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Geometric spectral functionals

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Abstract

The thesis consists of three papers introducing spectral functionals on manifolds, together with geometric interpretation and natural generalization to noncommutative geometry. This is one of the first attempts to define functionals generalizing metric, Einstein and torsion tensors to the noncommutative setting, which is carried out using Wodzicki residue. The explicit computations of several examples (compact manifold, noncommutative torus, almost-commutative $M \times \mathbb{Z}_2$ and more) illustrate the method.

Streszczenie

Niniejsza rozprawa składa się z trzech artykułów, w których wprowadzamy funkcjonały spektralne posiadające geometryczną interpretację na rozmaitościach i posiadające naturalne uogólnienie w geometrii nieprzemiennej. Prezentujemy tam jedną z pierwszych prób uogólnienia tensorów metrycznego, Einsteina i torsji na przypadki nieprzemienne, przy użyciu residuum Wodzickiego. Dla zobrazowania tej metody przeprowadzamy obliczenia dla kilku konkretnych przykładów, m.in. zwartej rozmaitości, torusa nieprzemiennego czy prawie przemiennej iloczynu $M \times \mathbb{Z}_2$.

List of articles

- [P1] L. Dąbrowski, A. Sitarz, P. Zalecki, *Spectral metric and Einstein functionals*, Adv. Math. Vol. 427, (2023) 1091286
- [P2] L. Dąbrowski, A. Sitarz, P. Zalecki, *Spectral metric and Einstein functionals for Hodge-Dirac operator*, arXiv:2307.14877, J. Noncommut. Geom. in print
- [P3] L. Dąbrowski, A. Sitarz, P. Zalecki, *Spectral Torsion*, arXiv:2308.01644, Comm. Math. Phys. in print

Contents

1	Introduction	5
1.1	Motivation of noncommutative geometry	5
1.1.1	Spectral triples	6
1.2	Towards physics	7
1.2.1	Spectral Action	7
1.2.2	Almost commutative geometries	8
1.3	Pseudodifferential operators and Wodzicki residue	9
2	Spectral functionals	11
2.1	Spectral metric and Einstein functionals	12
2.1.1	Remarks	50
2.2	Spectral metric and Einstein functionals for Hodge-Dirac operator	51
2.3	Spectral torsion	64
2.4	Outlook and generalization	77
2.4.1	Functionals for Laplace-type operators	77
2.4.2	Spectral functionals for a class of Dirac-type operators on manifolds	79
3	Conclusions	83

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Chapter 1

Introduction

This thesis includes three research articles that implement geometrical concepts linked to physics using the tools of noncommutative geometry (NCG). Alain Connes established NCG, a nascent branch of mathematics, in the 1980s [7] and since then both mathematicians and physicists have contributed significantly to the development of this promising topic. However, there are still uncharted areas to uncover and the presented work aims to propose a range of new ideas and results.

The first indication that noncommutativity is important in physics is the relationship between the quantum position and momentum operators. The phase space of quantum mechanics is an example of noncommutative Moyal space [23, 22], which has been studied as a toy model and an effective description of the tangent space of string theory in the presence of B field. NCG offers tools for developing and studying complex mathematical and physical models that go beyond traditional differential geometry. Various examples, including discrete set of points [21, 2], noncommutative manifolds (e.g., noncommutative torus [24], quantum spaces, quantum groups, quantum spheres [10] and κ -deformed Minkowski [20]), have been studied in terms of differential calculi, Riemannian geometry, spectral geometry, and physical models. From the point of view of physics, substituting classical space by noncommutative one is considered a step towards quantum gravity.

Another class of noncommutative spaces that physicists are particularly interested in is almost-commutative spaces, which are products of manifolds and finite algebras. This type of spaces naturally incorporates gauge fields into the model and gives them a geometrical interpretation. Almost-commutative geometry has been utilized to provide a noncommutative description of the Standard Model (see for example [19] and references therein).

1.1 Motivation of noncommutative geometry

The Gelfand-Naimark duality between (non)unital commutative C^* -algebras and (locally) compact Hausdorff spaces serves as the base for NCG. If we consider complex functions defined on a space, we can easily check, that together with pointwise addition and multiplication they form an algebra. We can evaluate functions at a point and such evaluation map preserves addition and multiplication (and conjugation) of functions and thus is a character of algebra of functions. It turns out that under certain assumptions, every character is of this type, implying some equivalence between the space and the set of characters. Gelfand and Naimark proved in [12] that every commutative C^* -algebra is isomorphic to the algebra of continuous functions on a topological space.

Table 1.1: Dictionary between corresponding objects in topology and algebra.

Topology	Algebra
locally compact Hausdorff topological space	nonunital C^* -algebra
compact Hausdorff topological space	unital C^* -algebra
homeomorphism	automorphism
continuous proper map	morphism
open (dense) subset	(essential) ideal
compactification	unitization
Stone–Čech compactification	multiplier algebra
Cartesian product	tensor product

The connection between spaces and algebras allows us to move the focus of geometry from spaces to algebras. In this scenario, we need to know which objects correspond to each other. Table 1.1 [26] provides an example of such a dictionary between algebra and topological space (for few more entries see also [14]).

When dealing with algebras, we can omit the commutativity condition and treat noncommutative algebras as counterparts of generalized, noncommutative "spaces". They differ from classical spaces in that they can contain no points in the usual sense, because points corresponds to characters and it is impossible to define functional preserving nonabelian multiplication (because complex numbers commute) unless commutators of all elements of the algebra are in kernels of these characters, which is a very restrictive condition. Nevertheless, if we manage to develop geometric methods on the side of function algebras, we can attempt to generalize them to noncommutative algebras and study the geometry of such generalized spaces.

Fortunately, we do not need to deal with abstract algebras. According to the noncommutative Gelfand-Naimark theorem, each C^* -algebra is isometrically $*$ -isomorphic to a C^* -subalgebra of bounded operators on a Hilbert space. Therefore, guided by the classical example, we may develop the geometric characteristics of noncommutative spaces using the Hilbert space.

1.1.1 Spectral triples

Of course, the C^* -algebra alone describes only a topology of the space whereas geometric structures are much richer. A possible technique for studying them is to introduce the concept of spectral triples. The prototype of spectral triples is a collection of objects defined on a spin manifold M consisting of the Hilbert space of L^2 -spinors, the Dirac operator, D , acting on the spinors and the algebra of smooth complex functions on the manifold, which is a dense subalgebra of C^* -algebra of continuous functions. This algebra is naturally represented on the Hilbert space by pointwise multiplication. In this scenario, we can easily reconstruct the manifold and its metric structure from the so defined triple. For example, the distance between two points A, B can be calculated using the formula [5]

$$d(A, B) = \sup_{\| [D, f] \| \leq 1} |f(A) - f(B)|, \quad (1.1)$$

where f is taken from the algebra of smooth functions on M .

In general, a spectral triple consists of a Hilbert space $\mathcal{H}$, an algebra $\mathcal{A}$, which is a dense subalgebra of a C^* -algebra, represented faithfully by π on $\mathcal{H}$ as a subalgebra of $B(\mathcal{H})$ and an unbounded, densely defined, selfadjoint operator D (called Dirac operator) on $\mathcal{H}$, such that commutators of D

Table 1.2: Dependence between the KO-dimension of the spectral triple and the signs of the relations involving J [6].

KO-dim mod 8	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

with representation of elements of $\mathcal{A}$ are bounded. Additional conditions, as given below, could be introduced, adding more structures.

The spectral triple is n -summable if for every $k > n$, $|D|^{-k}$ is of the trace-class.

We call it even if there exists a self-adjoint grading operator γ , such that

$$\gamma^2 = 1, \quad D\gamma = \gamma D, \quad \forall a \in \mathcal{A} : \gamma a = a\gamma. \quad (1.2)$$

We say that the spectral triple is real if there exists an antilinear operator J , such that

$$J^2 = \epsilon 1, \quad JD = \epsilon' DJ. \quad (1.3)$$

If the triple is also even, we demand that

$$J\gamma = \epsilon'' \gamma J, \quad (1.4)$$

where the constants $\epsilon, \epsilon', \epsilon'' = \pm 1$ and they are determined by the so-called KO-dimension of the spectral triple (table 1.2). Moreover, J determines a right representation of the algebra $\pi^\circ(a) = Ja^*J^{-1}$, which provides us with two additional conditions. First, the two representations must commute, that is, for all $a, b \in \mathcal{A}$,

$$[\pi(a), \pi^\circ(b)] = 0. \quad (1.5)$$

The next one, known as the first order condition, is

$$[[D, \pi(a)], \pi^\circ(b)] = 0. \quad (1.6)$$

This mimics in noncommutative case the classical situation that the Dirac operator is a differential operator of order one.

In the case of classical spinors, the grading operator is the chirality, usually called γ^5 in 4 dimensions, and J is the charge conjugation. Similarly to Gelfand-Naimark theorem, Connes proved, that a commutative spectral triple under certain assumptions, known as the 7 axioms of Connes' reconstruction theorem [6], is the canonical spectral triple of some spin manifold. In the noncommutative setting, the spectral triple defines a noncommutative "manifold" that does not exist as an independent object.

The commutator $[D, \pi(a)]$ defines the exterior derivative of a . We put $\pi(da) = [D, \pi(a)]$ and an $\mathcal{A} - \mathcal{A}$ -bimodule $\Omega^1(\mathcal{A})$ generated by $\pi(a)[D, \pi(b)]$ is called the bimodule of differential 1-forms.

1.2 Towards physics

1.2.1 Spectral Action

We saw that spectral triples correspond to spin manifolds, and the Dirac operator may be used to calculate distances. We can also use it to compute the action functional for physical models based

on spectral triples. The action is typically a sum of two terms [4]: the spectral action $\text{tr}f\left(\frac{|D|}{\Lambda}\right)$, where f is suitable positive function and Λ is a cutoff scale, and the fermionic action $\langle\psi|D|\psi\rangle$, where $\psi \in \mathcal{H}$ (see also [27] and [25] for spectral formulation of fermionic action).

To compute the spectral action we can utilize an expansion in Λ using Seeley-Gilkey-DeWitt coefficients a_j (the entire construction is available in [4]):

$$\text{tr}f\left(\frac{|D|}{\Lambda}\right) \approx \sum_j f_j a_j \left(\frac{|D|}{\Lambda}\right), \quad (1.7)$$

where constants f_j are determined by the used function f . Usually, we need only the first terms of the sum, which can be also computed using mathematical tools such as Wodzicki residue or heat kernel expansion [29].

Using the spectral action applied to a spin manifold, we obtain the Einstein-Hilbert action with cosmological constant for General Relativity using a manifold's canonical Dirac operator ([6, 16, 17]). Moreover, the full action of the Standard Model of elementary particles may be reconstructed using the spectral and fermionic action, after the gauge fields are included as fluctuations of the Dirac operator in a mild extension of spin geometry called almost-commutative geometry [8].

1.2.2 Almost commutative geometries

The term "almost commutative geometry" refers to a product of a spin manifold and its canonical spectral triple $(C^\infty(M), L^2(S), D_M, \gamma_M, J_M)$ with a finite spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F, \gamma_F, J_F)$ in which both the algebra and the Hilbert space are finite dimensional. The resultant Dirac operator is taken as

$$D = D_M \otimes 1_F + \gamma_M \otimes D_F \quad (1.8)$$

and the other objects are given by usual tensor products. With this choice of D we have

$$D^2 = D_M^2 \otimes 1_F + 1_M \otimes D_F^2. \quad (1.9)$$

In this picture, the Hilbert space contains all of the model's fermionic fields, while the gauge fields are encoded in the family of Dirac operators. We call them fluctuated Dirac operators, and they arise from the initial Dirac operator in the following manner

$$D_\omega = D + \omega + \epsilon' J \omega J^{-1}, \quad (1.10)$$

where ϵ' and is the same as in (1.3), and ω is a differential form, i.e. a sum of terms of the form $a[D, b]$ with $a, b \in \mathcal{A}$.

The most prominent example of this type of geometries is the Spectral Standard Model, with the finite part of the algebra being $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and the finite part of the Hilbert space consisting of all types of fermions (even particles with different chirality or color are treated as different particles). The finite matrix D_F encodes the masses of particles and the mixing angles.

Details of this model have evolved as there are still several choices to be made. Originally, the model was far from perfection, but many issues were addressed and resolved over time. Among others, the challenges include fermion doubling or quadrupling. This has origins in the fact that in the finite part of the Hilbert space we distinguish particles with different chirality (and also antiparticles), whereas Dirac spinors on the manifold side address the same degrees of freedom.

This problem can be solved by using a suitable projection or modifying the spectral triple. Another option is to limit the Hilbert space to physical degrees of freedom, but the consequence is that the spectral triple will not be of the product type. This solution appears to be closer to the classical formulation of the Standard Model, as studied in [1, 3].

Furthermore, there was a problem with the mass of the Higgs particle. Under certain assumptions it was calculated to be around 170 GeV, which is more than a third more than the measured mass. Although it is impressive that the model could predict the Higgs mass, the result was incorrect, which forced further modifications. Finally, we notice that the Spectral Standard Model contains some issues, such as the problem with its formulation in the Lorentzian signature, as up to now the spectral approach works in Euclidean signature. There is some progress in this subject [9], although the full Lorentzian formulation is yet unknown.

1.3 Pseudodifferential operators and Wodzicki residue

Pseudodifferential operators are a generalization of differential operators. They share similar idea to interchange of differential and algebraic equations by the Fourier transform. Let f be a C^∞ function on the real line (but it works similarly for manifolds) and $\hat{f}$ its Fourier transform (assuming that it exists). We have a well-known formula

$$\frac{d}{dx}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\xi \hat{f}(\xi) e^{ix\xi}. \quad (1.11)$$

Analogously, for a differential operator P we can write

$$(Pf)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sigma(P)(x, \xi) \hat{f}(\xi) e^{ix\xi}, \quad (1.12)$$

where $\sigma(P)$ is a function characteristic for the operator P , which may be created from P by altering ∂_a to $i\xi_a$, resulting in a polynomial in ξ . Allowing $\sigma(P)(x, \xi)$ to be a smooth function in both arguments instead of a polynomial in ξ results in pseudodifferential operators, provided that $\sigma(P)$ belongs to a certain symbol class. In our case it is sufficient to require

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(P)(x, \xi)| \leq C_{\alpha, \beta} (1 + \|\xi\|)^{k - |\beta|},$$

where α, β are multiindices, $C_{\alpha, \beta}$ are constants, and an integer k is called the order of P . For more possibilities and conditions see [13, 28, 15].

To work with pseudodifferential operators on the level of their symbols, we have to know how to compose them. The composition rule must be consistent with the composition of differential operators and can be derived from it. It is given by the following formula:

$$\sigma(PQ)(x, \xi) = \sum_{\alpha} \frac{(-i)^{|\alpha|}}{|\alpha|!} \partial_\alpha^\xi \sigma(P)(x, \xi) \partial_\alpha^x \sigma(Q)(x, \xi), \quad (1.13)$$

where α is a multiindex.

Usually it is convenient to expand a symbol into a series

$$\sigma(P)(x, \xi) = \sum_{i=0}^{\infty} \sigma_{k-i}(P)(x, \xi), \quad (1.14)$$

where k is the order of P and every $\sigma_j(P)(x, \xi)$ is homogeneous in ξ with degree j . The first term, $\sigma_k(P)(x, \xi)$, is called the principal symbol of P . Using this type of expansion, the composition rule for order- k operator P and order- l operator Q for the first three symbols is

$$\begin{aligned}\sigma_{k+l}(PQ) &= \sigma_k(P)\sigma_l(Q), \\ \sigma_{k+l-1}(PQ) &= \sigma_k(P)\sigma_{l-1}(Q) + \sigma_{k-1}(P)\sigma_l(Q) - i\partial_a^\xi \sigma_k(P)\partial_a^x \sigma_l(Q), \\ \sigma_{k+l-2}(PQ) &= \sigma_k(P)\sigma_{l-2}(Q) + \sigma_{k-1}(P)\sigma_{l-1}(Q) + \sigma_{k-2}(P)\sigma_l(Q) \\ &\quad - i\partial_a^\xi \sigma_k(P)\partial_a^x \sigma_{l-1}(Q) - i\partial_a^\xi \sigma_{k-1}(P)\partial_a^x \sigma_l(Q) - \frac{1}{2}\partial_{ab}^\xi \sigma_k(P)\partial_{ab}^x \sigma_l(Q).\end{aligned}\tag{1.15}$$

The Wodzicki residue is a unique trace of the algebra of pseudodifferential operators acting on sections of a complex vector bundle over a manifold, although it can be generalised to a suitable class of spectral triples. In the case of compact, oriented manifold M of dimension n , the formula for Wodzicki residue is up to a constant given by

$$Wres(P) = \int_M \int_{|\xi|=1} \text{tr } \sigma_{-n}(P)(x, \xi). \tag{1.16}$$

This work was inspired by the theorem which states that the Wodzicki residue can be used to produce the Einstein-Hilbert action on a Riemannian manifold M :

$$Wres(\Delta^{-\frac{n}{2}+1}) \propto \int_M R \sqrt{g} dx^n, \tag{1.17}$$

where Δ is the laplacian and R is the curvature scalar. Similarly, the usual integration of a function f can be formulated in this way as

$$Wres(f\Delta^{-\frac{n}{2}}) \propto \int_M f \sqrt{g} dx^n. \tag{1.18}$$

The Wodzicki residue allows us to find a noncommutative formulation of even more geometric quantities, not limited to scalars as R . Curvature tensors carry more information than just scalar curvature, but their definition for spectral triples has yet to be established. This thesis presents a consistent technique for introducing and generalizing tensor geometric objects using Wodzicki residue methods ([P1, P2, P3]).

Chapter 2

Spectral functionals

The presented series of articles focuses on geometric tensors such as the metric, torsion, and Einstein tensor. When working with manifolds, the metric is a fundamental quantity that allows us to construct curvature tensors (Einstein, Riemann, Ricci) as well as some canonical differential operators, such as Laplacian or Dirac operator.

In noncommutative geometry, however, the spectral triple is a fundamental object from which we must learn how to extract the geometric tensors. It is well-known that the scalar of curvature can be obtained in the spectral Einstein-Hilbert action [16], whereas, the other geometric tensors are yet to be defined in noncommutative setting (see [11] for an attempt to define the Ricci tensor). In the presented papers we used Wodzicki residue to define spectral functionals which can be used to obtain metric, torsion and Einstein tensors in the classical case. We can work with these functionals using only the Dirac operator, differential 1-forms and Wodzicki residue. This allows us to make a straightforward generalization to the noncommutative setting, as the Dirac operator and differential 1-forms are present for every spectral triple, and a generalization of Wodzicki residue exists for a large class of them. These functionals only recreate geometric tensors on manifolds, but in the noncommutative case, these quantities can be defined using them.

Comments on notation

In the discussed papers, we attempted to adopt a single consistent notation, but because the work was divided among three articles (plus one in preparation) over two years, the notation naturally evolved. For example, in [P1], we defined several metric-type and Einstein-type functionals for different operators and use subscripts and superscripts to denote which operator is utilized, ex. $\mathcal{G}^\Delta$, g_D . In [P3, P2], we only considered functionals defined on differential forms using the Dirac operator, therefore we omitted the subscript.

We frequently deal with inverse powers of Laplace-type operators L , or, more specifically, their symbols homogenous in ξ of the (up to) three highest degrees. We denote symbols with fraktur font, using $\mathfrak{a}_j$ for L , $\mathfrak{b}_j$ for L^{-1} and $\mathfrak{c}_j$ for L^{-k} . If $\sigma_k(P)$ is a symbol of homogeneous degree k in ξ of an operator P , then $\sigma_j(L) = \mathfrak{a}_j$. For the inverses of L we use a convention with "-" sign omitted. We write $\sigma_{-j}(L^{-1}) = \mathfrak{b}_j$ and $\sigma_{-2k-1}(L^{-k}) = \mathfrak{c}_{2k+1}$. Equations (A.8) and (A.9) of [P1] utilize symbols of different powers of order- k operator P , with the same letter. To distinguish symbols of the same order but different powers of an operator, as in this example, we consequently

use different subscripts, i.e. $\sigma_{lk}(P^l) = \mathfrak{r}_{lk}$ vs. $\sigma_{lk}(P^{l-1}) = \mathfrak{r}_{k(l-1)+2}$. We proceed analogously every time there is a danger of confusion.

In the commutative case we utilize normal coordinates and expand terms up to the appropriate order in x . Therefore, upper and lower indices are not always used as contravariant and covariant ones.

2.1 Spectral metric and Einstein functionals

In the first article [P1], we propose a choice of functionals based on vector fields, differential forms, and various Laplace-type or Dirac operators for manifolds. The densities of these functionals are proportional to the metric or Einstein tensors evaluated on these vector fields or differential forms (sometimes with some additional terms). The most important functionals defined in theorem 4.1 of [P1] are:

$$\begin{aligned} \mathfrak{g}_D(u, w) &:= Wres(\hat{u}\hat{w}D^{-n}), \\ \mathfrak{G}_D(u, w) &:= Wres[\hat{u}(\hat{w}D + D\hat{w})D^{-n+1}], \end{aligned} \tag{2.1}$$

where $\hat{u}, \hat{w}$ are Clifford multiplication by differential forms u, w . In the classical case

$$\begin{aligned} \mathfrak{g}_D(u, w) &\propto \int g(u, w) vol_g, \\ \mathfrak{G}_D(u, w) &\propto \int G(u, w) vol_g, \end{aligned} \tag{2.2}$$

and in the noncommutative case we can use them to define the metric and Einstein tensors as they use only the Dirac operator and differential forms.

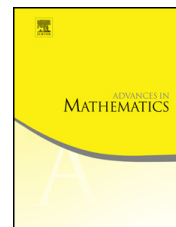
In the second part of [P1], we attempt to compute these functionals in some noncommutative examples while also investigating some general features of the metric and Einstein functionals for differential forms and Dirac operators.

The main example is the conformally rescaled noncommutative torus. In this case, we can also use laplacian and vector fields, which is not possible in general in contrary to the functionals with differential forms and Dirac operators. The second example is a product of a spectral triple over a spin manifold with two-point space.



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journal homepage: www.elsevier.com/locate/aimSpectral metric and Einstein functionals [☆]Ludwik Dąbrowski ^a, Andrzej Sitarz ^{b,*}, Paweł Zalecki ^b^a SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy^b Institute of Theoretical Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland

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ABSTRACT

We define bilinear functionals of vector fields and differential forms, the densities of which yield the metric and Einstein tensors on even-dimensional Riemannian manifolds. We generalise these concepts in non-commutative geometry and, in particular, we prove that for the conformally rescaled geometry of the noncommutative two-torus the Einstein functional vanishes.

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1. Introduction

Riemannian geometry flourishing for more than hundred and fifty years is continually at the core of modern mathematics with a wealth of applications and open problems. Its role as a key to understanding general relativity makes it ubiquitous in modern physics. The study of interconnection between the geometric objects on the manifolds and differential operators led to the development of spectral geometry and, with the seminal

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question “*Can one hear the shape of a drum?*”, was popularised by Mark Kac [36]. A further generalization was brought by noncommutative geometry [8], where spectral methods were proposed to study invariants of algebras, functionals on noncommutative algebras and geometric objects that extends the notion of curvature, leading, for example, to the version of Gauss-Bonnet theorem for noncommutative tori.

The aim of this paper is to propose a few new functionals that express other basic geometric objects, in particular the Einstein tensor, using the methods of the Wodzicki residue and its generalisation in noncommutative geometry.

1.1. An overview of the field

An eminent spectral scheme that generates geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives *prima facie* is the small-time asymptotic expansion of the (localised) trace of heat kernel [27,28]. This has many diverse applications in mathematics and in physics, mostly in the context of general relativity and its generalisations.

Using the Mellin transform, the coefficients of this expansion can be transmuted into certain values or residues of the (localised) zeta function of the Laplacian. In turn, they can be expressed using the Wodzicki residue $\mathcal{W}$ (also known as noncommutative residue), which is a unique (up to multiplication by a constant) tracial state on the algebra of pseudo-differential operators (Ψ DO) on a complex vector bundle E over a compact manifold M of dimension $n \geq 2$ [31,54]. For the oriented manifold M it is given up to multiplicative constant by a simple integral formula,

$$\mathcal{W}(P) := \int_M \left(\int_{|\xi|=1} \text{tr} \sigma_{-n}(P)(x, \xi) \mathcal{V}_\xi \right) d^n x, \quad (1.1)$$

where tr is the trace over endomorphisms of the bundle E at any given point of M , $\sigma_{-n}(P)$ is the symbol of order $-n$ of a pseudodifferential operator P and $\mathcal{V}_\xi$ denotes the volume form on the unit sphere. The name ‘residue’ comes from the fact that it is indeed a residue (at $z=1$) of the ζ -function associated to P [30]. In this paper, for simplicity, we focus only on closed oriented M of even dimension $n = 2m$. In this case, for a Riemannian manifold M equipped with a metric tensor g and the (scalar) Laplacian Δ one has,

$$\mathcal{W}(\Delta^{-m}) = v_{n-1} \text{vol}(M), \quad (1.2)$$

and a *localized* form, a functional of $f \in C^\infty(M)$,

$$\mathfrak{v}(f) := \mathcal{W}(f\Delta^{-m}) = v_{n-1} \int_M f \text{vol}_g, \quad (1.3)$$

where

$$v_{n-1} := \text{vol}(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)},$$

is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

A startling result regarding a higher power of the Laplacian was divulged by Connes in the early 1990s [9] and explicitly confirmed independently in [38] and in [37]. Namely, for $n > 2$,

$$\mathcal{W}(\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M R(g) \text{vol}_g, \quad (1.4)$$

is up to a constant a Riemannian analogue of the Einstein-Hilbert action functional of general relativity in vacuum. Here $R = R(g)$ is the scalar curvature, that is the g -trace $R = g^{jk} \text{Ric}_{jk}$ of the Ricci tensor with components Ric_{jk} in local coordinates, where g^{jk} are the raised components of the metric g .

A localised form of (1.4) for $n > 2$ is a functional on $C^\infty(M)$ (cf. [12])

$$\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M f R(g) \text{vol}_g. \quad (1.5)$$

For the Riemannian spin manifolds this can be expressed using such operators in spinor bundle as the spin Laplacian or the Dirac operator which are related by the Schrödinger-Lichnerowicz formula. In particular, for the Dirac operator one has:

$$\begin{aligned} \mathcal{W}(f|D|^{-n}) &= 2^m v_{n-1} \int_M f \text{vol}_g, \\ \mathcal{W}(f|D|^{-n+2}) &= -2^m \frac{n-2}{24} v_{n-1} \int_M f R(g) \text{vol}_g. \end{aligned} \quad (1.6)$$

In the noncommutative realm the spectral-theoretic approach to scalar curvature has been extended also to quantum tori in the seminal work of Connes and Tretkoff [15], expanded in [13] and then extensively studied by many authors [19–21, 18, 25, 39, 35, 42, 51, 43, 47], see also [10, 23, 40] for recent surveys. Therein, the pseudodifferential operators and symbol calculus introduced in [7] and extended to crossed product algebras in [1, 2], cf. also [52, 32, 33, 29] for detailed account, have been employed for computations of certain values and residues of ζ -functions of suitable Laplace type operators.

In these papers the analogue of conformal transformations of the standard (flat) geometry of the noncommutative 2-torus was worked out, with the conformal scaling (Weyl factor) taken as an invertible positive element from the underlying algebra A ; see also [11]

for a conformally non-flat quantum 4-torus. These achievements produced extremely interesting modular aspects, which, however, involve complicated, often computer-assisted, calculations with thousands of terms.

In this respect, the alternative use of Weyl factor from the copy A° of A in the commutant of A in [16] and the early nonconformally flat modification of the torus geometry by an element from A° in [17] allowed stripping the modular aspects and somewhat simplifying the computations, while still producing a nontrivial scalar curvature. One can also work with A° as the underlying algebra and modify the Dirac operator by an element from A instead. This still yields the usual spectral triple, while taking the other two combinations yields the twisted (modular) triples, so there are just two different types of situation [6].

Now, for quantum tori, and more generally for spectral triples with real structure J , there is a natural choice for A° as JAJ . Moreover, since A° and A coincide in the classical (commutative) case, also modifying the quantum torus geometry by elements from A or from A° appear a viable option and both of them should be considered. The latter leads to the usual (untwisted) spectral triple with the algebra A .

Concerning the spectral approach described above, the Wodzicki residue methods have already been used for quantum tori in dimensions $n = 2$ and $n = 4$ in [24] and [21], respectively, and have been extended to any dimension $n \geq 2$ in [41], see also [46], and applied in [18,47] for the scalar curvature on NC tori.

Altogether, the aforementioned spectral methods allowed so far to extract the scalar invariants built from the metric and the curvature tensors, while the understanding of the curvature tensor itself has not yet been achieved. Only recently has the Ricci curvature tensor Ric_{jk} been recovered using the Hodge-de Rham spectral triple, and more precisely from the difference of zeta functions of the Laplacian on functions and on one-forms, and applied to the noncommutative two-torus [25].

No doubt it would be extremely interesting to recover other important tensors in both the classical setup as well as for the generalised or quantum geometries. In this paper we accomplish this task for the metric tensor g itself, its dual, and for the Einstein tensor

$$G := Ric - \frac{1}{2}R(g)g,$$

which directly enters the Einstein field equations with matter, and its dual. For this purpose, we employ in fact the Wodzicki residue of a suitable power of the Laplace type operator multiplied by a pair of other differential operators. Notably, we demonstrate that the Wodzicki residue density recovers the tensors g and G as certain bilinear functionals of vector fields on a manifold M , while their dual tensors are recovered as the density of bilinear functionals of differential one-forms on M .

Using Connes' pseudodifferential calculus on noncommutative tori we also propose a conspicuous quantum analogue of these functionals and probe it on 2 and 4-dimensional noncommutative tori. Note, however, that a direct comparison with [25] is arduous not only due to the difference between the canonical and Hodge-de Rham spectral triples, but

more fundamentally, due to the unknown way how to extract from the Einstein tensor the Ricci tensor due to the ordering ambiguity (*quantum product*) of g and $R(g)$.

The aforementioned functionals are built using either the Laplace or Dirac operator (in case M is spin manifold), but before embarking on the latter one, we need actually first to settle other Laplace-type operators, acting on vector bundles of suitable rank over M . In fact, it will be advantageous to consider a general class of such operators and to control which tensors they produce. For this reason, our notation will systematically keep track of the dependence on a particular differential operator.

This will be also beneficial for another reason. Namely, one typically obtains spectrally the scalar curvature R of a manifold using the Laplace or Dirac operators, which are built openly from the Levi-Civita connection (so torsion-free). Classically, these operators can be also characterised by the fact that the so obtained R minimises the relevant functionals (as otherwise there is a non-negative contribution from the torsion). This property may be not satisfied in the noncommutative realm, as actually there is no implicit notion of torsion (see, however, other frameworks in e.g. [26,49,3–5]).

It turns out that already on the noncommutative 4-torus the most immediate quantum analogue of the conformally rescaled *flat* Laplace operator does not minimise the scalar curvature functional [50]. Thus, if no particular reference Laplace-type operator is declared to be torsion-free, it can be convenient to label also the spectrally obtained geometric tensors such as R , Ric or G by the concrete operator Δ which has been employed for their definition, and call them, e.g. “ Δ scalar curvature”, etc. By doing so, we will be able to determine which Laplace-type operators provide the same geometric invariant.

1.2. Organization of the article

We start with a brief overview of the normal coordinates and the expression of the Laplace operator and its symbol at a given point on the manifold in these coordinates and present the explicit results for the symbols of the negative powers of the Laplace operator. Then we prove the main theorems showing that functionals on vector fields (understood as differential operators) yield the metric (Theorem 2.3) and the Einstein tensor (Theorem 2.4) densities.

In Section 3 we demonstrate in Theorems 3.1 and 3.2 the value of the metric and Einstein functionals for Laplace-type operators on vector bundles, giving interesting examples of the spinor Laplacian (Proposition 3.4) and square of the Dirac operator (Proposition 3.5). In Section 4 we propose functionals on the space of differential one-forms on a spin_c manifold and prove that these functionals provide densities of the respective dual metric and Einstein tensors (Theorem 4.1).

In Section 5 we propose an extension of these functionals to the noncommutative realm, focusing in Section 5.1 on Laplacian on noncommutative tori, where outer derivations are interpreted as vector fields. We prove that the spectral Einstein tensor vanishes identically for the conformally rescaled geometry of the noncommutative 2-torus (Propo-

sition 5.2) and provide an explicit and compact formula for the spectral Einstein and metric tensors for the 4-torus. In Section 5.2 we propose the generalisation of the spectral metric and Einstein functionals on noncommutative differential forms for spectral triples on conformally rescaled noncommutative 2 and 4-tori, showing that also the contravariant spectral Einstein functional vanishes for the conformally rescaled spectral triple on noncommutative 2-torus (Proposition 5.10) and provide an explicit formula for the spectral metric and Einstein functionals for the noncommutative 4-torus. Furthermore, we discuss also these functionals for finitely summable and regular spectral triples and illustrate how these functionals behave under tensoring a finite summable regular spectral triple with the simplest non-trivial finite triple on $\mathbb{C}^2$.

The Appendix provides a few formulae and computations of pseudodifferential symbols, which are used in proofs.

1.3. Notation

Throughout the article we work with a closed, orientable Riemannian manifold of dimension $n = 2m$, with a fixed metric g , while in the case of functionals on differential forms we assume the existence of spin_c structure and fix the spinor bundle. We denote by γ^a the matrices that satisfy $\gamma^a \gamma^b + \gamma^b \gamma^a = 0$ if $a \neq b$, and by $(\gamma^a)^2 = 1$ for $a, b = 1, \dots, n$.

We denote the Einstein tensor on M by G . The spectral metric and Einstein functionals on vector fields we denote by $\mathfrak{g}^P, \mathfrak{G}^P$, where P is an appropriate Laplace-type operator, whereas for the spectral metric and Einstein functionals on differential forms we use the notation $\mathfrak{g}_D$ and $\mathfrak{G}_D$, where D is a Dirac-type operator.

The proofs are all based on the technique of normal coordinates $\mathbf{x}$ around a fixed point on the manifold and the expansion of all geometric objects (metric, orthonormal frames, connection, vector fields, etc.) up to the relevant order in $\mathbf{x}$.

2. The metric and the Einstein tensor densities

We consider an even-dimensional compact Riemannian manifold M with components of the metric g given in chosen local coordinates by g_{ab} . The Laplace operator, which is densely defined on $L^2(M, \text{vol}_g)$, is expressed as

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \partial_a (\sqrt{\det(g)} g^{ab} \partial_b), \quad (2.1)$$

where g^{ab} is the inverse of the matrix g_{ab} and we use here and in the following the summation convention over repeated indices.

The symbols of the differential operator Δ are:

$$\mathfrak{a}_2 = g^{ab} \xi_a \xi_b, \quad \mathfrak{a}_1 = \frac{-i}{\sqrt{\det(g)}} \partial_a (\sqrt{\det(g)} g^{ab}) \xi_b, \quad \mathfrak{a}_0 = 0. \quad (2.2)$$

As the next step, we will conveniently express the symbols using centered normal coordinates $\mathbf{x}$ [45] around a fixed point of M with $\mathbf{x} = 0$.

2.1. Laplace operator and its powers in normal coordinates

Let us recall that in the normal coordinates the metric has a Taylor expansion:

$$g_{ab} = \delta_{ab} - \frac{1}{3}R_{acbd}x^cx^d + o(\mathbf{x}^2), \quad (2.3)$$

and

$$\sqrt{\det(g)} = 1 - \frac{1}{6}\text{Ric}_{ab}x^ax^b + o(\mathbf{x}^2), \quad (2.4)$$

where R_{acbd} and Ric_{ab} are the components of the Riemann and Ricci tensor, respectively, at the point with $\mathbf{x} = 0$ and we use the notation $o(\mathbf{x}^k)$ to denote that we expand a function up to the polynomial of order k in the normal coordinates. The inverse metric is

$$g^{ab} = \delta^{ab} + \frac{1}{3}R_{acbd}x^cx^d + o(\mathbf{x}^2), \quad (2.5)$$

where on the right-hand side we still can keep the lower indices a, b as at the point with $\mathbf{x} = 0$ the metric is standard Euclidean and so the tensor indices are lowered and raised by the Kronecker symbols δ_{ab} and δ^{ab} .

Consequently, the symbols of the Laplace operator in normal coordinates are

$$\begin{aligned} \mathfrak{a}_2 &= (\delta_{ab} + \frac{1}{3}R_{acbd}x^cx^d)\xi_a\xi_b + o(\mathbf{x}^2), \\ \mathfrak{a}_1 &= \frac{2i}{3}\text{Ric}_{ab}x^a\xi_b + o(\mathbf{x}^2). \end{aligned} \quad (2.6)$$

Then, by a straightforward application of (2.5), (2.3), (2.4) to (A.6) one has:

Lemma 2.1. *In normal coordinates around a fixed point of the manifold M the symbols of the inverse of the Laplace operator read*

$$\begin{aligned} \mathfrak{b}_2 &= \|\xi\|^{-4}(\delta_{ab} - \frac{1}{3}R_{acbd}x^cx^d)\xi_a\xi_b + o(\mathbf{x}^2), \\ \mathfrak{b}_3 &= -\frac{2i}{3}\text{Ric}_{ab}x^a\xi_b\|\xi\|^{-4} + o(\mathbf{x}), \\ \mathfrak{b}_4 &= \frac{2}{3}\text{Ric}_{ab}\xi_a\xi_b\|\xi\|^{-6} + o(1). \end{aligned} \quad (2.7)$$

Next, we apply (Lemma A.1) to compute the three leading symbols of the powers of the pseudodifferential operator Δ^{-1} :

Proposition 2.2. *The first three leading symbols of the operator Δ^{-k} , $k > 0$*

$$\sigma(\Delta^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2} + \dots,$$

are given up to order respectively $\mathbf{x}^2, \mathbf{x}, \mathbf{1}$ in normal coordinates around a fixed point by,

$$\begin{aligned} \mathfrak{c}_{2k} &= ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\ \mathfrak{c}_{2k+1} &= \frac{-2ki}{3||\xi||^{2k+2}} \text{Ric}_{ab} x^b \xi_a + o(\mathbf{x}), \\ \mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3||\xi||^{2k+4}} \text{Ric}_{ab} \xi_a \xi_b + o(\mathbf{1}). \end{aligned} \quad (2.8)$$

Proof. We use (2.7) and (A.11) keeping only terms with the right order in $\mathbf{x}$, which for $\mathfrak{c}_{2k+2}$ yields only 4 terms instead of 10 in (A.11), i.e.

$$\begin{aligned} \mathfrak{c}_{2k+2} &= k(\mathfrak{b}_2)^{n-1} \mathfrak{b}_4 - i \frac{k(k-1)}{2} (\mathfrak{b}_2)^{k-2} \partial_a^\xi (\mathfrak{b}_2) \partial_a^x (\mathfrak{b}_3) - \\ &\quad - \frac{k(k-1)}{4} (\mathfrak{b}_2)^{k-2} \partial_a^\xi \partial_b^\xi (\mathfrak{b}_2) \partial_a^x \partial_b^x (\mathfrak{b}_2) - \\ &\quad - \frac{k(k-1)(k-2)}{6} (\mathfrak{b}_2)^{k-3} \partial_a^\xi (\mathfrak{b}_2) \partial_b^\xi (\mathfrak{b}_2) \partial_a^x \partial_b^x (\mathfrak{b}_2) + o(\mathbf{1}). \end{aligned}$$

We simplify it further by taking into account the properties of the Riemann tensor,

$$R_{abcd} \xi_a \xi_b = 0 = R_{abcd} \xi_c \xi_d,$$

leading to the above result. $\square$

2.2. Spectral functionals of vector fields

Let V, W be a pair of vector fields on a compact Riemannian manifold M , of dimension $n = 2m$. Using the Laplace operator, we define two functionals $\mathfrak{g}^\Delta(V, W)$ and $\mathfrak{G}^\Delta(V, W)$.

Theorem 2.3. *The functional:*

$$\mathfrak{g}^\Delta(V, W) := \mathcal{W}(VW\Delta^{-m-1}),$$

is a bilinear, symmetric map, whose density is proportional to the metric evaluated on the vector fields:

$$\mathfrak{g}^\Delta(V, W) = -\frac{v_{n-1}}{n} \int_M g(V, W) \text{vol}_g.$$

Proof. The product of two vector fields is a differential operator with a symbol

$$\sigma(VW) = \mathbf{v}_2 + \mathbf{v}_1 = -V^a W^b \xi_a \xi_b + iV^a \delta_a(W^b) \xi_b. \quad (2.9)$$

Then,

$$\mathcal{W}(VW\Delta^{-m-1}) = \int_M \int_{\|\xi\|=1} \sigma_{-2m}(VW\Delta^{-m-1}) \text{vol}_g,$$

where by (2.8)

$$\sigma_{-2m}(VW\Delta^{-m-1}) = -V^a W^b \xi_a \xi_b \|\xi\|^{-2m-2}.$$

Using integration over S^{2m-1} we get

$$\int_{\|\xi\|=1} \sigma_{-2m}(VW\Delta^{-m-1}) = -\frac{v_{n-1}}{n} V^a W^a,$$

which ends the proof. $\square$

Theorem 2.4. *The functional:*

$$\mathcal{G}^\Delta(V, W) := \mathcal{W}(VW\Delta^{-m}), \quad (2.10)$$

is a bilinear, symmetric map, whose density is proportional to the Einstein tensor G evaluated on the two vector fields:

$$\mathcal{G}^\Delta(V, W) = \frac{v_{n-1}}{6} \int_M G(V, W) \text{vol}_g.$$

Proof. Similarly to the previous theorem, we need to compute the symbol of order $-2m$ of the pseudodifferential operator $VW\Delta^{-m}$. Using (A.2) we have

$$\begin{aligned} \sigma_{-2m}(VW\Delta^{-m}) &= \mathbf{v}_2 \mathbf{c}_{2m+2} + \mathbf{v}_1 \mathbf{c}_{2m+1} - i\partial_a \mathbf{v}_2 \delta_a \mathbf{c}_{2m+1} \\ &\quad - i\partial_a \mathbf{v}_1 \delta_a \mathbf{c}_{2m} - \frac{1}{2} \partial_a \partial_b \mathbf{v}_2 \delta_a \delta_b \mathbf{c}_{2m}. \end{aligned} \quad (2.11)$$

As the normal coordinates $\mathbf{x}$ are 0 at the (arbitrary) fixed point of the manifold, we are interested only in terms that do not vanish at $\mathbf{x} = 0$. Since both $\mathbf{c}_{2m+1}$ and $\delta_a \mathbf{c}_{2m}$ vanish at $\mathbf{x} = 0$ we are left only with terms that depend on $\mathbf{v}_2$. We explicitly compute

$$\begin{aligned}
\mathfrak{v}_2 \mathfrak{c}_{2m+2} &= -\frac{m(m+1)}{3\|\xi\|^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d + o(\mathbf{1}), \\
-i\partial_a \mathfrak{v}_2 \delta_a \mathfrak{c}_{2m+1} &= -i(-V^a W^b \xi_b - V^b W^a \xi_b) \frac{-2mi}{3\|\xi\|^{2m+2}} \text{Ric}_{ca} \xi_c + o(\mathbf{1}) \\
&= \frac{2m}{3\|\xi\|^{2m+2}} V^a W^b \xi_c (\xi_a \text{Ric}_{bc} + \xi_b \text{Ric}_{ac}) + o(\mathbf{1}), \\
-\frac{1}{2} \partial_a \partial_b \mathfrak{v}_2 \delta_a \delta_b \mathfrak{c}_{2m} &= \frac{1}{2} (V^a W^b + V^b W^a) \frac{-m}{3\|\xi\|^{2m+2}} (R_{cadb} + R_{cbda}) \xi_c \xi_d + o(\mathbf{1}) \\
&= -\frac{2m}{3\|\xi\|^{2m+2}} V^a W^b \xi_c R_{cadb} \xi_d + o(\mathbf{1}).
\end{aligned} \tag{2.12}$$

As a result,

$$\begin{aligned}
\sigma_{-2m}(VW\Delta^{-m}) &= -\frac{m(m+1)}{3\|\xi\|^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d \\
&\quad + \frac{2m}{3\|\xi\|^{2m+2}} V^a W^b \xi_c (\xi_a \text{Ric}_{bc} + \xi_b \text{Ric}_{ac} - R_{cadb} \xi_d) + o(\mathbf{1}).
\end{aligned} \tag{2.13}$$

Integrating (2.13) over S^{2m-1} and substituting $\mathbf{x} = 0$ we have, for the first term,

$$\begin{aligned}
\int_{\|\xi\|=1} -\frac{m(m+1)}{3\|\xi\|^{2m+4}} V^a W^b \text{Ric}_{cd} \xi_a \xi_b \xi_c \xi_d &= \\
&= \frac{v_{n-1}}{2m(2m+2)} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \frac{-m(m+1)}{3} V^a W^b \text{Ric}_{cd} \\
&= -\frac{v_{n-1}}{12} V^a W^a R - \frac{v_{n-1}}{6} V^a W^b \text{Ric}_{ab},
\end{aligned}$$

and for the second term,

$$\begin{aligned}
\int_{\|\xi\|=1} \frac{2m}{3\|\xi\|^{2m+2}} V^a W^b \xi_c (\xi_a \text{Ric}_{bc} + \xi_b \text{Ric}_{ac} - R_{cadb} \xi_d) &= \\
&= \frac{v_{n-1}}{2m} \cdot \frac{2m}{3} V^a W^b (\delta_{ac} \text{Ric}_{bc} + \delta_{bc} \text{Ric}_{ac} - \delta_{cd} R_{cadb}) \\
&= \frac{v_{n-1}}{3} V^a W^b \text{Ric}_{ab}.
\end{aligned}$$

Combining them together and integrating over the manifold we get the result, which is obviously symmetric and bilinear. $\square$

Remark 2.5. A localized version of the functionals $\mathfrak{g}^\Delta$ and $\mathfrak{G}^\Delta$ is automatic since they satisfy,

$$\mathcal{W}(fVW\Delta^{-n-1}) = \mathfrak{g}^\Delta(fV, W) = \mathfrak{g}^\Delta(V, fW), \tag{2.14}$$

$$\mathcal{W}(fVW\Delta^{-n}) = \mathcal{G}^\Delta(fV, W) = \mathcal{G}^\Delta(V, fW) \quad (2.15)$$

for all $f \in C^\infty(M)$.

We will call g^Δ and $\mathcal{G}^\Delta$ respectively *metric* and *Einstein* (spectral) functionals.

3. Spectral functionals for the operators of Laplace type

In this section, we will demonstrate how the results vary if one passes to the Laplace-type operators, acting on sections of a vector bundle V of rank $\text{rk}(V)$. These operators generalize the scalar Laplacian in the sense that they have the same principal symbol (times the matrix unit), yet they may contain both some nontrivial connections and torsion.

For this purpose we assume that there is a connection ∇ on the vector bundle V , i.e. for any vector field X on M , we have a covariant derivative ∇_X on the module of smooth sections of V . Using the notation $\nabla_a := \nabla_{\partial_a}$ in local normal coordinates around a fixed point on the manifold we have:

$$\nabla_a = \partial_a - \mathbf{T}_a,$$

where each $\mathbf{T}_a$ is a $C^\infty(M)$ endomorphism of the sections of V . Using normal coordinates and expanding $\mathbf{T}_a$ around $\mathbf{x} = 0$ we have,

$$\mathbf{T}_a(x) = T_a + T_{ab}x^b + o(\mathbf{x}).$$

Thus, we can write the symbol of the generalized Laplace operator

$$\Delta_T = -g^{ab}(\nabla_a \nabla_b - \Gamma_{ab}^c \nabla_c),$$

in the normal coordinates as the sum of

$$\begin{aligned} \mathfrak{a}_2 &= (\delta_{ab} + \frac{1}{3}R_{acbd}x^c x^d)\xi_a \xi_b + o(\mathbf{x}^2), \\ \mathfrak{a}_1 &= \frac{2i}{3}\text{Ric}_{ab}x^a \xi_b + (2iT_a \xi_a + 2iT_{ab}x^b \xi_a) + o(\mathbf{x}), \\ \mathfrak{a}_0 &= T_{aa} - T_a T_a + o(\mathbf{1}). \end{aligned} \quad (3.1)$$

Then the parts of order $-2, -3, -4$ of the inverse of Δ_T expanded in $\mathbf{x}$ up to the order, respectively $2, 1, 0$, are

$$\begin{aligned}
\mathfrak{b}_2 &= \|\xi\|^{-4} \left(\delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{b}_3 &= -\frac{2i}{3} \text{Ric}_{ab} x^a \xi_d \|\xi\|^{-4} - (2iT_a \xi_a + 2iT_{ab} x^b \xi_a) \|\xi\|^{-4} + o(\mathbf{x}), \\
\mathfrak{b}_4 &= \frac{2}{3} \text{Ric}_{ab} \xi_a \xi_b \|\xi\|^{-6} - 4T_a T_b \xi_a \xi_b \|\xi\|^{-6} \\
&\quad - (T_{aa} - T_a T_a) \|\xi\|^{-4} + 4T_{ab} \xi_a \xi_b \|\xi\|^{-6} + o(\mathbf{1}).
\end{aligned} \tag{3.2}$$

Theorem 3.1. *The functional*

$$\mathfrak{g}^{\Delta_T}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_T^{-n-1})$$

does not depend on the connection $\mathbf{T}$, and

$$\mathfrak{g}^{\Delta_T}(V, W) = rk(V) \mathfrak{g}^{\Delta}(V, W).$$

Proof. This is evident, since the principal symbol does not depend on $\mathbf{T}$. $\square$

Theorem 3.2. *The functional*

$$\mathfrak{G}^{\Delta_T}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_T^{-n})$$

is equal to

$$\mathfrak{G}^{\Delta_T}(V, W) = \frac{v_{n-1}}{6} rk(V) \int_M G(V, W) vol_g + \frac{v_{n-1}}{2} \int_M F(V, W) vol_g, \tag{3.3}$$

where

$$F(V, W) = Tr V^a W^b F_{ab},$$

and F_{ab} is the curvature tensor of the connection $\mathbf{T}$. For $\mathbf{T} = 0$ it is equal to $rk(V) \mathfrak{G}^{\Delta}(V, W)$.

Proof. First we compute the leading symbols of $(\Delta_T)^{-m}$ up to the appropriate order in $\mathbf{x}$,

$$\begin{aligned}
\mathfrak{c}_{2m} &= \|\xi\|^{-2m-2} \left(\delta_{ab} - \frac{m}{3} R_{ajbk} x^j x^k \right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{c}_{2m+1} &= \frac{-2mi}{3} \|\xi\|^{-2m-2} \text{Ric}_{ak} x^k \xi_a - 2mi \|\xi\|^{-2m-2} (T_a \xi_a + T_{ab} x^b \xi_a) + o(\mathbf{x}) \\
\mathfrak{c}_{2m+2} &= \frac{m(m+1)}{3} \|\xi\|^{-2m-4} \text{Ric}_{ab} \xi_a \xi_b \\
&\quad - 2m(m+1) \|\xi\|^{-2m-4} T_a T_b \xi_a \xi_b + m(T_a T_a - T_{aa}) \|\xi\|^{-2m-2} \\
&\quad + 2m(m+1) \|\xi\|^{-2m-4} T_{ab} \xi_a \xi_b + o(\mathbf{1}).
\end{aligned} \tag{3.4}$$

Expanding in a similar way

$$\nabla_V = V^a(\partial_a - T_a - T_{ab}x^b) + o(\mathbf{x}), \quad \nabla_W = W^a(\partial_a - T_a - T_{ab}x^b) + o(\mathbf{x}),$$

for vector fields V and W , we compute only the terms of the symbol of $\nabla_V \nabla_W \Delta_T^{-m}$ of order $-2m$, which depend on T_a and T_{ab} at $x = 0$, as the remaining terms would be identical to the case considered earlier,

$$\begin{aligned} \sigma_{-2m}(\nabla_V \nabla_W \Delta_T^{-m}) &= (2m(m+1)T_a T_b \xi_a \xi_b + mT_{aa}||\xi||^2 - mT_a T_a ||\xi||^2 \\ &\quad - 2m(m+1)T_{ab} \xi_a \xi_b) V^j W^k \xi_j \xi_k ||\xi||^{-2m-4} \\ &\quad + 2mV^a(\partial_a W^b) T_c \xi_b \xi_c ||\xi||^{-2m-2} \\ &\quad - 2m(V^a W^b + V^b W^a) T_a T_c \xi_b \xi_c ||\xi||^{-2m-2} \\ &\quad + V^a W^b T_a T_b ||\xi||^{-2m} - V^a W^b T_{ba} ||\xi||^{-2m} \\ &\quad + 2mT_{ab}(V^c W^b + W^c V^b) \xi_a \xi_c ||\xi||^{-2m-2} \\ &\quad - V^a(\partial_a W^b) T_b ||\xi||^{-2m}. \end{aligned} \quad (3.5)$$

Integrating it over S^{2m-1} and setting $\mathbf{x} = 0$ we have the following result for the density of the Wodzicki residue (before taking the trace over endomorphisms)

$$\begin{aligned} w(\nabla_V \nabla_W \Delta_T^{-m}) &= v_{n-1} \left(\frac{1}{2} V^a W^b (T^2 \delta_{ab} + T_a T_b + T_b T_a + T_{cc} \delta_{ab} \right. \\ &\quad - T^2 \delta_{ab} - T_{cc} \delta_{ab} - T_{ab} - T_{ba}) + V^a (\partial_a W^b) T_b \\ &\quad - V^a W^b (T_a T_b + T_b T_a) + V^a W^b T_a T_b - V^a W^b T_{ba} \\ &\quad \left. + V^a W^b (T_{ab} + T_{ba}) - V^a (\partial_a W^b) T_b \right) \\ &= \frac{1}{2} v(S^{n-1}) V^a W^b (T_{ab} - T_{ba} + T_a T_b - T_b T_a) \\ &= \frac{1}{2} v(S^{n-1}) V^a W^b F_{ab}, \end{aligned} \quad (3.6)$$

where F_{ab} is the curvature tensor of the connection $\mathbf{T}$, as indeed in the normal coordinates:

$$F_{ab} = -(\partial_a \mathbf{T}_b - \partial_b \mathbf{T}_a) + [\mathbf{T}_a, \mathbf{T}_b] = T_{ab} - T_{ba} + [T_a, T_b] + o(1),$$

which finishes the proof. $\square$

Note that the functionals $\mathcal{G}^{\Delta_T}$ and $\mathcal{G}^{\Delta_T}$ are automatically ‘localised’ as in Remark 2.5.

We finish the section by demonstrating that any 0-order perturbation of the Laplace (type) operator Δ_T does not modify the metric functional, whereas the Einstein func-

tional is modified by a term whose density involves the metric functional multiplied by the trace of the 0-order term.

Lemma 3.3. *For the Laplace type operator $\Delta_{T,E} := \Delta_T + E$, where E is an endomorphism of the vector bundle, the metric functional does not depend on E ,*

$$g^{\Delta_{T,E}}(V, W) = g^{\Delta_T}(V, W), \quad (3.7)$$

whereas the Einstein functional

$$\mathcal{G}^{\Delta_{T,E}}(V, W) := \mathcal{W}(\nabla_V \nabla_W \Delta_{T,E}^{-m})$$

reads,

$$\mathcal{G}^{\Delta_{T,E}}(V, W) = \mathcal{G}^{\Delta_T}(V, W) + \frac{1}{2} \int_M (\text{Tr } E) g(V, W) \text{ vol}_g. \quad (3.8)$$

Proof. The first statement is obvious. For the second statement by computing the symbols of Δ^{-m} we first see from the formula (A.6) that E enters the symbol of order -4 linearly and, consequently, using (A.11) the only additional term that appears in the symbol of Δ^{-m} would appear in order $-2m - 2$ as

$$-mE||\xi||^{-2m-2}.$$

The only term depending on E that arises in the contribution to the respective symbol of order $-2m$ of the product $\nabla_V \nabla_W \Delta^{-m}$ would then be

$$mEV^a W^b \xi_a \xi_b ||\xi||^{-2m-2},$$

which after integrating over the sphere, taking trace and using (2.3) gives

$$\frac{1}{2} (\text{Tr } E) g(V, W). \quad \square$$

As follows from [28,27] the operators $\Delta_{T,E}$ are the most general Laplace-type operators on M .

3.1. Spectral functionals for the spin Laplacian

A particularly interesting example is the application of the above result to the case of the spinor bundle of rank 2^m (assuming that the manifold M has a spin structure, which we fix).

Recall that in terms of a (local) basis e_i of orthonormal vector fields on M the Levi-Civita covariant derivative reads

$$\nabla_{e_i} e_j = \alpha_{ijk} e_k, \quad (3.9)$$

where α_{ijk} is the Levi-Civita connection, expressed explicitly in the basis e_i through the structure constants c_{ijk} of the commutators of orthonormal vector fields,

$$\alpha_{ijk} := \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji}), \quad [e_i, e_j] = c_{ijk} e_k. \quad (3.10)$$

If M is a spin manifold, then the lift of the Levi-Civita covariant derivative to Dirac spinor fields reads

$$\nabla_{e_i}^{(s)} = e_i - \frac{1}{4} \alpha_{ijk} \gamma^j \gamma^k, \quad (3.11)$$

with γ^j as in Section 1.3. The spinorial Laplace operator is

$$\Delta^{(s)} := \nabla^{(s)*} \nabla^{(s)} = -\nabla_{e_i}^{(s)} \nabla_{e_i}^{(s)} + \nabla_{\nabla_{e_i} e_i}^{(s)} = -\nabla_{e_i}^{(s)} \nabla_{e_i}^{(s)} + \alpha_{iij} \nabla_{e_j}^{(s)}. \quad (3.12)$$

In the normal coordinates around a fixed point of the manifold one has then

$$e_i = \frac{\partial}{\partial x^i} - \frac{1}{6} R_{ijk\ell} x^j x^k \frac{\partial}{\partial x^\ell} + o(\mathbf{x}^2) \quad (3.13)$$

and

$$\alpha_{ijk} = -\frac{1}{2} R_{\ell ijk} x^\ell + o(\mathbf{x}), \quad (3.14)$$

and therefore,

$$\begin{aligned} \Delta^{(s)} &= -\partial_i \partial_i + \frac{1}{3} R_{ijk\ell} x^j x^k \partial_i \partial_\ell + o(\mathbf{x}^2) \\ &+ \frac{2}{3} R_{ij} x^i \partial_j + \frac{1}{4} R_{i\ell jk} x^\ell \gamma^j \gamma^k \partial_i + o(\mathbf{x}) \\ &+ o(\mathbf{1}), \end{aligned} \quad (3.15)$$

where in the three lines we collected terms of different order, expanding them up to the same order in normal coordinates $\mathbf{x}$ as their order.

By comparing the symbol with (3.1) it is easy to identify then $\Delta^{(s)}$ as a Laplace-type operator Δ_T , with the respective expansion of the spin connection,

$$T_a = 0, \quad T_{ab} = \frac{1}{8} R_{abjk} \gamma^j \gamma^k.$$

We have then

Proposition 3.4. *The metric and Einstein functional associated to the spin Laplacian are proportional to the functionals of the scalar Laplacian,*

$$\begin{aligned} g^{\Delta^{(s)}}(V, W) &:= \mathcal{W}(\nabla_V^{(s)} \nabla_W^{(s)} (\Delta^{(s)})^{-n-2}) = 2^m g^\Delta(V, W), \\ \mathcal{G}^{\Delta^{(s)}}(V, W) &:= \mathcal{W}(\nabla_V^{(s)} \nabla_W^{(s)} (\Delta^{(s)})^{-n}) = 2^m \mathcal{G}^\Delta(V, W). \end{aligned} \quad (3.16)$$

Proof. This is a consequence of Theorem 3.2. While the result for the metric functional is obvious for the Einstein functional we see the trace of the additional term, arising from the curvature of the connection vanishes,

$$\frac{1}{4} \text{Tr } V^a W^b (R_{abjk} \gamma^j \gamma^k - R_{bajk} \gamma^j \gamma^k) = 0,$$

due to the skew symmetry of the Riemann tensor in the two last indices. $\square$

3.2. Spectral functionals for the Dirac operator

An immediate application of Lemma 3.3 is the computation of the Einstein functional using the Dirac operator instead of the Laplacian. The Dirac operator is, in a local basis of orthonormal frames, a first-order differential operator

$$D = i\gamma^j \nabla_{e_j}^{(s)}, \quad (3.17)$$

and its square D^2 differs from the spinorial Laplacian only by a quarter of the scalar of curvature

$$D^2 = \Delta^{(s)} + \frac{1}{4} R.$$

More generally, if we consider a spin_c structure and the Dirac operator twisted by a $U(1)$ -connection,

$$D_A = D + A,$$

the respective formula reads,

$$D_A^2 = \Delta^{(s)} + \frac{1}{4} R + F,$$

where

$$F = \gamma^j \gamma^k F_{jk},$$

and F_{jk} is the curvature of A . Then as the consequence of Lemma 3.3 we have,

Proposition 3.5. *The spectral metric and Einstein functionals associated with the Dirac operator D_A do not depend on the connection A and read,*

$$\begin{aligned} g^{D_A^2}(V, W) &:= \mathcal{W}(\nabla_V^{(s)} \nabla_W^{(s)} |D_A|^{-n-2}) = 2^m g^\Delta(V, W), \\ \mathcal{G}^{D_A^2}(V, W) &:= \mathcal{W}(\nabla_V^{(s)} \nabla_W^{(s)} |D_A|^{-n}) = 2^m \left(\mathcal{G}^\Delta(V, W) + \frac{1}{8} \mathcal{R}(V, W) \right), \end{aligned} \quad (3.18)$$

where

$$\mathcal{R}(V, W) = v_{n-1} \int_M R(g) g(V, W) \text{vol}_g.$$

4. Metric and Einstein functionals of differential forms

In differential geometry besides functionals on vector fields, one can alternatively consider functionals on the dual bimodule of one-forms. In this section, we will investigate whether the Einstein tensor (or, more precisely, its contravariant version) can be obtained from such functional using the spectral methods. For this purpose, we need to represent differential forms as differential operators, and a suitable way is to employ Clifford modules. We assume thus that M is a $n = 2m$ dimensional spin_c manifold and use the Clifford representation of one-forms as 0-order differential operators, that is, endomorphisms of a rank 2^m spinor bundle.

As generating endomorphisms, we can work with Clifford multiplication by the local coframe basis e^j dual to the orthonormal oriented frame basis e_j , which simply amounts to multiplication by (constant) gamma matrices γ^j as in Section 1.3. Thus, let v, w , with the components with respect to local coordinates v_a and w_a , respectively, be two differential forms represented in such a way as endomorphisms (matrices) $\hat{v}$ and $\hat{w}$ on the spinor bundle.

Theorem 4.1. *The following spectral functionals of one-forms on a spin-c manifold M of dimension n*

$$\begin{aligned} g_D(v, w) &:= \mathcal{W}(\hat{v} \hat{w} D^{-n}), \\ \mathcal{G}_D(v, w) &:= \mathcal{W}(\hat{v}(D \hat{w} + \hat{w} D) D^{-n+1}) \\ &= \mathcal{W}((D \hat{v} + \hat{v} D) \hat{w} D^{-n+1}), \end{aligned} \quad (4.1)$$

read

$$\begin{aligned} g_D(v, w) &= 2^m v_{n-1} \int_M g(v, w) \text{vol}_g, \\ \mathcal{G}_D(v, w) &= 2^m \frac{v_{n-1}}{6} \int_M G(v, w) \text{vol}_g, \end{aligned} \quad (4.2)$$

where $g(v, w) = g^{ab}v_a w_b$ and $G(v, w) = (\text{Ric}^{ab} - \frac{1}{2}Rg^{ab})v_a w_b$, using the expressions $v = v_a dx^a$, $w = w_b dx^b$ in any local coordinates.

Proof. The proof of the formula for the metric functional g_D is easy and we skip it, concentrating on the Einstein functional, splitting it into two parts as follows:

$$\mathcal{G}_1(v, w) = \mathcal{W}(\hat{v}D\hat{w}DD^{-n}),$$

and

$$\mathcal{G}_2(v, w) = \mathcal{W}(\hat{v}\hat{w}D^{-n+2}).$$

We start with the first. Let us again work with normal coordinates x around a fixed point on the manifold M with $\mathbf{x} = 0$. We can rewrite the first three leading symbols of $D^{-n} = D^{-2m}$ (using (3.4) and the Lichnerowicz formula),

$$\begin{aligned} \mathfrak{c}_{2m} &= \|\xi\|^{-2m-2} \left(\delta_{ab} - \frac{m}{3} R_{ajbk} x^j x^k \right) \xi_a \xi_b + o(\mathbf{x}^2), \\ \mathfrak{c}_{2m+1} &= \frac{-2mi}{3} \|\xi\|^{-2m-2} \text{Ric}_{ak} x^k \xi_a - 2mi \|\xi\|^{-2m-2} (T_{ab} x^b \xi_a) + o(\mathbf{x}) \\ \mathfrak{c}_{2m+2} &= \frac{m(m+1)}{3} \|\xi\|^{-2m-4} \text{Ric}_{ab} \xi_a \xi_b - \frac{m}{4} R \|\xi\|^{-2m-2} + o(\mathbf{1}), \end{aligned} \quad (4.3)$$

where

$$T_{ab} = \frac{1}{8} R_{abjk} \gamma^j \gamma^k,$$

and we have used the antisymmetry of T_{ab} . The Clifford images of v and w have a local expansion around $\mathbf{x} = 0$ in normal coordinates,

$$\hat{v} = v_a \gamma^a + o(\mathbf{1}), \quad \hat{w} = w_a \gamma^a + w_{ab} \gamma^a x^b + o(\mathbf{x}),$$

where w_a, v_a and w_{ab} are constants. Note that here due to the metric in normal coordinates at $x = 0$ being Euclidean, the gamma matrices satisfy $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. Observe that the symbol of $\hat{v}D$ needs to be expanded to $o(\mathbf{1})$ whereas symbol of $\hat{w}D$ to $o(\mathbf{x})$, and so they read

$$\begin{aligned} \sigma(\hat{v}D) &= i v_a \gamma^a \gamma^j i \xi_j + o(\mathbf{1}), \\ \sigma(\hat{w}D) &= i w_a \gamma^a \gamma^j (i \xi_j + \frac{1}{8} R_{\ell jps} x^\ell \gamma^p \gamma^s) - i w_{ab} \gamma^a \gamma^j \xi_j x^b + o(\mathbf{x}). \end{aligned} \quad (4.4)$$

The constants w_{ab} arise from the dependence of the one-form w on the coordinates around $x = 0$.

The symbol of $\hat{v}D\hat{w}D$ up to order $o(\mathbf{1})$ in normal coordinates is

$$\begin{aligned}\sigma(\hat{v}D\hat{w}D) &= v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \xi_j \xi_k - \frac{1}{8} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^s R_{jkps} \\ &\quad - i v_a w_{bj} \gamma^a \gamma^j \gamma^b \gamma^k \xi^k + o(\mathbf{1}).\end{aligned}\tag{4.5}$$

We can omit additional terms with explicit dependence on normal coordinates as they will all vanish at $x = 0$.

The symbol of order $-n$ of the product $\hat{v}D\hat{w}DD^{-n}$ comes then (at $x = 0$) as

$$\begin{aligned}\sigma_{-n}(\hat{v}D\hat{w}DD^{-n}) &= \frac{m(m+1)}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \text{Ric}_{rs} \xi_j \xi_k \xi_r \xi_s \|\xi\|^{-2m-4} \\ &\quad - \frac{m}{4} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k R_{\xi_j \xi_k} \|\xi\|^{-2m-2} \\ &\quad - \frac{2m}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k (\text{Ric}_{rj} \xi_k \xi_r + \text{Ric}_{rk} \xi_r \xi_j) \|\xi\|^{-2m-2} \\ &\quad - \frac{m}{4} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^q (R_{rjpk} \xi_r \xi_k + R_{rkpj} \xi_r \xi_p) \|\xi\|^{-2m-2} \\ &\quad + \frac{m}{3} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k (R_{rjpk} \xi_r \xi_p + R_{rkpj} \xi_r \xi_p) \|\xi\|^{-2m-2} \\ &\quad - \frac{1}{8} v_a w_b \gamma^a \gamma^j \gamma^b \gamma^k \gamma^p \gamma^s R_{jkps} \|\xi\|^{-2m} + o(\mathbf{1}).\end{aligned}\tag{4.6}$$

We see at once that it does not contain any term with w_{ab} hence the result will be bilinear in the differential forms. Integrating over $\xi \in S^{n-1}$ and taking the trace over the matrices γ we obtain:

$$\mathfrak{G}_1(v, w) = 2^m \frac{v_{n-1}}{6} \int_M \left(G(v, w) + \frac{n-2}{4} g(v, w) R \right) \text{vol}_g.$$

The functional $\mathfrak{G}_2$ requires the computation of the symbol of D^{-n+2} up to $o(\mathbf{1})$ as the product of two forms is an operator of order zero. Then, using the explicit formula for the symbol of D^{-n+2} (4.3), we obtain

$$\mathfrak{G}_2(v, w) = -2^m \frac{v_{n-1}}{6} \int_M \frac{n-2}{4} g(v, w) R \text{vol}_g.$$

This shows our statement about $\mathfrak{G}_D = \mathfrak{G}_1 + \mathfrak{G}_2$. $\square$

5. Towards noncommutative metric and Einstein functionals

The spectral methods that we have proposed to obtain the metric and Einstein functionals are well suited for generalisation to the noncommutative case. So far, almost

exclusively scalar geometric quantities (like scalar curvature) were computed for noncommutative tori, with conformally rescaled Laplace and Dirac operators (see [22] for a review) and partial conformal rescaling [17,11]. As the tensors carry more information than scalar geometric objects their study on noncommutative level is certainly more interesting though definitely complicated, especially that a general algebra may have no outer derivations which are usually regarded as the noncommutative counterpart of vector fields. On the other hand, differential forms are naturally associated to any spectral triple, with the latter deemed to best encode the notion of a Riemannian manifold in the noncommutative case.

On the quantum tori we take advantage of both of these structures and correspondingly propose definitions of the associated spectral metric and Einstein tensors using the generalisation of the Wodzicki residue on the generalised algebra of symbols on noncommutative tori. We compute explicitly for 2- and 4-dimensional noncommutative tori the relevant functionals of outer derivations for the conformally rescaled Laplace operators and of one-forms with conformally rescaled Dirac operators. We also define analogous functionals for regular finitely summable spectral triples and study their product with the simplest nontrivial finite spectral triple.

5.1. The metric and Einstein functionals for the Laplacian on a noncommutative tori

The quantum tori are prominent examples of noncommutative manifolds which admit noncommutative analogues of many classical geometrical objects. In particular, there are outer derivations that act on the smooth algebra $\mathcal{A} = C^\infty(\mathbb{T}_\theta^n)$ and that can be interpreted as noncommutative vector fields, even though in general they form only a complex vector space rather than an $\mathcal{A}$ -bimodule. It is then straightforward to identify a noncommutative counterpart of the flat-metric Laplace operator. This can also be generalised to the case of conformally rescaled geometry, where the conformal factor is taken as a positive element of the algebra of the noncommutative torus.

We can therefore exploit our definition of the metric and Einstein spectral functionals to define the corresponding functionals for the noncommutative tori and compute them explicitly. For that purpose we use the pseudodifferential calculus of symbols as defined in [15] and then used and developed by many authors (see [22] for a review). The rules of the algebra of symbols are almost identical to the usual ones, however, with the partial derivatives replaced by derivations and the symbols valued in the algebra of noncommutative torus.

This algebra admits a natural generalisation of the Wodzicki residue, which is defined as the trace of the integral over the cosphere $||\xi|| = 1$ of the symbol of order $-n$ (where n is the dimension of the torus). The existence of such Wodzicki residue trace on the algebra of symbols over the noncommutative 2-torus was demonstrated and discussed in [24,41,50]. We denote it again by $\mathcal{W}$.

However, for the conformally rescaled $\mathbb{T}_\theta^n$ we find it convenient to work with an enlarged algebra $\hat{\mathcal{A}}$, which is generated by $\mathcal{A}$ and its copy $\mathcal{A}^\circ$ commuting with $\mathcal{A}$. Indeed

for the Laplace-type operators, the relevant Weyl factor in principle can be taken from $\hat{\mathcal{A}}$, though usually it is assumed to be in $\mathcal{A}$ and we adhere to this convention. For the spectral triple with conformally rescaled Dirac operator, the conformal factor is instead assumed to be from $\mathcal{A}^o$ as only in such a case the associated differential one-forms (generated by $\mathcal{A}$ and its commutators with the Dirac operator) are bounded operators, on which our functionals will be defined.

Now, the standard flat Dirac operator on $\mathbb{T}_\theta^n$ as well as its conformal rescaling are clearly first-order differential operators for the extended pseudodifferential calculus with symbols valued in the algebra $\hat{\mathcal{A}}$, since they are just built from the derivations on $\mathcal{A}$ which extend to derivations on $\hat{\mathcal{A}}$. Next, considering for simplicity only the *strictly irrational* [48] noncommutative torus (with the center of algebra equal to $\mathbb{C}$), there is an obvious factorized trace $\tau^\otimes$ on the enlarged algebra $\hat{\mathcal{A}}$ given by $\tau^\otimes(ab^o) = \tau(a)\tau(b^o)$. Moreover, since $\tau^\otimes$ is invariant under derivations like τ , we use it to define the tracial Wodzicki residue on $\hat{\mathcal{A}}$ -valued symbols as above, and still denote it by $\mathcal{W}$ and its density by w .

5.1.1. The metric and Einstein functionals for the conformally rescaled Laplacian on a noncommutative 2-torus

Since every two-dimensional Riemannian manifold has a vanishing Einstein tensor, it is natural to ask whether this holds also for the noncommutative torus. We shall demonstrate that it is true in the case of a conformally rescaled Laplace operator. We denote by $\mathcal{A} = C^\infty(\mathbb{T}_\theta^2)$ the algebra of smooth elements of the noncommutative two-torus and by τ the standard trace over its C^* -algebraic completion. By $\mathcal{H} = L^2(\mathbb{T}_\theta^2, \tau)$ we denote the standard Hilbert space obtained by the GNS construction for the tracial state τ . By δ_1, δ_2 we denote the basis of derivations of the algebra $\mathcal{A}$ implemented as densely defined operators on $\mathcal{H}$.

Definition 5.1. Let $h \in C^\infty(\mathbb{T}_\theta^2)$ be positive, invertible, with a bounded inverse. We define as the conformally rescaled Laplace operator for the noncommutative torus the following densely defined selfadjoint operator on $\mathcal{H}$:

$$\Delta_h = h^{-1} \Delta h^{-1}, \quad (5.1)$$

where

$$\Delta = \sum_{a=1,2} \delta_a^2. \quad (5.2)$$

This definition is motivated by the commutative case $\theta = 0$ where the operator Δ_h is unitarily equivalent to the operator $h^{-2} \Delta$, which is a densely defined self-adjoint operator on $\mathcal{H}_h = L^2(\mathbb{T}^2, \tau_h)$, where $\tau_h(a) = \tau(h^2 a)$.

Similarly, we define the vector fields as appropriate self-adjoint generalisations of operators unitarily equivalent to derivations,

$$V_h = \sum_{a=1,2} V^a h \delta_a h^{-1},$$

where $V^a \in \mathbb{C}$.

Proposition 5.2. *For the conformally rescaled Laplace operator on a noncommutative 2-torus the metric functional reads*

$$\mathcal{G}^{\Delta_h}(V_h, W_h) = \mathcal{W}(V_h W_h \Delta_h^{-2}) = \pi \tau(h^4) V^a W^b \delta_{ab},$$

whereas the spectral Einstein functional and its density vanish identically

$$\mathcal{G}^{\Delta_h}(V_h, W_h) = \mathcal{W}(V_h W_h \Delta_h^{-1}) = 0.$$

Proof. The metric functional is straightforwardly computed

$$\begin{aligned} \mathcal{G}^{\Delta_h}(V_h, W_h) &:= \mathcal{W}((hV^a \delta_a h^{-1})(hW^b \delta_b h^{-1})(h^{-1} \Delta h^{-1})^{-2}) \\ &= \mathcal{W}(hV^a W^b \delta_a \delta_b \Delta^{-1} h^2 \Delta^{-1} h) \\ &= \pi \tau(h^4 V^a W^b \delta_{ab}) = \pi \tau(h^4) V^a W^b \delta_{ab}. \end{aligned} \quad (5.3)$$

Note that the components of the noncommutative metric tensor functional scale as h^4 as in the classical situation.

Next, we compute the Einstein functional,

$$\begin{aligned} \mathcal{G}^{\Delta_h}(V_h, W_h) &:= \mathcal{W}((hV^a \delta_a h^{-1})(hW^b \delta_b h^{-1})(h^{-1} \Delta h^{-1})^{-1}) \\ &= \mathcal{W}(hV^a W^b \delta_a \delta_b \Delta^{-1} h) = \mathcal{W}(h^2 V^a W^b \delta_a \delta_b \Delta^{-1}). \end{aligned} \quad (5.4)$$

Since Δ is the standard (flat) Laplace operator and V, W are just $\mathbb{C}$ -linear combinations of the derivations, $V^a W^b \delta_a \delta_b \Delta^{-1}$ is an operator that has exclusively symbol of order 0. Since its symbol of order -2 vanishes, its product with any element from the algebra, like h^2 , has the same property, and thus the Wodzicki residue of it vanishes. $\square$

Note that by using the same argument and the trace property of the Wodzicki residue we also show that for any $a \in \mathcal{A}$ the localised functional vanishes

$$\mathcal{G}^{\Delta_h}(a, V, W) = \mathcal{W}(a V W \Delta_h^{-1}) = 0$$

(here aV , in general, does not implement an algebra derivation). Thus, the conformally rescaled geometry of the noncommutative 2-torus has indeed the same property as a 2-dimensional manifold - its Einstein tensor vanishes.

We finish this example by reiterating that the analysis does not change if h is taken from the larger algebra $\hat{\mathcal{A}}$.

5.1.2. The metric and Einstein functionals for the conformally rescaled Laplacian on a noncommutative 4-torus

As the analogue of the conformally rescaled Laplace operator over the noncommutative 4-torus (acting on $\mathcal{H} = L^2(\mathbb{T}_\theta^4, \tau)$) we simply take,

$$\Delta_h = \sum_{a=1}^4 \chi^{-1} \cdot \delta_a \cdot \chi \cdot \delta_a \cdot \chi^{-1}, \quad (5.5)$$

where $\chi = h^2$ is a positive element of the algebra $C^\infty(T_\theta^4)$ and δ_a are the standard derivations extended as operators on a dense subspace of $L^2(T_\theta^4, \tau)$. The deformation parameter θ is here, in fact, a matrix (such that the center of the algebra is $\mathbb{C}$). Note that the operator Δ_h is again a noncommutative generalisation of an operator that classically (for $\theta = 0$) is unitarily equivalent to a Laplace operator for the conformally rescaled metric on the four-torus.

Similarly, as in the case of 2-dimensional noncommutative torus we take two arbitrary vector fields V, W understood as h -rescaled linear combinations of derivations,

$$V_h = V^a \chi \delta_a \chi^{-1}, \quad W_h = W^b \chi \delta_b \chi^{-1}.$$

Proposition 5.3. *The spectral metric and Einstein functionals on derivations for a conformally rescaled Laplace operator over the noncommutative 4-torus are, respectively,*

$$\begin{aligned} g^{\Delta_h}(V_h, W_h) &= 2\pi^2 \tau(\chi^3) V^a W^b \delta_{ab}, \\ \mathcal{G}^{\Delta_h}(V_h, W_h) &= 2\pi^2 \tau \left(-\frac{1}{24} \chi (V^a \delta_a \chi) \chi^{-1} (W^b \delta_b \chi) - \frac{1}{24} \chi (W^a \delta_a \chi) \chi^{-1} (V^b \delta_b \chi) \right. \\ &\quad + \frac{5}{24} (V^a \delta_a \chi) \chi^{-1} (W^b \delta_b \chi) \chi + \frac{5}{24} (W^a \delta_a \chi) \chi^{-1} (V^b \delta_b \chi) \chi \\ &\quad - \frac{1}{24} (V^a \delta_a \chi) (W^b \delta_b \chi) - \frac{1}{24} (W^a \delta_a \chi) (V^b \delta_b \chi) \\ &\quad - \frac{1}{3} V^a W^b (\delta_{ab} \chi) \chi + \frac{1}{6} \chi V^a W^b (\delta_{ab} \chi) \\ &\quad + \left(-\frac{1}{24} (\delta_a \chi) \chi^{-1} (\delta_a \chi) \chi - \frac{1}{24} \chi (\delta_a \chi) \chi^{-1} (\delta_a \chi) \right. \\ &\quad \left. \left. - \frac{1}{24} (\delta_a \chi) (\delta_a \chi) + \frac{1}{12} \chi (\Delta \chi) + \frac{1}{12} (\Delta \chi) \chi \right) V^b W^b \right). \end{aligned} \quad (5.6)$$

We skip the proof, which is a straightforward and tedious computation of the respective symbols and the integration over the cosphere. Note that the above expression can be further simplified using the trace property of τ . Let us also note that the expression has

a well-defined commutative limit, in which case the spectral Einstein functional density becomes as follows:

$$\begin{aligned} w(VW\Delta_h^{-2}) = & 2\pi^2 \left(\frac{1}{4} V^a W^b (\delta_a \chi)(\delta_b \chi) - \frac{1}{6} V^a W^b (\delta_{ab} \chi) \chi \right. \\ & \left. - \frac{1}{8} (\delta_a \chi)(\delta_a \chi)(V^b W^b) + \frac{1}{6} (\Delta \chi) \chi (V^b W^b) \right), \end{aligned} \quad (5.7)$$

and is exactly equal to:

$$\frac{2\pi^2}{6} \sqrt{g} G_{ab} V^a W^b.$$

5.2. Spectral Einstein and metric tensor for spectral triples

Let $(\mathcal{A}, D, \mathcal{H})$ be a n -summable spectral triple and $\Omega_D^1(\mathcal{A})$ be the $\mathcal{A}$ bimodule of one-forms generated by $\mathcal{A}$ and $[D, \mathcal{A}]$, which by definition consists of bounded operators on $\mathcal{H}$. We assume that there exists a generalised algebra of pseudodifferential operators which contains $\mathcal{A}$, D , $|D|^\ell$ for $\ell \in \mathbb{Z}$, and there exists a tracial state $\mathcal{W}$ on it, still called a noncommutative residue, which identically vanishes on $T|D|^{-k}$ for any $k > n$ and a zero-order operator T (an operator in the algebra generated by $\mathcal{A}$ and $\Omega_D^1(\mathcal{A})$). We propose the following definition of metric and Einstein functionals for spectral triples.

Definition 5.4. The spectral metric functional on differential forms is

$$g_D(v, w) = \mathcal{W}(vw|D|^{-n}),$$

where $v, w \in \Omega_D^1(\mathcal{A})$, and the Einstein functional is

$$\mathcal{G}_D(v, w) = \mathcal{W}(v\{D, w\}D|D|^{-n}).$$

Note that since $\Omega_D^1(\mathcal{A})$ is a bimodule over $\mathcal{A}$ these functionals are already localised, as one can replace v by av . A simple consequence of the properties of $\mathcal{W}$ on the generalised pseudodifferential calculus for regular spectral triples are the following linearities for the spectral metric functional over one-forms,

$$g_D(vb, w) = g_D(v, bw), \quad g_D(av, w) = g_D(v, wa).$$

However, unlike in the classical case it is not entirely obvious whether any form of linearity over $\mathcal{A}$ in the second entry for the Einstein functional will hold in general. To ensure this property, we propose the following requirement on a class of spectral triples.

Definition 5.5. We say that a spectral triple with a trace on the generalised algebra of pseudodifferential operators is spectrally closed if for any zero-order operator T (as defined earlier) the following holds:

$$\mathcal{W}(TD|D|^{-n}) = 0.$$

Lemma 5.6. *The classical spectral triple over a closed oriented spin-c manifold M of dimension $n = 2m$ is spectrally closed in the above sense.*

Proof. Computing the symbol of $D|D|^{-n} = DD^{-2m}$ using the expansion (4.3) we see that it vanishes identically at $\mathbf{x} = 0$. Since T is an operator of order zero (does not depend on ξ) the product TDD^{-2m} also has a vanishing symbol of order $-n$, hence both the density of the Wodzicki residue and the residue itself vanish identically. $\square$

Theorem 5.7. *If a spectral triple with a noncommutative residue over the generalised pseudodifferential calculus is spectrally closed, then the localised Einstein functional satisfies*

$$\mathcal{G}_D(v\flat, w) = \mathcal{G}_D(v, \flat w),$$

for $\flat \in \mathcal{A}$.

Proof. We compute

$$\begin{aligned} \mathcal{G}_D(v, \flat w) &= \mathcal{W}(v\{D, \flat w\}D|D|^{-n}) \\ &= \mathcal{W}(v([D, \flat]w + \flat\{D, w\})D|D|^{-n}) \\ &= \mathcal{W}(v\flat\{D, w\}D|D|^{-n}) = \mathcal{G}_D(v\flat, w), \end{aligned}$$

since the first term in the second line contains a product of three one-forms and therefore is a 0-th order term, so by the assumption of spectral closedness it vanishes. $\square$

Remark 5.8. Note that in the case of spectral triple over a manifold, since one-forms commute with $\flat$, $\flat \in \mathcal{A}$, the Einstein functional has the same linearities over the algebra as a tensor.

Next, we present two important situations where the assumptions before Definition 5.4 are satisfied. The first is based on the algebra of pseudodifferential operators over noncommutative tori [7,15] and the second on the abstract algebra of pseudodifferential operators for regular n -summable spectral triples [14].

5.2.1. The metric and Einstein functionals for the conformally rescaled spectral triple on a noncommutative tori

We begin with the spectral triple on the noncommutative n -torus, $(\mathcal{A}, \mathcal{H}, D_k)$, where $\mathcal{A}$ is the δ_a -smooth subalgebra and $D_k := kDk$ is the conformal rescaling of the standard (flat) Dirac operator $D = \sum_a \gamma_a \delta_a$ with γ_a as in Section 1.3. As argued in [17] the conformal factor $k > 0$ has to be taken from the copy $\mathcal{A}^o$ of $\mathcal{A}$ in its commutant. The

bimodule of one-forms, generated by the commutators $[D_k, a]$, $a \in \mathcal{A}$, is a free left module generated by $k^2 \gamma^j$. With assumptions as in Section 5.1 using the extension of the calculus in [7] to $\hat{\mathcal{A}}$ -valued symbols and of the analogue of the Wodzicki residue, the calculations are very much similar to the case of functionals for the Laplace operator on noncommutative tori. We provide them explicitly for the two lowest even dimensional cases.

Example 1. Functionals of the conformally rescaled spectral triple on the noncommutative 2-torus.

As argued in [17] it is possible to construct a usual (untwisted) spectral triple with a conformally rescaled Dirac operator, however, the conformal factor $k > 0$ has to be taken from the commutant of the algebra. With $\mathcal{A} = C^\infty(\mathbb{T}_\theta^2)$ and $\mathcal{A}^o$ a copy of $\mathcal{A}$ in the commutant of $\mathcal{A}$, such a spectral triple is given by $(\mathcal{A}, D_k = kDk, \mathcal{H} \otimes \mathbb{C}^2)$, where $D = D_1$ and $k \in \mathcal{A}^o$ is the standard flat Dirac operator,

$$D = \begin{pmatrix} 0 & \delta_2 - i\delta_1 \\ \delta_2 + i\delta_1 & 0 \end{pmatrix}.$$

In fact D_k is an analogue of the classical Dirac operator for the flat metric rescaled conformally and unitarily transformed to act on Hilbert space with the volume measure of the flat metric.

The space of one-forms is a bimodule over $\mathcal{A}$ generated by all commutators $[D_k, a]$, $a \in \mathcal{A}$ and it can be shown that it is a free left module generated by $k^2 \sigma^j$, $j = 1, 2$, where σ^j are Pauli matrices.

First, we show,

Lemma 5.9. *The conformally rescaled spectral triple on the noncommutative 2-torus, $(\mathcal{A}, D_k = kDk, \mathcal{H} \otimes \mathbb{C}^2)$, is spectrally closed.*

Proof. We have

$$\sigma_{-2}(D_k^{-1}) = \sigma_{-2}(k^{-1}D^{-1}k^{-1}) = \sigma^p \left(\frac{\delta_{pq}}{\|\xi\|^2} - 2 \frac{\xi_p \xi_q}{\|\xi\|^4} \right) (-k^{-2} \delta_q k k^{-1}).$$

Integrating it over the $\|\xi\| = 1$ cosphere gives identically zero, therefore the Wodzicki residue of any expression that is a product of a zero-order operator with D_k^{-1} vanishes. $\square$

Next, we have the following.

Proposition 5.10. *For the conformally rescaled spectral triple over the noncommutative 2-torus the metric functional for $v = k^2 V^a \sigma^a$ and $w = k^2 W^a \sigma^a$, $V^a, W^a \in \mathcal{A}$, reads*

$$g_{D_k}(v, w) = \tau(V^a W^a),$$

whereas the spectral Einstein functional vanishes identically,

$$\mathcal{G}_{D_k}(v, w) = 0.$$

Proof. The computation of the metric functional is straightforward. Next, since the spectral triple is spectrally closed, it suffices to compute the spectral Einstein functional over the one-forms that generate the bimodule $\Omega_{D_k}(\mathcal{A})$,

$$\mathcal{G}_{j\ell} = \mathcal{W}(k^2 \sigma^j \{D_k, k^2 \sigma^\ell\} D_k D_k^{-2}).$$

The expression in curly brackets can be rewritten as

$$\sigma^j k^2 [D_k, k^2 \sigma^\ell] k^{-1} D^{-1} k^{-1} + 2 \sigma^j \sigma^\ell k^4.$$

Since the second term is of order zero, we only need to compute the symbol of order -2 of the first term, which can be rewritten as

$$\sigma^j k^3 [D, k^2 \sigma^\ell] D^{-1} k^{-1}.$$

Then, using the tracial property of $\mathcal{W}$ we get

$$\mathcal{G}_{j\ell} = \mathcal{W}(\sigma^j k^2 [D, k^2 \sigma^\ell] D^{-1}) = \mathcal{W}(\sigma^j k^2 [D, k^2] \sigma^\ell D^{-1}) + \mathcal{W}(\sigma^j k^4 [D, \sigma^\ell] D^{-1}),$$

which vanishes since both expressions have a vanishing symbol of order -2 . $\square$

It is rewarding to see that the conformally rescaled spectral triple of the noncommutative 2-torus shares indeed the same property with its commutative counterpart, namely, it has vanishing Einstein tensor.

Example 2. The functionals for the conformally rescaled spectral triple on the noncommutative 4-torus.

The situation is more complicated for the strictly irrational 4-dimensional noncommutative torus. Following the classical situation, given a positive invertible $k \in \mathcal{A}^o$ (which is in the commutant of $\mathcal{A}$), we take $D_k = k D k$ as the conformally rescaled and unitarily transformed to act on $L^2(\mathbb{T}_\theta^4, \tau) \otimes \mathbb{C}^4$, operator D . Let v and w be two arbitrary one-forms in the Clifford algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$,

$$v = k^2 V^a \gamma^a, \quad W = k^2 W^a \gamma^a,$$

where γ^a , $a = 1, \dots, 4$ are the standard gamma matrices, $\{\gamma^a, \gamma^b\} = 2\delta_{ab}$, and V^a, W^a are elements of $\mathcal{A}$ (which, of course, commute with k). First, we establish.

Lemma 5.11. *The conformally rescaled spectral triple for a 4-dimensional noncommutative torus is spectrally closed.*

Proof. The computations show that the integral over the cosphere of the symbol of D_k^{-3} of order -4 is:

$$\int_{||\xi||=1} \sigma_{-4}(D_k^{-3}) d\xi = \frac{i}{2} \gamma^a \left(k^{-2} [k^{-2}, \{k^{-1}, \delta_a k\}] k^{-2} \right). \quad (5.8)$$

It is easy to see that the trace of the above expression vanishes also if multiplied by any element of zero-order. This is so because any element of the algebra of operators of zero order T , contains only products of γ , powers of k and elements from $\mathcal{A}$, and thus it commutes with k . Therefore taking the trace over the algebra gives

$$\mathcal{W}(TD_k^{-3}) = 0. \quad \square$$

Next, we explicitly compute the metric and Einstein tensor functional for v, w .

Proposition 5.12. *The metric and the Einstein functionals for the conformally rescaled spectral triple on the noncommutative 4-torus are, respectively,*

$$\begin{aligned} g_{D_k}(v, w) &= \tau(W^a V^b k^{-4}), \\ \mathcal{G}_{D_k}(v, w) &= \tau \left(V^a W^b \left(\frac{1}{3} k^{-4} (\delta_a k) k^2 (\delta_b k) + \frac{2}{3} k^{-3} (\delta_a k) k^1 (\delta_b k) + k^{-2} (\delta_a k) (\delta_b k) \right. \right. \\ &\quad + \frac{2}{3} k^{-1} (\delta_a k) k^{-1} (\delta_b k) - \frac{4}{3} k (\delta_a k) k^{-3} (\delta_b k) - \frac{2}{3} k^2 (\delta_a k) k^{-4} (\delta_b k) \\ &\quad + \frac{2}{3} k^{-1} (\delta_a \delta_b k) + \delta_{ab} \left(\frac{1}{3} k^{-1} (\delta_c k) k^{-1} (\delta_c k) + \frac{1}{3} k^2 (\delta_c k) k^{-4} (\delta_c k) \right. \\ &\quad \left. \left. + \frac{2}{3} k^1 (\delta_c k) k^{-3} (\delta_c k) - \frac{2}{3} k^{-1} (\Delta k) \right) \right). \end{aligned} \quad (5.9)$$

We skip the proof which consists of a straightforward though tedious computation noting only that to get (5.9) we used the cyclicity of the trace.

Remark 5.13. The above expression has a well-defined commutative limit,

$$\begin{aligned} \mathcal{G}_{D_k}(v, w) &= \frac{1}{\text{vol}(\mathbb{T}^4)} \int_{\mathbb{T}^4} V^a W^b \left(\left(\frac{2}{3} k^{-2} k_a k_b + \frac{2}{3} k^{-1} k_{ab} \right) \right. \\ &\quad \left. + \left(\frac{4}{3} \delta_{ab} k^{-2} k_c k_c - \frac{2}{3} \delta_{ab} k^{-1} k_{cc} \right) \right), \end{aligned} \quad (5.10)$$

which recovers the classical formula for the Einstein tensor for a conformally deformed metric on the 4-torus with the metric $g_{ab} = k^{-4} \delta_{ab}$,

$$G_{ab} = 4(k^{-2} k_a k_b + k^{-1} k_{ab}) + 8\delta_{ab} k^{-2} k_c k_c - 4\delta_{ab} k^{-1} k_{cc}. \quad (5.11)$$

Note that the density of the functional involving the one-forms uses the contravariant Einstein tensor and the volume form, and scales with k like $k^4 k^4 \sqrt{k^{-16}}$ so the density in 4 dimensions scales exactly like the covariant Einstein tensor. The scaling factor comes from the convention that τ is normalised with $\tau(1) = 1$.

5.2.2. The metric and Einstein functionals for a regular finitely summable spectral triple

The second situation, which is different from the conformally rescaled geometries of noncommutative tori, is the case of regular, finitely summable spectral triples (for simplicity with simple dimension spectrum). In this case there is a PDO algebra and calculus of symbols as defined by Connes and Moscovici [14] (see also [34] and [53]) and there exists a tracial state (see [14] Proposition II.1) denoted $\mathscr{W}$ (as before). The definition and methods of computation of the spectral metric and Einstein functionals are, however, quite involved and explicit expressions are feasible only in some special cases of highly symmetric Dirac operators (like the flat Dirac operator on noncommutative tori or fully symmetric Dirac operators on spheres). One can study though some functorial properties, for example the behaviour of the functionals under tensor product of regular spectral triples, of which a particular instance is a finite spectral triple as the second factor. When the first factor is the classical spectral triple (corresponding to spin-c manifolds) this corresponds to *almost commutative geometries* which play a significant role as models for the geometry of physical interactions.

Below we present an example of the metric and Einstein functionals for the tensor product of a regular (not necessarily commutative) spectral triple with a simplest non-trivial spectral triple on two points, expressing them in terms of the functionals on the components.

Example. Functionals for the spectral triple on $\mathscr{A} \otimes \mathbb{C}^2$.

We assume that $(\mathscr{A}, D, \mathscr{H})$ is an even spectral triple with grading γ of dimension n satisfying the assumptions described in section 5.2, which is spectrally closed, and we consider $(\mathscr{A} \otimes \mathbb{C}^2, \mathscr{D}, \mathscr{H} \otimes \mathbb{C}^2)$, where

$$\mathscr{D} = \begin{pmatrix} D & \gamma c \\ \gamma c^* & D \end{pmatrix},$$

where $c \in \mathbb{C}$.

It is easy to see that the bimodule of one-forms, associated to $\mathscr{D}$ consists of the following operators,

$$\omega = \begin{pmatrix} w_+ & \gamma c \phi_+ \\ \gamma c^* \phi_- & w_- \end{pmatrix},$$

where $w_{\pm} \in \Omega_D^1(\mathscr{A})$ and $\phi_{\pm} \in \mathscr{A}$.

We can now state

Proposition 5.14. *Let ω, ω' be two one-forms in the spectral triple $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{D}, \mathcal{H} \otimes \mathbb{C}^2)$. Then,*

$$g_{\mathcal{D}}(\omega, \omega') = g_D(w_+, w'_+) + g_D(w_-, w'_-) + cc^*v(\phi_+\phi'_- + \phi_-\phi'_+),$$

and

$$\begin{aligned} \mathcal{G}_{\mathcal{D}}(\omega, \omega') &= \mathcal{G}_D(w_+, w'_+) + \mathcal{G}_D(w_-, w'_-), \\ &+ cc^*g_D(w_+ - w_-, w'_+ - w'_-) - \frac{n}{2}cc^*(g_D(w_+, w'_+) + g_D(w_-, w'_-)) \\ &+ cc^*(g_D(w_+, d\phi'_+) - g_D(d\phi_+, w'_-) + g_D(w_-, d\phi'_-) - g_D(d\phi_-, w'_+)) \\ &+ (cc^*)^2v((\phi_+ + \phi_-)(\phi'_+ + \phi'_-)), \end{aligned}$$

where $d\phi = [D, \phi]$ for $\phi \in \mathcal{A}$.

We skip the computational proof mentioning only that the condition of being spectrally closed does not need to be preserved in the tensor product with a finite spectral triple.

6. Final remarks

The concept that geometric objects like tensors (metric, torsion and curvature tensors) can be expressed using spectral methods provides an invaluable possibility to study them globally both for the manifolds as well as for various extensions of geometries like noncommutative geometry.

First of all, the metric functionals we constructed should be compared with various other concepts for the metric tensor proposed in the noncommutative realm on the algebraic level. Furthermore, possible relations with the notion of a distance (between states) and quantum metric spaces should be examined. The possibility of studying spectral functionals linked to connections leads to the possibility of defining an abstract notion of torsion and torsion-free connection. This would provide a natural contact with the various concepts of linear connections and possibly then with such notions as the Levi-Civita connection in the noncommutative case, for example [26,49,3–5].

The newly introduced Einstein functional can be further computed for a variety of spectral triples on interesting algebras. This includes, in particular, more general spectral triples for the noncommutative two-tori (cf. [16]) for which we expect the spectral Einstein functional to vanish. This can be further generalised as

Conjecture 6.1. *A regular spectral triple of dimension 2 has a vanishing Einstein spectral functional.*

Even if the regularity assumption should be suitably supplemented such a result will show the robustness of the noncommutative generalisation of manifolds.

Another direction is to follow a spectral definition of a noncommutative Einstein manifold (or, being more precise, an Einstein spectral triple).

Definition 6.2. A spectral triple is called an Einstein spectral triple if the spectral Einstein functional is proportional to the metric functional.

The study of these objects in both the context of almost commutative geometries and applications in mathematical physics can be very interesting. We also hope that the proposed functionals may allow for a global spectral view on the Ricci and curvature tensors themselves and rephrase in the spectral language the notion of flat manifolds. This, together with the Einstein tensor playing a significant role in physics, can prove helpful in applications to quantum field theory and various incarnations of quantum gravity. A possibility of having a direct approach not only to the action functional but also to a version of equations of motion is another appealing direction of studies that could possibly be linked to a more general variational formula.

Yet another direction to investigate might be connected with orbifolds and other singular generalisations of a Riemannian manifold, to see for example how our spectral functionals apprehend the singularities. Also their extension to the case of a twisted algebra of pseudodifferential operators as in [44] is an appealing task.

Another interesting problem remains the equivalence of the algebra of pseudodifferential operators ([7] and [14]) for the noncommutative tori with various Dirac operators. Comparison of the functionals discussed in this work can shed some light on this open problem.

Appendix A. Algebra of symbols of pseudodifferential operators

Suppose that P and Q are two pseudodifferential operators with symbols,

$$\sigma(P)(x, \xi) = \sum_{\alpha} \sigma(P)_{\alpha}(x) \xi^{\alpha}, \quad \sigma(Q)(x, \xi) = \sum_{\beta} \sigma(Q)_{\beta}(x) \xi^{\beta}, \quad (\text{A.1})$$

respectively, where α, β are multiindices. The composition rule for the symbols of their product takes the form [27].

$$\sigma(PQ)(x, \xi) = \sum_{\beta} \frac{(-i)^{|\beta|}}{|\beta|!} \partial_{\beta}^{\xi} \sigma(P)(x, \xi) \partial_{\beta} \sigma(Q)(x, \xi), \quad (\text{A.2})$$

where ∂_a^{ξ} denotes the partial derivative with respect to the coordinate of the cotangent bundle.

We start with computation of the three leading coefficients of the symbols of P^{-1} (we assume that the kernel of P is finite dimensional and can be neglected in the following) for a second-order differential operator P , with the symbol expansion,

$$\sigma(P)(x, \xi) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0. \quad (\text{A.3})$$

The inverse is a pseudodifferential operator P^{-1} , with a symbol of the form,

$$\sigma(P^{-1})(x, \xi) = \mathfrak{b}_2 + \mathfrak{b}_3 + \mathfrak{b}_4 + \dots, \quad (\text{A.4})$$

where $\mathfrak{b}_k$ is homogeneous in ξ of order $-k$. Inserting these expressions into (A.2) and taking homogeneous parts of order 0, -1 and -2 we get the following set of equations:

$$\begin{aligned} \mathfrak{a}_2 \mathfrak{b}_2 &= 1, \\ \mathfrak{a}_1 \mathfrak{b}_2 + \mathfrak{a}_2 \mathfrak{b}_3 - i \partial_a^\xi(\mathfrak{a}_2) \partial_a(\mathfrak{b}_2) &= 0, \\ \mathfrak{a}_2 \mathfrak{b}_4 + \mathfrak{a}_1 \mathfrak{b}_3 + \mathfrak{a}_0 \mathfrak{b}_2 - i \partial_a^\xi(\mathfrak{a}_1) \partial_a(\mathfrak{b}_2) \\ &\quad - i \partial_a^\xi(\mathfrak{a}_2) \partial_a(\mathfrak{b}_3) - \frac{1}{2} \partial_a^\xi \partial_b^\xi(\mathfrak{a}_2) \partial_a \partial_b(\mathfrak{b}_2) = 0, \end{aligned} \quad (\text{A.5})$$

which we solve recursively, obtaining

$$\begin{aligned} \mathfrak{b}_2 &= \mathfrak{a}_2^{-1}, \\ \mathfrak{b}_3 &= -\mathfrak{b}_2 \left(\mathfrak{a}_1 \mathfrak{b}_2 - i \partial_a^\xi(\mathfrak{a}_2) \partial_a(\mathfrak{b}_2) \right), \\ \mathfrak{b}_4 &= -\mathfrak{b}_2 \left(\mathfrak{a}_1 \mathfrak{b}_3 + \mathfrak{a}_0 \mathfrak{b}_2 - i \partial_a^\xi(\mathfrak{a}_1) \partial_a(\mathfrak{b}_2) \right. \\ &\quad \left. - i \partial_a^\xi(\mathfrak{a}_2) \partial_a(\mathfrak{b}_3) - \frac{1}{2} \partial_a^\xi \partial_b^\xi(\mathfrak{a}_2) \partial_a \partial_b(\mathfrak{b}_2) \right). \end{aligned} \quad (\text{A.6})$$

Next, we show a technical lemma that allows us to compute the symbol of the higher inverse power of the Laplace operator. This is a straightforward application of the iterated formula (A.2) and we include it only for completeness. To shorten the notation, we shall denote the derivatives with respect to the coordinates on the manifold by δ and use still ∂ for the partial derivatives with respect to the coordinates on the cotangent bundle. Further, we shorten the notation of the derivatives applied to a certain element of the product, which would be valid for all derivatives (both ∂, δ), for example:

$$\partial_a^{[k]}(y_1 y_2 \cdots y_n) = y_1 y_2 \cdots \partial_a(y_k) \cdots y_n.$$

Lemma A.1. *Given the first three leading symbols of an operator P (with the principal symbol being of order $-k$),*

$$\sigma(P) = \mathfrak{p}_k + \mathfrak{p}_{k+1} + \mathfrak{p}_{k+2} + \cdots,$$

we can express the first three leading symbols of $R = P^l$,

$$\sigma(R) = \mathfrak{r}_{lk} + \mathfrak{r}_{lk+1} + \mathfrak{r}_{lk+2} + \cdots,$$

in terms of $\mathfrak{p}_k, \mathfrak{p}_{k+1}, \mathfrak{p}_{k+2}$, as:

$$\begin{aligned} \mathfrak{r}_{lk} &= (\mathfrak{p}_k)^l, \\ \mathfrak{r}_{lk+1} &= \sum_{j=1}^l (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-j} - i \sum_{1 \leq j < w \leq l} \partial_a^{[j]} \delta_a^{[w]} (\mathfrak{p}_k)^l, \\ \mathfrak{r}_{lk+2} &= \sum_{j=1}^l (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+2} (\mathfrak{p}_k)^{l-j} \\ &\quad + \sum_{1 \leq j < s \leq l} (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{s-j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-s}, \\ &\quad - i \sum_{s=1}^l \sum_{1 \leq j < p \leq l} \partial_a^{[j]} \delta_a^{[p]} (\mathfrak{p}_k)^{s-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-s}, \\ &\quad - \frac{1}{2} \sum_{1 \leq j < p \leq l} \sum_{1 \leq r < s \leq l} \partial_b^{[r]} \delta_b^{[s]} \partial_a^{[j]} \delta_a^{[p]} (\mathfrak{p}_k)^l, \end{aligned} \tag{A.7}$$

Proof. We proceed by induction in l . The claim is obvious for $l = 1$. Assume now that it holds for $l - 1$, and consider $R = P^l = PP^{l-1}$. By (A.2) we can see, that

$$\begin{aligned} \mathfrak{r}_{lk} &= \mathfrak{p}_k \mathfrak{r}_{(l-1)k}, \\ \mathfrak{r}_{lk+1} &= \mathfrak{p}_k \mathfrak{r}_{(l-1)k+1} + \mathfrak{p}_{k+1} \mathfrak{r}_{(l-1)k} - i \partial_a \mathfrak{p}_k \delta_a \mathfrak{r}_{(l-1)k}, \\ \mathfrak{r}_{lk+2} &= \mathfrak{p}_k \mathfrak{r}_{(l-1)k+2} + \mathfrak{p}_{k+1} \mathfrak{r}_{(l-1)k+1} + \mathfrak{p}_{k+2} \mathfrak{r}_{(l-1)k} \\ &\quad - i \partial_a \mathfrak{p}_k \delta_a \mathfrak{r}_{(l-1)k+1} - i \partial_a \mathfrak{p}_{k+1} \delta_a \mathfrak{r}_{(l-1)k} - \frac{1}{2} \partial_a \partial_b \mathfrak{p}_k \delta_a \delta_b \mathfrak{r}_{(l-1)k}. \end{aligned} \tag{A.8}$$

The formula for $\mathfrak{r}_{lk}$ is obvious. Consider now the expression for $\mathfrak{r}_{lk+1}$ and assume that the formula holds for $l - 1$, then

$$\begin{aligned} \mathfrak{r}_{lk+1} &= \mathfrak{p}_k \left(\sum_{j=1}^{l-1} (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-1-j} - i \sum_{1 \leq j < s \leq l-1} \partial_a^{[j]} \delta_a^{[s]} (\mathfrak{p}_k)^{l-1} \right) \\ &\quad + \mathfrak{p}_{k+1} \mathfrak{p}_k^{l-1} - i \partial_a \mathfrak{p}_k \delta_a \mathfrak{p}_k^{l-1} \\ &= \left(\mathfrak{p}_k \sum_{j=1}^{l-1} (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-1-j} + \mathfrak{p}_{k+1} \mathfrak{p}_k^{l-1} \right) \end{aligned}$$

$$\begin{aligned}
& -i \left(\mathfrak{p}_k \sum_{1 \leq j < s \leq l-1} \partial_a^{[j]} \delta_a^{[s]} (\mathfrak{p}_k)^{l-1} + \partial_a \mathfrak{p}_k \delta_a \mathfrak{p}_k^{l-1} \right) \\
& = \sum_{j=1}^l (\mathfrak{p}_k)^{j-1} \mathfrak{p}_{k+1} (\mathfrak{p}_k)^{l-j} - i \sum_{1 \leq j < s \leq l} \partial_a^{[j]} \delta_a^{[s]} (\mathfrak{p}_k)^l,
\end{aligned}$$

where we have used only the reordering of terms and the Leibniz rule for δ_a :

$$\delta_a (\mathfrak{p}_k)^{l-1} = \sum_{s=1}^{l-1} \delta_a^{[s]} (\mathfrak{p}_k) l - 1.$$

The formula for $\mathfrak{r}_{lk+2}$ can be proved in a similar way. We can split the product in (A.8) into the sum with no derivatives, only first-order derivatives, and finally second-order derivatives. We skip the proof for the first two parts and illustrate only the last part. The component of $\mathfrak{r}_{lk+2}$ with second-order derivatives (denoted $\mathfrak{r}_{lk+2}(2)$) will come from the following contributions.

$$\mathfrak{r}_{lk+2}(2) = \mathfrak{p}_k \mathfrak{r}_{(l-1)k+2}(2) - i \partial_a \mathfrak{p}_k \delta_a \mathfrak{r}_{(l-1)k+1}(1) - \frac{1}{2} \partial_a \partial_b \mathfrak{p}_k \delta_a \delta_b \mathfrak{r}_{(l-1)k} = \dots \quad (\text{A.9})$$

which, after assuming the validity for $l-1$ gives,

$$\begin{aligned}
& \dots = \mathfrak{p}_k \left(-\frac{1}{2} \sum_{1 \leq j < p \leq l} \sum_{1 \leq r < s \leq n} \partial_b^{[r]} \delta_b^{[s]} \partial_a^{[j]} \delta_a^{[p]} (\mathfrak{p}_k)^l \right) \\
& - i \partial_b \mathfrak{p}_k \delta_b \left(-i \sum_{1 \leq j < w \leq l} \partial_a^{[j]} \delta_a^{[w]} (\mathfrak{p}_k)^l \right) \\
& - \frac{1}{2} \partial_a \partial_b \mathfrak{p}_k \delta_a \delta_b \mathfrak{p}_k^{l-1}.
\end{aligned} \quad (\text{A.10})$$

Again, using the Leibniz rule, we see that it is indeed the term from (A.7) with second-order derivatives split into the parts that first $\mathfrak{p}_k$ has no derivatives acted upon, one derivative, and two derivatives. $\square$

Corollary A.2. *In the special case of scalar symbols (a sufficient condition is that symbol $\mathfrak{p}_k$ is scalar) we have (with the same notation as above),*

$$\begin{aligned}
\mathfrak{r}_{lk} &= (\mathfrak{p}_k)^l, \\
\mathfrak{r}_{lk+1} &= l(\mathfrak{p}_k)^{l-1}\mathfrak{p}_{k+1} - i\frac{l(l-1)}{2}(\mathfrak{p}_k)^{l-2}\partial_a(\mathfrak{p}_k)\delta_a(\mathfrak{p}_k), \\
\mathfrak{r}_{lk+2} &= l(\mathfrak{p}_k)^{l-1}\mathfrak{p}_{k+2} + \frac{l(l-1)}{2}(\mathfrak{p}_k)^{l-2}(\mathfrak{p}_{k+1})^2 \\
&\quad - i\frac{l(l-1)}{2}(\mathfrak{p}_k)^{l-3}\left[\mathfrak{p}_k\left(\partial_a(\mathfrak{p}_{k+1})\delta_a(\mathfrak{p}_k) + \partial_a(\mathfrak{p}_k)\delta_a(\mathfrak{p}_{k+1})\right) \right. \\
&\quad \left. + (l-2)\mathfrak{p}_{k+1}\partial_a(\mathfrak{p}_k)\delta_a(\mathfrak{p}_k)\right] \\
&\quad - \frac{l(l-1)}{24}(\mathfrak{p}_k)^{l-4}\left(6(\mathfrak{p}_k)^2\partial_a\partial_b(\mathfrak{p}_k)\delta_a\delta_b(\mathfrak{p}_k) \right. \\
&\quad + 3(l-2)(l-3)\partial_a(\mathfrak{p}_k)\partial_b(\mathfrak{p}_k)\delta_a(\mathfrak{p}_k)\delta_b(\mathfrak{p}_k) \\
&\quad + 4(l-2)\mathfrak{p}_k\left[\partial_a(\mathfrak{p}_k)\partial_b(\mathfrak{p}_k)\delta_a\delta_b(\mathfrak{p}_k) \right. \\
&\quad \left. \left. + \partial_a(\mathfrak{p}_k)\partial_b\delta_a(\mathfrak{p}_k)\delta_b(\mathfrak{p}_k) + \partial_a\partial_b(\mathfrak{p}_k)\delta_a(\mathfrak{p}_k)\delta_b(\mathfrak{p}_k)\right]\right). \tag{A.11}
\end{aligned}$$

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2.1.1 Remarks

As this was the first article in the series, we learned some new facts about the functionals and we can now better understand our results. I present here some remarks and results that extend these in [P1].

Symmetry of functionals

In the commutative scenario both metric and Einstein functionals are balanced, i.e. $\mathcal{G}_D(u, av) = \mathcal{G}_D(au, v)$ for every differential 1-forms u, v and function a ¹. In the noncommutative case we indeed have $\mathcal{G}_D(u, av) = \mathcal{G}_D(ua, v)$, although in general this is not true for $\mathcal{G}_D$. The proof of theorem 5.7 shows that the Einstein functional is balanced in this sense iff $Wres(u[D, a]wD|D|^{-n}) = 0$ for all forms u, v and all $a \in \mathcal{A}$. We guarantee it by assuming that the spectral triple is spectrally closed (definition 5.5), but, in fact, this assumption is stronger than we need. Later, after introducing the torsion functional in [P3], we observe, that $Wres(u[D, a]wD|D|^{-n}) = \mathcal{T}_D(u, [D, a], w)$, as $[D, a]$ is also a differential 1-form. Thus, the Einstein functional is balanced iff the torsion functional identically vanishes. Even in the commutative case, if we use a Dirac-type operator with torsion, the resulting Einstein functional will no longer be C^∞ -linear in both entries.

Powers of pseudodifferential operators

In corollary A.2 we are focused on operators P , $R = P^l$ and their three leading symbols. The corollary is a special (commutative) case of lemma A.1, where l is a positive integer. It turns out that corollary A.2 holds for all integers l . We can prove that using the following formulas for $P = O^{-1}$. If the three leading symbols of O are $\mathfrak{o}_k$, $\mathfrak{o}_{k-1}$, and $\mathfrak{o}_{k-2}$, and $\mathfrak{o}_k$ is scalar, then adopting the same convention for derivatives ($\partial_a^\xi \equiv \partial_a$, $\partial_a^x \equiv \delta_a$) we have

$$\begin{aligned} \mathfrak{p}_k &= \mathfrak{o}_k^{-1}, \\ \mathfrak{p}_{k+1} &= \mathfrak{p}_k(-\mathfrak{o}_{k-1}\mathfrak{p}_k + i\partial_a\mathfrak{o}_k\delta_a\mathfrak{p}_k) = -\mathfrak{o}_k^{-2}\mathfrak{o}_{k-1} - i\mathfrak{o}_k^{-3}\partial_a\mathfrak{o}_k\delta_a\mathfrak{o}_k, \\ \mathfrak{p}_{k+2} &= \mathfrak{p}_k\left(-\mathfrak{o}_{k-1}\mathfrak{p}_{k+1} - \mathfrak{o}_{k-2}\mathfrak{p}_k + i\partial_a\mathfrak{o}_k\delta_a\mathfrak{p}_{k+1} + i\partial_a\mathfrak{o}_{k-1}\delta_a\mathfrak{p}_k + \frac{1}{2}\partial_{ab}\mathfrak{o}_k\delta_{ab}\mathfrak{p}_k\right). \end{aligned} \quad (2.3)$$

Before simplifying $\mathfrak{p}_{k+2}$, we can write two helpful terms:

$$\begin{aligned} \delta_a\mathfrak{p}_{k+1} &= -\mathfrak{o}_k^{-2}\delta_a\mathfrak{o}_{k-1} + 2\mathfrak{o}_k^{-3}\mathfrak{o}_{k-1}\delta_a\mathfrak{o}_k - i\mathfrak{o}_k^{-3}\delta_a\partial_b\mathfrak{o}_k\delta_b\mathfrak{o}_k \\ &\quad - i\mathfrak{o}_k^{-3}\partial_b\mathfrak{o}_k\delta_{ab}\mathfrak{o}_k + 3i\mathfrak{o}_k^{-4}\partial_b\mathfrak{o}_k\delta_a\mathfrak{o}_k\delta_b\mathfrak{o}_k, \\ \delta_{ab}\mathfrak{p}_k &= -\mathfrak{o}_k^{-2}\delta_{ab}\mathfrak{o}_k + 2\mathfrak{o}_k^{-3}\delta_a\mathfrak{o}_k\delta_b\mathfrak{o}_k. \end{aligned} \quad (2.4)$$

Now, we can write

$$\begin{aligned} \mathfrak{p}_{k+2} &= -\mathfrak{o}_k^{-2}\mathfrak{o}_{k-2} + \mathfrak{o}_k^{-3}\mathfrak{o}_{k-1}^2 - i\mathfrak{o}_k^{-3}(\partial_a\mathfrak{o}_{k-1}\delta_a\mathfrak{o}_k + \partial_a\mathfrak{o}_k\delta_a\mathfrak{o}_{k-1}) \\ &\quad + 3i\mathfrak{o}_k^{-4}\mathfrak{o}_{k-1}\partial_a\mathfrak{o}_k\delta_a\mathfrak{o}_k - \frac{1}{2}\mathfrak{o}_k^{-3}\partial_{ab}\mathfrak{o}_k\delta_{ab}\mathfrak{o}_k - 3\mathfrak{o}_k^{-5}\partial_a\mathfrak{o}_k\partial_b\mathfrak{o}_k\delta_a\mathfrak{o}_k\delta_b\mathfrak{o}_k \\ &\quad + \mathfrak{o}_k^{-4}(\partial_a\mathfrak{o}_k\partial_b\mathfrak{o}_k\delta_{ab}\mathfrak{o}_k + \partial_a\mathfrak{o}_k\partial_b\delta_a\mathfrak{o}_k\delta_b\mathfrak{o}_k + \partial_{ab}\mathfrak{o}_k\delta_a\mathfrak{o}_k\delta_b\mathfrak{o}_k). \end{aligned} \quad (2.5)$$

¹Analogously for metric functional, and the same functionals with vector fields and Laplace-type operators.

Next, we have to use corollary A.2 for such P and a positive l , and compare the results to those obtained by applying the same equations to O with l substituted by $-l$. Results for $\mathfrak{r}_{lk}$ are trivially in agreement. For $\mathfrak{r}_{lk+1}$ we get

$$\begin{aligned}\mathfrak{r}_{lk+1} &= l\mathfrak{o}_k^{-(l-1)}(-\mathfrak{o}_k^{-2}\mathfrak{o}_{k-1} - i\mathfrak{o}_k^{-3}\partial_a\mathfrak{o}_k\delta_a\mathfrak{o}_k) - i\frac{l(l-1)}{2}\mathfrak{o}_k^{-(l-2)}(-\mathfrak{o}_k^{-2}\partial_a\mathfrak{o}_k)(-\mathfrak{o}_k^{-2}\delta_a\mathfrak{o}_k) \\ &= -l\mathfrak{o}_k^{-l-1}\mathfrak{o}_{k-1} - i\frac{l(l+1)}{2}\mathfrak{o}_k^{-l-2}\partial_a\mathfrak{o}_k\delta_a\mathfrak{o}_k.\end{aligned}\tag{2.6}$$

The result is also consistent with the formula's direct application to O , as $l(l+1) = -l(-l-1)$. For $\mathfrak{r}_{lk+2}$ we also have the same result. We skip the computations, as they are similar to the ones above. There are ten types of terms which should be treated separately in comparison to only two in $\mathfrak{r}_{lk+1}$, which makes the computation long, but it is still the same idea, i.e. expressing $\mathfrak{p}_j$ in terms of $\mathfrak{o}_j$ and collecting similar terms.

This generalization of corollary A.2 can be used to reduce the steps involved in obtaining the negative powers of a laplace-type operators, which is a crucial step in calculation of each spectral functional we introduced.

2.2 Spectral metric and Einstein functionals for Hodge-Dirac operator

One of the most interesting questions that followed was if the results from [P1] hold when applied to the second natural choice of Dirac-type operator, namely the Hodge-Dirac operator $D = d + d^*$ defined on the space of differential forms, with d^* being the adjoint of the exterior derivative d . The main result of [P2] is that the metric and Einstein functionals defined by Hodge-Dirac operator remain the same as for the classical Dirac operator on the same manifold, up to a numerical constant, i.e. the dimension of the Hilbert space (spinors vs. differential forms).

It is also worth noting, that the Hodge-Dirac operator is spectrally closed, as defined in [P1], and hence torsion-free.

Algebra of $\lambda_{\pm}^k$ matrices

In the construction of the Hodge-Dirac operator in [P2], we introduce operators $\lambda_{\pm}^j$ that raise and lower the degree of differential forms. Because their definition may not be fully apparent in the article, it will be explained here. These operators are defined in the local coordinate system by their actions on the basis $dx^J = dx^{j_1} \wedge \dots \wedge dx^{j_l}$, $j_1 < \dots < j_l$:

$$\lambda_+^k dx^J = dx^k \wedge dx^J = \epsilon_K^{kJ} dx^K, \quad \lambda_-^k dx^J = \epsilon_{kK}^J dx^K,\tag{2.7}$$

where ϵ_B^A for some multiindices A, B is equal to $(-1)^{|\pi|}$ if $B = \pi(A)$ for some permutation π or 0 otherwise.

The action of λ_-^k operator on a form arises from the evaluation of this form on a vector ∂_k in its first entry. Thus, if we should keep track of the position of indices, we should write λ_k^- , although it is irrelevant in our situation. Given $\lambda_{\pm}^j$, we may define $\gamma^j = -i(\lambda_+^j - \lambda_-^j)$, which behaves like (flat²) Dirac gamma matrices.

²If we want $\{\gamma^j, \gamma^k\} = 2g^{jk}$ instead of $2\delta^{jk}$, we should take $-i(\lambda_+^j - g^{jk}\lambda_-^k)$.

SPECTRAL METRIC AND EINSTEIN FUNCTIONALS FOR THE HODGE-DIRAC OPERATOR

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ABSTRACT. We examine the metric and Einstein bilinear functionals of differential forms introduced in [8], for the Hodge-Dirac operator $d + \delta$ on an oriented, closed, even-dimensional Riemannian manifold. We show that they are equal (up to a numerical factor) to these functionals for the canonical Dirac operator on a spin manifold. Furthermore, we demonstrate that the spectral triple for the Hodge-Dirac operator is spectrally closed, which implies that it is torsion-free.

Keywords: *Noncommutative geometry, Einstein tensor, spectral geometry, Wodzicki residue.*

1. INTRODUCTION

Spectral geometry investigates relationships between geometric structures of manifolds and the spectra of certain differential operators. Its direct and inverse problems are inextricably linked to other areas of mathematics such as number theory, representation theory, and areas of mathematical physics such as quantum mechanics and general relativity. In this regard, starting with the Laplace-Beltrami operator on a closed Riemannian manifold, general Laplace-type operators have been extensively studied, and their spectra provide insights into the geometry and topology of the underlying space. The distribution of eigenvalues, for example, reveals information about the curvature or shape and global geometric properties such as diameter or volume, connectivity, or the presence of holes. In this vein, the Dirac-type operators have also been studied, beginning with the canonical Dirac operator on the spin manifold. When subsumed into Connes' concept of spectral triples [1, 2], they "can hear the shape of a drum" [14] in the sense that their equivalence (a suitably strengthened isospectrality) implies the isometricity of manifolds in virtue of the reconstruction theorem [4]. Furthermore, they allow for broad and captivating generalisations in noncommutative geometry.

Various (interrelated) spectral schemes that generate geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives are the small-time asymptotic expansion of the (localised) trace of the heat kernel [11, 12], certain values or residues of the (localised) zeta function of the Laplacian, the spectral action, and the Wodzicki residue $\mathscr{W}$ (also known as noncommutative residue). In this paper, we focus on the latter one, which is the unique (up to multiplication by a constant) tracial state on the algebra of pseudo-differential operators (Ψ DO) on a complex vector bundle E over a compact manifold M of dimension $n \geq 2$ [13, 19]. For the oriented manifold M it is given by an integral

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formula,

$$(1.1) \quad \mathcal{W}(P) := \int_M w(P)$$

where the density $w(P)$ is given in local coordinates by

$$(1.2) \quad \int_{|\xi|=1} \text{tr } \sigma_{-n}(P)(x, \xi) \mathcal{V}_\xi d^n x.$$

Here tr is the trace over endomorphisms of the bundle E at any given point of M , $\sigma_{-n}(P)$ is the symbol of order $-n$ of a pseudodifferential operator P and $\mathcal{V}_\xi$ denotes the volume form on the unit co-sphere.

When applied to the (scalar) Laplacian Δ on a Riemannian manifold M of dimension $n = 2m$ equipped with a metric tensor g it yields, in a *localized* form, a functional of $f \in C^\infty(M)$,

$$(1.3) \quad v(f) := \mathcal{W}(f\Delta^{-m}) = v_{n-1} \int_M f \text{vol}_g,$$

where

$$v_{n-1} := \text{vol}(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)},$$

is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

A startling result regarding a higher power of the Laplacian presented by Connes [3] in the early 1990s (see [15, 16] for explicit computations) states that

$$(1.4) \quad \mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M f R(g) \text{vol}_g,$$

which for $n > 2$ and $f = 1$ is, up to a constant, a Riemannian analogue of the Einstein-Hilbert action functional of general relativity in vacuum. Here $R = R(g)$ is the scalar curvature, that is the g -trace $R = g^{jk} R_{jk}$ of the Ricci tensor with components R_{jk} in local coordinates, where g^{jk} are the raised components of the metric g .

In the noncommutative realm, the spectral-theoretic approach to scalar curvature has been extended to quantum tori in the seminal work of Connes and Tretkoff [6] and extensively studied by many authors (see references in [8]).

In the recent paper [8] we accomplished the task of extracting two other important tensorial geometrical objects through spectral methods. These were the metric tensor g itself, its dual, and the Einstein tensor

$$G := \text{Ric} - \frac{1}{2} R(g) g,$$

which directly enters the Einstein field equations with matter and its dual. In fact, for this purpose, we employed the Wodzicki residue of a suitable power of the Laplace type operator or of the Dirac type operator, multiplied by a pair of other differential operators. Notably, we have recovered the tensors g and G as the density of certain bilinear functionals of vector fields on a manifold M , while their dual tensors are the density of bilinear functionals of differential one-forms on M . The latter functionals (up to a numerical factor) we have obtained also for the canonical Dirac operators (in case M is a spin manifold). Then, using Connes' and Moscovici's [5] generalisation of pseudodifferential calculus for noncommutative spectral triples, we introduced their conspicuous quantum analogue and probed it on 2 and 4-dimensional noncommutative tori.

The aim of this paper is, employing methods of the Wodzicki residue, to analyse the metric and Einstein functionals for another natural Dirac-type operator, namely the Dirac-Hodge operator $d+\delta$ acting on (complex) differential forms $\Omega(M)$ of arbitrary order on a oriented even-dimensional Riemannian manifold M . It is worth mentioning that the associated Hodge-Dirac spectral triple is characterised [7] by the fact that its dense Hilbert subspace of continuous forms provides a Morita equivalence $Cl(M) - Cl(M)$ bimodule, where $Cl(M)$ is the C^* -algebra of continuous sections of the bundle of Clifford algebras on M . As is well known, the canonical spectral triple on a spin manifold is instead characterised by the fact that its dense Hilbert subspace of continuous Dirac spinors provides a Morita equivalence $Cl(M) - C(M)$ bimodule, where $C(M)$ is the algebra of continuous complex functions on M . As our first main result, we demonstrate that these two different pivotal cases yield in fact equal spectral metric and Einstein functionals (up to a numerical factor). Moreover, as our second main result, we prove that the associated spectral triple is spectrally closed, that is, for any operator T of zero-order,

$$\mathcal{W}(TD|D|^{-n}) \equiv 0.$$

A forthcoming result [9] demonstrates that, as a consequence, the Hodge-Dirac operator has no torsion.

2. PRELIMINARIES

Let $n = 2k$ be the dimension of an oriented, closed, smooth Riemannian manifold M . We will use capital letters to denote increasing sequences of numbers between 1 and n , of fixed length $0 \leq \ell \leq n$. A differential ℓ -form $\omega = \sum_J \omega_J dx^J$ is determined by its coefficients ω_J , with respect to coordinates indicated by the multi-index J , where with a slight abuse of notation 0-forms (i.e. functions) will correspond to $J = \emptyset$.

We introduce the operators λ_+^j and λ_-^j which respectively raise/lower the degree of forms, with components given by

$$(\lambda_+^p)_J^I = \epsilon_{pJ}^I, \quad (\lambda_-^p)_J^I = \epsilon_J^{pI},$$

where $\epsilon_{pJ}^I = (-)^{|\pi|}$ if the juxtaposed index pJ is a permutation π of I and $\epsilon_{pJ}^I = 0$ otherwise, and similarly for ϵ_J^{pI} . They satisfy

$$\begin{aligned} \lambda_+^p \lambda_+^r + \lambda_+^r \lambda_+^p &= 0, \\ \lambda_-^p \lambda_-^r + \lambda_-^r \lambda_-^p &= 0, \\ \lambda_+^p \lambda_-^r + \lambda_-^r \lambda_+^p &= \delta_{pr} \text{id}, \end{aligned} \tag{2.1}$$

which follow from the relations (c.f. [18])

$$\begin{aligned} \sum_K \epsilon_{pK}^I \epsilon_{rJ}^K &= \epsilon_{prJ}^I, & \sum_K \epsilon_K^{pI} \epsilon_J^{rK} &= \epsilon_{rpJ}^I, \\ \sum_K \epsilon_{pK}^I \epsilon_J^{rK} &= \delta_{pr} \epsilon_J^I - \epsilon_{pJ}^{rI}, & \sum_K \epsilon_K^{rI} \epsilon_{pJ}^K &= \epsilon_{pJ}^{rI}, \end{aligned} \tag{2.2}$$

where the juxtaposed indices can be ordered using a signed permutation. We also introduce

$$\gamma^p = -i(\lambda_+^p - \lambda_-^p),$$

which satisfy the following Clifford algebra relation

$$\{\gamma^p, \gamma^r\} = 2\delta_{pr}.$$

In the rest of the paper, we employ normal coordinates x centred around some fixed point on the manifold. Recall that then the components of the metric tensor g , its covariant (raised) components, and the square root of the determinant of the matrix of the components of g and the components of the Christoffel symbols of the Levi-Civita connection have the following Taylor expansion around $x = 0$:

$$(2.3) \quad \begin{aligned} g_{ab} &= \delta_{ab} - \frac{1}{3}R_{acbd}x^cx^d + o(\mathbf{x}^2), \\ g^{ab} &= \delta^{ab} + \frac{1}{3}R_{acbd}x^cx^d + o(\mathbf{x}^2), \\ \sqrt{\det(g)} &= 1 - \frac{1}{6}\text{Ric}_{ab}x^ax^b + o(\mathbf{x}^2), \\ \Gamma_{bc}^a &= -\frac{1}{3}(R_{abcd} + R_{acbd})x^d + o(\mathbf{x}^2). \end{aligned}$$

Here R_{acbd} and Ric_{ab} are the components of the Riemann and Ricci tensors, respectively, at the point $x = 0$ and we use the notation $o(\mathbf{x}^k)$ to denote that we expand a function up to the polynomial of order k in the normal coordinates. The expansion in normal coordinates is more convenient notation to obtain the value of relevant quantities (symbols of operators) and their derivatives at a given point on the manifold, which we need to compute the products of symbols according to the rules of multiplication of symbols (A.2). The position of indices for $\lambda_{\pm}^p$, coordinates x and ξ , as well as Riemann and Ricci tensors at the chosen point is chosen for simplicity only (as we are using normal coordinates and the metric at this point is δ_{ab}). We use Einstein summation convention for repeated indices (independently of their position), by $\{ , \}$ we denote anticommutators.

2.1. Hodge-Dirac operator. We focus on the Hodge-Dirac operator $D = d + d^*$, where d is the exterior derivative and d^* is its (formal) adjoint. Using our notation, we compute (locally) the symbol of D ,

$$(2.4) \quad \sigma(D) = (i\lambda_+^p - ig^{pr}\lambda_-^r)\xi_p + \lambda_-^p\lambda_+^r\lambda_-^s\Gamma_{rt}^sg^{pt},$$

which in normal coordinates takes form

$$(2.5) \quad \sigma(D) = -\gamma^p\xi_p - \frac{1}{3}i\lambda_-^pR_{sapb}x^ax^b\xi_s - \frac{1}{3}\lambda_-^p\lambda_+^r\lambda_-^s(R_{srpa} + R_{spr a})x^a + o(\mathbf{x}^2).$$

We compute then the symbols of the Hodge-Dirac Laplacian D^2 in normal coordinates up to orders relevant for our purposes.

Lemma 2.1. *The three homogeneous symbols of D^2 read*

$$\begin{aligned} \mathfrak{a}_2 &= (\delta_{ab} + \frac{1}{3}R_{acbd}x^cx^d)\xi_a\xi_b + o(\mathbf{x}^2), \\ \mathfrak{a}_1 &= +\frac{2}{3}i\text{Ric}_{ab}\xi_ax^b - \frac{2}{3}i\lambda_+^p\lambda_-^r(R_{rpab} + R_{rapb})x^b\xi_a + o(\mathbf{x}^1), \\ \mathfrak{a}_0 &= +\frac{2}{3}\lambda_+^p\lambda_-^r\text{Ric}_{pr} + \frac{1}{3}\lambda_+^p\lambda_+^r\lambda_-^s\lambda_-^t(R_{tsrp} + R_{trsp}) + o(\mathbf{x}^0). \end{aligned}$$

Proof. The computation of the principal symbol $\mathfrak{a}_2$ is obvious for the symbol of order 1:

$$\begin{aligned}
\mathfrak{a}_1 &= -\frac{1}{3}i\{\lambda_+^t, \lambda_-^p \lambda_+^r \lambda_-^s\}(R_{srpa} + R_{spr a})x^a \xi_t + \frac{1}{3}i\{\lambda_-^t, \lambda_-^p \lambda_+^r \lambda_-^s\}(R_{srpa} + R_{spr a})x^a \xi_t \\
&\quad + \frac{1}{3}\gamma^p \lambda_-^r (R_{aprb} + R_{abrp})x^b \xi_a \\
&= -\frac{1}{3}i\lambda_+^r \lambda_-^s (R_{srta} + R_{stra})x^a \xi_t - \frac{1}{3}i\lambda_-^p \lambda_+^r (R_{trpa} + R_{tpra})x^a \xi_t \\
&\quad - \frac{1}{3}i\lambda_-^p \lambda_-^s (R_{stpa} + R_{spta})x^d \xi_t - \frac{1}{3}i\lambda_+^p \lambda_-^s (R_{tpsa} + R_{tas p})x^a \xi_t \\
&\quad + \frac{1}{3}i\lambda_-^p \lambda_-^s (R_{tpsa} + R_{tas p})x^a \xi_t \\
&= \frac{2}{3}iRic_{sa}x^a \xi_s - \frac{1}{3}i\lambda_+^r \lambda_-^s (R_{srta} + R_{stra} - R_{trsa} - R_{tsra} + R_{trsa} + R_{tasr})x^a \xi_t \\
&\quad - \frac{1}{3}i\lambda_-^r \lambda_-^s (R_{stra} + R_{srta} - R_{trsa} - R_{tasr})x^a \xi_t \\
&= \frac{2}{3}iRic_{sa}x^a \xi_s - \frac{2}{3}i\lambda_+^r \lambda_-^s (R_{srta} + R_{stra})x^a \xi_t.
\end{aligned}$$

and the order 0 symbol:

$$\begin{aligned}
\mathfrak{a}_0 &= -\frac{1}{3}i\gamma^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{sqrp} + R_{srqp}) \\
&= -\frac{1}{3}\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{sqrp} + R_{srqp}) \\
&= \frac{2}{3}\lambda_+^p \lambda_-^s Ric_{ps} + \frac{1}{3}\lambda_+^p \lambda_+^r \lambda_-^q \lambda_-^s (R_{sqrp} + R_{srqp}).
\end{aligned}$$

2.2. The inverse of D^2 and its powers. In this section, we present the results that can be applied to a more general situation than the Hodge-Dirac operator. Note, that since we work with pseudodifferential operators, we denote by the inverses of elliptic operator the corresponding parametrix. For this reason we can ignore the kernel of these operators. Let us start with the following lemma:

Lemma 2.2. *Let L be a Laplace-type operator with symbol*

$$\sigma(L) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0$$

expressed in normal coordinates as

$$\begin{aligned}
\mathfrak{a}_2 &= (\delta_{ab} + \frac{1}{3}R_{acbd}x^c x^d)\xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{a}_1 &= iP_{ab}\xi_a x^b + o(\mathbf{x}^1), \\
\mathfrak{a}_0 &= Q + o(\mathbf{x}^0).
\end{aligned}$$

Then the three leading symbols of $\sigma(L^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2}$ are:

$$\begin{aligned}
\mathfrak{c}_{2k} &= ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3}R_{acbd}x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{c}_{2k+1} &= -ik||\xi||^{-2k-2} P_{ab}\xi_a x^b + o(\mathbf{x}^1), \\
\mathfrak{c}_{2k+2} &= -k||\xi||^{-2k-2} Q + k(k+1)||\xi||^{-2k-4} \left(P_{ab} - \frac{1}{3}Ric_{ab} \right) \xi_a \xi_b + o(\mathbf{x}^0).
\end{aligned}$$

Proof. First, observe that we follow the notation of most papers (see [6]) and to simplify the notation indicate a negative order $-k$ by $k \geq 0$, so $\mathfrak{b}_2$ below denotes the symbol of order -2 . We start with computing leading symbols of the inverse of L , i.e. $\sigma(L^{-1}) = \mathfrak{b}_2 + \mathfrak{b}_3 + \mathfrak{b}_4$ using the fact, that $\sigma(LL^{-1}) = \sigma(1) = 1$. We have:

$$\begin{aligned}\mathfrak{b}_2 &= (\mathfrak{a}_2)^{-1} = \|\xi\|^{-4} \left(\delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\ \mathfrak{b}_3 &= \mathfrak{b}_2 (-\mathfrak{a}_1 \mathfrak{b}_2 + i \partial_\xi^a \mathfrak{a}_2 \partial_a \mathfrak{b}_2) = -i \|\xi\|^{-4} P_{ab} \xi_a x^b + o(\mathbf{x}^1), \\ \mathfrak{b}_4 &= \mathfrak{b}_2 \left(-\mathfrak{a}_0 \mathfrak{b}_2 - \mathfrak{a}_1 \mathfrak{b}_3 + i \partial_\xi^a \mathfrak{a}_2 \partial_a \mathfrak{b}_3 + i \partial_\xi^a \mathfrak{a}_1 \partial_a \mathfrak{b}_2 + \frac{1}{2} \partial_\xi^{ab} \mathfrak{a}_2 \partial_{ab} \mathfrak{b}_2 \right) \\ &= -\|\xi\|^{-4} Q + 2 \|\xi\|^{-6} \left(P_{ab} - \frac{1}{3} Ric_{ab} \right) \xi_a \xi_b + o(\mathbf{x}^0).\end{aligned}$$

To finish we apply Lemma A1 in [8] and compute the three leading symbols of the powers of the pseudodifferential operator L^{-k} .

Using the above lemma for $L = D^2$, with the Hodge-Dirac operator D (2.4), we get the following result.

Proposition 2.3. *The leading symbols of D^{-2k} are, up to the appropriate order in x ,*

$$\begin{aligned}\mathfrak{c}_{2k} &= \|\xi\|^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbc} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x}^2), \\ \mathfrak{c}_{2k+1} &= -\frac{2}{3} k i \|\xi\|^{-2k-2} Ric_{ab} x^b \xi_a + \frac{2}{3} k i \|\xi\|^{-2k-2} \lambda_+^r \lambda_-^s (R_{srba} + R_{sbra}) x^a \xi_b + o(\mathbf{x}^1) \\ (2.6) \quad \mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3} \|\xi\|^{-2k-4} Ric_{ab} \xi_a \xi_b \\ &\quad - \frac{2}{3} k(k+1) \|\xi\|^{-2k-4} \lambda_+^r \lambda_-^s (R_{srab} + R_{sarb}) \xi_a \xi_b \\ &\quad + \frac{1}{3} k \|\xi\|^{-2k-2} \lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{sqrp} + R_{srqp}) + o(\mathbf{x}^0).\end{aligned}$$

Proof. For $L = D^2$ we substitute in Lemma 2.2

$$\begin{aligned}P_{ab} &= \frac{2}{3} Ric_{ab} - \frac{2}{3} \lambda_+^r \lambda_-^s (R_{srab} + R_{sarb}), \\ Q &= -\frac{1}{3} \lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{sqrp} + R_{srqp}).\end{aligned}$$

3. SPECTRAL FUNCTIONALS

In [8], we defined two spectral functionals for finitely summable spectral triples, which for the canonical spectral triple over the spin manifold M allow to recover the metric and the Einstein tensors, viewed as bilinear functionals over a pair of one-forms. We recall the definition:

Definition 3.1 (cf. [8], Definition 5.4). If $(\mathcal{A}, D, \mathcal{H})$ is a n -summable spectral triple, let Ω_D^1 be the $\mathcal{A}$ bimodule of one forms generated by $\mathcal{A}$ and $[D, \mathcal{A}]$. Moreover, assume there exists a generalised algebra of pseudodifferential operators which contains $\mathcal{A}, D, |D|^\ell$ for $\ell \in \mathbb{Z}$ with a tracial state $\mathcal{W}$ over this algebra (called a noncommutative residue), which identically vanishes

on $T|D|^{-k}$ for any $k > n$ and a zero-order operator T (an operator in the algebra generated by $\mathcal{A}$ and $\Omega^1(\mathcal{A})$). Then, for $u, w \in \Omega_D^1(\mathcal{A})$, we call

$$\mathfrak{g}_D(u, w) := \mathfrak{W}(uw|D|^{-n}),$$

metric functional, and

$$\mathfrak{G}_D(u, w) := \mathfrak{W}(u\{D, w\}D|D|^{-n}).$$

Einstein functional.

3.1. Hodge-de Rham spectral triple. We compute now these functionals for $\mathcal{A} = C^\infty(M)$ and $D = d + \delta$, identifying for $\dim M = n \geq 2$,

$$\Omega_D^1(A) \simeq \Omega^1(M)$$

via the correspondence of local components

$$u = \gamma^p u_p \leftrightarrow U = u_p e^p,$$

with respect to an orthonormal coframe e^p . First, let us compute the metric functional $\mathfrak{g}$.

Proposition 3.2. *For $U, V \in \Omega^1(M)$, the metric spectral functional reads*

$$(3.1) \quad \mathfrak{g}(U, W) = 2^n v_{n-1} \int_M g(U, W) \text{vol}_g.$$

Proof. It can be seen that it is adequate to expand U, W , and so u, w , up to $o(\mathbf{x}^0)$ in normal coordinates:

$$u = \gamma^p u_p + o(\mathbf{x}^0), \quad w = \gamma^r w_r + o(\mathbf{x}^0).$$

We compute (locally) the density of $\mathfrak{W}(UW|D|^{-n})$ as

$$\int_{||\xi||=1} \text{Tr}(\gamma^p \gamma^r u_p w_r) \mathfrak{c}_n(D) d^n x = v_{n-1} \sqrt{g} \text{Tr}(\gamma^p \gamma^r u_p w_r) d^n x = 2^n v_{n-1} \sqrt{g} u_p w_p d^n x.$$

The factor 2^n comes from the trace of 1 over the space of differential forms.

Next, we have,

Proposition 3.3. *For $U, V \in \Omega^1(M)$, the Einstein functional reads*

$$\mathfrak{G}(U, W) = \frac{2^n}{6} v_{n-1} \int_M G(U, W) \text{vol}_g,$$

where G is the Einstein tensor for M .

Before we begin with the proof let us demonstrate some useful lemmas. The first computes $\mathfrak{W}(ED^{-2m+2})$ for two specific cases of endomorphism E .

Lemma 3.4. *If E is locally given by*

$$e^{(0)} + e_{pq}^{(2)} \gamma^p \gamma^q,$$

the density of the functional $\mathfrak{W}(ED^{-2m+2})$ reads locally

$$\frac{n-2}{24} 2^n v_{n-1} \sqrt{g} R (-e^{(0)} - e_{pp}^{(2)}) d^n x.$$

On the other hand, if $\tilde{E}$ is locally given by

$$\tilde{e}^{(0)} + \tilde{e}_{pq}^{(2)} \lambda_+^p \lambda_-^q,$$

the local density of the functional $\mathcal{W}(\tilde{E}D^{-2m+2})$ reads locally

$$-\frac{n-2}{24} 2^n v_{n-1} \sqrt{g} R \tilde{e}^{(0)} d^n x.$$

The proof is based on direct calculations using Proposition 2.3:

$$(3.2) \quad \text{Tr} \left(E \int_{\|\xi\|=1} \mathfrak{e}_{2(m-1)+2} \right) \mathcal{V}_\xi d^n x = \\ \frac{n-2}{12} v_{n-1} \text{Tr} \left(E \left(2(R_{srqp} + R_{sqrp}) \lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s + R - 2Ric_{qp} \lambda_+^p \lambda_-^q \right) \right) d^n x$$

Next, more generally, we state:

Lemma 3.5. *For P such that $\sigma(P) = F^{ab} \xi_a \xi_b + G^a \xi_a + H$, where locally*

$$F^{ab} = f^{(0)ab} + f_{pq}^{(2)ab} \gamma^p \gamma^q,$$

$w(PD^{-n})$ reads locally

$$v_{n-1} \left(\text{Tr} H + \frac{2^n}{24} R (-f^{(0)aa} - f_{pp}^{(2)aa}) \right) d^n x.$$

Also, for $\tilde{P}$ such that $\sigma(\tilde{P}) = \tilde{F}^{ab} \xi_a \xi_b + G^a \xi_a + H$, where locally

$$\tilde{F}^{ab} = \tilde{f}^{(0)ab} + \tilde{f}_{pq}^{(2)ab} \lambda_+^p \lambda_-^q,$$

$w(\tilde{P}D^{-n})$ reads locally

$$v_{n-1} \left(\text{Tr} H + \frac{2^n}{48} \left(-2R\tilde{f}^{(0)aa} - \tilde{f}_{pp}^{(2)aa} R + 2(\tilde{f}_{pq}^{(2)ab} + \tilde{f}_{pq}^{(2)ba}) R_{paqb} \right) \right) d^n x.$$

The proof follows directly by computation using Proposition 2.3 and Lemma A.3 applied to the explicit expression:

$$(3.3) \quad \int_{\|\xi\|=1} \sigma_{-2m}(PD^{-2m}) \mathcal{V}_\xi = \frac{1}{6} F^{aa} \left[(R_{srqp} + R_{bqrs}) \lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s + \frac{1}{2} R - R_{pq} \lambda_+^p \lambda_-^q \right] \\ + \frac{1}{6} F^{ab} [-R_{ab} + (R_{qapb} + R_{paqb}) \lambda_+^p \lambda_-^q] + H.$$

Proof. [Proof of Proposition 3.3] We begin with computing the symbol of $uDwD$ at $x = 0$, where it suffices to expand u and w as:

$$u = \gamma^p u_p + o(\mathbf{x}^0), \quad w = \gamma^s w_s + \gamma^s w_{sa} x^a + o(\mathbf{x}^1),$$

and thus locally:

$$uDwD = \gamma^p \gamma^q \gamma^r \gamma^s u_p w_r \xi_q \xi_s - i \gamma^p \gamma^q \gamma^r \gamma^s u_p w_{rq} \xi_s \\ - \frac{1}{3} i \gamma^p \gamma^q \gamma^r \lambda_-^s \lambda_+^t \lambda_-^z (R_{ztsq} + R_{zstq}) u_p w_r + o(\mathbf{x}^0).$$

Then, we use lemma 3.4 for $P = uDwD$ and $E = uw$. In this case we have

$$\begin{aligned} E &= u_p w_q \gamma^p \gamma^q, \\ H &= -\frac{1}{3} i \gamma^p \gamma^q \gamma^r \lambda_-^s \lambda_+^t \lambda_-^z (R_{ztsq} + R_{zstq}) u_p w_r \\ &= -\frac{2}{3} i \gamma^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{srqt} + R_{sqr t}) u_p w_t + \frac{1}{3} i \gamma^p \gamma^q \gamma^r \lambda_-^s \lambda_+^t \lambda_-^z (R_{ztsr} + R_{zstr}) u_p w_q, \\ F^{pq} \xi_p \xi_q &= \gamma^r \gamma^p \gamma^s \gamma^q u_r w_s \xi_p \xi_q = (2u_r w_p \delta_{qs} \gamma^r \gamma^s - u_r w_s \gamma^r \gamma^s \delta_{pq}) \xi_p \xi_q, \end{aligned}$$

where we used that $\gamma^p \gamma^q \xi_p \xi_q = \delta_{pq} \xi_p \xi_q$. Next, we see,

$$\begin{aligned} e^{(0)} &= 0, & e_{ab}^{(2)} &= u_a w_b, \\ f^{(0)} &= 0, & f_{cd}^{(2)ab} &= 2u_c w_a \delta_{bd} - u_c w_d \delta_{ab}. \end{aligned}$$

Finally, the contribution arising from E gives:

$$-\frac{n-2}{24} 2^n v_{n-1} R u_a w_a.$$

whereas the part from F is,

$$-\frac{2^n}{24} v_{n-1} R f_{ii}^{(2)aa} = \frac{n-2}{24} 2^n v_{n-1} R u_a w_a.$$

These two terms cancel each other, and we are left with terms that arise from H . The only possible terms in $\text{Tr } H$ are linear combinations of $u_a w_a R$ and $u_a w_b R i_{C_{ab}}$, thus, we know, that the result is symmetric in u_a, w_b . It allows us to simplify the second term in H :

$$H = -\frac{2}{3} i \gamma^p \lambda_-^q \lambda_+^s \lambda_-^t (R_{tsqb} + R_{tqsb}) u_p w_b + \frac{1}{3} i \gamma^r \lambda_-^q \lambda_+^s \lambda_-^t (R_{tsqr} + R_{tqsr}) u_a w_a + \dots,$$

where " $\dots$ " are terms antisymmetric in u_a, w_b , which we can neglect. We can also insert $-i\lambda_+$ instead of the remaining γ 's because the part with λ_- will be traceless. Now, using the lemma A.3 we get:

$$\text{Tr } H = \frac{1}{6} \text{Ric}_{ab} u_a w_b - \frac{1}{12} R u_a w_a = \frac{1}{6} G_{ab} u_a w_b.$$

This proves the result.

We deduce that for the spin_c manifolds the spectral functionals for the Hodge-de Rham spectral triple are equal, up to the rank of the vector bundles, to those for the canonical spin_c spectral triple.

3.2. Spectral closedness and torsion. In this section, we will prove that the Hodge-Dirac spectral triple has the property of being spectrally closed.

Theorem 3.6. *Let T be an operator of order 0 from the algebra generated by $a[D, b]$, $a, b \in C^\infty(M)$. Then,*

$$\mathcal{W}(TD|D|^n) = 0.$$

Proof. If we compute the symbol of TD at a chosen point on the manifold M in normal coordinates at $x = 0$ we obtain,

$$\sigma(TD) = T(-\gamma^p \xi_p).$$

Next, if we combine it with Proposition 2.3 we see that the symbol of order $-n$ of $TD|D|^n$ is:

$$\sigma_{-n}(TD|D|^n) = 0 + o(\mathbf{x}^0).$$

This ends the proof.

As a consequence, we demonstrate in [9] that the Hodge-Dirac spectral triple is torsion-free. It is interesting to study generalised Hodge-Dirac operators, which are defined through an arbitrary linear connection, are not metric compatible, and have a torsion. Also, the extension to noncommutative Hodge-de Rham calculi and a comparison with the approach of [10] would be of great interest. Note that we focused here on even-dimensional manifolds only for the sake of simplicity and the result easily extends to the case of odd-dimensional manifolds (for technical details that include the computation of symbols, compare [9]).

APPENDIX A. DETAILS OF COMPUTATIONS

We begin with a formula for the product of two pseudodifferential operators, P and Q , which have expansion in homogeneous symbols,

$$(A.1) \quad \sigma(P)(x, \xi) = \sum_{\alpha} \sigma(P)_{\alpha}(x, \xi), \quad \sigma(Q)(x, \xi) = \sum_{\beta} \sigma(Q)_{\beta}(x, \xi),$$

respectively, where α, β are multiindices (such that $|\alpha| = \sum_k \alpha^k$ and $|\beta|$ are bounded from above). The symbol $\sigma(P)_{\alpha}(x, \xi)$ is homogeneous of order $|\alpha|$ if for $r > 0$ $\sigma(P)_{\alpha}(x, r\xi) = r^{|\alpha|} \sigma(P)_{\alpha}(x, \xi)$. The composition rule for the symbols of their product takes the form [11].

$$(A.2) \quad \sigma(PQ)(x, \xi) = \sum_{\beta} \frac{(-i)^{|\beta|}}{|\beta|!} \partial_{\beta}^{\xi} \sigma(P)(x, \xi) \partial_{\beta} \sigma(Q)(x, \xi),$$

where ∂_a^{ξ} denotes the partial derivative with respect to the coordinate of the cotangent bundle.

Lemma A.1. *A direct computation of traces of products of λ matrices is based on the following recursive formula:*

$$\begin{aligned} & \text{Tr } \lambda_+^{p_1} \dots \lambda_+^{p_k} \lambda_-^{q_1} \dots \lambda_-^{q_k} \\ &= \frac{1}{2} \sum_{j=1}^k (-1)^{k-j} \delta^{p_1 q_j} \text{Tr } (\lambda_+^{p_2} \dots \lambda_+^{p_k} \lambda_-^{q_1} \dots \lambda_-^{q_{j-1}} \lambda_-^{q_{j+1}} \dots \lambda_-^{q_k}). \end{aligned}$$

In particular, we have

$$(A.3) \quad \begin{aligned} \text{Tr } (\lambda_+^p \lambda_-^q) &= 2^{n-1} \delta^{pq}, \\ \text{Tr } (\lambda_+^{p_1} \lambda_-^{q_1} \lambda_+^{p_2} \lambda_-^{q_2}) &= 2^{n-2} (\delta^{p_1 q_1} \delta^{p_2 q_2} + \delta^{p_1 q_2} \delta^{p_2 q_1}), \end{aligned}$$

and

$$(A.4) \quad \begin{aligned} \text{Tr } (\lambda_+^{p_1} \lambda_-^{q_1} \lambda_+^{p_2} \lambda_-^{q_2} \lambda_+^{p_3} \lambda_-^{q_3}) &= \\ &= 2^n \left(\frac{1}{8} \sum_{\sigma \in S_3} \delta^{p_1 q_{\sigma(1)}} \delta^{p_2 q_{\sigma(2)}} \delta^{p_3 q_{\sigma(3)}} - \frac{1}{4} \delta^{p_1 q_2} \delta^{p_2 q_3} \delta^{p_3 q_1} \right). \end{aligned}$$

Next, we present the results on traces of products of γ and λ matrices.

Lemma A.2.

$$(A.5) \quad \text{Tr } (\gamma^p \gamma^q \lambda_+^r \lambda_-^s) = 2^{n-2} (2\delta^{pq} \delta^{rs} + \delta^{ps} \delta^{qr} - \delta^{pr} \delta^{qs}) = 2^{n-1} (\delta^{pq} \delta^{rs} - \frac{1}{2} \varepsilon_{rs}^{pq}),$$

$$(A.6) \quad \text{Tr } (\gamma^p \gamma^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z) = 2^{n-2} \delta^{pq} (\delta^{rs} \delta^{tz} + \delta^{rz} \delta^{st}) - 2^{n-3} (\delta^{rs} \varepsilon_{tz}^{pq} + \delta^{st} \varepsilon_{rz}^{pq} + \delta^{tz} \varepsilon_{rs}^{pq} + \delta^{rz} \varepsilon_{st}^{pq})$$

We skip the computational proof, which is based on expressing γ -matrices in terms of λ -matrices,

$$\gamma^p \gamma^q = -\lambda_+^q \lambda_-^p + \lambda_+^p \lambda_-^q + \delta^{pq} + \dots,$$

and using the results of Lemma A.1.

As a consequence, we obtain the following identities for the geometric quantities:

Lemma A.3. *In normal coordinates around $x = 0$ we have the following identities*

$$\begin{aligned}
 (A.7) \quad & \text{Tr}(\lambda_+^p \lambda_-^q) \text{Ric}_{pq} = 2^{n-1} R, \\
 & \text{Tr}(\lambda_+^p \lambda_-^q) (R_{paqb} + R_{qapb}) = 2^n \text{Ric}_{ab}, \\
 & \text{Tr}(\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s) (R_{srqp} + R_{sqrp}) = -2^{n-2} R, \\
 & \text{Tr}(\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s) \text{Ric}_{rs} = 2^{n-2} (\delta_{pq} R + \text{Ric}_{pq}), \\
 & \text{Tr}(\gamma^p \gamma^q \lambda_+^r \lambda_-^s) \text{Ric}_{rs} = 2^{n-1} \delta_{pq} R, \\
 & \text{Tr}(\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s) (R_{rasb} + R_{sarb}) = 2^{n-2} (2\delta_{pq} \text{Ric}_{ab} + R_{qapb} + R_{paqb}), \\
 & \text{Tr}(\gamma^p \gamma^q \lambda_+^r \lambda_-^s) (R_{rasb} + R_{sarb}) = 2^n \delta_{pq} \text{Ric}_{ab}, \\
 & \text{Tr}(\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z) (R_{ztsr} + R_{zstr}) = 2^{n-3} (-R\delta_{pq} + 2\text{Ric}_{pq}), \\
 & \text{Tr}(\gamma^p \gamma^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z) (R_{ztsr} + R_{zstr}) = -2^{n-2} \delta_{pq} R.
 \end{aligned}$$

Proof. Direct computation using (A.3)-(A.5) and the properties of the Riemann and Ricci tensors.

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2.3 Spectral torsion

In the last article [P3] we introduce a new functional — spectral torsion — and compute it for some interesting examples of spectral triples. The torsion functional is defined as

$$\mathcal{T}_D(u, v, w) := \text{Wres}(\hat{u}\hat{v}\hat{w}D|D|^{-n}), \quad (2.8)$$

where D can be now a Dirac-type operator with torsion. In the classical case this functional is

$$\mathcal{T}_D(u, v, w) \propto \int u_a v_b w_c T_{abc}^A \text{vol}_g, \quad (2.9)$$

where T_{abc}^A is the totally antisymmetric part of the torsion tensor.

On spin manifolds, lifting the Levi-Civita connection to the spinor bundle yields the canonical Dirac operator, which is torsion-free and metric-compatible. In the case of spectral triples, we begin with the Dirac operator, for which we know nothing about its torsion. The spectral torsion functional is a simple method for examining the Dirac operator's torsion and determining if it vanishes.

SPECTRAL TORSION

LUDWIK DĄBROWSKI⁽¹⁾, ANDRZEJ SITARZ⁽²⁾, AND PAWEŁ ZALECKI⁽²⁾

ABSTRACT. We introduce a trilinear functional of differential one-forms for a finitely summable regular spectral triple with a noncommutative residue. We demonstrate that for a canonical spectral triple over a closed spin manifold it recovers the torsion of the linear connection. We examine several spectral triples, including Hodge-de Rham, Einstein-Yang-Mills, almost-commutative two-sheeted space, conformally rescaled noncommutative tori, and quantum $SU(2)$ group, showing that the third one has a nonvanishing torsion if nontrivially coupled.

1. INTRODUCTION

The existence and uniqueness of a metric-compatible linear connection with vanishing torsion is one of the fundamental theorems of Riemannian geometry. Torsion appears quite naturally as a vector-valued two-form in this approach, and the assumption that it identically vanishes leads to the Levi-Civita connection, which solely depends on the metric. In the background of general relativity lies the torsion-free condition of Riemannian geometry, which very accurately describes the gravitational interaction of bodies. Torsion, on the other hand, has a physical interpretation as the quantity that measures the twisting of reference frames along geodesics and has been considered an independent field in physics in Einstein-Cartan theory [4]. Torsion, unlike gravitational fields, must vanish in a vacuum and does not propagate; however, its existence causes nonlinear interactions of matter with spin and has the potential to change the standard singularity theorems of General Relativity (see [16, 21] for a review of physical theories).

The emergence of noncommutative geometry [5, 6], which generalises standard notions of differential geometry to an algebraic (or operator-algebraic) setup, has raised new questions about the concepts of linear connections, metric, and torsion. A transparent link between different approaches, ranging from purely algebraic noncommutative differential geometry to operator algebraic spectral triple formalism, and a unifying view of metric, linear connection, and torsion has yet to be established. Although the algebraic concepts of linear connections, torsion, and metric compatibility have been achieved, the existence of the Levi-Civita connection can only be proven in special cases (see [1, 2, 3] and the references therein).

On the other hand, the spectral triple approach, with the Dirac operator being the fundamental object of geometry, has not yet been able to determine whether the constructed Dirac operators correspond to the Levi-Civita connection or whether they contain a nonvanishing torsion. So far, for that purpose, one could only try to minimise the spectral functional corresponding to the integrated scalar curvature while keeping the metric defined by the Dirac operator unchanged.

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Even in the classical case of manifolds, this can be a difficult task. (see computations of spectral action for a nonvanishing torsion [15, 20, 18]), which becomes even harder in a genuinely noncommutative situation [22].

The aim of this paper is to propose a plain, purely spectral method that allows to determine the torsion as the density of the torsion functional and impose (if possible) the torsion-free condition for regular finitely summable spectral triples.

2. DIRAC OPERATOR WITH TORSION

Let us assume that M is a closed spin manifold of dimension n , with the metric tensor g_{ij} . In terms of a (local) basis e_i of orthonormal vector fields on M the covariant derivative can be written as

$$(2.1) \quad \nabla_{e_i} e_j = \omega_{ijk} e_k.$$

The requirement that ∇ is metric, that is for any vector fields X, Y, Z ,

$$\mathcal{L}_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

allows to determine the connection explicitly through the formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \\ &\quad + g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X), \end{aligned}$$

where T is the torsion tensor,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Therefore, the spin connection coefficients ω_{ijk} can be split into the Levi-Civita connection, ω_{ijk}^{LC} and the contorsion part, τ_{ijk} . The first can be expressed explicitly in the basis e_i through the structure constants c_{ijk} of the commutators of orthonormal vector fields,

$$(2.2) \quad \omega_{ijk}^{LC} := \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji}), \quad [e_i, e_j] = c_{ijk} e_k,$$

and the remaining part of the connection, the contorsion tensor, is related to the torsion of the connection,

$$\tau_{ijk} = \frac{1}{2}(T_{ijk} + T_{kij} + T_{kji}),$$

where $T(e_i, e_j) = T_{ijk} e_k$.

The lift of the covariant derivative to Dirac spinor fields is,

$$(2.3) \quad \nabla_{e_i}^{(s)} = e_i - \frac{1}{4} \omega_{ijk} \gamma^j \gamma^k,$$

with γ^j are the usual generators of Clifford algebra. The Dirac operator on a spin manifold is, in a local basis of orthonormal frames, a first-order differential operator,

$$(2.4) \quad D^{(s)} = i \gamma^j \nabla_{e_j}^{(s)},$$

and is unique for a given metric and contorsion tensor. The canonical Dirac operator, D , is the one for vanishing contorsion and we call it torsion-free Dirac operator. The torsion tensor can be expressed using the contorsion,

$$T_{ijk} = \tau_{ijk} - \tau_{jik}.$$

Note the antisymmetry $\tau_{ijk} = -\tau_{ikj}$ while $T_{ijk} = -T_{kij}$. The torsion tensor can be split into three parts, the vector part, totally antisymmetric part and the Cartan part, however the Cartan part does not contribute at all to the Dirac operator and the vector part has to vanish if the resulting Dirac operator has to be self-adjoint [13]. Therefore, out of the full torsion only the antisymmetric part appears in the selfadjoint Dirac operator and hence we shall assume that torsion (and contorsion) are antisymmetric tensors. Therefore, the Dirac operator with a fully antisymmetric torsion T is,

$$(2.5) \quad D_T = i\gamma^j (\nabla_{e_j}^{(s,LC)} - \frac{1}{8} T_{jkl} \gamma^k \gamma^l) = D - \frac{i}{8} T_{jkl} \gamma^j \gamma^k \gamma^l,$$

where D is the standard Dirac operator, which originates from the Levi-Civita connection. Note that since $[D_T, f] = [D, f]$ for any function $f \in C^\infty(M)$, the algebra of differential forms remains the same. We will denote $\hat{u}$ the Clifford multiplication by the one-form u . Let $\mathcal{W}$ denote the Wodzicki residue [25, 14]. We introduce the following definition of the torsion functional of which the name is explained by the subsequent torsion recovery theorem.

Definition 2.1. For D_T given by 2.5 the trilinear functional of differential one-forms u, v, w

$$(2.6) \quad \mathcal{T}(u, v, w) = \mathcal{W}(\hat{u}\hat{v}\hat{w}D_T|D_T|^{-n}),$$

is called torsion functional.

As our first main result we state,

Theorem 2.2. *The torsion functional for the Dirac operator D_T on a Riemannian closed spin manifold of dimension n is,*

$$(2.7) \quad \mathcal{T}(u, v, w) = -2^m i V(S^{n-1}) \int_M \text{vol}_g u_a v_b w_c T_{abc}.$$

where $m = \frac{n}{2}$ for even n and $m = \frac{n+1}{2}$ for odd $n > 1$.

An immediate consequence is,

Corollary 2.3. *The Dirac operator D_T is torsion free iff the torsion functional vanishes:*

$$T \equiv 0 \iff \mathcal{T}(\hat{u}, \hat{v}, \hat{w}) = 0.$$

To prove the theorem we employ the calculus of pseudodifferential operators and expansion of the symbols in normal coordinates.

2.1. Normal coordinates and symbols. Let us recall that in the normal coordinates [19] the metric and the Levi-Civita connection have a Taylor expansion at a given point ($\mathbf{x} = 0$) on the manifold,

$$(2.8) \quad \begin{aligned} g_{ab} &= \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x}^2), \\ g^{ab} &= \delta^{ab} + \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x}^2), \\ \sqrt{\det(g)} &= 1 - \frac{1}{6} \text{Ric}_{ab} x^a x^b + o(\mathbf{x}^2), \\ \Gamma_{bc}^a(x) &= -\frac{1}{3} (R_{abcd} + R_{acbd}) x^d + o(\mathbf{x}^2), \end{aligned}$$

where R_{acbd} and Ric_{ab} are the components of the Riemann and Ricci tensors, respectively, at the point with $\mathbf{x} = 0$ and we use the notation $o(\mathbf{x}^k)$ to denote that we expand a function up to the polynomial of order k in the normal coordinates.

As a next step, we compute the symbols of D_T and its inverses as pseudodifferential operators (see Appendix). We start with the symbol of D_T expanded in normal coordinates up to $o(\mathbf{1})$:

$$(2.9) \quad \sigma(D_T) = i\gamma^j \left(i\xi_j - \frac{1}{8} T_{jps} \gamma^p \gamma^s \right) + o(\mathbf{1}).$$

A Clifford multiplication by a differential form v can be expanded in local normal coordinates around a point on a manifold, as:

$$\hat{v} = v_a \gamma^a + o(\mathbf{1}).$$

We can now proceed with the proof of the theorem.

Proof of Theorem 2.2: First of all, observe that since each one-form is a zero-differential operator then effectively we need to compute the symbol of order $-n$ of $D_T |D_T|^{-n}$. We start with the even dimension case $n = 2m$, postponing the case of odd dimensions till later. To compute the two leading symbols of D_T^{-2k} , denoted by $\mathfrak{c}_{2m} + \mathfrak{c}_{2m+1}$ we need the two leading symbols of $\sigma(D_T^2) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0$.

$$(2.10) \quad \begin{aligned} \mathfrak{a}_2 &= ||\xi||^2 + o(\mathbf{x}), \\ \mathfrak{a}_1 &= -\frac{1}{8} i T_{abc} (\gamma^j \gamma^a \gamma^b \gamma^c + \gamma^a \gamma^b \gamma^c \gamma^j) \xi_j + o(\mathbf{1}). \end{aligned}$$

Then, using results from [11] (Lemma A.1) we have:

$$(2.11) \quad \begin{aligned} \mathfrak{c}_{2m} &= ||\xi||^{-2m} + o(\mathbf{x}), \\ \mathfrak{c}_{2m+1} &= \frac{1}{8} im ||\xi||^{-2m-2} T_{abc} (\gamma^j \gamma^a \gamma^b \gamma^c + \gamma^a \gamma^b \gamma^c \gamma^j) \xi_j + o(\mathbf{1}). \end{aligned}$$

We are now ready to compute the symbol of order $-n$ of $\hat{u}\hat{v}\hat{w}D_T |D_T|^{-n}$:

$$(2.12) \quad \begin{aligned} \sigma_{-n}(\hat{u}\hat{v}\hat{w}D_T |D_T|^{-n}) &= -\frac{1}{8} u_a \gamma^a v_b \gamma^b v_c \gamma^c \left(i T_{jps} \gamma^j \gamma^p \gamma^s ||\xi||^{-2m} \right. \\ &\quad \left. + (-\gamma^k) \left(-im ||\xi||^{-2m-2} T_{psr} (\gamma^j \gamma^p \gamma^s \gamma^r + \gamma^p \gamma^s \gamma^r \gamma^j) \xi_j \xi_k \right) \right) + o(\mathbf{1}). \end{aligned}$$

Then, we proceed with the calculation of the Wodzicki residue.

$$(2.13) \quad \begin{aligned} \mathcal{W}(\hat{u}\hat{v}\hat{w}D_T |D_T|^{-n}) &= \int_M \text{Tr}_{Cl} \int_{||\xi||=1} \sigma_{-n}(\hat{u}\hat{v}\hat{w}D_T |D_T|^{-n}) \\ &= -2^m i V(S^{n-1}) \int_M \text{vol}_g u_a v_b w_c T_{abc}. \end{aligned}$$

This ends the proof for even n . To see that the proof holds also for odd n we apply the usual technique to compute the symbol of $|D_T| = \sqrt{D_T^2}$ following [23, Chapter XIII §1]. It follows

that the symbol $|D_T|$ is, as a sum of homogenous symbols, of which we need only the first two (up to relevant order in normal coordinates):

$$\sigma(|D_T|) = \left(|\xi| + o(\mathbf{x}) \right) + \left(-\frac{1}{8} i T_{abc} (\gamma^j \gamma^a \gamma^b \gamma^c + \gamma^a \gamma^b \gamma^c \gamma^j) \xi_j |\xi|^{-1} + o(\mathbf{1}) \right) + \dots$$

Then the rest of the proof is the same as in the case of even dimensions. $\square$

Remark 2.4. Note that the lowest odd dimension in which there can be a nonvanishing torsion is $n = 3$. In the lowest possible even dimension $n = 4$ the torsion functional can also be written as a functional over a single one-form:

$$(2.14) \quad \mathcal{T}(u) = \mathcal{W}(\gamma \hat{u} D_T |D_T|^{-n}).$$

3. NONCOMMUTATIVE TORSION

The spectral definition of torsion, Def.2.1, can be readily extended to the noncommutative case of spectral triples. Let $(\mathcal{A}, D, \mathcal{H})$ be a n -summable unital spectral triple, $\Omega_D^1(\mathcal{A})$ be the $\mathcal{A}$ bimodule of one forms generated by $\mathcal{A}$ and $[D, \mathcal{A}]$, which by definition consists of bounded operators on $\mathcal{H}$. We assume that there exists a generalised algebra of pseudodifferential operators, which contains $\mathcal{A}$, D , $|D|^\ell$ for $\ell \in \mathbb{Z}$, and there exists a tracial state $\mathcal{W}$ on it, called a noncommutative residue. Note that these assumptions follow if the spectral triple is finitely summable and regular, (c.f. [17] and [24]). Moreover, we assume that the noncommutative residue identically vanishes on $T|D|^{-k}$ for any $k > n$ (c.f. [7]) and a zero-order operator T (i.e., an operator in the algebra generated by $\mathcal{A}$ and $\Omega_D^1(\mathcal{A})$). We introduce the following definition of torsion functional for spectral triples.

Definition 3.1. Torsion functional for the spectral triple $(\mathcal{A}, D, \mathcal{H})$ is a trilinear functional of $u, v, w \in \Omega_D^1(\mathcal{A})$ defined by,

$$\mathcal{T}(u, v, w) = \mathcal{W}(uvwD|D|^{-n}).$$

We say that the spectral triple is torsion-free (or, for simplicity, the Dirac operator D is torsion-free) if $\mathcal{T}$ vanishes identically.

We recall a stronger condition of spectrally closed spectral triples defined in [11, Definition 5.5] which, in particular, implies the torsion-free condition stated above.

Definition 3.2. We say that a spectral triple with a trace on the generalised algebra of pseudodifferential operators is spectrally closed if for any zero-order operator P (from the algebra generated by $\mathcal{A}$, and $\Omega_D^1(\mathcal{A})$) the following holds:

$$\mathcal{W}(PD|D|^{-n}) = 0.$$

Lemma 3.3 ([11, Lemma 5.6]). *The classical spectral triple over a closed spin-c manifold M of dimension $n = 2m$ is spectrally closed in the above sense and hence is torsion-free.*

4. EXAMPLES

The examples of spectral triples presented below are all significant cases of various noncommutative geometries to which we apply the proposed spectral definition of torsion and compute the torsion functional.

4.1. Hodge-Dirac. Let us start with a purely classical example of a Hodge-Dirac spectral triple over an oriented, closed Riemannian manifold, where the Hilbert space are square-summable differential forms and the Dirac is $D = d + \delta$, where $\delta = *d*$, with $*$ being the Hodge star operator. In [12] we demonstrated that such spectral triple is spectrally closed and therefore torsion-free.

4.2. Einstein-Yang-Mills. Consider a closed spin manifold M of even dimension $n = 2m$ and a spectral triple

$$\left(C^\infty(M) \otimes M_N(\mathbb{C}), \tilde{D}, L^2(S) \otimes M_N(\mathbb{C}) \right).$$

Here, the algebra $M_N(\mathbb{C})$ acts by left multiplications on $M_N(\mathbb{C})$ regarded as a Hilbert space with the scalar product $\langle M, M' \rangle = \text{Tr}(M^* M')$. Also,

$$\tilde{D} = D \otimes \text{id}_N + A + JAJ^{-1}$$

is a fluctuation of $D \otimes \text{id}_N$, where D is the standard Dirac operator on M ,

$$A = A^* \in \Omega_D^1 = \{ \alpha [D \otimes \text{id}_N, \beta] \mid \alpha, \beta \in C^\infty(M) \otimes M_N(\mathbb{C}) \},$$

and $J = C \otimes *$, with C being the charge conjugation on spinors in S . Note that A equals $i\gamma^a X_a$, where $X_a = -X_a^* \in C^\infty(M, M_N(\mathbb{C}))$ acts on $M_N(\mathbb{C})$ by left multiplication, while JAJ^{-1} equals $i\gamma^a$ composed with the right multiplication by X_a^* . Therefore, the fluctuation $A + JAJ^{-1}$ amounts to the action of $i\gamma^a \text{Ad}_{X_a}$. It follows that the center of this algebra, $C^\infty(M, \mathbb{C}1_N)$, acts trivially, so we just can consider X_a such that $\langle X_a, 1_N \rangle = 0$, that is, traceless matrix valued functions, $X_a \in C^\infty(M, su(N))$. Moreover, we need only the expansion of X_a and so of the operator $i\gamma^a \text{Ad}_{X_a}$ in normal coordinates up to $o(\mathbf{1})$.

It is known that this spectral triple is finitely summable and regular. To compute the torsion functional for $\tilde{D}$ we start (slightly abusing the notation) with its symbol expanded as:

$$(4.1) \quad \sigma(\tilde{D}) = \gamma^a (-\xi_a + i \text{Ad}_{X_a}) + o(\mathbf{1}).$$

It is not difficult to see that the bimodule $\Omega_{\tilde{D}}^1$ spanned by $\alpha [\tilde{D}, \beta]$, $\alpha, \beta \in C^\infty(M, M_N(\mathbb{C}))$, is also spanned just by the elements $u = \gamma^a v_a$, where we can expand $v_a \in M_N(\mathbb{C}) + o(\mathbf{1})$. We now compute the symbols of $\sigma(uvw\tilde{D}) = \mathfrak{s}_1 + \mathfrak{s}_0$ for $u, v, w \in \Omega_{\tilde{D}}^1$:

$$(4.2) \quad \begin{aligned} \mathfrak{s}_1 &= -\gamma^a \gamma^b \gamma^c \gamma^d u_a v_b w_c \xi_d, \\ \mathfrak{s}_0 &= i\gamma^a \gamma^b \gamma^c \gamma^d u_a v_b w_c \text{ad}_{X_d}. \end{aligned}$$

Next, to compute the two leading symbols of $\tilde{D}^{-2m}$, $\sigma(\tilde{D}^{-2m}) = \mathfrak{c}_{2m} + \mathfrak{c}_{2m+1}$, we need the two leading symbols of $\tilde{D}^2$, $\sigma(\tilde{D}^2) = \mathfrak{a}_2 + \mathfrak{a}_1 + \dots$,

$$(4.3) \quad \begin{aligned} \mathfrak{a}_2 &= \|\xi\|^2 + o(\mathbf{x}), \\ \mathfrak{a}_1 &= -2\text{Ad}_{X_a} \xi_a + o(\mathbf{1}), \end{aligned}$$

and the two leading symbols of $\tilde{D}^{-2}$, $\sigma(\tilde{D}^{-2}) = \mathfrak{b}_2 + \mathfrak{b}_3 + \dots$,

$$(4.4) \quad \begin{aligned} \mathfrak{b}_2 &= \|\xi\|^{-2} + o(\mathbf{x}), \\ \mathfrak{b}_3 &= +2\text{Ad}_{X_a} \xi_a \|\xi\|^{-4} + o(\mathbf{1}). \end{aligned}$$

Then, using [11, Lemma A.1] we have:

$$(4.5) \quad \begin{aligned} \mathfrak{c}_{2m} &= ||\xi||^{-2m} + o(\mathbf{x}), \\ \mathfrak{c}_{2m+1} &= n \text{Ad}_{X_a} \xi_a ||\xi||^{-n-2}. \end{aligned}$$

Therefore, since $\partial_a \mathfrak{c}_{2m} = 0$,

$$\sigma_{-n} \left(uvw \tilde{D} ||\tilde{D}||^{-n} \right) = \mathfrak{s}_0 \mathfrak{c}_n + \mathfrak{s}_1 \mathfrak{c}_{n+1},$$

where both terms on the right hand side contain linearly ad_{X_a} . It turns out however, that the trace of ad_X , $X \in M_N(\mathbb{C})$, considered as a linear operator on $M_N(\mathbb{C})$ vanishes. This could be checked explicitly, using the basis $E_{\alpha\beta}$, $\alpha, \beta = 1, \dots, N$ of $M_N(\mathbb{C})$, where the matrix $E_{\alpha\beta}$ has the α, β matrix entry 1 and otherwise 0. We compute:

$$\text{Ad}_{E_{\mu\nu}} E_{\alpha\beta} = E_{\mu\beta} \delta_{\alpha\nu} - E_{\alpha\nu} \delta_{\mu\beta} = K_{\mu\nu; \alpha\beta, \rho\tau} E_{\rho\tau},$$

where K as a matrix is,

$$K_{\mu\nu; \alpha\beta, \rho\tau} = \delta_{\mu, \rho} \delta_{\beta, \tau} \delta_{\nu, \alpha} - \delta_{\alpha, \rho} \delta_{\nu, \tau} \delta_{\mu, \beta}.$$

Thus,

$$\text{Tr ad}_{E_{\mu\nu}} = K_{\mu\nu; \alpha\beta, \alpha\beta} = N(\delta_{\mu, \alpha} \delta_{\nu, \alpha} - \delta_{\nu, \alpha} \delta_{\mu, \alpha}) = 0.$$

Therefore, the spectral torsion functional $\mathcal{T}(u, v, w)$ for $\tilde{D}$ vanishes and so $\tilde{D}$ is torsion free.

4.3. Almost commutative $\mathbf{M} \times \mathbb{Z}_2$. We assume that M is a closed spin manifold and $(C^\infty(M), D, \mathcal{H})$ is an even spectral triple of dimension n with the standard Dirac operator D and a grading γ . We know that it is spectrally closed, so, in particular, D is torsion-free. We consider the usual double-sheet spectral triple, $(C^\infty(M) \otimes \mathbb{C}^2, \mathcal{D}, \mathcal{H} \otimes \mathbb{C}^2)$, with the Dirac operator,

$$\mathcal{D} = \begin{pmatrix} D & \gamma\Phi \\ \gamma\Phi^* & D \end{pmatrix},$$

where $\Phi \in \mathbb{C}$. The bimodule of one forms, associated to $\mathcal{D}$ consists of the following operators,

$$(4.6) \quad \omega = \begin{pmatrix} w^+ & \Phi\gamma f^+ \\ \Phi^*\gamma f^- & w^- \end{pmatrix},$$

where $w^\pm \in \Omega_D^1(M)$ and $f^\pm \in C^\infty(M)$. We shall denote the diagonal part as ω^d and the off-diagonal part as ω^o .

In order to calculate $\mathcal{W}(\omega_1 \omega_2 \omega_3 \mathcal{D} \mathcal{D}^{-2m})$ for one-forms $\omega_1, \omega_2, \omega_3$ let us observe that due to linearity it suffices to do it separately for diagonal and off-diagonal parts. Moreover, the algebra rules in this extended Clifford algebra are obvious and we restrict ourselves to the four possible cases. Furthermore, since $\mathcal{D}^{-2m} = D^{-2m} \mathbf{1} (1 - 2m|\Phi|^2 D^{-2} + \dots)$ we note that in the Wodzicki residue formula $\mathcal{D}^{-2m}$ can be replaced by D^{-2m} .

By a straightforward calculation we get,

$$\begin{aligned}
 \mathcal{W}(\omega_1^d \omega_2^d \omega_3^d \mathcal{D} \mathcal{D}^{-2m}) &= 0, \\
 \mathcal{W}(\omega_1^d \omega_2^d \omega_3^o \mathcal{D} \mathcal{D}^{-2m}) &= \mathcal{W}(|\Phi|^2 (w_1^+ w_2^+ f_3^+ + w_1^- w_2^- f_3^-) D^{-2m}) \\
 &= |\Phi|^2 (g(w_1^+, w_2^+) f_3^+ + g(w_1^-, w_2^-) f_3^-), \\
 \mathcal{W}(\omega_1^d \omega_2^o \omega_3^o \mathcal{D} \mathcal{D}^{-2m}) &= \mathcal{W}(|\Phi|^2 (w_1^+ f_2^+ f_3^- + w_1^- f_2^- f_3^+) D D^{-2m}) = 0, \\
 \mathcal{W}(\omega_1^o \omega_2^o \omega_3^o \mathcal{D} \mathcal{D}^{-2m}) &= \mathcal{W}(|\Phi|^4 (f_1^+ f_2^- f_3^+ + f_1^- f_2^+ f_3^-) D^{-2m}) \\
 &= |\Phi|^4 \mathcal{V}(f_1^+ f_2^- f_3^+ + f_1^- f_2^+ f_3^-).
 \end{aligned}
 \tag{4.7}$$

where g is the metric functional on M and the functional $\mathcal{V}(f)$ is proportional to $\int_M f$. Note that the first equation is a consequence of D being torsion-free, whereas the third equation holds if $\mathcal{W}(\omega D D^{-2m}) = 0$, which is the consequence of the spectral triple on M being spectrally closed.

As a consequence, we see that the product of the standard spectral triple over M with a finite discrete one does not satisfy the torsion-free condition unless $\Phi = 0$. Note that the generalisation of the model with Φ being a complex-valued function on M does not change the conclusion.

4.4. Conformally rescaled noncommutative tori. Consider the spectral triple on the noncommutative n -torus, $(\mathcal{A}, \mathcal{H}, D_k)$, where $\mathcal{A} = C^\infty(\mathbb{T}_\theta^n)$ is the δ_j -smooth subalgebra for the standard derivations δ_j , $j = 1, \dots, n$. Next, $\mathcal{H} = L^2(\mathbb{T}_\theta^n, \tau)$, where τ is the standard trace which annihilates δ_j . Also, $D_k = kDk$ is the conformal rescaling of the standard (flat) Dirac operator $D = \sum_j \gamma^j \delta_j$ with γ^j being the usual gamma matrices and the conformal factor $k > 0$ is from the copy $\mathcal{A}^o$ of $\mathcal{A}$ in the commutant of $\mathcal{A}$. The bimodule of one-forms, generated by the commutators $[D_k, \alpha]$, $\alpha \in \mathcal{A}$, is a free left module generated by $k^2 \gamma^j$. Using the extension of the calculus of pseudodifferential operators over noncommutative tori, following [8], to $\hat{\mathcal{A}}$ -valued symbols (where $\hat{\mathcal{A}}$ is the algebra generated by $\mathcal{A}$ and $\mathcal{A}^o$) and of the analogue of the Wodzicki residue, we defined and compute various spectral functionals, as the metric and Einstein functionals [11, Section 5.2.1] in dimension 2 and 4. In particular, we demonstrated there that strictly irrational noncommutative 2- and 4-dimensional tori with a conformally rescaled Dirac operator are spectrally closed. We extend here this result to any dimension:

Theorem 4.1. *The spectral triple $(C^\infty(\mathbb{T}_\theta^n), \mathcal{H}, D_k)$ for strictly irrational noncommutative torus of dimension n is spectrally closed, that is for any T , operator of order 0 (that is generated by $a, [D_k, b]$, $a, b \in \mathcal{A}$)*

$$\mathcal{W}(TD_k |D_k|^{-n}) = 0.$$

Proof. The proof is based on a simple observation that the symbol of order $-n$ of $TD_k |D_k|^{-n}$ is a sum of terms that are of the form $T_{\alpha,j,\beta}(\xi) k^\alpha \delta_j(k) k^\beta$, for $j = 1, \dots, n$, some integers α, β and $C^\infty(\mathbb{T}_\theta^n)$ -valued functions $T_{\alpha,j,\beta}(\xi)$, which are homogeneous in ξ of degree $-n$. Since for strictly irrational tori the trace over the full algebra $\mathcal{A} \mathcal{A}^o$ factorizes, then it is sufficient to show that $\tau(k^\alpha \delta_j(k) k^\beta) = 0$ for any j, α, β . For that we write $k = e^h$ and observe that

$$\tau(k^\alpha \delta_j(k) k^\beta) = \tau(e^{(\alpha+\beta)h} \delta_j(e^h))$$

vanishes since for $\alpha + \beta \neq -1$ it equals

$$\frac{1}{\alpha + \beta + 1} \tau(\delta_j(e^{(\alpha+\beta+1)h}))$$

while for $\alpha + \beta = -1$ it equals $\tau(\delta_j(h))$. This ends the proof. $\square$

As an immediate consequence of spectral closedness, we have:

Corollary 4.2. *The Dirac operator D_k over the n -dimensional strictly irrational noncommutative torus $\mathbb{T}_\theta^n$ is torsion free.*

4.5. Quantum $SU(2)$.

We briefly recall the construction of the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ of [9]. Let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$ -algebra generated by a and b , subject to the commutation rules:

$$(4.8) \quad \begin{aligned} ba &= qab, & b^*a &= qab^*, & bb^* &= b^*b, \\ a^*a + q^2b^*b &= 1, & aa^* + bb^* &= 1, \end{aligned}$$

where $0 < q < 1$. The Hilbert space $\mathcal{H}$ has an orthonormal basis $|j\mu n\uparrow\rangle$ for $j = 0, \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^+, \dots, j^+$; together with $|j\mu n\downarrow\rangle$ for $j = \frac{1}{2}, 1, \dots$, $\mu = -j, \dots, j$ and $n = -j^-, \dots, j^-$, where $j^+ = j + \frac{1}{2}$ and $j^- = j - \frac{1}{2}$. For the explicit formulas defining the representation of $\mathcal{A}$ on $\mathcal{H}$ we refer to [9] or [10]. The equivariant Dirac operator is chosen to be isospectral to the classical Dirac over $SU(2)$;

$$(4.9) \quad D|j\mu n\uparrow\rangle = (2j + \tfrac{3}{2})|j\mu n\uparrow\rangle, \quad D|j\mu n\downarrow\rangle = -(2j + \tfrac{1}{2})|j\mu n\downarrow\rangle.$$

In [10] it was shown that the triple is regular and its dimension spectrum of the spectral triple was determined, together with an explicit formula that computes respective noncommutative residue. In particular, the following theorem was proven.

Theorem 4.3. *If $T \in \Psi^0(\mathcal{A})$ is in the algebra of pseudodifferential operators of order 0 (as defined in [10]) then for $P^\uparrow, P^\downarrow$, projections on the respective up and down part of the Hilbert space of spinors, the noncommutative integral of $P^{(\uparrow, \downarrow)}T|D|^{-2}$ could be explicitly computed as:*

$$(4.10) \quad \begin{aligned} \oint P^\uparrow T|D|^{-2} &= (\tau_1 \otimes \tau_0^\uparrow + \tau_0^\downarrow \otimes \tau_1)(\beta(T)), \\ \oint P^\downarrow T|D|^{-2} &= (\tau_1 \otimes \tau_0^\downarrow + \tau_0^\uparrow \otimes \tau_1)(\beta^0(T)), \end{aligned}$$

where the noncommutative integral is, as usually,

$$\oint T := \text{Res}_{z=0} \text{Tr}(T|D|^{-z}).$$

The map β^0 is a projection on the two first legs of the zero-degree part of a map ρ , which is a homomorphism

$$(4.11) \quad \rho : \mathcal{B} \rightarrow \mathcal{A}(D_q^2) \otimes \mathcal{A}(D_q^2) \otimes \mathcal{A}(\mathbb{S}^1),$$

where $\mathcal{B}$ is a subalgebra of $\Psi^0(\mathcal{A})$ generated by diagonal parts of the generators a, b and their commutators with D and $|D|$, $\mathcal{A}(D_q^2)$ are quantum discs. Using the symbol map $\sigma : \mathcal{A}(D_q^2) \rightarrow$

$\mathcal{A}(\mathbb{S}^1)$ and the standard irreducible representation of quantum discs π the functionals $\tau_1, \tau_0^\uparrow, \tau_0^\downarrow$ are defined for $x \in \mathcal{A}(D_q^2)$, as

$$(4.12) \quad \begin{aligned} \tau_1(x) &:= \frac{1}{2\pi} \int_{\mathbb{S}^1} \sigma(x), \\ \tau_0^\uparrow(x) &:= \lim_{N \rightarrow \infty} \text{Tr}_N \pi(x) - (N + \frac{3}{2})\tau_1(x), \\ \tau_0^\downarrow(x) &:= \lim_{N \rightarrow \infty} \text{Tr}_N \pi(x) - (N + \frac{1}{2})\tau_1(x). \end{aligned}$$

Using the above theorem we can state as a simple corollary,

Corollary 4.4. *The standard equivariant spectral triple over $\mathcal{A}(SU_q(2))$ is spectrally closed and, in particular, the Dirac operator (4.9) is torsion-free.*

Proof. We compute,

$$\oint T D |D|^{-3} = \oint T (P^\uparrow - P^\downarrow) |D|^{-3} = (\tau_1 \otimes (\tau_0^\uparrow - \tau_0^\downarrow) + (\tau_0^\downarrow - \tau_0^\uparrow) \otimes \tau_1)(\beta(T)),$$

where we used (4.10). Then, from the definition of traces we see that:

$$\tau_0^\uparrow(x) - \tau_0^\downarrow(x) = -\tau_1(x),$$

and therefore

$$\oint T D |D|^{-3} = (\tau_1 \otimes (-\tau_1) + (\tau_1) \otimes \tau_1)(\beta(T)) = 0. \quad \square$$

5. FINAL REMARKS

We introduced a tangible notion of torsion for finite summable, regular spectral triples with a generalised noncommutative trace via a spectral trilinear functional of Dirac one-forms. It allows a direct verification whether the spectral triple (or the Dirac operator) is torsion-free. Four of the discussed examples are indeed torsion-free except the simplest case of an almost-commutative geometry. We conjecture that the spectral triples over almost-commutative geometries with non-simple internal algebra, nontrivially coupled by D , have a non-vanishing torsion. We hope that in many other cases, the spectral torsion functional will enable a more advanced study of the families of Dirac operators.

DECLARATIONS

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2.4 Outlook and generalization

In previous papers, we discussed spectral functionals independently. The natural progression would be to investigate how non-vanishing torsion affects the other functionals. An article on this issue is still in preparation, however in this section, the important results in the commutative case will be sketched.

2.4.1 Functionals for Laplace-type operators

Let us start with the general Laplace-type operator L on some vector bundle with symbol $\sigma(L) = \mathbf{a}_2 + \mathbf{a}_1 + \mathbf{a}_0$. For Laplace-type operators we have $\mathbf{a}_2 = g^{ab}\xi_a\xi_b$. In normal coordinates we can write

$$\mathbf{a}_2 = \xi_a\xi_b \left(\delta_{ab} + \frac{1}{3}R_{acbd}x^cx^d \right) + o(\mathbf{x}^2), \quad \mathbf{a}_1 = i\xi_a(T_a + P_{ab}x^b) + o(\mathbf{x}), \quad \mathbf{a}_0 = Q + o(\mathbf{1}), \quad (2.10)$$

where T_a, P_{ab}, Q are some endomorphisms of the vector bundle in $x = 0$. Using corollary A.2 from [P1] generalized as in the end of section 2.1 we get $\sigma(L^{-k}) = \mathbf{c}_{2k} + \mathbf{c}_{2k+1} + \mathbf{c}_{2k+2} + \dots$ with

$$\begin{aligned} \mathbf{c}_{2k} &= \xi_a\xi_b ||\xi||^{-2k-2} \left(\delta_{ab} - \frac{k}{3}R_{acbd}x^cx^d \right) + o(\mathbf{x}^2), \\ \mathbf{c}_{2k+1} &= -ik\xi_a ||\xi||^{-2k-2} (T_a + P_{ab}x^b) + o(\mathbf{x}), \\ \mathbf{c}_{2k+2} &= -k ||\xi||^{-2k-2} Q + k(k+1) ||\xi||^{-2k-4} \left(P_{ab} - \frac{1}{3}Ric_{ab} - \frac{1}{2}T_aT_b \right) \xi_a\xi_b + o(\mathbf{1}). \end{aligned} \quad (2.11)$$

The functionals in question are of the type $Wres(OL^{-m})$, where $n = 2m$ is the dimension of the manifold and O is a differential operator of order ≤ 2 . The symbol of O in $x = 0$ can be written as $\sigma(O) = F^{ab}\xi_a\xi_b + iG^a\xi_a + H$. In this case we can write

Lemma 1. *For operators L, O as above we have*

$$\begin{aligned} Wres(OL^{-m}) &= \frac{v_{n-1}}{24} \int Tr [24H + 12G^aT_a + F^{aa}(-12Q + 6P_{bb} - 2R - 3T_bT_b) \\ &\quad + (F^{ab} + F^{ba})(-6P_{ab} + 2Ric_{ab} - 3T_aT_b)], \end{aligned} \quad (2.12)$$

where v_{n-1} is the volume of a unit $(n-1)$ -sphere.

Proof. To calculate the Wodzicki residue, we need to compute $\sigma_{-n}(OL^{-m})$ in $x = 0$, integrate it over $|\xi| = 1$ take a trace over the vector bundle and integrate it over the manifold. Here, we need to complete the first two steps. Using the symbol composition rule, we get six terms

1. $\int_{||\xi||=1} H \mathbf{c}_{2m} = v_{n-1}H + o(\mathbf{1}),$
2. $\int_{||\xi||=1} iG^a\xi_a \mathbf{c}_{2m+1} = \frac{1}{2}v_{n-1}G^aT_a + o(\mathbf{1}),$
3. $\int_{||\xi||=1} G^a\partial_a \mathbf{c}_{2m} = o(\mathbf{1}),$
4. $\int_{||\xi||=1} F^{ab}\xi_a\xi_b \mathbf{c}_{2m+2} = \frac{1}{24}v_{n-1}[F^{aa}(-12Q + 6P_{bb} - 2R - 3T_bT_b) \\ + (F^{ab} + F^{ba})(6P_{ab} - 2Ric_{ab} - 3T_aT_b)] + o(\mathbf{1}),$

5. $\int_{||\xi||=1} -i(F^{ab} + F^{ba})\xi_a \partial_b \mathfrak{c}_{2m+1} = -\frac{v_{n-1}}{2}(F^{ab} + F^{ba})P_{ab} + o(\mathbf{1}),$
6. $\int_{||\xi||=1} -\frac{1}{2}(F^{ab} + F^{ba})\partial_{ab} \mathfrak{c}_{2m} = \frac{v_{n-1}}{3}F^{ab} Ric_{ab} + o(\mathbf{1}).$

The sum of these terms gives us the desired result. $\square$

There are some special types of operator O , which are particularly interesting.

Corollary 2. *Taking $O = EL$ in lemma 1 for some bundle endomorphism E , we have $F^{ab} = \delta^{ab}$, $G^a = T_a$ and $H = Q$, hence*

$$\text{Wres}(EL^{-m+1}) = \frac{(n-2)v_{n-1}}{24} \int \text{Tr} [E(-12Q + 6P_{aa} - 2R - 3T_a T_a)]. \quad (2.13)$$

Using the sum $O + EL$, for O as in lemma 1, we get

$$\begin{aligned} \text{Wres}((O + EL)L^{-m}) &= \frac{v_{n-1}}{24} \int \text{Tr} [24H + 12G^a T_a \\ &\quad + (F^{aa} + (n-2)E)(-12Q + 6P_{bb} - 2R - 3T_b T_b) \\ &\quad + (F^{ab} + F^{ba})(-6P_{ab} + 2Ric_{ab} - 3T_a T_b)]. \end{aligned} \quad (2.14)$$

Dirac-type operators on manifolds

In most cases we take $L = D^2$ for some Dirac-type operator. Let us consider a Dirac-type operator $D = D_0 + B$, where B is an operator of order 0 and D_0 is another Dirac-type operator (usually a canonical one determined by metric on the manifold) with symbol $\sigma(D_0) = \mathfrak{d}_1 + \mathfrak{d}_0$. We require, that in normal coordinates $\mathfrak{d}_1 = -\gamma^a \xi_a + f_{bc}^a x^b x^c + o(\mathbf{x}^2)$ (without linear term in x) and $\mathfrak{d}_0 = g_a x^a + o(\mathbf{x})$ (only linear term), where f_{bc}^a, g_a are some endomorphisms.

The square of D_0 is a Laplace-type operator³ L_0 with $T_a^0 = 0$. We can see it as

$$\mathfrak{a}_1 = \mathfrak{d}_1 \mathfrak{d}_0 + \mathfrak{d}_0 \mathfrak{d}_1 - i \partial_a^\xi \mathfrak{d}_1 \partial_a^x \mathfrak{d}_1.$$

Each of these three terms vanish at $x = 0$, resulting in $T_a^0 = 0$.

We are particularly interested in two examples of D_0 : the canonical Dirac operator and the Hodge-Dirac operator. The latter is explicitly written in [P2], and it is clear that it meets the requirements on D_0 . The former can be easily derived in normal coordinates using the construction in [P3]. For $e_i = \partial_i - \frac{1}{6} R_{ijkl} x^j x^k \partial_l + o(\mathbf{x}^2)$ we get $c_{ijk} = \frac{1}{2} R_{ijak} x^a + o(\mathbf{x})$. Then $\omega_{abc}^{LC} = -\frac{1}{2} R_{iabc} x^i + o(\mathbf{x})$ and

$$\begin{aligned} \mathfrak{d}_1 &= -\gamma^a \xi_a + \frac{1}{6} \gamma^a R_{abcd} x^b x^c \xi_d + o(\mathbf{x}^2), \\ \mathfrak{d}_0 &= \frac{i}{8} R_{iabc} x^i \gamma^a \gamma^b \gamma^c + o(\mathbf{x}) = -\frac{i}{4} Ric_{ia} x^i \gamma^a + o(\mathbf{x}). \end{aligned} \quad (2.15)$$

The last equality can be obtained using the symmetries of the Riemann tensor and the algebra rules for γ 's.

$$\begin{aligned} R_{iabc} \gamma^a \gamma^b \gamma^c &= -(Ric_{ab} + Ric_{ba}) \gamma^a \gamma^b \gamma^c = -R_{iabc} (\gamma^b \gamma^c \gamma^a + \gamma^c \gamma^a \gamma^b) \\ &= -2R_{iabc} (\gamma^a \gamma^b \gamma^c + \delta^{ac} \gamma^b + \delta^{ac} \gamma^b - \delta^{ab} \gamma^c - \delta^{bc} \gamma^a) = -2R_{iabc} \gamma^a \gamma^b \gamma^c - 6Ric_{ia} \gamma^a. \end{aligned}$$

³It provides a condition on f_{bc}^a as $\mathfrak{d}_1^2 = \mathfrak{a}_2$. We have $(\{\gamma^a, f_{cd}^b\} + \frac{1}{3} R_{acbd}) \xi_a \xi_b x^c x^d = 0$.

It is clear that the canonical Dirac operator is also of the D_0 type.

Because we already know the metric, Einstein and torsion functionals for these two examples of D_0 , we will focus on calculating the corrections from B . First, we need to determine $L = D^2$.

$$L = D^2 = (D_0 + B)^2 = L_0 + B^2 + \{D_0, B\}, \quad (2.16)$$

where $L_0 = D_0^2$. In normal coordinates we need to know the symbols of L up to appropriate powers of x . We start with expanding $B = B_0 + B_a x^a + o(\mathbf{x})$. Next, we need to know the part proportional to ξ up to linear terms in x and the order zero part only in $x = 0$, therefore we can use $\{i\gamma^a \partial_a, B\}$ instead of $\{D_0, B\}$.

$$\{i\gamma^a \partial_a, B\} = i\{\gamma^a, B\} \partial_a + i\gamma^a (\partial_a B). \quad (2.17)$$

Finally, using notation from (2.10), we get

$$P_{ab} = P_{ab}^0 + i\{\gamma^a, B_b\}, \quad T_a = i\{\gamma^a, B_0\}, \quad Q = Q^0 + i\gamma^a B_a + B_0^2, \quad (2.18)$$

where Q^0, P_{ab}^0 are calculated for $L_0 = D_0^2$ and T_a^0 vanishes, as stated before.

Before proceeding further, we compute $Wres(OL^{-m})$ for one more example of interesting operators O , i.e. $O = ED$.

Corollary 3. *Taking $O = ED$ in lemma 1, where E is a bundle endomorphism, we have $F^{ab} = 0$, $G^a = iE\gamma^a$, $H = EB_0$, and*

$$Wres(EDD^{-2m}) = \frac{v_{n-1}}{2} \int \text{Tr}[E(2B_0 + i\gamma^a T_a)] = \frac{v_{n-1}}{2} \int \text{Tr}[E(2B_0 - \gamma^a \{\gamma^a, B_0\})]. \quad (2.19)$$

2.4.2 Spectral functionals for a class of Dirac-type operators on manifolds

Following the notation from [P1, P3], we will denote $\hat{u}$ as a multiplication by 1-form u represented on the vector bundle, with $\hat{u} = \gamma^a u_a$. We will now calculate explicitly the spectral functionals in the presence of B . The results will be valid for both D_0 operators, canonical Dirac, and Hodge-Dirac.

We assume that all the functionals considered below are known for D_0 , and we focus on the terms that depend on B (note subscripts D and D_0 in functionals), hence the results will be written as a sum of a functional with $B = 0$ (unless it vanishes as in (2.19)) and B -dependent terms.

Metric functional

As usual, the metric functional remains unchanged. We can see that by putting $O = \hat{u}\hat{w}$. We can put it as the following proposition:

Proposition 4. *The metric remains the same for all B s.*

$$\mathcal{g}_D(u, w) = \mathcal{g}_{D_0}(u, w) = v_{n-1} \text{Tr}(1) \int u_a w_a = v_{n-1} \text{Tr}(1) \int g^{ab} u_a w_b \sqrt{g}. \quad (2.20)$$

Torsion functional

To compute the torsion functional we can use (2.19) with $E = \hat{u}\hat{v}\hat{w}$.

$$\begin{aligned}\mathcal{T}_D(u, v, w) &= \frac{v_{n-1}}{2} \int u_i v_j w_k \text{Tr} [\gamma^i \gamma^j \gamma^k (2B_0 - \gamma^a \{\gamma^a, B_0\})] \\ &= v_{n-1} \int u_i v_j w_k \text{Tr} [(\gamma^i \gamma^k \gamma^j - \gamma^j \gamma^k \gamma^i) B_0].\end{aligned}\tag{2.21}$$

It is easy to check, that the combination $\gamma^i \gamma^k \gamma^j - \gamma^j \gamma^k \gamma^i$ is totally antisymmetric in i, j, k .

We can also rewrite the result in more compact form, i.e.

Proposition 5. *The torsion functional for general B is*

$$\mathcal{T}_D(u, v, w) = \frac{v_{n-1}}{2} \int u_i v_j w_k \text{Tr} ([\gamma^j, \gamma^i] \{\gamma^k, B_0\}).\tag{2.22}$$

We keep here the result with B , and depending on the situation we can use concrete form of B . For Dirac-type operators with torsion, B is determined by the torsion, as shown in [P3].

Single 1-form functional

We can obtain another functional as a byproduct of the torsion functional. If we use only one 1-form in (2.19) instead of three, i.e. $E = \hat{u}$, we get

$$\text{Tr} [u_b \gamma^b (2B_0 - \gamma^a \{\gamma^a, B_0\})] = 0.\tag{2.23}$$

We can state this result as follows:

Proposition 6. *The functional $\mathcal{X}_D(u)$ defined on manifolds as follows:*

$$\mathcal{X}_D(u) := \text{Wres}(\hat{u} D D^{-2m})\tag{2.24}$$

vanishes identically for every B .

We obtained nontrivial functional which vanishes identically in the commutative case. However, this is not guaranteed in the noncommutative setting. It would be intriguing to investigate whether the vanishing of this functional is related to any interesting properties of spectral triples.

Einstein-Hilbert functional

If we use (2.13) with $E = 1$ or, more generally, $E = f$ for a function f , we get

$$\begin{aligned}\mathcal{R}_D(f) = \text{Wres}(f D^{-2m+2}) &= \frac{(n-2)v_{n-1}}{24} \int f \text{Tr} \left(-12Q^0 - 12i\gamma^a B_a - 12B_0^2 + 6P_{aa}^0 \right. \\ &\quad \left. + 6i\{\gamma^a, B_a\} - 2R + 3\{\gamma^a, B_0\}\{\gamma^a, B_0\} \right).\end{aligned}\tag{2.25}$$

Because

$$\text{Tr} \left(\{\gamma^a, B_0\}\{\gamma^a, B_0\} \right) = 2\text{Tr} \left(\gamma^a \{\gamma^a, B_0\} B_0 \right),$$

we have

Proposition 7. *The Einstein-Hilbert functional for general B on manifolds is*

$$\mathcal{R}_D(f) = \mathcal{R}_{D_0}(f) + \frac{(n-2)v_{n-1}}{4} \int f \operatorname{Tr} (-2B_0^2 + \gamma^a \{\gamma^a, B_0\} B_0), \quad (2.26)$$

where $\mathcal{R}_{D_0}(f)$ is the same functional for $B = 0$.

Einstein functional

The correction to the Einstein functional is by far the most tedious to obtain, therefore we start by stating the result before presenting the proof.

Proposition 8. *The Einstein functional for general B on manifolds is*

$$\begin{aligned} \mathcal{G}_D(u, w) = \mathcal{G}_{D_0}(u, w) + \frac{v_{n-1}}{2} \int \operatorname{Tr} \Big(iu_a(\partial_c w_b)[\gamma^a, \gamma^b]\{\gamma^c, B_0\} + \frac{i}{2}u_a w_b[\gamma^a, \gamma^b]\{\gamma^c, B_c\} \\ + u_a w_b[(\delta^{ab} B_0 - \gamma^a \{\gamma^b, B_0\})(\{\gamma^c, B_0\}\gamma^c - 2B_0)] \Big), \end{aligned} \quad (2.27)$$

where $\mathcal{G}_{D_0}(u, w)$ is the same functional with $B = 0$.

We see that the correction depends on the derivative of w_b . The term with this derivative is proportional to the same combination of γ 's and B_0 as in the torsion functional (2.22). This follows from theorem 5.7 in [P1], which was previously addressed in section 2.1.1.

Proof. We have to compute $\operatorname{Wres}[(\hat{u}D\hat{w}D + \hat{u}\hat{w}D^2)D^{-n}]$ using (2.14). Let us now examine operator $O = \hat{u}D\hat{w}D$. This operator consists of 4 parts:

1. $O_0 = \hat{u}D_0\hat{w}D_0 = -u_c w_d \gamma^c \gamma^a \gamma^d \gamma^b \partial_a \partial_b - u_c (\partial_b w_d) \gamma^c \gamma^b \gamma^d \gamma^a \partial_a + \frac{1}{4}u_c w_d \gamma^c \gamma^a \gamma^d \gamma^b \operatorname{Ric}_{ab} + o(\mathbf{1})$,
2. $\hat{u}D_0\hat{w}B = iu_c \gamma^c \gamma^a \gamma^b (w_b B_0 \partial_a + (\partial_a w_b) B_0 + w_b B_a) + o(\mathbf{1})$,
3. $\hat{u}B\hat{w}D_0 = iu_c w_b \gamma^c B_0 \gamma^b \gamma^a \partial_a + o(\mathbf{1})$,
4. $\hat{u}B\hat{w}B = u_a w_b \gamma^a B_0 \gamma^b B_0 + o(\mathbf{1})$.

The first part follows from equation (4.5) in [P1]. It gives raise to F_0^{ab} , G_0^a , H_0 . The full F^{ab} , G^a , H are

$$\begin{aligned} F^{ab} &= F_0^{ab} = u_c w_d \gamma^c \gamma^a \gamma^d \gamma^b = u_c w_d \gamma^c (2\delta^{ad} - \gamma^d \gamma^a) \gamma^b, \\ G^a &= G_0^a + iu_c w_b \gamma^c (\gamma^a \gamma^b B_0 + B_0 \gamma^b \gamma^a) = iu_c w_b \gamma^c (\gamma^a \gamma^b B_0 + B_0 \gamma^b \gamma^a), \\ G_0^a &= -u_c (\partial_b w_d) \gamma^c \gamma^b \gamma^d \gamma^a, \\ H &= H_0 + u_a w_b \gamma^a B_0 \gamma^b B_0 + iu_c \gamma^c \gamma^a \gamma^b ((\partial_a w_b) B_0 + w_b B_a). \end{aligned} \quad (2.28)$$

In (2.14) we need F^{aa} and $F^{ab} + F^{ba}$. It is easy to check, that

$$\begin{aligned} F^{aa} &= (2-n)u_a w_b \gamma^a \gamma^b, \\ F^{ab} + F^{ba} &= 2u_c w_d \gamma^c (\delta^{ad} \gamma^c + \delta^{bd} \gamma^a - \delta^{ab} \gamma^d). \end{aligned} \quad (2.29)$$

We can see, that the term $(F^{aa} + (n-2)E)$ in (2.14) vanishes, as $E = u_a w_b \gamma^a \gamma^b$.

Now we can write the Einstein functional.

$$\mathcal{G}_D(u, w) = \mathcal{G}_{D_0}(u, w) + \frac{v_{n-1}}{8} \int \text{Tr} [8(H - H_0) + 4G^a T_a + (F^{ab} + F^{ba})(-2(P_{ab} - P_{ab}^0) - T_a T_b)], \quad (2.30)$$

where $F^{ab} = F_0^{ab}$ and

$$\mathcal{G}_{D_0}(u, w) = \frac{v_{n-1}}{12} \int \text{Tr} [12H_0 + (F^{ab} + F^{ba})(-3P_{ab}^0 + Ric_{ab})]. \quad (2.31)$$

The trace in (2.30) is quite complicated. To simplify it, we can start with terms linear in B . We have

$$\begin{aligned} & \text{Tr} [8iu_c \gamma^c \gamma^a \gamma^b ((\partial_a w_b) B_0 + w_b B_a) - 4iu_c (\partial_b w_d) \gamma^c \gamma^b \gamma^d \gamma^a \{\gamma^a, B_0\} \\ & \quad - 4iu_c w_d \gamma^c (\delta^{ad} \gamma^b + \delta^{bd} \gamma^a - \delta^{ab} \gamma^d) \{\gamma^a, B_b\}] \\ & = 4iu_a \text{Tr} [(\partial_c w_b) (2\gamma^a \gamma^c \gamma^b B_0 - \gamma^a \gamma^c \gamma^b \gamma^d \{\gamma^d, B_0\}) \\ & \quad + w_b (2\gamma^a \gamma^c \gamma^b B_c - \gamma^a \gamma^c \{\gamma^b, B_c\} - \gamma^a \gamma^c \{\gamma^c, B_b\} + \gamma^a \gamma^b \{\gamma^c, B_c\})] \\ & = 4iu_a \text{Tr} [2(\partial_c w_b) (\gamma^a \gamma^b \gamma^c - \gamma^c \gamma^b \gamma^a) B_0 + w_b (\gamma^a \gamma^c \gamma^b B_c - \gamma^b \gamma^a \gamma^c B_c - 2\gamma^a B_b + \gamma^a \gamma^b \{\gamma^c, B_c\})] \\ & = 4iu_a \text{Tr} [(\partial_c w_b) [\gamma^a, \gamma^b] \{\gamma^c, B_0\} + w_b (-\gamma^a \gamma^b \gamma^c B_c - \gamma^b \gamma^a \gamma^c B_c + \gamma^a \gamma^b \{\gamma^c, B_c\})] \\ & = 2iu_a \text{Tr} [2(\partial_c w_b) [\gamma^a, \gamma^b] \{\gamma^c, B_0\} + w_b [\gamma^a, \gamma^b] \{\gamma^c, B_c\}]. \end{aligned}$$

The quadratic part can be written as

$$\begin{aligned} & \text{Tr} [8u_a w_b \gamma^a B_0 \gamma^b B_0 - 4u_c w_b \gamma^c (\gamma^a \gamma^b B_0 + B_0 \gamma^b \gamma^a) \{\gamma^a, B_0\} \\ & \quad + 2u_c w_d \gamma^c (\delta^{ad} \gamma^b + \delta^{bd} \gamma^a - \delta^{ab} \gamma^d) \{\gamma^a, B_0\} \{\gamma^b, B_0\}] \\ & = 2u_a w_b \text{Tr} [4\gamma^a B_0 \gamma^b B_0 - 2\gamma^a (\gamma^c \gamma^b B_0 + B_0 \gamma^b \gamma^c) \{\gamma^c, B_0\} \\ & \quad + \gamma^a (\gamma^c \{\gamma^b, B_0\} \{\gamma^c, B_0\} + \gamma^c \{\gamma^c, B_0\} \{\gamma^b, B_0\} - \gamma^b \{\gamma^c, B_0\} \{\gamma^c, B_0\})]. \end{aligned}$$

To simplify this expression we use the trace property, the anticommutation rules of γ 's and the fact, that $(\gamma^c)^2$ is proportional to identity (thus commutes with everything). We want every term to start with $\gamma^a \gamma^b B_0$, $\gamma^b \gamma^a B_0$, $\gamma^a B_0 \gamma^b$ or $\delta^{ab} B_0$. We get

$$\begin{aligned} & 2u_a w_b \text{Tr} [(-2\gamma^a B_0 \gamma^b + \gamma^b \gamma^a B_0 - \gamma^a \gamma^b B_0) \{\gamma^c, B_0\} \gamma^c + 4\gamma^a B_0 \gamma^b B_0 - 2\gamma^b \gamma^a B_0^2 + 2\gamma^a \gamma^b B_0^2] \\ & = 4u_a w_b \text{Tr} [(\delta^{ab} B_0 - \gamma^a \{\gamma^b, B_0\}) (\{\gamma^c, B_0\} \gamma^c - 2B_0)]. \end{aligned}$$

The proof is completed by summing the linear and quadratic terms. □

Chapter 3

Conclusions

The functionals introduced in the papers are very promising tools for generalizing tensor qualities from Riemannian geometry to noncommutative geometry. We demonstrated that explicit calculations of these functionals are feasible for at least some of the spectral triples considered in the literature, justifying the belief that they are more than simply abstract concepts but practical computational tools. Thus, further study in this topic may be very intriguing, especially since it is still in its early stages and there is much work to be done in both commutative and noncommutative settings.

In the commutative case, the two most essential points appear to be discovering more geometrically relevant functionals and generalizing them to a larger class of manifolds and operators. To begin, we believe that it is still possible to find analogous functionals involving different objects such as the Riemann tensor, or different functionals that result in similar expressions. In the last situation, it would be very interesting to determine if the results are still similar in the noncommutative case.

Furthermore, in the commutative case, we almost exclusively worked with even-dimensional compact manifolds with no boundary, leaving computation of these functionals in a more general setting for the future. We can also consider different and more general operators and check how they affect the known functionals, as partially done in section 2.4. Note that the research in this direction is currently realized by other groups [18, 30, 31].

In the noncommutative case, it would be beneficial to see additional examples of direct computations and learn about certain general properties of the functionals. Yet, even in the present state, spectral functionals can be extremely useful in understanding noncommutative spaces and their geometry.

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