Jagiellonian University

Faculty of Physics, Astronomy and Applied Computer Science

Phase structure of 1+1 Causal Dynamical Triangulation with matter

Hongguang Zhang

Thesis written under the supervision of Prof. Jerzy Jurkiewicz and Dr. Andrzej Görlich, presented to the Jagiellonian University for the PhD degree in Physics

Krakow 2015

Abstract

The definition of a path integral of Quantum Gravity requires a regularization of the space-times and performing a summation over all regularized space-time geometries joining the in- and out- states. In the approach presented in the thesis a UV cut-off in a form of a random lattice was introduced to regularize geometries and the path integral becomes a sum over distinct triangulations of a given topological manifold. We consider a toy model, where space-time is two-dimensional. In all our considerations we assume that time can be Wick-rotated. Definition of a path integral requires additional topological restrictions on a class of allowed manifolds. One possible approach would be a minimal requirement, to restrict the summation to all manifolds with a fixed topology of space-time (usually assumed to be spherical, with possible boundaries corresponding to the boundary states). In a lattice regularization this leads to the model of Euclidean Dynamical Triangulations (EDT), where the space-time manifold is constructed by gluing flat regular simplices (equilateral triangles) along the edges with a universal length a. In this approach a distinction between time and space is lost, but can be reconstructed dynamically. The EDT geometry may be coupled to matter fields. In the continuum limit ($a \to 0$) the model is expected to be described by the two-dimensional Liouville Conformal Field Theory with a conformal charge of the matter field c. Type of matter fields (choice of c) determines the critical behaviour. Liouville theory predicts that for $c > 1$ geometry of space-times becomes unstable with complex critical exponents. The lattice EDT version can, in this case, be studied numerically and shows that a typical geometry becomes that of branched polymers with the Hausdorff dimension $d_H = 2$ and the spectral dimension $d_S = 4/3$. For $c < 1$ critical behaviour of the theory depends on c and, in particular, for $c = 0$ (pure gravity) is characterized by a Hausdorff dimension $d_H = 4$ and $d_S = 2$. An alternative version of the model is a model of Causal Dynamical Triangulations, where the topology of admissible space-time geometries is additionally reduced by the requirement of the existence of a global time foliation for each configuration. In effect critical properties of the model change. The only case studied analytically

(both in the continuum and in the lattice formulation) is the case of pure gravity, where the geometry is characterized by $d_H = 2$ and $d_S = 2$. Also for CDT we may include interactions of geometry with matter, but it was not clear if the analogue of a $c = 1$ barrier is present in this case.

The thesis consists of four papers, in which the effects of the coupling between matter and geometry for two-dimensional quantum gravity models are analysed. The introductory part of the thesis contains a general formulation of the problems studied and a short discussion of methods and results.

In the first paper we analyse the so called "Lee-Yang edge" model in the EDT formulation. The model is interesting because naively there is a disagreement between continuous critical exponents (in this case $c = -22/5$) and the lattice result, obtained using the Random Matrix Theory. In the paper we show that the effect of the operator mixing explains the difference and leads to a full agreement of the exponents. This article was prepared during a one-year stay at the Niels Bohr Institute in Copenhagen.

The remaining three papers present results of the numerical Monte Carlo analysis of the CDT 1+1 geometry coupled to one or more copies of the scalar field. In the first article we consider d copies of a massless scalar field, which in the EDT version would correspond to a geometry interacting with matter with a central charge $c = d$. We study systems with periodic boundary conditions in time and space, where the period in the time direction T is fixed (and relatively large). We show that including matter ($d > 1$) has also in this case a very important physical effect, although different than in EDT. We observe a development of the de Sitter-like semi-classical blob with the Hausdorff dimension $d_H = 3$. This effect may be interpreted as a breakdown of the original toroidal geometry to form a collapsed "spherical" one, although a value of d_H indicates that quantum effects lead to a non-trivial scaling of geometry. This geometry is further analysed in the second paper, where we show that the spectral dimension remains $d_S = 2$ also for systems with the collapsed geometry. A transition between the two regimes: that with a toroidal geometry with $d_H = 2$ for $d = 0$ and a "spherical" geometry with $d_H = 3$ for $d > 1$ may be interpreted as a phase transition. This cannot be checked, even numerically, with a discrete integer set of d. In the third paper we propose to study a possible transition using a CDT system interacting with $d = 4$ copies of a massive scalar field. Introducing a new parameter in a form of the mass we may "switch off" the field degrees of freedom, freezing them to zero (for large mass). We show that for a finite value of the mass parameter we observe a transition between the two regimes. The character of this transition suggests that it is a phase transition of the second (or higher) order.

Streszczenie

Definicja całki po trajektoriach w kwantowej grawitacji wymaga regularyzacji czaso-przestrzeni oraz wykonania sumowania po wszystkich zregularyzowanych geometriach czaso-przestrzennych łączących stan wejściowy i wyjściowy. W podejściu użytym w pracy ultra-fioletowe obcięcie jest zrealizowane przez wprowadzenie przypadkowej sieci, a całka po trajektoriach staje się sumą po różnych triangulacjach topologicznej rozmaitości. Rozważamy model-zabawkę, w którym geometria czaso-przestrzeni jest dwuwymiarowa. We wszystkich rozważanych w tej pracy modelach czas został obrócony (w sensie obrotu Wicka). Definicja całki funkcjonalnej wymaga dodatkowego zdefiniowania klasy rozmaitości, uwzględnionych w sumowaniu. Jedno z możliwych podejść używa minimalnego ograniczenia, w którym uwzględnia się rozmaitości o ustalonej globalnej topologii (zazwyczaj sferycznej, z możliwymi brzegami, odpowiadającymi stanom wejściowym i wyjściowym). Dla regularyzacji sieciowej realizacją jest model Euklidesowych Dynamicznych Triangulacji (EDT), gdzie czaso-przestrzeń staje się symplicjalną rozmaitością zbudowanąną z regularnych sympleksów (trójkątów), sklejonych wzdłuż krawędzi o długości a. W tym podejściu traci się rozróżnienie między kierunkiem czasowym i przestrzennym, ale to rozróżnienie może być odzyskane dynamicznie. Geometria EDT może sprzęgać się z polami materii. W granicy ciągłej ($a \to 0$) spodziewamy się, że model może być opisany przez ciągłą dwuwymiarową Konforemną Teorię Pola Liouville'a z konforemnym polem materii z ładunkiem centralnym c. Rodzaj pola materii (ładunek c) determinuje zachowanie krytyczne. Teoria Liouville'a przewiduje, że dla $c > 1$ geometria staje się niestabilna, scharakteryzowana przez zespolone wykładniki krytyczne. Wersja sieciowa EDT może, w tym przypadku, być zbadana metodami numerycznymi i wykazuje, że typowa geometria odpowiada układowi polimerów rozgałęzionych o wymiarze Hausdorffa $d_H = 2$ i wymiarze spektralnym $d_S = 4/3$. Dla $c < 1$ krytyczne zachowanie teorii jest funkcją c, w szczególności dla czystej grawitacji ($c = 0$) charakteryzuje się wymiarem Hausdorffa $d_H = 4$ i $d_S = 2$. Alternatywna wersja modelu nosi nazwę modelu Kauzalnych Dynamicznych Triangulacji (CDT). W tym modelu dopuszcza

się tylko geometrie pozwalające na zdefiniowenie globalnej foliacji czasowej dla każdej konfiguracji. W konsekwencji własności krytyczne ulegają zmianie. Jedynym przypadkiem zbadanym metodami analitycznymi (zarówno w wersji ciągłej, jak sieciowej) jest czysta grawitacja, gdzie $d_H = 2$ i $d_S = 2$. Dla CDT również możemy badać sprzężenia z materią, ale nie jest oczywiste czy bariera $c = 1$ istnieje również dla tego modelu.

Rozprawę doktorska stanowią cztery publikacje, w których badane są sprzężenia materii w modelach dwuwymiarowej grawitacji kwantowej. Rozprawa zawiera też część wstępną, w której przedstawiamy zakres badanych problemów oraz krótkie omówienie metod i uzyskanych wyników.

W pierwszej publikacji badamy tzw. model "krawędzi Lee-Yanga" w sformułowaniu EDT. Model jest interesujący z tego względu, że naiwnie wydaje się prowadzić do niezgodności między ciągłym modelem Liouville'a (w tym wypadku z $c = -22/5$), a przewidywaniami sieciowymi, uzyskanymi w ramach Teorii Macierzy Przypadkowych. W artykule wykazujemy, że uwzględnienie efektu mieszania operatorów daje pełną zgodność przwidywań dotyczących eksponent krytycznych. Ta publikacja powstała w czasie rocznego pobytu na stażu w Instytucie Nielsa Bohra w Kopenhadze.

Pozostałe trzy publikacje przedstawiają rezultatu numerycznych analiz (metodami Monte Carlo) modelu CDT w 1+1 wymiarach, sprzężonego z jedną lub więcej kopiami pola skalarnego. W pierwszym artykule rozważamy model zawierający d kopii bezmasowego pola skalarnego. W wersji EDT odpowiadałoby to sprzężeniu geometrii z materią o ładunku centralnym $c = d$. W naszych obliczeniach warunki brzegowe są periodyczne w kierunkach czasowym i przestrzennym, przy czym periodyczność w kierunku czasowym jest ustalona i duża. Pokazujemy, że wprowadzenie materii ($d > 1$) ma bardzo istotny wpływ na fizyczne własności modelu, chociaż efekt ten jest inny niż w EDT. Obserwujemy pojawienie się semi-klasycznej geometrii tła, podobnej do geometrii de Sittera, o wymiarze $d_H = 3$. Możemy interpretować ten efekt jako załamanie wyjściowej toroiaelnej topologii do skolapsowanej "sferycznej" geometrii, chociaż wartość d_H wskazuje na nietrywialny wpływ efektów kwantowych. Dalsza analiza "sferycznej" geometrii przeprowadzona jest w drugiej publikacji, gdzie pokazujemy, że wymiar spektralny pozastaje równy $d_S = 2$ Przejście między dwoma zachowaniami z $d_H = 2$ dla czystej grawitacji ($d = 0$) i $d_H = 3$ dla $d > 1$ może być interpretowany jako przejście fazowe. Nie można jednak tego sprawdzić, nawet stosując metody numeryczne, ze względu na dyskretny charakter d. W trzeciej publikacji proponujemy zbadanie takiego przejścia w modelu, w którym wprowadzamy oddziaływanie czterech pól skalarnych z niezerową masą. Wprowadzając parametr masowy możemy "wyłączyć" pole, zamrażając polowe stopnie swobody do zera (dla dużych mas). Wykazujemy, że dla skończonej wartości parametru masowego obserwujemy przejście między dwoma reżimami. Charakter przejścia sugeruje, że jest to przejście fazowe drugiego (lub wyższego) rzędu.

Preface

The discovery of gravity goes back to Newton's theory in 17th century when the phenomenon of gravity was explained and given a mathematical foundation. For over 200 years it was believed to be a full and complete theory, describing the gravitational interactions between massive objects both on Earth and on a cosmic scale. In the beginning of the 20th century, Einstein's theoretical work on Special Relativity and General Relativity provided a different, profound understanding of space, time, mass, and a relation between the curvature of space-time geometry and the gravitational force.

Another revolutionary development in theoretical physics which also happened in the beginning of the 20th century, related to micro-physics at the atomic and molecular scale, was the discovery of Quantum Mechanics. A relation between the large-scale phenomena described by General Relativity and Quantum Mechanics at a microscopic scale is not obvious, since at a quantum scale the gravitational effects play no role. Since that time both the General Relativity and Quantum Mechanics were tested in a large number of experiments. The theoretical attempts to unify the two theories became important once it was realized that such a unification may be crucial to understand the evolution of the Early Universe at the Planck scale, where geometric fluctuations become so large that their effect cannot be neglected.

In the second half of the 20th century theoretical physics had a great success, when in the language of Quantum Field Theory it formulated a Standard Model, unifying the three existing types of interactions in Nature: weak, electromagnetic and strong. An obvious next step would be to construct a theory including also Quantum Gravity as one of it's elements. Although many attempts in this direction were made, the problem remains open to this day.

The basic instrument of theoretical physics is a perturbation theory. In a conventional approach to Quantum Field Theory, infinities which appear in the perturbative expansion around an UV fixed point [2] can be absorbed in physical quantities, using methods of the Renormalization Group. This approach does not work for a naive quantization of General Relativity. The first problem is the background geometry. In a standard Quantum Field Theory geometry provides a scene on which physical phenomena live. In General Relativity the geometry itself is a physical degree of freedom to be quantized. A separation of the metric into the background and quantum fluctuations leads to a non-renormalizable theory. The reason is that the Newton's gravitational constant G, playing the role of a coupling constant, in space-time dimension 4 has a mass dimension $[G] = -2$. As a consequence the perturbative approach can be used at most as the effective description, since the gravitational constant is small in value. From a fundamental point of view, a theory which requires introduction of infinitely many cut-off parameters cannot be accepted as fundamental.

A fundamental theory of Quantum Gravity should predict the existence of the background semiclassical geometry without the need to put in there by hand. It should probably have a non-Gaussian UV fixed point, where a perturbative theory can be defined. This idea was formulated by Steven Weinberg as a postulate of Asymptotic Safety [22]. In the IR limit, on large scales, Quantum Gravity must agree with a classical General Relativity. The basic difficulty in a construction of such a theory is the lack of experimental tools, which could provide hints on the correct ways to proceed. One is limited to use the indirect information, coming from the observations in astronomy and tests of the consistence.

Some years ago it seemed that String Theory [23] may be a tool explaining the structure of the Universe at all scales. A great theoretical effort was done in this direction. String Theory requires introducing new degrees of freedom in a form of supersymmetry, which so far were not discovered experimentally. It also seems that so far it's predictive power remains restricted.

A number of attempts was made to formulate a Quantum Theory of Geometry, in the language of Quantum Field Theory. One of the important formulations is Loop Quantum Gravity [24], where a big progress was made in recent years. The approach reformulates the basic degrees of freedom using the set of Ashtekar's variables. The interesting tests of this approach are related to the simplified theory, Loop Quantum Cosmology, with a reduced number of degrees of freedom. The results indicate the important role of quantum effects play in the evolution of the Early Universe, in particular in the existence (or it's lack) of a Big Bang.

An alternative attempt is to define a quantum theory through a regularized formulation in the language of Feynman path integrals. The idea to regularize the continuous geometry by a locally flat manifold built from simplices originates from the work of Regge [3]. It was originally a way to discretize the problems in classical General Relativity, but was soon recognized as a method to discuss path integrals, reduced to a summation over simplicial manifolds. A large number of works include papers on *spin foams* [25] and Dynamical Triangulations [4]. In this work we concentrate on the latter approach.

A standard way to proceed in theoretical physics is to start by considering simplified models, with fewer degrees of freedom or in a lower dimension. From the point of view of Quantum Gravity such a simplified approach is to study the

Quantum Gravity in two dimensions. Such a simplification has many convenient features. In the space-time dimension $d = 2$ Quantum Gravity becomes renormalizable, so one may hope that a solution in this case may, by the ϵ expansion, provide information about $d = 4$. In $d = 2$ one also may formulate a continuous Conformal Field Theory, in particular in the Euclidean approach, with imaginary time. In this case quantum theory becomes a statistical theory of random manifolds. Such a theory produces a number of important predictions, when coupled to a conformal matter with the central charge $c \leq 1$. The critical parameters in this case were derived by Knizhnik, Polyakov and Zamolodchikov [11] (KPZ exponents). These exponents become imaginary for the matter central charge $c > 1$.

Realizing the ideas of Regge, two-dimensional manifolds can be triangulated, leading to a theory of dynamical lattices, possibly with matter fields. Using a Wick rotated Euclidean formulation (Euclidean Dynamical Triangulations - EDT [4]) one has a set of theories characterized by the lattice spacing a with or without matter. In the continuum limit ($a \to 0$) one expects to obtain a Conformal Field Theory with a particular value of the central charge c and the criticality described by the KPZ exponents (at least for $c \leq 1$). For a number of physically important cases this can be explicitly checked. An important example is the case of a pure 2d EDT ($c = 0$) which can be mapped on a large-N Hermitean random matrix model which can be solved. Analytic solutions exist also for other matrix models (like the Ising model, corresponding to $c = 1/2$). Analytic methods fail for $c > 1$. A lattice theory can then be studied by numerical methods, which show that above the $c = 1$ barrier typical triangulated surfaces behave like fractal Branched Polymers (BP) with the Hausdorff dimension $d_H = 2$ and the spectral dimension $d_S = 4/3$. Another interesting case corresponds to a dimers model on a random lattice, known as the Lee-Yang edge singularity. Although the model can also be formulated as a matrix model, the critical behaviour predicted by a solution does not agree with the value resulting from the KPZ analysis. This problem will be the subject of one of the papers [12] presented in this thesis, where we explain the difference by the operator mixing.

The EDT formulation, although very interesting from a theoretical point of view, has one basic problem: the relation between the scaling for space and time is different. This is the case even for pure gravity and can be attributed to a formation of baby universes, in other words to a topological instability of the spatial Universe. This problem can be cured by restricting the class of topologies considered in a path integral to those, satisfying the requirement of causality. The lattice version of such a theory was formulated by Ambjorn and Loll and is known as Causal Dynamical Triangulation model [14].

The remaining three papers in this thesis are related to a problem of the role played by the $c = 1$ barrier in CDT models coupled to matter fields. It is not obvious if the existence of the $c = 1$ barrier is relevant for CDT. In the first paper [18] we couple d copies of the massless scalar field to a CDT geometry. In the EDT formulation this would correspond to a central charge $c = d$ and for $d > 1$ would lead to a BP phase. We show that in this case CDT models develop a semiclassical background geometry of a de Sitter-like blob with $d_H = 3$. This situations is similar to that observed in numerical studies of a four-dimensional pure CDT gravity [26] in one of it's phases .

In the second paper [19] we show that the spectral dimension for a model with $d = 4$ massless scalar fields remains $d_S = 2$, the same as for pure gravity ($d = 0$), although the nature of the background geometry (and Hausdorff dimension) changes.

The last paper [20] studies the effect of introducing the mass in the model for $d = 4$ scalar fields. One can predict that for a large mass the model behaves as pure gravity, since the scalar degrees of freedom get frozen. For a finite mass there must be a transition between the $d_H = 2$ and $d_H = 3$ regimes. We show the arguments that the transition is a continuous phase transition.

The last three papers use numerical simulations as a tool to obtain information about the behaviour of the geometry.

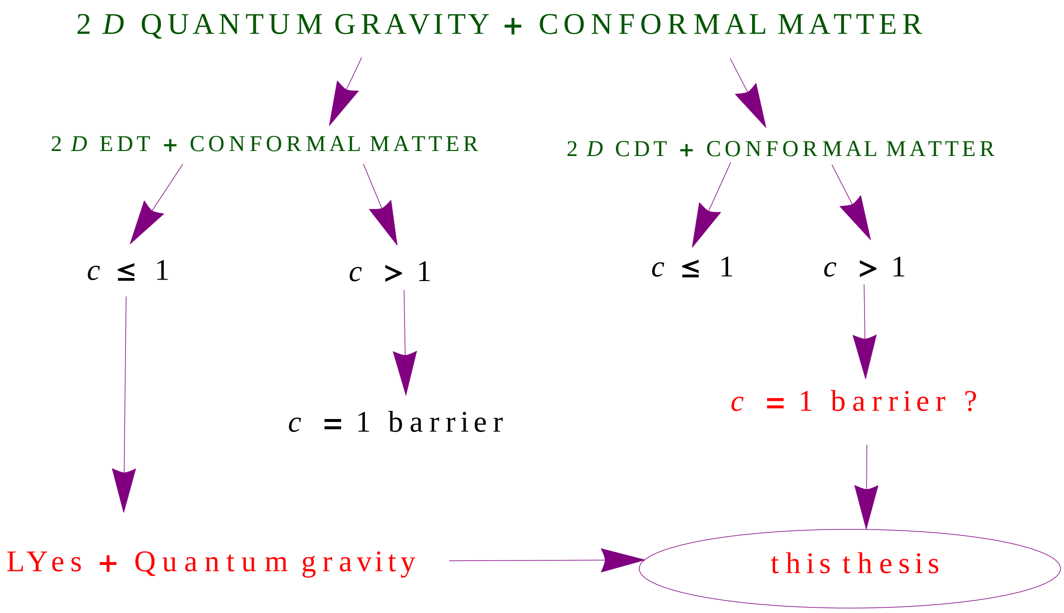

Figure 1: Thesis structure: the red one is the work of this thesis. LYes is "Lee-Yang edge singularity".

List of Publications

[1]. Pseudo-topological transitions in 2D gravity models coupled to massless scalar fields.
Authors: J. Ambjorn, A. T. Goerlich, J. Jurkiewicz, H.-G. Zhang
Nuclear Physics B 863 (2012) 421
[arXiv: hep-th/1201.1590] [18]

[2]. A note on the Lee-Yang singularity coupled to 2d quantum gravity
Authors: J. Ambjorn, A. Goerlich, A. Ipsen, H. -G. Zhang
Phys.Lett. B735 (2014) 191
[arXiv:hep-th/1406.1458] [12]

[3]. Spectral dimension in 2D Causal Dynamical Triangulations model coupled to massless scalar fields
Authors: J. Ambjorn, A. T. Goerlich, J. Jurkiewicz, H.-G. Zhang
[arXiv: gr-qc/1412.3434] [19]

[4]. A c=1 phase transition in two-dimensional CDT/Horava-Lifshitz gravity?
Authors: J. Ambjorn, A. T. Goerlich, J. Jurkiewicz, H.-G. Zhang
[arXiv: gr-qc/1412.3873][20]

Other publications

[1]. Exploring trans-Planckian physics and the curvature effect by primordial power spectrum with WMAP five-year data
Authors: Jie Ren, Xin-He Meng, Hong-Guang Zhang
Journal-ref:Int.J.Mod.Phys.D18:1343-1353 2009 [27]

[2]. Exact solutions of embedding the 4D Universe in a 5D Einstein manifold
Authors: Jie Ren, Xin-He Meng, Hong-Guang Zhang
Journal-ref:Int.J.Mod.Phys.D17:257-263 2008 [28]

Acknowledgments

- I am very grateful to my supervisor Professor Jerzy Jurkiewicz for introducing me to a fascinating topic of Quantum Gravity. For scientific inspiration, constant help and support which made this thesis possible.

- I acknowledge Professor Jan Ambjørn for help and a fruitful collaboration in a nice atmosphere and for a hospitality at the Niels Bohr Institute.

- I am specially thankful to Andrzej Görlich for the great help in research and life both in Krakow and Copenhagen.

- Last but not least, I wish to thank my family, especially my mother and my girl friend, for a support and encouraging me to finish the thesis.

Contents

Chapter 1

Quantum gravity and path integral

An internally consistent theory reconciling General Relativity (classical theory of gravity) and Quantum Mechanics, is expected to describe interactions at the scale shorter than Planck scale and to have the correct behaviour at the classical gravitational scale. The path integral method in Quantum Gravity is one of the ways to satisfy the probabilistic property of Quantum Mechanics from the beginning and to capture the geometry of classical geometric observables. From a microscopic physics to a macroscopic geometry, the path integral is supposed to provide weights for a superposition of all possible trajectories, which is the essence of Quantum Mechanics and Quantum Field Theory.

The path integral amplitude or a partition function of Quantum Gravity may be defined as a formal integral over all spacetime geometries (and matter fields)

$$Z = \int \mathcal{D}_{\mathcal{M}}[g]\, e^{i(S_{EH}[g]+S_M[g])}. \tag{1.1}$$

where $S_{EH}[g]$ and $S_M[g]$ are the Einstein-Hilbert action for geometric degrees of freedom and the action for matter fields respectively. The integral over geometries $[g]$ is over the equivalence classes of spacetime metrics g with respect to the diffeomorphism group $Diff_{\mathcal{M}}$ on manifold $\mathcal{M}$. In this thesis, we use the natural Planck units $c = \hbar = 1$. The domain of integration should specify also the class of topologies of the space-time to be included. The partition function (1.1) depends on two coupling constants G and Λ.

The form of the geometric Einstein-Hilbert action is:

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^D x \sqrt{-\det g}\,(R - 2\Lambda)\,, \tag{1.2}$$

where G is the gravitational constant, Λ is the cosmological constant, and $\mathcal{M}$ is the space-time manifold equipped with a pseudo-Riemannian metric $g_{\mu\nu}$. R

denotes the associated Ricci scalar curvature. Since the determinant of g and R are both scalars in a tensorial sense and therefore independent of the coordinate system, S_{EH} is a well-defined function. The path integral is however not well-defined, because the choice of the space of admissible geometries G and the measure on these geometries need yet to be defined.

One may use a less ill-defined theory after a Wick rotation, though it is unclear how one could give, even in principle, a coordinate-invariant prescription to go from Lorentzian to Euclidean geometries. Following the Euclidean quantum gravity approach introduced by Hawking [1], we take the Euclidean version of (1.1) as follows

$$Z = \int \mathcal{D}_{\mathcal{M}}[g]\, e^{-(S_{EH}[g]+S_M[g])}. \tag{1.3}$$

with S_{EH} and S_M being a Euclidean versions of the corresponding actions. The integral is over Riemannian metrics g_{ab} on a D-dimensional compact manifold M and the Euclidean Einstein-Hilbert action is given by

$$S_{EH}[g_{\mu\nu}] = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x \sqrt{\det g}\,(R-2\Lambda)\,. \tag{1.4}$$

The integral has a thermal system interpretation with $S_{EH}[g]$ playing the role of the energy in Boltzmann weight.

1.1 Lattice quantum gravity

One of the ways to provide a nonperturbative definition of the measure in Quantum Gravity is to introduce a UV cut-off or a space-time lattice following a similar step in QCD [2]. The difference is that now the geometry is a degree of freedom and the lattice regularization has to take this into account. In the discretized formulation the path integral becomes a sum (integral) taken over distinct triangulations of a given topological manifold M. The link lengths are fixed to the same value a, which implies that each abstract triangulation is associated with a piecewise flat geometry called a "simplicial manifold". Different geometric realizations correspond to different ways flat simplices are glued together.

The natural realization of the Einstein-Hilbert action (1.1) on piecewise linear manifolds goes back to Regge [3]. Performing the discretized path integral corresponds to a summation over the set of abstract triangulations on M with a gravitational action $S[T]$ for each piecewise linear geometry T. For pure gravity (without matter and with Euclidean time) the partition function gives

$$Z = \sum_T \frac{1}{C_T}\, \mathrm{e}^{-S[T]}, \tag{1.5}$$

where $S[T]$ is the action of geometry and C_T is a symmetry factor of the graph T, the order of the automorphism group of T. Introducing a triangulation of space-time, we did not break the diffeomorphism-invariance, since all observables can be expressed by geometric invariants (edge lengths).

The relation between the lattice version of a theory and a continuum version requires a check if the limit $a \to 0$ exists. Existence of the limit means that observables can be calculated by

$$\langle \mathcal{O} \rangle_a = \frac{1}{Z_a} \sum_T \frac{1}{C_T} \mathrm{e}^{-S[T]} \mathcal{O}[T]. \tag{1.6}$$

In the continuum limit $a \to 0$, we regularize the observables using the standard scaling relations

$$\langle \mathcal{O} \rangle_a = a^{-\Delta} \langle \mathcal{O} \rangle_{cont} + O(a^{-\Delta+1}). \tag{1.7}$$

In general such a scaling will not be possible for the lattice action $S[T]$ with an arbitrary choice of bare coupling constants. However, if there is a fixed point, the correlation lengths of certain correlators may diverge in terms of the number of lattice spacings, and this makes the scaling possible.

Chapter 2

Euclidean Dynamical Triangulations

Euclidean dynamical triangulation (EDT) model [4] is a particular regularization of the lattice quantum gravity. The integral over the space of all geometries is replaced by a sum over flat equilateral simplices (see Fig. 2.1 of flat equilateral triangles in two dimensions) with fixed edge length which is the analogue to the lattice spacing in a lattice field theory. In this case the Regge action simplifies to a linear combination of the numbers of simplices of various dimension. In the following we will restrict the discussion to the two-dimensional case. The space-time manifold is obtained by gluing the equilateral triangles (simplices) along their faces (edges of triangles). Each edge belongs to two neighbouring triangles.

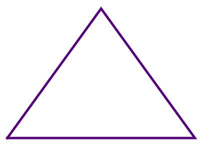

Figure 2.1: A visualization of fundamental building block for 2D EDT: equilateral triangle.

Since the path integral sums over all geometries, one needs to find a way of encoding the curvature. In 2D, the number of triangles around each vertex characterizes the local curvature at that vertex. The curvature can be expressed by the deficit angle when moving around that vertex. With equilateral triangles, flat space-time means that six triangles meet at a vertex. By removing or adding one or more triangles, the deficit angle changes and the curvature increases or decreases. In effect, the curvature is positive or negative respectively (see Fig. 2.2). There is no upper limit for the number of triangles around a vertex, but the limit exits from below. In principle the smallest number may be 1, in which case two sides of a single triangle are glued together. The case of 2 triangles means that these triangles must have the same vertices. In a standard approach these two pathological cases are rejected and one demands only "regular triangulations" with the number of triangles meeting at a vertex being 3 or more.

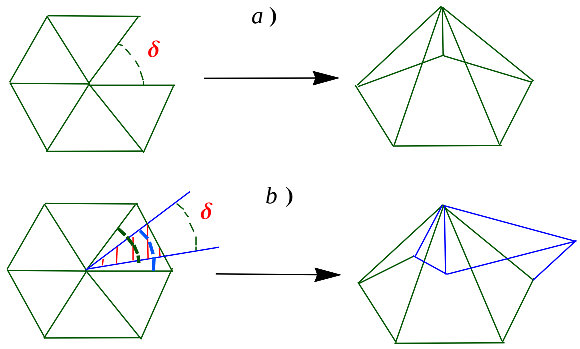

Figure 2.2: Regge angles in 2D. a): positive deficit angle. b): negetive defict angle.

For a pure gravity in 2D the sum over triangulations can be performed using a relation to a perturbative expansion of a random matrix model [6]. The essential idea in a similar context goes back to the work [5] about the large N limit of QCD . For pure gravity the sum over triangulations can be realized with a matrix model using a diagrammatic expansion (see Fig. 2.3)

$$Z = \int \mathcal{D}M \ \mathrm{e}^{-NTr(\frac{1}{2}M^2 - \frac{g}{3}M^3)}, \qquad (2.1)$$

where M is a $N \times N$ hermitian matrix and the integral is defined by the analytic continuation in the coupling g. The order g^n in a power series expansion of $N^2F = \log Z$ (free energy or a sum of connected diagrams) counts the number of diagrams constructed with n 3-point vertices. In this formulation the pathological triangulations mentioned above are included, but a finite renormalization of the coupling constant is sufficient to eliminate them and to obtain a theory containing only regular triangulations but belonging to the same universality class. The size of the random matrix N controls (through the $1/N^2$ expansion) the topological expansion in the number of genera. The leading order corresponds to planar surfaces with a spherical topology (only planar diagrams contribute).

For a fixed topology the matrix model can be analytically solved for any $g < g_c$ where g_c is a critical value of the coupling. Approaching g_c corresponds to

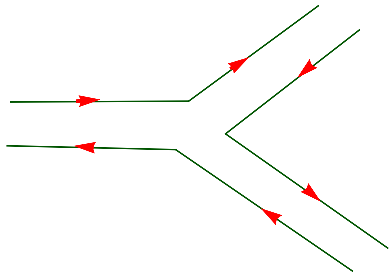

Figure 2.3: Random triangulation of pure gravity: three point vertex of matrix model.

a continuum limit $a \to 0$. The critical behaviour fully agrees with the predictions of the Liouville theory.

There is a number of important questions one can formulate, the most important being the question of universality. If we replace the term $\frac{g}{3} TrM^3$ by $\frac{g}{4} TrM^4$ in (2.1) we obtain a discretization of the 2d manifolds using squares instead of triangles. The model can be solved also in this case and the critical behaviour is the same as in the model using triangles. One can show that the generic critical behaviour is universal for a generalized model built from an arbitrary combination of polygons with positive weights g_i. This means that the continuum limit of the discretized theory is indeed that described by Liouville theory. A typical 2D random geometry is characterized by a Hausdorff dimension $d_H = 4$ and a spectral dimension $d_S = 2$. One can also study the role played by the topology of the manifold. The expansion in $1/N^2$ is an asymptotic expansion and terms in this expansion grow typically as $\Gamma(h)$ where h is the number of handles. This suggests that to get a well-behaved theory a summation over topologies should be restricted.

2.1 2D EDT coupled to matter

In a continuum, the 2D quantum gravity coupled to a minimal rational conformal field can be formulated as Liouville theory, and the analytical results exist as long as the central charge c of the conformal theory is less than or equal to 1. For a discretized theory with triangles as building blocks we can introduce matter, in the simplest version locating the matter fields at the centres of triangles (see Fig. 2.4). We expect that in the continuum limit we will get the behaviour described by a corresponding version of Liouville theory. In the general case it is not possible to solve the discretized theory, however there are some important cases, where we can also map a theory on a random matrix model. The important example is the Ising model, where the field takes two values $s = \pm 1$. In the dual formulation this corresponds to a situation where the vertices are coloured and the model can be mapped on a solvable two-matrix model with

$$Z = \int \mathcal{D}M_1 \mathcal{D}M_2 \ \mathrm{e}^{-NTr(\frac{1}{2}(M_1^2+M_2^2)+c\ M_1 M_2 - \frac{g}{4}(M_1^4+M_2^4))}, \tag{2.2}$$

where $c = \exp(-\beta)$ and β corresponds to the inverse temperature and where the building blocks are assumed to be squares. Similar model can be written for triangles. Both models have the same critical behaviour, and have two phases: the ordered phase for small β and the disordered phase for large β with a third-order phase transition between the phases at $\beta = \beta_c$. At the phase transition a critical behaviour of the model corresponds to a Liouville theory with $c = 1/2$. It can easily be shown that the model is in fact a fermionic massless Majorana model on a random surface. The continuum limit is obtained for $g \to g_c(\beta)$. The critical behaviour with $c = 1/2$ is realized for $\beta = \beta_c$.

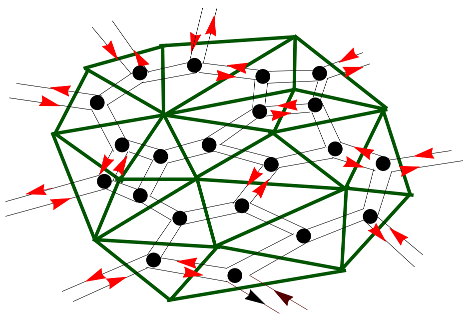

Figure 2.4: Random triangulation of a surface: each of the triangular faces is dual to a three point vertex of a quantum mechanical matrix model.

The Ising model can be solved in the case when the spins couple to a magnetic field h. For the model defined by (2.2) it would correspond to a replacement $\frac{g}{4}(M_1^4 + M_2^4) \to \frac{g}{4}(e^h M_1^4 + e^{-h} M_2^4)$.

2.2 Lee-Yang edge singularity on a random surface

The original model of the Lee-Yang edge singularity [7] is formulated in the Ising model defined above its critical temperature ($\beta < \beta_c$), where at a finite value of a non-zero, purely imaginary magnetic field $h = iH_c$ the model undergoes a phase transition. We will consider the Lee-Yang edge singularity coupled to a random surface (2D EDT). The surprising effect in this case is that the prediction of the matrix model disagrees with the KPZ exponent.

In [8] the so called hard dimers model on a random surface (EDT) was described by the partition function of a two matrix model

$$Z_{matrix} = \int \mathcal{D}\phi \mathcal{D}M \; \mathrm{e}^{NTr[-\frac{1}{2}\phi^2+\frac{1}{4}g\phi^4-\frac{1}{2}M^2+g\sqrt{\xi}M\phi^3]}, \tag{2.3}$$

where ϕ, M are $N \times N$ hermitian matrices, ξ is the dimers activity and g is the coupling constant. In the $N \to \infty$ limit only planar diagrams survive, and it is easily seen that the class of connected diagrams in order g^n corresponds precisely to the set of all planar 4-regular random graphs with n four-valent vertices, each has at most one dimers attached (see 2.5). The corresponding Feynman rules are shown in 2.6.

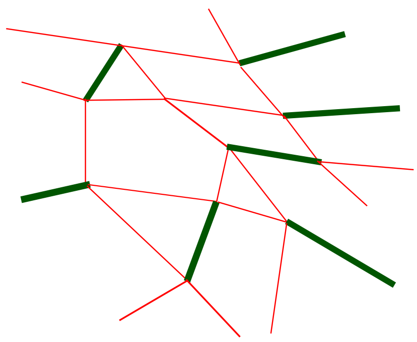

Figure 2.5: Dimers model coupled to random surface.

Performing the integral over M one obtains a one-matrix model

$$Z_{matrix} = \int \mathcal{D}\phi \; \mathrm{e}^{NTr[-\frac{1}{2}\phi^2+\frac{g}{4}\phi^4+\frac{g^2\xi}{2}\phi^6]} \tag{2.4}$$

where the irrelevant constants have been suppressed. In the paper [9], Staudacher calculated the critical exponents from (2.4) obtaining a string susceptibility $\gamma = -1/3$ and a magnetization exponent $\sigma = 1/2$ by the orthogonal polynomials method.

It was shown in [10] that the critical behaviour of the Lee-Yang edge singularity (or the hard dimer model) could be associated with the $\mathcal{M}(2,5)$ (minimal) conformal field theory. As we explain in the Appendix 1, the magnetization exponent is given by $\sigma = \frac{\Delta}{1-\Delta}$. For the Lee-Yang edge singularity on a regular lattice

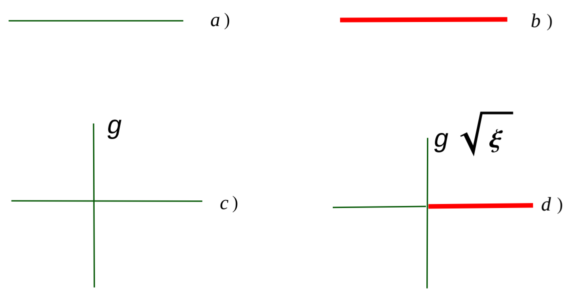

Figure 2.6: a) ϕ-propagator. b) M-propagator. c) unoccupied vertex. d) vertix occupied by dimer

with $c = -22/5$ (non-unitary model) and $\Delta_0 = -1/5$, we know the corresponding $\sigma_0 = -1/6$, which agrees with numerical determinations of σ_0.

For the case of the Lee-Yang edge singularity coupled to quantum gravity, one finds from the KPZ formula [11] in Liouville theory the string susceptibility $\gamma = -3/2$, and the dressed conformal weight $\Delta = -1/2$, and thus $\sigma = -1/3$. Clearly the values of critical exponent γ and σ are at odds with the Staudacher's result. More details of the calculation can be found in the Appendix 1.

The disagreement between the critical exponents calculated from the hard dimer model coupled to a random surface by the matrix model and in quantum Liouville theory coupled to $M(2,5)$ conformal field theory, is a threat to the identity between the lattice theory and the continuum theory. In [12], we consider the Lee-Yang "magnetization" calculated in matrix models following the progress of [13], and find the missed conformal weight and the correct string susceptibility from the contribution of the mixing operator.

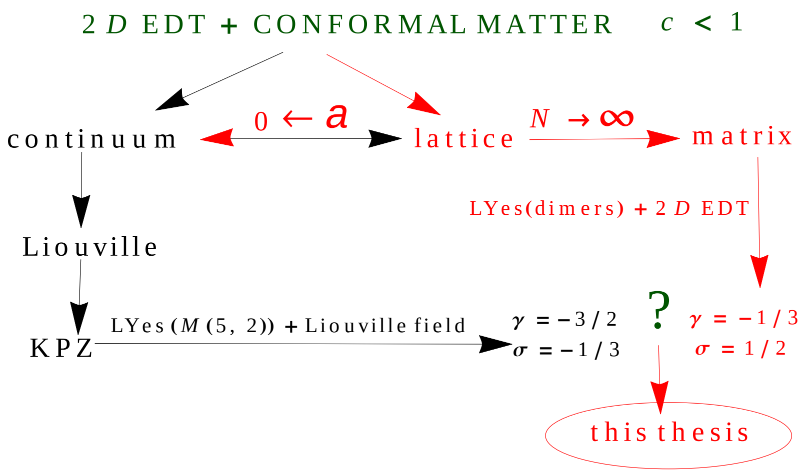

Figure 2.7: Critical exponent of Lee-Yang edge singularity coupled to a random surface: the red critical exponents γ and σ come from the random matrix model (lattice), the black from KPZ formula in Liouville theory (continuum).

Chapter 3

Causal Dynamical Triangulations

The presence of baby universes at the cut-off scale in EDT (see 3.1) is a somewhat unappealing feature, and this is one of the reasons for incorporating causality. Lorentzian quantum gravity is a hope to suppress such non-causal configurations, and thus to prevent the singularity of the time arrow. The Causal Dynamical Triangulations (CDT) model proposed in [14], is in an attempt to resemble better the causal structure present in the Lorentzian path integral, by introducing a global time foliation in the configurations of EDT.

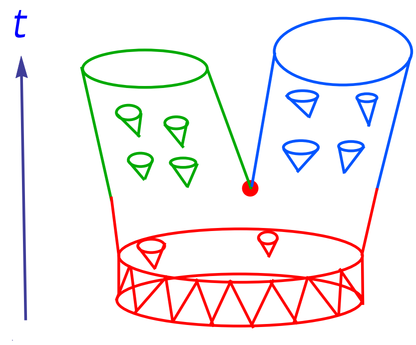

Figure 3.1: Baby universe creation and the light cone singularity. In CDT, the spatial slices are not allowed to split.

In the general case the building blocks of CDT are simplices with the lengths in the space-like directions a_s and in the time-like direction a_t. They are constant but not necessarily equal to each other (see Fig. 3.2 for the building blocks between the layers t and $t{+}1$). Let us denote the asymmetry factor between the two lengths $a_t^2 = \alpha a_s^2 = \alpha a^2$ ($\alpha < 0$ for Lorentzian case). The time in CDT is taken to be the mean proper time. Discrete Lorentzian geometries can be characterized as "globally hyperbolic" D-dimensional simplicial manifolds with a sliced structure, where $(D-1)$-dimensional "spatial hypersurfaces" are connected by suitable sets of D-dimensional simplices on each time-slice. Abstract triangulation on each time-slice can be constructed by gluing together $(D-1)$-simplices with all of spatial lengths $a_s = a$.

The standard rotation from Lorentzian to Euclidean signature in quantum field theory $t_L \mapsto -it_E$ in the complex lower half-plane implies $iS_L \to -S_E$ for the Einstein-Hilbert action. Though there is no suitable analytic continuation for a general metric, it turns out that our particular geometries in CDT do allow for a continuation from a piecewise linear geometry with Lorentzian signature to one with Euclidean signature.

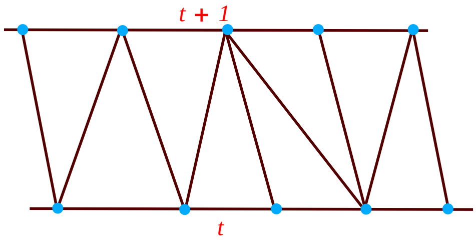

Figure 3.2: Visualization of fundamental building blocks in CDT between the foliation leaves t and $t+1$.

After a Wick rotation, the lattice quantum gravity model can be considered as a standard statistical model taking a form of the sum over a class of Euclidean geometries with positive weights (the unboundedness of the Euclidean action is suppressed in CDT [15]). Tools and techniques in statistical field theory can be applied in this Euclidean theory, and the theory of critical phenomena can be used to search for fixed points where the lattice formulation may have a continuum interpretation as in the standard theory of critical phenomena. The Lorentzian explanation of the observables requires to perform later an inverse Wick rotation, this is possible configuration by configuration but may be not obvious how to do it once the continuum theory is reached. We will not touch the problem of inverting the Wick rotation in this thesis.

We will also restrict the discussion to a two-dimensional case, where the spatial topology is that of a closed circle S^1 (see the figure 3.3). For pure gravity the model can be analytically solved [14]. We will not repeat the discussion here. Let us only mention that the properties of pure gravity in CDT are very different than those in EDT, in particular in the continuum limit. In CDT the scaling of space and time is the same, the Hausdorff dimension $d_H = 2$ and the spectral dimension $d_S = 2$, which is closer to the expected behaviour of a quantum theory than in the EDT case. For a CDT geometry we may either use a construction based on triangles, like in the original [14] paper, or an equivalent dual lattice, where vertices are located in the centres of triangles. One can interpret the time variable in this case to be shifted by half unit with respect to the direct (integer) time of the triangulated lattice. In the dual case each vertex has exactly three neighbours: two at the same time level and one at a later or earlier time. The dual formulation is very convenient for numerical simulations and for discussing a model coupled to matter, since matter fields can most naturally be located at dual vertices (see fig 3.3).

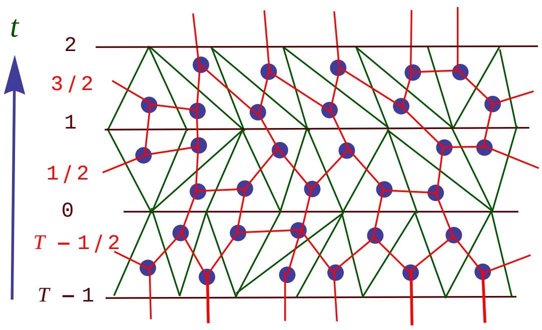

Figure 3.3: 2D CDT manifold built from triangles and it's dual lattice structure. The vertices on a dual lattice can be interpreted as lying in the half-integer times. In case we couple matter fields to the CDT geometry, the fields are located in vertices of a dual lattice.

3.1 CDT coupled to matter

The geometry of the CDT ensemble for the pure gravity case differs from the Euclidean case. Since the ensemble of CDT geometries can be viewed as a subclass of the EDT ensemble of geometries, permitting a global time foliation, it is interesting to check if a coupling to the analogue of conformal matter will show similarity to that in the EDT case. Namely we need to check what is the role played by the $c = 1$ barrier (if any). This is not obvious, since the CDT geometry is different than EDT. In the continuous Euclidean 2D Liouville theory, the KPZ exponents become complex for $c > 1$, which signals a breakdown of the concept of a surface on which the conformal matter lives. Numerical studies of the EDT lattice in this case indicated that a typical geometry becomes unstable and is characterized by that of branched polymers.

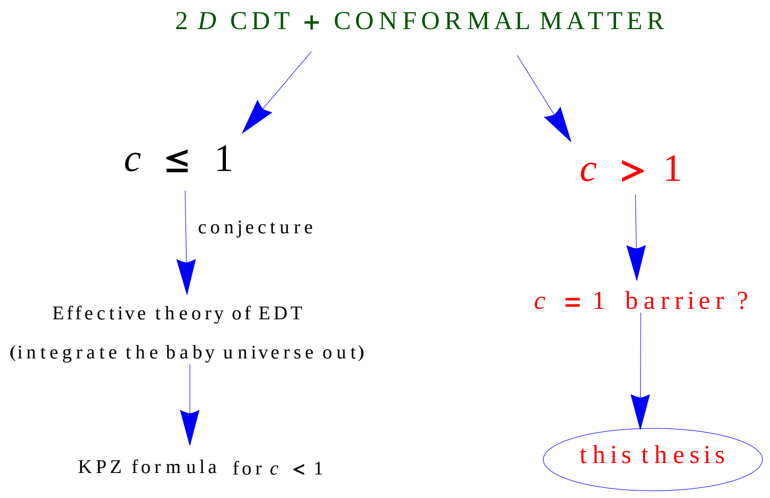

Figure 3.4: The work of 2D CDT coupled to matter in this thesis: we will mainly focus on the implications of the $c = 1$ barrier.

The question of a role the matter plays in CDT was addressed earlier in [16], where the case for $c > 1$ was studied numerically, using a model, where the CDT geometry is coupled to n copies of the Ising field. In the critical point of a similar model in EDT, each Ising field corresponds to a central charge $c = 1/2$, so if the number of copies n is larger than two we realize a system with $c = n/2$, crossing the $c = 1$ barrier. In their study the authors considered the case of 8 Ising spins (corresponding naively to $c = 4$ in EDT) coupled to a CDT geometry. it was observed that the average geometry was indeed very different from the pure gravity CDT geometry. The quantitative analysis in this case was however difficult, requiring fixing the value of the Ising coupling constant to its critical value. The methods developed later for studies of the CDT geometry in higher dimensions were also not known at that time, so the detailed analysis of scaling was not possible.

In a series of papers [18], [19] and [20] we used a different field coupled to CDT. In the paper [18], we used a system with d copies of the massless scalar field. A single scalar field for the EDT case corresponds to a central charge $c = 1$ so d copies would lead to $c = d$, again crossing the barrier for $d > 1$. Although the step in c is twice larger than for the Ising field, the advantage is that the massless scalar field is always critical and we do not have any new coupling constant. We analyzed further the properties of this system in [19]. In the last paper [20] we generalized the model to include the mass parameter for the scalar field. This new coupling constant allows one to analyse a transition between the pure gravity geometry and the geometry of CDT with the massless matter.

As we mentioned earlier, it is natural to place the matter fields at the centres of triangles (see Fig.3.3). The action associated with a given triangulation T is

$$S_m(T, x^\mu) = \sum_{\mu=1}^{d} \left[\sum_{<i,j>} (x_i^\mu - x_j^\mu)^2 + \sum_{<i>} m^2 (x_i^\mu)^2 \right] + \Lambda N(T) \tag{3.1}$$

where the summation is over d matter fields, and over the neighbouring triangles $< i, j >$. For the massless case we have $m^2 = 0$ and the action has a trivial translational zero mode. The last term is the geometric action, analogous as in pure gravity case. Λ is the bare cosmological constant. For pure gravity the field part of the action is absent.

The path integral is formally

$$Z = \sum_T \frac{1}{C_T} \int \prod_{i,\mu} dx_i^\mu \; e^{-S(T)}, \tag{3.2}$$

where C_T is a symmetry factor of the graph T . In the massles case the zero mode of the field action can be eliminated by freezing a field value of one of the vertices. The model is defined for the bare cosmological constant $\Lambda > \Lambda_c$. The approach $\Lambda \to \Lambda_c$ corresponds to a situation where the average number of vertices $\bar{N}$ of the system approaches infinity.

The model of CDT coupled to matter cannot be solved analytically. We will analyse its properties using numerical methods to be described below. Numerical methods introduce restrictions on the parameters of the model: we must consider finite volumes and a precise definition of the boundary conditions. We will always use a dual lattice to represent geometry. As was explained above each configuration in (3.2) is characterized by a particular way vertices are linked in the configuration. Each vertex has two neighbours at the same time level t and one neighbour above or below. The number of links between two time slices must be at least one, so at each time slice lies at least two vertices. The number of vertices at a slice t is denoted by $L(t)$. Each slice is assumed to have a topology of S^1 (a closed circle built from $L(t)$ links). In our simulations we assume the system to be periodic in time with a period T (see Fig. 3.5).

The global topology is in effect that of a torus ($T^2 = S^1 \times S^1$) and is preserved in numerical simulations. For each configuration we have

$$\sum_t^T L(t) = N \tag{3.3}$$

where N is the total volume (number of vertices). In our analysis we check the scaling in the continuum limit by considering systems with a fixed period T and

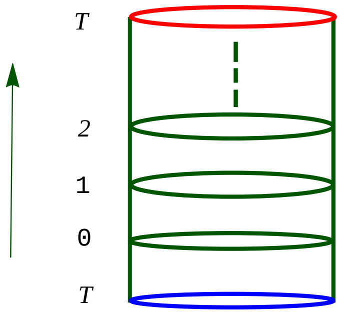

Figure 3.5: The boundary condition for simulation: torus with periodic boundary conditions.

a growing sequence of $\bar{N}$ - the average total number of (dual) vertices. To fix the total volume we use a modified action

$$S_m(T) \to S_m(T) + \epsilon(N(T) - \bar{N})^2. \tag{3.4}$$

In effect the average volume of the system fluctuates around $\bar{N}$. The parameter ϵ controls the size of fluctuations and in practice has no physical relevance.

Numerical simulations are used to produce a finite sequence of statistically independent configurations. In a Monte Carlo approach these configurations appear with probabilities

$$\mathcal{P}(T) \propto e^{-S_m(T,\ x_i^\mu)} \tag{3.5}$$

These configurations are used to get the estimate of the observable in question by averaging it's value over the set of configurations. In our analysis presented in the thesis we are interested only in geometric observables. This means that in practice the field degrees of freedom are integrated out.

3.2 Physical questions

In the first of the series of papers [18] we considered the case of 2D CDT coupled to d (d=0,1,2..) massless scalar fields. $d = 0$ corresponds to pure gravity. The observable measured in numerical experiments is the average volume profile $\langle L(t)\rangle$. The average is taken over systems with a fixed d and the periodicity T, and a sequence of volumes $\bar{N}$. This is the way to analyze the possible scaling properties of the distribution when approaching a continuum limit. We found that for $d > 1$ configurations break the time-shift invariance of the action: A typical volume distribution consists of a blob with a macroscopic volume and a cut-off size stalk, required by the boundary conditions (see Fig. 3.6). To analyze this structure we use the method applied earlier to study volume profiles in a 4D CDT for pure gravity. Individual configurations are shifted in time in such a way that the average position of the center of volume is between $T/2$ and $T/2+1$. The time $T/2+1/2$ is defined as a zero reference time $\bar{t}$. Shifted volume distributions are averaged to obtain the average profile.

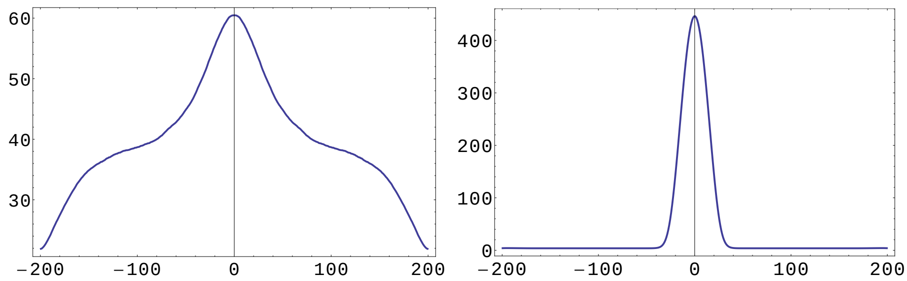

Figure 3.6: The dependence of the semi-classical volume distribution on t for 2D CDT: $d=0$ (pure gravity) and $d=4$, volume 16k. In both cases we use $T=400$.

The main result of the paper [18] is the observation that for $d>1$ the averaged profiles $\bar{N}^{1/d_H-1}L(\bar{t}/N^{1/d_H})$ are very regular. In the blob range the almost perfect universal fit corresponds to a Hausdorff dimension $d_H=3$. The blob distribution after scaling can be fitted with a function

$$f(\tau)=\alpha\cos^2(\alpha\tau),\quad \tau=\bar{t}/N^{1/3}. \tag{3.6}$$

The width α depends on d and grows with d (the distributions get more narrow). A value of a period T has no effect on the blob distribution, provided T is big enough. This is contrasted with the behaviour for $d=0$ (pure gravity). There our method of shifting the profile produces a pseudo-maximum around $\bar{t}=0$, but the averaged volume profile becomes more flat for the growing T and in fact we have to scale T together with a volume $\bar{N}$ to get a universal distribution. In this case the scaling of the volume profile corresponds to $d_H=2$. For $d=1$ we see a transition between the two regimes.

Another quantity measured in the paper is the correlator of volume fluctuations. Following the methods used in 4D CDT this permits to determine the "effective action" (or rather its second derivative), which surprisingly has a pseudo-diagonal form, suggesting the existence of the pseudo-local action and the existence of a transfer matrix. This problem will be discussed in a forthcoming publication.

The second paper [19] is devoted to a study of the spectral dimension d_S of the system described above. In the presented article we check if the spectral dimension changes from $d_S=2$ for pure gravity if massless scalar fields are present.

We analyze the model with $d = 4$ fields, where the blob structure is very well visible. The spectral dimension is measured performing, for a number of statistically independent configurations, a diffusion process, starting from a random vertex located at a slice with the maximal volume. Return probability measured in the units of a diffusion time gives information about d_S. Our result indicates that although the measured quantity has a behaviour typical for a system with a spherical geometry, a value of d_S is consistent with $d_S = 2$. This means that systems with $d > 1$ show non-trivial quantum effects and although $d_H = 3$, the spectral dimension is $d_S = 2$. Increasing d we might see a transition between the $d_H = 2$ regime of pure gravity and $d_H = 3$ regime for $d > 1$. With a discrete set of d the nature of this transition cannot be established.

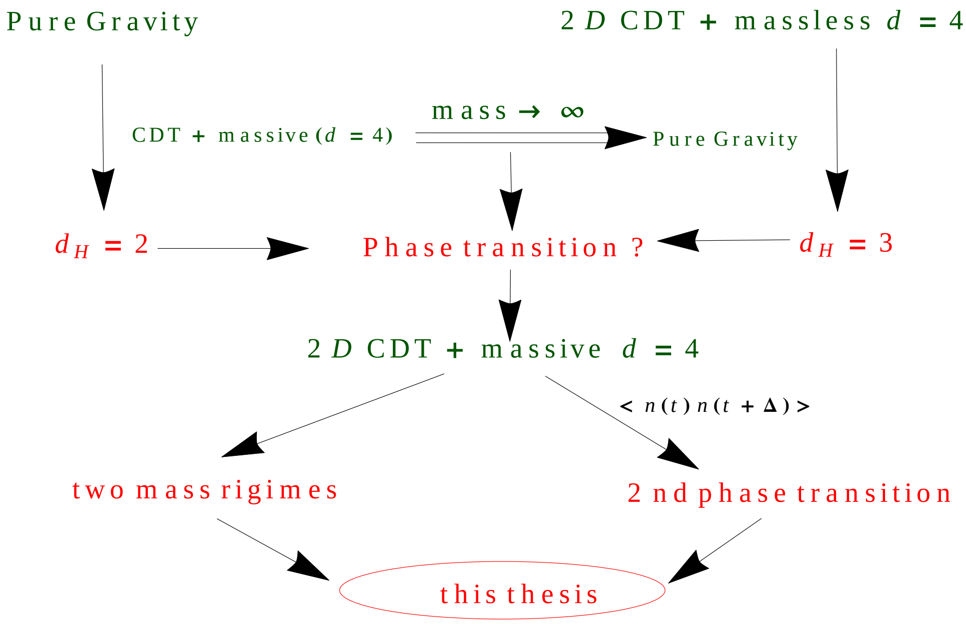

Figure 3.7: The 2D CDT part in this thesis: it is shown that the difference of d_H and d_S from pure gravity and massless $d > 1$ implies the probability of a phase transition.

In the paper [20] we try to address this question by considering a system of four scalar fields with a mass m^2, coupled with a CDT geometry. For a large mass the system approaches the pure gravity system, since field fluctuations become small. For small mass we expect a transition to the $d_H = 3$ regime. We analysed the scaling of volume profiles for a range of mass values, and indeed we see a transition at a finite value of m^2, but the nature of the transition was difficult to be determined only using the scaling of volume profiles. Qualitatively one may see a transition as the effect of the interaction between the blob and tha stalk: approaching the transition from the small mass limit, the blob gets broader, at some point the stalk disappears. We analyzed the behaviour of a volume-volume correlator defined by

$$corr(\Delta) = \langle \frac{1}{N} \sum_t L(t)L(t+\Delta) \rangle \tag{3.7}$$

for a maximal value $\Delta = T/2$. This quantity does not require fixing the center of volume. For a fixed T we expect it to have a different scaling in the two regimes, approaching a constant for small masses and growing linearly with $\bar{N}$ for large m^2. This behaviour is indeed confirmed. For a finite value of $m^2 \approx 0.135 \pm 0.005$ we observe a sharp transition, with a derivative growing as $\bar{N}^\alpha$ and $\alpha \approx 1.48 \pm 0.12$. This suggests that the transition is probably a phase transition of a higher order.

3.3 Implementation of the Monte Carlo simulation

As there are no analytic methods know to solve the model of the scalar field coupled to CDT, we use numerical method of Monte Carlo simulations to calculate expectation values of observables. The implementation of the Monte Carlo algorithm [17] works obviously only for the Wick-rotated system with a Euclidean signature. As explained above, we use the dual lattice picture with matter fields placed at the centres of triangles. As shown in Fig. 3.3, in the dual graph, each vertex has two "space-like" links (horizontal links) and one "time-like" (vertical link) pointing either up or down. The triangles are then represented as vertices and the spatial line at half-integer times has the integer length $L(t)$.

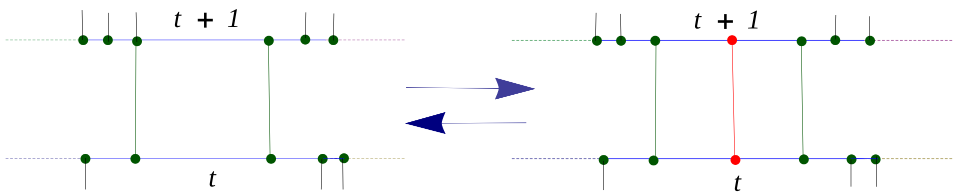

Figure 3.8: Geometric moves of 2D CDT in a dual lattice

The Monte Carlo algorithm used in simulations consists of two elements: the update of geometry and the update of fields. For the update of geometry we use a move (and its inverse) adding a new link (and two new vertices) between the slices at t and $t+1$ (see Fig. 3.8). To properly implement the Monte Carlo algorithm, the detailed balance condition must be fulfilled:

$$P[A]W[A \to B] = P[B]W[B \to A]. \tag{3.8}$$

New values of scalar fields at new vertices are generated using the heat bath algorithm. The inverse move deletes a random link joining two time layers. The restriction on this move is that the number of vertices in each layer must be at least two (one pointing up in time and one pointing down). The geometric move is repeated many times, producing a sweep of the lattice. The second element of the algorithm is the global update of fields. It is performed after each sweep, all fields are updated a number of times in a random order, using a heat bath algorithm.

Conclusions

The presented thesis covers four papers. The first concerns the problem in 2D Euclidean Dynamical Triangulations (EDT) coupled to a conformal matter. It is known under the name “the Lee-Yang edge singularity problem”. We investigated the critical exponents of the “magnetization” of the hard dimer model coupled to a random surface, and found that the contribution of a mixing operator permits to get the same critical exponents as those from the quantum Liouville theory coupled to a $\mathcal{M}(2,5)$ conformal field theory. The price of this agreement is that the naive separation between geometric and matter degrees of freedom which might seem self-evident for models of spins living on dynamical graphs can not be taken for granted. The analysis is performed using the random matrix model.

The remaining three papers discuss properties of a CDT model in two dimensions, where the quantum geometry is coupled to matter. The main conclusion we obtain is the fact that matter plays a very important role also in CDT (at least in 2D). Similar problem in the EDT formulation resulted in the instability of geometry and a development of branched polymer structures above the $c = 1$ barrier. We show that this barrier is present also in CDT, but the consequence is different: the system develops the extended de Sitter-like semi-classical background geometry with a Hausdorff dimension $d_H = 3$.

We analysed various observables in this phase, but obviously many questions remain, even in a relatively simple system. There are at least two unresolved questions. The author would like to list here and point out the most interesting:

- What is the microscopic picture of the phase transition? Is there a transfer matrix for the scalar matter coupled to CDT such as in four dimensional pure gravity? If there is, what about the eigenvalues? Is there a hint for the phase transition or the explanation of a different behaviour of volume profiles?
- How about other observables for scalar matter coupled to CDT? Can we observe the phase transition directly?

A physically relevant theory must describe matter fields. The work on incorporating multiple massive scalar fields and a massive point particle is already in progress. We developed the efficient method of calculating field propagator on a random lattice based on a diffusion process. An important question is

- How to observe gravitational interaction between matter in our model and how to establish the relation between the effective coupling constant and the conventional Newton's gravitational constant?

Appendix A

Critical exponents for Lee-Yang edge singularity

A.1 Critical exponents: conformal matter

In conformal theory, minimal model are specified by a pair of relatively prime integers (p, q). In this paper, we assume $p > q$. The central charge $c_{p,q}$ and conformal weight $h_{p,q}$ can be written as:

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \tag{A.1}$$

$$h_{p,q} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} = \frac{n^2 - (p-q)^2}{4pq} \tag{A.2}$$

where $1 \leq r \leq q-1$ and $1 \leq s \leq p-1$, and n here is $|pr - qs|$. In statistical physics, the magnetization exponent σ is given by

$$\sigma = \frac{1}{\delta} = \frac{D-2+\eta}{D+2-\eta} = \frac{\eta}{4-\eta} = \frac{\Delta}{2-\Delta} \tag{A.3}$$

where $\Delta = h + h'$ is the scaling dimension and D is the dimension of the system. For special case $D = 2$, we have

$$\sigma = \frac{h}{1-h}. \tag{A.4}$$

A.2 Critical exponents from Liouville theory

From Liouville theory, we have string susceptibility and the dressed conformal weight (see [21] for the derivation)

$$\gamma_{\text{str}} = \frac{1}{12}\left(c - 1 - \sqrt{(c-25)(c-1)}\right). \tag{A.5}$$

$$\Delta = \frac{\sqrt{1-c+24\Delta_0} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}, \tag{A.6}$$

where c is the central charge and Δ_0 is the conformal weight of the conformal field.

First, by inserting (A.1) into (A.5), we get a string susceptibility exponent for a general (p, q) model

$$\gamma_{\text{str}} = \frac{-2|p-q|}{p+q-|p-q|} = 1 - \frac{p}{q} \tag{A.7}$$

Second, by inserting (A.1) and (A.2) into (A.6), we get the dressed conformal weight for a general (p, q) model

$$\Delta_{p,q} = 1 - \frac{p+q-|pr-qs|}{2q} = 1 - \frac{p+q-n}{2q} \tag{A.8}$$

where $n = |pr - qs|$. We use the convention that $p > q$ in this thesis.

A.3 Critical exponents: Ising model and Lee-Yang edge singularity

- The critical exponents of Ising model $\mathcal{M}(4,3)$ and Lee-Yang edge singularity $\mathcal{M}(5,2)$ (non-unitary) from the conformal theory (before it was coupled to gravity) are shown in Tab. A.1.

- The critical exponents of Ising model $\mathcal{M}(4,3)$ and Lee-Yang edge singularity $\mathcal{M}(5,2)$ coupled to Liouville field from the KPZ formula are shown in Tab. A.2.

- The critical exponents of Ising model and Lee-Yang edge singularity (dimers model) coupled to a random matrix model (Staudacher's calculation) are shown in Tab. A.3

Matter	(p,q)	$c_{p,q}$	n	Δ	σ
Ising	(4,3)	1/2	$n_\sigma = 2$	$\Delta = h_\sigma$ =1/16	1/15
LYS	(5,2)	-22/5	n_σ =1	$\Delta = h_\sigma$ = -1/5	-1/6

Table A.1: Conformal weight and the scaling dimension for Ising model and Lee-Yang edge singularity.

Matter + Liouville field	γ_{KPZ}	Δ_{KPZ}	σ_{KPZ}
Ising	-1/3	1/6	1/5
LYS	-3/2	-1/2	-1/3

Table A.2: Critical exponents for conformal matter Ising model $\mathcal{M}(4,3)$ and Lee-Yang edge singularity $\mathcal{M}(5,2)$ coupled to quantum gravity from KPZ formula in Liouville theory.

Matter + Lattice	γ_{matrix}	Δ_{matrix}	σ_{matrix}
Ising	-1/3	1/6	1/5
LYS	-1/3	1/3	1/2

Table A.3: Critical exponents for conformal matter Ising model $\mathcal{M}(4,3)$ and Lee-yang edge singularity $\mathcal{M}(5,2)$ coupled to quantum gravity from matrix (critical exponent of Lee-Yang edge singularity shown here was obtained by Staudacher [9]).

Appendix B

Dimers model

Dimer models are defined on bipartite graphs which consist of vertices either black or white with edges connecting them together (see Fig.B.1 a)). Hard dimers model is the one of perfect matching without the dimers touching each other (see Fig.B.1 b)).

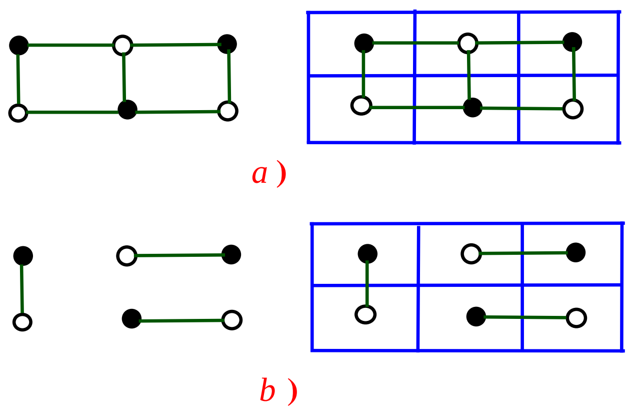

Figure B.1: a) an example of a bipartite graph, with three black vertices and three white vertices connected by edges. b) an example hard dimers model without the dimers touching each other.

Bibliography

[1] S. W. Hawking, *The Path-Integral Approach to Quantum Gravity* in *General relativity: an Einstein centenary survey* ed. S. W. Hawking and W. Israel, Cambridge University Press (1979).

[2] G. 't Hooft, M. Veltman, *One loop divergencies in the theory of gravitation,* Ann. Inst. Poincare **20**, 69 (1974). M. H. Goroff, A. Sagnotti, *The ultraviolet behavior of Einstein gravity*, Nucl. Phys. B **266**, 709 (1986).

[3] T. Regge, *General relativity without coordinates,* Il Nuovo Climento 19 (1961) 558-571.

[4] J. Ambjorn B. Durhuus, and J. Fröhlich, *Discreases of of triangulated random surface models and possible cures,* Nuclear Physics B 257 (1985) 433-449. F. David, *Randomly triangulated surfaces in 2 dimensions,* Physics Letters B 159 (1985) 303-306.

[5] G. 't Hooft, *A planar diagram theory for strong interactions,* Nucl. Phys. B **72**, 461 (1974).

[6] P. Di Francesco, P. Ginsparg, J. Zinn-Justin: *2D Gravity and Random Matrices*, Journal-ref: Phys.Rept.254:1-133,1995,[hep-th/9306153].

[7] C.N. Yang and T.D. Lee, Phys . Rev . **87** (1952) 404.
T.D. Lee and C.N. Yang, Phys. Rev . **87** (1952) 410.

[8] V.A. Kazakov : *Ising model on a dynamical planar random lattice: Exact solution*, Phys.Lett. A119 (1986) 140-144

[9] Matthias Staudacher: *The Yang-lee Edge Singularity On A Dynamical Planar Random Surface*, Nucl.Phys. B336 (1990) 349.

[10] J .L . Cardy, Phys. Rev. Lett. **54** (1985) 1354.

[11] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov: *Fractal Structure of 2D Quantum Gravity*, Mod. Phys. Lett. A **3** (1988) 819.

[12] J. Ambjørn, A. Görlich, A. C, Ipsen, H.-G. Zhang: *A note on the Lee-Yang singularity coupled to 2d quantum gravity,* Phys.Lett. B735 (2014) 191 [arXiv:hep-th/1406.1458].

[13] C. Crnkovic, P. H. Ginsparg and G. W. Moore, *The Ising Model, the Yang-Lee Edge Singularity, and 2D Quantum Gravity,* Phys. Lett. B **237** (1990) 196. *G. W. Moore, N. Seiberg and M. Staudacher,* Nucl. Phys. B **362** (1991) 665. A. A. Belavin and A. B. Zamolodchikov, *On Correlation Numbers in 2D Minimal Gravity and Matrix Models,* J. Phys. A **42** (2009) 304004 [arXiv:0811.0450 [hep-th]].

[14] J. Ambjorn and R. Loll, *Nonperturbative Lorentzian quantum gravity, causality and topology change,* Nucl.Phys.B **536** (1998) 407-434, [hep-th/9805108].

[15] J. Ambjorn, A. Goerlich, J. Jurkiewicz, R. Loll, *Nonperturbative Quantum Gravity,* Physics Reports 519, 2012, 127-210 arXiv:1203.3591

[16] J. Ambjorn, K.N. Anagnostopoulos and R. Loll: *A new perspective on matter coupling in 2d quantum gravity*, Phys. Rev. D **60** (1999) 104035 [hep-th/9904012].

[17] J. Smit Introduction to quantum fields on a lattice: a robust mate, Cambridge University Press, 2002.

[18] J. Ambjørn, A. Görlich, J. Jurkiewicz, H.-G. Zhang: *Pseudo-topological transitions in 2D gravity models coupled to massless scalar fields,* Nucl.Phys. B863 (2012) 421-434 [arXiv: hep-th/1201.1590].

[19] J. Ambjørn, A. Görlich, J. Jurkiewicz, H.-G. Zhang: *The spectral dimension in 2D CDT gravity coupled to scalar fields,* [arXiv: gr-qc/1412.3434]

[20] J. Ambjørn, A. Görlich, J. Jurkiewicz, H.-G. Zhang: *A c=1 phase transition in two-dimensional CDT/Horava-Lifshitz gravity ?,* [arXiv: gr-qc/1412.3873]

[21] F. David, *Conformal Field Theories Coupled to 2D Gravity in the Conformal Gauge,* Mod. Phys. Lett. A **3** (1988) 1651.
Conformal Field Theory and 2D Quantum Gravity Or Who's Afraid of Joseph Liouville? J. Distler and H. Kawai, Nucl. Phys. B **321** (1989) 509.

[22] S. Weinberg, *Ultraviolet divergences in quantum theories of gravitation* General relativity: an Einstein centenary survey, pp. 790-831 (1979).

[23] J. Polchinski, *String Theory*, Cambridge University Press (2005).

[24] A. Ashtekar, *New Variables for Classical and Quantum Gravity*, Phys. Rev. Lett. **57**, 2244 (1986). A. Ashtekar and J. Lewandowski, *Background independent quantum gravity: a status report*, Class. Quantum Grav. **21**, R53 (2004) [gr-qc/0404018]. C. Rovelli, *Quantum Gravity*, Cambridge University Press (2004).

[25] A. Perez, *Spin foam models for quantum gravity*, Class. Quantum Grav. **20**, R43 (2003) [gr-qc/0301113].

[26] J. Ambjorn, J. Jurkiewicz and R. Loll, *A non-perturbative Lorentzian path integral for gravity,* Phys. Rev. Lett. **85**, 924 (2000) [hep-th/0002050]. J. Ambjørn, J. Jurkiewicz and R. Loll, *Dynamically triangulating Lorentzian quantum gravity,* Nucl. Phys. B **610**, 347 (2001) [hep-th/0105267].

[27] Jie Ren, Xin-He Meng, Hong-Guang Zhang, *Exploring trans-Planckian physics and the curvature effect by primordial power spectrum with WMAP five-year data,* Int.J.Mod.Phys.D18:1343-1353 2009

[28] Jie Ren, Xin-He Meng, Hong-Guang Zhang, *Exact solutions of embedding the 4D Universe in a 5D Einstein manifold,* Int.J.Mod.Phys.D17:257-263 2008