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Relation on the Thesis:

Geometric spectral functionals

by Paweł Zalecki

Geometry is one of the most ancient disciplines in the history of science. It was started with the need to measure fields (the one with crops, not the ones like the Higgs field), and in its multi millennial history it has branched in various directions. Classical mechanics, with the study of the phase space, can be seen as a branch of geometry. This view received a dramatic shake up with the advent of *quantum mechanics*. Due to Heisenberg principle and its generalisations, it became impossible to discuss of phase space as an ordinary space made of points. It became necessary to consider position and momenta as *noncommuting* operators, and the state of a particle had to be generalised to a state on the algebra of these noncommuting operators. The quantum phase space had become a *noncommutative geometry*.

The description of a noncommutative geometry via a noncommutative algebra, represented as operators, is the main topic of the thesis of Paweł Zalecki. The thesis opens with a preamble where it is described how a series of mathematical results, mainly the Gelf'and-Naijmark theorem and its consequences, show that the description of a noncommutative geometry is via a C^* -algebra, represented as operators on a Hilbert space. This leads to a *spectral approach*, pioneered by Alain Connes and others, which has several physical applications, including a description of the standard models of fundamental interactions.

Although Connes' application of noncommutative geometry to physics is based on the spectral action, another approach to physics is based on the Wodzicki residue. This is in some sense an algebrization of pseudodifferential operators, and is particularly apt at the description of geometry used for the theory of gravitation.

General relativity is the theory of curved spacetime, and its roots are in Riemannian geometry. The dynamical variable of the theory is spacetime itself, via the metric tensor and its functions like curvature etc. A quantization of gravity cannot be performed via a

straightforward quantization of the metric tensor as a field, due to nonrenormalizability. It therefore is very likely that a theory of quantum gravity will pass through the quantization of spacetime itself, in a noncommutative geometry. It is therefore important to understand how the spectral approach relates to Einsteinian relativity, and in particular the Einstein tensor.

This is, as I will partly describe below, the main thread of this thesis. I note in passing that the candidate has a larger production, which parallels these developments, but is more concerned with fields. The three, three extremely interesting, articles collected form a coherent corpus, and the added material expands some aspects, and connects them. The thesis has a strong emphasis on the differential calculus, which is one of the trademarks of the Krakow school on noncommutative geometry.

The first article (*L. Dabrowski, A. Sitarz, P. Zalecki, Spectral metric and Einstein functionals, Adv. Math. Vol. 427, (2023)1091286*), lays the ground work. It deals with the definition of bilinear vector fields in the algebraic algebraic context, it can be seen as a prelude to the full noncommutative geometry. Several geometric relations are described in terms of the Wodzicki residue.

In my opinion the most interesting result of the paper is Theorem 4.1, where the Einstein metric tensor G is expressed as a Wodzicki residue. The final part is fully noncommutative, with applications to the standard example of the noncommutative torus. In this way the geometry of the previous sections is rendered fully noncommutative.

While the topological aspects of a noncommutative geometry are captured by the algebra, its states and representation, the all important metric aspects are encoded in the Dirac operator. The first article dealt with the usual representation of it on the spin bundle. There is however an alternative representation on the space of forms, the Dirac Hodge operator $d + \delta$. Since this operators is represented on the space of forms, its use is even more natural in the context fo the Wodzicki residue. The analysis of the Einstein tensor and the other relavant quantity in general relativity is the main topic of the second article (*L. Dabrowski, A. Sitarz, P. Zalecki, Spectral metric and Einstein functionals for Hodge-Dirac operator, arXiv:2307.14877, J. Noncommut. Geom. in print*). The results are coherent with the first article, denoting the robustness of the approach.

The third and final paper collected in the thesis: (*L. Dabrowski, A. Sitarz, P. Zalecki, Spectral Torsion, arXiv:2308.01644, Comm. Math. Phys. in print*), deals with the quantization of spaces with torsion. Although torsion is a possible characteristic for a curved

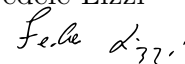
space, it has not received much attention in the literature, possibly because of the intricacies of the calculations. It is nevertheless of great physical interest, not last because the recent observations of gravitational waves may lead to experimental verifications. In this sense the quantization of spaces with torsion is very interesting. In addition, the paper adds another relevant entry in the spectral algebrization of the usual geometry, and is valuable for this. Again the tool is the Wodzicki residue. The introduced torsion is studied on a variety of spaces relevant for the physical applications of noncommutative geometry: the two sheeted manifold, Hodge-deRham, Einstein-Yang-Mills, the noncommutative torus, and the quantum group $SU(2)$. In most spaces the usual Dirac operator is torsion free, except the two sheeted manifold, opening the way for interesting coupling of gravity to the Higgs field.

As I said the results presented in this thesis are interesting, and I enjoyed reading it, also for the candidate sections which introduced, connected, and further explained the results. From what I read it emerges a very competent mathematical physicist, with showing a broad knowledge, superior, in my opinion, to that of a typical doctoral candidate. The results are original solution of physical problems, and they demonstrate a skilled physicist. I may add that all of the papers are (or will be) published in highly reputable journals.

I, the undersigned, hereby certify that the reviewed doctoral dissertation of Pawel Zalecki meets the statutory requirements set in the Polish Act - Law on Higher Education and Science for doctoral theses in the fields of natural and exact sciences within the discipline of physical sciences. I request its acceptance by the Discipline Council for Physical Sciences at Jagiellonian University and its authorization for public defence.

Furthermore, given the overall excellent and original quality of the thesis, both in terms of the results it contains, but also in terms of the care and attention that was taken in presenting these results in a coherent and complete manner, I recommend to award the thesis with distinction.

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