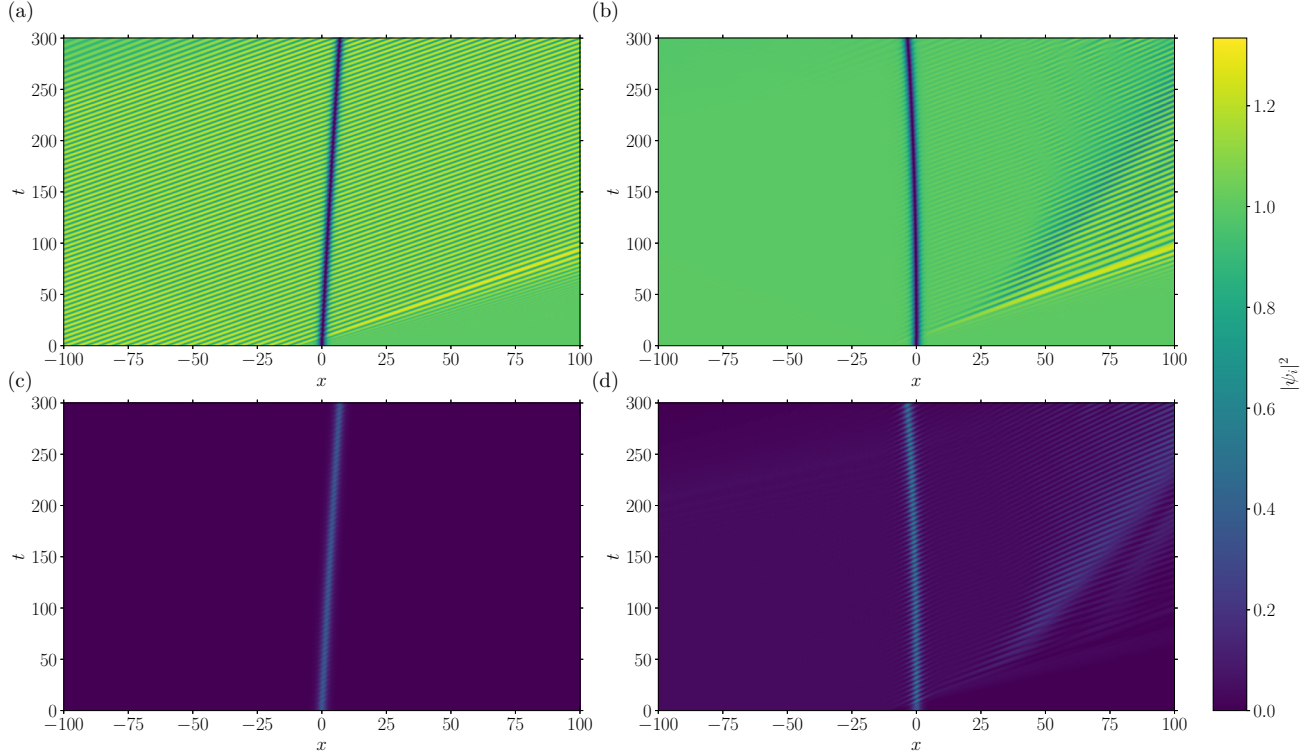


FIG. 17. Evolution of the dark-bright soliton with $\mu = 1$, $\kappa = 0.9$, and $g_{12} = 0.7$, interacting with a wave in the (a) and (c) dark and (b) and (d) bright component coming from the left with frequencies corresponding to $\tilde{\omega} = -1.4$ and amplitude $a = 0.2$. The probability densities of [(a) and (b)] the dark and [(c) and (d)] the bright components are presented.

$$\begin{aligned}
 \psi_1(x, t) &\xrightarrow{x \rightarrow +\infty} a\beta \left(-\frac{k_1^2}{2} - \tilde{\omega} \right) e^{-i\omega_1^+ t} t_1 e^{ik_1 x} + a\beta e^{-i\omega_1^- t} t_1^* e^{-ik_1 x} + \sqrt{\mu} e^{-i\mu t}, \\
 \psi_2(x, t) &\xrightarrow{x \rightarrow -\infty} a e^{-i\omega_2^+ t} e^{ik_2^+ x} + a e^{-i\omega_2^+ t} r_2^+ e^{-ik_2^+ x} + a e^{-i\omega_2^- t} r_2^- e^{-ik_2^- x}, \\
 \psi_2(x, t) &\xrightarrow{x \rightarrow +\infty} a e^{-i\omega_2^+ t} t_2^+ e^{ik_2^+ x} + a e^{-i\omega_2^- t} t_2^- e^{ik_2^- x},
 \end{aligned} \tag{73}$$

with β the same as in Eq. (50). Similarly to before, we can derive the force exerted on the soliton

$$\begin{aligned}
 F = a^2 [&k_1^2 (R_1 - T_1) + (k_2^+)^2 (R_2^+ - T_2^+) + (k_2^-)^2 (R_2^- - T_2^-) \\
 &+ (k_2^\pm)^2],
 \end{aligned} \tag{74}$$

where $R_1 = |r_1|^2$, $T_1 = |t_1|^2$, $R_2^\pm = |r_2^\pm|^2$, and $T_2^\pm = |t_2^\pm|^2$.

Again, we compute the values of the reflection and transmission coefficients numerically and compare the resulting acceleration with the full PDE simulations. In this case we observe the positive radiation pressure for all the values of parameters considered, and everything is described well by the linearized model provided the amplitude is small (Figs. 13–16). Examples of the evolution for waves in the both components are presented in Fig. 17.

D. Dark-bright solitons in a harmonic trap

To verify how well the above results apply in realistic situations, let us include a harmonic trapping potential

$$V_1(x) = V_2(x) = \frac{1}{2} \omega_x^2 x^2 \tag{75}$$

in Eq. (3), solve it numerically, and compare with the linearized approximations obtained without a trap. In this section we use physical units, unless we specifically refer to our units, defined in Sec. II A. In order for the parameters to be possible to achieve in experiments, we choose the trapping frequencies used in [36], i.e., $\omega_x = 2\pi \times 14$ Hz and $\omega_\perp = 2\pi \times 425$ Hz. Similarly, we normalize ψ_i to the numbers of atoms $N_1 \approx 152\,500$ and $N_2 \approx 1700$ in the dark and bright components, respectively. Again, N_1 was inspired by [36], while the proportion $N_2 \approx 0.01 N_1$ is similar to that in [33]. We choose $a_{11} = a_{22} \approx 100 a_0$ and $a_{12} \approx 110 a_0$, where a_0 is the Bohr radius. With these parameters, we use the gradient flow method and find a dark-bright soliton, presented in Fig. 18.

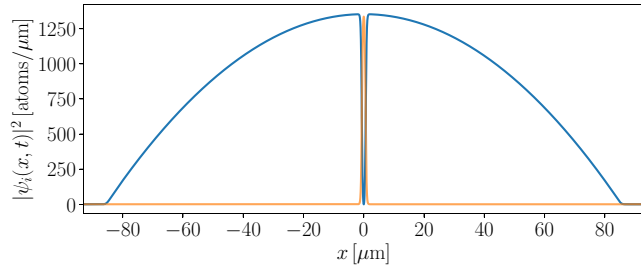


FIG. 18. Dark-bright soliton in a harmonic trap with a frequency $\omega_x = 2\pi \times 14$ Hz and $\omega_\perp = 2\pi \times 425$ Hz in a CNLSE with $a_{12} \approx 110a_0$ (in our units $g_{12} = 1.1$). The numbers of atoms are about $N_1 \approx 152\,500$ and $N_2 \approx 1700$ in the dark [blue (darker) line] and bright [orange (lighter) line] components, respectively, which correspond to $\mu = 35\,000$ and $\kappa = 20$ in our units.

We simulated collisions of the DB soliton with monochromatic waves of several frequencies, incoming from both the dark and the bright component. The amplitudes of these waves were equal to about 10% of the amplitude of the DB soliton's dark component, corresponding to $a = 20$ in our units. Despite the fact that the linearized approximation does not include a trap, the measured accelerations followed somewhat similar curves (Fig. 19). The accelerations were measured only in the short time interval (about first 4 ms), since for

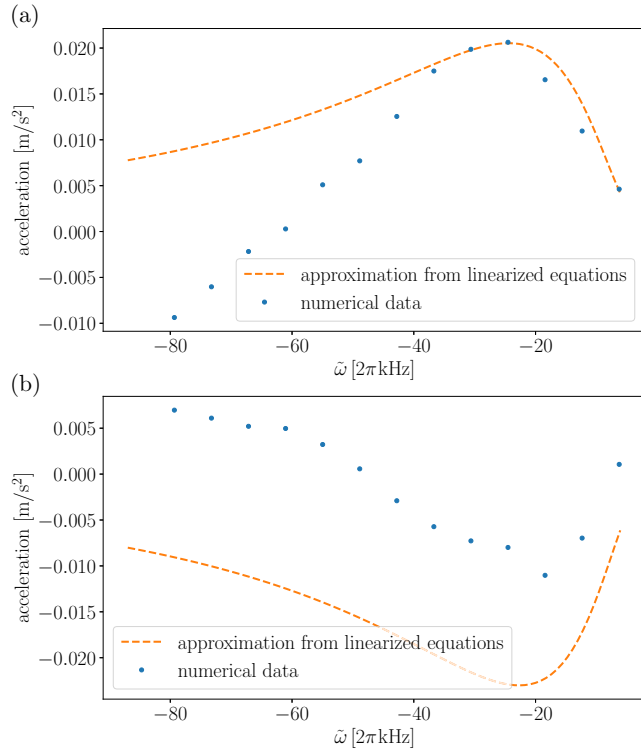


FIG. 19. Acceleration of the dark-bright soliton (Fig. 18) in a harmonic trap under the influence of waves in the (a) dark and (b) bright components, coming from the left with different frequencies and amplitude $a \approx 0.1\sqrt{\mu}$, compared with a linear approximation without a trapping potential.

longer times the force gets affected more by the harmonic potential (see an example of trajectories in Fig. 20).

VI. CONCLUSION

Using the Newtonian approximation and staying in the linear regime, we have successfully described the acceleration of the scatterer due to the action of the radiation pressure of a wave scattering on a dark and a dark-bright soliton. This simple model agrees with numerical simulations for a wide range of parameters. We have shown that a collision of a scalar dark soliton with a wave in the second component of the condensate always results in a positive radiation pressure. For dark-bright solitons, however, we found that the radiation pressure is negative if the wave is incoming from the dark component and positive otherwise. Our results provide a quantitative description for idealized homogenous BECs and a qualitative model for more realistic trapped condensates.

The mechanism responsible for NRP in this model relies on the fact that the soliton is present in both components. Otherwise, the equations separate, and from the conservation of energy it follows that the reflection and transmission coefficients sum to one. This implies that the force is always non-negative, as we have seen explicitly for the scalar soliton case. If the soliton is a vector soliton, then the constraints from the energy conservation allow for both positive and negative signs of the force. Furthermore, the dispersion relations (i.e., the wave numbers k_1 and $k_2^\pm$) play an important role in determining this sign (see Table I).

The discrepancies between our effective linear model and the full PDE simulations are completely expected for the larger amplitudes of the incoming wave, since the linearization relies on it being small. However, the disagreement for large frequencies of the wave incoming from the dark component and hitting the dark-bright soliton is currently not well understood within the scope of this paper. This could be an opportunity for further research, especially combined with a detailed study of the nonlinear effects, which can play a role here. Another interesting possibility would be to investigate NRP on other solitons in a two-component BEC, such as bright-bright and dark-dark solitons.

It is also worth mentioning that there is a correspondence between dark (dark-bright) solitons and Néel (Bloch) walls in the parametrically driven nonlinear Schrödinger equation [75–82], although with different dynamics. Moreover, the problem described in this paper shows some similarities to the wall-on-wall scattering described in [82], which is an interesting topic for further study.

The described setups can in principle be reproduced experimentally. Hopefully, in the future, this article could help to promote NRP from being a purely theoretical concept to an observable physical phenomenon.

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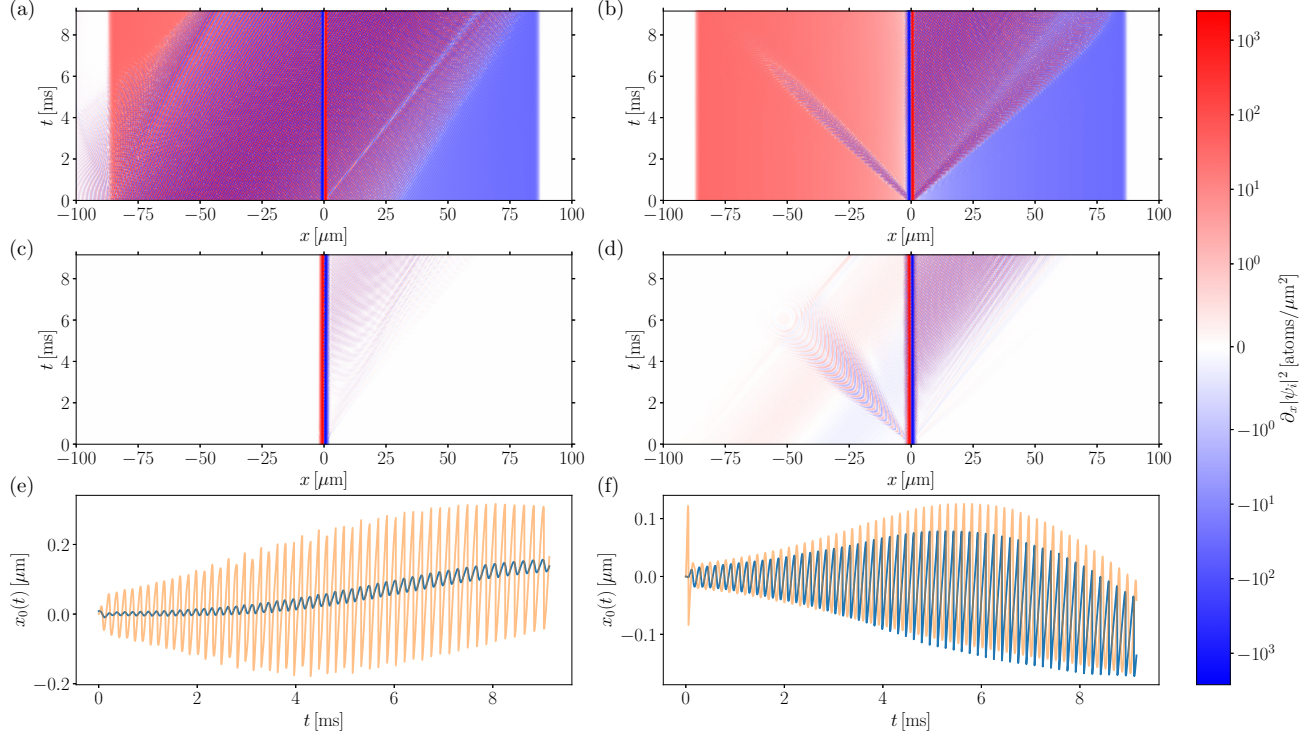


FIG. 20. Evolution of the dark-bright soliton in a harmonic trap (Fig. 18), interacting with a wave in the [(a), (c), and (e)] dark and [(b), (d), and (f)] bright components coming from the left with frequencies $\omega_1^+ = 2\pi \times 12\,351$ Hz, $\omega_1^- = -2\pi \times 174$ Hz, and $\omega_2^+ = 2\pi \times 12\,386$ Hz (i.e., $\tilde{\omega} = -2\pi \times 6263$ Hz) and amplitude $a \approx 0.1\sqrt{\mu}$. Spatial derivatives of the probability density of [(a) and (b)] the dark and [(c) and (d)] the bright components are presented as functions of time and space. (e) and (f) Additionally, the positions of the dark [blue (darker) line] and bright [orange (lighter) line] parts of the soliton are shown as functions of time.

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APPENDIX A: DERIVATION OF TOTAL ENERGY AND MOMENTUM

Using the Lagrangian density (4), we can derive the energy-momentum tensor

$$T_v^\mu = \sum_{i=1,2} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i)} \partial_v \psi_i + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i^*)} \partial_v \psi_i^* \right) - \mathcal{L} \delta_v^\mu, \quad (\text{A1})$$

where $\mu, v = 0, 1$; $\partial_0 = \partial_t$; and $\partial_1 = \partial_x$. Since we consider CNLSEs without the trapping potential, the Lagrangian density (4) is invariant under translations, and then from the Noether theorem it follows that such a tensor is a conserved current, meaning that it obeys

$$\sum_{\mu=1,2} \partial_\mu T_v^\mu = 0. \quad (\text{A2})$$

The total energy and momentum are defined

$$E = \int_{-\infty}^{\infty} T_0^0 dx, \quad P = - \int_{-\infty}^{\infty} T_1^0 dx, \quad (\text{A3})$$

respectively. Computing the energy-momentum tensor and integrating, we obtain their explicit form (5) and (6). Then Eqs. (7) and (8) follow from (A2).

APPENDIX B: NUMERICAL METHODS

In all of the simulations of soliton dynamics in the full PDE, we have used the second-order split-step method [83].

TABLE I. Example of reflection and transmission coefficients (derived from linearized equations) and the formulas present in the effective forces (72) and (74). Kronecker deltas are such that we set $i = 1$ for the wave incoming from the dark component and $i = 2$ for the bright. The parameters of the soliton and the scattered wave are $\mu = 1$, $\kappa = 0.9$, $g_{12} = 0.7$, and $\tilde{\omega} = -1.4$. Note that the wave numbers change the sign of the total force.

Coefficient	Wave from dark	Wave from bright
R_1	1.511×10^{-6}	4.792×10^{-6}
T_1	9.930×10^{-1}	9.325×10^{-3}
R_2^+	2.666×10^{-6}	1.601×10^{-4}
T_2^+	5.188×10^{-3}	9.929×10^{-1}
$\delta_{i1} + R_1 - T_1$	6.962×10^{-3}	-9.321×10^{-3}
$\delta_{i2} + R_2^+ - T_2^+$	-5.185×10^{-3}	7.279×10^{-3}
$k_1^2(\delta_{i1} + R_1 - T_1)$	1.003×10^{-2}	-1.343×10^{-2}
$(k_2^+)^2(\delta_{i2} + R_2^+ - T_2^+)$	-1.343×10^{-2}	1.885×10^{-2}

This method requires periodic boundary conditions; therefore, at large x the dark soliton (and the dark part of the dark bright soliton) was glued to the antisoliton to achieve $\psi_1 = -\sqrt{\mu}$ at the right boundary. The spatial step was $\Delta x = 0.1$, while the temporal step was in the range from $\Delta t = 0.0001$ to $\Delta t = 0.0003$, depending on a particular simulated configuration. We used $x \in [-500, 1000]$ or $x \in [-1000, 2000]$. Other methods with other boundary conditions are possible to implement in CNLSEs (see, e.g., [84]).

Linearized equations were solved using sparse matrices. The derivatives were discretized using the five-point stencil. The grid was $x \in [-20, 20]$ with the step $\Delta x = 0.01$. The size of the grid in this problem does not need to match the size of the spatial grid used in solving the full PDE. In fact, we verified that the solutions to linearized equations do not depend on the grid size, provided it is sufficiently large, such that the solutions achieve the expected asymptotic form.

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Letter

Negative radiation pressure in the abelian Higgs model

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ABSTRACT

Interactions of small-amplitude monochromatic plane waves with domain walls in (1+1) dimensional abelian Higgs model with a sextic potential were studied. The effective force exerted on a domain wall was derived from a linearized equation and compared with numerical simulations of the original model. It was shown that the domain walls always accelerate in one direction, regardless of the direction of the incoming wave. This implies that in some cases an effect called negative radiation pressure is observed, i.e. instead of pushing, the wave pulls the soliton.

1. Introduction

Solitons are a subject of extensive experimental and theoretical studies in many areas of physics, such as condensed matter physics [5], nuclear physics [28], Bose–Einstein condensates [18], nonlinear optics [26], cosmology [24], particle physics [14] and more. Kinks [38], also called domain walls,¹ are one of the simplest (1+1) dimensional topological solitons [27], existing in scalar field theories such as sine-Gordon, ϕ^4 , ϕ^6 , double sine-Gordon [6], Christ–Lee [8] and many other models. Due to their simplicity, they are often studied in various scenarios, e.g. kink-antikink collisions [1–3,6,7,20,29,37]. Domain wall solutions are also present in some gauge theories, especially in (1+1) dimensional version of the abelian Higgs model [13,17,23,33]. Due to its simplicity, the abelian Higgs model is often considered a textbook example of the Higgs mechanism (see e.g. Manton and Sutcliffe [27]) and it is related to the well-known Ginzburg–Landau theory of superconductivity [21]. Moreover, some recent studies propose its physical realization in quantum simulations using ultracold atoms [12,19,22,25,30–32,40]. In this paper, we use the (1+1) dimensional abelian Higgs model, albeit with a sextic potential instead of a usually used quartic potential, in order to research the interaction of domain walls with a radiation consisting of small-amplitude plane waves.

Usually, when a plane wave is scattered on a soliton, it pushes the soliton in the direction of propagation: we refer to this as the *positive*

radiation pressure (PRP). However, it has been shown that, in some systems, the incoming plane wave can exert a pulling force on the soliton. This effect is called a *negative radiation pressure* (NRP) and was found in many models, including kinks in ϕ^4 model [16,34], domain walls in ϕ^6 model [11,36], vortices in Gross–Pitaevskii model [15], dark-bright solitons in coupled nonlinear Schrödinger equations [9] and other [35,39]. Different mechanisms are responsible for this effect. For example, in the ϕ^4 model the nonlinear coupling creates a double frequency wave, which, combined with the reflectionlessness in the linear order, exerts NRP. However, a more common mechanism involves transfer of energy from slower to faster modes mainly in the linear order, due to two different dispersion relations of these modes. This can be realized by using two coupled fields (see Ciurla et al. [9], Romańczukiewicz [35]), non-topological solitons rotating in phase (such as Q-balls), or by asymmetry of the solitons (see Romańczukiewicz [36]). The latter mechanism is responsible for NRP in the model studied in this paper: in such a case, waves exert PRP or NRP, depending on the side from which they come. This motivates the choice of the sextic potential, which has the asymmetric domain wall solutions.

In this letter, we study NRP exerted on a domain wall. We consider monochromatic plane waves in the gauge field, oscillating only in one direction orthogonal to the direction of propagation, which allows us to reduce the problem to (1+1) dimensions. The incoming waves are treated as linear perturbations of the domain wall, which gives a good

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¹ Strictly speaking, a kink is a soliton in (1+1) dimensional spacetime, while a domain wall is a higher-dimensional object [38]. However, since we are interested in interactions of a domain wall with waves perpendicular to it, and neglect its shape across other directions, we can treat the domain wall as a kink embedded in more dimensions and use the names interchangeably.

description, provided their amplitude is sufficiently small. The slowly moving domain wall, on the other hand, is approximated as a Newtonian particle. Effective force derived from these assumptions can then be compared with numerical solutions of a partial differential equation (PDE) describing the full theory.

The paper is organized as follows. In Sec. 2 the considered model, its domain wall solution, and the conserved charges are presented. Then, in Sec. 3 small perturbations of the domain wall are considered, and the S -matrix for scattering on the soliton is calculated. These results are applied in Sec. 4 to calculate the effective force exerted by a radiation on a domain wall. Finally, the effective force is compared with the numerical simulations in Sec. 5.

2. The model

Let us consider the Higgs model, i.e. a complex scalar field ϕ in (3+1) dimensions coupled with an abelian gauge field A_μ :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*D^\mu\phi - U(|\phi|^2), \quad (1)$$

where $D_\mu = \partial_\mu + ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. However, we will use a sextic potential

$$U(|\phi|^2) = |\phi|^2(|\phi|^2 - 1)^2 \quad (2)$$

instead of a commonly used quartic potential. The vacua in this model form a union of a circle $\mathcal{V}_2$ and a point $\mathcal{V}_1$. More precisely, $\mathcal{V}_1 = \{(\phi, A_\mu) : \phi = A_\mu = 0\}$ and $\mathcal{V}_2 = \{(\phi, A_\mu) : |\phi| = 1, A_\mu = 0\}$. Due to this structure, topological defects (solitons) are present: our focus will be on domain walls, interpolating between $\mathcal{V}_1$ and $\mathcal{V}_2$. It is, however, worth mentioning, that the theory has also vortices $\mathcal{V}_2 \rightarrow \mathcal{V}_2$ with a winding number as a topological charge. The well-known Higgs mechanism is present in this model: at the vacuum $\mathcal{V}_2$, after fixing the phase of the scalar field, the vector field admits mass $m = \sqrt{2}e$.

We are interested in a special case of simple solutions, which can be derived from a restricted (1+1) dimensional model, obtained from the (3+1) dimensional theory with the following assumptions. First, let the gauge field have nonvanishing component only in the direction perpendicular to x , and denote it $A_\perp(x, t)$. Second, fix the phase of the scalar field, making it real, and assume that the scalar fields depend only on the x coordinate and time, i.e. $\phi = \phi(x, t)$. Then, the full theory (1) reduces to a (1+1) dimensional restricted model

$$\mathcal{L} = \frac{1}{2}(\partial_t A_\perp)^2 - \frac{1}{2}(\partial_x A_\perp)^2 + (\partial_t \phi)^2 - (\partial_x \phi)^2 - e^2 A_\perp^2 \phi^2 - U(\phi^2). \quad (3)$$

The simplified equations of motion are

$$A_{\perp,tt} - A_{\perp,xx} + 2e^2 A_\perp \phi^2 = 0, \quad (4)$$

$$\phi_{tt} - \phi_{xx} + e^2 A_\perp^2 \phi + U'(\phi^2)\phi = 0,$$

where

$$U'(\phi^2) = 1 - 4\phi^2 + 3\phi^4. \quad (5)$$

Looking for static solutions with $A_\perp = 0$, we obtain the second order ordinary differential equation (ODE)

$$-\phi_{xx} + U'(\phi^2)\phi = 0 \quad (6)$$

with the known domain wall (kink) solution

$$\phi_D(x) = \sqrt{\frac{1 + \tanh x}{2}}. \quad (7)$$

It is worth noting that the domain wall also obeys the first order ODE

$$\partial_x \phi_D = \sqrt{U(\phi_D^2)}. \quad (8)$$

Interactions of the static solutions with both scalar and vector fields can be studied. The interactions of the domain walls with a radiation in the scalar field without the gauge potential were studied in

Romańczukiewicz [36]. In this paper, we focus on small waves in the gauge field instead.

Finally, we can derive the conservation laws in the simplified model. Using the Noether theorem, the energy-momentum tensor can be obtained:

$$\begin{aligned} T^0_0 &= \frac{1}{2}(\partial_t A_\perp)^2 + \frac{1}{2}(\partial_x A_\perp)^2 + (\partial_t \phi)^2 + (\partial_x \phi)^2 + e^2 A_\perp^2 \phi^2 + U(\phi^2), \\ T^1_1 &= -\frac{1}{2}(\partial_t A_\perp)^2 - \frac{1}{2}(\partial_x A_\perp)^2 - (\partial_t \phi)^2 - (\partial_x \phi)^2 + e^2 A_\perp^2 \phi^2 \\ &\quad + U(\phi^2), \end{aligned} \quad (9)$$

$$T^0_1 = -T^1_0 = \partial_t A_\perp \partial_x A_\perp + 2\partial_t \phi \partial_x \phi$$

and the total energy and momentum are

$$E = \int_{-\infty}^{\infty} T^0_0 dx, \quad P = -\int_{-\infty}^{\infty} T^0_1 dx. \quad (10)$$

Then, from the conservation law

$$\partial_\mu T^\mu_\nu = 0 \quad (11)$$

we conclude that the change of the total momentum in time depends only on the asymptotic form of T^1_1 , namely

$$\partial_t P = T^1_1 \Big|_{-\infty}^{\infty}. \quad (12)$$

3. Linearized equations

In order to study the interactions of solitons with a radiation, we consider small perturbations around the domain wall:

$$\begin{aligned} \phi(x, t) &= \phi_D(x) + b\delta\phi(x, t), \\ A_\perp(x, t) &= a\delta A_\perp(x, t), \end{aligned} \quad (13)$$

where a and b are small real parameters. After substitution to (4), the equations for $\delta\phi(x, t)$ and $\delta A_\perp(x, t)$ in the leading order separate, and since we are interested only in interactions with gauge field waves in the linear regime, we put $b = 0$. Then, we can separate the variables:

$$\delta A_\perp(x, t) = \eta(x)e^{i\omega t}, \quad (14)$$

obtaining a linear Schrödinger-like equation

$$-\eta''(x) + V(x)\eta(x) = \omega^2\eta(x) \quad (15)$$

with an effective potential

$$V(x) = e^2(1 + \tanh(x)). \quad (16)$$

We are interested in the scattering modes, which asymptotically are plane waves propagating from the left to the right or vice versa. Such solutions have a wavenumber ω at $x \rightarrow -\infty$ and

$$k = \sqrt{\omega^2 - 2e^2} \quad (17)$$

at $x \rightarrow +\infty$. Therefore, we are looking for solutions η_+ and η_- , such that

$$\begin{aligned} \eta_+(x \rightarrow -\infty) &= e^{ix\omega} + S_{11}e^{-ix\omega}, \\ \eta_+(x \rightarrow +\infty) &= S_{21}e^{ikx}, \\ \eta_-(x \rightarrow -\infty) &= S_{12}e^{-ix\omega}, \\ \eta_-(x \rightarrow +\infty) &= e^{-ikx} + S_{22}e^{ikx}, \end{aligned} \quad (18)$$

where $e^{i\omega t}\eta_+(x)$ describes a wave moving from $-\infty$ to $+\infty$, $e^{i\omega t}\eta_-(x)$ is a wave moving in the opposite direction, and S_{ij} are elements of the so-called S -matrix. Importantly, the waves at $+\infty$ are not propagating if $\omega^2 \leq 2e^2$. The wavenumber (17) can be understood as a gauge field acquiring mass $m = \sqrt{2}e$ at $|\phi| = 1$ vacuum due to the Higgs mechanism.

The solutions with the asymptotics discussed above can be found exactly (see Barut et al. [4]):

$$\eta_{\rightarrow}(x) = e^{ix\omega} {}_2F_1\left(-\frac{1}{2}i(k-\omega), \frac{1}{2}i(k+\omega); i\omega+1; -e^{2x}\right) - \alpha e^{-ix\omega} {}_2F_1\left(\frac{1}{2}i(k-\omega), -\frac{1}{2}i(k+\omega); 1-i\omega; -e^{2x}\right), \quad (19)$$

$$\eta_{\leftarrow}(x) = \beta e^{-ix\omega} {}_2F_1\left(\frac{1}{2}i(k-\omega), -\frac{1}{2}i(k+\omega); 1-i\omega; -e^{2x}\right),$$

where

$$\alpha = \frac{\Gamma(1+i\omega)\Gamma\left(-\frac{i\omega}{2}-\frac{ik}{2}\right)\Gamma\left(-\frac{i\omega}{2}-\frac{ik}{2}+1\right)}{\Gamma(1-i\omega)\Gamma\left(\frac{i\omega}{2}-\frac{ik}{2}\right)\Gamma\left(\frac{i\omega}{2}-\frac{ik}{2}+1\right)}, \quad (20)$$

$$\beta = \frac{\Gamma\left(-\frac{ik}{2}-\frac{i\omega}{2}\right)\Gamma\left(-\frac{ik}{2}-\frac{i\omega}{2}+1\right)}{\Gamma(-ik-2)\Gamma(1-i\omega)(2e^2+3ik-\omega^2+2)}.$$

This allows us to find the values of the S -matrix elements, and especially their moduli squared:

$$|S_{11}|^2 = |S_{22}|^2 = \frac{\sinh^2\left(\frac{1}{2}\pi(\omega-k)\right)}{\sinh^2\left(\frac{1}{2}\pi(\omega+k)\right)}, \quad (21)$$

$$|S_{21}|^2 = \frac{\omega}{k}(1-|S_{11}|^2),$$

$$|S_{12}|^2 = \frac{k}{\omega}(1-|S_{11}|^2),$$

which will be used in the computation of the effective force exerted on the domain wall.

4. Effective dynamics

To study the domain wall dynamics for low velocities, the wall can be approximated as a Newtonian particle. More precisely, we assume that its shape remains constant, and we treat its center position x_0 as a function of time, which obeys

$$M\ddot{x}_0(t) = F. \quad (22)$$

Let P_v be the total momentum of the boosted domain wall with a velocity v . Using equation (10), we obtain the effective mass:

$$M = \left.\frac{\partial P_v}{\partial v}\right|_{v=0} = \frac{1}{2}. \quad (23)$$

The effective force is

$$F = \langle \partial_t P \rangle_T, \quad (24)$$

where P is the total momentum of the domain wall and the scattered wave, and $\langle \cdot \rangle_T$ means the average over the period of the oscillations.

Now we can present the specific results describing the interaction between the domain wall and the radiation. We assume that a sinusoidal plane wave of the form

$$\delta A_{\perp}(x, t) = \frac{a}{2i} (e^{-i\omega t} \eta(x) - c.c.) \quad (25)$$

is scattered on a soliton, where η is the asymptotic form of $\eta_{\rightarrow}$ or $\eta_{\leftarrow}$ from the equation (18) and the amplitude a is sufficiently small, in order to use the linear regime. Then, we can apply the formula for the effective force (24) together with the equation for the time derivative of the total momentum (12) to the setup consisting of a domain wall and the sinusoidal wave (25). However, we cannot use the formula for $\partial_t P$ directly, since T^1_1 for plane waves does not have well-defined limits at infinities. Instead, we use the formula (12) for the finite interval with arbitrary size and then average over the period of oscillations. This average is independent of the length of the chosen interval, and gives the effective forces:

$$F_{\rightarrow} = \frac{1}{2}a^2 \left[\omega^2 (1 + |S_{11}|^2) - k^2 |S_{21}|^2 \right], \quad (26)$$

$$F_{\leftarrow} = \frac{1}{2}a^2 \left[-k^2 (1 + |S_{22}|^2) + \omega^2 |S_{12}|^2 \right].$$

Below the threshold (i.e. $\omega \leq \sqrt{2}e$), the waves at $+\infty$ cannot propagate, therefore the waves coming from the left are fully reflected, i.e. $|S_{11}| = 1$ and $|S_{21}| = 0$ for $F_{\rightarrow}$. For the case above the threshold, we know the explicit formulas (21) for the S -matrix elements. Using these data, we get

$$F_{\rightarrow} = \begin{cases} a^2 \omega^2 & \omega \leq \sqrt{2}e, \\ a^2 \omega \frac{-k \sinh(\pi\omega) \sinh(\pi k) + \omega \cosh(\pi\omega) \cosh(\pi k) - \omega}{\cosh(\pi(k+\omega)) - 1} & \omega \geq \sqrt{2}e, \end{cases} \quad (27)$$

$$F_{\leftarrow} = a^2 k \frac{\omega \sinh(\pi\omega) \sinh(\pi k) - k \cosh(\pi\omega) \cosh(\pi k) + k}{\cosh(\pi(k+\omega)) - 1}.$$

Both forces are always positive (or zero), regardless of the direction of the incoming wave. This means that if the wave is coming from the left, we observe the positive radiation pressure (PRP) and when it is coming from the right, we see the negative radiation pressure (NRP). For $\omega \rightarrow \infty$:

$$F_{\rightarrow} = F_{\leftarrow} = \frac{a^2 e^2}{2}, \quad (28)$$

which is not surprising, since then also $k \rightarrow \omega$ and the waves on both sides become more similar to each other. For $\omega \rightarrow \sqrt{2}e$ waves with different directions behave differently:

$$F_{\rightarrow} = 2a^2 e^2, \quad (29)$$

$$F_{\leftarrow} = 0,$$

while $F_{\rightarrow}$ vanishes for $\omega = 0$.

5. Numerical results

To check the validity of the approximate effective forces (27), the equations (4) were solved numerically, and the results were compared. The initial condition was the domain wall $\phi = \phi_D$ and the ‘wave train’

$$A_{\perp}(x, t) = a \sin(\omega x - \omega t) \frac{\tanh(x - 20 + L - v_1 t) - \tanh(x + 20 - v_1 t)}{2} \quad (30)$$

for the wave going from left to right or

$$A_{\perp}(x, t) = a \sin(-kx - \omega t) \frac{\tanh(x - 20 + v_2 t) - \tanh(x + 20 - L + v_2 t)}{2} \quad (31)$$

for the wave propagating from right to left, where $v_1 = 1$, $v_2 = k/\omega$ are group velocities and $x \in [-L, L]$ with $L = 250$. The RK4 method was used, with space and time steps $\Delta x = 0.01$ and $\Delta t = 0.005$ respectively. Accelerations were measured by fitting the quadratic function to the position of the soliton in the time range 20 to 200.

The results, presented in Figs. 1, 2 and 3, agree well with the approximate formulas derived from the linearized equations. An example of the soliton evolution with NRP can be seen in the Fig. 4. As expected, the accuracy of the approximation drops for larger amplitudes (Fig. 2), and charges (Fig. 3), due to the nonlinear effects.

6. Conclusions

Using the linearized Newtonian approximation and numerical simulations, we have shown that the domain walls always accelerate towards $\phi = 0$ vacuum. Therefore, we observe both PRP and NRP, depending on the direction of the incoming wave. This behavior is very similar to the

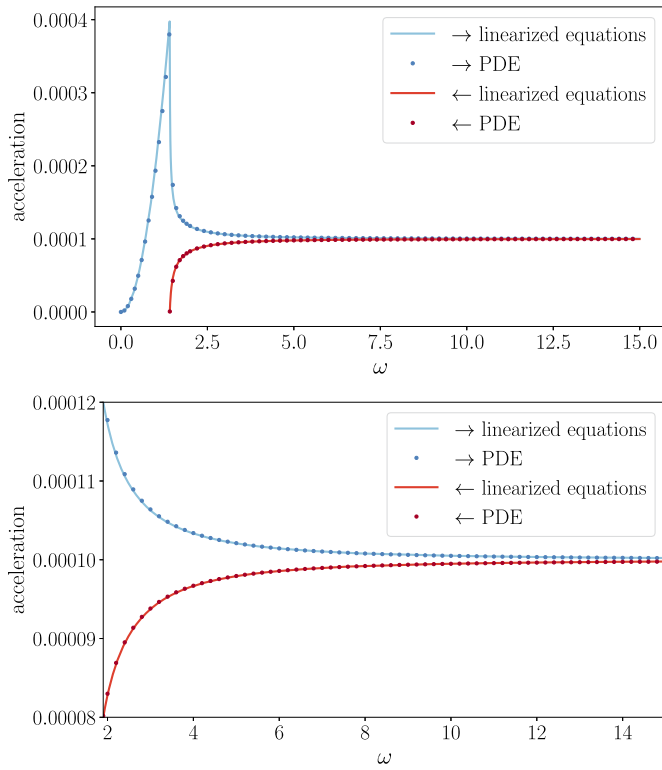


Fig. 1. Acceleration of the domain wall interacting with a wave coming from the left (upper, blue) or right (lower, red) with an amplitude $a = 0.01$, a charge $e = 1$ and different frequencies.

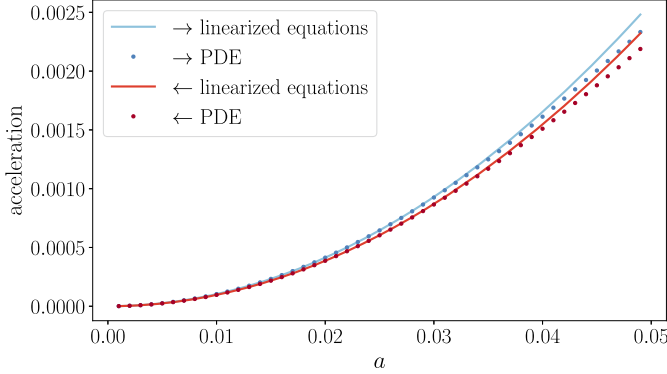


Fig. 2. Acceleration of the domain wall interacting with a wave coming from the left (upper, blue) or right (lower, red) with a frequency $\omega = 4$, a charge $e = 1$ and different amplitudes.

one observed in the ϕ^6 model without a gauge field [36], and we expect that it would play a similar role in multikink collisions, although it requires further study.

In this paper, we chose the sextic potential, since it is the simplest one with the asymmetric domain wall solutions, allowing for the simple NRP mechanism described well by the linear order approximation. It is, however, possible to study radiation pressure in the more usual abelian Higgs model with quartic self-interaction, although that would probably involve similar mechanism to the one found in the ϕ^4 model [16,34], and therefore would require computations in higher orders. Moreover, it could also be interesting to investigate higher order effects for waves with higher amplitudes in the sextic potential case.

Another possibility for further research would be to study oscillons and their interactions with kinks. Oscillons can play an important role in

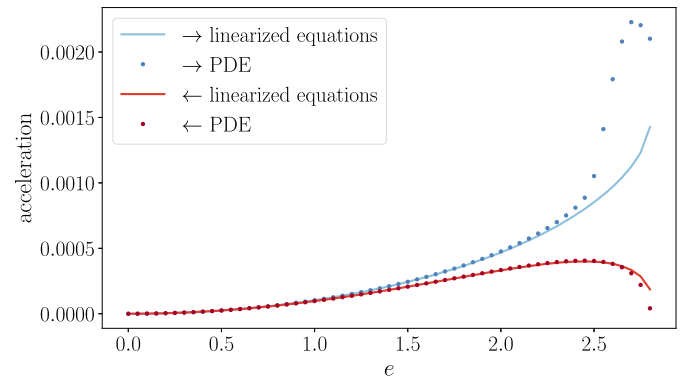


Fig. 3. Acceleration of the domain wall interacting with a wave coming from the left (upper, blue) or right (lower, red) with an amplitude $a = 0.01$, a frequency $\omega = 4$ and different charges e .

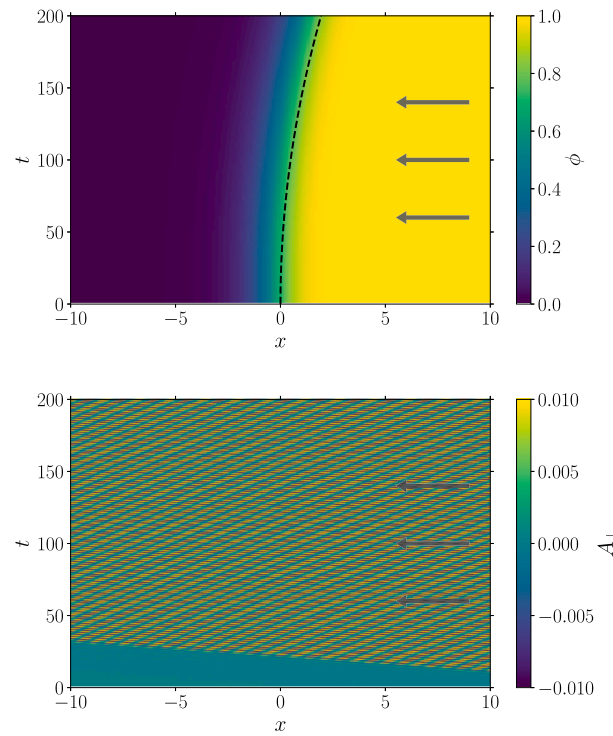


Fig. 4. Example of an evolution of the domain wall interacting with a gauge field wave coming from the right with an amplitude $a = 0.01$, a frequency $\omega = 4$ and a charge $e = 1$. Both scalar field (upper) and gauge field (lower) are presented. The dashed line represents the soliton trajectory with acceleration obtained from the linearized approximation. Arrows mark the direction of the incoming wave.

many physical processes and have interesting, nontrivial dynamics, and moreover, they can store enough energy to bounce back kinks [10]. Furthermore, radiation pressure exerted on more complicated topological solitons, i.e. vortices, can be studied in this model.

Observation of negative radiation pressure in (1+1) abelian Higgs model with a sextic potential is yet another indication that this phenomenon is widespread in theoretical physics. Although the results of this paper are not applicable to experiments directly, the studied system can be understood as an important simple toy model, expanding the mathematical tools needed to find NRP in nature. We believe that studying NRP can lead to better understanding of many physical scenarios, e.g. the dynamics in systems of many solitons, in which the radiation is present.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Dominik Ciurla reports a relationship with National Science Centre Poland that includes: funding grants. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Perturbations of Q-balls: from spectral structure to radiation pressure

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ABSTRACT: We investigate Q-balls in a 1+1 dimensional complex scalar field theory. We find that the relaxation of a squashed Q-ball is dominated by the decay of a normal mode through nonlinear coupling to scattering modes and a long-lasting quasi-normal mode. We also study how these Q-balls behave when exposed to scalar radiation, finding that for certain conditions they can experience negative radiation pressure.

KEYWORDS: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons

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Contents

1	Introduction	1
2	A class of Q-balls in 1+1 dimensions	2
2.1	The model	2
2.2	Stationary solutions	4
3	Spectral structure	5
3.1	Linearization	5
3.2	Zero modes	6
3.3	Half-propagating and quasi-normal modes	7
3.4	Bound modes	8
3.5	Mode decay	11
4	Radiation pressure	13
5	Conclusions	15
A	Metastable Q-balls	16

1 Introduction

Q-balls are non-topological solitons which arise in various non-linear scalar field theories possessing an unbroken continuous global symmetry [1–3]. They carry a Noether charge associated with this symmetry, and are stationary localized field configurations with an explicitly time-dependent phase. Typical examples of Q-balls in flat 3+1 dimensional Minkowski spacetime are spherically symmetric solutions of a model with a single complex scalar field and a suitable self-interaction potential [3], or solitons of the renormalizable Friedberg-Lee-Sirlin two-component model with a symmetry breaking potential [2]. Physically, they can be considered as a condensate of a large number of scalar quanta which, for a fixed value of the charge Q , yields an extremum of the effective energy functional. In such a context, the charge can also be interpreted as the particle number.

Q-balls have received considerable attention over the last three decades (for a review, see, e.g., [4–6]). However, most works address the stability of stationary Q-balls and the domains of their existence. There have only been a few studies of the dynamics of Q-balls [7–10], which appears to be very different from the usual dynamics of topological solitons. The reasons for this are related to phase-dependent force of interaction between the Q-balls, their non-topological charge which can be transferred in collisions [10], and the complicated spectrum of their excitations, which as we will show below may include both normal and quasinormal (QNM) modes (for discussions of QNM see [11–13] in the context of black holes, [14] in relation with excitations of a non-Abelian monopole, and [15]). Furthermore, in some cases these excitations cannot be considered as linearized perturbations, since at least second order corrections must be taken into account in order to understand them fully [16].

There is some similarity between Q-balls and oscillons [17–19], extremely long-lived, spatially localized, almost periodic non-linear field configurations. It was pointed out that a Q-ball can be roughly viewed as a system of two interacting oscillons, associated with real components of a complex scalar field [10]. However, while Q-balls are non-radiating stationary solutions, oscillons slowly radiate energy [20–22]. A peculiar feature of the radiation of an oscillon is its resonant character [23–25]: the radiating oscillon can pass through a sequence of quasi-stable Q-ball-like configurations. The relation between Q-balls and oscillons is supported by the existence of a so-called adiabatic invariant that is approximately conserved [26–28].

Important information about the properties of solitons can be obtained from the study of their interactions with external perturbations. In particular, one can consider a small amplitude incoming wave moving towards a soliton. It has been observed that in a large class of models with kink solutions the radiation pressure exerted on the kink can be negative [29–31]. Similar effects were found in Bose-Einstein condensates [32]. Scalar radiation may also strongly influence the dynamics of soliton collisions [33], so a natural question arises as to what the effect of the interaction of Q-balls with incoming scalar radiation might be.

In simple cases in (1+1) dimensions, analytical solutions for stationary Q-balls are known, as discussed, for example, in [9]. The goal of this paper is to study perturbative excitations of some of these Q-balls. In particular, we reconsider squashing perturbations of a single Q-ball [9] and examine the effects of radiation pressure. The paper is organized as follows. In section 2, we define the model and some important properties of the Q-balls. Section 3 is devoted to the linearization problem and spectral structure of the Q-balls. We analyze scattering and bound modes, as well as the so-called half-propagating modes, investigating in particular their connection with quasinormal modes. We analyze how the spectral structure, and especially the bound and quasinormal modes, influences the long-term evolution of a perturbed Q-ball. In section 4, we focus on the motion of Q-balls interacting with a monochromatic wave. We find that for a certain set of parameters, it is possible for Q-balls to accelerate towards the source of radiation, which is an indication of the negative radiation pressure. After our conclusions in section 5, we comment in appendix A on the results of [9] for a metastable Q-ball.

2 A class of Q-balls in 1+1 dimensions

2.1 The model

The (1+1)-dimensional scalar field theory we will study is defined by the following Lagrangian [7, 9]:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(|\phi|^2), \quad (2.1)$$

where the asterisk denotes complex conjugation and the rescaled potential of the self-interacting complex scalar field ϕ is

$$V(|\phi|^2) = |\phi|^2 - |\phi|^4 + \beta |\phi|^6. \quad (2.2)$$

This potential is shown for various values of the parameter β in figure 1. There is a minimum at $\phi = 0$ which for $\beta > 1/4$ is the global minimum, while for $\beta < 1/3$ there is a second minimum at $|\phi|^2 = 1 + \sqrt{1 - 3\beta}$.

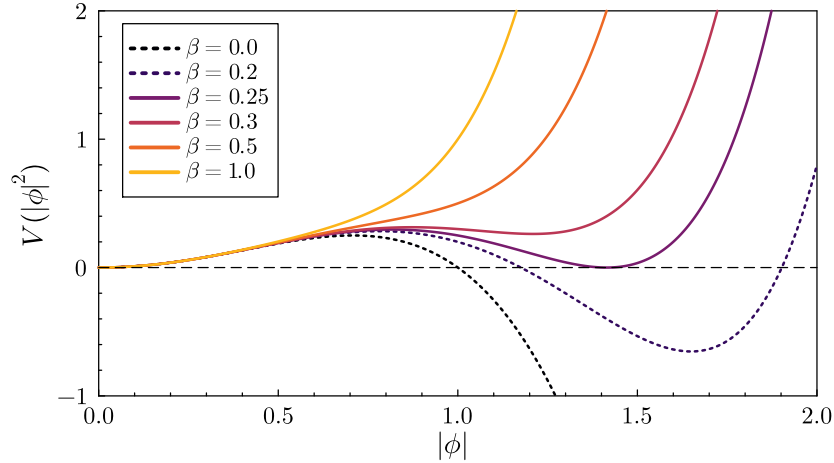


Figure 1. The field theory potential $V(|\phi|^2)$ defined by (2.2) for various values of β . For the potentials shown with dotted lines, the vacuum at $|\phi| = 0$ is metastable.

The global U(1) symmetry of the Lagrangian (2.1) corresponds to the conserved current

$$j_\mu = i(\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi), \quad \partial^\mu j_\mu = 0. \quad (2.3)$$

Variation of the Lagrangian (2.1) with respect to the scalar field leads to the equation of motion

$$\phi_{tt} - \phi_{xx} + \frac{\partial V}{\partial |\phi|^2} \phi = 0 \quad (2.4)$$

A stationary (1+1)-dimensional Q-ball configuration with harmonic time dependence can be parameterised as

$$\phi(x, t) = e^{i\omega t} f(x) \quad (2.5)$$

where $f(x)$ is a real profile function which satisfies the first order equation

$$\frac{df}{dx} = \pm \sqrt{\tilde{V}(f^2)} \quad (2.6)$$

with the boundary conditions $f(\pm\infty) = 0$, and

$$\tilde{V}(f^2) = V(f^2) - \omega^2 f^2 \quad (2.7)$$

is the effective potential for the Q-ball configuration. Further specifying $f_x(0) = 0$ centres the profile function on the origin.

The Noether charge of the stationary configuration is

$$Q = \int dx j_0 = 2\omega \int dx f^2 = 2\omega N, \quad (2.8)$$

where N is the L^2 norm of the scalar field. The total energy of the Q-ball can be written as

$$E = \int dx \left\{ (f_x)^2 + \omega^2 f^2 + V(f^2) \right\} = \int dx \left\{ (f_x)^2 + \tilde{V}(f^2) \right\} + \omega Q. \quad (2.9)$$

2.2 Stationary solutions

Equation (2.6) admits the analytic solution

$$f(x; \omega) = \frac{\sqrt{2}\omega'}{\sqrt{1 + \sqrt{1 - 4\beta\omega'^2} \cosh(2\omega'x)}}, \quad (2.10)$$

where $\omega' = \sqrt{1 - \omega^2}$ is the complementary frequency [9]. For the sake of simplicity, we will drop ω and write a shorthand version $f(x)$ in most cases, except for one paragraph in section 3.2.

Localized soliton solutions of the model (2.1) exist only for a limited range of values of the angular frequency, $\omega_{\min} \leq \omega \leq \omega_{\max}$. Here the upper bound corresponds to mass of the linearized scalar excitations, $\omega_{\max} = V''(0)/2 = 1$, and the lower bound depends on the value of the parameter β as [9]

$$\omega_{\min}(\beta) = \sqrt{1 - \frac{1}{4\beta}}. \quad (2.11)$$

Note that the potential (2.2) possesses a unique global minimum at $\phi = 0$ if $\beta > \frac{1}{4}$. If $\beta < \frac{1}{4}$, the local minimum at $\phi = 0$ becomes a false vacuum. In such a case, the configuration (2.10) will be metastable. Finally, in the marginal case $\beta = \frac{1}{4}$ the vacuum $V(|\phi|^2) = 0$ is two-fold degenerate between $\phi_0 = 0$ and $|\phi_1| = \sqrt{2}$ and the model supports topological solitons, or kinks. In this limit, the minimal value of the angular frequency is zero.

More generally, as the angular frequency approaches the minimal value $\omega_{\min}$ (2.11), the effective potential $\tilde{V}(f^2)$ takes the form of the standard ϕ^6 potential

$$\tilde{V}(f^2) = \beta f^2 \left(f^2 - \frac{1}{2\beta} \right)^2 \quad (2.12)$$

with minima at $\tilde{V}(0) = \tilde{V}\left(\frac{1}{2\beta}\right) = 0$ and the Q-ball splits into a pair of kink-like solutions interpolating between these vacua [34]

$$\phi_K(x, t) = \frac{e^{i\omega_{\min}t}}{2\sqrt{\beta}} \sqrt{1 \pm \tanh\left(\frac{x}{2\sqrt{\beta}}\right)}. \quad (2.13)$$

The energy of a kink is $E_K = \frac{1}{8}\beta^{-3/2}$.

Substitution of the ansatz (2.10) into expressions (2.8) and (2.9) for the energy and the charge of the configuration gives [9]

$$E = \frac{4\omega\omega' + Q(4\beta - 1 + 4\beta\omega^2)}{8\omega\beta}; \quad Q = \frac{4\omega}{\sqrt{\beta}} \operatorname{arctanh}\left(\frac{1 - \sqrt{1 - 4\beta\omega'^2}}{2\omega'\sqrt{\beta}}\right). \quad (2.14)$$

Note, it is possible to consider Q-balls for $\beta < \frac{1}{4}$, but in such cases they are excitations of a false vacuum and are only metastable solutions. Large perturbations can lead to a collapse to a true vacuum. In [9] the authors considered a relaxation process of a weakly perturbed Q-ball for $\beta = 0$. In this paper we will mostly assume that $\beta > \frac{1}{4}$, although some results for $\beta = 0$ will be discussed in the appendix.

3 Spectral structure

3.1 Linearization

Our aim now is to analyse the spectrum of linearized perturbations of these Q-balls. By analogy with the analysis of stability of the solutions in [16, 35], we consider a perturbative expansion of the scalar field

$$\phi(x, t) = f(x)e^{i\omega t} + A\xi(x, t) + \dots, \quad (3.1)$$

where the real parameter A is the amplitude of the small perturbation of the Q-ball, and substitute it into the field equation (2.4). In the linear approximation in A we obtain

$$\ddot{\xi} - \xi'' + [V'(f^2) + V''(f^2)f^2]\xi + V''(f^2)f^2\xi^* = 0. \quad (3.2)$$

For oscillating modes it is consistent to consider perturbations ξ of the form

$$\xi(x, t) = e^{i(\omega+\rho)t}\eta_1(x) + e^{i(\omega-\rho)t}\eta_2(x), \quad (3.3)$$

where the parameter ρ encodes the possible eigenfrequencies of the perturbation. The linearized equation (3.2) can then be written as a set of two coupled second order ordinary differential equations for the components η_1 and η_2^* :

$$L \begin{bmatrix} \eta_1 \\ \eta_2^* \end{bmatrix} = 0 \quad (3.4)$$

$$L = - \begin{bmatrix} (\omega + \rho)^2 & \\ & (\omega - \rho)^2 \end{bmatrix} + \begin{bmatrix} D & S \\ S & D \end{bmatrix} \quad (3.5)$$

where

$$D = -\frac{d^2}{dx^2} + U + S \quad (3.6)$$

and the potentials of perturbations $U(x)$ and $S(x)$ are (see figure 2)

$$U = V'(f^2) = 1 - 2f^2 + 3\beta f^4 \quad (3.7)$$

$$S = f^2 V''(f^2) = f^2(-2 + 6\beta f^2) \quad (3.8)$$

Note that $U(x) \rightarrow 1$ and $S(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In these limits the equations for η_1 and η_2 decouple.

Assuming $\omega > 0$, we consider the following cases:

- $\rho \in (0, 1 - \omega)$: there are no traveling waves, however, bound modes may exist, depending on the form of the potential.
- $\rho \in (1 - \omega, 1 + \omega)$: only the component η_1 is asymptotically propagating, with $e^{-ik_1 x}$ the right-moving mode and $e^{ik_1 x}$ the left-moving mode, where the wavenumber is $k_1 = \sqrt{(\omega + \rho)^2 - 1} > 0$. The second component η_2 remains exponentially localized on the Q-ball, $k_2^2 = (\omega - \rho)^2 - 1 < 0$. Throughout this paper we will refer to this state as the half-propagating mode. Such modes are crucial for understanding quasinormal modes in other models, including the 't Hooft-Polyakov monopole [14].

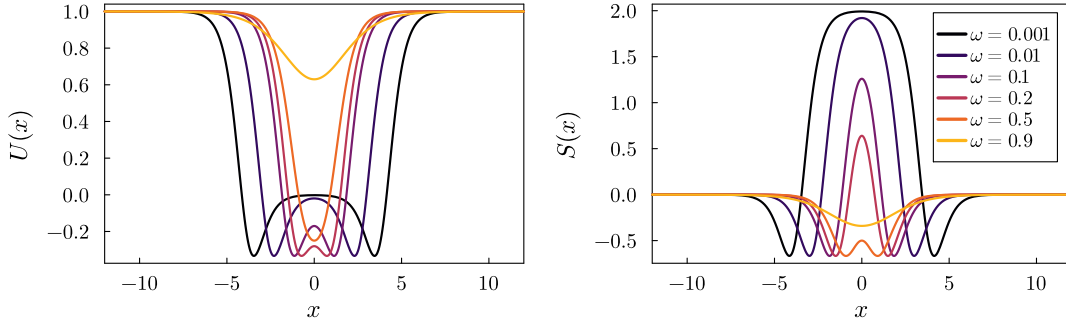


Figure 2. Potentials of linearized perturbations $U(x)$ and $S(x)$, defined in equations (3.7) and (3.8), for selected values of ω and $\beta = \frac{1}{4}$.

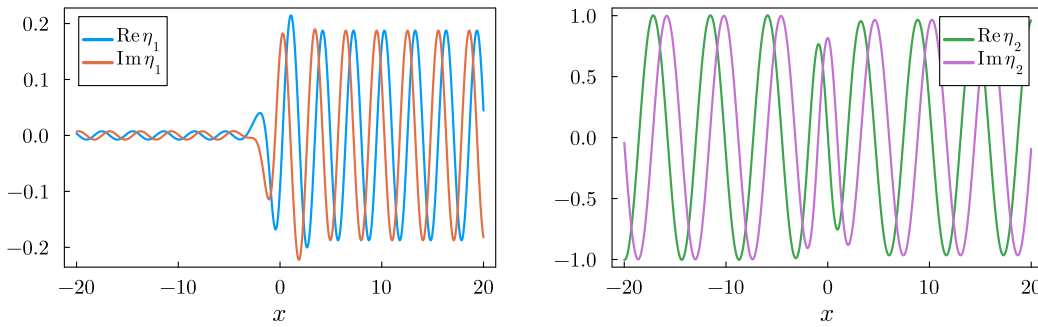


Figure 3. Example solution to the linearized problem for $\beta = \frac{1}{4}$, $\omega = 0.8$ and $\rho = 2$ with a wave travelling to the right in the second channel.

- $\rho > 1 + \omega$: both components are propagating, but the component η_2 oscillates with *negative* frequency, which means that for $k_2 > 0$ the term e^{ik_2x} describes a wave moving to the right, $k_2 = \sqrt{(\omega - \rho)^2 - 1} > 0$. An example of such a mode is presented in figures 3 and 4.

3.2 Zero modes

There are two zero modes that satisfy the equation (3.4) with $\rho = 0$. One of them is the translational mode, which shifts the position of the Q-ball,

$$\eta_0^{(tr)}(x) = \partial_x f(x); \quad (f(x) - x_0 \eta_0^{(tr)}(x)) e^{i\omega t} \approx f(x - x_0) e^{i\omega t}. \quad (3.9)$$

The other one is the phase shifting zero mode

$$\eta_0^{(ph)}(x) = f(x); \quad (f(x) + i\alpha \eta_0^{(ph)}(x)) e^{i\omega t} \approx f(x) e^{i\omega t + i\alpha}, \quad (3.10)$$

which corresponds to the global $U(1)$ symmetry of the model (2.1).

However, apart from these two modes, there are two more solutions of the linearized equation (3.2), of the form $\xi(x, t) = \eta(x, t) e^{i\omega t}$ with the η factor also time dependent: a stationary mode $\eta^{(Q)}(x, t)$, which transforms the frequency of the Q-ball ω as $\omega + \delta\omega$, and

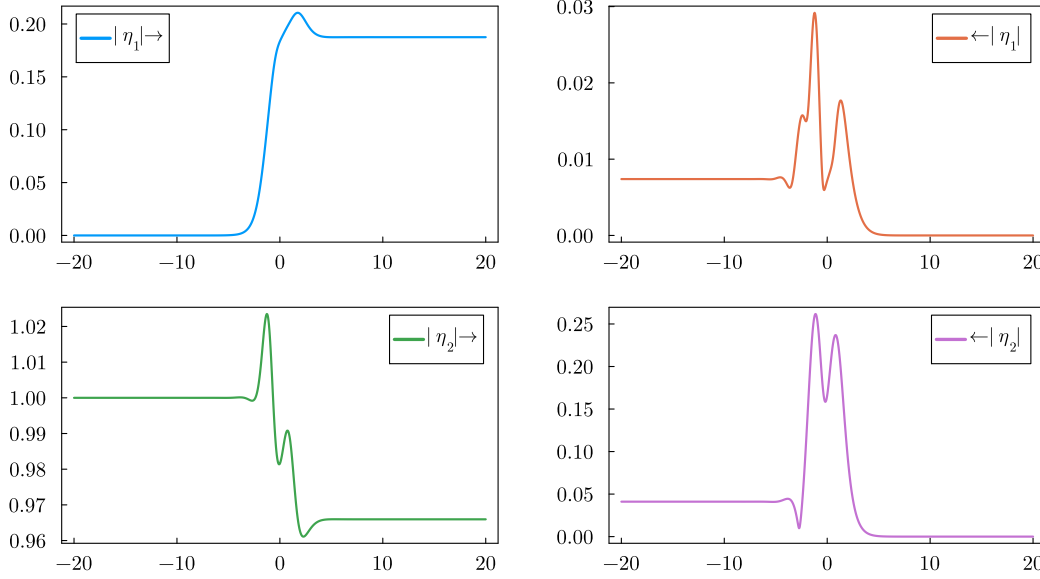


Figure 4. Decomposition of the wave into scattered components for $\beta = \frac{1}{4}$, $\omega = 0.8$ and $\rho = 2$ with a wave travelling right boundary conditions.

a mode corresponding to Lorentz symmetry. Recall that the profile function $f(x; \omega)$ (2.10) depends on the frequency, therefore excitation of this mode affects both the frequency and the profile of the Q-ball, thus

$$\eta^{(Q)}(x, t) = \partial_\omega f(x; \omega) + it f(x; \omega); \quad (f(x; \omega) + \delta\omega \eta^{(Q)}(x, t))e^{i\omega t} \approx f(x; \omega + \delta\omega)e^{i(\omega + \delta\omega)t}. \quad (3.11)$$

The Lorentz symmetry mode is

$$\eta^{(Lor)}(x, t) = t\partial_x f(x) + i\omega x f(x), \quad (3.12)$$

which follows from the expansion of a boosted Q-ball solution expanded up to $\mathcal{O}(v)$ terms:

$$f(\gamma(x - vt))e^{i\gamma\omega(t - vx)} \approx [f(x) - v(t\partial_x f(x) + i\omega x f(x))]e^{i\omega t}. \quad (3.13)$$

Note that translational, phase shifting and Lorentz zero modes preserve the charge of the Q-ball while the frequency transforming mode changes it.

3.3 Half-propagating and quasi-normal modes

For ρ within the range $(1 - \omega, 1 + \omega)$ only one component of the linearized solution propagates, $k_1^2 > 0$, whereas the other component cannot, $k_2^2 < 0$. In such a case, the second component η_2 has to decay as $x \rightarrow \pm\infty$, as $e^{\mp|k_2|x}$.

In our numerical calculations we fixed

$$\eta_1'(0) = \eta_2'(0) = 0 \quad (3.14a)$$

for a symmetric solution, normalised by setting

$$\eta_2(0) = 1, \quad (3.14b)$$

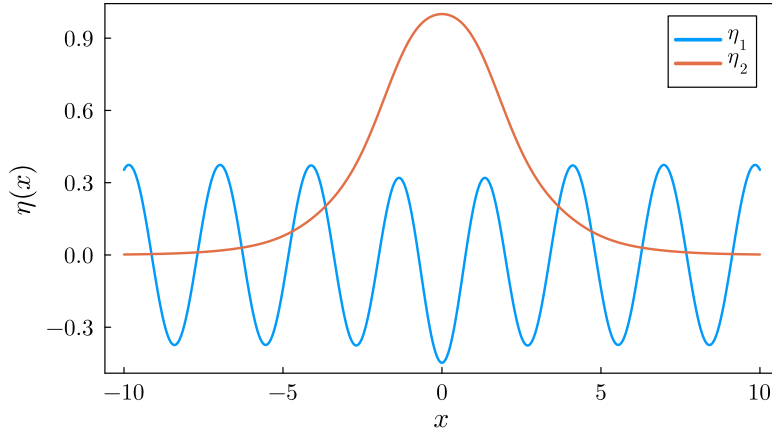


Figure 5. An example of symmetric half-propagating mode for $\beta = \frac{1}{4}$ and $\omega = \frac{\sqrt{3}}{2}$, $\rho = 1.54$.

and then adjusted the value of $\eta_1(0)$ so as to satisfy the remaining boundary condition

$$\eta_2'(x_f) + |k_2|\eta_2(x_f) = 0 \quad (3.14c)$$

at some large distance $x_f > 0$. As a result of these conditions, far from the Q-ball the profile of the first, propagating, component has the form

$$\eta_1(x) = A_{\text{rad}} \cos(k_1|x| + \delta). \quad (3.15)$$

An example of such a symmetric half-propagating solution is presented in figure 5.

The consequences of the excitation of the half-propagating mode are twofold. First, the non-propagating part $\eta_2(x)$ has an impact on the profile of the Q-ball. Second, the propagating component of this mode $\eta_1(x)$ radiates away the energy of the perturbed configuration. We can expect that the modes with the largest amplitude would decay faster, and conversely, the modes with the smallest amplitude would stay longer. In figure 6 we show the amplitude A_{rad} of the propagating component as a function of ρ for $\beta = \frac{1}{4}$ and $\omega = \frac{\sqrt{3}}{2}$. There are two important features. First, around $\rho = 1.57$ this amplitude grows almost to infinity, which indicates that our boundary conditions (3.14) cannot be simultaneously fulfilled, which basically means that $\eta_2(0)$ should vanish. Second, before this discontinuity, the amplitude of the propagating mode becomes very small, $A_{\text{rad}} = 0.0034979$ for $\rho = 1.538789$. Such a minimal radiation tail can, and in this case does, indicate the presence of a quasinormal mode [20]. Quasinormal modes satisfy purely outgoing boundary conditions, which break the Hermiticity of the operator L in (3.4). This allows the eigenfrequency to be complex. And indeed, we have found a quasinormal mode at $\rho = 1.538789 + 1.180 \cdot 10^{-5}i$. The half-life of this mode is $T_{1/2} = \frac{1}{\text{Im}\rho} \log(2) \approx 58700$ which explains the longevity of the oscillations. The profile of the half-propagating mode with minimal radiation tail is shown in figure 7.

3.4 Bound modes

For perturbation frequencies $\rho \in (0, 1 - \omega)$ neither of the components can propagate, and the profiles have only exponential tails. The corresponding normalizable modes are bound

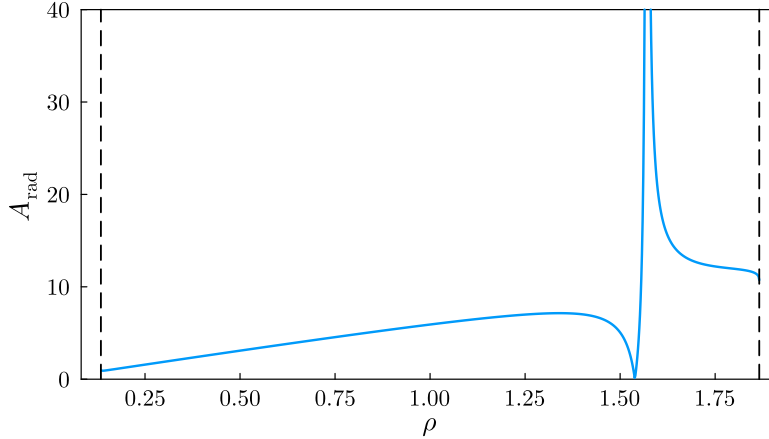


Figure 6. Radiation tails of the symmetric half-propagating modes as a function of ρ for $\beta = \frac{1}{4}$ and $\omega = \frac{\sqrt{3}}{2}$.

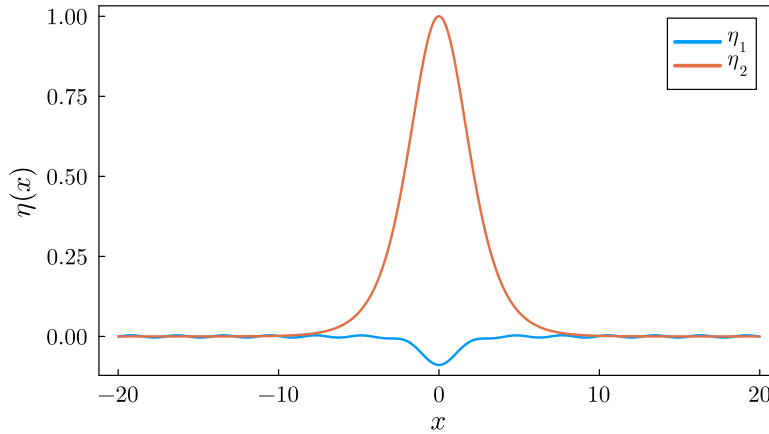


Figure 7. profile of the half-propagating mode with minimal radiation tail for $\beta = \frac{1}{4}$ and $\omega = \frac{\sqrt{3}}{2}$ and $\rho = 1.538789$.

to the Q-ball. We found various examples of these modes, including¹ in the case $\beta = 0$ analysed in [9]. As mentioned above, there always is a translational zero mode and one bound mode for $\rho = 0.1336$, which is very close to the frequency of one of the peaks in the spectrum. The component for $\omega - \rho$ is localized at $x = 0$, the other $\omega + \rho$ is very close to the threshold, but $\eta_2(0)$ is much smaller than $\eta_1(0)$, which explains why there is no peak near the threshold in the power spectrum of $\phi(0, t)$.

For $\beta = \frac{1}{4} + \epsilon$ the Q-ball frequency can be small and the Q-ball profile looks like two weakly bound kinks. In the spectrum there are two modes that can be interpreted as symmetric and antisymmetric combinations of translational modes of the kinks. The symmetric combination, which has frequency $\rho = 0$, is the translational mode of the Q-ball.

¹For most of our analysis we consider stable Q-balls with $\beta \geq \frac{1}{4}$, but we devote appendix A to the relaxation problem for $\beta = 0$.

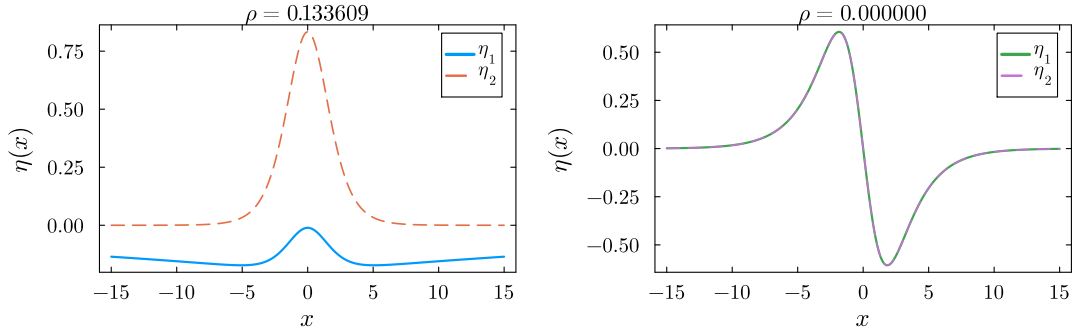


Figure 8. Bound modes for $\beta = \frac{1}{4}$ and $\omega = \frac{\sqrt{3}}{2}$.

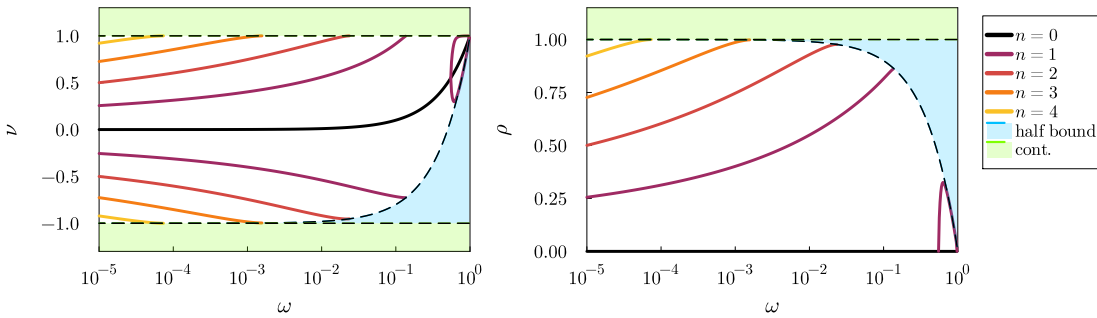


Figure 9. Eigenfrequencies of bound modes for $\beta = 1/4$ as a function of ω .

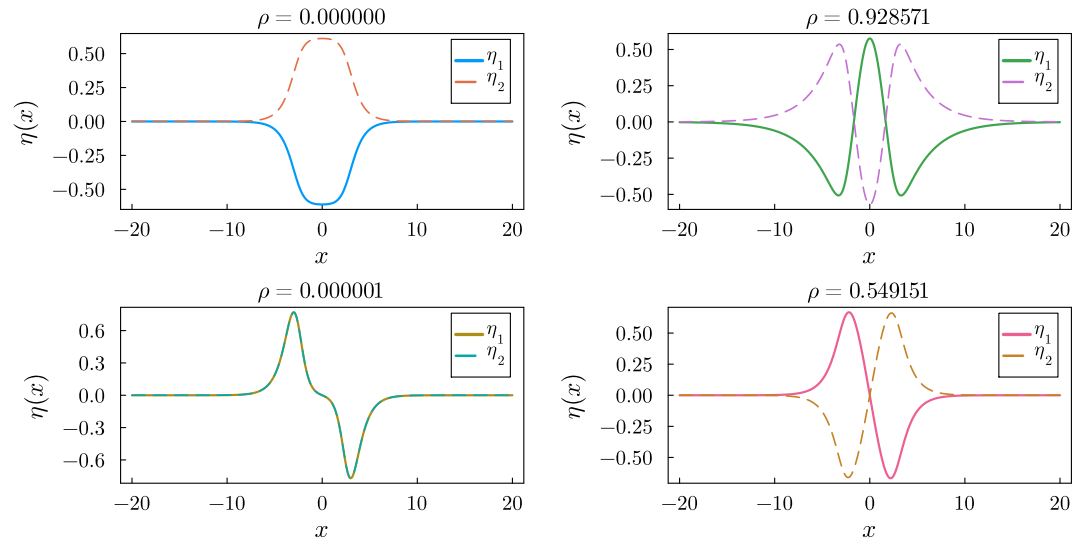


Figure 10. Bound modes of a small frequency Q-ball $\beta = 1/4$, $\omega = 0.01$.

The antisymmetric combination, with $\rho > 0$, represents the oscillation of the width of the Q-ball. Moreover, as seen on the right-hand side of figure 9, more modes appear as $\omega \rightarrow 0$, since the potential generated by the Q-ball has a wide well, which can support many modes. An example of such a case is shown in figure 10.

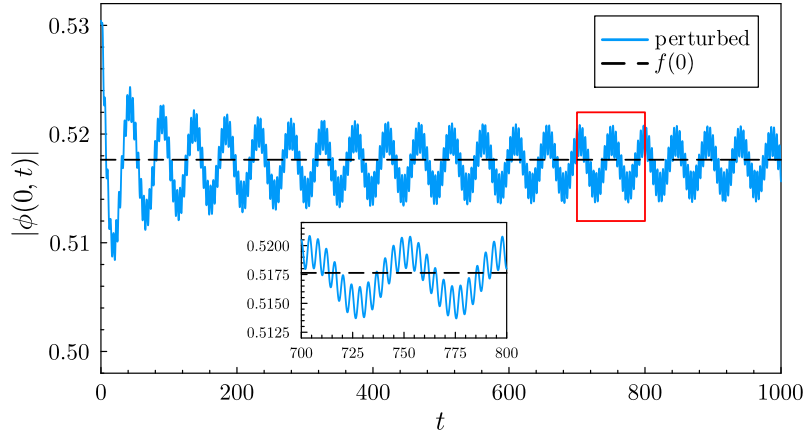


Figure 11. Evolution of the amplitude of the perturbed Q-ball $|\phi(0, t)|$ at the centre of the configuration for $\beta = \frac{1}{4}$, $\omega = \frac{\sqrt{3}}{2}$ and $\lambda = 1.05$.

3.5 Mode decay

Considering perturbations of the Q-ball we repeat the same procedure as in [9], i.e. we examine a one-parameter squashing/stretching perturbation of the form

$$\phi \rightarrow \phi_\lambda = \sqrt{\lambda} e^{i\omega t} f(\lambda x), \quad (3.16)$$

where λ is a positive parameter. Note that these deformations do not affect the charge of the Q-ball. Hereafter we consider the most interesting case of a twofold degenerate vacuum and take $\beta = \frac{1}{4}$, $\lambda = 1.05$ and $\omega = \frac{\sqrt{3}}{2}$. The decay of the perturbation is shown in figure 11. In the appendix we present and comment on a similar plot for the $\beta = 0$ case studied in [9]. Considering the longer time evolution of the initially squashed Q-ball, we see that the perturbation decays with time, although the decay rate slows down. However, there is a pattern, which is similar to the Manton-Merabet law [36] for the decay of the shape mode of the ϕ^4 kink: $A(t) \sim t^{-1/2}$.

In figure 12 we show the power spectra of the field ϕ at the centre and at $x = 10$, and, following [9], the power spectrum of the magnitude of the field $|\phi(0, t)|$. Both the spectra of the field values and of their absolute values carry valuable information. Below we examine explicitly what are the differences between these two approaches.

Let us suppose that the field can be represented as a sum of modes oscillating with frequencies ν_n :

$$\phi(0, t) \approx \sum_n c_n e^{i\nu_n t}. \quad (3.17)$$

This yields

$$|\phi(0, t)|^2 = \phi\phi^* \approx \sum_{n,m} c_n c_m^* e^{i(\nu_n - \nu_m)t}. \quad (3.18)$$

Thus, there are configurations with the same absolute value of the field $|\phi(0, t)|$ but different frequencies.

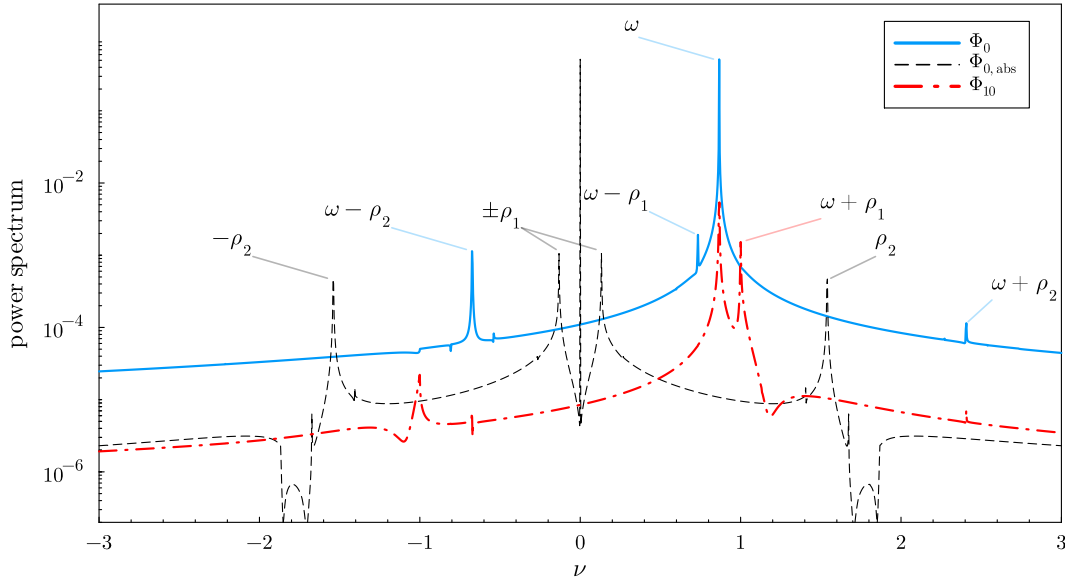


Figure 12. Power spectra of the fluctuations of the field ϕ (blue solid line) and of its absolute value $|\phi|$ (black dashed line) at the origin $x = 0$, and of the field ϕ at $x = 10$ (dash-dotted red line), all with $\beta = \frac{1}{4}$.

First, we consider the features of the power spectrum of perturbations at the center of the Q-ball, see the blue curve in figure 12. The most prominent peaks in the spectrum correspond to the frequencies (in ascending order) -0.673 , 0.734 , 0.866 and 2.405 .

- Clearly, the highest peak at $\nu = 0.866$ corresponds to the frequency of the stationary Q-ball, $\omega = \sqrt{3}/2$.
- The peak at $\nu = 0.734$ corresponds to the oscillational mode with $\nu = \omega - \rho_1$ with $\rho_1 = 0.1336$.
- At $\nu = \omega + \rho_1 = 0.9996 < 1$ there is a peak (barely visible at $x = 0$ but quite prominent at $x = 10$ — red dashed line) corresponding to the second frequency of the mode. Although $\nu = \omega + \rho_1$ is very close to 1 it is easy to verify numerically that it is still below the threshold, see figure 8.
- The peak at $\nu = -0.673$ corresponds to the lower real part of the quasinormal mode with the real part of the frequency $\rho_2 = 1.539$.
- The second frequency of the quasinormal mode, corresponding to the propagating component, is visible at $\nu = \omega + \rho_2 = 2.405$.

The mode $\omega - \rho_1$ is bounded while the mode $\omega - \rho_2$ is half-propagating. Indeed, considering the power spectrum of fluctuations at $x = 10$, see the red dash-dotted curve in figure 12, we can clearly identify the Q-ball frequency ω and threshold frequencies ± 1 . Also, small peaks at $\omega \pm \rho_2$ are visible there.

The power spectrum of fluctuations of absolute value of the field at the origin is displayed in figure 12, black curve. The most prominent peak is at $\omega = 0$ which corresponds to the net contribution of all frequencies, $\omega_n - \omega_n = 0$. Another peak, at ± 0.132 , corresponds to $\pm(\omega_3 - \omega_2)$ and the peaks at ± 1.541 correspond to $\pm(\omega_3 - \omega_1)$.

4 Radiation pressure

In this section, we investigate how Q-balls move when exposed to scalar radiation. We expect that for small amplitudes (much smaller than the amplitude of the Q-ball) the waves would scatter according to linearized equations. The scattered waves can have different momenta than the initial wave, and the excess of the momentum carried by waves can be related to the force acting on the Q-ball. In particular, we will show that the radiation can both push (positive radiation pressure, PRP) or pull (negative radiation pressure, NRP) the Q-ball depending on the composition of the incoming wave.

Following [31, 32], let us consider a field within a finite interval $x \in [-L, L]$, where $L \gg 1$, containing a Q-ball and a wave scattered off the Q-ball. The total momentum of the field in this interval is given by

$$P = \int_{-L}^L \mathcal{P} dx = - \int_{-L}^L T_{tx} dx = - \int_{-L}^L (\phi_t \phi_x^* + \phi_t^* \phi_x) dx, \quad (4.1)$$

where $\mathcal{P}$ is the momentum density, and T is the energy-momentum tensor. A moving Q-ball then has momentum $P = E\gamma v$. A wave scattered off the Q-ball carries momentum which can be obtained from the conservation law $\partial_\mu T^{\mu\nu} = 0$

$$\partial_t \mathcal{P} = -\partial_x T_{xx} = -\partial_x \left[\partial_t \phi \partial_t \phi^* + \partial_x \phi \partial_x \phi^* - V(|\phi|^2) \right]. \quad (4.2)$$

Integrating the left-hand side of the above expression and averaging over a period gives the rate of change of the momentum within the interval. The right hand is a total derivative, so integration gives only boundary terms. Assuming that the wave asymptotically (for $x \approx \pm L$) has the form $\phi = A(x)e^{i(\nu t - kx)}$, where $A(x)$ is a slowly changing function, we find that the rate of change of the momentum inside of the interval is equal to the flux of momentum through the boundaries of the interval:

$$\frac{dP}{dt} = \int_{-L}^L \partial_t \mathcal{P} dx = 2k^2 \left[A^2(-L) - A^2(L) \right]. \quad (4.3)$$

Assuming that the amplitudes of the waves are small compared to that of the Q-ball, we can identify $\frac{dP}{dt}$ as a force F acting on the Q-ball.

A single mode consists of two components with frequencies $\nu_1 = \omega + \rho$ and $\nu_2 = \omega - \rho$. Therefore, the asymptotic form of the field for which the incident wave propagates in channel j is

$$\phi(x \rightarrow -\infty) = A \sum_{l=1,2} (\delta_{lj} + r_j) e^{i(\nu_l t + k_l x)} \quad (4.4)$$

$$\phi(x \rightarrow \infty) = A \sum_{l=1,2} t_l e^{i(\nu_l t - k_l x)} \quad (4.5)$$

The rate of momentum transfer to the Q-ball is the sum of contributions coming from all scattered waves:

$$F_j = 2A^2 \sum_{l=1,2} k_l^2 (\delta_{lj} + R_l - T_l) \quad (4.6)$$

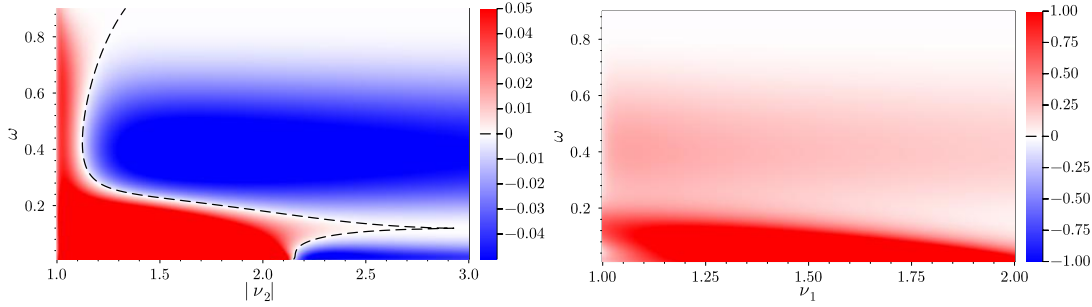


Figure 13. Force (divided by A^2) exerted on Q-balls for $\beta = \frac{1}{4}$ as a function of ν_2 and ω in the case of incident wave in the second sector.

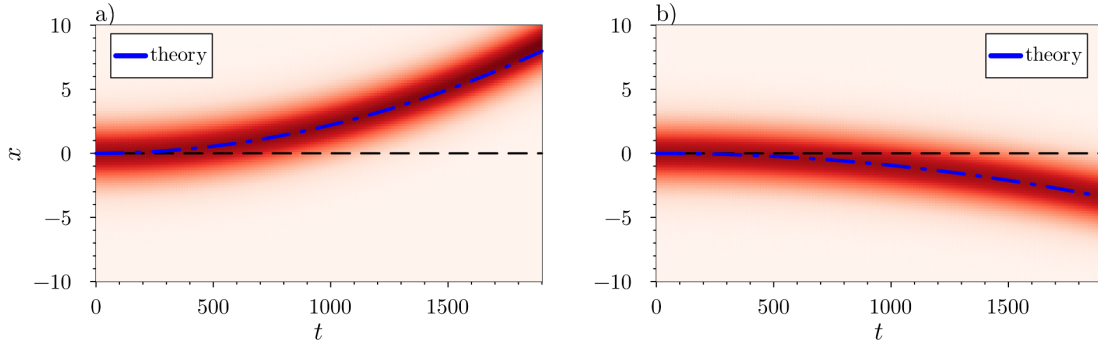


Figure 14. Motion of Q-balls under the influence of a wave in a) first channel and b) second channel. $\beta = 0.26$, $\omega = 0.6$, $\rho = 2.1$, $A = 0.01$. Colour intensity indicates the amplitude of the field.

where $R_l = |r_l|^2$ and $T_l = |t_l|^2$ are reflection and transmission coefficients in each channel and k_l are wave numbers. Negative radiation pressure can occur if the transmission coefficient from a smaller to larger wave number channel is large enough. The only such scenario is when the incident wave corresponds to a frequency $\nu_2 = \omega - \rho < -1$. The other frequency is equal to $\nu_1 = 2\omega - \nu_2$, so $|k_2| < |k_1|$. Figure 13 shows the force calculated from the solution of the linearized equation exactly in such a case. For values of $|\nu_2|$ close to the threshold positive radiation pressure is expected, while for larger $|\nu_2|$ there are regions where negative radiation pressure should be visible. The red colour represents PRP and the blue, NRP. In the other case, with an incident wave in the first component, the radiation pressure was always positive. An example of both positive and negative radiation pressure for $\beta = 0.26$ and $\omega = 0.4$ is shown in figure 14. The accelerations predicted from the effective force (4.6) agree well with the accelerations computed from numerical simulations of the full PDE, as presented in figure 15.

In the same way as we calculated the force from the energy and momentum conservation law, we can calculate the rate of change of the charge of a Q-ball by integrating (2.3)

$$\frac{dQ}{dt} = \int dx \partial_x j_x = 2A^2 \sum_{l=1,2} |k_l| (\delta_{lj} - R_l - T_l) \quad (4.7)$$

Numerically we found this value to be of order 10^{-15} , which means that to linear order the wave does not change the charge of the Q-ball. The result is not surprising because it follows from the unitarity of the S matrix.

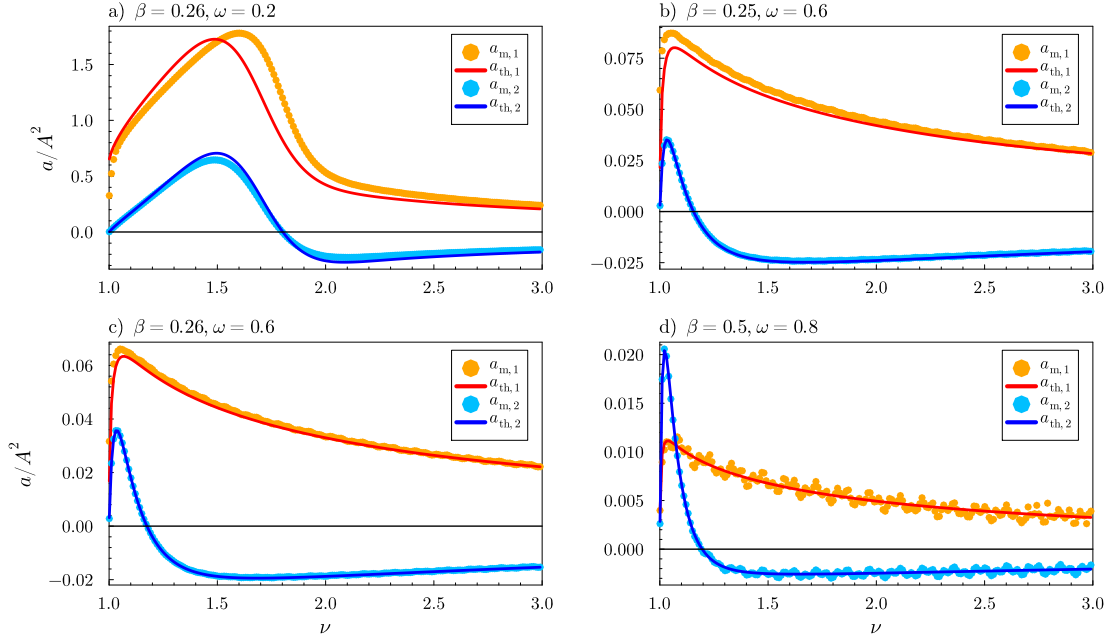


Figure 15. Comparison of measured acceleration (dots) with theoretical prediction (lines).

5 Conclusions

In this paper we have discussed how external perturbations affect Q-balls in (1+1) dimensions. Two examples are considered: (i) squashing perturbations of the Q-ball and (ii) interaction with incoming radiation. Considering the power spectrum of linearized perturbations about the Q-ball we find that there are both translational and phase shifting zero modes, as well as bounded, half-propagating and quasi-normal modes. We have shown that, depending on the nature of the incoming wave, the Q-ball can be pulled towards the source of the incident radiation, or it can be pushed away from it.

An interesting extension of this work would be to study perturbative excitations of Q-balls in (3+1) dimensions. Another interesting question, which we hope to be addressing in the near future, is to investigate the decays of Q-balls caused by external perturbations.

We note, finally, that stationary rotating Q-balls can induce superradiant amplification for scattered outgoing waves [37]. An important assumption in [37] is that the incoming wave does not excite the translational mode of the Q-ball. As a result its position does not change during the process of superradiant emission of radiation, related to the resonance energy transfer from the stationary Q-ball to the scattered wave. However, since, as we have seen, radiation pressure on the Q-ball pushes it to move, one may wonder whether this will affect the criteria for superradiance to occur, and it would be interesting to investigate this question further.

Acknowledgments

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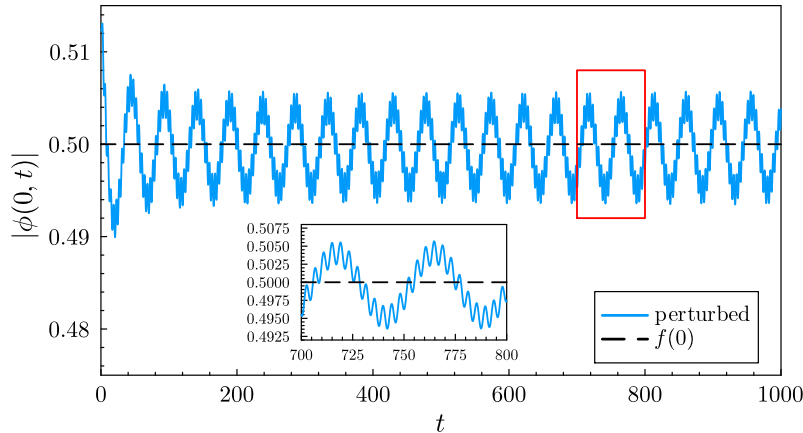


Figure 16. Evolution of the amplitude of the perturbed Q-ball $|\phi(0, t)|$ at the centre of the configuration for $\beta = 0$, $\omega = \frac{\sqrt{3}}{2}$ and $\lambda = 1.05$.

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A Metastable Q-balls

In the main part of this paper we studied the behavior of Q-balls for models with $\beta \geq \frac{1}{4}$, when potentials have one ($\phi = 0$ for $\beta > \frac{1}{4}$) or two vacua ($|\phi| = 0, \sqrt{2}$ for $\beta = \frac{1}{4}$). For $\beta < \frac{1}{4}$ the minimum at $\phi = 0$ is just a local minimum, and therefore it is a false vacuum. However, certain restricted configurations, if not excited too much, can exist around the false vacuum, at least in the classical theory. In [9] the decay of such a configuration for the model with $\beta = 0$ was discussed. We have repeated the same analysis as in section 3.5 but for $\beta = 0$. In this appendix we report on these results.

We present the decay in figure 16, while the power spectra are presented in figure 17. A direct comparison with figure 12 shows a very similar spectral structure with a single bound and one quasi-normal mode. The bound mode is for $\rho_1 = 0.1317$ and the QNM is for $\rho_2 = 1.5150692 + 9.96 \cdot 10^{-5}i$. Measured peaks: $-0.650, 0.731, 0.997, 2.383$ are consistent with the bound mode $\omega + \rho_1 = 0.9977$, $\omega - \rho_1 = 0.7343$ and the QNM $\omega + \text{Re } \rho_2 = 2.3811$ and $\omega - \text{Re } \rho_2 = -0.64904$. In [9] the authors claim that the persisted oscillations are due to the bound state of two Q-balls, some form of a breather. However, since the frequencies match so well with the bound and the quasinormal mode, we believe that their claim is overstated.

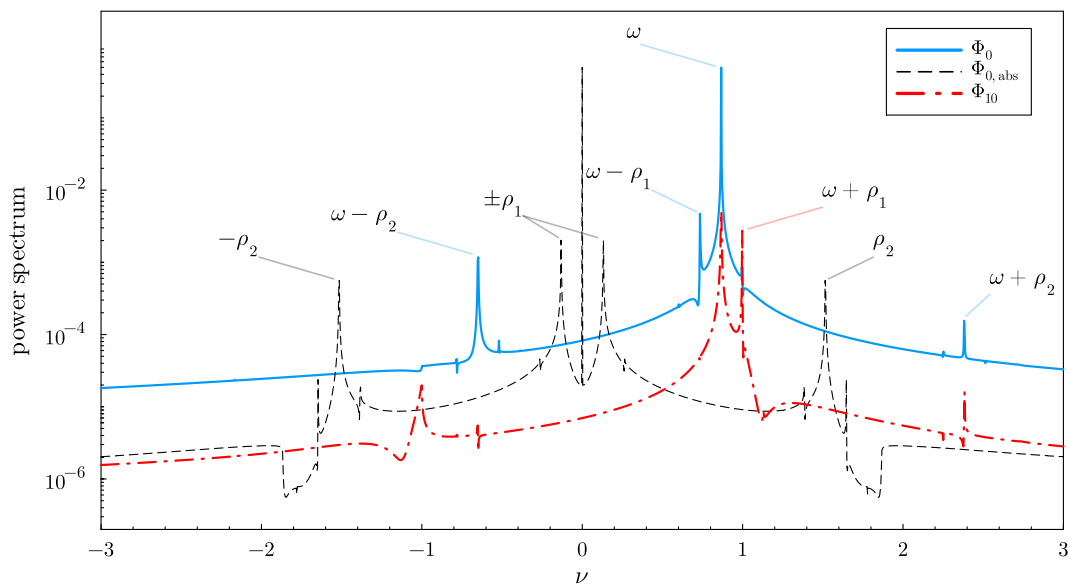


Figure 17. Power spectra of the fluctuations of the field ϕ (blue solid line) and of its absolute value $|\phi|$ (black dashed line) at the origin $x = 0$, and of the field ϕ at $x = 10$ (dash-dotted red line), all with $\beta = 0$.

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