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Faculty of Mathematics

Roland Donniger
University of Vienna
Faculty of Mathematics
Oskar-Morgenstern-Platz 1
1090 Vienna, Austria

March 15, 2022

Review of PhD thesis “Nonlinear Schrödinger equations with trapping potentials in higher dimensions” by Filip Ficek

To whom it may concern,

The thesis under review is concerned with the study of a Schrödinger equation with a Hartree nonlinearity and a trapping potential. Physically, the nonlocal nonlinearity arises from coupling the Schrödinger equation to Newtonian gravity, i.e., the convolution-type nonlinearity emerges from the solution formula of the Poisson equation. Furthermore, the trapping potential is the one from the harmonic oscillator. That is why the equation is called the “Schrödinger-Newton-Hooke (SNH) system”. In addition, the SNH system arises as a nonrelativistic limit in the study of perturbations of the AdS spacetime in general relativity. This serves as the main physical motivation. The main topic of the thesis are the existence, uniqueness, and stability of standing wave solutions or solitons in higher dimensions. In low dimensions the existence of solitons for such equations is usually studied by variational methods. However, in higher dimensions the necessary Sobolev embeddings fail and this approach is no longer available. However, ground states are typically spherically symmetric so it is no loss of generality to restrict oneself to the radial case. Then the corresponding elliptic problems become ODEs which can be studied by more elementary methods. This is the strategy of the present thesis.

After an introduction in Chapter 1, Chapter 2 starts with a discussion of the SNH system as a nonrelativistic limit of AdS perturbations. The derivation given there is of a formal nature but strongly motivates the study of the SNH system.

In Chapter 3, some basic properties of the SNH equation are discussed, in particular symmetries, conservation laws, and the scaling heuristics of dispersive equations. Then it is briefly described how to prove the existence of solitons using the calculus of variations and the limitations of this approach are discussed.

Chapter 4 constitutes the heart of the thesis and is concerned with the existence and uniqueness of stationary solutions. First, the spherical reduction of the SNH system is

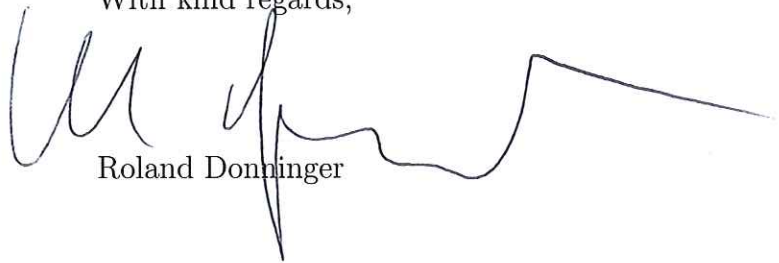
provided. Then the frequencies of the stationary solutions with a fixed value at the origin are computed numerically. This requires precise asymptotics of the numerical solutions which cannot be obtained directly by just solving the initial value problem. Luckily, the asymptotic equation can be solved explicitly in terms of hypergeometric functions. Thus, by gluing the explicit asymptotics to the numerical solution obtained by bisection, it is possible to read off the asymptotics of the solution and hence the frequency. These numerical findings are then taken as a starting point for a rigorous proof of existence. The proof relies on a shooting method and continuity arguments. Roughly speaking, one exploits certain structural properties of the equation and continuity arguments to exclude one unwanted scenario after the other. As a result, the existence of a ground state is obtained. This solution is characterized by the fact that it is strictly positive. However, the shooting technique can also be used to prove the existence of *excited states* with a prescribed number of zeros, which is demonstrated next. Then, the behavior of the solution is studied when the value at zero goes to infinity. The strategy here is to properly rescale and derive a limiting equation which is then analyzed. Furthermore, by exploiting some special features of the SNH equation it is also possible to prove the uniqueness of the stationary solutions. In the final part of this chapter, the dependence of the frequency on the value at zero, denoted by b , is studied. For small and large values of b it is possible to study this problem perturbatively but for intermediate values one has to rely on numerics and graphs for different dimensions are obtained.

In the final Chapter 5, dynamical properties of the SNH system are investigated. The spectral stability of a stationary solution is determined by the spectrum of the corresponding linearized operator that is derived first. This is very similar to the standard NLS case and already there, this operator is not fully understood from a rigorous point of view. For the stability of the ground state one can make use of the Vakhitov-Kolokolov criterion. Asymptotically, for small values of b , a perturbative approach is possible but for larger values of b one has to rely on numerics. Also the spectral stability of the excited states is investigated asymptotically as $b \rightarrow 0$ but numerics for larger values of b seem to be challenging in this case. Finally, the dynamics of the system are studied. Due to the trapping potential the long-time asymptotics are very challenging to compute numerically. A promising approach is to restrict oneself to the *resonant approximation* that is supposed to describe the long-time behavior accurately. The corresponding resonant system is derived and some of its properties and its Hamiltonian structure are discussed. The thesis ends with four technical appendices where auxiliary results are provided.

My overall impression of the thesis is excellent. It is very carefully written and a lot of effort has been devoted to intuitive explanations. As a result, the thesis is fairly easy to read and the results are presented in a clear manner. The thesis studies the problem from the theoretical physics point of view. Some of the results are rigorous (or can easily be turned into rigorous mathematics), some are formal, and some are numerical. Even though the thesis is written in physics style (no definition-theorem-proof structure, no ϵ - δ notation), it is always clearly spelled out which parts are meant to be rigorous and which parts are more of a formal nature or rely on numerics. This mixture of rigorous proofs, formal arguments, and numerics is typical for the field and proved very fruitful in the investigation of new phenomena. The thesis is not complete in the sense that the questions posed in the beginning are answered in the affirmative. It rather lays the

groundwork for many exciting investigations yet to come. I see this as a strength and as an ideal starting point for a possible scientific career. The candidate clearly meets the criteria stipulated in the act on academic degrees and title in the Arts (Dz. U. of 2016 item 882).

With kind regards,

A handwritten signature in dark ink, consisting of several loops and a long horizontal stroke extending to the right.

Roland Donniger

