
Exact Standing Wave Spacetimes

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A doctoral thesis submitted to:

Faculty of Physics, Astronomy and Applied Computer Science, Jagiellonian University,

Kraków

To my parents
and Amma and Abba for selling eggs at a village called Naraicha . . .

“Be patient toward all that is unsolved in your heart and try to love the questions themselves, like locked rooms and like books that are now written in a very foreign tongue.

Do not now seek the answers, which cannot be given you because you would not be able to live them.

And the point is, to live everything. Live the questions now. Perhaps you will then gradually, without noticing it, live along some distant day into the answer.”

-Rainer Maria Rilke

Declaration

This thesis is a result of my individual effort, and it incorporates contributions that arise from the collaborative work done under my supervisor, Dr. hab. Sebastian Szybka, Prof. UJ. Every aspect of this thesis is original and has not been presented for any other academic qualification. The completion of this work was carried out under the guidance of my supervisor.

The following chapters are based on articles that have been peer-reviewed and published, or are in preparation:

- **Chapter 2**

Freely falling bodies in a standing-wave spacetime

Sebastian J. Szybka and Syed U. Naqvi. Phys. Rev. D **103**, 024011 – Published 6 January 2021, arXiv:2010.12549.

- **Chapter 3**

Standing electromagnetic waves coupled to gravity

Syed U. Naqvi. In preparation.

- **Chapter 4**

Chaos and Einstein-Rosen gravitational waves

Sebastian J. Szybka and Syed U. Naqvi. Phys. Rev. D **108**, L081501 – Published 6 October 2023, arXiv:2305.04873.

Except where stated otherwise by reference or acknowledgment, the work presented is entirely my own. The results in chapter 3 do not qualitatively differ from those in chapter 2, and as a result, they have not been published. I intend to publish them later in accordance with the ideas outlined in the conclusion section of chapter 3.

At the onset of my doctoral research, I delved into the theoretical aspects of the gravitational wave memory effect. Eager to explore its observational implications, I initiated a collaboration to acquire expertise in gravitational wave data analysis techniques. Currently, I am actively engaged in a collaboration, working optimization of techniques, specifically employing particle swarm optimization and differential

evolution, to analyze eccentric gravitational wave signals. This work is done using PyCBC software package used in the analysis of gravitational wave data. We checked the efficacy of our optimization code on simulated detector data and are now looking into real detector data. As this work falls outside the scope of this thesis, I have not included it here.

Syed Umair Husain Naqvi
December 2023

Acknowledgements

“Nie raz poniosłem w życiu porażkę i dlatego mi się udało. ”

For 5 years spent in Poland, I should at least learn one quote. It's about failures and successes. My failures are my own but I owe my successes (how so ever small) to a lot of people. Gravity is something that I had fallen in love with in my class 10th and it is unbelievable, for me at least, to imagine a person born in Nainital, India will get a chance to work on it 6000 km away in Kraków, Poland.

To make this happen, first of all, I thank my supervisor Dr. hab. Sebastian Szybka, Prof. UJ. He not only found time to clear my doubts but also my stupid questions about Physics. His constant guidance in helping me understand general relativity (GR) has always kept me motivated. I may not have fully mastered the intricacies of GR just yet, but I'm grateful to him for taking me closer to grasp the fundamentals so far. I would like to express my gratitude to Dr. Bhooshan Gadre, my collaborator, for his invaluable assistance in helping me comprehend the complexities of data analysis and observational aspects.

I thank my teachers (from Nainital, Lucknow, Delhi to Marseille and Kraków) for allowing me to embrace the 'idiot' in me and never stop asking questions. The same goes for my friends (from Indira Nagar, Gomti Nagar, Vijay Nagar, and Zakir Nagar) ¹ who provided emotional support, discussed non-Physics stuff, and controlled their urge to kill me for my poor jokes.

To Agam Sir for handing me my first research project during my Bachelors's. To Venkat Sir for instilling the passion. To Sen Ma'am and Ehsan Sir for helping me appreciate constructive criticism about my approach to Physics. To Eric Jullo, Anjan, Sanjay and Tabish Sir for making me aware of the hardships involved in research.

¹'Nagar' here means the area in a city in India. It can also mean a surname, like my first Physics teacher in school Mr. Nagar, Thank you to you too.

This Ph.D. journey would be incomplete without the amazing group of people at the observatory. Especially the ‘OAUJ-Mafia’ group, where we did novice presentations, questioned freely the concepts of Physics, and helped each other understand our topics better. Though ‘Chai’ in the fort was always good, more so were the outreach events we carried out. A special thanks from the bottom of my heart (and at the bottom of the page in their respective languages) to each of them.

At the observatory, I would like to thank Ela, Alek, and Tomasz for their administrative and other help.

I thank my mom and dad for not worrying much if I will become a mad scientist (though I still might have to give the Indian Civil Services exam after defending). To Tanya and Zubair for continuing to believe in me that I will become a Physicist and not a Physician or a cook (though there has been a significant increase in my cooking skills). To my wife, Tania for standing by and helping me maintain my curiosity in the hardest times.

Finally to some strangers for their kindness.² And thanks to ‘Biblioteki Narodowej w Warszawie’ where this thesis began during the pandemic.

धन्यवाद
வஹைர மஹி
மிகக் நன்றி
তোমাকে অনেক ধন্যবাদ
ବହୁତ୍ ବହୁତ୍ ଧନ୍ୟବାଦ୍
ماں اور باپ کا شکریہ
Dziękuję bardzo

²Including the border guard police who gave me a timely extension to complete my thesis formalities.

Abstract

This thesis explores standing gravitational wave solutions to Einstein's Field Equations. In addition to well-known exact solutions like black holes, cosmological models, and gravitational waves, standing gravitational waves provide insights into the inherent non-linearities of Einstein's field equations. This research focuses on investigating spacetimes which are toy models of standing gravitational waves in three distinct settings.

First, a standing wave vacuum solution with three-torus topology. I examine free-falling observers in the T^3 polarized Gowdy model. The geodesic deviation equation reveals attraction at the anti-nodes in the longitudinal direction of neighboring test particles confirmed by the geodesic equation. In the transverse direction, we found an analogue of memory effect. I analyze the spacetime dynamics using the Newman-Penrose formalism and study the Weyl scalars.

Second, I delve into an electrovacuum solution featuring standing electromagnetic waves. This investigation aims to comprehend the interplay between electromagnetic waves and gravity, as well as the characteristics of the resulting standing gravitational waves.

Finally, in a cylindrical standing gravitational wave spacetime, we identified chaotic geodesics and conducted a thorough analysis of their properties. Our investigation included examining fractal structures, Poincaré sections, and the presence of homoclinic and heteroclinic networks. The conclusive evidence of chaos in this Einstein-Rosen wave spacetime establishes a novel connection to other disciplines studying chaos.

Notation

c	Speed of light ($2.997 \times 10^8 \text{ m/s}$)
$\hbar$	Planck's constant ($6.62607015 \times 10^{-34} \text{ kgm}^2/\text{s}$)
$\otimes$	Tensor product
$*$	Hodge dual
GR	General Relativity
GW	Gravitational wave
EM	Electromagnetic wave
$E.F.E$	Einstein Field Equations
$\cup$	$A \cup B$ denotes union of sets A and B
$\cap$	$A \cap B$ denotes intersection of sets A and B
$\in$	$p \in A$ denotes that p is an element of A
$\emptyset$	the empty set
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^n$	the set of n -tuples real numbers
$\mathbb{C}$	the set of complex numbers
C^n	the set of n -times continuously differentiable functions
C^∞	the set of infinitely continuously differentiable functions
$\wedge$	The exterior product of two vectors u and v , denoted by $u \wedge v$
$\mapsto$	$f : A \rightarrow B$ denoted f is a map from the set A to the set B

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Chapter 1

Introduction

“It is important to remember that the physical interpretation of the mathematical notions occurring in a physical theory must be compatible with the equations of the theory.”

Andrzej Trautman

The theory of general relativity has been proven successful with an abundance of observational evidence. This evidence has spanned from the 1919 solar expedition to the Hulse-Taylor binary and recent gravitational wave detections [3]. The study of exact solutions, particularly those with gravitational radiation, continues to inspire and deepen our understanding [4]. Current efforts focus on exploring the observational, astrophysical, and quantum aspects of gravitational waves, pushing the boundaries of our knowledge in this fascinating field.

Gravitational wave detections have provided a new astrophysical window to analyze our universe. We get information about the inherent characteristics of the source, such as its mass, spin, and other related parameters. We also get to know the extrinsic parameters like luminosity distance, time of coalescence, etc [5]. These may assist further with binary black hole (BBH) population studies, which can be important inputs into galactic and cosmological investigations [6]. They can also serve as a cosmological window in the future to help us examine the dynamics that drive the expansion of our universe [6]. To date, around 100 detections have been made.¹ In addition to captivating astrophysical and cosmological inquiries,

¹Apart from binary coalescences, the recent discovery of gravitational wave background [7] has rebooted the stochastic GW searches.

there are still intriguing theoretical questions to explore in the realm of gravitational waves. Studying the interactions of gravitational waves provides an opportunity to explore beyond the linear regime and uncover their intricacies through exact solutions. As standing wave phenomena are commonly studied in both mechanical and electromagnetic settings, studying standing gravitational waves is an interesting research topic to explore. So delving into the dynamics of standing gravitational waves we can deepen our understanding of the fundamental workings of gravity.

Historically, there exist many exact solutions to Einstein field equations [4]. Many represent gravitational radiation but their physical properties are unclear. There are also non-trivial issues like – meaning of gravitational energy, what are the coordinate independent quantities associated with a spacetime, the formation of singularities, etc. But we also know that it was the exact solution study of gravitational waves that helped in solving many unanswered questions like [8] — What is the definition of a plane gravitational wave in the full theory? Do such waves carry energy? Does the full theory admit solutions corresponding to the gravitational waves emitted by bounded sources? After the pioneering Bondi-Pirani-Robinson papers [9, 10] and the Trautman-Robinson papers [11, 12], many of those questions were answered and the search for gravitational waves became more conclusive. There are still questions surrounding the interaction of gravitational and electromagnetic radiation, such as how they couple and what radiation in other spacetimes represents. As a step forward, we first need to decipher spacetimes, which can help us investigate what happens when two gravitational waves meet. This thesis aims to contribute to this study with the help of exact standing wave spacetimes.

The concept of standing waves in general relativity is still not well-defined, making it a non-trivial problem. While the aspect of traveling gravitational waves has been extensively studied in GR, particularly from bounded sources and in the context of exact solutions, standing waves are not yet well understood due to the scarcity of exact solutions that exhibit them. Moreover, the physically studied scenarios of standing GWs are also limited. Toy models of standing gravitational waves can provide some insight into their nature through the behavior of test masses.

Standing gravitational waves have been studied in linear settings [10] where you can superimpose gravitational wave solutions. A standing wave exact solution study can provide in-depth analysis at the non-linear level. It can help us understand the non-linear effects arising due to the interaction of gravitational waves. Also, as chaos forms an intriguing aspect in gravitational waves, the study of chaotic geodesics in these standing wave spacetimes can open up new possibilities for investigation.

The primary objective of this work is to gain a comprehensive theoretical understanding of the interaction between gravitational waves — namely in a standing wave setting. To achieve this, we investigate standing wave spacetimes using standard tools of general relativity. Initially, we examined vacuum spacetimes with standing gravitational waves. The results were published in a publication [13]. This was followed by the exploration of standing electromagnetic waves coupled with gravity. Additionally, our research efforts have led us to uncover chaos within cylindrical standing gravitational wave spacetimes by analyzing geodesics. Through the utilization of Poincaré maps, we further investigate the underlying non-linearity, revealing a fractal structure and intricate interplay between homoclinic and heteroclinic crossings and these results were published in [2]. These findings offer invaluable insights into the dynamics of gravitational wave interactions and invoke interest in standing gravitational waves.

First, we present a little context of colliding plane waves. Then we write about the study of chaos in general relativity. Then we discuss the phenomena of standing waves in other branches of physics following standing waves studies in general relativity. We end with a brief overview of general relativity and Einstein's equation and of the tools used to study dynamical systems. Therein we establish the basic mathematics and theorems that form the foundation of our work.

This thesis describes the work done in two published PRD papers and one unpublished work. First on freely falling observers in T^3 Gowdy spacetime [13]. Second on chaotic geodesics in Einstein-Rosen gravitational waves [2].

1.1 Historical context and motivation

We begin by establishing the historical context surrounding the study of gravitational wave interactions, which sets the stage for our first investigation on T^3 Gowdy spacetimes and our subsequent exploration of the electrovacuum case. Furthermore, we delve into the study of chaos in general relativity, laying the foundation for our third study on cylindrical standing gravitational waves.

1.1.1 Exact solutions involving wave interaction

A wave is usually assumed to be any disturbance in the medium propagating energy from one end to another. Sound travels via vibrations of air molecules as longitudinal

waves.² Electromagnetic waves are disturbances in the electromagnetic fields as transverse waves. But unlike a traveling wave, one also has standing waves which do not appear to travel. These are formed, in the simplest setting, by the superposition of two traveling waves having the same frequency and amplitude. Whether it's a guitar string plucked and producing sound waves or trapping particles by lasers with electromagnetic waves, standing waves arise in many instances. As gravitational waves have been detected, a natural question about their interaction arises and the phenomena of standing gravitational waves come across as an interesting study.

Gravitational waves can be considered as small ripples or disturbances in a larger background. When analyzing these waves in the linearized regime, they are typically treated as perturbations to the background. For example, astrophysical gravitational waves are studied as perturbations to flat spacetime or primordial gravitational waves which are studied as perturbations to the cosmological background. Suppose you have a pair of in-spiraling black holes or other compact objects that are very far from the detectors on Earth. The region around the detectors can be approximated as a region of flat spacetime. Then the incoming gravitational waves can be treated as perturbations assuming that the wavelength is short compared to the scale of the variation of the curvature. The detectors employ mirrors that reflect light back and forth, measuring the distance between them.³ This distance is influenced by perturbations that encode the nature of gravitational waves. The alterations in the mirror distance generate an interference pattern. Analyzing this interference pattern enables us to accurately determine the influence of gravitational waves. By analyzing this interference pattern, we can determine the effect of gravitational waves.

In principle, one can also look for exact solutions to Einstein Field Equations admitting gravitational waves.⁴ This was done in the cylindrical case by Einstein and Rosen⁵ who discovered a cylindrical gravitational wave solution. Apart from cylindrical symmetry, plane wave solutions have been also studied in [9, 17]

²Although sound waves have a characteristic longitudinal nature, in [14], it was theoretically and experimentally demonstrated that airborne sound can possess artificial transversality in an acoustic micropolar metamaterial.

³The constancy of the speed of light, c , is used to determine the time it takes for photons to travel back and forth. It is important to note that the gravitational wave (GW) wavelength is significantly longer than the wavelength of the light used in the detector. Therefore, it can reasonably be assumed that the incoming GW does not affect this light within the detector, allowing us to extract valuable information about the change in distance between the mirrors from the time it takes for the photons to traverse this path.

⁴One of the first wave-like solutions was found by H. W. Brinkmann [15].

⁵Earlier found by Gudio Beck [16].

Gravitational waves have been studied by various methods. They can be studied in a linearized regime or as exact solutions as highlighted above. There are other well-established methods like Post-Newtonian expansions and numerical relativity — for these, we refer the readers to other sources [18].⁶ Each of these methods has been successfully implemented in establishing the proof of gravitational waves through exhaustive data-analysis techniques [20]. However these methods have not been used to study interaction of GWs. Nonetheless we have exact solutions were one can probe the interactions of waves.

Before going further we highlight the intricacies involved with radiation in space-times. We consider electromagnetic waves in a vacuum as an example below. This is just to establish the difference between the linear and non-linear case. It's important to keep these technical aspects in mind while studying radiation in any spacetime.

In Einstein's theory, the curvature is symbolized by the left-hand side of the Einstein field equations, while the quantity of matter and energy creating that curvature is expressed on the right-hand side. For an EM wave in Einstein's theory, there is the contribution to the right-hand side due to the sources generating the EM field plus the EM field itself. One can then choose to study a simpler version of electromagnetic waves in Einstein's theory on a fixed background. In this approximation, the contribution of the electromagnetic field energy to the overall gravitational field can be neglected. Therefore omitting the coupling between EM and gravitational field, one can study electromagnetic fields in Einstein's theory. As the superposition of two EM waves can be studied in Maxwell's theory, one can also study the superposition of two GW waves in Einstein's theory at a linear level (where the contribution of GW to the background curvature can be neglected).

Now we consider the case when electromagnetic waves are coupled to gravity and the contribution of the EM field is taken into account. We know that an electromagnetic field has energy, so in Einstein's theory, it curves spacetime. Hence the electromagnetic fields are coupled to the perturbations in the space-time curvature. To deal with this one has to address Einstein-Maxwell equations which are nonlinear. So if electromagnetic waves pass through each other, they will have some non-linear interaction owing to the gravitational fields related to them. We see that even for electromagnetic waves in some spacetime, one has to deal with the non-linearities. A similar situation arose when people were trying to understand gravitational wave

⁶There is also the Effective-one-body (EOB) formalism. The EOB approach is an analytical method for solving the problem of two massive objects interacting through gravitational forces in GR. It describes the different phases of the two-body dynamics by mapping the problem to that of a test particle moving in an effective metric [19].

interactions in general relativity. Now we move to exact solutions focusing on the interaction of GWs.

The exact solution of Einstein's equations representing the mutual scattering of gravitational waves was first studied by Khan and Penrose [21]. They studied the interaction of two-plane impulsive gravitational waves. They concluded that after the collision, there is no longer a plane wave, and there is a focusing effect of each wave which results in a singularity. It was suggested that the presence of singularities may be a feature of spacetimes resulting from the collision of gravitational waves. This issue was also dealt with by Szekeres [22] and Matzner and Tipler [23]. They described the collision part with elementary functions and investigated the global nature of the singularities encountered. In an alternative study Chandrasekhar and Ferrari [24] made use of Ernst equations to analyze spacetimes with two spacelike Killing vectors which led them to Nutku-Halil solution [25] representing colliding impulsive gravitational waves with uncorrelated polarization. We refer the reader for details to the books [26, 4].

Apart from the interaction of plane waves in flat backgrounds, there have been studies involving colliding waves but in non-flat backgrounds. Gowdy cosmologies [27], having plane symmetry, represent a closed universe with interacting gravitational waves. These gravitational waves emerge from the initial 'big bang' singularity. Hayward [28] studied plane-symmetric spacetimes which were asymptotically flat in the past but had future curvature singularity. Centrella and Matzner [29] analyzed the collision of plane gravitational waves in a non-flat background. It is shown that, if the background spacetime is assumed to be expanding, then future singularity can be avoided. In such spacetime, the interaction of waves can be studied in detail. Though they have non-standard topologies, still they help in providing valuable information for models that cannot be studied otherwise.

In this thesis, one of the spacetimes studied is called Gowdy spacetimes. Our first work [13] deals with a special case of Gowdy spacetime, namely the three-torus solution, which is used to investigate the standing wave nature. These solutions can be probed with relative ease using just pen and paper, and are useful for studying the dynamics of Einstein's equations. Since they allow for gravitational waves of arbitrary wavelength, they provide a unique setting to study standing gravitational waves. This topic is addressed in the second chapter of the thesis. The third chapter will focus on the study of electrovacuum solution in a similar topology.

1.1.2 Exact solutions involving chaos

The Einstein field equations reveal to us the non-linear nature of gravity. They have been studied in various forms, including as a dynamical system. We know that deterministic non-linear dynamical systems can also exhibit complex behaviors like chaos. So, when we study the E.F.E as a dynamical system, the emergence of chaotic phenomena becomes a natural premise that has been extensively explored for quite some time.

We know that chaos as a phenomenon can occur even in simple settings like in discrete-time dynamical systems such as Hénon maps, logistic maps, etc. You also witness chaos in systems with differential equations in continuous-time dynamical systems like the Lorenz system, double pendulum, van der Pol oscillator, etc. And as in GR one usually deals with a set of coupled non-linear differential equations, chaos is expected. Heuristically one can imagine chaotic behavior in any spacetime arising due to non-linear coupling of metric functions via the differential equations. So chaos might manifest in the geodesics of a spacetime. In our work, we explored the nature of chaotic geodesics for a spacetime with a gravitational wave nature. Extensive research has explored chaos in various spacetimes, employing diverse settings and methodologies. Below, we present a concise overview of these investigations.

One of the earliest concepts used to study dynamical systems was the idea of entropy by Kolmogorov [30]. Here the ‘dynamical entropy’ was used to obtain a measure of how the distinct solution trajectories are generated in a system over long time intervals. This method was used by Barrow [31] to investigate chaos in Einstein’s equations in a cosmological setting.⁷ The relation between chaotic behavior and a sign of sectional curvature with the Jacobi metric was studied in [34] to probe chaos in Mixmaster cosmological models. Chaos was also studied in FLRW (Friedmann–Lemaître–Robertson–Walker metric) cosmology with inflation modeled using cosmological constant along with coupled matter fields [35].

There are already many known studies of chaos in astronomical studies starting from the famous three-body problem [36] to the non-linear motion of a star around a galactic center also known as Hénon–Heiles system [37], from the rings of Saturn [38] to the red spot of Jupiter [39]. In general relativity for one of the simplest setting, the motion of a test massive particle in black hole spacetime displays chaotic

⁷The setting was a diagonal Bianchi type-IX or “Mixmaster” universe which was studied by [32] and [33]. Mixmaster universe consists of gravitational waves with sufficient degrees of freedom to allow the spatial three-curvature to be anisotropic. The propagation of GWs generates the three-curvature anisotropy which has a backreaction upon their motion which is the reason for chaos in this spacetime.

behavior. In [40], they studied a criterion for the chaos of a test particle motion around an N -black hole system and found that a local instability determined by the Riemann curvature tensor provides a sufficient test for chaos. Then in [41] the motion of a spinning test particle in Schwarzschild spacetime was analyzed, using the Poincaré map and the Lyapunov exponent.⁸ Chaos was also studied for test particles moving around a slowly rotating black hole with a dipolar halo in [44]. Chaos for the Kerr case was studied in [45]. Through a numerical study, it was discovered that chaos exists in the geodesic motion of perturbed Kerr spacetime [46]. Also, the case for chaos in black hole shadow in $f(R)$ gravity was taken up in [47].¹⁰

Chaos was also found in the study of a vacuum gravitational collapse [48]. They found the presence of a fractal basin boundary at the critical surface in five-dimensional vacuum Einstein equations.

We now state a rudimentary notion of chaos in studies of GWs. Additionally, we will discuss the existing literature on chaos in GWs. A gravitational wave's basic idea is of a time-dependent gravitational potential keeping in mind that space and time are themselves the dynamic variables. Consider a double pendulum, which we know shows a sensitive dependence on the initial conditions of the oscillating angles; the system being non-integrable. Two uncoupled pendulums typically do not exhibit sensitive dependence on initial conditions and, therefore, are not chaotic. However, in the case of a double pendulum, the presence of coupling and the nonlinearity in its equations of motion result in an excess of degrees of freedom beyond the constants of motion, rendering the system non-integrable and prone to chaos. Nevertheless, there are instances of non-integrable systems, such as damped or driven oscillators, that do not display chaotic behavior. Likewise, if gravitational waves are studied as a perturbation to flat spacetime and if this perturbed system is non-integrable then chaos may or may not appear. But if one encounters sensitive dependence of some feature, say geodesics, on initial conditions, then one might expect chaos in such a setting.

In the work of Podolský, and Veselý [49] the geodesics studied in a non-homogeneous vacuum pp-waves demonstrated chaotic behavior. In [50] they studied a modification

⁸Lyapunov exponent characterizes the rate of separation of infinitesimally close trajectories in phase space. Also in [42], another method called the Melnikov method⁹ was used to identify chaos in nearly integrable systems to relativistic particle motion around a Schwarzschild black hole. This quantity has an issue with being *invariant* and more invariant tools to study chaos were studied in [43]

¹⁰The generalization of Einstein's theory of relativity known as $f(R)$ gravity results from relaxing the assumption that the gravitational field's Hilbert-Einstein action is strictly linear in the scalar of Ricci curvature R . The standard GR action $\int R\sqrt{-g}d^4x$ is generalised as $\int f(R)\sqrt{-g}d^4x$.

of the famous Hénon-Heiles potential the orbits of free test particles in plane-fronted gravitational waves spacetimes and found chaos for specific values of energies.

In the linear GW setting, the Melnikov method was used to find chaos for test particles in Schwarzschild black holes perturbed by a particular class of gravitational waves.¹¹ Shingo Suzuki and Kei-ichi Maeda found that if a spinning test particle's spin is greater than some critical value, it will exhibit chaotic behavior while orbiting a Schwarzschild black hole. They used the quadrupole formula to evaluate the gravitational wave emission. In [52], the gravitational waveforms from chaotic orbits were studied from the observational perspective. Also a new method was proposed in [53] for detecting the gravitational waves using the dynamic chaos signatures of a non-linear system perturbed by a gravitational wave.

We study cylindrical gravitational waves as a special case for the general Einstein-Rosen class. We found for the linearly polarised case, chaotic geodesics are present in the cylindrically symmetric standing gravitational waves. The periodic forcing provided by the standing wave in this spacetime gives rise to a chaotic homoclinic and heteroclinic network. This will form the basis of Chapter 4 in this thesis. Issues with invariant definitions to study chaos are also discussed in Chapter 4.

1.2 Standing waves in general physics

In this section, we delve into the core features of a stationary wave in one, two, and three dimensions. This is just to revise the readers with some of the well-known concepts. A few parts have been followed from [54]. Additionally, we briefly discuss non-linear stationary waves on water, known as Faraday waves, and the usual stationary waves detected in electromagnetic situations. This is to offer some background information for our research on chaos and stationary electromagnetic waves.

A wave is a vibration of particles around their mean position, that transfers energy through a medium or vacuum. We are all familiar with the notion of a wave in some medium- waves on water, sound waves, and waves in vacuum- electromagnetic waves that help us see things around us, etc. It can be characterized by its wavelength, amplitude, frequency, and speed.

There can be traveling waves where the entire waveform moves in one direction. It can also happen that a pair of superimposed periodic waves of the same frequency

¹¹ The particular type was studied in [51] where field equations governing perturbations of a spherical star in general relativity were solved using Born approximation.

travel in opposite directions. Then the wave may appear to be still forming a standing wave. A standing wave has fixed regions where the amplitude of vibrations goes to almost zero and some fixed regions where the vibrations are the maximum. These fixed regions are called node and antinode respectively.

A simple way to generate a standing wave is to produce oscillations in a string fixed at two ends. The waves generated at one end travel along the string and are reflected at the other, generating a continuous superposition that gives rise to interference patterns characteristic of standing waves. Standing waves are a fairly common phenomenon. They are found in sound waves, water surfaces, atomic orbitals, and in the famous Chladni patterns.¹²

Basic feature of standing wave: We follow the book by Rayleigh [55] for the definition of a standing wave which was also used by Chandrashekar [56] to define standing gravitational waves. For a standing wave, the vibrations are *stationary*. Here the motion of each particle of the system is proportional to some function of the time and this motion is the same for all the particles. Let's start with the wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } a \text{ is constant}$$

Assume $y = XT$, where X is a function of x and T is a function of t only. We get

$$\frac{1}{T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} = -m^2, \text{ } m \text{ is a constant}$$

The well known solution of such a differential equation is $T = C_1 \cos(mat) + C_2 \sin(mat)$ and $X = C_3 \cos(mx) + C_4 \sin(mx)$. So the vibrations turn out to be simple harmonic but of arbitrary period. The value of y , after some simplification, is

$$y = C \cos(mat - \epsilon_1) \cos(mx - \epsilon_2)$$

where $C, \epsilon_1, \epsilon_2$ are constants comprising C_i , where $i = 1, 2, 3, 4$. This shows that a simple form of stationary vibrations can be thought of as a superposition of equal progressive vibrations with opposite directions of propagation.

A standing wave is a product of pure spatial and time functions with regions of nodes and antinodes. We show this in the figure below.

¹²Chladni figures, also known as Chladni patterns, refer to the different modes of vibration on a rigid surface. When a plate or membrane resonates, it is divided into regions that vibrate in opposite directions. As a result, salt or other particles tend to clump toward the areas bounded by lines where no vibration occurs (nodal lines).

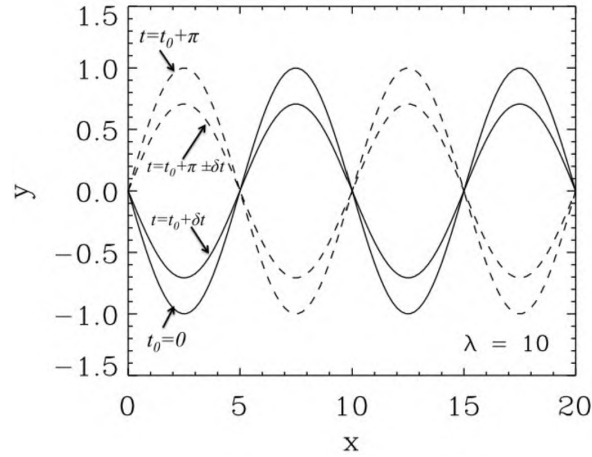


Fig. 1.1 An example of a standing wave with two full wavelengths shown at some instant of time. The solid curves represent $t_0 = 0$ and $t = t_0 + \delta t$. The dashed curves show where the waves will be at $t = t_0 + \pi$ and $t = t_0 + \pi \pm \delta t$. We see the stationary points where the wave is crossing the $y = 0$ axis; they are the nodes. Figure taken from [57].

In the realm of general relativity, investigating standing waves entails grappling with several complexities. These standing waves, much like their counterparts in classical physics, exhibit typical spatial and temporal dependencies. However, delving into finer details, such as amplitude and frequency, becomes notably intricate, especially when considering gravitational wave (GW) energy. On one hand, we may think the metric function governing spacetime behave in a standing wave manner [58]. Yet, comprehending the accumulation of GW energy at antinodes and nodes poses challenges due to the absence of a precise definition for GW energy in this context. We might speculate that akin to how standing electromagnetic waves have been used to trap particles at antinodes or how Chladni patterns reveal particle locations at nodes, there could be analogous behaviors in standing GWs. One could attempt to quantify this ‘trapping’ behavior at antinodes, given the propensity for GW energy accumulation to attract particles. However, the extent to which these behaviors hold for standing waves in diverse spacetime settings remains uncertain. A comprehensive study, incorporating both longitudinal and transverse effects, may shed light on the nature of standing gravitational waves within the framework of general relativity.

Standing waves in mechanical and electromagnetic setting: Various conditions can lead to the formation of standing waves. The more usual setting for demonstrating them is in a medium that is finite or bounded. Simple examples are a guitar string (starting from the base to the top and divided into many frets), a

drum surface (bounded by the rims stretching it), and air in a room (bounded by the walls). In such bounded mediums when the wavelength meets its correct reflection, standing waves occur. One of the well-known phenomena involved is resonance when the driving frequency is equal to the natural frequency of the system producing a standing wave. Also, a system producing standing waves has numerous natural frequencies known as harmonics—the simplest known as the fundamental or first harmonic, and so on.

The usual interesting part of a standing wave is not it is still, but rather the concept of getting a larger amplitude than the one driving the standing wave. We know that a wave of one particular frequency can amplify over other frequencies which makes it seem that we are getting more for less. This is what is used in non-electrical musical instruments, where if you pluck a string or blow a flute you are creating vibrations covering a spectrum of frequencies. The result you hear is the amplification of the fundamental frequency and its multiples.

For the one-dimensional case – There are various systems that we can consider, each with different boundary conditions. For example, a guitar has two fixed ends. A flute has two free ends, and a wine glass has one fixed and one free end. In the flute case, the simplest scenario for standing waves occurs, characterized by one node in the middle, effectively resulting in half a wavelength. To obtain the next possible standing wave pattern, we require an antinode positioned between two nodes, which results in one and a half wavelengths. This is shown in the middle part of Fig.(1.2) along with the other cases. Further details can be found in most high-school Physics textbooks.

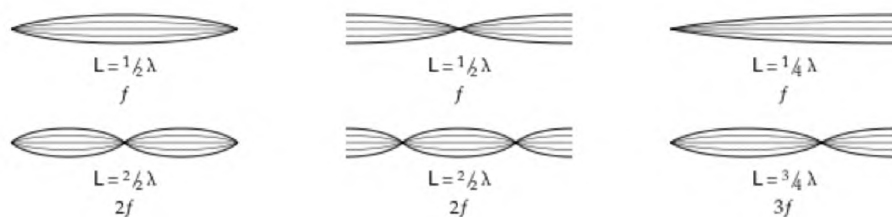


Fig. 1.2 1D example of nodes antinodes on a string. Fundamental node and first harmonic for two fixed ends (left), two free ends (middle), and one fixed end and one free end (right).

For the two-dimensional case – Here we focus on a vibrating circular membrane. Consider the vibrating membrane as an open disk $\mathbb{D}$ with a radius of a . At any given time t , the any point on the disk is given by a point (x, y) on it. The vertical

displacement is represented by $u = u(x, y, t)$ or in polar coordinates as $u = u(r, \theta, t)$. Such a configuration can be modelled as a wave equation in polar coordinates as $\partial^2 u / \partial t^2 = K^2 \nabla^2 u$ for $x^2 + y^2 < a^2$ and $u(x, y, t) = 0$ for $x^2 + y^2 = a^2$; where ∇ is the Laplace operator in polar coordinates and K is a constant. As a more familiar equation, we have

$$\frac{\partial^2 u}{\partial t^2} = K^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

Here, $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, and K is a constant that determines the wave speed based on the membrane. Using the separation of variables approach $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ and some initial and boundary conditions the solution is:

$$u_{mn}(r, \theta, t) = \sum_{m,n} \left(J_m(\lambda_{mn}r) \right) \left(A_m \cos(m\theta) + B_m \sin(m\theta) \right) \left(\cos(K\lambda_{mnt}t) \right)$$

where $m = 0, 1, \dots$ and $n = 1, 2, \dots$ and $\lambda_{mn} = \alpha_{mn}/a$, with α_{mn} being the n th positive root of the Bessel function of first kind J_m . Each solution represents a two-dimensional standing wave. The constants A_m, B_m are determined by the initial conditions. Trivial case of no vibration is for m, n both zero, for other cases you have vibrational modes of the membrane. So for example $u_{0,1}$ would imply only a vibrating central portion of membrane representing an antinode and the rim acting as a node. This is analogous to the fundamental mode of a vibrating string. Higher modes can exhibit intricate patterns composed of both circles and straight lines. For instance, the example of the $(0, 3)$ mode on a circular drum is shown in (1.3). It also shows the Chladni patterns are formed by particles on a vibrating surface representing a combination of nodes and antinodes.

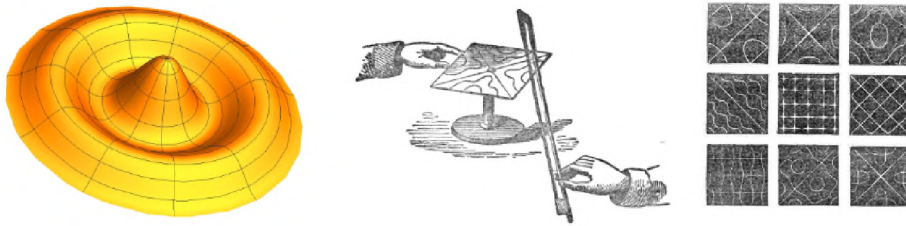


Fig. 1.3 Left: $(0,3)$ vibration mode of a circular membrane. Middle: Generating Chladni patterns on a rectangular tile. Right: Sample Chladni patterns generated. Image is taken from *The Project Gutenberg EBook of Sound*, by John Tyndall.¹³

For the three-dimensional case – Standing waves can also be studied for example on a cube or a sphere. Keep in mind that for the one-dimensional case, the nodes

were points, two-dimensional nodes were curves, and in three-dimensional, the nodes are two-dimensional surfaces. One of the physical examples is the orbital of electrons around an atom. From a quantum-mechanical perspective, we know how the wave function of an electron reveals its dynamics in terms of probability densities around the nucleus.¹⁴ So as a guitar string is constrained at two ends, so is the electron constrained around a nucleus. Owing to the constraint, vibrations are produced only at specific frequencies.¹⁵

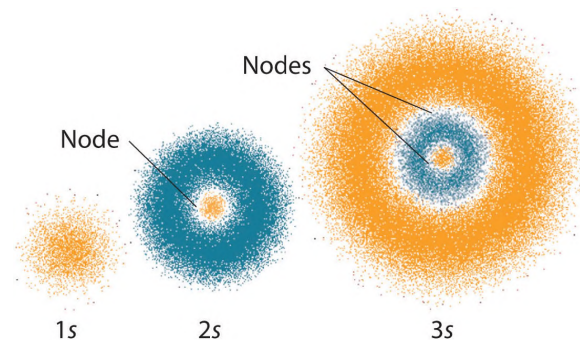


Fig. 1.4 Here we show the electron probability densities for the hydrogen 1s, 2s, and 3s orbitals. We see that the 2s and 3s orbitals feature nodes, representing regions with a 0% probability of finding an electron. The presence of nodes is akin to the standing waves discussed earlier. The alternating colors in the 2s and 3s orbitals denote regions with different phases, a crucial factor in chemical bonding.

About our work, we would like to mention the *Faraday wave* and *standing waves in an electromagnetic setting*.

If you have fluid on a vertically vibrating plate, then at some frequency you will have standing waves on the surface discovered by Michael Faraday. Interestingly, at a critical frequency (known as Faraday's instability), the surface of the water shows instability and there exist non-linear standing waves. And fluid droplets (of a certain size and at a particular frequency) [60] can be made 'walk' randomly on the surface. This phenomenon has been used to perform various hydrodynamical quantum analog experiments.¹⁶ Two walking fluid droplets on a vibrating surface can interact on a macroscopic level, orbiting each other at stable, discrete distances. This configuration

¹⁴Here we solve the time independent Schrödinger equation $-\hbar^2/2m\nabla^2\Psi + V\Psi = E\Psi$, where $\Psi = \Psi(r, \theta, \phi)$ is the wave-function associated with an electron of mass m and ∇ is Laplace operator in spherical coordinates.

¹⁵ The time-independent Schrödinger equation has solutions only for specific energy values, and these values are referred to as energy *eigenvalues*; each eigenvalue is associated with an *eigenfunction*. As these are discrete, an atom can only absorb or emit energy in quantas [59].

¹⁶The droplets are known as *Couder* droplets. The phenomenon is widely used to explore pilot wave theory.

analogously mirrors a bound state in a quantum mechanical system, with the discrete distances akin to discrete energy levels. A partial motivation for our third work was whether there would be classically quantized orbits at the antinodes of cylindrical gravitational waves. This study led us to find chaotic orbits in Einstein-Rosen waves.

Optical tweezers, employ a focused laser beam to trap and manipulate microscopic and sub-microscopic objects, such as atoms, nanoparticles, and droplets, resembling tweezers. It uses the properties of standing electromagnetic waves to trap particles [61].¹⁷ As done by [62], one can consider an electromagnetic resonator in flat space-time in general relativity and then study the standing gravitational waves generated by such a configuration. We investigated an electrovacuum solution featuring standing electromagnetic waves. In this particular unpublished work, our objective was to delve into the intricate dynamics and properties of the electrovacuum system under the influence of these stationary electromagnetic waves. By exploring this unique scenario, we aimed to gain insights into the interplay between electromagnetic fields and the gravitational aspects within the vacuum.

1.3 Standing waves in general relativity

Although there is currently no explicit definition for standing waves in the context of GR; exact standing wave spacetimes may help to elucidate their nature. We start with basic attempts to define standing gravitational waves and then outline proposed definitions from existing literature. The proposed definitions are not invariant and the standing GW concept in full GR is still elusive. We briefly comment on the concept of standing waves emerging at the linear level. We also discuss the methods proposed to define standing waves in full non-linear GR.

The dynamics of a spacetime are represented by Einstein field equations. Simply put, spacetime is a differentiable manifold with many structures build on it. One of the important structures, the metric functions represent the different configurations one can study. For example, the wave nature of Einstein field equations, where both the left and right hand side of the equations have to be taken into account. This often leads to dealing with non-linearity when considering exact solutions [63]. In a very simple case, the wave nature can be studied by either analyzing the form of the metric functions or the matter part of the Einstein equations. Therefore, standing

¹⁷The interference pattern created by the laser beams helps establish stable trapping regions. The distribution of light intensity in the standing wave pattern creates regions of higher and lower intensity, contributing to the forces that trap particles.

waves can be understood in terms of what oscillatory behavior is shown by the metric functions. Or the perturbation produced by a specific stress-energy tensor may have a standing-wave nature, i.e., a mixture of the spatial and temporal parts. It should be noted that the interpretation of the aforementioned concepts can only be considered as a heuristic approach to understanding a standing gravitational wave.

Initially, the nature of standing waves in general relativity was probed by analyzing the superposition of waves in the linear regime. Then restrictions on metric functions were put in place to further analyze standing waves in the context of exact solutions. One also had a ‘gravitational waves in a box’ class of solutions to find more about standing waves [64].

We can explore several areas of interest related to exact standing wave spacetimes having purely standing gravitational waves. One area of interest is the behavior of gravitational waves under linear superposition which has been explored by Bondi [10] and Grischuk [65]. Another important aspect is defining standing gravitational waves in a covariant manner. One can also seek to understand the nonlinear effects that may arise when two gravitational waves interact within a complete exact solution. Additionally one also needs to establish how can one define nodes and antinodes in the context of standing gravitational waves.

In the context of a full exact solution study, the above-mentioned area of interest poses challenges. Although the complete picture is not clear for standing gravitational waves, one expects non-linear effects to arise. This thesis attempts to study toy models of standing GWs to investigate test masses behavior and address those non-linearities.

One of the first solutions to Einstein’s field equations admitting wave-like nature was given by Brinkmann [15] in 1925.¹⁸ Cylindrical gravitational wave solutions were studied by Gudio Beck [16] and later by Einstein and Rosen [66].¹⁹ After the work of Einstein and Rosen, these cylindrical gravitational waves were considered, aside from the weak-field limit, as proof of the existence of GWs in full GR. They are also known as *Einstein-Rosen waves*. This solution is part of a broader family known as *generalized Einstein-Rosen spacetimes*, characterized by metrics dependent on both time and a spatial coordinate. Their cosmological interpretation can be categorized into three families: (i) spatially homogeneous isotropic (Robertson-Walker) models;

¹⁸He introduced the pp-waves (plane-fronted waves with parallel propagation) solutions to study radiation.

¹⁹Gudio Beck found it while studying time-independent axisymmetric fields [67].

(ii) spatially homogeneous models; and (iii) spatially inhomogeneous anisotropic models [68].²⁰

From the whole class of Einstein-Rosen spacetimes, the particular class of solutions, the Einstein-Rosen waves have been studied quite ind detail. Their non-trivial polarisations was studied numerically by Piran [71] and analytically by Halilsoy [72].²¹ In his paper [56], Chandrasekhar focused on cylindrical waves in General Relativity (GR). He refers to Rayleigh's standard definition of a standing wave and interprets his solution as a stationary monochromatic wave.²² Although Einstein-Rosen wave metrics do not represent GWs from bounded sources, they have been a studied in various interesting settings [73].²³

Gravitational waves can be formulated by considering them as perturbations of Minkowski spacetime. Depending on the perturbation and the symmetry of the source, one can study the type of gravitational waves accordingly. In 1975, Grishchuk and Sazhin [65] discussed the concept of standing gravitational waves within the linearized regime. They considered an alternating toroidal electromagnetic field, which acted as a time-dependent stress-energy momentum term, leading to the generation of standing gravitational waves. These standing gravitational waves, produced at the center of the toroid, exhibited a cylindrical nature. In the linearized limit, they evaluated the gravitational strain and its amplitude. Furthermore, they analyzed whether electromagnetic waves could be generated by these cylindrical standing gravitational waves and whether they could be detected.

Gravitational waves can also be studied in the context of cosmological models. One of the ways to study them is as a perturbation to the Friedman-Lemaître-Robertson Walker (FLRW) metric. Beyond the homogeneous and isotropic FLRW model, Lifshitz, Khalatnikov, and Belinski conducted studies on anisotropic and inhomogeneous cosmological models. In these models, when the metric exhibits a non-trivial dependence on spatial coordinates in addition to time, we encounter generalized wave solutions reminiscent of the Einstein-Rosen type. These models also serve as valuable frameworks for investigating gravitational waves in the early universe [69]. A particular

²⁰They can also be studied as stationary axisymmetric spacetimes [56, 69, 70]. We found in the literature that sometimes people refer to Einstein-Rosen wave along with a scalar or EM field as generalized Einstein-Rosen spacetimes also.

²¹We have found the presence of chaotic geodesics in linearly polarised Einstein-Rosen case. Our future work would be to probe the cross-polarised Halilsoy solution.

²²The definition of stationary wave, see Rayleigh 1945, vol. 1, p. 227. [55].

²³We more information on cylindrical spacetimes we refer to [74].

transformation²⁴ of Einstein Rosen metrics called Gowdy transformations is useful for studying the inhomogeneous three-torus universe; which is the spacetime we study in our first work [13]. Charach [75] studied such an inhomogeneous universe with interacting electromagnetic and gravitational waves. Their motivation was to look into the problem of coupling between gravitational and electromagnetic radiation in the early universe; our second work is based on this electrovacuum spacetime.

Standing gravitational waves were also studied on how they affect the power spectrum of temperature anisotropies in the cosmic microwave background (CMB).²⁵ While studying relic gravitational waves in CMB radiation, Grishchuk [65] showed that the wave-equation for a gravitational wave in a homogeneous isotropic universe with the scale factor can be written as a Schrodinger-like equation. There he mentions standing waves being created in the early universe.²⁶

Gowdy, Misner, and Berger studied the three-torus vacuum model representing a closed universe consisting only of linearly polarised gravitational waves. As this spacetime is the basis for the first work [13] we explore this in detail in Chapter 2.

Bondi took the issue of superposition in linear approximation when examining unsymmetrical body gravitational waves [10]. The wave zone was studied but a deeper analysis revealed the formation of singularities. For vacuum solutions, the metric functions determine the dynamics of the spacetime one is analyzing. Almost two decades ago Hans Stephani [4] probed the question, *do Einstein's equations admit standing wave solutions?* He proposed to look for exact solutions with i) metric functions depending on a periodic timelike coordinate and a spacelike coordinate and ii) the existence of a divergence-free gravitational analogue of the Poynting vector. However, the conditions were not invariantly formulated and gave opposite results when applied to some spacetimes.

Gogberashvili, Myrzakul, and Singleton [76] studied a standing gravitational wave nonvacuum solution in the presence of a domain wall and a massless scalar field. Their setting is similar to an electromagnetic standing wave between two infinite conducting planes. They found discrete frequencies for their standing gravitational waves.²⁷

²⁴Here the transitivity surface element is, however, associated with the temporal, rather than with the spatial, variable.

²⁵The power spectrum represents the distribution of temperature fluctuations across different angular scales in the CMB.

²⁶The main idea was to see conditions when a weak gravitational wave in a non-stationary isotropic universe can be amplified. The term superadiabatic amplification of gravitational waves was given to denote the possibility of the creation of gravitons.

²⁷Also there was an experiment [77] to investigate discrete quantum properties in matter. The main aim was to observe quantum states in a gravitational field due to its weakness compared to other

Also in [78], the phenomena of parametric resonance excitation of photons by a gravitational wave was studied.²⁸ There a standing gravitational wave was required to cause parametric resonance.

Standing waves were also probed in quasi-stationary approximations in binary black hole inspirals. Periodic solutions to Einstein equations having standing waves were introduced by Detweiler [79] in the weak field zone. To find out more about the inspiral epoch for the last few orbits in a binary black hole, various periodic standing-wave solutions in weak field limits were investigated [80]. For example, for the case of Post-Minkowski computations in [81], for non-linear scalar fields in [82] finally for the full GR case in [83].

We mention now the high-frequency limit in GR, which was later utilized to propose a covariant definition of standing gravitational waves. The high-frequency expansion of a vacuum gravitational field in powers of its small wavelength was studied by Isaacson [84]. He considered the lowest-order non-linearities and obtained a gauge-invariant averaged stress for the effective energy localized in the high-frequency GWs. Analogous to the electromagnetic null field, the analogue Poynting vector was used to investigate the flow of energy and momentum by GWs. Then in [85] Burnett has studied gravitational waves having small amplitudes and short wavelengths (high frequencies) and proved a theorem that a particular “effective stress-energy” tensor acts as a source of curvature of the smooth background spacetime. In [86], Szybka and Cieřlik, a similar formalism was used about nonvacuum spacetimes to propose a covariant definition of standing GWs.²⁹ They mentioned a spacetime (M, g) will contain a standing gravitational wave if

- (i) it belongs to a one-parameter family of spacetimes $(M, g(\lambda))$ satisfying the Green-Wald assumptions [87] (the background spacetime is denoted as $(M, g^{(0)})$),
- (ii) the Ricci tensor of the background metric $g^{(0)}$ is of Segre type $[(11)1,1]$. In the algebraic classification of the traceless Ricci tensor, one examines an eigenvalue equation. The presence of a Lorentz metric introduces a complex

forces and present an experiment demonstrating gravitational quantum bound states of neutrons falling in a confined potential well.

²⁸It is a vibrational state where the energy is injected into a system from an external source at resonance. This leads to a notable amplification of the system’s response. They examined when the parametric resonance instability of the photon field will occur in the presence of a gravitational wave.

²⁹Green and Wald in [87] were investigating whether non-linear terms in the Einstein equation can mimic Dark Energy and found a zeroth order “effective stress-energy tensor” associated with the small scale density and metric fluctuations. They concluded that small-scale density inhomogeneities cannot be associated with cosmic acceleration.

algebraic structure, resulting in the eigenvalue equation containing terms of various orders. Segre notation is used to denote these orders, with round brackets indicating that some eigenvalues coincide. Specifically, one eigenvalue equals zero, while the remaining eigenvalues are 1 and -1.

Following the above Segre type, one can define an effective energy-momentum tensor $t^{(0)} = \frac{1}{2}\rho(x^\alpha)(k_+^b \otimes k_+^b + k_-^b \otimes k_-^b)$.³⁰ Here $\rho(x^\alpha)$ is the energy density and $k_\pm$ are null vectors oriented in opposite spatial directions. This effective energy-momentum tensor $t^{(0)}$ can be treated as a superposition of null dust with equal amplitudes, but moving in opposite directions. In this respect, they quantified the definition of an effective standing wave spacetime. But this definition proposed by Szybka and Cieřlik [86] has to be verified on more examples and extended to two and three dimensional standing waves. Moreover, in the high-frequency limit information is lost, so it could be that other types of spacetimes mimic in this limit standing gravitational waves. In other words, the status of this proposition needs further investigation.

Although the definition of a standing gravitational wave remains elusive, this thesis explores spacetimes with standing wave interpretations to clarify the behavior of test masses in their presence. This thesis comprises the work done on the Gowdy T^3 , the electromagnetic Gowdy, and the Einstein-Rosen metric. We conclude with a brief symbolic mention of the spacetimes that have been the focus of this thesis. A more detailed background is given at the beginning of the following chapters.

- Chapter 2 : T^3 Gowdy metric

$$ds^2 = e^f(-dt^2 + dz^2) + t(e^p dx^2 + e^{-p} dy^2)$$

where $0 \leq z < 2\pi$, $t > 0$, $0 \leq x, y < 2\pi$, $f = f(t, z)$ and $p = p(t, z)$.

The results for this work are published here [13].

- Chapter 3 : Electromagnetic Gowdy

$$ds^2 = \frac{e^f}{\sqrt{t/T}} p^2(-dt^2 + dx^2) + \frac{t}{T}(p^2 e^{-w} dy^2 + p^{-2} e^w dz^2)$$

where $0 \leq x < 2\pi$, $t > 0$, $0 \leq y, z < 2\pi$, T is constant, $f = f(t, x)$, $p = p(t, x)$, $w = w(t, x)$

The results of this work are included in an unpublished draft.

- Chapter 4 : Einstein Rosen cylindrical gravitational wave

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi} dz^2$$

³⁰The musical isomorphism $\flat$ corresponds to the standard index-lowering. In differential geometry, the musical isomorphism is a mapping between the tangent and cotangent bundles of a pseudo-Riemannian manifold, induced by its metric tensor (further discussed in the next section).

where $\rho > 0$, $-\infty < t, z < \infty$, $0 \leq \varphi < 2\pi$ and $\psi = \psi(t, \rho)$ and $\gamma = \gamma(t, \rho)$
 The results for this work are published here [2].

1.4 Brief review of general relativity

Einstein's special relativity elevated the absolute nature of space and time into a unified notion of the spacetime continuum. This idea was extended and gravity was treated not as a force, but as the bending of spacetime. The new theory – general relativity proposed by Einstein states the following: The spacetime metric describes observer-independent properties of spacetime. This spacetime metric instead of being flat (as in special relativity) is curved. The curvature (or the deviation from flatness) is responsible for the physical effects of a gravitational field. Furthermore, the matter (which is presented by the stress-energy momentum tensor) is related to this curvature by the Einsteins field equations.

We do not delve into an exhaustive formulation of GR here and refer the readers to the standard textbooks. Instead, in this thesis, we present only the essential definition pertinent to our discussion. We give a basic definition of manifolds and maps between manifolds. After defining tangent vectors, derivatives, and relevant curvature tensors we arrive at Einstein equations. We also mention differential forms and Cartan equations used in our work. The notion of null tetrad is provided and Petrov classification is also mentioned. The definitions and notation follow [88] and [89].

We need the notion of *topological spaces* to begin defining manifolds but only provide an informal definition. A topological space defines point relationships via open sets. Also, we mention the idea of *homeomorphism*³¹ which is a mapping preserving essential structure between two topological spaces. We give the explicit definition of both in A.

Differentiable Manifolds: The stage on which general relativity (GR) is formulated is the mathematical object called *manifold*. Locally it looks like $\mathbb{R}^n$ but globally it has complex structures. The idea is to formulate spacetime as a manifold and to understand the structures or the mathematical constructions that live on it. An *n-dimensional differentiable manifold* is a topological space³² M such that

³¹Its a map between topological spaces which is *injective*, *surjective* and has a function and its inverse as continuous.

³²Specifically the manifolds considered here are Hausdorff and paracompact. For more details we refer the reader to Appendix A of [88].

- M is locally homeomorphic³³ to $\mathbb{R}^n$: for each $p \in M$, there is an open set O such that $p \in O$ and homeomorphism $\phi : O \rightarrow U$ with U being an open subset of $\mathbb{R}^n$
- Consider two open subsets O_μ and O_ν that overlap such that $O_\mu \cap O_\nu \neq \emptyset$. We require that the corresponding maps $\phi_\mu : O_\mu \rightarrow U_\mu$ and $\phi_\nu : O_\nu \rightarrow U_\nu$ are *compatible*, meaning that $\phi_\mu \circ \phi_\nu^{-1} : \phi_\nu(O_\mu \cap O_\nu) \rightarrow \phi_\mu(O_\mu \cap O_\nu)$ is *smooth* (also known as infinitely differentiable or C^∞).

The maps ϕ_μ are called *charts*; a collection of charts is defined as an *atlas*. Each chart provides a coordinate system to label the region O_μ of M . The coordinate associated to $p \in O_\mu$ is :

$$\phi_\mu(p) = (x^1(p), \dots, x^n(p))$$

A manifold M can have multiple atlases, and we can freely add new charts to an atlas as long as they are compatible with existing ones. When two atlases are compatible, meaning all their charts work together, they define the same differentiable structure on the manifold. After the concept of manifold is formulated, we can talk about maps between manifolds.

Maps between Manifolds: A function $f : M \rightarrow \mathbb{R}$ is smooth, if the map $f \circ \phi^{-1} : U \rightarrow \mathbb{R}$ is smooth for all charts ϕ . Similarly, we have a map $f : M \rightarrow N$ between two manifolds M and N is smooth if the map $\psi \circ f \circ \phi^{-1} : U \rightarrow V$ is smooth for all charts $\phi : M \rightarrow U \subset \mathbb{R}^{\dim(M)}$ and $\psi : N \rightarrow V \subset \mathbb{R}^{\dim(N)}$.

We also require the notion of *diffeomorphism* which is crucial in GR as it preserves the smoothness and differentiable structure of spacetime. A *diffeomorphism* is defined to be a smooth homeomorphism with a smooth inverse $f : M \rightarrow N$. It enables an invertible, smooth map between manifolds M and N that has a smooth inverse. To keep in mind homeomorphic manifolds can be continuously transformed into each other, while diffeomorphism demands smoothness in both the map and its inverse. Next, we can define the necessary mathematical structures, such as tangent vectors and spaces, to introduce tensors in GR.

Tangent Vectors and Spaces: Now in order to perform calculus on manifolds we need the help of tangent spaces. As an example, let us look into differentiation. Say, we want to differentiate a function $f : M \rightarrow \mathbb{R}$ at some point p . For this we introduce a chart $\phi = (x^1, \dots, x^n)$ in the neighbourhood of p . With this we can construct the map $f \circ \phi^{-1} : U \rightarrow \mathbb{R}, U \subset \mathbb{R}^n$. As we know how to differentiate functions on $\mathbb{R}^n$, we can

³³In general, topological spaces can be interpreted as geometric objects, and homeomorphism intuitively implies continuous stretching and bending of objects into a new shape. In this way, square and circle are homeomorphic with one another unlike a sphere and a torus.

get an idea to differentiate functions on M ,

$$\left. \frac{\partial f}{\partial x^\mu} \right|_p := \left. \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} \right|_{\phi_p}$$

This depends on the choice of chart ϕ and coordinates x^μ . We require a coordinate independent definition of differentiation. For this, we first define *tangent vectors* and *tangent space*. Consider smooth functions over a manifold M and denote the set of all smooth functions as $C^\infty(M)$. A *tangent vector* X_p is an object that differentiated functions at point $p \in M$, $X_p : C^\infty(M) \rightarrow \mathbb{R}$ satisfying

- Linearity : $X_p(f + g) = X_p(f) + X_p(g)$ for all $f, g \in C^\infty(M)$.
- $X_p(f) = 0$ when f is constant.
- Leibniz rule : $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(M)$

The set of all tangent vectors at point p forms an n -dimensional vector space known as *tangent space* $T_p(M)$. The tangent vectors $\partial_\mu|_p$ give a basis (this is a coordinate basis, we deal with non-coordinate basis later) for $T_p(M)$. Hence we can write any tangent vector as

$$X_p = X^\mu \partial_\mu|_p$$

where $X^\mu = X_p(x^\mu)$ are the components of the tangent vector in this basis.

We can also have objects in which there is a choice of tangent vector for every point $p \in M$ known as *fields*. We briefly talk about vector fields and one-forms to help us form the notion of tensors. A smooth assignment of a tangent vector X_p to each $p \in M$ is defined as a *vector field*. It is a map $X : C^\infty(M) \rightarrow C^\infty(M)$. For a given coordinate basis, we can expand any vector field as $X = X^\mu \frac{\partial}{\partial x^\mu}$, where X^μ are smooth functions on M . We also state the notion of *dual vector space* which we will use later.

At each point $p \in M$, there exists a vector space $T_p(M)$. The dual of this space, $T_p^*(M)$ is called the *cotangent space* at p (its elements called cotangent vector/covector). For a given basis $\{e_\mu\}$ of $T_p(M)$, we have a dual basis $\{\theta^\mu\}$ for $T_p^*(M)$ defined by $\theta^\nu(e_\mu) = \delta_\mu^\nu$. They can be used to expand any co-vector as $\omega = \omega_\mu \theta^\mu$.

We may also create cotangent fields, known as *one-forms*, by smoothly selecting a member from the tangent space $T_p^*(M)$ at each point p . These one-forms, in general, map vector fields to real numbers. The set of all one-forms on M is denoted as $\Lambda^1(M)$.

Note that we spoke earlier about the differentiation of a function, but there also exists an important concept of how to differentiate a vector field. This involves the concept of *Lie derivatives* along with two crucial concepts being the *pullback*

and *pushforward* of mappings³⁴ between manifolds. We skip this for brevity's sake, avoiding excessive detail. The notion of differentiation we will require is a *covariant derivative* which we define later. Now we proceed to define tensors.

Tensor and tensor fields: A *tensor of rank (r, s)* at a point $p \in M$ is defined to be a multi-linear map

$$T : \overbrace{T_p^*(M) \times \cdots \times T_p^*(M)}^r \times \overbrace{T_p(M) \times \cdots \times T_p(M)}^s \rightarrow \mathbb{R}$$

Such a tensor is said to have a total rank of $r + s$. Now we can define a *tensor field* as a smooth assignment of a (r, s) tensor to every point $p \in M$. So given a basis $\{e_\mu\}$ for vector fields and a dual basis $\{\theta^\mu\}$ for one-forms, the components of the tensor are defined to be

$$T^{\mu_1, \dots, \mu_r}_{\nu_1, \dots, \nu_s} = T(\theta^{\mu_1}, \dots, \theta^{\mu_r}, e_{\nu_1}, \dots, e_{\nu_s})$$

Several operations can be performed on tensor fields to generate additional tensors. We refer the reader to other differential geometry books for more details on tensors. Since we have utilized differential forms and exterior derivatives in our work, we define them now.

Differential forms and exterior derivatives: Differential forms are mathematical constructs used to express and manipulate multivariate functions over manifolds. Exterior derivatives are operations applied to differential forms, extending the concept of differentiation while capturing geometric and topological properties of the underlying space. Again for brevity, we only mention the basic definition here, for other properties and details we refer the reader to [89].

As we mentioned covectors and one-forms before, let's elaborate a bit further in order to arrive at exterior derivative. A simple way to construct a one-form is to take function $f \in C^\infty(M)$ by $df(X) = X(f)$. This can be further used to build a basis for $\Lambda^1(M)$. We can introduce coordinates x^μ on M with the corresponding basis $e_\mu = \partial/\partial x^\mu$ of vector fields (written shorthand as $\partial/\partial x^\mu \equiv \partial_\mu$). Then we simply take the functions $f = x^\mu$ which, using the aforementioned definition gives $dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu_\nu$. Hence dx^μ provided a basis for $\Lambda^1(M)$, dual to the coordinate basis $\partial/\partial x^\mu$. An arbitrary one-form $\omega \in \Lambda^1(M)$ can be expanded as $\omega = \omega_\mu dx^\mu$. Whereas in such a basis one-form df is $df = \frac{\partial f}{\partial x^\mu} dx^\mu$.

An interesting class of totally anti-symmetric $(0, p)$ tensor fields is known as *p-forms*. The set of all *p-forms* over a manifold M is denoted as $\Lambda^p(M)$. If I have

³⁴We refer the readers to [88] for more details.

a p -form ω and q -form η , then their tensor product will provide a $(p+q)$ -tensor. Anti-symmetrizing it gives us a $(p+q)$ -form. The construction is called *wedge product* defined as

$$(\omega \wedge \eta)_{\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1, \dots, \mu_p} \eta_{\nu_1, \dots, \nu_q]}$$

where [...] represents antisymmetrization over all indices.

The definition we just stated a while back of one-form can be extended to higher forms. An exterior derivative is then essentially a map $d : \Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$. For local coordinates, it acts as

$$(d\omega) = \frac{1}{p!} \frac{\partial \omega_{\mu_1, \dots, \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

Metric, Connection, and Curvature: Finally, we are in a position to introduce the main protagonists – the *metric tensor*, the *connection* to map vector spaces and finally *curvature*. The idea is to encapsulate the notion of measuring distances between points, keeping in mind the subtleties involved in comparing a tensor at two infinitesimally nearby points on a manifold.

We just mention a few things about curvature before proceeding. Our intuitive understanding of curvature often comes from two-dimensional surfaces in ordinary three-dimensional space. We refer the readers to Chapter 9 of [88] to know more about extrinsic curvature for surfaces in higher-dimensional spaces. We are currently focused on investigating spacetime curvature and the notion of curvature we will focus will be on intrinsic curvature that applies to any manifold, regardless of any higher-dimensional context.

Intuitively, a metric is designed to quantify the ‘infinitesimal squared distance’ associated with an ‘infinitesimal displacement.’ This concept of an ‘infinitesimal displacement’ aligns precisely with the idea of a tangent vector defined before. A *metric* g is a $(0,2)$ tensor field which is *symmetric* ($g(X,Y) = g(Y,X)$) and *non-degenerate* (if, for any $p \in M$, $g(X,Y)|_p = 0$ for all $Y \in T_p(M)$ then $X_p = 0$) For a given choice of coordinates, it is often written as:

$$ds^2 = \sum_{\mu, \nu} g_{\mu\nu} dx^\mu \otimes dx^\nu \quad (1.1)$$

In most applications of differential geometry, our focus lies on *Riemannian manifolds*. The simplest instance of a Riemannian manifold is Euclidean space $\mathbb{R}^n$ (in cartesian coordinates it is equipped with metric $g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$). In the context

of general relativity, we will be dealing with a manifold which is *Lorentzian manifold*. A simple example would be *Minkowski space*, it is with the metric $\eta = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \dots + dx^{n-1} \otimes dx^{n-1}$, where its components are $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$. Importantly, at any point P on a Lorentzian manifold, we can find a local orthonormal basis e_μ for the tangent space $T_P(M)$, making the metric resemble, upto first derivative, as the Minkowski metric, i.e, $g_{\mu\nu}|_P = \eta_{\mu\nu}$.

In GR, we know the issue arises when differentiating involves comparing a tensor at two infinitesimally nearby points on the manifold. The problem being that tensors at different points belong to distinct spaces. To address this challenge, a mathematical structure known as a *covariant derivative* or *connection* is introduced. For a manifold M , a *covariant derivative/connection* ∇ is a map which sends a pair of smooth vector fields X, Y to a smooth vector field $\nabla_X Y$ with properties of linearity, Leibniz rule, etc.³⁵ With the help of the covariant derivative, the manifold is endowed with more structures. Now we can have in a basis $\{e_\mu\}$, the *connection components*, $\Gamma_{\rho\nu}^\mu$ defined as

$$\nabla_{e_\rho} e_\nu = \Gamma_{\rho\nu}^\mu e_\mu \quad (1.2)$$

Once certain connection, *Levi-Civita connection*³⁶ is defined, these connection components are known as *Christoffel symbols*.³⁷ The connection links tangent spaces and this mapping is known as parallel transport. This mapping is essential for defining differentiation.

Now equipped with a derivative operator, we can define *parallel transport* of a vector along a curve C . So if you have a vector field X along this curve with coordinates $x^\mu(\tau)$, such that, $X^\mu|_C = \frac{dx^\mu}{d\tau}$. Then a tensor field T is *parallelly transported* along C if $\nabla_X T = 0$. Parallel transport exhibits path dependency, relying on both the connection and the specific path, characterized by the vector field X in this instance.

We have arrived till now to the notion that for any given metric (regardless of its signature), there exists a distinct definition of parallel transport that conserves the

³⁵For X, Y, Z as vector fields and f, g as some functions we have

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z,$
- $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z,$
- $\nabla_X (fY) = f\nabla_X Y + (\nabla_X f)Y,$

where action of ∇ is defined as $\nabla_X f = X(f)$.

³⁶On a manifold with a metric, the metric singles out a preferred connection. **Theorem:** For a manifold M with a metric g , there exists a unique torsion-free connection ∇ , called *Levi-Civita connection* such that the metric is covariantly constant ($\nabla g = 0$).

³⁷In the coordinate basis components of the Christoffel symbols as $\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

inner products of all parallelly transported vector pairs. Consequently, the presence of a metric leads to the concept of parallel transport and, subsequently, an intrinsic notion of the manifold's curvature. This is precisely the concept of curvature within the spacetime $(M, g_{\mu\nu})$ that will be our interest.

Riemann, Weyl and Einstein tensor: Now with the help of metric, connection and parallel transport we can arrive at the notion of curvature. In the previous section, we discussed how a derivative operator allows us to parallel transport vectors along curves. However, this parallel transport's outcome depends on the chosen connecting curve, and we can leverage this path dependence to define curvature. In this section, we define the *Riemann curvature tensor* based on the non-commutativity of successive differentiations. Conceptually its related to the path-dependent behavior of parallel transport, especially how it governs a vector's failure to return to its original state when transported around a closed loop. We do not explicitly derive it here as it is well known in standard GR textbook. One can show that there exists a tensor field $R_{\mu\nu\rho}{}^\sigma$ (called the *Riemann curvature tensor*) such that for all dual vector fields ω_ρ we have $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \omega^\rho = -R_{\mu\nu}{}^\rho{}_\sigma \omega^\sigma$.³⁸

Now we establish a few properties of Riemann tensor to help us define Weyl tensor and finally arrive at Einstein equations.

1. $R_{\mu\nu\rho}{}^\sigma = -R_{\nu\mu\rho}{}^\sigma$
2. $R_{[\mu\nu\rho]}{}^\sigma = 0$
3. For the derivative operator, say ∇_δ , associated with the metric,
 $\nabla_\delta g_{\mu\nu} = 0$, $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$
4. Bianchi identity $\nabla_{[\mu} R_{\nu\rho]\sigma}{}^\delta = 0$
5. $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

We can decompose the Riemann tensor into a 'trace part', giving the *Ricci tensor*, and the 'trace free part', giving the *Weyl tensor*. The trace of the Riemann tensor over its first two or last two indices is zero via the antisymmetry properties (1) and (3). The trace over the second and fourth (or first and second) indices help us define the *Ricci tensor*, $R_{\mu\rho}$ by contraction over one of the indices as $R_{\mu\rho} = R_{\mu\sigma\rho}{}^\sigma$. From the symmetry property (5), we can have $R_{\mu\rho}$ satisfy $R_{\mu\rho} = R_{\rho\mu}$. We can go further and contract over the left-over two indices as $R = g^{\rho\mu} R_{\rho\mu}$ to give as the *Ricci scalar* R .

³⁸Its usual coordinate basis form is $R_{\rho\mu\nu}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\mu\rho}^\sigma + \Gamma_{\nu\rho}^\lambda \Gamma_{\mu\lambda}^\sigma - \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\sigma$.

The Weyl tensor, $C_{\mu\nu\rho\sigma}$, is the “trace free part” is defined for manifolds of dimension $n \geq 3$ as :

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$$

Intuitively, the Ricci tensor gives information about the change in the volume of a body by the tidal forces. The Weyl tensor details how the shape of a body is distorted by tidal forces. So the tidal forces experienced by a freely-falling body are expressed by the Weyl tensor.

We can contract Bianchi identity (4) to get an important equation satisfied by $R_{\mu\nu}$

$$\nabla_\mu R_{\nu\rho\sigma}{}^\mu + \nabla_\nu R_{\rho\sigma} - \nabla_\rho R_{\nu\sigma} = 0 \quad (1.3)$$

One raising the index σ with the metric and contracting over ν and σ leads to $\nabla_\mu R_\rho{}^\mu + \nabla_\nu R_\rho{}^\nu - \nabla_\rho R = 0$. This can be written as $\nabla^\mu G_{\mu\nu} = 0$ where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Here : $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is the *Einstein tensor*.

We have briefly seen how general relativity views gravity as curved spacetime and describes the laws of physics within this framework. The key aspect left in general relativity is the equation governing the spacetime metric. This equation, influenced by Mach’s principle, does not prescribe spacetime geometry in advance.³⁹ Instead, it states that spacetime geometry is shaped by the matter distribution in the universe, making the spacetime metric not just a backdrop for physics but also a dynamic variable that responds to the matter in spacetime. With this as essential for an accurate description of gravity, we just need to answer as to which equation governs the connection between spacetime geometry and the distribution of matter.

In summary, we’ll briefly address the question and touch on the derivation of the *Einstein Field Equations* without going into extensive detail. The important clue lies in the comparison of tidal forces in Newtonian gravity and GR. In Newtonian theory we have the relation between gravitational potential ϕ and the tidal acceleration of two nearby particles. In GR, the information about tidal acceleration of two nearby particle is encoded in the Riemann tensor. So, crudely stating there is a correspondence between the Riemann tensor and the gravitational potential. We know, via Poisson equation the gravitational potential ϕ is related to mass density ρ . But from SRT and GR we know the energy properties of matter are derived by the stress-energy tensor $T_{\mu\nu}$. So there is a unique equation between $R^\mu{}_{\nu\alpha\beta}$ and $T_{\mu\nu}$

³⁹See section 1.4 of [88] for more details.

which encapsulate the essence of GR as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \quad (1.4)$$

These are the *Einstein Field Equations* (in natural units $G = c = 1$) which relate the curvature of spacetime, represented by the Einstein tensor, to the distribution of mass and energy, described by the energy-momentum tensor (which would be zero for vanishing $T_{\mu\nu}$ then called *vacuum Einstein Field Equations*). They are fundamental equations that govern how gravity works and have been validated through experiments and observations, playing a pivotal role in our understanding of the universe's structure and behavior.

After arriving at E.F.E., we now proceed to discuss Cartan structure equations, which we have used in our work. Then, we mention the Newman-Penrose formalism, which was also employed to analyze the dynamics of the spacetime we studied. Finally, we conclude with the Petrov classification to gain insight into the classifications of gravitational fields.

Cartan's Structure Equations: Computing the components of the Riemann tensor can be a straightforward yet laborious task. However, there exists an alternative approach that involves reorganizing the connection, torsion, and curvature tensors using the language of *forms*. We define *connection one-form*, *curvature two-form* and arrive at Cartan's equations which we have used in our work. For full derivation we refer the readers to [88, 89].

Apart from the usual coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, we can also have other non-coordinate basis which prove useful at times in computing curvature tensor. For example, a basis such as $e_{\hat{a}} = e_a^\mu \partial_\mu$.⁴⁰ This is the basis such that on a Lorentzian manifold we have $g(e_{\hat{a}}, e_{\hat{b}}) = g_{\mu\nu} e_a^\mu e_b^\nu = \eta_{ab}$. In 4-dimension, such a basis is sometimes called a *tetrad*. Note that they are not unique (given one set of tetrads you can find another set by Lorentz transformations). Now you also have the dual basis of one-forms $\hat{\theta}^a$ defined as $\theta^{\hat{a}}(e_{\hat{b}}) = \delta_b^a$. Their relation with coordinate basis is by $\theta^{\hat{a}} = e^a_\mu dx^\mu$.

For a non-coordinate basis $\{e_{\hat{a}}\}$, the components of a connection is defined as $\nabla_{e_{\hat{c}}} e_{\hat{b}} = \Gamma_{\hat{c}\hat{b}}^{\hat{a}} e_{\hat{a}}$. So the connection 1-forms (using the Levi-Civita connection) are defined as

$$\omega_{\hat{b}}^{\hat{a}} = \Gamma_{\hat{c}\hat{b}}^{\hat{a}} \theta^{\hat{c}}$$

⁴⁰Here μ, ν indices are raised/lowered with the metric $g_{\mu\nu}$ while a, b indices are raised/lowered with the flat metric η_{ab} or δ_b^a . So Greek μ, ν tell us that we're in the coordinate basis, or Roman a, b tells us we're in the some tetrad basis.

To compute connection one-forms, particularly in the case of a torsion-free connection, we can use the first of the two Cartan structure relations.

$$d\theta^{\hat{a}} + \omega^{\hat{a}}_{\hat{b}} \wedge \theta^{\hat{b}} = 0 \quad (1.5)$$

One can compute the components of the Riemann tensor in a non-coordinate basis $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = R(\theta^{\hat{a}}, e_{\hat{c}}, e_{\hat{d}}, e_{\hat{b}})$. One can use the anti-symmetry of the last two indices and arrive at curvature two-forms, $\Omega^{\hat{a}}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} \theta^{\hat{c}} \wedge \theta^{\hat{d}}$. The second Cartan structure relation expresses this in terms of the curvature one-form as follows:

$$\Omega^{\hat{a}}_{\hat{b}} = d\omega^{\hat{a}}_{\hat{b}} + \omega^{\hat{a}}_{\hat{c}} \wedge \omega^{\hat{c}}_{\hat{b}} \quad (1.6)$$

Here the notation we are using of ‘caret’ or ‘hat’ above a symbol denote an orthonormal non-coordinate basis. It can be above the symbol ($\hat{\Omega}^a_b$) as used in literature or above the index ($\Omega^{\hat{a}}_{\hat{b}}$) which is what we have followed in our paper [13].

Newman-Penrose formalism: Apart from the usual coordinate component method there is the orthonormal basis (tetrad) method which we outlined above. Instead of an orthonormal basis, there exists a *null tetrad* (introduced by Newman and Penrose [90]). Here one considers two null vectors k^a and l_a with $k^a l_a = 1$. To complete the basis, one picks two orthonormal spacelike vectors x^a and y^a orthogonal to k^a and l^a and defines a complex vector $m^a = (x^a + iy^a)/\sqrt{2}$ and its complex conjugate $\bar{m}^a$. The metric tensor in terms of null tetrad components becomes $g_{ab} = -k_a l_b - l_a k_b + m_a \bar{m}_b + \bar{m}_a l_b$.⁴¹ There are ten independent components of the Ricci are determined by the scalar quantities.⁴² For us the important part are the ten independent components of the Weyl tensor determined by five complex scalar functions defined as:

$$\begin{aligned} \Psi_0 &= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} m^{\hat{\lambda}} k^{\hat{\mu}} m^{\hat{\nu}}, \\ \Psi_1 &= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} l^{\hat{\lambda}} k^{\hat{\mu}} m^{\hat{\nu}}, \\ \Psi_2 &= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} m^{\hat{\lambda}} \bar{m}^{\hat{\mu}} l^{\hat{\nu}}, \\ \Psi_3 &= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} l^{\hat{\kappa}} k^{\hat{\lambda}} k^{\hat{\mu}} \bar{m}^{\hat{\nu}}, \\ \Psi_4 &= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} l^{\hat{\kappa}} \bar{m}^{\hat{\lambda}} l^{\hat{\mu}} \bar{m}^{\hat{\nu}}. \end{aligned}$$

⁴¹Please note here that the indices a, b here are for the null tetrad which we have used in our paper as $\hat{a}$.

⁴²In the literature they are usually written as $\Phi_{00} = \frac{1}{2} R_{\mu\nu} k^\mu k^\nu$, $\Phi_{22} = \frac{1}{2} R_{\mu\nu} l^\mu l^\nu$ and so on. We refer to [4] for more details.

These Weyl scalars convey how the deformation of spacetime occurs in the presence of a gravitational wave. Szekeres [91] used the idea of a ‘gravitational compass’ to analyze the free gravitational field. He found that it splits into superposition of transverse, longitudinal, and ‘Coulomb’ fields. So the Weyl scalars encode this information as:

- Ψ_2 representing the ‘Coulomb’ type effect of the radiation
- Ψ_1 and Ψ_3 representing the ingoing and outgoing ‘longitudinal’ radiation terms
- Ψ_0 and Ψ_4 representing the ingoing and outgoing ‘transverse’ radiation terms

If spacetime is asymptotically flat having radiation, then Ψ_1 and Ψ_2 can be made to be zero by an appropriate choice of the null tetrad. For gravitational waves in the linearized regime, the information about the deformation of spacetime in the transverse direction is conveyed by the Ψ_4 .⁴³

The above interpretation is, in general, not valid for all spacetime. For example, in cylindrical spacetimes having radiation, it was shown that along with the ‘Coulomb’ part of Weyl Ψ_2 the transverse part Ψ_0 and Ψ_4 also exists [92].⁴⁴

Petrov Classification: The classification of gravitational fields is commonly done using the Petrov-Penrose classification of the Weyl tensor. This classification scheme relies on the number of distinct principal null directions associated with the Weyl tensor and the repetition of these directions. Recall that the Weyl tensor describes the curvature of spacetime in general relativity and plays a crucial role in characterizing gravitational fields. So the Petrov-Penrose classification focuses on the properties of the Weyl tensor to categorize different types of gravitational fields.

In this classification, the principal null directions of the Weyl tensor are the special null directions along which the curvature associated with the Weyl tensor behaves in a simple manner. The number of distinct principal null directions and the number of times these directions are repeated determine the Petrov-Penrose classification. For example, in the simplest case, if there is only one distinct principal null direction and it is repeated, the gravitational field is classified as Petrov type I. This classification corresponds to gravitational fields with cylindrical or plane symmetry.

⁴³It’s related to the strain amplitude as $\Psi_4 = h_+ - ih_\times$, where $h_+, h_\times$ are the ‘plus’ and ‘cross’ polarisation of a GW.

⁴⁴In terms of the Weyl tensor, the Einstein-Rosen metric does not possess multiple principal null directions (They represent the eigenvectors of the Riemann tensor helping determine for directions along the light cone. More information provided in [93]) with equal multiplicities, which is a characteristic feature of algebraically special spacetimes. Instead, the Weyl tensor in the Einstein-Rosen metric exhibits a specific algebraic type known as Petrov type D, which indicates a single principal null direction with a repeated multiplicity.

Following [26] we define the *principal null direction* of the gravitational field for a null vector $\mathbf{k}$ if it satisfies

$$k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0 \quad (1.7)$$

The null tetrad $k^a, l^a, m^a, \bar{m}^a$ defined in the section above has some transformation rules (Eq. 2.1, 2.2, 2.3 in [26]). The component Ψ_0 of the Weyl tensor transforms according to Eq. 2.2 in [26] as⁴⁵

$$\Psi_0 = \Psi'_0 - 4K\Psi'_1 + 6K^2\Psi'_2 - 4K^3\Psi'_3 + K^4\Psi'_4 \quad (1.8)$$

which, in order for $\mathbf{k}$ to be a principal null direction implies $\Psi_0 = 0$. This leads us to a quartic equation in K such that $\Psi'_0 - 4K\Psi'_1 + 6K^2\Psi'_2 - 4K^3\Psi'_3 + K^4\Psi'_4 = 0$. There are four (complex) roots to this equation, each corresponding to a principal null direction. The multiplicity of each principal null direction is the same as the multiplicity of the corresponding root. When a root has a particular multiplicity, it indicates that there are multiple principal null directions related to that specific eigenvalue. These principal null directions represent the paths along which gravitational waves propagate. Therefore, the statement emphasizes the correspondence between the *number of principal null directions* and the *multiplicity of the corresponding roots*, highlighting the relationship between the *geometric properties of spacetime* and the *eigenvalues of the Weyl tensor*. Overall it can be demonstrated that for a null direction $\mathbf{k}$ of multiplicity 1, 2, 3 or 4 we have

$$\begin{aligned} k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0 &\implies \psi_0 = 0, & \psi_1 \neq 0, \\ C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0 &\implies \psi_0 = \psi_1 = 0, & \psi_2 \neq 0, \\ C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\mu = 0 &\implies \psi_0 = \psi_1 = \psi_2 = 0, & \psi_3 \neq 0, \\ C_{\kappa\lambda\mu\nu}k^\mu = 0 &\implies \psi_0 = \psi_1 = \psi_2 = \psi_3 = 0, & \psi_4 \neq 0, \end{aligned}$$

The spacetime admitting four distinct principal null directions (*pnds*) is classified as *algebraically general* or type I, otherwise it is *algebraically special*. So for our first work [13] in Gowdy T^3 model we found it is *algebraically general*. We summarize the distinct algebraic types:

⁴⁵ $\mathbf{k}' = \mathbf{k} + K\bar{\mathbf{m}} + \bar{K}\mathbf{m} + K\bar{K}\mathbf{l}$, $\mathbf{l}' = \mathbf{l}$, $\mathbf{m}' = \mathbf{m} + K\mathbf{l}$ where K is complex.

type I :	four distinct <i>pnds</i>
type II :	one <i>pnd</i> of multiplicity 2, others distinct
type D :	two distinct <i>pnds</i> of multiplicity 2
type III :	one <i>pnd</i> of multiplicity 3, other distinct
type N :	one <i>pnd</i> of multiplicity 4
type O :	conformally flat

These various types defined above have the following physical interpretation. *Type D* regions correspond to the gravitational fields generated by isolated massive bodies like stars. Specifically, these regions manifest as the outer field surrounding a gravitating object, and their defining attributes are the object's mass and angular momentum. *Type III* regions are connected to a specific form of longitudinal gravitational radiation. *Type N* regions are linked to transverse gravitational radiation, the same type of radiation that astronomers have detected using instruments like LIGO (Laser Interferometer Gravitational-Wave Observatory). In these regions, the quadruple principal null direction aligns with the wave vector, which denotes the direction of propagation for this particular radiation. *Type II* regions combine the effects observed in types D, III, and N, resulting in a complex and nonlinear combination of these characteristics. *Type O* regions, also known as conformally flat regions, correspond to locations where the Weyl tensor completely vanishes.⁴⁶

The information on how a gravitational field behaves near the source and how it is detected on Earth far from the source is encoded in the properties of the Weyl tensor. Following [90], one can divide this region between source and detector into five regions; a near zone, three transition zone, and a radiation zone. In the near zone, all terms are important implying the full non-linear deformation of spacetime. In transition zone ψ_0, ψ_1, ψ_2 become negligible. In the radiation zone, only ψ_4 is important, the Riemann tensor is essentially null and that is the transverse deformation left behind which is detected on Earth. The fourth zone is generally essentially type III and the third is “algebraically special” or usually type II. The gravitational radiation emitted from an isolated system will usually not be algebraically special. The peeling theorem describes the way in which, as one moves

⁴⁶In these regions, the Weyl tensor, which describes the curvature of spacetime, exhibits a unique property: it identically disappears. This implies that the gravitational field within Type O regions possesses a distinct conformal symmetry, meaning that the geometry of spacetime can be transformed by a conformal transformation without altering the underlying physics.

farther away from the source of the radiation, the various components of the radiation field “peel” off, until finally, only type N radiation is noticeable at large distances.⁴⁷

1.5 Brief review of dynamical systems and chaos

The dynamics involving chaos were first glimpsed by Henri Poincaré while studying the orbits of three celestial bodies under mutual gravitational attraction (e.g Sun Moon Earth system). Instead of focusing on *individual* orbits, Poincaré focused on how a set of initial points would evolve. The behavior of orbits from those sets of initial points displayed chaotic nature. Early mathematical works on chaos were done by Birkhoff [95] in 1920s, Cartwright and Littlewood [96] in the 1940s, Smale [97] and Kolmogorov [98] in the 1960s. Most of the earlier advancements were obscured by mathematical technicalities. Now the situation has changed drastically, especially with the advent of digital computers in finding numerical solutions to dynamical systems. We follow the notations and definitions given in [1] and [99].

A *dynamical system* is a deterministic mathematical prescription for evolving the state of a system forward in time. Time can be treated as a continuous variable or a discrete integer-valued variable. An example of a dynamical system of N first-order, autonomous, ordinary differential equations is:

$$\begin{aligned} dx^{(1)}/dt &= F_1(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \\ dx^{(2)}/dt &= F_2(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \\ &\vdots \\ dx^{(N)}/dt &= F_N(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \end{aligned} \tag{1.9}$$

This can be expressed in vector form as

$$d\mathbf{x}(t)/dt = \mathbf{F}[\mathbf{x}(t)] \tag{1.10}$$

where $\mathbf{x}$ is an N -dimensional vector (here it means the standard Euclidean vector, representing a quantity with a magnitude and a direction). The terms are written as:

⁴⁷Known as the Goldberg-Sachs peeling theorem that describes the behavior of gravitational waves near null infinity defined in [94].

$$\mathbf{x} = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(N)} \end{bmatrix}, \quad \mathbf{F}(\mathbf{x}) = \begin{bmatrix} F^{(1)}(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \\ F^{(2)}(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \\ \vdots \\ F^{(N)}(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \end{bmatrix}$$

One can take any initial state of the system $\mathbf{x}(0)$ and in principle solve the equations to obtain future state $\mathbf{x}(t)$ for $t > 0$. Such systems are coined as dynamical systems. Fig.(1.5) represents *phase space* depicting the essential features of a dynamical system, namely, *orbit/trajectory* and *flow* for the case of a simple undamped pendulum system.⁴⁸

Apart from the continuous case, there are also dynamical systems with discrete, integer-valued time (with n as the time variable, $n = 0, 1, 2, 3, \dots$). Such systems are maps and can be written in vector form as

$$\mathbf{x}_{n+1} = \mathbf{M}(\mathbf{x}_n) \quad (1.11)$$

where $\mathbf{x}_n$ is N -dimensional, $\mathbf{x}_n = (x_n^1, x_n^1, x_n^2, \dots, x_n^N)$. So given an initial state $\mathbf{x}_0$, we obtain the state at time $n = 1$ by $\mathbf{x}_1 = \mathbf{M}(\mathbf{x}_0)$ and so on. Thus, given any initial condition $\mathbf{x}_0$, one can generate an orbit of discrete systems. This is particularly useful when reducing a continuous-time system of dimensionality N to a discrete-time map of dimensionality $N - 1$ using the Poincaré surface-of-section technique.

In large-dimensional systems, it is reasonable to think that the orbit structure will be more complex. The required dimensionality, N , for chaos to emerge is known.⁴⁹ For the case of N first-order autonomous ordinary differential equations $N \geq 3$.

⁴⁸By flow we mean the trajectories generated by all the initial conditions analogous to the paths followed by particles of a flowing fluid.

⁴⁹So if $N = 2$ chaos can be ruled out due the Poincaré-Bendixon theorem [100].

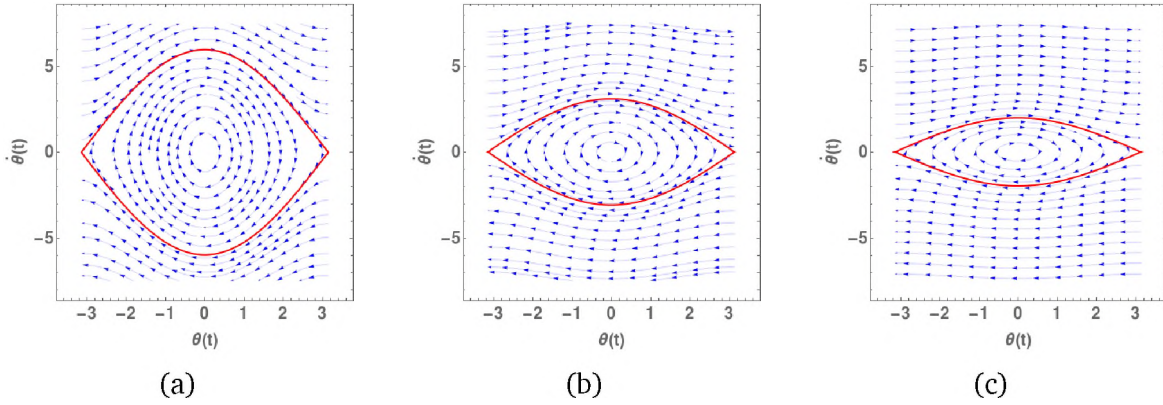


Fig. 1.5 Phase space or portraits for a pendulum whose motion is governed by the differential equation $\ddot{\theta} + \frac{g}{L} \sin \theta = 0$, where L is the length of the pendulum, g is the acceleration of gravity acting on it and ‘ $\cdot$ ’ is differentiation with respect to time. The three plots are for different pendulum lengths. In red we have a sample of one trajectory. The blue ones span the entire phase space indicating different energy levels. The arrows indicate the direction of streamlines depicting the system’s trajectories.

A useful technique to reduce a continuous time system (or flow) to a discrete-time map is called the Poincaré surface of section method. Consider N first-order autonomous ordinary differential equation Eq.(1.9). The reduction of the N -dimensional flow to an $(N - 1)$ is called a ‘*Poincaré map*’. For the simple case take $N = 3$ as shown in the figure below. We consider a solution of Eq.(1.9). Now, choose an appropriate $(N - 1)$ -dimensional surface (‘surface of section’) in the N -dimensional phase space. The intersections of the orbit are observed. In the figure, point A and point B are two successive crossings of the surface of the section. Since A can be used as an initial condition in Eq.(1.9), it uniquely determines point B . Similarly, B uniquely determines A by reversing time in Eq.(1.9). Therefore, the Poincaré map is an invertible two-dimensional map transforming the coordinates $(x_n^1, \dots)$ of the n th piercing of the surface of section to the coordinate $(x_{n+1}^1, \dots)$ at $n + 1$.

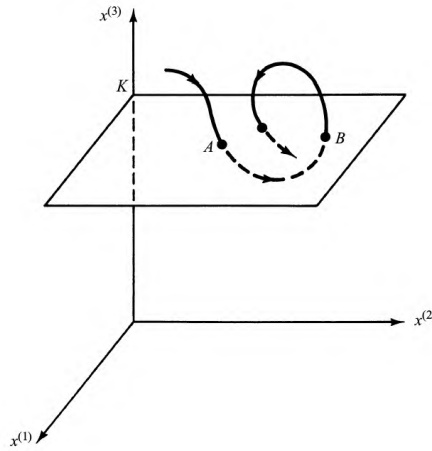


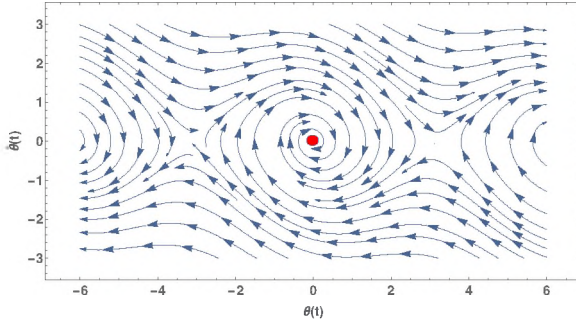
Fig. 1.6 A schematic plot showing a Poincaré section. Here, for convenience, we have a surface of section in the plane $x^{(3)} = K$. The successive crossings of this surface of sections are the point A and point B . Point A uniquely determines point B , as it can be used as an initial condition in Eq.(1.9) to calculate B . Similarly, B uniquely determines A by reversing time in Eq.(1.9) and using B as the initial condition. Figure from [1].

We now discuss the notion of *attractors* and *fractals* to formalize the tools used to study chaos in a system. We then mention the *stable* and *unstable* manifold helpful in describing the *homoclinic* and *heteroclinic* intersections.

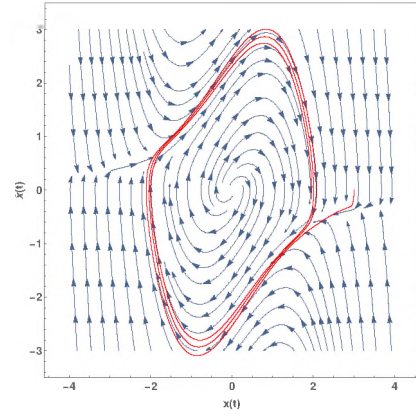
In Hamiltonian systems, like those described by Newton's equations for frictionless particle motion, certain phase space variable choices result in the preservation of phase space volumes over time. Consider an N -dimensional $\mathbf{x}$ -phase space. We choose the initial closed $(N - 1)$ -dimensional surface S_0 in the N -dimensional $\mathbf{x}$ -phase space, and evolve each point on the surface S_0 forward in time by using them as initial conditions given by Eq.(1.10). Then the closed surface S_0 evolves to a closed surface S_t at some later time t and the N -the dimensional volume of the region enclosed by S_0 is $V(0)$ and enclosed by S_t is $V(t)$. We call a volume-preserving system *conservative* if $V(t) = V(0)$. On the other hand, if the flow does not preserve volume and no change of variables can help in doing so, then we say the system is *non-conservative*. Using the divergence theorem we have that

$$dV(t)/dt = \int_{S_t} \nabla \cdot \mathbf{F} d^N x \quad (1.12)$$

where $\int_{S_t}$ represents the integral over the volume interior to the surface S_t and $\nabla \cdot \mathbf{F} \equiv \sum_{i=1}^N \partial F_i(x^{(1)}, \dots, x^{(N)}) / \partial x^{(i)}$.⁵⁰ If $\nabla \cdot \mathbf{F} < 0$ in a specific phase space region, indicating volume contraction in that region, we categorize the system as *dissipative*. An important characteristic of dissipative systems is the presence of attracting sets or *attractors* in the phase space. In a dynamical system, these are bounded subsets to which regions of initial conditions of nonzero phase space volume asymptote as time increases.



(a) Trajectory in phase space for damped harmonic oscillator.



(b) Trajectory in phase space for van der Pol equation (an oscillator with non-linear damping).

Fig. 1.7 In (a) the attractor is the red point at the origin. In (b) the attractor is the red curve.

In Fig.(1.7a)(a), the orbit spirals into the origin. In Fig.(1.7)(b) we have a limit cycle around which some initial condition circulates in periodic motion. The first example is one in which the attractor of a continuous dynamical system is a set of dimension zero (a single point) and the second in which the attractor is a set of dimension one (a closed curve). But the resulting attractors have a more intricate structure in phase space for chaotic dynamics.

The attractors for such chaotic dynamical systems have dimensions which is not an integer. In the terminology of Mandelbrot, such geometrical objects are *fractals*. An attractor which is a fractal is called a *strange attractor*. In essence, a fractal attractor is considered strange, with “chaos” referring to its dynamics and “strangeness” to its geometry. However, it’s important to note that chaotic attractors may not always

⁵⁰ For e.g, for forced damped pendulum equation $\frac{d^2\theta}{dt^2} + \nu \frac{d\theta}{dt} + \sin\theta = A \sin(2\pi f t)$ with the strength of the damping ν , and the amplitude A and the frequency f of the external force, we have $\nabla \cdot \mathbf{F} = \nu$. From Eq.(1.12), we have V decreases exponentially with time, $V(t) = \exp(-\nu t)V(0)$.

be strange and attractors can be strange without being chaotic. In cases involving differential equations, strangeness and chaos often coexist.⁵¹

The dynamics of an attractor is *chaotic* when it displays exponentially sensitive dependence on initial conditions. Consider two nearby initial conditions $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0) = \mathbf{x}_1(0) + \Delta(0)$, Δ is a small change. Now evolve them forward in time by a continuous time dynamical system yielding orbits $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ as shown in Fig.(1.8).

At the time t , the separation between two orbits is defined as $\Delta(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t)$. In the limit $|\Delta(0)| \rightarrow 0$, and large t , if the orbits remain bounded⁵² and the difference between the solutions $|\Delta(t)|$ grows exponentially (i.e. $|\Delta(t)|/|\Delta(0)| \approx \exp(ht)$, $h > 0$), then the system displays sensitive dependence on initial conditions and is chaotic.

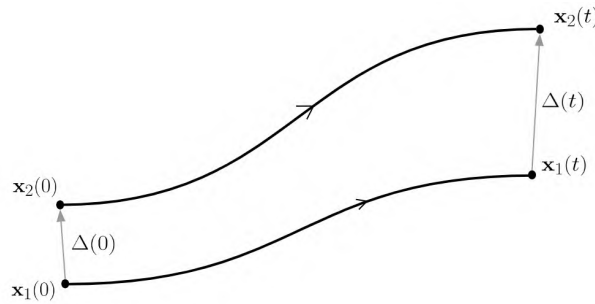


Fig. 1.8 Two nearby orbits evolving in phase space. Figure from [1].

The condition for orbits to remain bounded is necessary, since if the orbits go to infinity then the distances can diverge exponentially in a simple manner. An example is, $dx/dt = x$, a single autonomous linear, first-order differential equation. Here $d[x_2(t) - x_1(t)] = [x_2(t) - x_1(t)]$, so $\Delta(t) \approx \exp(t)$. So the requirement of bounded solutions eliminates such trivial cases. A driven damped pendulum⁵³ in first-order autonomous form is $x^{(1)} = d\theta/dt, x^{(2)} = \theta, x^{(3)} = 2\pi ft$, which are its phase space variables. But, $x^{(3)}$ is unbounded since is proportional to t . The reason for this system being chaotic is that $x^{(3)}$ occurs as an argument of sine, hence in the phase space

⁵¹For the modified driven pendulum equation, $\ddot{x} + \gamma\dot{x} = p_1 \sin(\omega_1 t) + p_2 \sin(\omega_2 t) - (x + \beta \sin(2\pi x))$, it was shown in [101] that for an irrational ratio of ω_1/ω_2 , the driving is quasiperiodic. And for the values of the parameters β, γ, p_1 and p_2 , the attractor of the system can either be strange and nonchaotic or strange and chaotic. For more details we refer readers to [102].

⁵²By bounded, we mean there is some region $\mathbf{x} < R < \infty$, which the solutions never leave. So if the motion is on an attractor, then the attractor lies in $|\mathbf{x}| < R$.

⁵³Equations for driven damped pendulum, $\frac{d^2\theta}{dt^2} + \nu \frac{d\theta}{dt} + \sin\theta = A \sin(2\pi ft)$, where θ is the angular displacement, ν is damping coefficient and A and f are the amplitude and frequency of the external driving force.

$x^{(1)}, \bar{x}^{(2)}, \bar{x}^{(3)}$, where $\bar{x}^{(2,3)} \equiv (x^{(2,3)} \text{ modulo } 2\pi)$ one can analyze the chaotic behavior. As the variables $\bar{x}^{(2)}$ and $\bar{x}^{(3)}$ lie between 0 and 2π , they are bounded.

Though there are other geometrical aspects of chaos like characterizing the fractal dimension etc., we will focus on the time evolution dynamics of chaotic orbits. Namely, we touch upon *stable* and *unstable* manifold and explain their *homoclinic* and *heteroclinic* intersections. This was the signature of chaos mentioned by Poincaré when he was trying to visualize the three-body problem. We also state the necessary tools to encapsulate the final result of our third work [2].

We begin with explaining the linear stability of a system and how the eigenvalues lead to the concept of *stable* and *unstable* subspace. And then how the eigenvectors help us construct the *stable* and *unstable* manifold.

For a system of real first-order differential equations $dx/dt = \mathbf{F}(x)$, a steady state is a point $\mathbf{x} = \mathbf{x}^*$ at which $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$. To examine the behaviour of orbits near $\mathbf{x}^*$, consider a small perturbation $\boldsymbol{\eta}(t)$. Substituting this into $dx/dt = \mathbf{F}(x)$ and expanding $\mathbf{F}(x)$ to first order in $\boldsymbol{\eta}(t)$ we have

$$\mathbf{F}(\mathbf{x}^* + \boldsymbol{\eta}) = \mathbf{F}(\mathbf{x}^*) + \mathbf{DF}(\mathbf{x}^*) \cdot \boldsymbol{\eta} + O(\boldsymbol{\eta}^2)$$

where, since $\mathbf{x}^*$ is a steady state, $\cdot$ is the scalar product, $\mathbf{F}(\mathbf{x}^*) = 0$ and $\mathbf{DF}$ denotes the Jacobian matrix of partial derivatives. The $\mathbf{x}$ and $\mathbf{F}(\mathbf{x})$ are defined as usual and Jacobian is represented as:

$$\mathbf{DF}(\mathbf{x}) = \begin{bmatrix} \partial F^{(1)}/\partial x^{(1)} & \partial F^{(1)}/\partial x^{(2)} & \dots & \partial F^{(1)}/\partial x^{(N)} \\ \partial F^{(2)}/\partial x^{(1)} & \partial F^{(2)}/\partial x^{(2)} & \dots & \partial F^{(2)}/\partial x^{(N)} \\ \vdots & & & \\ \partial F^{(N)}/\partial x^{(1)} & \partial F^{(N)}/\partial x^{(2)} & \dots & \partial F^{(N)}/\partial x^{(N)} \end{bmatrix}$$

Now we obtain the following equation for the time dependence of the perturbation of $\mathbf{x}$ from the steady state

$$d\boldsymbol{\eta}/dt = \mathbf{DF}(\mathbf{x}^*) \cdot \boldsymbol{\eta} + O(\boldsymbol{\eta}^2) \quad (1.13)$$

Neglecting terms of order $\boldsymbol{\eta}^2$ we get the linear stability problem whose form is

$$d\mathbf{y}/dt = \mathbf{A} \cdot \mathbf{y} \quad (1.14)$$

where $\mathbf{y}$ is a real N -dimensional vector and $\mathbf{A}$ is real time-independent $N \times N$ matrix. The solution to Eq.(1.14) is of the form $\mathbf{y}(t) = \mathbf{e} \exp(st)$ and can be reformulated as

the eigenvalue equation

$$\mathbf{A} \cdot \mathbf{e} = s \cdot \mathbf{e} \quad (1.15)$$

This equation has nontrivial solutions for values of s satisfying the N th order polynomial equation as

$$D(s) = \det[\mathbf{A} - s\mathbf{I}] = 0, \quad (1.16)$$

where $\mathbf{I}$ is $N \times N$ identity matrix. For a simple case, we can consider only cases where $D(s) = 0$ thereby having N *distinct* roots $s = s_k$ for $k = 1, 2, \dots, N$ (i.e., $s_k \neq s_j$ if $k \neq j$). Now for each such root, there is an eigenvector $\mathbf{e}_k$, and time evolution is expressed as

$$\mathbf{y}(t) = \sum_{k=1}^N A_k \mathbf{e}_k \exp(s_k t) \quad (1.17)$$

where A_k are constant coefficients (which can be complex) which can be found out from the initial condition $\mathbf{y}(0) = \sum_{k=1}^N A_k \mathbf{e}_k$. As the coefficients of the polynomial $D(s)$ are real, the eigenvalues s_k are either real or else occur in complex conjugate pairs. Now the type of eigenvalues and the associated eigenvectors helps in the construction of stable and unstable manifolds as we will describe now.

We take the case when eigenvalues are real or complex.

For complex conjugate of eigenvalues $s_l = s_{l+1}^* = \alpha_l - i\beta_l$. Also take $\mathbf{e}_l = \mathbf{e}_{l+1}^* = \mathbf{e}_l^R + i\mathbf{e}_l^I$. Here $*$ denotes complex conjugate and $\alpha_l, \beta_l, \mathbf{e}_l^R$ and $\mathbf{e}_l^I$ are all real. We get two linearly independent real solutions,

$$\begin{aligned} \mathbf{g}_l(t) &= \mathbf{e}_l^R \exp(\alpha_l t) \cos(\beta_l t) + \mathbf{e}_l^I \exp(\alpha_l t) \sin(\beta_l t), \\ \mathbf{g}_{l+1}(t) &= \mathbf{e}_l^I \exp(\alpha_l t) \cos(\beta_l t) - \mathbf{e}_l^R \exp(\alpha_l t) \sin(\beta_l t). \end{aligned}$$

For the case when s_l is real, we have $\mathbf{g}_l(t) = \mathbf{e}_l \exp \alpha_l t$, where $\mathbf{e}_l$ is real. Therefore Eq.(1.17) can be simply written as

$$\mathbf{y}(t) = \sum_{l=1}^N B_l \mathbf{g}_l(t) \quad (1.18)$$

where all the B_l is real and we can further explore the behavior by just focusing on the eigenvalues.

The case for $\text{Re}(s_l) = \alpha_l < 0$ corresponds to solution $\mathbf{g}_l(t)$ tends to origin asymptotically in time, either spiraling the plane spanned by $\mathbf{e}_l^R$ and $\mathbf{e}_l^I$ for $\text{Im}(s_l) \neq 0$ or moving along the line through the origin in direction of real eigenvector $\mathbf{e}_l$ if $\text{Im}(s_l) = 0$. When $\text{Re}(s_l) = \alpha_l > 0$ the corresponding solution $\mathbf{g}_l$ diverges from the origin exponentially in

time, either by spiraling or by moving linearly. Now the solutions that are going away from the origin exponentially with time are *unstable* and those moving exponentially towards origins are *stable*. The situation is depicted in Fig.(1.9).

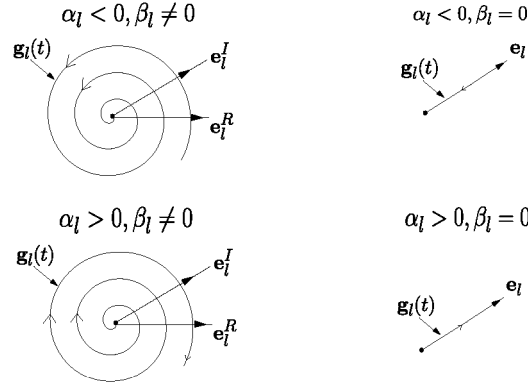


Fig. 1.9 Orbits for stable ($\alpha_l < 0$) and unstable ($\alpha_l > 0$) solutions. Figure from [1]

We need the notion of an invariant subspace to help us define the *stable*, *unstable* subspace which leads to manifold definition. Note for $\beta_l = 0$ and initial condition lying in the subspace spanned by the vectors $\mathbf{e}_l^R$ and $\mathbf{e}_l^I$ remain in that subspace for all time. So that subspace is invariant under the flow $d\mathbf{y}/dt = \mathbf{A}\cdot\mathbf{y}$.⁵⁴ Now all the independent vectors spanning invariant subspace corresponding to the unstable solution ($\alpha_l > 0$) can be denoted as $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n_u}\}$. And all the independent vectors spanning invariant subspace corresponding to the stable solution ($\alpha_l < 0$) can be denoted as $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n_s}\}$. For the eigenvalues whose real part are zero ($\alpha_l = 0$), the corresponding set of independent vectors spanning this subspace can be denoted as $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n_c}\}$. These three $\mathbf{u}$'s, $\mathbf{v}$'s, $\mathbf{w}$'s span the whole phase space : $n_u + n_v + n_c$. Now we can define the *unstable subspace* as

$$E^u = \text{span}[\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n_u}].$$

which implies the space spanned by the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n_u}$. The *stable subspace* is

$$E^s = \text{span}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n_s}].$$

The *center subspace* is

$$E^c = \text{span}[\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n_c}].$$

⁵⁴Also for real eigenvalue, for some initial condition, one can have a set originating from $\mathbf{e}_j$ forming its scalar multiples resulting in an invariant subspace.

We show an example of subspaces in Fig.(1.10).

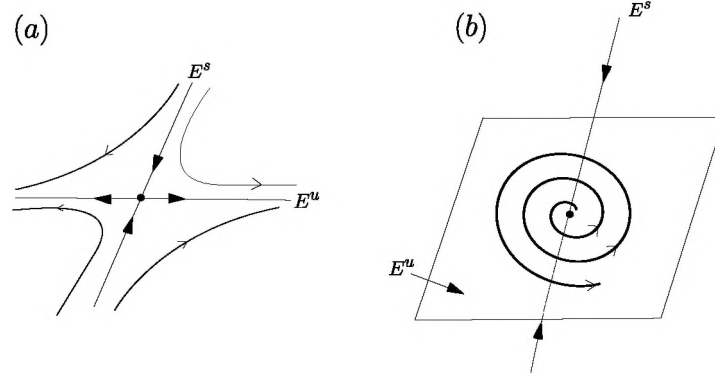


Fig. 1.10 Sample of stable and unstable subspaces. (a) Here $N = 2, n_u = n_s = 1$ where the stable space corresponds to two *real* eigenvalues. (b) Here $N = 3, n_s = 1, n_u = 2$ where the unstable subspace corresponds to a pair of complex conjugate eigenvalues. Figure based on [1].

Now for the steady states and periodic orbits of a smooth dynamical system, one can formulate the notion of stable and unstable manifolds. If the forward orbit starting from $\mathbf{x}$ approaches the steady state or closed curved traced out by the periodic orbit, then those set of points $\mathbf{x}$ form the *stable manifold* of that steady state or periodic orbit. Similarly, the set of points $\mathbf{x}$ such that the orbit going backward in time starting from $\mathbf{x}$ approaches the steady state or the closed curve traced out by periodic orbit, constitute the *unstable manifold* of that steady state or periodic orbit. We denote the stable manifold W^s and unstable manifold W^u . They have the same dimensionality as the linear subspace E^s and E^u and are tangent to them, $\dim(W^s) = n_s$, $\dim(W^u) = n_u$.

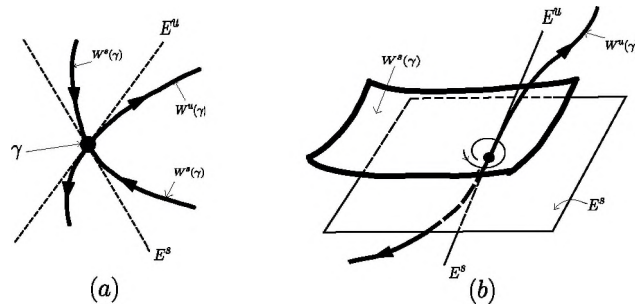


Fig. 1.11 This figure illustrates a scenario in a two-dimensional map with a fixed point γ , featuring one stable and one unstable direction. The figure also depicts a situation where $n_s = 2$ and $n_u = 1$ for a fixed point γ in a flow, indicating an orbit $W^s(\gamma)$ spiraling into the fixed point γ when two stable eigenvalues are complex conjugates. Figure based on [1].

Fig.(1.11) shows the situation for a two-dimensional map with fixed point γ having one stable and one unstable direction. The figure also shows a situation where $n_s = 2$ and $n_u = 1$ for a fixed point γ of a flow. That is an orbit $W^s(\gamma)$ spiraling into the fixed point γ when two stable eigenvalues are complex conjugates.

The main motivation for the previous definitions is to arrive at the *intersection* of stable and unstable manifolds. This ‘*intersection*’ was the glimpse of chaos witnessed by Poincaré for the first time when he was solving the restricted three-body problem. One of our main results concerns such an intersection while studying cylindrical standing gravitational waves. It can be demonstrated that the stable manifold of one fixed point cannot cross itself or another fixed point’s stable manifold. However stable manifolds can cross or intersect unstable manifolds which leads to chaos.

If the crossings of stable and unstable manifolds occur at the same fixed points, the points are homoclinic points (and are called *homoclinic intersection*). If the crossing happens at different fixed points, they are heteroclinic points (called *heteroclinic intersection*). In figure (1.12) we see the complicated crossing of homoclinic and heteroclinic intersections.

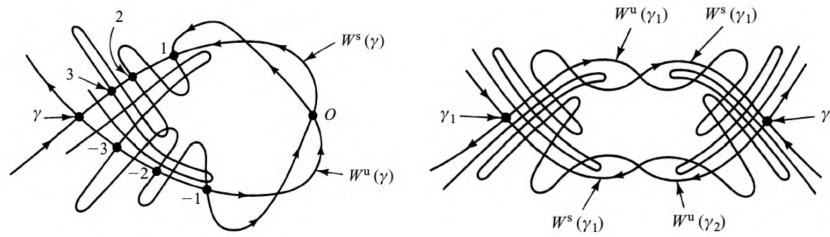


Fig. 1.12 Left: Sample of homoclinic crossing. Right: Heteroclinic crossing. The complexity arises in both figure from the necessity for stable and unstable manifolds to intersect an infinite number of times once they have crossed. In the figure, intersection point O on $W^s(\gamma)$ and $W^u(\gamma)$ persists in subsequent iterates. Forward iterations approach γ along the stable manifold, while backward iterations converge to γ along the unstable manifold. Figure drawn in Inkspace based on [1].

The type of crossing is important. If $W^s(\gamma)$ and $W^u(\gamma)$ cross *tangentially* then they form a single smooth homoclinic manifold *separatrices*.⁵⁵ More importantly, if they cross *transversely*, then they do not form the same smooth manifold but have an infinite number of crossings like a tangled web. This is the *homoclinic (or heteroclinic) tangle* where $W^s(\gamma)$ and $W^u(\gamma)$ are folded in an intricate manner resulting in chaotic orbits.

⁵⁵A boundary which separates the phase space into two distinct areas, each with a distinct type of motion.

Here the complexity of such intersections lies in the fact that once a stable and unstable manifold cross, they must intersect an infinite number of times. In the figure (1.12) there is an intersection O labeled which lies on $W^s(\gamma)$ and $W^u(\gamma)$. Now the next iterate, both forward and backward, will also lie on $W^s(\gamma)$ and $W^u(\gamma)$. The idea is that intersection points are mapped into intersection points. This will keep happening for future iterates as $W^s(\gamma)$ and $W^u(\gamma)$ are invariant sets. The points O when iterated forward will approach γ along the stable manifold successively mapping to points 1, 2, and 3 as shown in figure (1.12). Similarly the backward iterations of point O in time will come to point γ along the unstable manifold-this time mapping points -1,-2 and -3.

The complicated nature of Fig.(1.12) indicates the intricate dynamics whenever homoclinic or heteroclinic intersection happened transversally. This was shown by Smale [97].⁵⁶ Just to mention a bit more for a mathematical perspective, one can relate this to a *horseshoemap* which is a type of chaotic map for squares.⁵⁷ Here the map squishes the square, stretches it into a strip, and folds it into a horseshoe shape. Mainly there are these points which remain inside a square and form a fractal set within the map's invariant set. This set results from the intersection of a Cantor set of vertical lines with a Cantor set of horizontal lines, which has uncountable infinite points, but has measure zero (it is a set of points located on a line segment. Initially, you start with the entire interval, and then, in each iteration, you remove the middle third of each remaining segment while keeping the endpoints.).⁵⁸

This intricate dynamics of the homoclinic and heteroclinic intersection is what will constitute the final result of our third work.

⁵⁶Birkhoff-Smale Theorem [103]: Let $f : M \rightarrow M$ be a diffeomorphism. If f has a transverse homoclinic point q , then f has a periodic point in any neighborhood of q .

⁵⁷Introduced by Stephen Smale when studying orbits in a van der Pol oscillator (a non-conservative, oscillating system with non-linear damping).

⁵⁸For more details about Horseshoe map and Cantor set we refer to the standard dynamical systems text.

Chapter 2

Standing gravitational waves in vacuum

“ Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone.”

Leopold Infeld

The main results presented in this chapter are published in our first paper ‘*Freely falling bodies in a standing-wave spacetime*’ [13]. All calculations were independently conducted by both co-authors, and separate independent numerical codes were developed to cross-verify the results. In this chapter the results provided are generated using my independent *Mathematica* code (also the figures presented are generated with my version of the code). In this chapter, I present a more in-depth exploration, providing supplementary details and discussions to go along with the content presented in our paper.

2.1 Introduction

Standing waves are crucial in various physics branches, but the nonlinear nature of Einstein’s equations complicates their study in gravitational contexts. The absence of a precise definition hinders our understanding of gravitational standing waves. Stephani addressed this issue by presenting an exact standing-wave spacetime example in [4]. An alternative definition was proposed in [86], focusing on the

intuitive concept of standing waves involving alternating energy flows through nodes. However, localizing gravitational wave energy and establishing a covariant averaging procedure pose challenges, making a covariant definition problematic. To circumvent this, [86] suggests considering the high-frequency limit, which corresponds to an effective spacetime with a Ricci tensor of the Serge type which we have discussed in Chapter 1.

This work on standing waves in vacuum seeks to elucidate how test particles behave in a standing-wave spacetime. Standing-wave optical/acoustic traps are widely employed in science and industry. Hence, an important inquiry is how gravitational standing waves impact the paths of freely falling test masses.

The exact standing-wave spacetime discussed in papers [4, 86] (also [72]) can be interpreted in two ways. Originally, it is solution which belongs to the Einstein-Rosen class solution with cylindrical symmetry. Additionally, it can be transformed to a specific case of the three-torus Gowdy model, as explored in detail in [104]. We now explain the issues pertaining to standing gravitational waves, specifically the gravitational energy part. Then explore the features and drawbacks of the Einstein-Rosen metric as a standing gravitational wave spacetime, finally focussing on its T^3 Gowdy interpretation.

In the case of a standing wave, the maximum amplitude of the wave occurs at the antinodes, indicating the region of maximum energy. However, this concept cannot be straightforwardly applied to gravitational standing waves. Naively, one might assume that if the metric functions oscillate at the antinodes, the gravitational energy would be at its maximum at the antinode of a strong gravitational wave. This suggests that the geodesic equation should have stationary solutions at antinodes. One may show that this is not true for cylindrical standing waves (the Einstein-Rosen waves) discussed by Stephani [4]. This solution can be seen as a “nonlinear” combination of incoming gravitational waves and their reflections from the symmetry axis, with decreasing wave amplitudes in the radial direction. The highest gravitational energy concentration (measured by Thorne’s C-energy [105]) is at the symmetry center, resembling a gravitational geon but without strict confinement to a finite space due to cylindrical symmetry. To an external observer studying the paths of falling objects, the entire system appears as an infinitely massive structure extending along the symmetry axis. Consequently, this system differs significantly from the typical scenarios where electromagnetic or acoustic standing waves are explored.

Beyond cylindrical symmetry one can study Gowdy spacetimes which are better suited to study freely-falling observers in a standing GW setting. After imposing

few boundary conditions on the generalized Einstein-Rosen metrics we can arrive at inhomogeneous universes which solve Einstein's vacuum field equations exactly. The spatial sections of these universes exhibit either S^3 , $S^1 \times S^2$ and T^3 (three-torus) topologies. These are useful toy models formulated in [104, 106, 27] to study standing gravitational waves. The *three-torus solution*, in particular, can be regarded as standing-gravitational-wave inhomogeneous generalization of spatially flat Kasner universes [107]. They are also useful test cases for validating numerical simulations of Einstein's field equations [108–111].

In this chapter, we study the Gowdy-type interpretation of the standing-wave spacetime. In this interpretation, the amplitude of gravitational waves remains constant in spatial direction but decreases over time as the universe expands. Our work demonstrates that the geodesic equation indeed possesses stationary solutions at antinodes. To study the motion of freely falling bodies, we examine them relative to stationary observers positioned at antinodes. These investigations are conducted using the geodesic deviation equation. Further, we used the Newman-Penrose formalism to explore the dynamics further in terms of the Weyl scalars.

2.2 Formalism of standing waves in Gowdy spacetimes

We give a basic motivation followed by Gowdy to arrive at the metric which forms the basis of this chapter. This section follows from [64]. Gowdy spacetimes defines gravitational waves within a box by establishing isometry and group orbits with the aid of symmetries. A specific coordinate system is chosen, followed by the application of Einstein's equations. After simplifying the equations, the metric is derived.

Einstein's general relativity can be stated with a pen-and-paper calculation by analyzing the Gowdy spacetimes. They represent a simple case of gravitational waves in a box. Consider a rectangular box. On joining the two adjacent sides, we will get a cylinder. Further joining its two ends will result in a 2-torus (see Figure (2.1)). Similarly, if one takes a cube and associates each side to be connected with each adjacent side, one gets a three-torus. Now, one can have, say, a wave in the $+z$ -direction, and in the three-torus it will interact with itself from the wave coming from the $-z$ -direction. This is the T^3 configuration analyzed in our first work [13].

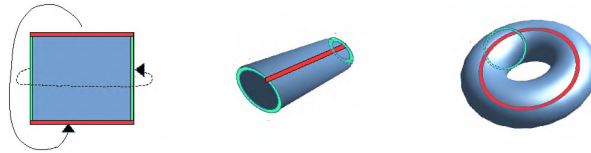


Fig. 2.1 Sample image of how a rectangle tile can be turned into a 2-torus. The solid-line deformation is followed by the dashed line.

Robert H. Gowdy formulated the Gowdy spacetimes and now we motivate them. This part is taken from his work [112] and other references mentioned there. By isometry group we mean group of motions that leaves the metric of a geometry unchanged. And group orbit means an orbit of an isometry group that is the set of points which can be reached by letting all operations of the group act on a single point.

The main idea is to understand G_2 spacetimes with the help of *isometry groups* and *group orbits*. G_2 group is defined by existence of two killing fields. And G_2 spacetimes have two-parameter Lie groups as their isometries. For the orbits to be closed and orientable, the G_2 group with an effective action on the manifold is $SU(1) \times SU(1)$ ¹ and the allowed orbits are only tori or circles. The limited variety facilitates the determination of all possible structures of these spacetimes. For example, connected, group-invariant spacelike hypersurfaces in these spacetimes are homeomorphic to the three-torus T^3 or three-handle $S^1 \times S^2$ or three-sphere S^3 .

Choosing coordinates properly allows the metric tensor in a coordinate patch on a G_2 spacetime with closed orbits to always be brought into a special form. One can take the $SU(1) \times SU(1)$ group coordinates $x^2 = \sigma, x^3 = \delta$ as coordinates on each orbit. The range for these coordinates is 0 to 2π . And each orbit is labeled by a space coordinate $x^0 = t, x^1 = \theta$. Here is an example of a $t = \text{constant}$ picture of the situation.

¹ $SU(n)$, is the Lie group of $n \times n$ unitary matrices with determinant 1. The ‘ $\times$ ’ is the *direct product*, an operation that takes two groups G and H and constructs a new group, denoted $G \times H$.

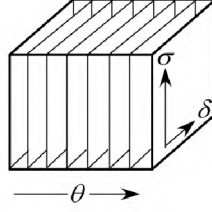


Fig. 2.2 Image showing constant t slice. Figure taken from [64].

So one can write the metric intrinsic to the two-dimensional orbit labeled by t, θ as

$$ds^2 = e^{2a}(d\theta^2 - dt^2) + C_{33}d\sigma^2 + 2C_{34}d\sigma d\delta + C_{44}d\delta^2 \quad (2.1)$$

where the functions $a, C_{33}, C_{34}, C_{44}$, depend only on t and θ . For the given metric, Einstein's Field Equations result in two equations directly integrable for the function a and a complex system of nonlinearly coupled wave equations for the other functions. It turns out that for a combination of $R = \sqrt{C_{33}C_{44} - (C_{34})^2}$ simplify the Einstein equations.²

After imposing a reflection symmetry within each group orbit, the form of the metric is:

$$ds^2 = e^{2a}(-dt^2 + d\theta^2) + R(e^{2\psi}d\sigma^2 + e^{-2\psi}d\delta^2) \quad (2.2)$$

where a, ψ are functions of t and θ only.³ The choice of R and periodic boundary conditions leads to various Gowdy topologies (T^3 or $S^1 \times S^2$ or S^3).

For orbits (wavefronts) to be regular everywhere at a given time, one solution is $R = t$. This would represent a universe expanding from an initial singularity $t = 0$. After imposing periodic boundary conditions, one arrives at T^3 polarised Gowdy.⁴

²The combination $\sqrt{C_{33}C_{44} - (C_{34})^2}d\sigma d\delta$ is the unit area of the group orbit.

³At this point all nonlinear couplings vanish, leaving an simple system of linear wave equations $\partial^2 R / \partial \theta^2 - \partial^2 R / \partial t^2 = 0$ and $\partial / \partial \theta (R \partial \psi / \partial \theta) - \partial / \partial t (R \partial \psi / \partial t) = 0$

⁴A Gowdy spacetime where the reflections in σ and δ are both symmetries is polarized Gowdy spacetime.

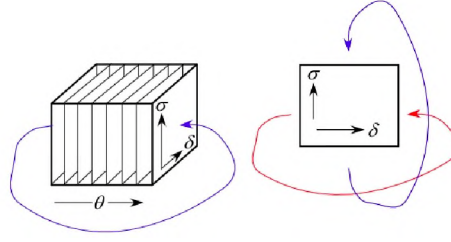


Fig. 2.3 Making the space geometry compact by imposing periodic boundary conditions. This is related to joining opposite sides of a cube and forming a three-torus as shown by the colored lines. Figure taken from [64].

The metric we study in our first work has the following form

$$\hat{g} = e^f(-dt^2 + dz^2) + t(e^p dx^2 + e^{-p} dy^2) \quad (2.3)$$

where z is identified with θ in the earlier metric form in Eq (2.2). Similarly for coordinates x and y identified with σ and δ and the two parameters f and p with a and ψ .

For the above metric $t > 0$ is a cosmic time function. The wave is in z -direction, $0 \leq z < 2\pi$ and $0 \leq x, y < 2\pi$. The metric functions are $f = f(t, z)$ and $p = p(t, z)$. The coordinates are ordered as $x^\alpha = (t, z, x, y)$. The particular solution we are interested in belongs to the class studied in the paper [75] and is given by

$$\begin{aligned} p &= -\ln t + 2\beta\sqrt{\lambda}J_0\left(\frac{t}{\lambda}\right)\sin\left(\frac{z}{\lambda}\right), \\ f &= \frac{\beta^2}{\lambda}t^2 \left[J_0^2\left(\frac{t}{\lambda}\right) + J_1^2\left(\frac{t}{\lambda}\right) - 2\frac{\lambda}{t}J_0\left(\frac{t}{\lambda}\right)J_1\left(\frac{t}{\lambda}\right)\sin^2\left(\frac{z}{\lambda}\right) \right] \\ &\quad - 2\beta\sqrt{\lambda}J_0\left(\frac{t}{\lambda}\right)\sin\left(\frac{z}{\lambda}\right), \end{aligned} \quad (2.4)$$

where J_i are the Bessel functions of the first kind and i th order. From the periodicity of $f(t, 0) = f(t, 2\pi)$, $p(t, 0) = p(t, 2\pi)$ we get $\lambda = 1/n$ with $n \in \mathbb{Z}$. The constant β is arbitrary. As $\beta \rightarrow 0$, the function $f \rightarrow 0$ and $p \rightarrow -\ln t$ and the metric corresponds to Minkowski spacetime. β and n are two parameters controlling the amplitude and the number of waves. Without loss of generality, we assume $\beta > 0$. There is a curvature singularity at $t = 0$. The nature of the singularity has been studied in [113]. For polarised Gowdy spacetimes, the strong cosmic censorship holds [114]. After the Big Bang, the model expands anisotropically. The metric functions are plotted in Fig.(2.4).

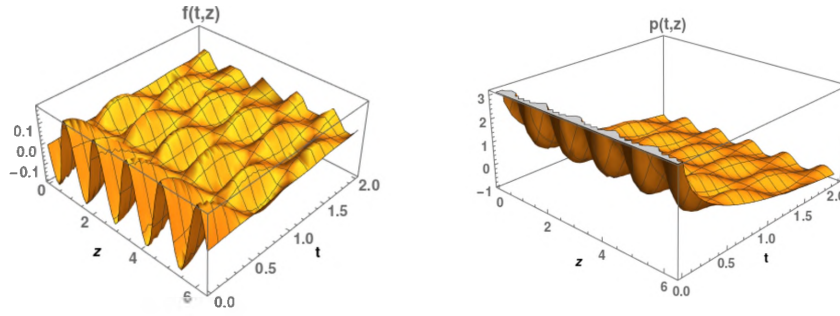


Fig. 2.4 Plot for the metric functions $f(t,z)$ and $p(t,z)$ in Eq.(2.4) for $\beta = 0.2$, $\lambda = 0.1$. One can change the parameters β and n to control the amplitude and the number of waves. In this expanding spacetime, the amplitude of wave decreases with time as we see in the figure.

2.3 Analysing test masses behavior

We first establish stationary observers at the antinodes. The freely falling observers at the antinodes help us identify the gravitational wave nature in both the transverse and longitudinal directions through the use of the geodesic deviation equation. We employ the geodesic equation to gain a coordinate-dependent yet preliminary insight into regions near the antinodes.

Stationary observers at antinodes

We investigate the geodesic equation under the assumption that $x = x_0$, $y = y_0$, where x_0 , y_0 are constants. The reduced geodesic equation is (we derive it in the later section as Eq.(2.19)):

$$\begin{aligned} \ddot{t} + \frac{1}{2} (f_{,t} \dot{t}^2 + 2f_{,z} \dot{t} \dot{z} + f_{,t} \dot{z}^2) &= 0, \\ \ddot{z} + \frac{1}{2} (f_{,z} \dot{z}^2 + 2f_{,t} \dot{t} \dot{z} + f_{,z} \dot{t}^2) &= 0, \end{aligned}$$

The first integral gives us:

$$\dot{t}^2 - \dot{z}^2 = e^{-f}.$$

Now we will establish that for particular values of z we get stationary solution, i.e, $\dot{z} = 0$. For $\dot{z} = 0$, the above equations reduce to:

$$0 = \ddot{t} + \frac{1}{2}f_{,t} \dot{t}^2, \quad (2.5)$$

$$\begin{aligned} 0 &= f_{,z}, \\ \dot{t} &= e^{-\frac{1}{2}f}, \end{aligned} \quad (2.6)$$

Here Eq.(2.6) is the integral of Eq.(2.5). The main nontrivial condition is $f_{,z} = 0$. Using Eq.(2.5), we have $f_{,z} = (\dots)(\cos(\frac{z}{\lambda}))$. This expression would be zero for $z = \lambda\pi(\frac{1}{2} + k)$, where $k \in \mathbb{Z}$. Hence the curve γ_k

$$x^\mu = [t(\tau), \lambda\pi(\frac{1}{2} + k), x_0, y_0] \quad (2.7)$$

is a future-directed timelike geodesic and a stationary solution to the geodesic equation for $k \in \mathbb{Z}$ and $t(\tau)$ given by Eq.(2.6). The odd and even values of k lead to different values of the metric functions, but without loss of generality we can assume $k = 0$. We study the behavior of massive particles around such stationary worldlines with the help of the geodesic deviation equation.

2.3.1 Geodesic deviation equation

To gauge a more detailed description, we explore the geodesic deviation equation. It reveals the true nature of how test masses are affected in the presence of a gravitational field. When analyzed in an orthonormal frame, it helps in defining coordinate-independent notions.

Consider a one-parameter family of geodesics $\gamma_p(\tau)$, for each $p \in \mathbb{R}$ and parameterized by affine proper time τ , with tangent T^μ and deviation vector ξ^μ . The vector field $T^\mu = (\partial/\partial\tau)^\mu$ is tangent to the family of geodesics and $\xi^\mu = (\partial/\partial p)^\mu$ is the displacement to an infinitesimally nearby geodesic.

One can define a rate of change along a geodesic of the displacement to an infinitesimally nearby geodesic as $v^\mu = T^\nu \nabla_\nu \xi^\mu$. This is the relative velocity of an infinitesimally nearby geodesic. Following the same way we can arrive at the relative acceleration of an infinitesimally nearby geodesic as $a^\mu = T^\sigma \nabla_\sigma v^\mu = T^\sigma \nabla_\sigma (T^\nu \nabla_\nu \xi^\mu)$. On simplifying the right-hand side of the previous equation related a^μ to the Riemann tensor as:

$$a^\alpha = -R^\alpha_{\sigma\beta\rho} \xi^\beta T^\sigma T^\rho \quad (2.8)$$

This is the *geodesic deviation equations* which gives us a characterization of curvature revealing the true effect a gravitational wave has on test masses. It has a simpler form in a ‘freely falling frame’. Hence in order to avoid ‘coordinate effects’, we pick a nonholonomic basis. So for our spacetime (M, g) with the metric given in coordinate basis by Eq.(2.3), we introduce an orthonormal tetrad $\{e^{\hat{\alpha}}(p)\} \subset T^p M$ as

$$\begin{aligned} e_{\hat{0}} &= e^{-f/2} \partial_t, \\ e_{\hat{1}} &= e^{-f/2} \partial_z, \\ e_{\hat{2}} &= e^{-p/2} / \sqrt{t} \partial_x, \\ e_{\hat{3}} &= e^{p/2} / \sqrt{t} \partial_y. \end{aligned} \quad (2.9)$$

where “” is used to denote orthonormal frame indices. Its dual basis is denoted by $\{\theta^{\hat{\alpha}}(p)\} \subset T_p^* M$ given as

$$\begin{aligned} \theta^{\hat{0}} &= e^{f/2} dt, \\ \theta^{\hat{1}} &= e^{f/2} dz, \\ \theta^{\hat{2}} &= e^{p/2} \sqrt{t} dx, \\ \theta^{\hat{3}} &= e^{-p/2} \sqrt{t} dy. \end{aligned} \quad (2.10)$$

Proof of $\{e_{\hat{\alpha}}\}$ as a freely falling frame

We show that the orthonormal tetrad $\{e_{\hat{\alpha}}\}$ is a freely falling frame of stationary observers at antinodes along the geodesics γ_k . This can be established using the Fermi-Walker transport as well but we show it explicitly here. ⁵The tangent vector is $u = e_{\hat{0}}$ and the antinode positions $z = \lambda \pi(1/2 + k)$ where $k \in \mathbb{Z}$. So along γ_k we show

$$\nabla_{e_{\hat{0}}} e_{\hat{\alpha}} = 0, \quad \nabla_{e_{\hat{0}}} \theta^{\hat{\alpha}} = 0.$$

In order to prove this we take help of *connection one forms*, $\omega_{\hat{\alpha}}^{\hat{\beta}}$, which are related to the orthonormal vectors as $\nabla_{e_{\hat{0}}} e_{\hat{\alpha}} = \omega_{\hat{\alpha}}^{\hat{\beta}}(e_{\hat{0}}) e_{\hat{\beta}}$. We use the Cartan’s first structure equation to read out the connection one forms :

$$d\theta^{\hat{\alpha}} + \omega_{\hat{\beta}}^{\hat{\alpha}} \wedge \theta^{\hat{\beta}} = 0. \quad (2.11)$$

⁵One can also prove this using Fermi-Walker transport. It is employed to establish a coordinate system or reference frame where any curvature within the frame arises exclusively from the presence of mass/energy density, eliminating the effects of arbitrary spin or frame rotation [115].

From the dual-basis Eq (2.10), we have :

$$\begin{aligned}
d\theta^{\hat{0}} &= -\frac{1}{2}e^{\frac{f}{2}}f_{,z}dt \wedge dz = -\frac{1}{2}e^{-\frac{f}{2}}f_{,z}\theta^{\hat{0}} \wedge \theta^{\hat{1}}, \\
d\theta^{\hat{1}} &= \frac{1}{2}e^{\frac{f}{2}}f_{,t}dt \wedge dz = \frac{1}{2}e^{-\frac{f}{2}}f_{,t}\theta^{\hat{0}} \wedge \theta^{\hat{1}}, \\
d\theta^{\hat{2}} &= \frac{1}{2}e^{\frac{p}{2}}\sqrt{t} \left[\left(\frac{1}{t} + p_{,t} \right) dt \wedge dx + p_{,z} dz \wedge dx \right] \\
&= \frac{1}{2}e^{-\frac{f}{2}} \left[\left(\frac{1}{t} + p_{,t} \right) \theta^{\hat{0}} \wedge \theta^{\hat{2}} + p_{,z} \theta^{\hat{1}} \wedge \theta^{\hat{2}} \right], \\
d\theta^{\hat{3}} &= \frac{1}{2}e^{-\frac{p}{2}}\sqrt{t} \left[\left(\frac{1}{t} - p_{,t} \right) dt \wedge dy - p_{,z} dz \wedge dy \right] \\
&= \frac{1}{2}e^{-\frac{f}{2}} \left[\left(\frac{1}{t} - p_{,t} \right) \theta^{\hat{0}} \wedge \theta^{\hat{3}} - p_{,z} \theta^{\hat{1}} \wedge \theta^{\hat{3}} \right].
\end{aligned}$$

We can compare this with Eq (2.11) and find out the connection one forms:

$$\begin{aligned}
\omega^{\hat{0}}_{\hat{1}} &= \frac{1}{2}e^{-\frac{f}{2}} \left[f_{,z} \theta^{\hat{0}} + f_{,t} \theta^{\hat{1}} \right], \\
\omega^{\hat{0}}_{\hat{2}} &= \frac{1}{2}e^{-\frac{f}{2}} (1/t + p_{,t}) \theta^{\hat{2}}, \\
\omega^{\hat{0}}_{\hat{3}} &= \frac{1}{2}e^{-\frac{f}{2}} (1/t - p_{,t}) \theta^{\hat{3}}, \\
\omega^{\hat{1}}_{\hat{2}} &= -\frac{1}{2}e^{-\frac{f}{2}} p_{,z} \theta^{\hat{2}}, \\
\omega^{\hat{1}}_{\hat{3}} &= \frac{1}{2}e^{-\frac{f}{2}} p_{,z} \theta^{\hat{3}}.
\end{aligned} \tag{2.12}$$

As $f_{,z}|_{\gamma_k} = 0$, it follows that $\nabla_{e_{\hat{0}}} e_{\hat{\alpha}} = 0$. This, in conjunction with the cobasis duality relation and the Leibniz rule for the covariant derivative, leads to $\nabla_{e_{\hat{0}}} \theta^{\hat{\alpha}} = 0$. In summary, our nonholonomic basis $e_{\hat{\alpha}}$ serves as a freely falling frame for observers traveling along geodesics γ_k (those positioned at antinodes).

Geodesic Deviation equation in a freely-falling frame

In a freely-falling frame, we get a simplified version of the geodesic deviation equation as:

$$\frac{d^2 \xi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}_{\hat{0}\hat{\beta}\hat{0}} \xi^{\hat{\beta}} \tag{2.13}$$

where ξ is the deviation vector along γ_k and τ is the proper time of the observer.⁶ Now one can use Cartan's second structure equation to find out the curvature two forms $\Omega_{\hat{\beta}}^{\hat{\alpha}}$,

$$\Omega_{\hat{\beta}}^{\hat{\alpha}} = d\omega_{\hat{\beta}}^{\hat{\alpha}} + \omega_{\hat{\sigma}}^{\hat{\alpha}} \wedge \omega_{\hat{\beta}}^{\hat{\sigma}}. \quad (2.14)$$

The curvature two forms are related to the components of the Riemann tensor as $\Omega_{\hat{\beta}}^{\hat{\alpha}} = \frac{1}{2} R_{\hat{\beta}\hat{\sigma}\hat{\delta}}^{\hat{\alpha}} \theta^{\hat{\sigma}} \wedge \theta^{\hat{\delta}}$. Also, from the formula, we see not all curvature two forms are needed as some of them will be zero. From the connection one forms (2.12) we found earlier, we have

$$\begin{aligned} \Omega_{\hat{1}}^{\hat{0}} &= \frac{1}{2} e^{-f} (f_{,tt} - f_{,zz}) \theta^{\hat{0}} \wedge \theta^{\hat{1}}, \\ \Omega_{\hat{2}}^{\hat{0}} &= \frac{1}{4} e^{-f} \left[(p_{,t} + \frac{1}{t})^2 - f_{,t}(p_{,t} + \frac{1}{t}) + 2(p_{,tt} - \frac{1}{t^2}) - f_{,z}p_{,z} \right] \theta^{\hat{0}} \wedge \theta^{\hat{2}} \\ &\quad + \frac{1}{4} e^{-f} \left[(p_{,t} + \frac{1}{t})(p_{,z} - f_{,z}) - f_{,t}p_{,z} + 2p_{,tz} \right] \theta^{\hat{1}} \wedge \theta^{\hat{2}}, \\ \Omega_{\hat{3}}^{\hat{0}} &= \frac{1}{4} e^{-f} \left[(p_{,t} - \frac{1}{t})^2 + f_{,t}(p_{,t} - \frac{1}{t}) - 2(p_{,tt} + \frac{1}{t^2}) + f_{,z}p_{,z} \right] \theta^{\hat{0}} \wedge \theta^{\hat{3}} \\ &\quad + \frac{1}{4} e^{-f} \left[(p_{,t} - \frac{1}{t})(p_{,z} + f_{,z}) + f_{,t}p_{,z} - 2p_{,tz} \right] \theta^{\hat{1}} \wedge \theta^{\hat{3}}. \end{aligned}$$

where $f_{,i} = \frac{df}{di}$ and $f_{,ii} = \frac{d^2f}{di^2}$.

Now we can read out the necessary components of the Riemann tensor as

$$\begin{aligned} R_{\hat{0}\hat{1}\hat{0}}^{\hat{1}} &= \frac{1}{2} e^{-f} (f_{,zz} - f_{,tt}), \\ R_{\hat{0}\hat{2}\hat{0}}^{\hat{2}} &= -\frac{1}{4} e^{-f} \left[(p_{,t} + \frac{1}{t})^2 - f_{,t}(p_{,t} + \frac{1}{t}) + 2(p_{,tt} - \frac{1}{t^2}) - f_{,z}p_{,z} \right], \\ R_{\hat{0}\hat{3}\hat{0}}^{\hat{3}} &= -\frac{1}{4} e^{-f} \left[(p_{,t} - \frac{1}{t})^2 + f_{,t}(p_{,t} - \frac{1}{t}) - 2(p_{,tt} + \frac{1}{t^2}) + f_{,z}p_{,z} \right]. \end{aligned} \quad (2.15)$$

⁶In this thesis [116], a bilocal geodesic operator (BGO) formalism was proposed to study propagation of light curves in curved spacetime. It would be interesting to probe the problem of null geodesics using that formalism for this spacetime.

Also the metric functions take form ($\sigma = 1$ for $k \in 2\mathbb{Z}$ and $\sigma = -1$ for $k \in 2\mathbb{Z} + 1$)

$$\begin{aligned} p|_{\gamma_k} &= -\ln t + 2\sigma\beta\sqrt{\lambda}J_0\left(\frac{t}{\lambda}\right), \\ f|_{\gamma_k} &= \frac{\beta^2}{\lambda}t^2 \left[J_0^2\left(\frac{t}{\lambda}\right) + J_1^2\left(\frac{t}{\lambda}\right) - 2\frac{\lambda}{t}J_0\left(\frac{t}{\lambda}\right)J_1\left(\frac{t}{\lambda}\right) \right] - 2\sigma\beta\sqrt{\lambda}J_0\left(\frac{t}{\lambda}\right), \\ f_{,t}|_{\gamma_k} &= \frac{2\beta}{\lambda}J_1\left(\frac{t}{\lambda}\right) \left[\sigma\sqrt{\lambda} + t\beta J_1\left(\frac{t}{\lambda}\right) \right]. \end{aligned}$$

We notice that the components of $R^{\hat{\alpha}}_{\hat{0}\hat{\beta}\hat{0}}$ vanish for $\hat{\alpha} \neq \hat{\beta}$, hence the equations decouple and we are only left with $R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}}$, $R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}}$, $R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}}$. After substituting these components into the geodesic deviation equation Eq.(2.13), we can obtain along γ_k , the geodesic equation. The equations have a simpler form in terms of coordinate time t than in terms of the proper time of the observer τ . For the left side of Eq.(2.13) we can use the normalization of $e_{\hat{0}}$ ($\frac{dt}{d\tau} = e^{-f/2}$) to write $\frac{d\xi}{d\tau}$ as $\frac{d\xi}{dt} \frac{dt}{d\tau} = e^{-f} \frac{d\xi}{dt}$. So the geodesic deviation equation is :

$$\begin{aligned} \frac{d^2\xi^{\hat{1}}}{dt^2} - \frac{1}{2}f_{,t} \frac{d\xi^{\hat{1}}}{dt} &= \frac{1}{2}(f_{,tt} - f_{,zz})\xi^{\hat{1}} = -\frac{\beta}{\lambda t}J_1\left(\frac{t}{\lambda}\right) \left[\sqrt{\lambda} + t\beta J_1\left(\frac{t}{\lambda}\right) \right] \xi^{\hat{1}}, \\ \frac{d^2\xi^{\hat{2}}}{dt^2} - \frac{1}{2}f_{,t} \frac{d\xi^{\hat{2}}}{dt} &= \frac{1}{4t} [f_{,t} + (p_{,t}(2 - tf_{,t}) + 2tp_{,tt})] \xi^{\hat{2}} + \dots \\ &= -\frac{\beta}{\lambda^{3/2}} \left[J_0\left(\frac{t}{\lambda}\right) - \frac{\lambda^2}{4\beta^2 t} J_1^{-1}\left(\frac{t}{\lambda}\right) f_{,t}^2 \right] \xi^{\hat{2}} + \dots, \\ \frac{d^2\xi^{\hat{3}}}{dt^2} - \frac{1}{2}f_{,t} \frac{d\xi^{\hat{3}}}{dt} &= \frac{1}{4t} [f_{,t} - (p_{,t}(2 - tf_{,t}) + 2tp_{,tt})] \xi^{\hat{3}} \\ &= \frac{\beta}{\lambda^{3/2}} \left[J_0\left(\frac{t}{\lambda}\right) - \frac{\lambda}{2} J_1\left(\frac{t}{\lambda}\right) f_{,t} \right] \xi^{\hat{3}}, \end{aligned} \tag{2.16}$$

The ... in the $\xi^{\hat{2}}$ and $\xi^{\hat{3}}$ component has terms $-\frac{1}{4t^2}(-1 + t^2 p_{,z}^2 - 2tf_{,t} + t^2 p_{,t}^2)$ and $f_{,z}$, $p_{,z}$. These terms are zero when we use one of the Einstein equations $G_{\hat{0}\hat{0}} = 0$, and that $f_{,z}|_{\gamma_k} = 0 \implies p_{,z}|_{\gamma_k} = 0$ for our solution Eq.(2.4). Equation (2.16) is a decoupled system of autonomous second-order linear ordinary differential equations with variable coefficients. The decoupling helps in analyzing the motion of test masses in *transverse* and *longitudinal* direction separately.

Now we discuss the numerical numerical solutions to the geodesic deviation equation. They were found using *Mathematica*.

The nature of gravitational wave in linear regime is *transverse*. But there exists other behavior, for e.g *longitudinal effects*, when studied in full nonlinear context. To exploit the transverse feature we use a ring of test particles at the antinodes as freely

observed at the antinodes. The aim here is to see the usual ‘stretching and squeezing’ behavior in the XY plane for the wave in the Z direction. But also since this is a full exact solution we also analyze what happens in the z direction itself to scope the non-linearities in the longitudinal direction.⁷

Geodesic Deviation in Transverse direction

We can consider a ring of test massive particles located at one of the antinodes denoted by $z = \lambda(\frac{\pi}{2} + k)$, where k is an integer. Without loss of generality, we can choose $k = 0$. This choice is equivalent in our formulas for all even values of k . Although the odd values of k lead to slightly different equations, the qualitative behavior of the metric functions remains similar for large t . Therefore, analyzing the odd antinodes can be thought of as a time-shifted analysis of the even antinodes.

As the model expands the amplitude of waves decreases with time. We will use the help of Tissot plots to show the results and the geodesic deviation equation first used to study the memory effect for plane GW in [117]. Tissot plots helps to visualize the distortions of a map projection.⁸ They can be used to study the effect of gravitational waves on a ring of freely falling particles. With them, one can visualize through a sequence of time-frames, the compression and expansion of the ring as the wave propagates through it.

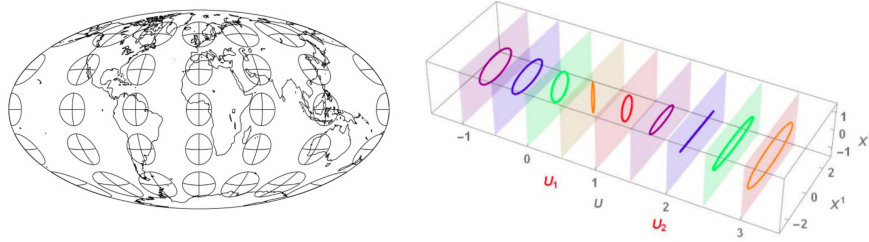


Fig. 2.5 Left: A sample indicatrix map [119]. Right: Usage of Tissot plots to show memory effect for plane GWs [117]. A ‘sandwich wave’ was considered with U having the profile and X_1, X_2 representing the transverse plane coordinates.

The initial conditions set how the shape of the Tissot diagram would be — forming the crucial aspect for interaction with the first wave Fig.(2.6) illustrates the trans-

⁷For GWs detected on earth, as they are far from source such a non-linear longitudinal effect is ‘peeled off’.

⁸Nicolas Auguste Tissot, a French cartographer, developed a method in the nineteenth century for analyzing distortion in map projections [118]. By employing various projection types, Tissot projected small circles from the Earth’s surface. The resulting indicatrix, an ellipse of distortion, vividly illustrates the magnitude and nature of distortion at its corresponding location.

formation of two initially stationary rings of particles into deformed “corrugated tubes” and eventually into ellipses (solid and dashed). The motion of these particles follows geodesics. The left (yellow) corresponds to $t_0 = 1.073$ and right (green) one to $t_0 = 1.073 + 0.31$. Here $t_0 = 1.173$.

The particles are at rest relative to a stationary observer (central) freely falling at antinode $d\xi^{\hat{\alpha}}/d\tau|_{t_0} = d\xi^{\hat{\alpha}}/dt|_{t_0} = 0$. This configuration does not though imply being stationary in our coordinates, hence the test particles in general do not move along γ_k . For the two figures, we see that there is a residual effect left behind.

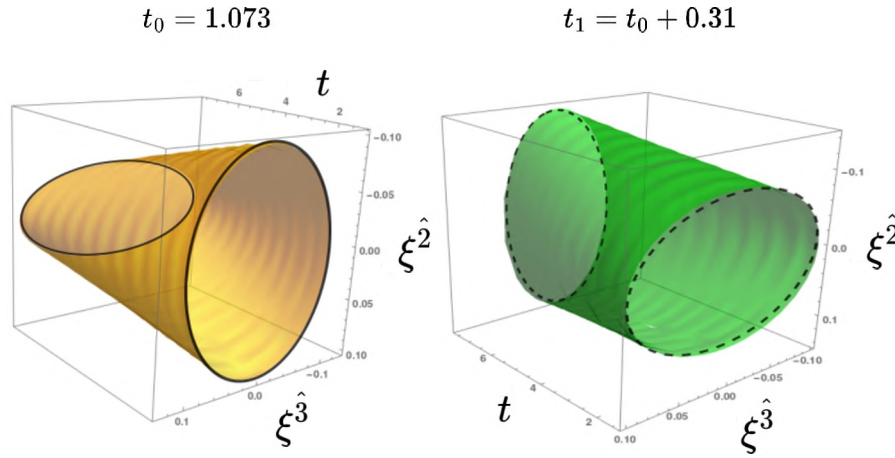


Fig. 2.6 On the left, you can see a Tissot plot that transforms from a ring shape to being stretched in the X -direction during a specific phase of the standing waves in the Z -direction. On the right, there is another Tissot plot that transforms from a ring shape to being stretched in the Y -direction during a slightly different phase of the standing waves in the Z -direction.

From Fig.(2.7) we get the polarization manifested in this spacetime, specifically identified as the ‘plus’ polarization. The case for cross-polarized was investigated here [120] where again this residual deformation was found.⁹ Here the standing GWs do not have compact support so it is not clear how to define the residual effect also termed in the literature as *velocity memory effect*.¹⁰ Our result can be interpreted as a velocity memory effect in the following sense: for the particles released at a zero-initial velocity, information about the phase of standing GW is encoded through the trajectories of test particles. As we see, for zero-velocity the particles released

⁹ $ds = e^f(-dt^2 + dz^2) + te^p(dx + \omega dy)^2 + te^{-p}dy^2$, where $t > 0$, $0 \leq x, y, z < 2\pi$ and $f = f(t, z)$, $p = p(t, z)$, $\omega = \omega(t, z)$. The additional polarisation is in the metric function $\omega = \omega(t, z)$.

¹⁰A permanent displacement/velocity left behind by the GW is termed as a *gravitational wave memory effect*. Discovered in linear theory by Braginski, Grishchuck [121], Thorne [122], Zeldovich [123] and in non-linear theory by Christodoulou, Blanchet and Damour [124], [125], it has been recently gained attention in the context of BMS symmetries [126] and astrophysics domain [127].

at two different configurations (here it means with a difference in initial time), the information about the phase is left behind as a ‘memory’. This can be seen clearly from the long-range trajectories for particles represented by solid and dashed rings.

Superimposed rings with slightly different initial time

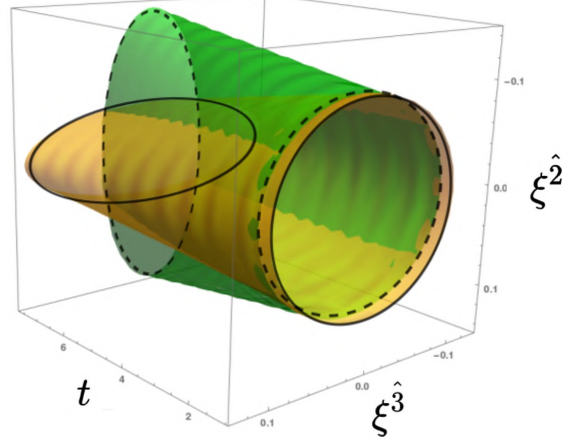


Fig. 2.7 In the Tissot diagram, two rings (solid and dashed) of initially stationary test particles relative to the central observer in a freely falling frame are depicted. The solid ring corresponds to a distinct phase of the standing gravitational wave compared to the dashed ring, resulting in different evolutions. The interaction with the first wave plays a crucial role, causing the test particles to follow specific profiles through a caustic point (line) that is only slightly influenced by subsequent waves. This long-range effect of the first wave on the bundle of test particle trajectories bears a resemblance to the velocity memory effect. The figure is for parameters: $\lambda = 1/10$, $\beta = 1/5$, the radius at $t_0 = 1.173$ of the solid ring is $r = 0.1$. The dashed ring of test particles is released at $t'_0 = t_0 + 0.31$. The rings are evolved for $\Delta\tau = 5$.

The redefinition of the time function's orientation does not affect our equations, except for a change in the sign of $dt/d\tau$. However, it alters the interpretation of the initial conditions, allowing us to examine the motion of test particles in an expanding universe with decreasing amplitude of standing waves or explore the behavior of test particles in a contracting spacetime with emerging standing waves. In the contraction model, we observed that the movement of test bodies exhibits qualitative similarities to the behavior of free bodies in the expanding model.

For the contracting model, the universe is heading towards $t = 0$ singularity. The amplitude of the wave is small at the beginning and increases towards the end. As compared to the expanding model, we get a larger effect of the corrugation of “tubes” at the end of evolution. We again observe the velocity memory since a shift in the

initial conditions alters the long-range behavior of trajectories and depicts the “plus polarization”.

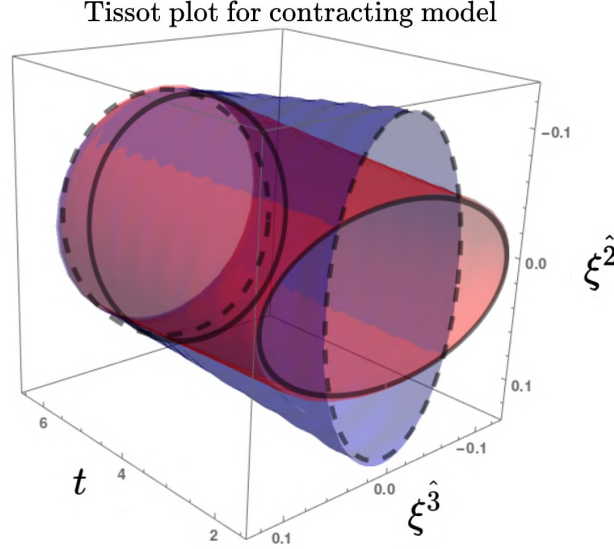


Fig. 2.8 The Tissot diagram for a contracting model (t is a decreasing function of the proper time τ). The parameters: $t_0 = 6.51$ (the solid ring), $t'_0 = t_0 - 0.31$ (the dashed ring). The remaining parameters are the same as for the expanding model.

Geodesic Deviation in Longitudinal direction

Now we utilize the $\hat{\xi}^3$ or the z component of Eq (2.16) to study the motion of particles along e_1 direction. Note this corresponds in the coordinate frame to the direction of ∂_z . We begin with test particles having a single nonzero component of the deviation vector $\hat{\xi}^1$. Likewise, in line with the previous investigations of the transverse effect, we analyze test particles that are initially at rest with respect to the stationary observer at an antinode with $k = 0$. For a choice of initial t_0 and the initial phase of the standing wave, the numerical analysis revealed an initial repulsion and a subsequent attraction to antinodes. More precisely, certain particles initially repelled from the antinodes eventually shift towards them, as the dominating attraction takes effect over time. This behavior aligns with our analysis in the next subsection where we quantify the behavior in terms of the Riemann tensor and the figures therein. After the initial interaction, the particles move almost freely, passing the anti-node, and fly away, due to the decreasing amplitude of the force (which goes as $1/t$) in the expanding model. We can interpret this behavior as a memory effect of the first interaction with the wave. This is shown in Fig.(2.9) where trajectories of several particles starting at different t_0 are shown.

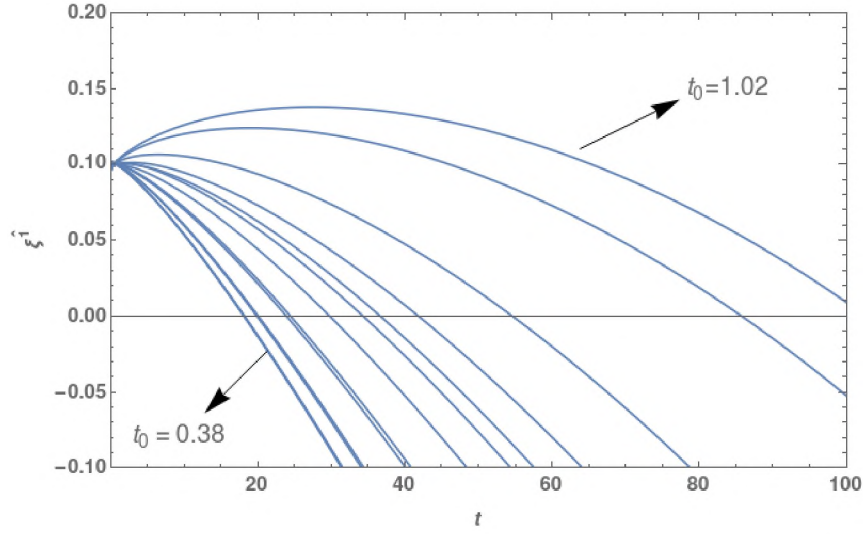


Fig. 2.9 The trajectories of test particles initially at rest near the antinode in the expanding model. Parameters: $\lambda = 1/10$, $\beta = 1/5$, t_0 varies uniformly from 0.38 to 1.02, $\xi^1(t_0) = 0.1$, $d\xi^1/d\tau(t_0) = 0$.

We can do a similar examination of freely falling bodies in the longitudinal direction for the collapsing model. A few typical trajectories of the initially stationary particle at different initial times t_0 . As the model collapses, the particle is attracted towards the antinode, but near the final crunch, a rapid movement is seen.

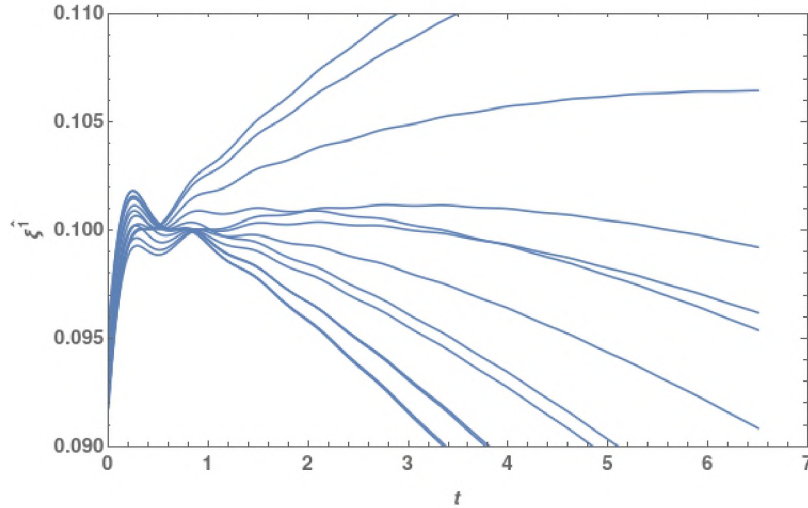


Fig. 2.10 In the contracting model, the trajectories of stationary test particles near antinodes released at different t_0 . Parameters: $\lambda = 1/10$, $\beta = 1/5$, t_0 varies uniformly from 0.38 to 1.02, $\xi^1(t_0) = 0.1$, $d\xi^1/d\tau(t_0) = 0$.

2.3.2 Geodesic equation

The “straightest possible paths” taken by a freely falling inertial observer are geodesics. For a given covariant derivative operator, ∇_μ , a geodesic is a curve whose tangent vector is parallel propagated along itself. It is a curve whose tangent, L^μ , satisfies $L^\mu \nabla_\mu L^\nu = 0$. An affinely parameterized geodesic can be mapped into a curve $x^\mu(t)$ in $\mathbb{R}^n$. Then one can arrive at the familiar *geodesic equation*:

$$\ddot{x}^\mu + \Gamma_{\sigma\rho}^\mu \dot{x}^\sigma \dot{x}^\rho = 0 \quad (2.17)$$

where ‘.’ represents derivative with respect to proper time τ . This represents a coupled system of n second-order differential equations. So given a metric $g_{\mu\nu}$ one can find out the Christoffel symbols Γ 's.

Alternatively, one can also consider the Lagrangian approach for geodesics. We mention in brief the approach taken from [128]. The main idea is to consider the metric as a Lagrangian $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, where $\dot{x}^\mu = dx^\mu/d\tau$ taking proper time τ as the affine parameter.

We know from classical mechanics that a system can be described in terms of a set of generalized coordinates x^μ which are functions of time t , which can define a space with a line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. This in classical mechanics is the *configuration space* of the system. One can arrive at the *Lagrangian* for the system having kinetic and potential energies, $\mathcal{L} = T - V = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - V(x)$, where $\dot{x}^\mu = dx^\mu/dt$. Now for the *action*¹¹ being stationary for small variations in the functions $x^\mu(t)$, one can arrive at the usual Euler-Lagrangian equations $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$. More importantly, the equations of motion have the form:

$$\ddot{x}^\mu + \Gamma_{\sigma\rho}^\mu \dot{x}^\sigma \dot{x}^\rho = -g^{\mu\sigma} \partial_\sigma V.$$

These equations represent an affinely parameterized geodesic with a force term on the right-hand side. The symbols $\Gamma_{\sigma\rho}^\mu$ denote the metric connections of the configuration space. When the forces are absent, Lagrange's equations dictate that “free” particles follow geodesics within the configuration space.

Now by analogy, for an arbitrary pseudo-Riemannian manifold, we can have equations for an affinely parameterized (null or non-null) geodesic $x^\mu(u)$ (for proper time taken as the affine parameter u will be τ) by considering the ‘Lagrangian’ $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, where $\dot{x}^\mu = dx^\mu/du$. So one can have the Euler-Lagrangian equations

¹¹Action: $S \int_{t_i}^{t_f} \mathcal{L} dt$ where t_f, t_i are final and initial time respectively.

as $\frac{d}{du} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$. The Euler-Lagrange equations offer a valuable method for deriving geodesic equations, allowing the extraction of connection coefficients from the geodesic equations themselves.

For e.g, in our case, we can take, for the affine parameter being proper time τ :

$$\mathcal{L} = e^f (-\dot{t}^2 + \dot{z}^2) + t(e^p \dot{x}^2 + e^{-p} \dot{y}^2) \quad (2.18)$$

where ‘dot’ is $d/d\tau$. Then the Euler-Lagrange equation gives :

$$\begin{aligned} \ddot{t} + \frac{1}{2} \left\{ f_{,t} \dot{t}^2 + 2f_{,z} \dot{t} \dot{z} + f_{,t} \dot{z}^2 + e^{-f} [e^p \dot{x}^2 (1 + p_{,t} t) + e^{-p} \dot{y}^2 (1 - p_{,t} t)] \right\} &= 0, \\ \ddot{x} + \dot{x} (\dot{t}/t + p_{,t} \dot{t} + p_{,z} \dot{z}) &= 0, \\ \ddot{y} + \dot{y} (\dot{t}/t - p_{,t} \dot{t} - p_{,z} \dot{z}) &= 0, \\ \ddot{z} + \frac{1}{2} \left[f_{,z} \dot{z}^2 + 2f_{,t} \dot{t} \dot{z} + f_{,z} \dot{t}^2 + t p_{,z} e^{-f} (-e^p \dot{x}^2 + e^{-p} \dot{y}^2) \right] &= 0, \end{aligned} \quad (2.19)$$

We can use the normalization of a four-velocity vector to give us the first integral of the form:

$$-\varepsilon = e^f (-\dot{t}^2 + \dot{z}^2) + t(e^p \dot{x}^2 + e^{-p} \dot{y}^2),$$

Here constant ε is equal to 1 or 0 for timelike or null geodesics, respectively. The Killing fields¹² ∂_x, ∂_y give two more quantities c_x, c_y that are conserved along geodesics

$$\begin{aligned} \dot{x} e^p t &= c_x, \\ \dot{y} e^{-p} t &= c_y. \end{aligned} \quad (2.20)$$

Now with the geodesic equations and the first integrals, we can study the trajectories of test particles along z direction as discussed in the following.

Behaviour in the longitudinal direction for different number of waves - expanding model

The number of waves in this T^3 universe is controlled by n . As the number of waves increases, so will the antinode formation. To see this we set up 31 parallel geodesics and analyzed their behavior. In all the following figures the dashed black lines are those particles that are kept exactly at the antinodes.

First, let's consider two waves. In Fig.(2.11), we see the formation of one antinode at $z = \pi/4$.

¹²Killing vector field that preserves the metric.

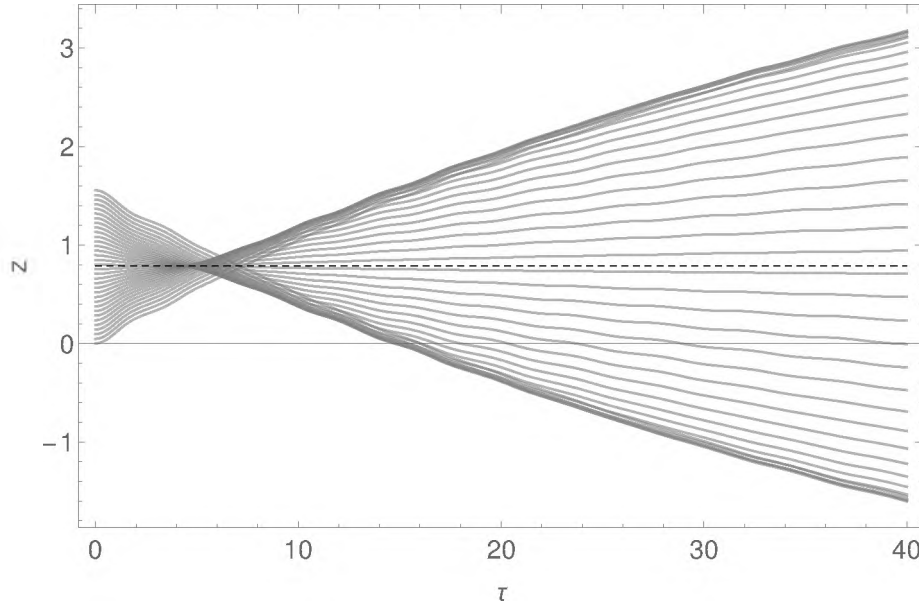


Fig. 2.11 Geodesic equation for z versus proper time τ . Parameters : $\lambda = 1/2$, $\beta = 1/5$. The final time $t(\tau = 40) \approx 32.4$. The dashed black line represents antinode at $z = \pi/4$

As we increase the number of waves, the number of antinodes increases. Here we see for $n = 6$, two antinodes at $z = \pi/4$ and $z = 5\pi/4$. We know that between two antinodes lies a node. So here it may seem that at $z = 3\pi/4$ (the second dashed black line) is a node. But this is not true since our analysis is only at antinodes hence we are not sure of it being a node. Also, all the antinodes are shown for *odd* values of k , but one changes the initial time t_0 and gets the same antinodes at the even position.

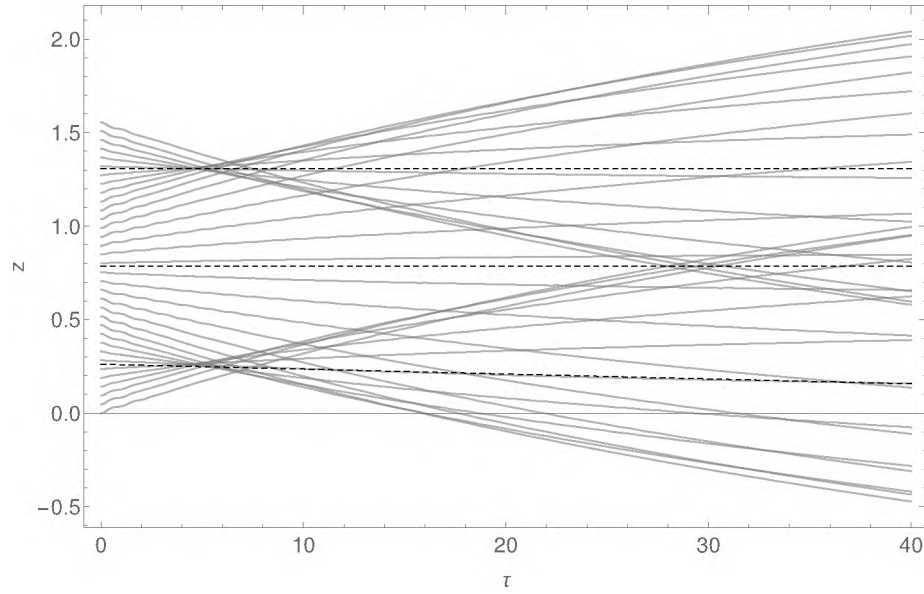


Fig. 2.12 Geodesic equation for z versus proper time τ . Parameters : $\lambda = 1/6$, $\beta = 1/5$. The initial time $t(\tau = 0) = 0.1$. The final time $t(\tau = 40) \approx 32.4$. The dashed black line represents antinode at $z = \pi/4, 5\pi/4$.

For $n = 50$, we see the formation of many antinodes(at $z = \pi/4, 3\pi/4, \dots, 31\pi/4$).

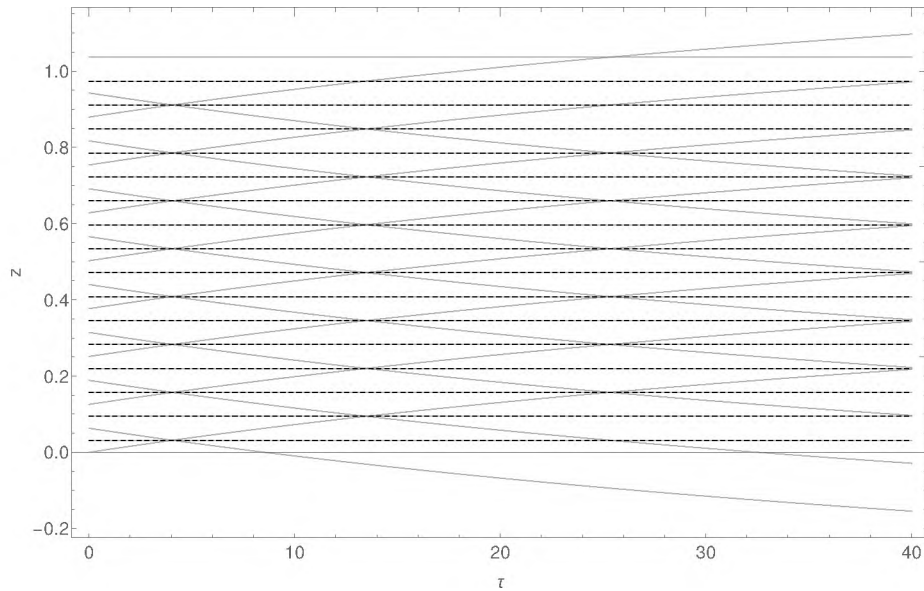


Fig. 2.13 Geodesic equation for z versus proper time τ . Parameters : $\lambda = 1/50$, $\beta = 1/5$. The initial time $t(\tau = 0) = 0.1$. The final time $t(\tau = 40) \approx 32.4$. The dashed black line represents antinode at $z = \pi/4, \dots, 31\pi/4$.

The effect in the longitudinal direction is evident from the results obtained via geodesic deviation and the subsequent explanation using the Newman-Penrose formalism. The main outcome of this study is illustrated in Fig.(2.13), which shows the attraction of geodesics depicted by the blue region in Fig.(2.14). This finding was used in our paper [13]. From the figure, it can be inferred that as geodesics seem to cross at antinodes, there might be some attraction at antinodes. But we know that some of the results for the geodesic equation are coordinate independent and moreover there lack of a definition for gravitational energy makes it challenging to explicitly state the reason for particle attraction in that region.

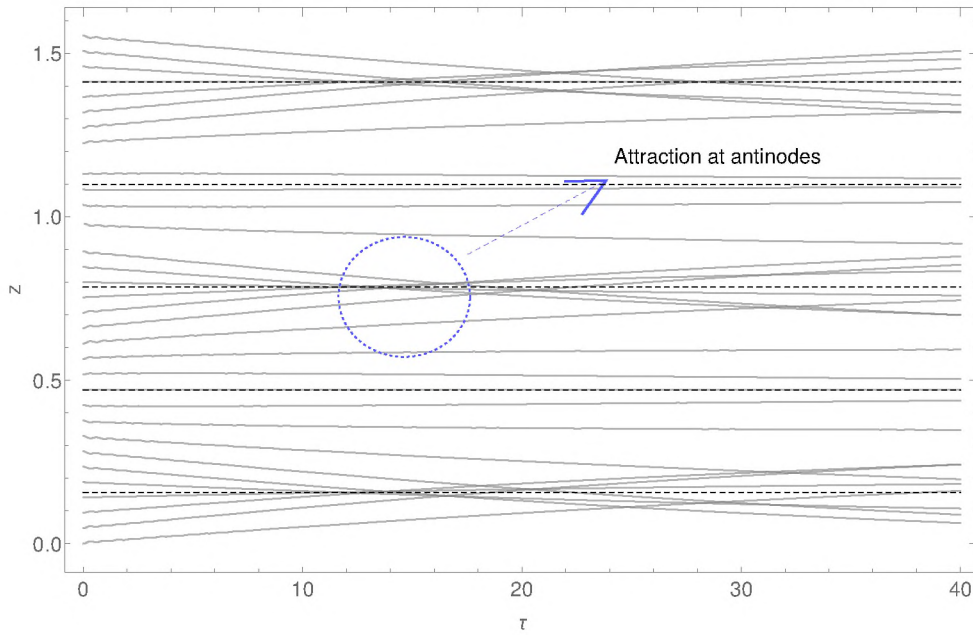


Fig. 2.14 Geodesic equation for z versus proper time τ . Parameters : $\lambda = 1/10$, $\beta = 1/5$. The initial time $t(\tau = 0) = 0.1$. The final time $t(\tau = 40) \approx 32.4$. The dashed black line represents antinodes. The figure used in the publication [13].

Just for completion, we show a plot for the geodesic equation for a smaller and larger β value:

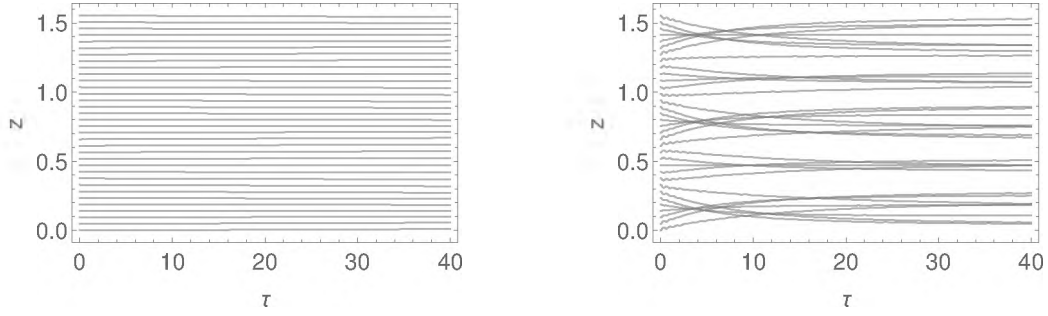


Fig. 2.15 Left plot for $\beta=0.01$ implying less amplitude so no antinodes are seen. Right plot for $\beta=0.6$, moderate amplitude and formation of antinode. For both case the final time $t(\tau = 40) \approx 32.4$ and $\lambda = 1/10$.

Behaviour in the longitudinal direction for different number of waves - contracting model

We perform the analysis for the contracting model, i.e. $t \rightarrow -t$. So now this T^3 universe is contracting. As the amplitude of waves decreases with time in a Gowdy model hence in the contracting model the amplitude will be small at the beginning. Hence the antinode formation is not precisely visible in this case Fig.(2.16).

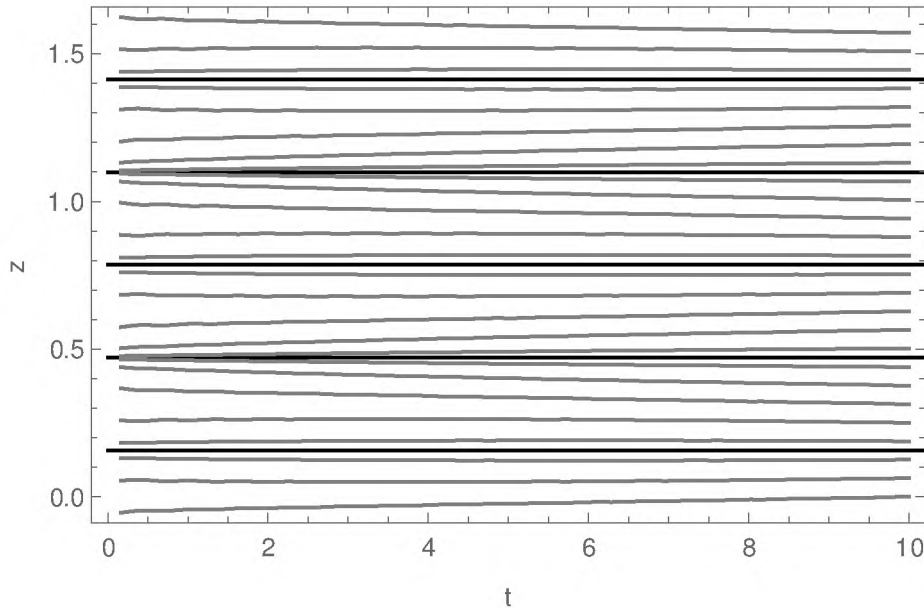


Fig. 2.16 Geodesic equation for z versus t in contracting model. Parameters : $\lambda = 1/10$, $\beta = 1/5$. The initial time $t(\tau) = 10$.

From Fig.(2.14) and (2.16), we can observe the presence of two distinct types of antinodes. These antinodes correspond to odd and even values of the parameter 'k' which defines the stationary geodesics γ_k according to Eq.(2.7). While there are differences between these two types, they are not fundamentally essential, and their roles can be interchanged by making an alternate choice of t_0 .

In the expanding model, the geodesics we have presented intersect the antinodes and traverse through the hypothetical observers that were the subject of our geodesic deviation studies. To gain further insights, it is instructive to compute the relative Lorentz factors and relative velocities in this context.

The four-velocity of the stationary observer at the antinode in our coordinate basis is given by

$$u_{obs} = e^{-f/2} \partial_t .$$

The Lorentz factor of test particles, as observed by stationary observers located at antinodes, is determined by

$$\gamma = u \cdot u_{obs} ,$$

where u is a four-velocity of test particles. The numerical analysis showed that for particles presented in Fig.(2.14) the velocities are nonrelativistic. They range from $0.0014c$ to $0.0066c$.

2.4 Dynamics of standing wave spacetime - Petrov types

The standing waves in our spacetime can be interpreted as a nontrivial superposition of two waves moving in opposite directions. To encapsulate this we first introduce the Newman-Penrose tetrad with the null vectors k_μ, l_μ that are aligned along the direction of propagation of the waves. We decompose the Weyl tensor into Newman-Penrose tetrad components and analyze the behavior of gravitational waves. We follow Stephani [4], and adapt the original notation of Newman and Penrose [90] to the signature +2. The complex null tetrad $w_{\hat{\mu}} = \{k, l, m, \bar{m}\}$ using the orthonormal tetrad $\{e_{\hat{\alpha}}\}$ is as follows:

$$\begin{aligned} k &= \frac{1}{\sqrt{2}}(e_{\hat{0}} + e_{\hat{1}}), & l &= \frac{1}{\sqrt{2}}(e_{\hat{0}} - e_{\hat{1}}), \\ m &= \frac{1}{\sqrt{2}}(e_{\hat{2}} - ie_{\hat{3}}), & \bar{m} &= \frac{1}{\sqrt{2}}(e_{\hat{2}} + ie_{\hat{3}}), \end{aligned} \tag{2.21}$$

where k, l are real vectors and $m, \bar{m}$ are two complex conjugate null vectors. The components of the metric associated with this null tetrad are

$$g_{\check{\alpha}\check{\beta}} = -2k_{(\check{\alpha}}l_{\check{\beta})} + 2m_{(\check{\alpha}}\bar{m}_{\check{\beta})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

where the inverted hat symbol in ' $\check{\alpha}$ ' is used to denote null tetrad indices. All vector products are zero, except for the following :

$$k \cdot l = -1, \quad m \cdot \bar{m} = 1.$$

With the help of the null tetrad, the 10 independent components of the Weyl tensor can be defined by the five complex coefficients $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$.

A Lorentz transformation can give rise to different null tetrads, hence the choice presented above is not unique. We want to adapt the Newman-Penrose null tetrad to our problem of standing waves. So we assume an additional condition. The concept of a standing gravitational wave can be understood as a non-trivial superposition of gravitational waves moving in the opposite spatial direction. As in the spacetime we study, the wave along $e_{\hat{1}}$ (which is one of the spatial vectors for our orthonormal freely falling frame) interacts with the wave from the opposite direction. Therefore, we assume $e_{\hat{1}} \cdot k = -e_{\hat{1}} \cdot l$. By this, we do not define the tetrad uniquely, but it is sufficient for our analysis.

So, the 10 independent components of the Weyl tensor are encoded into five complex Weyl-NP scalars whose physical interpretation was given by Szekeres [91] as :

$$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4 \longrightarrow \begin{pmatrix} \Psi_2 \text{ is a "Coulomb term"} \\ \Psi_1, \Psi_3 \text{ represent the ingoing and outgoing "longitudinal" radiation} \\ \Psi_0, \Psi_4 \text{ represent the ingoing and outgoing "transverse" radiation} \end{pmatrix}$$

As in vacuum spacetimes, $C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} = R_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}}$, for our study we have:

$$\begin{aligned}
\Psi_0 &:= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} m^{\hat{\lambda}} k^{\hat{\mu}} m^{\hat{\nu}} = C_{\hat{0}\hat{2}\hat{0}\hat{2}} \\
&= \frac{1}{4} (C_{\hat{0}\hat{2}\hat{0}\hat{2}} - C_{\hat{0}\hat{3}\hat{0}\hat{3}} + C_{\hat{1}\hat{2}\hat{1}\hat{2}} - C_{\hat{1}\hat{3}\hat{1}\hat{3}}) + \frac{1}{2} (C_{\hat{0}\hat{2}\hat{1}\hat{2}} - C_{\hat{0}\hat{3}\hat{1}\hat{3}}) \\
\Psi_1 &:= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} l^{\hat{\lambda}} k^{\hat{\mu}} m^{\hat{\nu}} = C_{\hat{0}\hat{1}\hat{0}\hat{2}} = 0, \\
\Psi_2 &:= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} k^{\hat{\kappa}} m^{\hat{\lambda}} \bar{m}^{\hat{\mu}} l^{\hat{\nu}} = C_{\hat{0}\hat{2}\hat{3}\hat{1}} = \frac{1}{4} (-C_{\hat{0}\hat{2}\hat{0}\hat{2}} - C_{\hat{0}\hat{3}\hat{0}\hat{3}} + C_{\hat{1}\hat{2}\hat{1}\hat{2}} + C_{\hat{1}\hat{3}\hat{1}\hat{3}}) \\
\Psi_3 &:= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} l^{\hat{\kappa}} k^{\hat{\lambda}} l^{\hat{\mu}} \bar{m}^{\hat{\nu}} = C_{\hat{1}\hat{0}\hat{1}\hat{3}} = 0, \\
\Psi_4 &:= C_{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} l^{\hat{\kappa}} \bar{m}^{\hat{\lambda}} l^{\hat{\mu}} \bar{m}^{\hat{\nu}} = C_{\hat{1}\hat{3}\hat{1}\hat{3}} \\
&= \frac{1}{4} (C_{\hat{0}\hat{2}\hat{0}\hat{2}} - C_{\hat{0}\hat{3}\hat{0}\hat{3}} + C_{\hat{1}\hat{2}\hat{1}\hat{2}} - C_{\hat{1}\hat{3}\hat{1}\hat{3}}) - \frac{1}{2} (C_{\hat{0}\hat{2}\hat{1}\hat{2}} - C_{\hat{0}\hat{3}\hat{1}\hat{3}}).
\end{aligned}$$

We can simplify the above expression by noting that $C_{\hat{0}\hat{3}\hat{1}\hat{3}} = -C_{\hat{0}\hat{2}\hat{1}\hat{2}}$, $C_{\hat{1}\hat{3}\hat{1}\hat{3}} = -C_{\hat{0}\hat{2}\hat{0}\hat{2}}$, $C_{\hat{1}\hat{2}\hat{1}\hat{2}} = -C_{\hat{0}\hat{3}\hat{0}\hat{3}}$. Using these together with the Weyl tensor being traceless, we get

$$\begin{aligned}
\Psi_0 &= \frac{1}{2} (C_{\hat{0}\hat{2}\hat{0}\hat{2}} - C_{\hat{0}\hat{3}\hat{0}\hat{3}}) + C_{\hat{0}\hat{2}\hat{1}\hat{2}} = \frac{1}{2} (R_{\hat{0}\hat{2}\hat{0}}^{\hat{2}} - R_{\hat{0}\hat{3}\hat{0}}^{\hat{3}}) + C_{\hat{0}\hat{2}\hat{1}\hat{2}}, \\
\Psi_4 &= \frac{1}{2} (C_{\hat{0}\hat{2}\hat{0}\hat{2}} - C_{\hat{0}\hat{3}\hat{0}\hat{3}}) - C_{\hat{0}\hat{2}\hat{1}\hat{2}} = \frac{1}{2} (R_{\hat{0}\hat{2}\hat{0}}^{\hat{2}} - R_{\hat{0}\hat{3}\hat{0}}^{\hat{3}}) - C_{\hat{0}\hat{2}\hat{1}\hat{2}}, \\
\Psi_2 &= -\frac{1}{2} (C_{\hat{0}\hat{2}\hat{0}\hat{2}} + C_{\hat{0}\hat{3}\hat{0}\hat{3}}) = \frac{1}{2} C_{\hat{0}\hat{1}\hat{0}\hat{1}} = \frac{1}{2} R_{\hat{0}\hat{1}\hat{0}}^{\hat{1}},
\end{aligned} \tag{2.22}$$

where components of the Riemann tensor are given by equations Eq.(2.15) and where $C_{\hat{0}\hat{2}\hat{1}\hat{2}} = O(1/\sqrt{t})$ in general, but $C_{\hat{0}\hat{2}\hat{1}\hat{2}} = 0$ at the antinodes. Therefore, at antinodes $\Psi_0 = \Psi_4$ is true.

Now we proceed with the physical interpretation. Though evaluated explicitly above, one can also see that the symmetries of the spacetime 2.3 imply $\psi_1 = \psi_3 = 0$. Following the physical interpretation mentioned above by Szekres [91], the longitudinal effect of waves in k^μ and l^μ directions is encoded in ψ_1 , ψ_3 . The transverse deformation is conveyed by ψ_0 , ψ_4 propagating in k^μ , l^μ directions. The Coulomb-like contribution is associated with ψ_2 . For our solution, we see that the longitudinal forces in the geodesic deviation equation (2.8) arise from the Coulomb part of the Weyl tensor and not from the longitudinal effect of the waves as $\psi_1 = \psi_3 = 0$. The transverse forces are mainly due to ψ_0 , ψ_4 but the Coulomb component ψ_2 also contributes to them.

We can utilize the fact that $\psi_1 = \psi_3 = 0$ to determine the Petrov type using the method outlined in the paper [129]. We keep l^μ fixed and carry out a null rotation

taking k^μ into a principal null vector. For the new tetrad $\psi'_0 = 0$. As the transformation rules of ψ_0 are known we get Eq.(1.8) defined in Chapter 1 [1.4]. The coincidences of the principal null vector defining the Petrov type are encoded in the multiplicity of the roots of Eq.(1.8). The problem is simplified considerably for $\psi_1 = \psi_3 = 0$: the only possibility is only four distinct or two double roots which correspond to Petrov types I and D, respectively. Let us denote Δ as $\Delta = 4(9\psi_2^2 - \psi_0\psi_4)$. By analyzing the data using a computer algebra system, we find that, in most cases, Δ does not equal zero. However, it is possible for Δ to vanish on certain hypersurfaces. Despite this possibility, the investigated spacetime is generally categorized as the Petrov type I and is not considered algebraically special.¹³

We can rewrite the geodesic deviation equation (2.8) in terms of Ψ 's. Using Eq.(2.22), we have we find $R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} = 2\Psi_2$, $R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}} = -\Psi_2 + \frac{1}{2}(\Psi_0 + \Psi_4)$, $R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}} = -\Psi_2 - \frac{1}{2}(\Psi_0 + \Psi_4)$. At antinodes, $\Psi_4 = \Psi_0$ and these equations reduce there to

$$\begin{aligned} R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} &= 2\Psi_2, \\ R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}} &= -\Psi_2 + \Psi_0, \\ R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}} &= -\Psi_2 - \Psi_0. \end{aligned} \quad (2.23)$$

The Coulomb contribution, ψ_2 , has the effect of compressing a sphere of test particles into an ellipsoid, deforming it along the e_1 direction. On the other hand, the transverse contribution, Ψ_0 , distorts a ring of test particles (in a plane stretched by vectors e_2, e_3) into an ellipse. We show the effect heuristically in the figure (2.17).

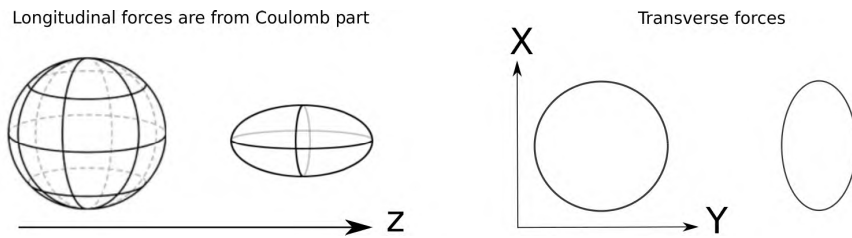


Fig. 2.17 Here we show what are the effects of Weyl components in a spacetime with the help of a cartoon figure. Left: A sphere of test masses will be elongated in the z -direction due to the Coulomb component Ψ_2 . Right: A ring of test masses will be permanently deformed in the *transverse* direction due to Ψ_0 and Ψ_4 but also has contributions from the Coulomb component.

¹³Apart from the trivial flat solution for $\beta = 0$ and the Kasner solution, which falls under the class described by the line element 2.3 but necessitates the extension of the metric functions 2.4, all other cases are considered in the analysis.

We write down the full formulas for Ψ 's and then study their asymptotic behavior. For late times $t \gg \lambda$, we obtain

$$\begin{aligned}\Psi_{0,4} &\simeq \mp \frac{ae^{-a^2t}}{4} \frac{1}{\sqrt{t}} \left\{ \mp \frac{2}{t} \cos\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) \sin\left(\frac{z}{\lambda}\right) \pm \frac{3a}{\sqrt{t}} \left[1 + \sin\left(2\frac{t \pm z}{\lambda}\right) \right] \right. \\ &\quad \left. + \frac{4}{\lambda} \cos\left(\frac{\pi}{4} + \frac{t \pm z}{\lambda}\right) + a^2 \left[\cos\left(\frac{\pi}{4} + 3\frac{t \pm z}{\lambda}\right) - 3 \sin\left(\frac{\pi}{4} + \frac{t \pm z}{\lambda}\right) \right] \right\}, \quad (2.24) \\ \Psi_2 &\simeq - \frac{ae^{-a^2t}}{4} \frac{1}{t} \left\{ \frac{2}{\sqrt{t}} \cos\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) \sin\left(\frac{z}{\lambda}\right) + a \left[\cos\left(\frac{2z}{\lambda}\right) + \sin\left(\frac{2t}{\lambda}\right) \right] \right\},\end{aligned}$$

where $a = \sqrt{\frac{2}{\pi}}\beta$ and the top and bottom signs correspond to Ψ_0 , Ψ_4 , respectively.

In general, Ψ 's are almost periodic functions with alternating signs. For late times $t \gg \lambda$ we have

$$\Psi_0 = O(1/\sqrt{t}), \quad \Psi_4 = O(1/\sqrt{t}), \quad \Psi_2 = O(1/t).$$

This asymptotic behavior corresponds to the decay condition of the Riemann tensor, which is anticipated in the cosmological analogy [68] of the cylindrical variant of the well-known peeling theorem [130]. Nevertheless, it is important to highlight that the standing waves in both the cylindrical Einstein-Rosen model and the Gowdy form do not adhere to the outgoing wave condition (nor any suitable cosmological counterpart of it). These standing waves encompass an equal proportion of incoming and outgoing radiation, implying that distinct peeling characteristics of the Weyl tensor are to be anticipated. Indeed, during late times, ψ_0 and ψ_4 overwhelmingly governed the Weyl tensor, resulting in a solution that resembles a superposition of two equivalent waves propagating in opposite directions. The nonlinear artifact arising from this superposition, ψ_2 , becomes negligible.

Using formulas (2.24), for late times $t \gg \lambda$ at antinodes $z = \lambda\pi(1/2 + k)$, where $k \in \mathbb{Z}$,

$$\begin{aligned}\Psi_0 = \Psi_4 &\simeq \sigma \frac{ae^{-a^2t}}{4} \frac{1}{\sqrt{t}} \left\{ \frac{2\sigma}{t} \cos\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) - \frac{3a}{\sqrt{t}} \left[1 - \sin\left(\frac{2t}{\lambda}\right) \right] \right. \\ &\quad \left. + \frac{4}{\lambda} \sin\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) + a^2 \left[3 \cos\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) - \sin\left(\frac{\pi}{4} + 3\frac{t}{\lambda}\right) \right] \right\}, \quad (2.25) \\ \Psi_2 &\simeq \frac{ae^{-a^2t}}{4} \frac{1}{t} \left\{ -\frac{2\sigma}{\sqrt{t}} \cos\left(\frac{\pi}{4} + \frac{t}{\lambda}\right) + a \left[1 - \sin\left(\frac{2t}{\lambda}\right) \right] \right\},\end{aligned}$$

where $a = \sqrt{\frac{2}{\pi}}\beta$ is a new auxiliary constant and $\sigma = 1$ if k is even, $\sigma = -1$ if k is odd. The components ψ_0 , ψ_4 , and ψ_2 exhibit oscillations around 0, yet in the case of ψ_2 , we observe

$$\Psi_2 \gtrsim -\frac{a}{2\sqrt{2}t^{3/2}}e^{-a^2t}$$

and (for $m \in \mathbb{Z}$)

$$\max_{\pi\lambda(m-1) < t < \pi\lambda m} \Psi_2 \gtrsim \frac{a^2}{2t}e^{-a^2t}.$$

The average value of ψ_2 is positive. When considering Eqs.(??) together, it implies that attraction towards antinodes prevails along $e_{\hat{1}}$. This finding holds true even during considerably early times, as we will explore in the following section. On average, freely falling particles are drawn towards antinodes, which stands as the primary outcome of our research. Additionally, it is worth mentioning that the repelling effect of ψ_2 at antinodes along $e_{\hat{2}}$ and $e_{\hat{3}}$ is overshadowed by the dominant contribution of $\psi_0 = \psi_4$.

The quantity of waves present on the three-torus is determined by $n = 1/\lambda$. Upon examining Eq.(2.25) and Eq.(2.24), we notice that during late times, the magnitude of the Coulomb component ψ_2 remains independent of the number of waves, whereas the magnitude of the transverse effect $\psi_{0,4}$ does vary with the number of waves.

For completeness, we provide asymptotics of Ψ 's at antinodes for the early times $t \ll \lambda$,

$$\begin{aligned} \Psi_0 = \Psi_4 &= 3\sigma \frac{e^{2\sigma\beta\sqrt{\lambda}}\beta}{4\lambda^{3/2}} + O(t^2), \\ \Psi_2 &= \sigma \frac{e^{2\sigma\beta\sqrt{\lambda}}\beta}{4\lambda^{3/2}} + O(t^2). \end{aligned}$$

In terms of the components of the Riemann tensor we obtain

$$2R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} + O(t^2) = 2R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}} + O(t^2) = -R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}} + O(t^2).$$

The behavior of the model is “asymptotically velocity term dominated” near the initial singularity.

The validity of our analysis regarding the asymptotic behavior of the components of the Riemann tensor $R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}}$, $R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}}$, $R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}}$ for late times and close to the initial singularity can be confirmed through plots obtained by evaluating the complete formulas for a specific set of parameters. Figures (2.18) and (2.19) provide confirmation of our overall findings.

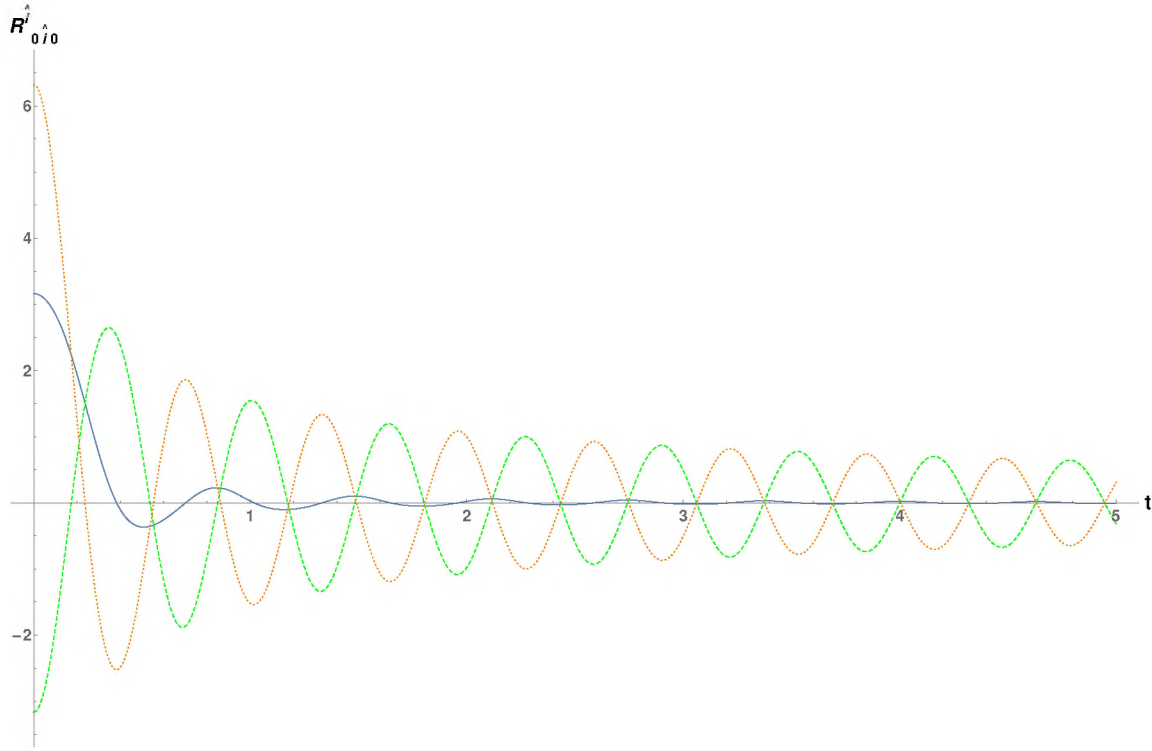


Fig. 2.18 The components of the Riemann tensor: $R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}}$ (blue line), $R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}}$ (orange line), $R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}}$ (green line) at the antinode which determines (via Eq.(2.13)) the force acting on test particles along directions ∂_z , ∂_x , ∂_y , respectively. The positive values imply attraction and negative repulsion. (The parameters: $\lambda = 1/10$, $\beta = 1/5$.)

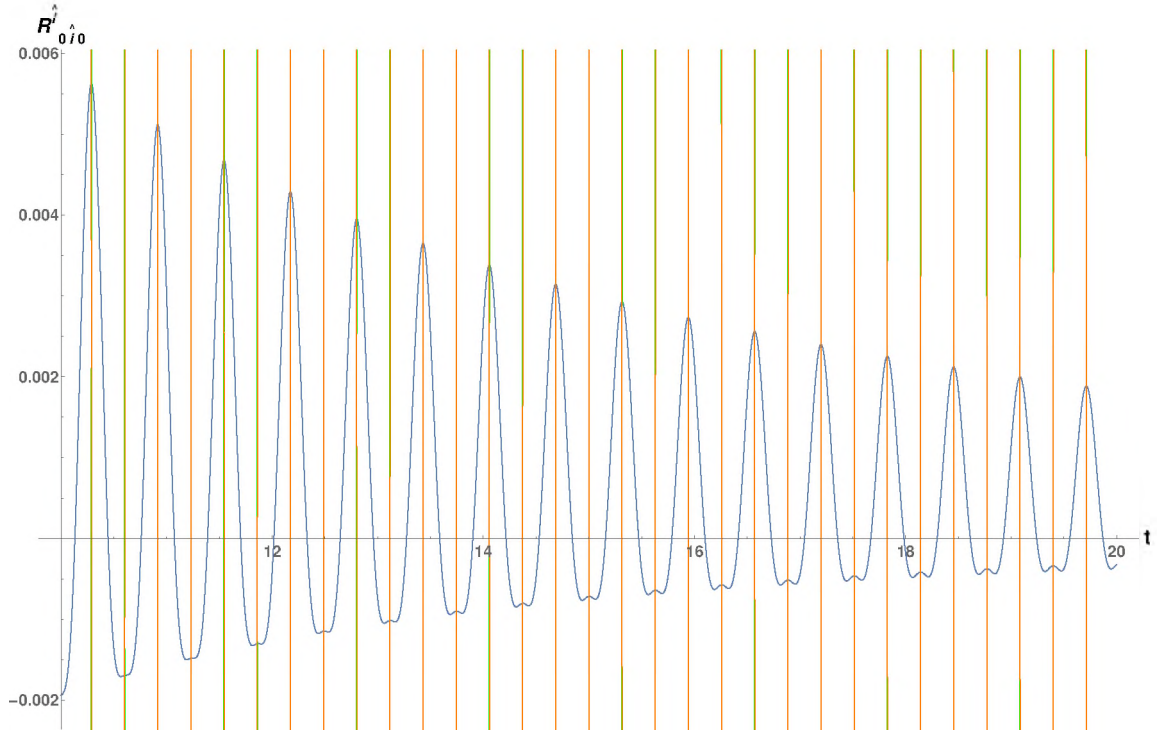


Fig. 2.19 The components of the Riemann tensor are shown for a different range while retaining all other details as depicted in Fig.(2.18). Following the initial phase, the transverse components of the force oscillate around zero without showing a distinct preference for a particular sign, as shown in Fig.(2.18). In contrast, the expansion of spacetime leads to a prevailing attraction towards the antinodes in the longitudinal direction.

2.5 Conclusion

In this part of the thesis, we conducted an examination of freely falling objects in a spacetime characterized by standing waves. We focused on a specific spacetime, namely, the polarized three-torus Gowdy model. Through our investigation, we demonstrated the existence of stationary timelike geodesics at antinodes. Furthermore, we thoroughly explored the geodesic deviation equation at these antinodes and provided evidence that test particles tend to be attracted, on average, towards antinodes along the direction of wave propagation.

The geodesic equation provided an intuitive confirmation that energy is concentrated in regions with significant geometry fluctuations. However, this result is incomplete since we lack a strict definition of gravitational energy. While metric

functions oscillate the most at the antinodes and particles are attracted to them, we cannot conclusively state that this attraction is due to gravitational energy.

To understand this phenomenon better, we analyzed the components of the Weyl tensor in the Newman-Penrose tetrad adapted to the symmetries of the standing-wave spacetime. The results obtained from this analysis confirmed our initial findings and shed light on the underlying mechanisms governing the behavior of particles in such spacetimes.

While our research is based on a specific exact solution, we believe that the properties of standing waves discovered here have broader applicability. The Szekeres theorem [91] supports this notion by indicating that there are no vacuum solutions to Einstein's equations with $\Psi_0 \neq 0$, $\Psi_4 \neq 0$, and $\Psi_1 = \Psi_2 = \Psi_3 = 0$. This implies that two transverse gravitational waves cannot be trivially superposed, leading to the presence of additional effects induced by standing gravitational waves. In our study, these effects were represented by the Coulomb component ψ_2 and the attraction to antinodes.

With this understanding, we are prompted to propose a hypothesis: The Weyl tensor of any standing-wave spacetime contains non-zero, periodic, or almost periodic components ψ_0 , ψ_2 , and ψ_4 in such a way that at antinodes, in an appropriate Newman-Penrose tetrad, $\psi_0 = \psi_4$. However, we acknowledge that this hypothesis requires further investigation and examination to validate its accuracy and general applicability.

Chapter 3

Standing gravitational waves in electrovacuum

“Knowing what is big and what is small is more important than being able to solve partial differential equations.”

Stanisław Ulam

This chapter is based on an unpublished work. I provide my independent calculations in this chapter. I have written my independent *Mathematica* code (which is also used to generate the images). I plan to submit this work following the ideas outlined in the conclusion.

3.1 Introduction

After working on a vacuum case for standing gravitational waves, we wanted to probe the non-vacuum case. So we undertook the study of an electrovacuum case.

An electrovacuum solution represents the exact solution of the Einstein field equation where only the electromagnetic (EM) field is present. So the field energy corresponding to the EM field is the only non-gravitational mass energy present. They satisfy the source-free Maxwell equations appropriate to the given symmetry. Electrovacuums are sometimes called source-free Einstein-Maxwell solutions.

The geometric setting is similar; you have a Lorentzian manifold, representing the curved spacetime denote by (M, g) . The only addition is the electromagnetic field

defined by the electromagnetic field tensor $F_{\mu\nu}$ on the Lorentzian manifold. For an electrovacuum solution, the following should be satisfied :

- i) The source-free curved spacetime Maxwell field equations $F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0$ and $F^{\mu\nu}_{;\nu} = 0$ are satisfied by the electromagnetic field tensor.
- ii) The Einstein tensor and electromagnetic tensor must be related as: $G_{\mu\nu} = 2(F^{\mu\sigma}F^{\nu}_{\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\eta}F_{\rho\eta})$

Our study involves spacetime which is the electromagnetic version of Gowdy. So the topology is as the previous work (T^3) but now we have an EM field contributing the stress-energy momentum tensor on the right-hand side of the Einstein field equations. This chapter is based on the unpublished work. As we did not get any novel results in comparison to the first work, we did not proceed with it. But we are studying other tools which can help elucidate the behavior of the coupling for electromagnetism and gravitational waves in this spacetime

In general relativity, Wheeler [131] had investigated a ‘gravitational electromagnetic entity’; where the EM/GW waves are confined in a region by virtue of its own field energy. The interaction of GWs and EM waves has been studied and also the conversion of EM→GWs or GWs→EMs. Also, the coupling between a GW and EM was being studied to investigate new methods to detect gravitational waves. For our first work, the spacetime represented standing gravitational waves in a vacuum. So in our next work, we probed a spacetime consisting of standing electromagnetic waves. Intuitively the questions we wanted to answer were: what would be the result of the coupling between electromagnetism and gravity? what would be the nature of standing gravitational waves produced due to the curvature oscillations due to standing electromagnetic waves? These would provide a glimpse into the non-linearities in gravity for this spacetime.

Previous studies of interaction of gravitational and electromagnetic fields:

At a linear level, the coupling between GWs and EM waves was studied as photon-to-graviton conversion by Gertsenshtein who showed that EMs under the presence of an external magnetic field in curved space could be converted into gravitational waves [132]. The standing electromagnetic waves in the context of general relativity were linearly addressed by Braginsky and Grishchuk [62]. They studied the gravitational waves generated at the symmetry center of a toroidal resonator. The symmetry center was taken as flat spacetime and effect of the standing EM waves was studied. As these standing EM waves will have a corresponding stress-energy momentum tensor; they were treated as perturbations to the flat spacetime considered at the symmetry

center. Interestingly the form of gravitational waves they found was of standing wave nature. The formula for the perturbation h in their paper represents cylindrical standing gravitational wave. The unperturbed solution of a strong electromagnetic field the gravitational and electromagnetic waves interacting with each other was studied by Alekseev and Sibgatullin [133]. They found the transmutation of EM and GWs waves.¹

In GR, the exact solutions of colliding plane waves consisting of GWs and EMs have been studied in different settings with a review provided in [26]. But most of the solutions are plane-fronted waves with parallel rays on a Minkowski background. Here there exists mutual focussing of colliding waves resulting in curvature singularities for the wave interaction region; thereby preventing a complete picture of the wave dynamics. We now know that a completed picture is provided by expanding spacetimes where one does not encounter such singularities.

Misra and Radhakrishnan [136] obtained the famous Einstein-Rosen solution but with electromagnetic energy distribution.² In the seminal paper of Thorne [105], where he defined the notion of energy for a system invariant under rotation and for translation along a symmetry axis (called *C-energy*), he studied this definition of energy in the case of vacuum and in presence of electromagnetic waves. Around the same time dynamics of cylindrical electromagnetic spacetimes were being studied by Melvin [138]. The subclass of that solution for gravitational and electromagnetic (also scalar) waves was taken up by Charach in [75] and in [69].

Our work focuses on this subclass. In this class of solutions of Einstein-Maxwell equations, there exists a dependence between the transverse part and the non-vanishing components of EM potential. One finds that the dynamical equations attributed to these fields reduce to one linear equation identical to the Einstein-Rosen vacuum case. The solution of Charach and Malin [75] was used for the construction of standing waves in cosmological models in [139] and [140]. We now describe how we rediscovered this solution.

We begin by working with a solution by Alekseev [141] which represents a plane-fronted non-solitonic electromagnetic wave.

$$\hat{g} = \frac{1}{\sqrt{t}} e^{4B(t-x)} (-dt^2 + dx^2) + t(dy^2 + dz^2), \quad (3.1)$$

¹The interaction between electromagnetic and gravitational waves in an external stationary magnetic field was also investigated by [134] who studied $EM \rightleftharpoons GWs$. Similar studies were already being done by Sibgatullin [135].

²Their solution was the plane-symmetric case of null electromagnetic fields obtained by Taub [137].

where $-\infty < x, y, z < +\infty$, $t > 0$ (t is a cosmic time function), $B(u) = \int \phi'(u)^2 du$, $u = t - x$ and $\phi(u)$ is an arbitrary real function. This is an exact solution to Einstein-Maxwell equations with electromagnetic four-potential A , such that $A^b = \phi(u)dz$. Note that the coordinates are ordered as $x^\alpha = (t, x, y, z)$ with the wave in x -direction.

This solution had an advantage over the electromagnetic Einstein-Rosen as it provides information about the interaction region. The electromagnetic Einstein-Rosen solutions have a linear dynamical equation, which means that when waves from this class collide, the resulting wave that emerges doesn't carry any information about the other wave it interacted with. The interaction in this class only affects the longitudinal part of the metric and their scattering. Moreover for the electromagnetic Einstein-Rosen solution, as the gravitational and electromagnetic parts of waves are in a fixed functional dependence of each other, the information about interaction is lost [141].

For metric (3.1), if $B(u) = \text{const}$ we have a Kasner universe with *Kasner exponents* $(-1/3, 2/3, 2/3)$.³ Through the function $\phi(u)$, we see that the EM wave is coupled to gravity, but it does not induce standard transverse deformations of space-time. So for now the EM wave is not accompanied by a GW wave in the traditional sense (transverse deformations).

Now we can allow the transverse components in order to have a superposition of two EM waves. So we break the homogeneity of the hypersurface formed by the Killing fields ∂_x , ∂_y and generalize the metric $\hat{g}$. We focus on Abelian G_2 solutions with hypersurface-orthogonal Killing vectors. Now the general metric which belongs to this class can be written as

$$g = \frac{1}{\sqrt{t/T}} e^{4[B(t,x)-A(t,x)]} (-dt^2 + dx^2) + \frac{t}{T} (e^{4A(t,x)} dy^2 + e^{-4A(t,x)} dz^2), \quad (3.2)$$

where $A(t, x)$ is a new auxiliary function and T is a constant. The electrovacuum case of the above metric was studied in a three-torus topology by Charach [75].

³The Kasner metric is an exact solution describing an anisotropic universe without matter. It has the form : $ds^2 = -dt^2 + \sum_{i=1}^{D-1} t^{p_i} (dx^i)^2$, having $D-1$ constants p_i known as *Kasner exponents*. This metric has spatially flat equal-time slices but the space is expanding/contracting at different rates and in different directions depending on the value of p_i .

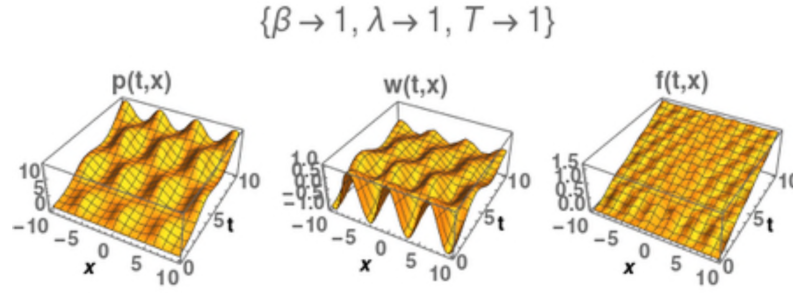


Fig. 3.1 This plot shows the metric functions defined in Eq.(3.1). We see $p(t,x)$, $w(t,x)$ and $f(t,x)$ for a specific value of β , λ and T .

Following the ideas presented by Charach, we consider a simple solution that belongs to the class studied there. Considering a simple profile of the EM wave, after the redefinition of coordinates we arrive at:

$$g = \frac{e^f}{\sqrt{t/T}} p^2 (-dt^2 + dx^2) + \frac{t}{T} (p^2 e^{-w} dy^2 + p^{-2} e^w dz^2), \quad (3.3)$$

where

$$\begin{aligned} p(t,x) &= 1 + e^w \frac{t}{T}, \\ w(t,x) &= \beta J_0(t/\lambda) \cos(x/\lambda), \\ f(t,x) &= \left(\frac{\beta}{2\lambda} \right)^2 t \left[-2\lambda J_0(t/\lambda) J_1(t/\lambda) \cos^2(x/\lambda) + t (J_0(t/\lambda)^2 + J_1(t/\lambda)^2) \right]. \end{aligned}$$

Here the parameters β , λ are arbitrary constants and J_k denote Bessel functions of the first kind. The metric Eq.(3.3) satisfies Einstein-Maxwell equations with the electromagnetic potential which in the Lorentz gauge $\nabla_\alpha A^\alpha = 0$ is given by

$$A^b = \frac{1}{2} \tanh[\ln(\sqrt{t/T}) + \frac{1}{2} \beta J_0(t/\lambda) \cos(x/\lambda)] dz. \quad (3.4)$$

This electromagnetic potential looks like this:

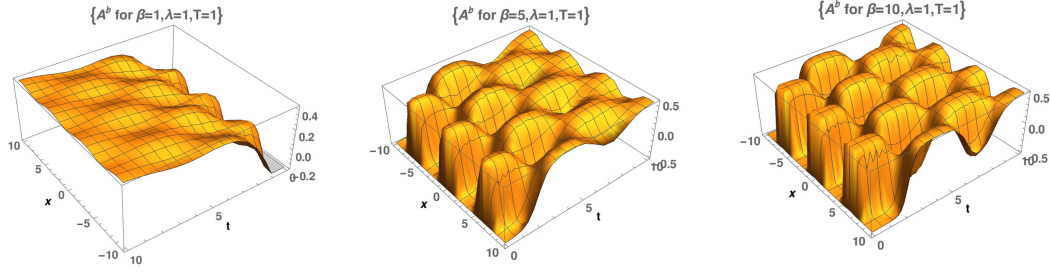


Fig. 3.2 The electromagnetic potential in Eq.(3.4) for increasing amplitude of wave β . The value of λ and T is constant.

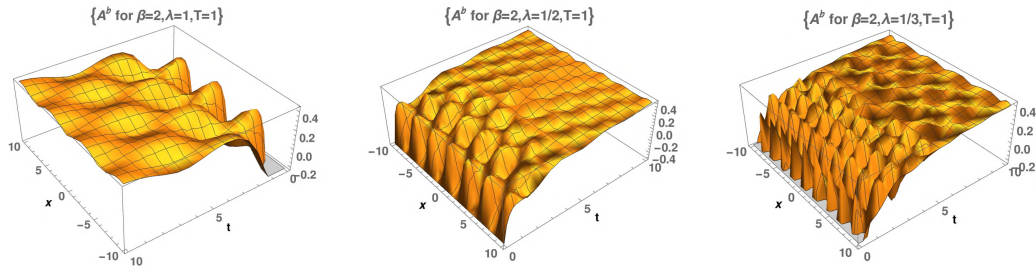


Fig. 3.3 As $\lambda = 1/n$, where n is the number of waves, so here we see the EM potential for increasing n and constant β and T .

We can use this metric and the EM potential in Einstein equations and get coupled differential equations involving them. The whole dynamics is majorly governed by the coupled nonlinear wave equations for functions $f(t, x)$ and $p(t, x)$ (corresponding to the transverse part of the gravitational waves and the electromagnetic vector potential) and $A^b(t, x)$. And since the field equations are non-linear, these two fields are the source for a longitudinal part of gravitation.

3.2 Analysing test masses behavior

3.2.1 Geodesic equation

One can obtain the geodesic equations as mentioned in previous Chapter 2 by following the Lagrangian approach or directly implementing the geodesic equation. Here we directly state⁴ the geodesic obtained for $\dot{t}, \dot{x}, \dot{y}, \dot{z}$.

⁴We make a distinction for the metric used in the previous chapter, there wave was in z -direction, hence we focused on the geodesic equation for z and t . In this spacetime, the wave is in x -direction.

The trajectory of test particles along ∂_x under the assumption $y = y_0$ and $z = z_0$ (where $y = y_0, z = z_0$ are constants) is of the form:

$$\begin{aligned}
& \ddot{t} + (i^2 + \dot{x}^2)\Delta + 2\dot{x}i\eta \\
& + e^{-f} \left(\frac{t}{T} \right)^{3/2} p^{-2} \left(e^{-w} \dot{y}^2 p^2 \left(\frac{1}{2t} + \frac{p_{,t}}{p} - \frac{w_{,t}}{2} \right) + e^w \dot{z}^2 p^{-2} \left(\frac{1}{2t} - \frac{p_{,t}}{p} + \frac{w_{,t}}{2} \right) \right) = 0 \\
& \ddot{x} + (i^2 + \dot{x}^2)\eta + 2\dot{x}i\Delta \\
& + e^{-f} \left(\frac{t}{T} \right)^{3/2} p^{-2} \left(-2 \frac{p_{,x}}{p} + w_{,x} \right) (e^{-w} \dot{y}^2 p^2 - e^w \dot{z}^2 p^{-2}) = 0 \\
& \ddot{y} + \dot{y} \left(\frac{\dot{t}}{t} + \left(\frac{2p_{,t}}{p} - w_{,t} \right) \right) \dot{t} + \left(\frac{2p_{,x}}{p} - w_{,x} \right) \dot{x} = 0 \\
& \ddot{z} + \dot{z} \left(\frac{\dot{t}}{t} + \left(w_{,t} - \frac{2p_{,t}}{p} \right) \right) \dot{t} + \left(\frac{w_{,x} - 2p_{,x}}{p} \right) \dot{x} = 0
\end{aligned} \tag{3.5}$$

where overdot represents differentiation⁵ with respect to proper time τ . And $\Delta = \frac{p_{,t}}{p} + \frac{f_{,t}}{2} + \frac{1}{4}$ and $\eta = \frac{f_{,x}}{2} + \frac{p_{,x}}{p}$.

The normalization of the four-velocity/wave vector gives rise to the first integral of the form

$$-\varepsilon = \frac{e^f p^2}{\sqrt{t/T}} (-i^2 + \dot{x}^2) + \frac{t}{T} (p^2 e^{-w} \dot{y}^2 + p^{-2} e^w \dot{z}^2),$$

where the constant ε is equal to 1 or 0 for timelike or null geodesics, respectively. This time the Killing fields are ∂_y, ∂_z and we obtain two more quantities c_y, c_z that are conserved along geodesics:

$$\begin{aligned}
\dot{y} \frac{t}{T} p^2 e^{-w} &= c_y, \\
\dot{z} \frac{t}{T} p^{-2} e^w &= c_z.
\end{aligned} \tag{3.6}$$

We investigate the geodesic equation under simplifying assumption $y = y_0, z = z_0$ where y_0, z_0 are constants and get the stationary observers at antinodes.

⁵For $a = a(t, x)$ then $\dot{a} = \frac{da(t, x)}{dt} = \frac{da}{dt} \dot{t} + \frac{da}{dx} \dot{x} = a_{,t} \dot{t} + a_{,x} \dot{x}$

Stationary observers at Antinodes

We investigate the geodesic equation under the assumption that $y = y_0, z = z_0$, where y_0, z_0 are constants. This reduces the geodesic equation Eq.(3.5) as:

$$\begin{aligned} \ddot{t} + (\dot{t}^2 + \dot{x}^2)\Delta + 2\dot{x}\dot{t}\eta &= 0 \\ \ddot{x} + (\dot{t}^2 + \dot{x}^2)\eta + 2\dot{x}\dot{t}\Delta &= 0 \end{aligned} \quad (3.7)$$

where $\Delta = \frac{p_t}{p} + \frac{f_t}{2} + \frac{1}{4}$ and $\eta = \frac{f_x}{2} + \frac{p_x}{p}$. Also from the first integral we have:

$$(\dot{t}^2 - \dot{x}^2) = e^{-f} p^{-2} \sqrt{t/T}.$$

To get stationary solution, i.e $\dot{x} = 0$ the above equations reduce to:

$$0 = \ddot{t} + \left(\frac{p_t}{p} + \frac{f_t}{2} + \frac{1}{4}\right)\dot{t}^2, \quad (3.8)$$

$$\begin{aligned} 0 &= \left(\frac{f_x}{2} + \frac{p_x}{p}\right), \\ \dot{t} &= e^{-\frac{1}{2}f} p^{-1} (t/T)^{1/4}, \end{aligned} \quad (3.9)$$

Here Eq.(3.9) is the integral of Eq.(3.8). We have the non-trivial condition $\frac{f_x}{2} + \frac{p_x}{p} = 0$ which is true when $x = \lambda \pi k$. We have $f_x = (\dots)(c \sin(\frac{x}{\lambda}))$ and $p_x = (\dots)(\sin(\frac{x}{\lambda}))$. This is zero for $x = \lambda \pi k$, where $k \in \mathbb{Z}$. In this case our future-directed timelike geodesic curve γ_k is

$$x^\mu = [t(\tau), \lambda \pi k, y_0, z_0] \quad (3.10)$$

which is a stationary solution to the geodesic equation for $k \in \mathbb{Z}$ and $t(\tau)$ given by Eq.(3.9). We note here that compared to the vacuum T^3 Gowdy case where antinodes condition was $f_z = 0$, here we have $\frac{f_x}{2} + \frac{p_x}{p} = 0$ and we have the solution vanishing at integer values $\lambda \pi, 2\lambda \pi, \dots$ instead of $\frac{1}{2}\lambda \pi, \frac{3}{2}\lambda \pi, \dots$. Here again the odd and even values of k will result in different metric functions, but without loss of generality, we can assume $k = 0$. Later for the odd and even values of k , we can set up various antinodes.

The geodesic equation is analyzed in a coordinate frame. This will not provide the full picture since the analysis is coordinate dependent. The stationary observers at antinodes move along timelike geodesics given by Eq.(3.10). Now we evaluate Eq.(2.5) numerically. For some initial moment t_0 , the geodesic equation is computed

for 31 test particles that are stationary in the coordinate frame.

$$\begin{aligned} x^\alpha(\tau=0) &= (t_0, x_m, y_0, z_0), \\ u &= e^{-f(t_0, x_i)/2} p^{-1}(t/T)^{1/4} \partial_t, \end{aligned} \quad (3.11)$$

The constants y_0, z_0 may be set to zero without loss of generality. The initial value of the zero component u^0 of the particle's four-velocity u is derived from the normalization condition $u \cdot u = -1$. We take the values x_m to be uniformly spaced over spatial periods of the metric functions as $x_m = \frac{2\pi}{\lambda} \frac{m}{10}$. The remaining parameters λ, β , and t_0 are provided in the figure (3.4). The choice of initial conditions Eq.(3.11) and the first integrals Eq.(3.8) implies that the test particles only move in the longitudinal direction ∂_z .

We show the plot of the geodesic equation. We see slight attraction of geodesics and for other parameters we observe the attraction to antinodes. We do not show them here as they are not qualitatively different from our previous result. We thought of considering charge particles instead of neutral test particles, but then the EM forces will overcome the gravitational attraction. So this is still investigation.

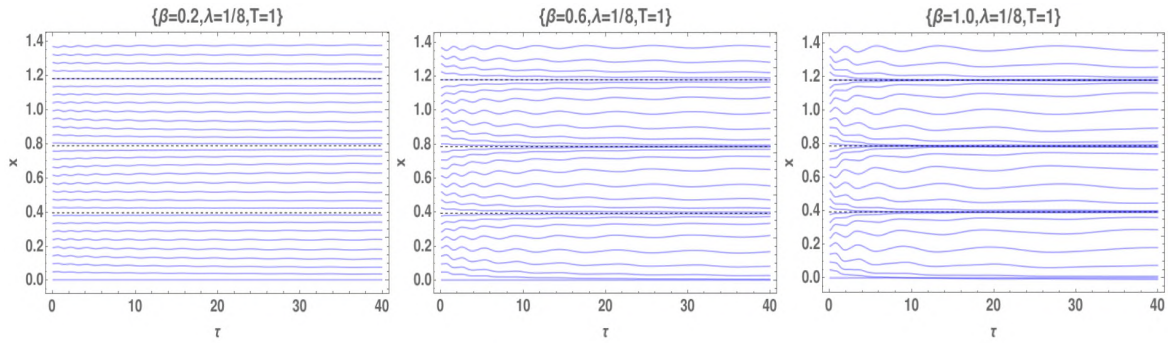


Fig. 3.4 This is the plot for the geodesic equation in x -direction for different values of β . $\lambda = 1/8$ and $T = 1$. We did not find anything interesting like in our previous chapter.

3.2.2 Geodesic deviation equation

In this section we just display the geodesic deviation equation in terms of the Weyl scalars as the structure is similar to our first work.

Let ξ be a deviation vector, then along γ_k in a freely falling frame

$$\frac{d^2 \xi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}_{\hat{0}\hat{\beta}\hat{0}} \xi^{\hat{\beta}}, \quad (3.12)$$

where τ is a proper time of the observer. In vacuum, the Riemann and Weyl tensors are equal so $R^{\hat{\alpha}}_{\hat{\alpha}\hat{\beta}\hat{\alpha}} = C^{\hat{\alpha}}_{\hat{\alpha}\hat{\beta}\hat{\alpha}}$. It is instructive to rewrite the geodesic deviation equation in terms of the complex Weyl coefficients. The relevant non-zero components of the Riemann tensor are

$$\begin{aligned} R^{\hat{1}}_{\hat{0}\hat{1}\hat{0}} &= 2\text{Re}(\Psi_2) , \\ R^{\hat{2}}_{\hat{0}\hat{2}\hat{0}} &= \frac{1}{2}\Re[-2\Psi_2 + \Psi_0 + \Psi_4] , \\ R^{\hat{3}}_{\hat{0}\hat{3}\hat{0}} &= \frac{1}{2}\Re[-2\Psi_2 - \Psi_0 - \Psi_4] , \\ R^{\hat{2}}_{\hat{0}\hat{3}\hat{0}} &= \frac{1}{2}\Im[\Psi_0 - \Psi_4] , \end{aligned} \tag{3.13}$$

where $\Re, \Im$ denote real and imaginary parts. For the solution studied, we have $\Psi_1 = \Psi_3 = 0$. In addition at antinodes: $\Psi_4 = \Psi_0$, hence $R^{\hat{2}}_{\hat{0}\hat{3}\hat{0}} = R^{\hat{3}}_{\hat{0}\hat{2}\hat{0}} = 0$. Ψ_0 and Ψ_2 are real (not only at antinodes). The geodesic deviation equation can be written at antinodes as

$$\begin{aligned} \frac{d^2\xi^{\hat{0}}}{d\tau^2} &= 0 , \\ \frac{d^2\xi^{\hat{1}}}{d\tau^2} &= -2\Psi_2\xi^{\hat{1}} , \\ \frac{d^2\xi^{\hat{2}}}{d\tau^2} &= (\Psi_2 - \Psi_0)\xi^{\hat{2}} , \\ \frac{d^2\xi^{\hat{3}}}{d\tau^2} &= (\Psi_2 + \Psi_0)\xi^{\hat{3}} . \end{aligned} \tag{3.14}$$

Here you can see the deviation on the X, Y, Z directions on the left side depending on the Weyl scalars. You have the longitudinal effect in the X direction and the transverse effect in YZ direction. These equations have a similar form as in our previous work in Chapter 2, therefore the Tissot plots we obtained did not reveal qualitatively different characteristics. Hence we do not show them.

3.3 Conclusion

In this part of thesis, our aim was to extend our previous work on vacuum T^3 Gowdy to electrovacuum T^3 Gowdy. We worked out the geodesic equation and geodesic deviation equation. We get new unique results, but they were qualitatively the same as old results.

As the form of the equations is same in the previous chapter, the analysis of geodesic and geodesic deviation did not reveal any new results. So we decided not to publish this work and in the meantime look at other tools to study this electrovacuum spacetime. One of the ideas is to look at the Bel-Robinson tensor to get an understanding of the energy at antinodes.⁶ Stephani [4] proposed the use of a *gravitational Poynting vector* analog for understanding standing gravitational waves, emphasizing the challenges of defining gravitational energy density and flow due to the nonlocal nature of gravity in general relativity. Despite this limitation, one can employing the concept of a super-Poynting vector and superenergy as mathematical tools in analyzing standing gravitational waves, constructing the analog using the Bel-Robinson tensor [143] (one can also possibly look into other canonical superenergy tensors in general relativity as defined by Garecki [144]). This was done for the case of cross-polarized standing gravitational waves, also known as the cylindrical Halilsoy solution [120]. One may also look into super-energy tensor of the electromagnetic field defined Chevreton [145]. While the results obtained may not be fully covariant due to the absence of a precise definition of gravitational energy, they can still contribute to the extension of our research endeavors.

⁶In trying to answer the problem posed by Synge [142] on ‘how to define a gravitational ‘energy density’ with the properties of being positive definite and zero only if the curvature vanishes?’, Louis Bel formulated the Bel-Robinson tensor.

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\mu\gamma}{}^{\nu} C_{\delta\nu\beta}{}^{\mu} + \star C_{\alpha\mu\gamma}{}^{\nu} \star C_{\delta\nu\beta}{}^{\mu}$$

where C is Weyl tensor and $\star$ is Hodge dual ($\star C_{\alpha\beta\gamma\delta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} C^{\mu\nu}{}_{\gamma\delta}$ with η being the canonical volume form). In GR, as defining local energy for the gravitational field is not unique, the Bel-Robinson tensor is a potential definition, since it remains divergence-free when the Ricci tensor vanishes (in vacuum).

Chapter 4

Cylindrical standing gravitational waves and chaotic geodesics

“ A scientist in his laboratory is not a mere technician: he is also a child confronting natural phenomena that impress him as though they were fairy tales.”

Marie Curie

The main results of this chapter were included in on our second paper ‘*Chaos and Einstein-Rosen gravitational waves*’ [2]. Each co-author independently carried out the calculations, and distinct numerical codes were created for mutual verification of the results. The results presented in this chapter are derived from my individual *Mathematica* code, and the figures displayed are also produced using my version of the code. Furthermore, I employed the PLgrid supercomputer (Grant number: PLG/2022/015759) to execute jobs. The fractal structure was generated using the co-authors’ *C++* code. I generated it using my independent *Python* code. In this chapter, I delve deeper into the subject matter, offering additional insights and discussions that complement the content presented in our paper.

4.1 Introduction

In chapter 1, we have described the studies involving chaos in general relativity. In this chapter, we briefly mention a few issues with the tools to measure chaos. Then

we proceed to explore cylindrical standing wave spacetimes, explaining our results which constituted the paper [2] and the future studies to be performed.

We know that general relativity is a covariant theory wherein the dynamics supersede the notion of rigid Newtonian concepts of absolute space and universal time. As the concept of time and space become relative, the standard measure of chaos becomes observer-dependent as they rely on the metric properties of phase space. Hence there is a need for observer-independent notions to characterize chaos. Einstein's equations, which are a set of coupled non-linear differential equations have been known to demonstrate chaotic behavior. To quantify this chaotic behavior, we need invariant quantities of non-linear dynamics.

One approach commonly used to study chaos in general relativity is through Lyapunov exponents, which quantify the separation rate of closely spaced trajectories. However, the coordinate-dependence of Lyapunov exponents in general relativity poses challenges, as the concept of time and space can vary between different observers and coordinate systems. Efforts have been made to formulate Lyapunov exponents invariantly [146],[147]. Additionally, invariant measures such as fractal dimensions, symbolic dynamics, and manifold crossings provide alternative means to investigate chaos in general relativity. In our investigation, we will focus on these measures to study chaotic behavior in spacetime, specifically by analyzing the strange repeller or strange saddle sets, which exhibit non-integer multi-fractal dimensions and feature orbits that are unstable in certain directions and stable in others.

At the beginning chaotic behavior was studied in general relativity encompassing two primary lines of research.

One in regards to inhomogeneous cosmological models, as in the Bianchi IX model [148]. The issue was with the invariance of the relativistic theory with respect to the concept of chaos in this approach [149]. In the context of the Bianchi IX model, a gauge-invariant approach grounded in Maupertuis' principle has been proposed in [150] and subsequently adapted with minor modifications [151]. The method above reveals the presence of localized instability in the cosmological model, particularly in proximity to its singularity. This further contributes to an abundance of early analysis and numerical investigations (as referenced in [4]), which strengthens the growing consensus that Bianchi IX exhibits chaotic behavior in a significant manner. Also, invariant Lyapunov exponents and chaos in cosmology were studied in [152].

The alternative approach involves considering a predetermined geometry and exploring the chaotic behavior of geodesic motion within that framework. In this context, geometry can be either an approximate solution or an exact solution of

Einstein's equations. Instances of chaotic geodesic motion have been studied in the context of exact geometry in [153], [154], [155]¹ and approximate geometry [42], [157].

It was argued in [158], that presence of a positive Lyapunov exponent does not symbolize chaos. The issue is with the nature of the time parameter used to define the exponent. In [158] a formulation of chaos, following [159] was given to characterize chaos in terms of "chaotic invariant subset" for the case of null geodesic flow in the two-black-hole spacetime.

The problem of characterizing chaos in general relativity has been successfully addressed in a series of papers [35], [52], which introduced and demonstrated the effectiveness of fractal dimensions and symbolic codings as invariant descriptors of chaos. A key concept in both methods is the existence of a chaotic invariant set of orbits [160]. A similar approach was employed in solving the problem of describing chaos in quantum mechanics by focusing on periodic orbits [161].

Our focus is on cylindrical standing wave spacetime. To establish chaos in this spacetime the following route was taken: Sensitive dependence of geodesics to initial conditions → Fractal structure → Poincaré sections → homoclinic and heteroclinic tangle.

4.2 Cylindrical symmetric spacetimes

The spacetimes which admit two commuting spacelike Killing vectors and one of the Killing vectors has closed orbits, meaning the trajectories it defines loop back on themselves, forming closed curves, are cylindrically symmetric spacetimes. On the other hand, if the orbits of the second Killing vector are open, then the trajectories extend indefinitely without looping back. This is in contrast to spacetimes representing plane-fronted gravitational waves, where the orbits of both Killing vectors are open, indicating trajectories that extend infinitely without closing back on themselves. The cylindrical symmetric spacetimes are the simplest metrics that have known exact gravitational wave solutions ([4, 105, 162]). While the lack of asymptotic flatness limits the utility of these spacetimes for studying realistic objects, there is hope that investigating the properties of exact gravitational waves can contribute to our understanding of various aspects of gravitational radiation. Although cylindrically

¹Here Majumdar–Papapetrou geometries were being studied, which extend the axially symmetric solutions of Einstein-Maxwell equations originally discovered by Hermann Weyl to a fully nonsymmetric and general scenario. Hartle and Hawking [156] demonstrated the possibility of "gluing" together these solutions to construct multi-black hole configurations involving charged black holes.

symmetric waves do not provide an exact description of radiation from bounded sources, they have proven valuable in addressing various complex matters. These include testing quasilocal mass-energy [163], verifying numerical relativity codes [164], studying cosmic censorship [165], and exploring quantum gravity [166]. For more information and references, refer to [167].

Following [168], we first look at the static cylindrically symmetric Levi-Civita family [169]:

$$ds^2 = b^2 \rho^{(d^2-1)/2} (d\rho^2 - dt^2) + \rho^{1+d} d\varphi^2 + \rho^{1-d} dz^2 \quad (4.1)$$

where b and d are arbitrary parameters², ρ is radial coordinates and φ is the polar angle, $0 < \varphi < 2\pi$. It includes flat spacetime $d = 1$. In the context of cylindrical symmetry, this solution serves a similar role to the Schwarzschild solution in the case of spherical symmetry. However, it exhibits an infinite naked singularity at the symmetry axis $\rho = 0$, (except for $d^2 = 1$) and there is no event horizon to conceal it.

A proposed physical interpretation for the metric is the following. In the Newtonian limit, Levi-Civita [169] found $d \approx (1 + 2M)/(1 - 2M)$, where M is the relativistic mass per unit length $GM/c^2 R$ for a cylinder. Therefore, for $d > 1$, the metric 4.1 can be interpreted as representing the external field of an infinitely long cylinder with a specific mass per unit length. For $d < 1$, the test particles feel a repulsive force far from the cylinder and then it has to be treated as a cylinder with negative mass. In the case of $d \neq 1$, the metric exhibits a curvature singularity that is not hidden by an event horizon. To address this, one can refer to [170] in which a stress tensor on the axis is induced compatible with the above interpretation. For $M = 0$ ($d = 1$), the space is flat and for $2M \rightarrow 1$ d grows without bound. The flat spacetime is only recovered when $d = 1$ and $b^2 = 1$. For arbitrary b one has flat space minus a wedge (if $b^2 > 1$) which is a deficit angle and may be interpreted as a gravitational wave created by a straight gauge cosmic string [171].³

Unlike the Schwarzschild solution, the Levi-Civita solution is not the most general exterior solution for a cylinder. The Kompaneets metric, developed by Kompaneets [173] and later discussed by Jordan et al. [174], is the most general cylindrically symmetric line element.⁴ It can also be interpreted as gravitational waves in a vacuum

²This is Kasner metric adapted to cylindrical symmetry.

³Recently GW signals from cosmic strings have been looked at the LIGO/Virgo data [172].

⁴In [74] it was pointed out that this metric does not include the rotating cylindrical GW spacetimes studied by Mashhoon and Quevedo, $ds^2 = e^{2(\gamma-\psi)}(dt^2 - dr^2) - W^2 e^{-2\psi}(d\varphi + \omega dt)^2 - e^{2\psi} dz^2$, where γ, ψ, W, ω are function of t, ρ ([175]). For some issues with defining a general cylindrical spacetimes and their rigorous definition we refer to the same reference.

with cylindrical symmetry. It's metric is written as

$$ds^2 = e^{2(\gamma-\psi)}(d\rho^2 - dt^2) + \rho^2 e^{-2\psi} d\varphi^2 + e^{2\psi} (dz + \omega d\varphi)^2 \quad (4.2)$$

where the functions γ , ψ and ω depend on t and ρ . Also, the two polarization of gravitational waves are described by the time-dependent functions $\psi(t, \rho)$ and $\omega(t, \rho)$ which describe the transverse modes. When $\omega(t, \rho) = 0$, the metric reduces to Einstein-Rosen [66] diagonal form.⁵ This is the form of metric which has been considered in our third work

The cylindrically symmetric metric with two hypersurface orthogonal Killing fields ∂_z , ∂_φ may be written in the form [72]⁶

$$g = e^{2(\gamma-\psi)} (-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\varphi^2 + e^{2\psi} dz^2, \quad (4.3)$$

where $\rho > 0$, $-\infty < t, z < \infty$, $0 \leq \varphi < 2\pi$ and the metric functions ψ and γ depend on t and ρ only. Hereafter, we consider the standing cylindrical wave (*linearly polarised*) of the form

$$\begin{aligned} \gamma &= \frac{\rho^2}{8\lambda^2} \left[J_0^2\left(\frac{\rho}{\lambda}\right) + J_1^2\left(\frac{\rho}{\lambda}\right) - 2\frac{\lambda}{\rho} J_0\left(\frac{\rho}{\lambda}\right) J_1\left(\frac{\rho}{\lambda}\right) \cos^2\left(\frac{t}{\lambda}\right) \right], \\ \psi &= \frac{1}{2} J_0\left(\frac{\rho}{\lambda}\right) \cos\left(\frac{t}{\lambda}\right), \end{aligned} \quad (4.4)$$

where λ is a constant and J_i are the Bessel functions of the first kind and i -th order. The metric (4.3) with the auxiliary functions defined above constitutes the regular exact solution to the vacuum Einstein equations.

4.3 Investigating the chaos

We first explore the sensitive dependence of geodesics on initial conditions. Then we move to figure the fractal structure present in this spacetime. Further, with the help of Poincaré maps, we probe the homoclinic and heteroclinic intersections. Finally, we witness the stable and unstable manifold crossings in terms of the tangle of Chaos which Poincaré has mentioned while tackling with the three-body problem.

⁵We just like to mention a notable characteristic of cylindrical spacetime is the possibility of defining a notion of gravitational energy within it. Thorne in [105] had introduced the concept of C-energy. Additionally, Thorne formulated a covariant C-energy flux vector denoted as P^μ , which follows the conservation law $P^\mu_{;\mu} = 0$.

⁶We are considering the case or $\alpha \rightarrow 0$ for the cross-polarised Halilsoy solution [72].

In between we mention the well known example of chaos, namely the double pendulum. These are only mentioned to be rudimentary aids to grasp our results.

4.3.1 Chaotic orbits

Let $u = u^\alpha \partial_\alpha = \dot{t} \partial_t + \dot{\rho} \partial_\rho + \dot{\phi} \partial_\phi + \dot{z} \partial_z$ denote the four-velocity of a massive test particle parameterized by the proper time τ (the dot denotes differentiation in τ). We have $g(u, u) = -1$. The spacetime symmetries lead to the implication that

$$g(\partial_z, u) = u_z, \quad g(\partial_\phi, u) = u_\phi = \rho^2 e^{-2\psi} \dot{\phi} = l$$

are conserved along the geodesics. The constant l assumes the role of angular momentum per unit mass, and without loss of generality, we may assume that l is greater than zero. To simplify matters, we focus on timelike geodesics confined to the plane $z = \text{const}$, which consequently results in $\dot{z} = u^z = u_z = 0$.

We know that the geodesic equation is a second-order differential system ($\ddot{x}^\mu + \Gamma^\mu_{\sigma\rho} \dot{x}^\sigma \dot{x}^\rho = 0$), so there will be terms of $\ddot{\rho}$, $\ddot{t}$. We will now illustrate a second-order differential equation and subsequently present the simplified system of first-order differential equations. Here is the equation for $\ddot{\rho}$:

$$\ddot{\rho} = -(\dot{t}^2 + \dot{\rho}^2) (\gamma_{,\rho} - \psi_{,\rho}) - 2\dot{t}\dot{\rho} (\gamma_{,t} - \psi_{,t}) - e^{-2(\gamma-\psi)+2\psi} (\psi_{,\rho} - \frac{1}{\rho}) \frac{l^2}{\rho^2}$$

Here ' $_{,\rho}$ ' and ' $_{,t}$ ' are derivative with respect to ρ and t respectively. We can consider $\dot{\rho} = P$. Similarly we will have a second-order differential system in t where we consider $\dot{t} = T$. With this our second-order differential system for ρ as a system of two first order differential system is $\dot{\rho} = P$ and $\dot{P} = -(T^2 + P^2)(\gamma_{,\rho} - \psi_{,\rho}) - 2TP(\gamma_{,t} - \psi_{,t}) - e^{-2(\gamma-\psi)+2\psi} (\psi_{,\rho} - \frac{1}{\rho}) \frac{l^2}{\rho^2}$.

Hence our geodesic equation as a system of first-order differential equation system is:

$$\begin{aligned} \dot{t} &= T, \\ \dot{T} &= -(T^2 + P^2)\Delta_{,t} - 2TP\Delta_{,\rho} + e^{-2\Delta+2\psi} \psi_{,t} \frac{l^2}{\rho^2}, \\ \dot{\rho} &= P, \\ \dot{P} &= -(T^2 + P^2)\Delta_{,\rho} - 2TP\Delta_{,t} - e^{-2\Delta+2\psi} (\psi_{,\rho} - \frac{1}{\rho}) \frac{l^2}{\rho^2}, \end{aligned}$$

where $\Delta = \gamma - \psi$ is a new auxiliary function and $\varphi(\tau)$ may be found via quadratures $\varphi = \int l/\rho^2 e^{2\psi} d\tau$ once the system above is solved. The equations above have an apparent singularity (the coordinates are not well-defined when $\rho = 0$). The singularity holds no significance in our studies due to the vanishing of singular terms when $l = 0$, and for $l \neq 0$, the preservation of angular momentum prevents particles from reaching $\rho = 0$.

As a rudimentary analogy, we use the concept of a double pendulum. In Fig.(4.1) we see the sensitive dependence of the trajectories of pendulum for a small change in the release angle. A similar line of thought is followed for the chaotic geodesics where a slight change in the initial velocity gives us a totally different trajectory after some time.

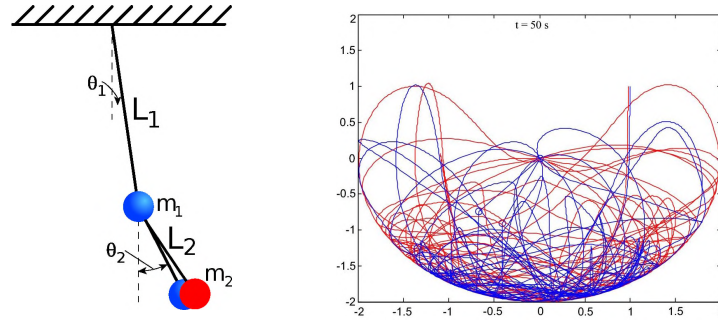


Fig. 4.1 This figure is adapted from [176, 177]. The pendulum is initially released for some θ_1 in blue. Then it is released from a slightly different θ_2 in red. The corresponding trajectories are shown for $t = 50$ sec where the diverging path is clearly visible.

While the initial integral $g(u, u) = -1$ could be employed for simplification purposes, we choose to retain it as a constraint that serves as a consistency check. All numerical computations presented in this article were conducted with the values $l = 1/3m$ and $\lambda = 1/(2\pi)m$. For cross-verification of the results, we employed Wolfram Mathematica, C++, and an independent Python code. In the C++ ODE solver, we set the absolute and relative tolerances to 10^{-6} and 10^{-10} respectively. Default tolerances were used in Wolfram Mathematica.

The presence of cylindrical symmetry leads to the conclusion that the curved cylindrically symmetric spacetime cannot exhibit asymptotic flatness. Additionally, within the studied spacetime, the $2 + 1$ dimensional Lorentzian submanifolds defined by $z = \text{const}$ and endowed with their naturally induced metric are also not asymptotically flat. The metric function ψ remains bounded as ρ approaches infinity, but γ diverges to infinity due to the asymptotic behavior of the Bessel functions. This divergence

of γ indicates that the Riemannian metrics with $t = \text{const}$ are not asymptotically flat [165]. This observation aligns with Thorne's concept of C-energy, as introduced in [105], where the C-energy is equal to $\gamma/4$.

The given system is described by two first integrals: $g(u, u) = -1$ and $\dot{\phi} = e^{2\psi} l / \rho^2$, which can be combined into a single equation:

$$-1 = e^{2(\gamma-\psi)}(-\dot{t}^2 + \dot{\rho}^2) + e^{2\psi} \frac{l^2}{\rho^2}, \quad (4.5)$$

This combined equation, together with the asymptotic behavior of the metric functions, reveals that all timelike geodesics remain confined within a bounded region in ρ . Notably, timelike geodesics with non-zero angular momentum ($l \neq 0$) exhibit a circular motion around the symmetry center.

The standing wave's amplitude attains its maximum at the symmetry axis and gradually decreases radially with ρ^{-2} . At a considerable distance from the center, the gravitational field resembles that of an infinite strut. Near the center, the test particles' orbits undergo perturbations due to the oscillations of the standing gravitational waves. These oscillations exert significant influences on the trajectories of the test particles, leading to observable disturbances in their motion. These waves act as periodic driving forces, significantly influencing the particle trajectories. As a consequence of this intricate behavior, chaotic dynamics are expected to emerge in this setting.

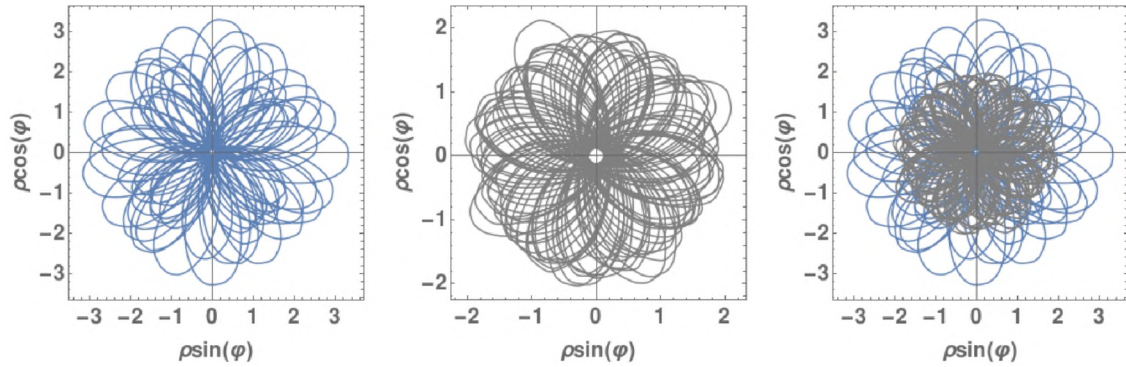


Fig. 4.2 Two timelike geodesics starting at slightly different initial points differ dramatically revealing sensitivity to initial conditions. The left figure corresponds to $t_0 = 0$, $\rho_0 = 1.13062m$, $P_0 = 0.83928$, $\tau_0 = 0$, $\varphi_0 = 0$, and $\tau_{max} = 3000m$. The central figure corresponds to the same parameters and initial conditions with P_0 shifted by $+6 \cdot 10^{-5}$. The darker shades of gray correspond to the initial part of the trajectories. The right figure is the superposed image, showing how the path of the article has completely changed after some time.

Two timelike geodesics, represented in the left and right figures, originate from nearly identical initial points but exhibit vastly different paths due to their sensitivity to initial conditions. In both figures, we consider the following parameters: $t_0 = 0$, $\rho_0 = 1.13062m$, $P_0 = 0.83928$, $\tau_0 = 0$, $\varphi_0 = 0$, and $\tau_{max} = 3000m$. The initial part of the trajectories is depicted in darker shades of gray. In the left figure, we observe the geodesic corresponding to the specified initial conditions. On the other hand, the right figure corresponds to the same set of parameters, except for a slight shift in the value of P_0 by $+6 \cdot 10^{-5}$. Despite this minute difference in the initial P_0 values, the geodesics exhibit remarkable divergence, emphasizing the chaotic nature of the system. The tiniest deviation in the initial conditions leads to significant deviations in the resulting paths, illustrating the butterfly effect at work in this dynamical system. It is crucial to acknowledge that even though the initial segments of the trajectories might seem similar, as time progresses, the two geodesics will evolve independently and follow entirely distinct trajectories.

4.3.2 Fractal structure

Fractals are unending, infinitely intricate patterns that exhibit self-similarity across various scales. They emerge from repetitive processes in feedback loops, capturing an important essence of chaos. Nature is rich with fractals seen in trees, rivers, coastlines, and other phenomena. Additionally, computers can generate abstract fractals, such as the Mandelbrot Set, by iteratively calculating simple equations.

To explore fractals, a simple example involves investigating the fractal structure of a double pendulum. Instead of tracing the trajectories of each pendulum to observe sensitive dependence, the focus shifts to calculating the time until the first flip-over occurs. By integrating the equations of motion until this flip-over event, various initial conditions can be explored, ranging from θ_1 and θ_2 values between $\pi/4$ and $3\pi/4$. A Python program with a resolution of different number of points for both θ_1 and θ_2 can be executed and the resulting time of the first flip-over is then plotted as a function of the initial angles, as demonstrated in Figure (4.3). This approach follows the methodology outlined in [178].

In the Fig.(4.3), several stable islands are depicted, representing areas where the pendulum remains unflipped (the dark blue regions). Beyond these stable regions lie areas of instability, but intriguingly, they encompass smaller islands of stability as well. These miniature islands are essentially distorted replicas of the primary stable region.

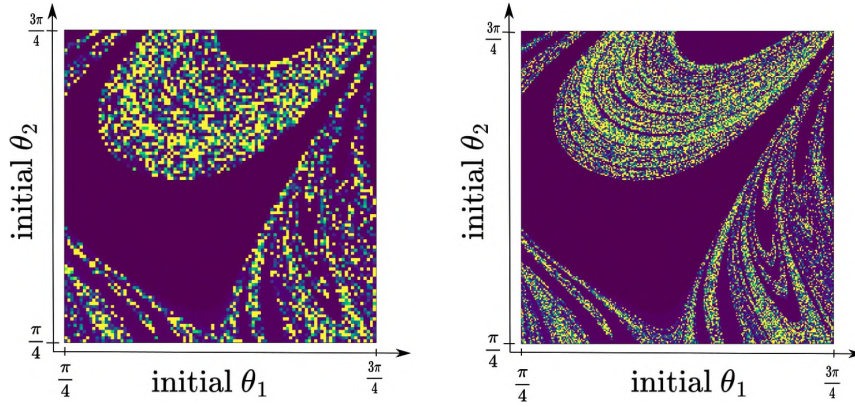


Fig. 4.3 One can take a resolution of 100×100 , meaning 100000 initial conditions from which the pendulum is released. Left: At 250×250 the fractal structure is appearing. Right: 1000×1000 points the fractal structure is evident (takes about 25 CPU hours) . Both figures are for θ_1, θ_2 between $\{\pi/4, 3\pi/4\}$. Figure adapted from [178].

Now we move on to our system. The numerical evidence for the sensitivity to initial conditions and chaos can be strengthened by systematic studies of the phase space. As we saw, in the case of a double pendulum, how the phase space of different initial θ 's evolved with the flipping, similarly we plot how φ will evolve for different values of ρ (radial distance) and $\dot{\rho}$ (also denoted as P) (radial velocity). One can similarly get the fractal by plotting other variables (say $\rho(\tau)$ for different initial conditions).

We first used Mathematica and then C++ and Python to get a more detailed fractal structures. In Fig.(4.4), we show how $\varphi(\tau = 200m)$ where we show a blurred fractal structure appearing. The left plot is for a grid of initial points ρ, P in the range 0.9 to 1.4. The right plot is for the range of 0.9 to 1.5. We went till $\tau = 100m$ for both the plots.

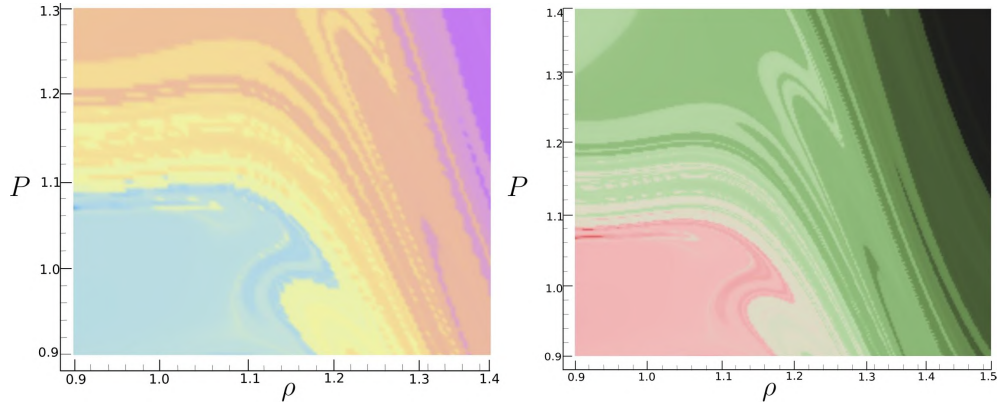


Fig. 4.4 We take grid of points in ρ and P , evolve φ and look for fractal structure. Left: For 8181 points we see where the values of $\varphi(\tau = 100m)$ have evolved. Right: The figure for 48542 points for same φ . Both were plotted in *Mathematica*. The purpose of this demonstration is to provide a heuristic comparison to the double pendulum mentioned earlier. As we increase the number of initial points, the fractal boundaries become increasingly apparent.

For a larger number of points, we used *SUNDIALS*, a suite of nonlinear and differential/algebraic equation solvers [179], and submitted jobs to PLGrid Infrastructure (grant number PLG/2022/015759). The figure 4.5 is adopted from our paper [2]. Here we focus on the region ρ 1.1 to 1.23 and P 0.84 to 0.9. The absolute and relative tolerances in the C++ ODE solver were set to 10^{-6} and 10^{-10} , respectively. The default tolerances were used in Wolfram Mathematica.

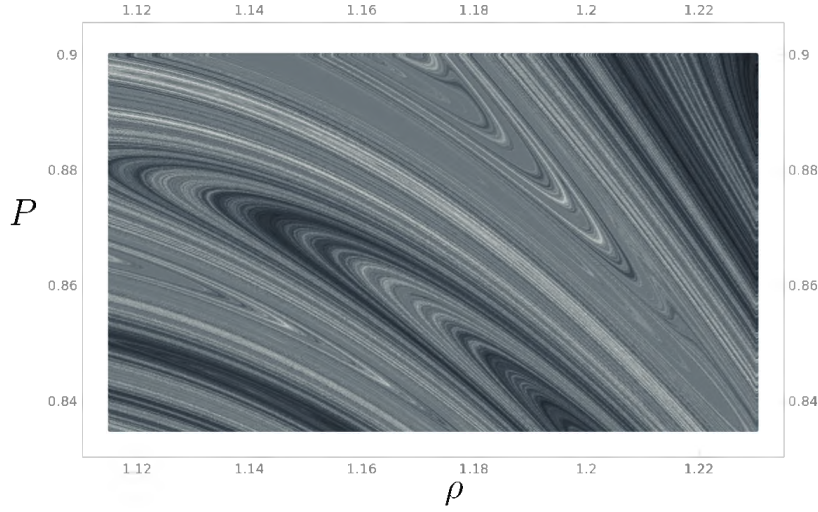


Fig. 4.5 The graph illustrates the variation of ϕ at $\tau = 400m$ concerning different initial points with $\phi_0 = 0$. The darkest point corresponds to $\phi(400m) = 85.87$, while the brightest point corresponds to $\phi(400m) = 247.66$. These points exhibit winding numbers of 13 and 39, respectively, revealing a clear fractal pattern in the data. This figure is used in [2].

Figure (4.5) demonstrates that the sensitivity to initial conditions depicted in Figure (4.2) is not a chance occurrence; instead, it characterizes the typical behavior of the system across a significant portion of the phase space. Although not visible in Figure (4.5), certain regions in the phase space do not exhibit chaotic behavior. For instance, in areas far from the symmetry center, standing waves lack the strength to significantly perturb the orbits of test particles, thus preventing the induction of chaotic behavior. This can be seen in the HD image of the fractal shown in Fig.(4.6).

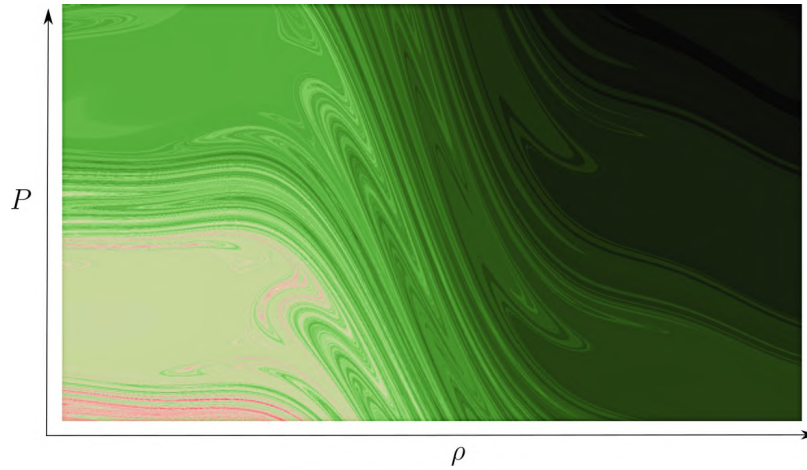


Fig. 4.6 This is a HD image of the fractal. An HD image as $1,280 \times 720$ pixels, approximately 1 million total pixels. We need around 15 hours on 7 cores to generate the full HD image (attached below) on a supercomputer. Here you see the evolution of $\phi(400m)$ for a grid of ρ and P consisting of one million point.⁷ Towards the right-most side there is no evident fractal structure as that region is far from symmetry region and the wave amplitude is significantly low to perturb the test particles.

Instead of evaluating the fractal dimension, our main focus was on examining other concrete signatures of chaos, specifically the homoclinic and heteroclinic intersections. These features offered additional valuable insights into the chaotic behavior we were investigating.

4.3.3 Poincaré maps

Many tools exist to analyze dynamical system. Understanding a dynamic system becomes obscure when there exist a lot of degrees of freedom. A state of a system can be studied as a point in high-dimensional space. For e.g, the double pendulum in a plane, the dimensional space is four.⁸ However, visualizing a point moving in four dimensions is non-trivial. Here Poincaré sections are helpful. A simple explanation is as follows. One can select a two-dimensional surface that intersects with the motion of points in phase space. Then choose a point on that surface and track its movement until it reaches the surface again. This point where it intersects is marked with a dot. This can be interpreted as a sort of ‘mapping of points’ and is called the Poincaré map. One can continue this process to further map the motion of points in your space. With the help of Poincaré maps, one can study different behaviors – quasi-periodic

⁸For a double pendulum as shown in Fig.(4.1), there are two angles (θ_1, θ_2) and their conjugate momenta (ω_1, ω_2) forming a four-dimensional phase space.

orbits, resonances, and chaos. Below, we present the visualization of a Poincaré map, inspired by the methodology introduced in reference [180], along with a rudimentary example showcasing a Poincaré section for a double pendulum system.

In integrable systems, phase space trajectories are regular, confined to n -dimensional tori (n is degrees of freedom). These tori are uniquely defined by n constants of motion and can be examined using Poincaré surfaces of section – intersections with 2D planes [181].⁹ But we know of integrable system displaying chaotic features and their tori are more complex.¹⁰ In the Poincaré sections of such systems, the tori's are surrounded by regions where chaotic phenomena arises. We just mention basic definitions of different types of orbits one can have in a Poincaré sections.

In the figure (4.7), on left we see a snapshot of dynamics on the surface of a torus, with different frequencies for the two directions ('longitude' and 'latitude'). This is a simple example of system having only periodic orbits. The central figure has different orbits. In light blue we have *quasiperiodic orbits*, defined by irrational frequency ratios, elegantly trace out intricate curves on the plane, and multiple such orbits can be nested within each other. At low energies, the double pendulum exhibits numerous quasiperiodic orbits with irrational frequency ratios. In an integrable system, the phase space is composed of nested tori with distinct frequency ratios, resulting in a Poincaré section featuring nested curves.

There also exist *resonances* where one can also have rational frequency ratios which lead to Poincaré map recurrence, with every n th dot at the same spot, where n is the resonance order. Near an exactly resonant orbit of order n , we may encounter nearby *nonlinear resonances*. These are quasiperiodic orbits with irrational frequencies that nearly recur every n th time they intersect the section. These orbits form small loops around the resonances, hitting each of the n loops once before returning to the initial one (in the figure there are with orders 3 (purple) and 5 (green)).

⁹To recall that in contrast, non-integrable chaotic systems typically have fewer integrals than degrees of freedom, allowing both regular and chaotic orbits to coexist in a single system's phase space [182].

¹⁰ The Hénon-Heiles system, classically integrable, demonstrates chaos when perturbations disrupt its regular motion, serving as an example of the transition from regularity to chaos. The Duffing oscillator, integrable in its undriven state, can display chaotic behavior when subjected to external forcing, especially at resonance frequencies.

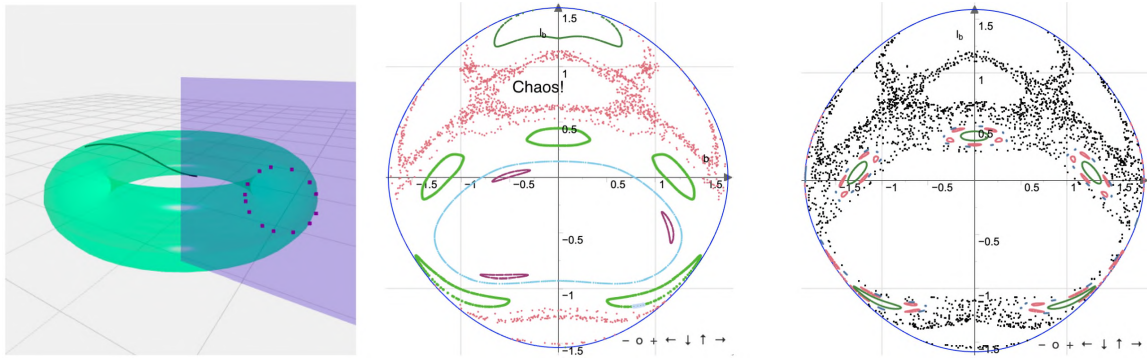


Fig. 4.7 Left: A snapshot of dynamics on the surface of a torus, where a sample trajectory (shown by a line) is traversing a cross-section. Centre: Different orbits arising. Right: Appearance of chaos with small islands of islands of regularity submerged in it. Figure taken from [180].

A fascinating aspect of dynamical systems lies in the coexistence of chaotic and quasiperiodic regions within their phase space in close proximity. Displayed on the right side is an ordinary quasiperiodic orbit (green) encircled by nonlinear resonances (red and blue dots), with several copies of the latter visible. The presence of islands within islands, forming a recursive pattern, is a characteristic feature of fractal behavior. Upon exploration, it becomes evident that these islands represent regions of regularity amidst a vast expanse of chaotic dynamics (black dots). This part was just to familiarize with physical notion of already known Poincaré maps and now we move on to our system.

In the preceding sections, numerical evidence of chaos has been provided. In this section, we employ the method of Poincaré sections to unveil the underlying mechanism responsible for inducing the system's intricate dynamics. The geodesic equation relies solely on the coordinate t when considering the term $\cos(t/\lambda)$ and its derivatives. Hence, it becomes convenient to interpret t/λ as an angle. Figure 4.8 illustrates the projection of the trajectory in the phase space.

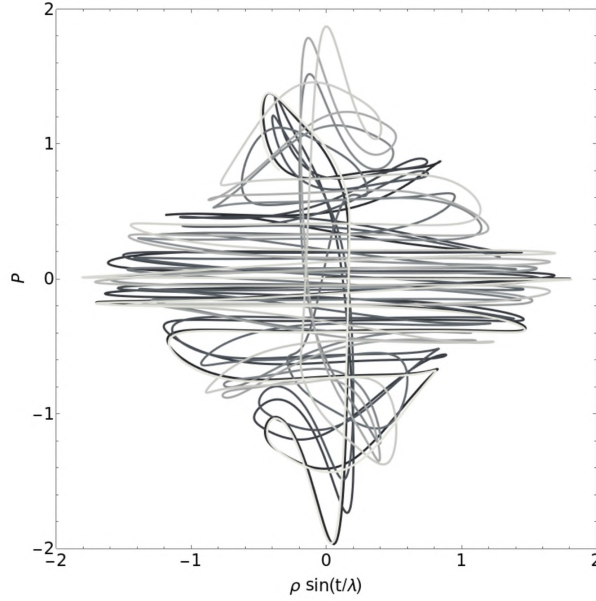


Fig. 4.8 The trajectory in the phase space is depicted with the angular variable t/λ as its projection. The trajectory initiates at the point $t_0 = 0$, $\rho_0 = 1.8$, and $P_0 = 0.01$, while the proper time τ ranges from 0 to 63. In the visualization, the initial part of the trajectory is represented by darker shades of gray as shown in[2]

To gain a deeper understanding of the system's dynamics, we can achieve a simplification of the three-dimensional flow by utilizing a two-dimensional Poincaré map denoted as Φ .¹¹ This is made possible by employing the Poincaré section $t \bmod 2\pi\lambda = 0$ as illustrated in the accompanying figure (4.9). This reduction significantly enhances the clarity of the system's behavior, making it easier to analyze and interpret.

¹¹Note: In our notation curly phi, ϕ , is the azimuthal angle and capital bold phi, Φ is the Poincaré map.

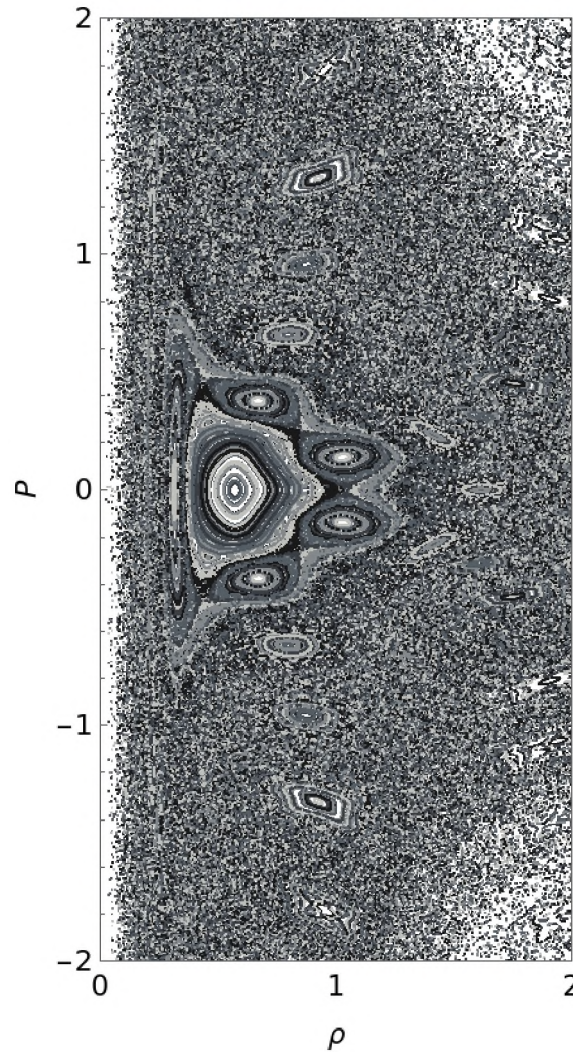


Fig. 4.9 Here we have the Poincaré section of the trajectories along $t/\lambda = 2\pi k$, where $k = 0, 1, 2, \dots$. We remove $T = i$ from the geodesic equation by applying constraint (4.5). Different shades of gray distinguish various trajectories, with periodic and quasi-periodic orbits standing out amidst a backdrop of chaotic ones. Figure used in our paper [2].

The Poincaré section uncovers the existence of numerous periodic points encircled by Kolmogorov-Arnold-Moser tori, commonly known as KAM tori.¹² These KAM tori play a crucial role in shaping the system's behavior and stability.

Poincaré-Birkhoff theorem states that resonant tori destruction results in an even number of periodic trajectories, half stable and half unstable. Phase space structure

¹²KAM tori (Kolmogorov-Arnold-Moser tori) are stable, quasiperiodic orbits in a dynamical system that exist under certain conditions when a system is perturbed from an integrable system. They represent regions of phase space where motion remains quasi-periodic.

is displayed using a Poincaré surface of the section, where invariant tori appear as circles around stable fixed points forming the main island of stability. Near destroyed resonant tori, smaller islands of stability arise around stable periodic points, forming Birkhoff chains with unstable points in between. Unstable periodic points give rise to asymptotic manifolds with stable and unstable branches. These structures create complex phase space patterns, driving deterministic chaos. A similar behaviour is what is depicted in figure (4.9). Now we probe our system deeper to analyze the details.

The periodic points of the system can be classified into two types: fixed points and points exhibiting 5-periodicity. For the sake of clarity, we will refer to the 5-periodic version of the system as Φ^5 instead of Φ . Consequently, all 5-periodic points will be considered as fixed points of the Φ^5 Poincaré map. These periodic points are the points where the particle goes around the symmetry center and enters the cross-section at the same point after 5 revolutions.

Within the central portion of the section presented in Fig.(4.9), there is a group of fixed points. Among these fixed points, one is encircled by 5 elliptic fixed points. Although one of these elliptic fixed points is barely visible, it lies on the axis $P = 0$, within the KAM tori that elongate along the P axis. These fixed points have emerged as a result of annihilated resonant tori and, in accordance with the Poincaré-Birkhoff theorem [35], are accompanied by 5 hyperbolic fixed points labeled as $p_{i=0,\dots,4}$ which are located at the junctions of the KAM tori. In the subsequent section, we will demonstrate the significance of these 5 hyperbolic fixed points for the behavior of the system in the central region of the phase space. At these hyperbolic points is where one encounters the intricate web of chaos.

4.3.4 The tangle of chaos

The origin of sentence dependence or the emergence of a fractal structure in a system exhibiting chaotic phenomena can be traced back to the vicinity of a hyperbolic or saddle point. When closely examined within this hyperbolic point, one encounters the intricate tangle of chaos. Now we define the stable and unstable manifolds¹³ for our system and show the homoclinic and heteroclinic intersections and finally the web of chaos.¹⁴

¹³Note: The manifold is related to the eigenvalue and eigenvector of the dynamical system (as that can be written down as a matrix) and not the usual manifold used in GR.

¹⁴Interestingly while dealing with the 3-body problem Poincaré had attempt to visualize it and he says *[I]f one seeks to visualize the pattern formed by these two curves and their infinite number of*

Let $S(p_i)$ and $U(p_i)$ represent the stable and unstable manifolds of the hyperbolic fixed point p_i , respectively. Among the five hyperbolic points, we focus on $p_0 \simeq (1.041334, 0)$, a small real constant denoted by ε , and an open set R containing p_0 . For any point p' within R , the Euclidean distance between p' and p is less than ε . The subsequent iterations of sets $\Phi^{-5}(R)$ and $\Phi^5(R)$ unveil the structures of the stable and unstable manifolds of the hyperbolic fixed point p_0 .

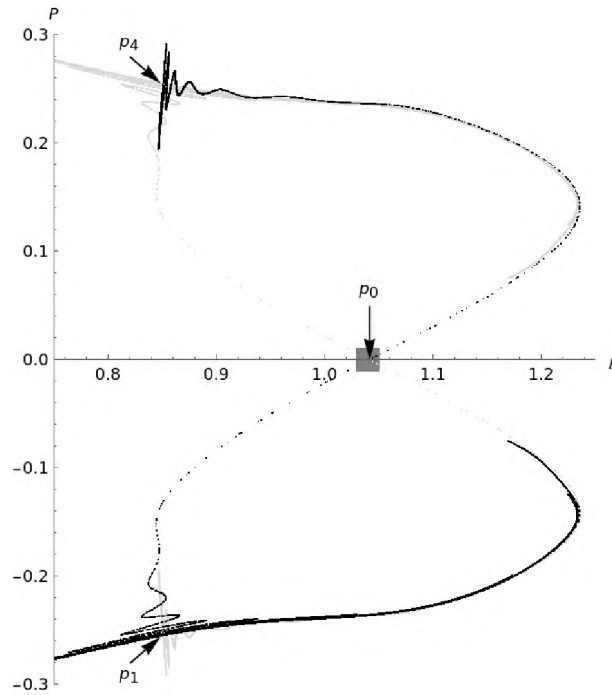


Fig. 4.10 The hyperbolic fixed point p_0 gives rise to intersecting stable and unstable manifolds, creating two homoclinic tangles located near p_1 (at the bottom) and p_4 (at the top), denoted as $S(p_0) \cap U(p_0) \neq \emptyset$. The darker shade illustrates the unstable manifold $U(p_0)$, while the region R (represented by the shaded rectangle) encompasses the fixed point p_0 . Figure from our paper [2].

In Figure (4.10), we present evidence of the intersection between the stable manifold and unstable manifold of p_0 ($S(p_0) \cap U(p_0) \neq \emptyset$). This intersection gives rise to the Poincaré homoclinic tangle, a widely recognized phenomenon that signifies the presence of chaotic trajectories.

intersections ... these intersections form a kind of lattice work, a weave, a chain-link network of infinitely fine mesh;...

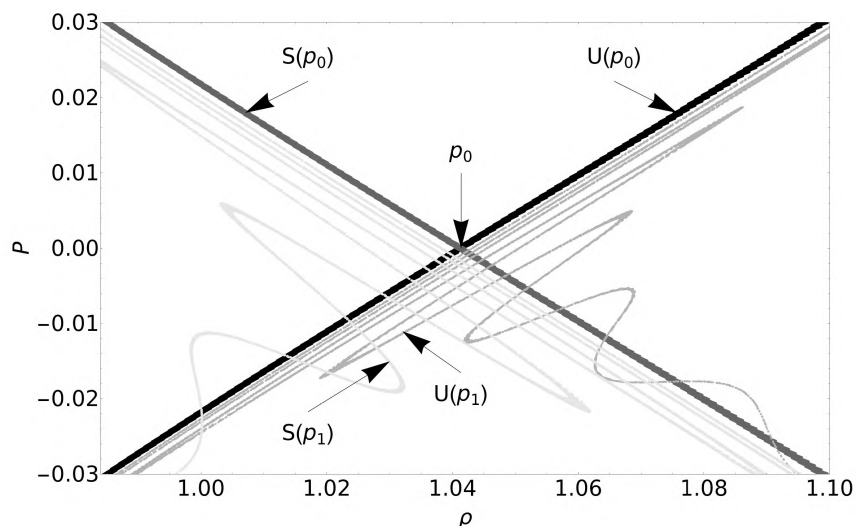


Fig. 4.11 The hyperbolic fixed points p_0 and p_1 have stable and unstable manifolds that intersect, creating heteroclinic intersections near p_0 and p_1 , specifically denoted as $S(p_0) \cap U(p_1) \neq \emptyset$ and $U(p_0) \cap S(p_1) \neq \emptyset$. Additionally, the homoclinic intersection is also visible, as represented by $S(p_1) \cap U(p_1) \neq \emptyset$. The manifolds are distinguished by different shades of gray. This was also used in our paper [2].

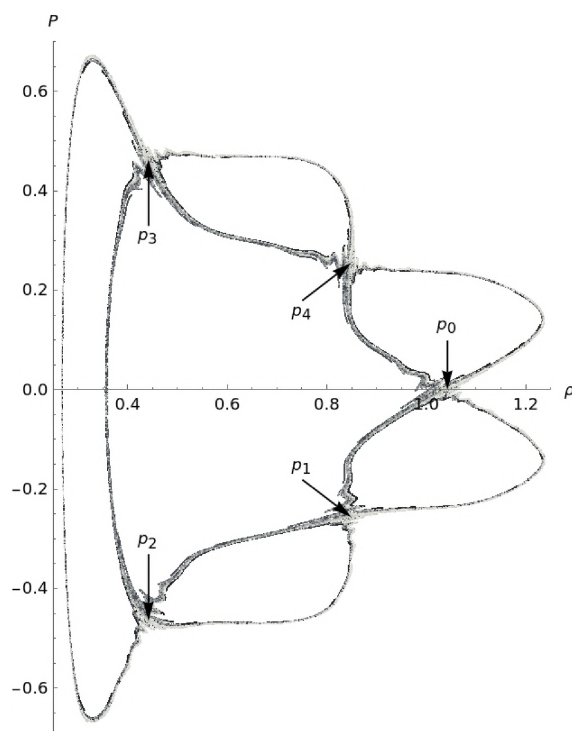


Fig. 4.12 The forward and backward iterations of the set R with ϕ unveil a complex and chaotic heteroclinic network. The trajectories are depicted using varying shades of gray to differentiate between them. Plot from our paper [2].

Homoclinic intersections, where stable and unstable manifolds of the same hyperbolic fixed point cross, are often accompanied by heteroclinic intersections. Heteroclinic intersections involve the stable and unstable manifolds of different hyperbolic fixed points coming together in a similar crossing fashion. In Figure (4.11), we showcase heteroclinic intersections between the stable and unstable manifolds of two hyperbolic fixed points, p_0 and $p_1 = (0.848293, -0.255512)$. These intersections are characterized by the non-empty crossing of $S(p_0)$ and $U(p_1)$, as well as $U(p_0)$ and $S(p_1)$.

In Figure (4.12), we observe the emergence of similar structures formed by the stable and unstable manifolds of all hyperbolic fixed points p_i . These structures combine to create a complex five-component chaotic heteroclinic network, which becomes evident through successive iterations of $\Phi(R)$ and $\Phi^{-1}(R)$.

Chaotic heteroclinic network

Heteroclinic networks are vital in dynamical systems, forming intricate connections between equilibrium states. These networks unveil unexpected behaviors as systems transition between steady states. They vary in complexity, from a few connections to numerous intertwined equilibrium points. Their significance spans physics, biology, and engineering, shedding light on phase transitions and complex phenomena [1]. Mathematically, they involve techniques like phase portraits and bifurcation diagrams, and in some cases, chaos arises, enriching our grasp of complex ¹⁵ dynamics.

Chaotic heteroclinic networks emerge from the intricate dynamics within a dynamical system containing multiple equilibrium points and heteroclinic connections. These networks represent complex, aperiodic behaviors as the system transitions unpredictably between stable states. Their formation hinges on the system's sensitivity to initial conditions, where small changes can lead to vastly different trajectories and switching patterns. In essence, chaotic heteroclinic networks showcase the intricate interplay of nonlinear dynamics, yielding unpredictable and complex behaviors within the network of transition pathways between equilibrium points.

¹⁵Heteroclinic networks can be used as essential mathematical models in computational neuroscience, explaining transitions between stable neural states, such as in pattern recognition and memory retrieval [183]. They also shed light on neural synchronization and plasticity. These networks are critical tools for understanding complex cognitive functions emerging from neural interactions within the brain [184, 185].

We digress over here for a crude analogy and consider the example of a nematode or a roundworm *C. elegans*¹⁶ which is about 1 mm in length and generally found in temperate soil environments. They have limited behavioral states and the stochastic switching between dominant states is often modelled as Markov Chains.¹⁷ Interestingly their behavior has been proposed to be owing to a chaotic heterocyclic network cite. It was shown in [185] that chaotic heteroclinic networks are effective models for switching dynamics. By manipulating the eigenvalues of saddle fixed points in these networks and the geometry of stable and unstable manifolds, they can quantitatively control switching dynamics. It was interesting to discover that their three-point chaotic network and our five-point chaotic heteroclinic network share a common foundation in regards to the mathematical tools of dynamical systems.

We show below the symbolic representation of the chaotic heteroclinic network found in the spacetime and end this section briefly mentioning about heteroclinic computing.

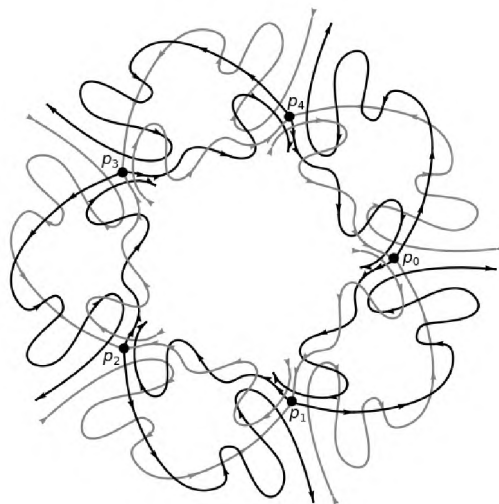


Fig. 4.13 The symbolic representation of the chaotic heteroclinic network is showcased in the figure. The unstable manifolds are represented by a darker color, allowing for a clear distinction between the different trajectories within the network.

¹⁶*C. elegans*, or *Caenorhabditis elegans*, is a microscopic roundworm often used in scientific research as a model organism. Its importance lies in its simple and well-understood biology, making it an ideal subject for studying fundamental biological processes. Researchers use *C. elegans* to investigate topics such as development, genetics, neurobiology, and aging, providing valuable insights into complex biological phenomena that extend beyond this tiny worm's size.

¹⁷Markov Chains are mathematical models used to study and predict sequential processes where future states depend solely on the current state, with no consideration of past states. They are essential in various fields like statistics, physics, and computer science, helping analyze and understand dynamic systems and make predictions based on probabilistic transitions.

In [186] the authors were trying to answer the question can complex networks of states in nonlinear dynamical systems solve computational tasks, and if so, which systems are suitable and how can computations be performed? They showed that the dynamics involved in heteroclinic networks can play a role in generating systems are capable of computing arbitrary logic operations.

4.4 Conclusion

We have offered compelling numerical support for the presence of chaotic geodesics within the cylindrical standing gravitational wave spacetime belonging to the Einstein-Rosen class. Utilizing Poincaré sections, we have revealed a chaotic heteroclinic network that governs the system's dynamics. This intricate interaction of the standing gravitational wave with test particles bears similarities to the familiar phenomenon of standing mechanical waves in Newtonian physics.

Lastly, it is worth highlighting the intriguing formal alignment of mathematical structures. Heteroclinic networks and chaotic heteroclinic networks, critical systems, find diverse applications in the modeling of cognitive processes and neural dynamics [184, 187]. These networks possess computational capabilities known as heteroclinic computing, allowing them to process information in a manner similar to neural networks [183, 186].

Chapter 5

Summary and Future Outlook

“The mathematicians know a great deal about very little and the physicists very little about a great deal.”

Stanisław Ulam

This thesis deals with toy models of standing wave spacetimes using the standard tools of general relativity (GR). The aim was to understand standing gravitational waves and how they govern the motion of test masses. We study standing gravitational waves in vacuum in three-torus topology. Then we investigated standing gravitational waves in an electrovacuum in the same topology. We found the existence of chaotic geodesics in cylindrical standing gravitational waves using the tools of dynamical systems.

Chapter 1 provides a historical background, motivation, and a brief review of the mathematical framework used in our work. It describes how wave interaction and chaos phenomena were studied in the context of GR. Then we state the basics of standing waves in general Physics. We discuss attempts made to find an exact solution corresponding to standing gravitational waves. A brief introductory review of general relativity is given along with the standard mathematical definitions and tools specifically used in our work. We end Chapter 1 with a brief review of dynamical systems and the description of basic concepts involved in chaos studies.

Chapter 2 deals with standing gravitational waves in vacuum in three-torus topology. In this spacetime, also known as T^3 Gowdy spacetime, we found that freely falling observers remain stationary at antinodes. To study coordinate-independent effects, we used the geodesic deviation equation in a non-holonomic basis. In the

longitudinal direction we found attraction at antinodes. In the transverse direction we found a permanent velocity effect of ring of test masses as an analogue to the gravitational memory effect. The geodesic equation also confirmed heuristically the attraction to antinodes in the longitudinal direction. We used the Newman-Penrose formalism to confirm the standing wave in this spacetime correspond to a non-trivial superposition of two waves. The main results of this chapter were published here [13].

Chapter 2 attempts to explain the behavior of freely falling observers at the antinodes of a standing GW but its full analysis and also behavior at the nodes is still unclear. As defining energy is non-trivial in GR [188–190], an invariant way to is needed to probe what is happening at the antinodes and nodes. Constructing new invariants and analyzing this spacetime may lead to a more precise formulation of standing waves in GR. Moreover, test masses were spinless but the case for a spin is open to investigation. Though spin is more tricky to handle in GR, given that the point mass approximation is unclear [191–193] — still open questions for investigation are if there is an accumulation of GW energy at antinodes then what will spin look like [193], and will it be unaffected at nodes? There are also other topologies in Gowdy to be explored, namely $S^1 \times S^2, S^3$ [194]. One can study the existence of freely-falling observers and probe whether any memory effects are preset or not, like how it has been done for Kundt geometry [195, 196].¹

Chapter 3 describes an electrovacuum solution in which standing electromagnetic waves are coupled to standing gravitational waves. The topology of this spacetime is also three-torus. To understand the effect of the coupling between EM and GW radiation we used the same mathematical tools as in Chapter 2. However, the results of the analysis geodesic equation and the geodesic deviation equation were the same as in Chapter 2 due to the same topology. As we did not get any qualitatively novel results, this work is unpublished and we hope to tackle this study in the future with other tools in GR.

For the spacetime studied in chapter 3 one can study the motion of charged particles and possibly look into electromagnetic memory [198] in this spacetime. Similar to gravitational wave (GW) memory, an electromagnetic (EM) wave induces a lasting alteration in the relative velocity of charged particles connected to a detector situated in the 2-D surface perpendicular to the wave's propagation direction as it travels through the detector. This was shown for an arbitrary curved spacetime in

¹Kundt spacetimes are Lorentzian manifolds with a geodesic null congruence and optical scalars (expansion, twist, and shear) that vanish. The pp-wave is a notable member of this Kundt class [197, 4].

[199]. We plan to continue this project and publish it with further innovative findings in the future.

In chapter 4 we studied the existence of chaos in cylindrical standing gravitational waves. After probing the sensitive dependence on the initial conditions of the chaotic geodesics, we further cemented the case for chaos by analyzing the fractal structure and Poincaré sections. Through the Poincaré maps, we found elliptic and hyperbolic fixed points. The crossing of stable and unstable manifolds at the hyperbolic fixed points reveals the origin of chaos in this spacetime. The homoclinic and heteroclinic intersections of stable and unstable manifolds of fixed points were investigated and the presence of a chaotic heteroclinic network was found. The results of this chapter are included in this paper [2].

Further lines of investigation in work done in chapter 4 consist of– will all spacetime with cylindrical symmetry with radiation have chaotic geodesics [200]? how will polarisation affect the chaotic structure in this spacetime [174]? These unanswered questions will be studied in the future.

The concept of a standing wave has captivated researchers in both mechanical and electromagnetic contexts. However, within the realm of General Relativity (GR), a definitive definition of standing waves is still elusive. A theoretical exploration, considering non-linearities, may hold the potential to inform and guide the design of future experimental setups. This thesis explored exact standing wave spacetimes. The results we found while probing the test masses behavior under the influence of standing gravitational waves paves the way for future exploration of standing waves in full GR. While the presence and significance of gravitational waves in GR and astronomy are becoming increasingly apparent through mounting observations, this thesis strives to stimulate theoretical inquiries into the realm of standing gravitational waves via toy models.



²Credit: Cartoon drawn by C.V Vishveshwara [201].

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Appendix A

Additional Mathematical Preliminaries

“Before I came here I was confused about this subject. Having listened to your lecture I am still confused. But on a higher level.”

Enrico Fermi

Topological space

We state here the two definitions about *topological spaces* and *homeomorphism* between topological spaces. This act is as a prerequisite in formulating the idea of a manifold and follows from [88, 89]. Rudimentarily a topological space can be defined as a domain where every point can be regarded as residing within a local vicinity of other points, helping us formulate notions like continuity and convergence.

Definition: A topological space, denoted as M , comprises a collection of points equipped with a topology denoted as T . This consists of a set of open subsets $\{O_a \subset M\}$ which obey:

- The set M and the empty set $\emptyset$ are open subsets: $M \in T$ and $\emptyset \in T$.
- There is an open set defined as the intersection of a finite number of open sets. If $O_1 \in T$ and $O_2 \in T$ then $O_1 \cap O_2 \in T$.
- The union of any number, including possibly an infinite number, of open sets remains an open set. If $O_\alpha \in T$ then $\cup_\alpha O_\alpha \in T$.

For a point $p \in M$, we have $O \in T$ defined as a neighborhood of p if $p \in O$. With this we can have, for two distinct points, a neighborhood that contains one but not the other. Therefore, for any $p, q \in M$ with $p \neq q$, there exists $O_1, O_2 \in T$ such that $p \in O_1$ and $q \in O_2$ and $O_1 \cap O_2 = \emptyset$. Such a criterion is followed by *Hausdorff* topological spaces which will help in defining *differentiable manifolds*.

Homeomorphism

Definition : A *homeomorphism* between topological spaces (M, T) and $(\tilde{M}, \tilde{T})$ is a map $f : M \rightarrow \tilde{M}$ that is

- Injective (one-to-one): $p \neq q, f(p) \neq f(q)$
- Surjective (onto): $f(M) = \tilde{M}$, for each $\tilde{p} \in \tilde{M}$ there exists $p \in M$ such that $f(p) = \tilde{p}$.
Bijjective functions, being both injective and surjective, guarantee the existence of an inverse.
- Bi-continuous: meaning that both the function and its inverse are continuous. In order to define a notion of continuity, we need topology. So f is continuous if, for all $\tilde{O} \in \tilde{T}$, $f^{-1}(\tilde{O}) \in T$.