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Philosophiae Doctor Dissertation

GEOMETRY OF NONCOMMUTATIVE THREE-DIMENSIONAL FLAT MANIFOLDS

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Contents

Introduction

Until the end of the twentieth century the ways of thinking of geometry were based on the ideas developed in the Riemannian approach. This more or less is to treat geometry as study specified by its unique object - a set locally looking like a piece of $\mathbb{R}^n$. The aim of this investigation is to give possibly the most general quantitative description of this object using the tools intended for this purpose. These include the concepts and methods developed in topology, differential geometry, algebraic topology, theory of elliptic operators etc.

A novelty was Gelfand-Naimark theorem and the observation of duality between locally compact Hausdorff spaces and commutative C^*-algebras. Having a manifold M we can define an algebra of complex valued functions vanishing at infinity. The closure of this algebra in a supremum norm, denoted $C_0(M)$, is commutative C^*-algebra. This procedure shows that for every locally compact Hausdorff space there exists a corresponding C^*-algebra. What is not trivial is that the converse is also true. Israel Gelfand and Mark Naimark in 1943 proved a theorem which states that for every commutative C^*-algebra $\mathcal{A}$ there exists a locally compact Hausdorff space X, such that $\mathcal{A} \simeq C_0(X)$. Thus there is one to one correspondence between the category of locally compact Hausdorff spaces and the category of commutative C^*-algebras. This raised a question if it is possible to redefine old topological and geometrical concepts in a new C^*-algebraical language. It was also natural to ask if those notions could be extended to the case of noncommutative C^*-algebras. The basic ground for noncommuative geometry was set.

A motivation for noncommuative geometry came also from the physics with the discovery of quantum mechanics. One of the crucial ingredients in classical mechanics is the configuration manifold, which is the set of all possible values of positions and momenta a physical system can have at the same time. Because of the uncertainty principle this notion is not well defined in a quantum world - the simultaneous determination of position and momentum does not correspond to any physical situation. Thus all concepts defined "locally", such as point, tangent space, etc., are useless in a search of a quantum counterpart to the configuration manifold. In 1930 Paul Dirac gave the description of a quantum theory in terms of the Hilbert space and a theory of operators. In this approach the position and momentum are represented by two hermitian operators $\hat{x}$ and $\hat{p}$. The classical physical variables, which are functions on a coordinate manifold, are now replaced with the observables - the selfadjoint elements of the $*-$algebra generated by $\hat{x}$ and $\hat{p}$.

The example of quantum mechanics together with the general form of Gelfand-Naimark theorem gave rise to the concept of the not necessarily commutative C^*-algebra as, roughly speaking, "the algebra of functions on a noncommutative manifold". This concept caused an interest in noncommutative C^*-algebras with possible geometrical interpretations. One idea, which we shall only mention here, was to adopt directly this quantum procedure and to deal with the algebras generated by the selfadjoint "noncommuative coordinates". This lead to various independent considerations, such as quantum groups and related Hopf algebras, fuzzy sphere, $\kappa-$Minkowski space time, doubly special relativity to mention few of them.

Revolutionary, in a story which we plot here, was the concept of spectral triples defined by Alain Connes. In the 1980's he constructed the noncommutative differential calculus and the cyclic cohomology - noncommutative counterpart to de Rham (co)homology. He tied it to the concept of Fredholm modules, which serve as a representation of the differential calculus. This, in turn, gave rise to the concept of unbounded Fredholm modules or spectral triples defined in 1994 in his book on "Noncommutative Geometry" [12, 13]. We shall not go into more details here as the rest of our dissertation serves better understanding of this notion.

Since noncommutative geometry is still very recent, at least to mathematical standards, and rapidly developing theory there is a great need for further development of its basic tools and concepts. The procedure is more or less always the same - one attempts to redefine the topological concept in the language of commutative C^*-algebras and then check whether the definition continues to make sense in the case of noncommuative C^*-algebras. Usually one also would like to have the way backwards, i.e. a way to assign (having a commutative C^*-algebra) to a given C^*-algebraical concept a unique geometrical interpretation as in the case of Gelfand-Naimark theorem. We shall now list "the basic vocabulary" of some of concepts of noncommutative geometry with its classical counterparts. We restrict our attention mostly to such notions which are connected with the computation carried out in our dissertation.

Further analogues could be found in numerous textbooks and review pa-

GEOMETRY		NONCOMMUTATIVE GEOMETRY
locally compact Hausdorff spaces	–	C^*-algebras
compact Hausdorff spaces	–	unital C^*-algebras
regular Borel measures	–	states
quotient manifolds	–	fixed point algebras
vector bundles	–	finitely generated projective modules

pers (see for example [26]).

Having a spin$^{\mathbb{C}}-$ manifold one can easily construct a commutative spectral triple, moreover one gets different spectral triples for different spin^{C} structures. In 2008 in [17] Connes proved a celebrated reconstruction theorem. It states that for any spectral triple over a commutative pre$-C^*-$algebra $\mathcal{A}$ there exists a smooth oriented and compact spin$^{\mathbb{C}}-$manifold M, such that $\mathcal{A} \simeq C^\infty(M)$. This result is a milestone in the study of noncommuative generalisation of spin structures. Classically the definition of a spin structure, which is roughly speaking a further refinement of orientability, involves such concepts as tangent spaces, Clifford bundles, double coverings of spaces etc. As we already said these concepts no longer make sense in the case of noncommutative C^*-algebras. On the other hand, spectral triple remains well defined in both commutative and noncommutative situations. The reconstruction theorem shows that the notion of spectral triple appears to be "the correct" generalisation of spin structure in an algebraic language.

In this dissertation we are mostly interested in the following correspondence:

$$\begin{array}{ccc} \text{noncommutative} & \leftrightarrows & \text{classes of equivalence of} \\ \text{spin structures} & & \text{irreducible real spectral triples} . \end{array}$$

In 2003 Andrzej Sitarz in [50] elaborated the concept of equivariant real spectral triples and later, in 2006 with Mario Paschke in [44] he proved that the number of inequivalent equivariant irreducible spectral triples over noncommutative two torus is exactly the same as the number of spin structures over topological two torus. This result was generalised for any dimension in 2010 by Jan Jitse Venselaar in [55]. We would like to investigate these notions in the case of noncommutative generalisation of flat compact orientable three-dimensional manifolds. Classically each such manifold is homeomorphic to one of six three-dimensional Bieberbach manifolds, which are quotient of three torus by the action of a finite group. The classification of spin structures over three-dimensional Bieberbach manifolds was done by Frank Pfäffle in [45] in 2000. The main aim of this dissertation is to give a noncommutative generalisation of Bieberbach manifolds and to classify the flat real spectral triples over them.

We shall now briefly sketch the main content of our dissertation. The reader would find all necessary preliminaries in Chapter 1. This include the basics of $K-$theory, Fredholm modules and $K-$homology. Chapter 2 is devoted to the presentation of the definition of spectral triple. First we recall the classical case, i.e. the definition of spin structure and the corresponding classical Dirac operator as the differential operator acting on the sections of spinor bundle. After this we shall present the formulation of the definition of real spectral triples as quite natural generalisation of the classical case. We end this part with the definition of equivariant spectral triple. Chapters 3 contains the definition of noncommutative spin structure as a class of unitarily equivalent real spectral triples. We examine this notion on the example of the noncommutative three-dimensional torus, recalling the results of Venselaar. In this chapter we shall also introduce three different definitions of the irreducibility of spectral triple, illustrating the necessity for them with a toy model $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$.

Chapter 4 opens part of the dissertation directly referring to the noncommutative Bieberbach manifolds. First we discuss their classical definition and classification in dimension three, then following Pfäffle [45] we recall the classification of spin structures over three-dimensional Bieberbach manifolds. In Chapter 5 we also propose their noncommutative generalisation, using the fact that classically Bieberbach manifolds are quotients of tori by finite groups.

In Chapter 5 we compute the $K-$theory of noncommutative Bieberbach manifolds. First we discuss the Morita equivalence of noncommutative Bieberbach spaces and the crossed product of noncommutative three torus by the action of cyclic group. The we present three methods of computation on our toy model, which is Klein bottle. After choosing the most transparent of them we proceed to computation of the $K-$theory of noncommutative Bieberbach spaces.

Chapter 6 and Chapter 7 are devoted to the computation of the spectral triples and spectral action over noncommutative Bieberbach spaces. The classification of real spectral triples over noncommutative Bieberbach spaces, which come from restriction of $\mathbb{Z}_N-$equivariant flat spectral triples over noncommutative torus is presented in Chapter 7. We discuss the three definitions of reducibility introduced in previous part. Then we show that reducibility up to bounded perturbation of Dirac operator appears to be "the best suited". The number of irreducible real flat spectral triples classified in this approach agrees with the number of classical spin structures computed by Pfäffle in each case. We also discuss the equality of the spectra of Dirac operator computed classically and through the classification of equivariant and then irreducible spectral triples. The same applies to eta invariants of Dirac operators. In the last chapter we present the computation of the spectral action. We show that its nonperturbative part equals (up to multiplication) the spectral action functional over torus, while the perturbative expansion in energy scale parameter Λ differs by a constant number proportional to the eta invariant of the $\mathbb{Z}_N-$equivariant Dirac operator.

Chapter 1

Preliminaries

While in differential geometry the main objects are manifolds, i.e. sets equipped with a suitable structure, the approach of noncommutative geometry is based on algebras. Similarly, as a set is not enough and we need to introduce topology, differentiability, and local morphisms with open set in $\mathbb{R}^n$, the algebra itself is not enough and we need to have corresponding structures. We shall begin with recalling basic facts concerning C^*-algebras. Most of proofs, which we omit here, could be found in any textbook on C^*-algebras (see for example [29]).

1.1 C^*-algebras

Definition 1.1. *We say that $\|\cdot\|\colon \mathcal{A} \to \mathbb{R}$ is a norm on an algebra $\mathcal{A}$ if $\mathcal{A}$ is a normed complex vector space and*

$$\|1\| = 1, \qquad \|ab\| \leq \|a\|\,\|b\|.$$

Definition 1.2. *An involutive algebra is an algebra $\mathcal{A}$, together with an antilinear map $^*\colon \mathcal{A} \ni a \to a^* \in \mathcal{A}$, called involution, which for all $a, b \in \mathcal{A}$ satisfies:*

$$(ab)^* = b^* a^*, \quad (a^*)^* = a.$$

Definition 1.3. *We say that $\mathcal{A}$ is a pre$-C^*-$algebra if it is a normed involutive algebra such that for all $a \in \mathcal{A}$, $\|a^*\| = \|a\|$ and C^*-identity holds:*

$$\|aa^*\| = \|a\|^2.$$

If $\mathcal{A}$ is moreover complete we say that it is a C^*-algebra.

We recall here some basic facts about C^*-algebras...

Lemma 1.4. *Let* $f\colon \mathcal{A} \to \mathcal{B}$ *be an isomorphism of involutive algebras:*

$$f : \mathcal{A} \to \mathcal{B}, \quad f(a)^* = f(a^*), \ \ \forall a \in \mathcal{A}.$$

If $\mathcal{A}$ *and* $\mathcal{B}$ *are* C^**-algebras then* f *is an isometry:*

$$\|f(a)\|_{\mathcal{B}} = \|a\|_{\mathcal{A}} \quad \forall a \in \mathcal{A}.$$

Proof. See [29]. □

An immediate consequence of that lemma is that a C^*-algebra norm is uniquely determined, i.e. if $(\mathcal{A}, \|\cdot\|_1)$ and $(\mathcal{A}, \|\cdot\|_2)$ are both C^*-algebras, then $\|a\|_1 = \|a\|_2$.

Example 1.5 (Commutative pre-C^*-algebra)**.** *We say that function* f *over a locally compact set* M *vanishes at infinity if for all* $\epsilon > 0$ *there exists a compact subset* $N \subseteq M$ *such that for any* $x \notin N$, $|f(x)| < \epsilon$. *Let* $C_0^\infty(M)$ *denote the algebra of smooth functions over* M *which vanish at infinity. Then* $C_0^\infty(M)$ *is a pre-*C^**-algebra with pointwise addition and multiplication and involution defined as*

$$(f^*)(x) = \overline{f(x)}.$$

The norm is uniquely determined by the algebraic operation to be a supremum norm:

$$\|f\| = \sup_{x \in M}\{|f(x)|\}.$$

Example 1.6 (Commutative C^*-algebra)**.** *Let us take* $C_0^\infty(M)$ *from the previous example then* $\overline{C_0^\infty(M)}$, *the* C^*- *completion in the supremum norm is a* C^*-*algebra. It is easy to see that* $\overline{C_0^\infty(M)} = C_0(M)$ *is an algebra of continuous functions vanishing at infinity.* $C_0(M)$ *is unital if and only if* M *is compact.*

Example 1.7 (C^*-algebra of bounded operators)**.** *Let* $\mathcal{H}$ *be a complex Hilbert space and let* $\mathcal{A} = B(\mathcal{H})$ *be the algebra of bounded operators over* $\mathcal{H}$. *Then we take usual multiplication and addition and for any* $T \in B(\mathcal{H})$ *define:*

- *involution as the formal adjoint of* T, *i.e.*

$$\langle T^\dagger \eta, \theta \rangle = \langle \eta, T\theta \rangle.$$

- *norm as the supremum operator norm, i.e.*

$$\|T\|_{B(\mathcal{H})} = \sup_{\langle \eta, \eta \rangle = 1}\{\sqrt{\langle T\eta, T\eta \rangle}\}.$$

Example 1.8 (Noncommutative C^*-algebra)**.** *Let us take the Hilbert space and the algebra of bounded operators from the previous example. Any algebra $\mathcal{A}$ isomorphic to the subalgebra of $B(\mathcal{H})$ which is self adjoint and norm closed is a C^*-algebra. The norm and involution of $\mathcal{A}$ has to be compatible with $B(\mathcal{H})$ through the isomorphism $\pi\colon \mathcal{A} \to B(\mathcal{H})$:*

$$\pi(a^*) = \pi(a)^\dagger, \quad ||a||_{\mathcal{A}} = ||\pi(a)||_{B(\mathcal{H})} \quad \forall a \in \mathcal{A}.$$

The latter example evokes an important concept of representation which we shall recall here.

Definition 1.9. *Let $\mathcal{A}$ be a $*-$algebra and let $\mathcal{H}$ be a Hilbert space. Then a $*-$morphism $\pi : \mathcal{A} \to L(\mathcal{H})$ is called a representation of $\mathcal{A}$ on the space $\mathcal{H}$. Moreover we shall say that representation is:*

- ***reducible*** *if and only if there exists a nontrivial π invariant subspace of $\mathcal{H}$, i.e. $\mathcal{H}' \subsetneq \mathcal{H}$ such that $\mathcal{H}' \neq 0$ and:*

$$\pi(a)\psi \in \mathcal{H}' \quad \forall a \in \mathcal{A}, \psi \in \mathcal{H}'.$$

- ***irreducible*** *if and only if the only π invariant subspaces are 0 and $\mathcal{H}$;*

- ***faithful*** *if and only if π is a monomorphism.*

1.2 Gelfand-Naimark Theorem and GNS Representation

The Gelfand-Naimark theorem states that every commutative C^*-algebra is in fact the algebra of continuous complex functions over locally compact Hausdorff space. On the other hand the Gelfand-Naimark-Segal theorem provides a method to assign to every C^*-algebra (commutative or not) a representation on the Hilbert space.

Theorem 1.10 (Gelfand-Naimark-Segal, [29])**.** *Let $\mathcal{A}$ be a C^*-algebra, then it is isomorphic to norm closed (in the operator norm) C^*-subalgebra of the algebra of bounded linear operators on the separable Hilbert space.*

We shall now briefly recall those results - first the commutative case and then the representation theorem.

1.2.1 Commutative Case

Definition 1.11. *If $\mathcal{A}$ is a complex algebra (not necessarily C^*-algebra) then any non-zero morphism $\chi\colon \mathcal{A} \to \mathbb{C}$, i.e. a multiplicative linear functional shall be called a character of $\mathcal{A}$. The set of characters of $\mathcal{A}$ shall be denoted $\Omega(\mathcal{A})$*

Lemma 1.12. *For a commutative unital complex algebra there is one to one correspondence between the set of character $\Omega(\mathcal{A})$ and the set of maximal ideals through the relation:*

$$\mathcal{I} = \ker \chi.$$

It is a well known fact that a pointwise limit of characters is again a character, thus $\Omega(\mathcal{A})$ is a closed subset of the dual space $\mathcal{A}^*$.

Lemma 1.13. *Let $\mathcal{A}$ be a C^*-algebra, then $\Omega(\mathcal{A})$ is a locally compact Hausdorff space, moreover it is compact if and only if $\mathcal{A}$ is unital.*

Now we define a Gelfand transform. We use the fact, that $\Omega(\mathcal{A})$ is a locally compact Hausdorff space and thus $C_0(M)$ is a commutative C^*-algebra.

Definition 1.14. *For any $a \in \mathcal{A}$ we define a function $\hat{a}\colon \Omega(\mathcal{A}) \ni \chi \to \hat{a}(\chi) = \chi(a) \in \mathbb{C}$. The map Γ defined by:*

$$\Gamma\colon \mathcal{A} \ni a \to \hat{a} \in C_0(\Omega(\mathcal{A}))$$

shall be called a Gelfand transform.

Theorem 1.15 (Gelfand-Naimark, [29])**.** *Let $\mathcal{A}$ be a commutative C^*-algebra and let $\Omega(\mathcal{A})$ denote the space of maximal ideals of $\mathcal{A}$. $\Omega(\mathcal{A})$ has a structure of locally compact Hausdorff space. Moreover the Gelfand transform $\Gamma\colon \mathcal{A} \to C_0(\Omega(\mathcal{A}))$ is an isomorphism of C^*-algebras.*

1.2.2 Noncommutative Case

Definition 1.16. *We say that a linear map $\phi\colon \mathcal{A} \to \mathbb{C}$ is a positive linear functional on a C^*-algebra $\mathcal{A}$ if it is positive $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. If moreover $\phi(1) = 1$ we shall call ϕ a state.*

If ϕ and ψ are states over on a C^*-algebra, then $t\phi + (1-t)\psi$ is also a state if only $0 \leq t \leq 1$. Thus the set of states is a convex set.

Definition 1.17. *Let ϕ be a state. We say that ϕ is a pure state if there are no such states ϕ_1, ϕ_2 and a real number t that $\phi = t\phi_1 + (1-t)\phi_2$.*

Lemma 1.18. *For any selfadjoint element a of a C^*-algebra $\mathcal{A}$ there exists a pure state ϕ such that $\phi(a) = ||a||$.*

Example 1.19. *$C(M)$ is a commutative C^*-algebra of complex functions over a compact manifold M. Let μ be a Borel measure on M. Then :*

$$\phi_\mu(f) = \int_M f(x)\mu(\mathrm{d}x)$$

is a positive linear functional on $C(M)$. If we assume that $\int_M \mu(\mathrm{d}x) = 1$, then ϕ_μ is a state. The only pure states on $C(M)$ are characters - for a fixed element $x \in M$:

$$\phi_x(f) = f(x),$$

which corresponds to δ_x measure, such that $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise.

Any state and pure state over a commutative unital C^*-algebra is exactly one of the described in previous example. To see this we use:

Theorem 1.20 (Riesz-Markov, [28])**.** *Let M be a compact Hausdorff space. For any state ϕ on $C(M)$ there exists a unique regular Borel probability measure μ_ϕ such that:*

$$\phi(f) = \int_M f(x)\mu_\phi(\mathrm{d}x) \quad \forall f \in C(M).$$

Thus the states on $C(M)$ and the regular Borel probability measures on M are in one to one correspondence. In particular if M is a manifold, which is a metrizable set, then the states on $C(M)$ and the Borel probability measures on M are in one to one correspondence (see [28]).

Example 1.21. *Let $\mathcal{A} = M(n, \mathbb{C})$ and choose a positive matrix p such that $\mathrm{tr}(p) = 1$. Then*

$$\phi_p(a) = \mathrm{tr}(ap)$$

is a state on $M(n, \mathbb{C})$. In physics, especially in quantum mechanics, such matrices and their infinite-dimensional analogues are called a density matrices and corresponds to quantum states of a physical system.

Lemma 1.22. *Let $\mathcal{A}$ be a C^*-algebra and let ϕ be state on $\mathcal{A}$, then:*

$$\langle a, b\rangle_\phi = \phi(b^* a)$$

defines an inner product over $\mathcal{A}$ treated as a vector space.

The set $\mathcal{I} = \{a \in \mathcal{A} : \phi(a^*a) = 0\}$ is a closed left ideal of $\mathcal{A}$. The inner product $\langle \cdot, \cdot \rangle_\phi$ is well defined on the $\mathcal{A}/\mathcal{I}$.

Lemma 1.23. *The completion of $\mathcal{A}/\mathcal{I}$ in the norm induced by the scalar product $\langle \cdot, \cdot \rangle_\phi$ is a Hilbert space. We shall denote it by $\mathcal{H}_\phi$.*

We shall now define function $m\colon \mathcal{A} \times \mathcal{A}/\mathcal{I} \ni (a, [b]) \to m(a, [b]) = [ab] \in \mathcal{A}/\mathcal{I}$. For any $a \in \mathcal{A}$ the operator $m(a, \cdot)$ is densely defined on $\mathcal{H}_\phi$. We shall denote by π_ϕ its unique extension to $\mathcal{H}_\phi$.

Lemma 1.24. *$\pi_\phi\colon \mathcal{A} \to L(\mathcal{H}_\phi)$ is a representation of $\mathcal{A}$. We shall call it a GNS representation associated to $\mathcal{A}$ by the state ϕ. Moreover π_ϕ is irreducible if and only if ϕ is a pure state.*

Example 1.25. *Let $\mathcal{A} = C(M)$ for a compact manifold M and let us take the Lebesgue measure ω, then $\phi_\omega(f) = \int_M f \mathrm{d}\omega$, where $\mathrm{d}\omega$ is a volume form. We conclude that $\phi_\omega(f^*f) = 0$ if and only if $f \equiv 0$ on a dense subset of M, so $\mathcal{H}_\omega = l^2(M)$ as a completion of $C(M)$ in the integral norm and the inner product:*

$$\langle \eta, \theta \rangle = \int_M \bar{\eta}(x)\theta(x)\mathrm{d}\omega \quad \forall \eta, \theta \in l^2(M).$$

The representation π_ω acts on $l^2(M)$ through the pointwise multiplication, i.e.

$$(\pi_\omega(f)\eta)(x) = f(x)\eta(x) \quad \forall f \in C(M), \eta \in l^2(M).$$

This representation if faithful.

Example 1.26. *Similarly $\mathcal{A} = C(M)$ but let us take $\phi_y(f) = f(y)$ for a chosen $y \in M$. As we have already said it is a pure state. From $\phi_y(f^*f) = 0$ we conclude that $f(y) = 0$, so $\mathcal{I}_y = \ker\{C(M) \ni f \to f(y) \in \mathbb{C}\}$ is a maximal ideal of $C(M)$. The $\mathcal{H}_y \simeq \mathbb{C}$ with the "inner product" :*

$$(\eta, \theta) = \bar{\eta}\theta \quad \forall \eta, \theta \in \mathbb{C}.$$

The representation of $C(M)$ on $\mathbb{C}$ is a multiplication by the value in point y:

$$\pi_y(f)\eta = f(y)\eta \quad \forall f \in C(M), \eta \in \mathbb{C}.$$

As we see this representation has to be irreducible, as it is one-dimensional. It is a general statement that for a commutative algebras the only irreducible representations are one-dimensional.

1.3 C^*-dynamical systems

Definition 1.27. *Let $\mathcal{A}$ be a C^*-algebra and let G be a compact abelian group. We say that α is an action of G on $\mathcal{A}$ if for any $g \in G$ there exists an automorphism α_g of $\mathcal{A}$, such that:*

$$\alpha_g \circ \alpha_h = \alpha_{gh} \quad \forall g, h \in G.$$

In such situation we shall also introduce the concept of a C^*-dynamical system, i.e a triple $(\mathcal{A}, G; \alpha)$. Moreover we shall define a crossed product algebra (for simplicity we shall restrict only to dicrete groups G).

Definition 1.28 ([31])**.** *Let $(\mathcal{A}, G; \alpha)$ be a C^*-dynamical system and let G be discrete. The C^*-crossed product algebra $\mathcal{A} \rtimes_\alpha G$ is defined as the enveloping C^*-algebra of $l^1(G; \mathcal{A})$, the set of all Bochner summable $\mathcal{A}$-valued functions on G equipped with the following Banach $^*-$stucture:*

$$(xy)(g) = \sum_{h \in G} x(h) \alpha_h \left(y(h^{-1} g) \right),$$

$$x^*(g) = \alpha_g \left(x(g^{-1}) \right),$$

$$\|x\| = \sum_{g \in G} \|x(g)\|.$$

The crossed product can be equally redefined as a C^*-algebra generated by pairs $(a, g) \in \mathcal{A} \times G$ with multiplication:

$$(a, g) \cdot (b, h) = (a \alpha_g(b), gh) \quad \forall a, b \in \mathcal{A}, g, h \in G.$$

We shall use this convention especially for finite groups.

For a C^*-dynamical system the action of group defines a gradation on the algebra $\mathcal{A}$. Let $\hat{G}$ be a dual group to G (the group of characters on G), for any $p \in \hat{G}$ we define:

$$\mathcal{A}(p) = \{a \in \mathcal{A} | \alpha_g(a) = p(g) a \ \forall g \in G\}.$$

Then the family $\{\mathcal{A}(p) | p \in \hat{G}\}$ satisfies (see [31]):

$$\mathcal{A}(p)\mathcal{A}(q) \subset \mathcal{A}(pq) \quad \forall p, q \in \hat{G} \quad \text{and} \quad \mathcal{A} = \overline{\bigoplus_{p \in \hat{G}} \mathcal{A}(p)}.$$

Definition 1.29. *The C^*-closure of $\mathcal{A}(e)$, where e is a neutral element of $\hat{G}$, is a C^*-algebra. We shall call it a fixed point algebra of $\mathcal{A}$ under the action of G and denote it $\mathcal{A}^G$.*

1.3.1 Fixed Point Algebras of $C(M)$

Consider a discrete group G acting on a compact manifold M. We can introduce the equivalence relation:

$$x \sim_G y \Leftrightarrow \exists g \in G | x = g \triangleright y.$$

Then we shall recall that M/G is a topological manifold consisting of orbits of G, i.e. the classes of equivalence of the relation $\sim_G$.

Let $C(M)$ be a C^*-algebra of continuous functions over a compact manifold M. Then we have the C^*-dynamical system $(C(M), G; \alpha)$ where:

$$(\alpha_g(f))(x) = f(g^{-1} \triangleright x) \quad \forall f \in C(M), g \in G.$$

Lemma 1.30. *Let $(C(M), G; \alpha)$ be a C^*-dynamical system based on the α action of a discrete abelian group G on a compact manifold M. Then we have an isomorphism of C^*-algebras:*

$$C(M)^G \simeq C(M/G).$$

Proof. For any $f \in C(M)^G$ let us define $t_f \in C(M/G)$ such that:

$$f(x) = t_f([x]) \quad \forall x \in M.$$

It is easy to see, that $t : C(M)^G \ni f \to t_f \in C(M/G)$ is an isomorphism of algebras. From the uniqueness of C^*-norm we conclude that it is also an isomorphism of C^*-algebras. □

1.4 $K-$theory in a Nutshell

Most of differential geometry is concentrate around vector budles, which start to appear early, with the construction of tangent and cotangent bundles. A natural mathematical question was to aks about classification of them. The mathematical setup, similar to a homology theory was laid by Atiyah [1] and later grew into a substantial part of mathematics.

The formulation of $K-$theory in the language of projections and projective modules made it possible to extend the tools to the realm of algebras (when it developed into algebraic $K-$theory), while it became more interesting for C^*-algebra (known as $K-$theory of operator algebras).

In such version it became a significant tool in all studies of C^*-algebras and one of the basic ingredients of noncommutative geometry. We shall briefly recall the main definitions here, as one of the results presented is the computation of K-theory groups for noncommutative Bieberbach algebras.

Let $\mathcal{A}$ be a unital C^*-algebra. We define $M_\infty(\mathcal{A}) = \bigcup_{n=1}^\infty M_n(\mathcal{A})$. There exist a natural embedding algebra morphisms of projections and unitaries from $M_q(\mathcal{A})$ to $M_{q+p}(\mathcal{A})$ for all $p, q > 0$:

- if e is a projection, i.e. $e^2 = e$, then we define:

$$M_q(\mathcal{A}) \ni e \to \mathrm{diag}(e, 0_p) \in M_{q+p}(\mathcal{A});$$

- and if t is a unitary:

$$M_q(\mathcal{A}) \ni t \to \mathrm{diag}(t, \mathbb{1}_p) \in M_{q+p}(\mathcal{A}).$$

This allows to identify any projection or unitary from $M_p(\mathcal{A})$ as am element of $M_{p+q}(\mathcal{A})$ and thus an element of $M_\infty(\mathcal{A})$.

Definition 1.31. *We say that two projections $e_0, e_1 \in M_\infty(\mathcal{A})$ are unitarily equivalent if there exists a unitary $u \in M_\infty(\mathcal{A})$ such that $ue_0u^* = e_1$. Similarly we say that e_0 and e_1 are homotopy equivalent if there exists a continuous path of projections $[0,1] \ni t \to e_t \in M_\infty(\mathcal{A})$ such that each e_t is a projection. We shall say that two projections are Murray-von Neumann equivalent if they are unitarily equivalent or homotopy equivalent.*

It is nontrivial fact that for a C^*-algebra all the above conditions are equivalent to each other. It is much easier to see that they define an equivalence relation.

Definition 1.32. *We say that two unitaries $u, v \in M_\infty(\mathcal{A})$ are equivalent if there exists a continuous path of unitaries $[0,1] \ni t \to u_t \in M_\infty(\mathcal{A})$ such that $u_0 = u$ and $u_1 = v$.*

Lemma 1.33. *The set $V(\mathcal{A})$ consisting of equivalence classes of projections $e \in M_\infty(\mathcal{A})$ is a semigroup with addition:*

$$[e] + [f] = [\mathrm{diag}(e, f)] \quad \forall e, f \in M_\infty(\mathcal{A}).$$

Definition 1.34. *Let $\mathcal{A}$ be a unital C^*-algebra then the unique Grothendieck extension of the semigroup $V(\mathcal{A})$ is denoted $K_0(\mathcal{A})$.*

Lemma 1.35. *The set of equivalence classes of unitaries in $M_\infty(\mathcal{A})$ forms a commutative group with addition defined through:*

$$[u] + [v] = [uv] \quad \forall u, v \in M_\infty(\mathcal{A}).$$

We shall denote it $K_1(\mathcal{A})$.

The above definitions are not really intuitive, and, unfortunately, so is $K-$theory. However, at least in the case of $K_0(\mathcal{A})$ we have a nice geometric picture. It uses *finitely generate projective module* over $\mathcal{A}$. Recall that a module is projective, if it is a direct summand of a free module, the latter isomorphic to $\mathcal{A}^n$ for some $n > 0$.

Lemma 1.36. *Let $\mathcal{M}$ be a finitely generated projective module over $\mathcal{A}$, then there exists a projection $e \in M_q(\mathcal{A})$ such that $\mathcal{M} \simeq \mathcal{A}^{\otimes q} e$.*

Using it, it becomes rather easy to see that K_0 group tells us about the (stable) isomorphism classes of finitely projective modules over $\mathcal{A}$. The word *stable* means that we need to consider two projective modules equivalent also when their direct sum with some free modules are isomorphic to each other.

There is no simple geometric intepretation of $K_1(\mathcal{A})$, the closest one can get is the group parametrising connected components of the group of invertible elements in $M_\infty(\mathcal{A})$.

1.5 Fredholm Modules in a Nutshell

Since (roughly speaking) $K-$theory is a *homology theory* (satisfying Eilenberg - Steenrod axioms [51, Chapter 5]), it was expected that a dual theory will have also a natural geometrical realisation. It was the ingenious step made by Atiyah, who realised that the theory dual to $K-$theory (in the case of manifolds, or commutative algebras using the language of K-theory for operator algebras) has as basic objects elliptic operators. This led Kasparov to the theory of generalised operators of that type and, consequently, to the present formulation of K-homology in the language of Fredholm modules.

Fredholm Modules and $K-$homology

Definition 1.37. *A triple $(\mathcal{A}, F, \mathcal{H})$, consisting of a pre$-C^*-$algebra, represented on a Hilbert space $\mathcal{H}$, a selfadjoint, unitary operator F, is called a Fredholm module if $[\pi(a), F]$ is a compact operator for any $a \in \mathcal{A}$. Moreover, if there exists another selfadjoint unitary operator γ, such that $\pi(a)\gamma = \gamma\pi(a)$ and $\gamma F = -F\gamma$ call the Fredholm module even, otherwise it is odd.*

The model for such construction is given (in the case of an algebra of continuous functions on a manifold) by a triple $(C^\infty(M), l^2(E), F)$. where $C^\infty(M)$ is a pre$-C^*-$algebra of smooth functions over M, $l^2(E)$ square summable sections of a hermitian vector bundle E and F - sigh of an elliptic differential operator on E. In particular, E could be the bundle of spinors and $F = \mathrm{sign}(D)$, where D is a classical Dirac operator (see Sections 2.1.4 and 2.2).

Definition 1.38. *Let $(\mathcal{A}, F_1, \mathcal{H})$ and $(\mathcal{A}, F_2, \mathcal{H})$ be Fredholm modules. We shall say that $(\mathcal{A}, F_1, \mathcal{H}) \sim_h (\mathcal{A}, F_2, \mathcal{H})$, i.e. are homotopy equivalent, if there is a continous path of operators $[0,1] \ni t \to F(t)$ such that $F(0) = F_0, F(1) = F_1$ and $(\mathcal{A}, F(t), \mathcal{H})$ is a Fredholm module for each $t \in [0,1]$. Similarly we shall say that $(\mathcal{A}, F_1, \mathcal{H}) \sim_u (\mathcal{A}, F_2, \mathcal{H})$, i.e are unitarly equivalent, if there exists a unitary operator $u \in L(\mathcal{H})$ such that $uF_0u^* = F_1$. We say that two Fredholm modules are equivalent if they are homotopy equivalent of unitarly equivalent.*

Definition 1.39. *Let us consider the set of all (even or odd) Fredholm modules over a C^*-algebra $\mathcal{A}$. Then $K^0(\mathcal{A})$ (respectively $K^1(\mathcal{A})$) is the set of its equivalence classes of even (respectively odd) Fredholm modules.*

Lemma 1.40. *The sets $K^i(\mathcal{A})$ have a group structure with addition defined by $[(\mathcal{A}, F_1, \mathcal{H}_1)] + [(\mathcal{A}, F_2, \mathcal{H}_2)] = [(\mathcal{A}, F_1 \oplus F_2, \mathcal{H}_1 \oplus \mathcal{H}_2)]$.*

That all this is well-defined is a highly nontrivial fact, for details and proofs we refer to [51].

On the Use of Fredholm Modules

Apart from the natural use of Fredholm modules as representatives of K-homology (which allows us to compute explicitly pairing with $K-$theory) Fredholm modules give interesting geometrical constructions.

First of all, using Fredholm module one can obtain a noncommutative counterpart to a graded algebra $\Omega_F^* = \bigoplus_{k=0}^n \Omega_F^k$.

We define $\Omega_F^0 = \mathcal{A}$ and Ω_F^k is defined for $k > 0$ as a linear span of operators:

$$a_0[F, a_1] \dots [F, a_k] \quad \forall a_i \in \mathcal{A},$$

with product in Ω^* defined as a product of operators.

If we define a differential:

$$\mathrm{d}\colon \Omega^k \ni \omega \to F\omega - (-1)^k \omega F \in \Omega^{k+1},$$

then $\Omega_F^*(\mathcal{A})$ becomes a cochain complex and d is a coboundary map.

1.5.1 Pairing between $K-$theory and $K-$homology

The functor which assigns to any C^*-algebra group K^i $i = 0, 1$, is homology dual to the $K-$theory functor. This allows us to formulate an abstract pairing between the $K-$theory and $K-$homology. Using the demonstrated presentations of the representatives of both groups it is, however, possible to construct an explicit formula for the pairing, which uses the index of Fredholm operators. The latter are operators on a separable Hilbert space such their kernels and cokernels are finite dimensional subspaces of the Hilbert space.

Theorem 1.41 ([11]). *Let $\mathcal{A}$ be an involutive unital algebra, $(\mathcal{A}, \mathcal{H}, F)$ a Fredholm module over $\mathcal{A}$ and for $q \in \mathbb{N}$ let $(\mathcal{A}, \mathcal{H}_q, F_q)$ be the Fredholm module over $M_q(\mathcal{A}) = \mathcal{A} \otimes M_q(\mathbb{C})$ given by*

$$\mathcal{H}_q = \mathcal{H} \otimes \mathbb{C}^q,\ F_q = F \otimes \mathbb{1}\ ,\ \pi_q = \pi \otimes \mathbb{1}.$$

(a) Let $(\mathcal{A}, \mathcal{H}, F)$ be even, with $\mathbb{Z}_2$ grading γ, and let $e \in Proj(M_q(\mathcal{A}))$. Then the operator $\pi_q^-(e)F_q\pi_q^+(e)$ from $\pi_q^+(e)\mathcal{H}_q^+$ to $\pi_q^-(e)\mathcal{H}_q^-$ is a Fredholm operator. An additive map ϕ of $K_0(\mathcal{A})$ to $\mathbb{Z}$ is determined by

$$\phi([e]) = \mathrm{Index}(\pi_q^-(e)F_q\pi_q^+(e)).$$

(b) Let $(\mathcal{A}, \mathcal{H}, F)$ be odd and let $E_q = \frac{1}{2}(\mathbb{1} + F_q)$. Let $u \in GL_q(\mathcal{A})$. Then the operator $E_q\pi_q(u)E_q$ from $E_q\mathcal{H}_q$ to itself is a Fredholm operator. An additive map of $K_1(\mathcal{A})$ to $\mathbb{Z}$ is determined by

$$\phi([u]) = \mathrm{Index}(E_q\pi_q(u)E_q).$$

1.5.2 Unbounded Fredholm Modules

As the origins of K-homology and the theorey of Fredholm modules are in fact related with the theory of elliptic operators, it is no wonder that this led to the following definition, which was the starting point of the theory of spectral triples:

Definition 1.42 ([11]). *Let $\mathcal{A}$ be a C^*-algebra represented as a bounded operators on a Hilbert space $\mathcal{H}$. An unbounded Fredholm module is $(\mathcal{A}, \mathcal{H}, D)$, where D is an unbounded selfadjoint operator with compact resolvent such that $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$.*

The role of unbounded Fredholm modules (with some extra data as the $\mathbb{Z}_2$-grading γ etc.)is just to give us Fredholm modules:

Example 1.43 ([12]). *Let* $(\mathcal{A}, D, \mathcal{H})$ *be an unbounded Fredholm module, then* $(\mathcal{A}, F, \mathcal{H})$ *is a Fredholm module if we define* $F = \operatorname{sgn}(D)$.

An unbounded Fredholm module is a prototype of a spectral triple - but there is more to spectral triples, which we shall see next.

Chapter 2

Spectral Triples

2.1 Spin Structures

In contrast to the classical differential geometry, where the notion of spin structure over a Riemannian manifold is well-established, in noncommutative geometry there is no clear and straightforward definition. One reason is that the classical picture involves principal fibre bundles with $SO(n)$ and $Spin(n)$ groups, an element, which is totally missing in the noncommutative approach.

To see the classical construction and show the links to the noncommutative case we later study, we present the key elements of the definitions and crucial lemmas. We skip most of the proofs, as they can be found in the numerous literature.

2.1.1 Clifford Algebras

Definition 2.1 (see [36]). *Let V be an n-dimensional real vector space, i.e. $V \simeq \mathbb{R}^n$, equipped with an inner product $Q : V \times V \to \mathbb{R}$, which for any $t, u, v \in V$ and $\alpha \in \mathbb{R}$ is:*

$$\begin{aligned}
&\text{symmetric}: \quad Q(u,v) = Q(v,u),\\
&\text{linear}: \quad Q(\alpha u, v) = \alpha Q(u,v), \quad Q(u+t,v) = Q(u,v) + Q(t,v),\\
&\text{positive}: \quad Q(u,u) \geq 0.
\end{aligned}$$

The Clifford algebra $\mathrm{Cliff}(V)$ *is the universal unital associative algebra over* $\mathbb{R}$ *generated by all $v \in V$ subject to the relation:*

$$v \cdot w + w \cdot v = -2Q(v,w), \quad \forall v, w \in V.$$

Remark 2.2. *Clifford algebra is, in principle, a* **real** *algebra. Its complexification,* $\mathrm{Cliff}(V) \otimes_{\mathbb{R}} \mathbb{C}$ *shall be denoted by* $\mathbb{C}\mathrm{liff}(V)$.

Example 2.3 (see [34]). *For low dimensional cases we have:*

$$\mathrm{Cliff}(\mathbb{R}) = \mathbb{C}, \quad \mathrm{Cliff}(\mathbb{R}^2) = \mathbb{H}, \quad \mathrm{Cliff}(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}.$$

In general:

$$\dim(\mathrm{Cliff}(n)) = 2^{\dim(V)}.$$

The complexified Clifford algebras are easier to handle:

$$\mathbb{C}\mathrm{liff}(\mathbb{R}^{2k}) = M_{2^k}(\mathbb{C}), \quad \mathbb{C}\mathrm{liff}(\mathbb{R}^{2k+1}) = M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}).$$

Lemma 2.4 (see [36]). *The algebras $\bigwedge^* V$ (exterior algebra of V) and* $\mathrm{Cliff}(V)$ *are isomorphic as vector spaces (though not isomorphic as algebras).*

Lemma 2.5 (see [34]). *The algebra* $\mathrm{Cliff}(V)$ *is* $\mathbb{Z}_2-$*graded,*

Proof. Let us define a linear map $\gamma : V \ni v \to -v \in V$. It extends to the automorphism $\gamma : \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$. Since $\gamma^2 = id$, there is a decomposition:

$$\mathrm{Cliff}(V) = \mathrm{Cliff}^0(V) \oplus \mathrm{Cliff}^1(V),$$

where $\mathrm{Cliff}^a(V) = \{\phi \in \mathrm{Cliff}(V) \colon \gamma(\phi) = (-1)^a \phi\}$. □

Real Conjugation over Complex Clifford Algebras

Definition 2.6. *Let us take a complexified Clifford algebra* $\mathbb{C}\mathrm{liff}(V)$ *and consider the following map:*

$$(v_1 \cdot v_2 \cdots v_k)^* = v_k \cdots v_2 \cdot v_1, \ \ \forall v_i \in V.$$

Its antilinear extension to all of $\mathbb{C}\mathrm{liff}(V)$ *is called* **an involution**.

The composition of the involution with the $\mathbb{Z}_2$-grading γ is called (in physics) **charge conjugation**, which we denote by $C(x)$. It is again, an antilinear antiautomorphism of the complexified Clifford algebra.

2.1.2 $SO(n)$ and $Spin(n)$ Groups

Let T be an element of linear transformations of V which leaves the inner product Q invariant. Such elements form a group called orthogonal group $O(n)$.

Definition 2.7. *A special orthogonal group $SO(n)$ is a subgroup of $O(n)$ consisting of elements with determinant equals 1, i.e. for any $u, v \in V$ and $T \in SO(n)$:*

$$Q(u, v) = Q(Tu, Tv) \quad \text{and} \quad \det T = 1.$$

Definition 2.8. *The group Spin*(n) *is defined as a universal covering group of* $SO(n)$*, for* $n > 2$ *(and for* $n = 2$ *we set* $Spin(2) = U(1)$*). Since the first homotopy groups of* $SO(n)$ *are*

$$\pi_1(SO(n)) = \mathbb{Z}_2,$$

then the covering is a double covering and the following is an exact sequence of groups:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1.$$

Clifford Algebra and the Spin Group

Since the Clifford algebra contains the information about the inner product on a real vector space, it comes as no surprise that its group of invertible elements may contain a lot of subgroups related to that inner product.

Definition 2.9 (see [34]). *Let us define the following subset of the Clifford algebra:*

$$Spin(V) = \{v_1 \cdot v_2 \cdot \ldots v_{2r} \in \mathrm{Cliff}(V) \colon v_i \in V,\ Q(v_i, v_i) = 1,\ \forall i = 1, \ldots 2r\}.$$

Then for $V \sim \mathbb{R}^n$*,* $Spin(V)$ *is isomorphic to the* $Spin(n)$ *group.*

If we take the complexified Clifford algebra $\mathbb{C}\mathrm{liff}(V)$ then we have:

Definition 2.10 (see [34]). *Let us define the following subset of the complexified Clifford algebra:*

$$Spin^c(V) = \{v_1 \cdot v_2 \cdot \ldots v_{2r} \in \mathbb{C}\mathrm{liff}(V) : v_i \in V \otimes \mathbb{C}, Q(\bar{v}_i, v_i) = 1, \forall i = 1, \ldots 2r\}.$$

Then $Spin^c(V)$ *is a group and we have the following exact sequence of groups:*

$$1 \to U(1) \to Spin^c(V) \to SO(V) \to 1.$$

Lemma 2.11 (see [26]). *The subgroup of* $Spin^c(V)$*, which is invariant under charge conjugation is the group* $Spin(V)$*.*

2.1.3 Representation of the Clifford Algebra

Since complexified Clifford algebra have a rather simple structure (as seen in the Example 2.3) their representation theory is straightforward:

Lemma 2.12 (see [36]). *For* $n = 2k$ *there exists a unique irreducible representation of* $\mathbb{C}\mathrm{liff}(\mathbb{R}^n)$ *on* $\mathbb{C}^{2^k}$ *whereas for* $n = 2k + 1$ *there are exactly two irreducible representations of* $\mathbb{C}\mathrm{liff}(\mathbb{R}^n)$ *on* $\mathbb{C}^{2^k}$*.*

It is easy to see that these representations provide us with the representations of the groups $Spin^c(n)$ and $Spin(n)$. To obtain in turn their irreducible representations requires little work and discussing separately the even and the odd case.

Lemma 2.13. *In the even case ($n = 2k$) the irreducible representation of* $\mathbb{C}\mathrm{liff}(\mathbb{R}^n)$ *splits into $S^+ \oplus S^-$, which are both isomorphic to $\mathbb{C}^{2^{k-1}}$ and are irreducible and inequivalent representations of $Spin^c(n)$.*

In the odd case ($n = 2k+1$) each of the irreducible representations of $\mathbb{C}\mathrm{liff}(\mathbb{R}^n)$ *gives an irreducible representation of the $Spin^c(n)$ group, which are equivalent to each other.*

The proof can be found, for instance, in [26], pp. 192-194.

What we are interested, however, is a more general construction: a representation of the Clifford algebra together with an implementation of the charge conjugation.

Definition 2.14. *Let us consider a representation ρ of the complex Clifford algebra on $\mathcal{H}$, such that there exists an antilinear operator C on $\mathcal{H}$, which implements the involution on* $\mathbb{C}\mathrm{liff}(\mathbb{R}^n)$*:*

$$\rho(x^*) = C^{-1}\rho(x)C, \quad \forall x \in \mathbb{C}\mathrm{liff}(\mathbb{R}^n).$$

It is a nontrivial fact that such C exists, for details of the explicit construction see [26], for example.

2.1.4 Spin Structures and Bundles

Clifford Algebra Bundles and Clifford Bundles

As a next step we need to have a global picture: instead of working at a fixed point of the manifold, consider all points and all Clifford algebras constructed from the bundle of tangent spaces at all $x \in M$.

Definition 2.15. *A Clifford algebra bundle over a Riemannian manifold M is an algebra bundle, such that the fibre is isomorphic to* $\mathrm{Cliff}(T_xM)$*, $x \in M$ and the pointwise product of two C^∞ sections is again a C^∞ section.*

Definition 2.16. *A Clifford bundle over a Riemannian manifold M is a real vector bundle, such that its C^∞-sections are a left module over the Clifford algebra bundle over M for the pointwise product.*

The definitions extend naturally if we take any other bundle E equipped with an inner product (instead of taking TM).

Spin structure

We begin with the classical definition of the spin structure.

Definition 2.17 (see [34]). *Let E be an oriented n-dimensional vector bundle over a Riemannian manifold M, and let $P_{SO}(E)$ be its bundle of oriented orthonormal frames. Suppose $n \geq 2$, then a spin structure on E is a principal $Spin(n)$-bundle, $P_{Spin}(E)$, together with a 2−sheeted covering*

$$\xi \colon P_{Spin}(E) \to P_{SO}(E)$$

such that $\xi(gp) = \xi_0(g)\xi(p)$ for all $p \in P_{Spin}(E)$ and all $g \in Spin_n$, where $\xi_0 : Spin(n) \to SO(n)$ is a connected double covering of $SO(n)$.

One can reformulate this definition, saying that a *spin structure* is a principal $Spin(n)$ bundle such that the associated bundle with fibre $\mathbb{R}^n$ is isomorphic with the tangent bundle [4, Proposition,3.34].

As we can see, the definition is phrased in the terms of principal fibre bundles. We shall briefly sketch how to make a link between this definition and the approach based on Clifford algebras. The element we need are vector bundles, which are associated bundles. Since there exists a 1 : 1 correspondence between principal fibre bundles over M with a structure group G and vector bundles with a fibre isomorphic to a representation space of the group G ([30]) we might study instead such vector bundles.

So, a **spinor bundle** for a given *spin structure* is defined as an associated bundle for the $P_{Spin}(E)$ principal bundle which arises from an irreducible representation of $Spin(n)$. Having such a bundle, we might recover the principal bundle, which will give us the spin structure (in the original meaning). There may be, on one hand, topological obstructions to the existence of spin structure, while - if they exist - they may be several, distinct from each other. It is shown that the existence is equivalent to the vanishing of the second Stieffel-Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$, whereas the structures are parametrized by the elements of $H^1(M, \mathbb{Z}_2)$ (see [4, Definition,3.33])

Although in the definition we take E to be any vector bundle, we are mostly concerned with $E = TM$, the tangent bundle of the manifold.

The definition extends naturally (with obvious modifications) to the case of Spin$n^{\mathbb{C}}$ group, then we have spin$^{\mathbb{C}}$-structures.

From Clifford Bundles to Spin Structures and Back

Now, the link between the description of the spin structures using the language of principal fibre bundles (or equivalently, associated bundles) and the description of Clifford modules and Clifford algebra bundles becomes obvious.

We begin by the relation with spin$^{\mathbb{C}}$ structure. It has been shown by Plymen [48]:

Lemma 2.18 ([48]). *If M is an oriented Riemannian manifold then it admits a spin$^{\mathbb{C}}$ structure if there exists a complex vector bundle S over M such that for all $x \in M$ we have that S_x is an irreducible representation space for $\mathbb{C}liff(T_xM, g)$.*

Furthermore one has that every Clifford module is, in fact, isomorphic to a twisted spinor bundle $S \otimes W$, for a complex vector bundle W. (see [4, Proposition 3.34], [25, Lemma 2.35]).

The passage to spin (and not spin$^{\mathbb{C}}$) structures is (as expected) based on real Clifford algebra bundles and real Clifford modules and is discussed in [34, 26].

For us the crucial element, however, is not the real Clifford bundle (or a real Clifford module), but the complexified Clifford bundle together with the Clifford algebra involution (or the charge conjugation operation). Once such structure exist, then we can construct a real spinor bundle and, equivalently, a spin structure over a manifold. Of course, not all Clifford modules (for complex Clifford bundles) admit the involution operation. For example, the involution in the Clifford algebra exchanges the twisted spinor bundles (which are line bundles with a fixed magnetic monopole charge c) over a two-sphere unless $c = 0$ (see [54], p.18). However, if the involution exists then we have a spin structure.

Now, let us see that by that passage we can pass from the language of principal fibre bundles into the language of Clifford bundles or modules and various *algebraic* operations on them.

It remains only one gap to be bridged: the metric. Indeed, to define a Clifford algebra one needs a Riemannian metric defined on the tangent space (or any real vector bundle). Here appears the Dirac operator, which, in an ingenious way allows to give us the Clifford algebra bundle without any reference to the metric tensor.

This motivated the axiomatic formulation of the *real spectral triples* as we shall see later.

2.2 Classical Dirac Operator

Throughout this section E is a vector bundle over a compact Riemanian manifold (M, g) and $\Gamma(E)$ is a set of smooth sections of E. The Serre-Swan theorem [52] assures that $\Gamma(E)$ is a finitely generated projective module over $C^\infty(M)$. We take $\Omega^1(M) = \Gamma(T^*M)$ to be a bimodule of differential forms over M (sections of the bundle of smooth one-forms over TM). Moreover we will use a short notation of tensor product $\cdot \otimes_\bullet \cdot$ instead of $\cdot \otimes_{C^\infty(M)} \cdot$.

Definition 2.19. *A linear connection ∇ over a vector bundle E is a linear map $\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$ such that:*

$$\nabla(f\omega) = \mathrm{d}f \otimes_\bullet \omega + f\nabla(\omega) \quad \forall f \in C^\infty(M), \omega \in \Gamma(E).$$

Definition 2.20. *Let $E = TM$, then $\Gamma(E) \simeq \Omega^1(M)$. We say that the connection ∇ is torsion free if for any $\omega \in \Omega^1(M)$:*

$$\pi(\nabla(\omega)) = 0,$$

where $\pi : \Omega^1(M) \otimes_\bullet \Omega^1(M) \ni \omega_1 \otimes_\bullet \omega_2 \to \omega_1 \wedge \omega_2 \in \Omega^2(M)$.

Definition 2.21. *We say that ∇^g, a connection over $\Omega^1(M)$, is metric if for all $\omega_1, \omega_2 \in \Omega^1(M)$:*

$$\mathrm{d}g(\omega_1, \omega_2) = (g \otimes \mathrm{id})(\omega_1 \otimes_\bullet \nabla(\omega_2)) + (\mathrm{id} \otimes g)(\nabla(\omega_1) \otimes_\bullet \omega_2).$$

To see the application of the above definitions, let us give an example:

Example 2.22. *If $\Omega^1(M)$ is a free module over $C^\infty(M)$ (which means that the cotangent bundle is trivial), with the basis ω^i, $i = 1, 2, \dots n$, then any linear connection is fully determined on the generating one forms:*

$$\nabla(\omega^i) = \sum_{j,k} \Gamma^i_{jk} \omega^j \otimes_\bullet \omega^k.$$

We shall call functions Γ^i_{jk} Christoffel symbols.

A linear connection with all $\Gamma^i_{jk} = 0$ is called *flat*. In terms of Christoffel symbols we say that connection is torsion-free is those symbols are symmetric, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$, and that it is metric if $\partial_i g^{jk} = g^{jl}\Gamma^k_{li} + g^{kl}\Gamma^j_{il}$.

Lemma 2.23. *Let ∇ be a torsion-free metric connection over a cotangent bundle T^*M, then ∇ is unique. We shall call ∇ a Levi-Civita connection.*

The uniqueness of Levi-Civita connection is easily expressed through an explicit formula for the Christoffel symbols (valid globally in the case of a trivial cotangent bundle):

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{li}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$

From now on if not stated otherwise we shall always assume here that the connection is torsion-free and any connection ∇^g is assumed to be the Levi-Civita connection. Though in principle we work with real bundles and real-valued functions, all considerations extend naturally to the complex bundles (sections of which are modules over complex-valued functions), hermitian connections etc.

Recall that in previous section we have defined a spinor bundle as an irreducible representation of Clifford bundle $\mathbb{C}\mathrm{liff}(TM)$. We assume that the manifold is spin and denote the (complex) bundle of spinors $\mathcal{S}(M)$.

Let $\Gamma(\mathcal{S})$ be the set of smooth sections of $\mathcal{S}(M)$. Introducing the following inner product of two sections ψ, ϕ,

$$(\phi, \psi) = \int_M \langle \phi(x), \psi(x) \rangle \mathrm{d}\omega,$$

where $\langle \phi(x), \psi(x) \rangle$ is a standard inner product in the fibre S_x over point $x \in M$. We shall denote by $\Sigma(M)$ the closure of $\Gamma(\mathcal{S})$ in the resulting norm.

Since the Clifford bundle and the bundle of differential forms are isomorphic (as vector bundles) we shall use in the construction the canonical isomorphism map $c : \Omega^*(M) \to \mathbb{C}\mathrm{lif}(TM)$ called *Clifford multiplication.* Next we can have:

Definition 2.24. *We say that* $\nabla^S \colon \mathcal{S}(M) \to \Omega^1(M) \otimes_\bullet \mathcal{S}(M)$ *is a* **spin connection** *if it is a connection over the vector bundle* $\mathcal{S}(M)$*,*

$$\nabla(f\psi) = f\nabla^S(\psi) + \mathrm{d}f \otimes_* \psi \quad \forall f \in C^\infty(M), \psi \in \mathcal{S}(M),$$

and for all $\omega \in \Omega^1(M), \psi \in \mathcal{S}(M)$ *the following condition is met:*

$$\nabla^S(\gamma(\omega) \cdot \psi) = ((\mathrm{id} \otimes_\bullet \gamma)\nabla^g(\omega))(\psi) + \gamma(\omega) \cdot \nabla^S(\psi).$$

The existence of the spin connection for spin manifolds is a consequence of the fact that the spin structure data amount to lifting the structure group to *Spin*(n). Finally we have:

Definition 2.25. *Let us define an operator* D *on the smooth sections* $\psi \in S(M)$ *as:*

$$D\psi = (c \otimes_\bullet \mathrm{id})\nabla^S(\psi) \quad \forall \psi \in S(M).$$

The closure of D *in* $\Sigma(M)$ *is called a Dirac operator.*

Lemma 2.26 ([34], p.117). *The Dirac operator over a Riemannian manifold is formally selfadjoint,i.e.:*

$$(D\phi, \psi) = (\phi, D\psi) \quad \forall \phi, \psi \in \Sigma(M).$$

Moreover $\ker D = \ker D^2$ *and is finite dimensional.*

Theorem 2.27 ([34]). *Let M be a closed Riemannian manifold and denote by $\mathcal{S}p(D)$ the spectrum of its Dirac operator. Then the following holds:*

- *The set $\mathcal{S}p(D)$ is a closed subset of $\mathbb{R}$ consisting of an unbounded discrete sequence of eigenvalues.*
- *Each eigenspace of D is finite dimensional and consists of smooth sections of $S(M)$.*
- *The eigenspaces of D form a complete orthonormal decomposition of $\Sigma(M)$,*
- *The set $\mathcal{S}p(D)$ is unbounded on both sides of $\mathbb{R}$ and, if moreover $n \neq 3 \bmod 4$, then it is symmetric about the origin.*

Finally, let us state a theorem on the asymptotical behaviour of the eigenvalues:

Theorem 2.28. *Let M be a closed Riemannian manifold of dimension n. Then for large k, the k-th eigenvalue, λ_k of $|D|$ (ordered in a non-decreasing sequence, counted with multiplicities) is related to k through the Weyl formula [25]:*

$$k \sim (\lambda_k)^{\frac{n}{2}} \mathrm{Vol}(M).$$

2.3 Real Spectral Triple – Definition

The definition of *what is the spectral triple* appears not to be fixed yet. The first idea, which appeared in late 1980s [11] used rather the notion of *unbounded Fredholm modules*. Later, some more precise definitions appeared, to be finally presented by Connes [13] as a set of *axioms*.

For a long time, this formultion was accepted, however - apart from the classical case of Riemannian spin manifolds there were very few examples of objects satisfying all of the axioms.

Moreover, attempts to prove the reconstruction theorem (show that any commutative spectral triple satisfying all the axioms necessarily **is** the one coming from a Riemannian spin manifold) were unsuccesful for a long time. It appears that the final version of the theorem, proved by Connes used a slightly changed version of some axioms.

For completeness we present all original axioms, stressing their importance for our results and the limits of their applications. We alway assume that we are dealing with unital algebras.

2.3.1 Axioms

We will consider a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of the unital pre$-C^*-$algebra $\mathcal{A}$ faithfully represented as bounded operators on a separable Hilbert space $\mathcal{H}$ and operator D called a Dirac operator.

Axiom (0)**.** *We will demand of D to be unbounded and selfadjoint to ensure that D^2 is a positive operator. Moreover, we assume the kernel of D has a finite dimension.*

We set D^{-1} to be the inverse of D restricted to orthogonal complement of the kernel of D. In many cases one can restrict oneself to the situation when D has an empty kernel.

Axiom (I. - Dimension)**.** *The operator D has a compact resolvent, i.e. D^{-1} is a compact operator. Then its spectrum is discrete.*

Furthermore, there exists a nonnegative integer n called a metric dimension *of spectral triple, such that series of λ_k the eigenvalues of $|D|^{-1}$ arranged in a decreasing order are:*

$$\lambda_k \in O(k^{-n}).$$

This uniquely determines number n. If n is even we will call spectral triple even or graded, respectively odd or ungraded if n is odd. Note that although we assume here that $n \in \mathbb{Z}, n \geq 0$, there are known examples of spectral triples with fractional metric dimensions (in particular, over fractal sets).

Axiom (II. - Regularity/ Smoothness)**.** *For all $a \in \mathcal{A}$ operators $[D, a]$ are bounded and moreover for a derivation $\delta(T) = [|D|, T]$ we demand that both a and $[D, a]$ belong to $\mathrm{Dom}(\delta^m)$ for any integer m.*

The first condition is a part of the most known requirement for the spectral triple: that the commutators with D are bounded. Classically, this is (roughly speaking) equivalent to the fact that the algebra consists at least of functions, which are differentiable.

The remaining part assures that, in fact, we should be dealing with an algebra of smooth functions.

Axiom (III. - Finiteness)**.** *The algebra $\mathcal{A}$ is a pre$-C^*-$algebra. The space of smooth vectors $\mathcal{H}^\infty = \cap_m \mathrm{Dom}(|D|^m)$ is a finitely generated projective module over $\mathcal{A}$. Moreover it bears a Hermitian structure:*

$$\langle \eta, \theta \rangle = \int_{Dix} (\eta, \theta)|D|^{-n},$$

where $\int_{Dix}$ is a noncommutative ingeral (expressed in terms of Dixmier trace).

Axiom (IV - Odd and even spectral triples). *A spectral triple is called* even *if there exists an operator* $\gamma = \gamma^\dagger$*, such that* $\gamma^2 = 1$*,* $[\gamma, a] = 0$ *for each* $a \in \mathcal{A}$ *and* $D\gamma + \gamma D = 0$*. In other words* γ *is a* $\mathbb{Z}_2$*-grading of the Hilbert space, such that all* $a \in \mathcal{A}$ *are even and* D *is an odd operator.*

If there is no γ*, we call the spectral triple* odd.

Axiom (V - Reality condition an KR-dimension). *There exists an antilinear unitary element* J*, such that:*

$$J^2 = \zeta_J, \quad JD = \zeta_D DJ, \quad J\gamma = \zeta_\gamma J\gamma,$$

where ζ_J*,* ζ_D *and* ζ_γ *equals* ± 1 *depending on* $\mathbb{N} \ni n_{KR} \bmod 8$ *and are listed in the table below:*

n_{KR} *mod 8*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
ζ_D	+	−	+	+	+	−	+	+
ζ_J	+	+	−	−	−	−	+	+
ζ_γ	+		−		+		−	

The number n_{KR} *is even for even spectral triples (with* γ*) and odd for the odd ones. It is called* KR*- (or* K*-) dimension and is determined modulo* 8.

Furthermore, we assume that for any $a, b \in \mathcal{A}$ *the following identities are true:*

$$[a, JbJ^{-1}] = 0,$$

$$[[D, a], JbJ^{-1}] = 0.$$

The first condition means that conjugation by J maps the algebra $\mathcal{A}$ to its commutant, while the second one states that it is, at the same time, commutant of the one forms. Classically this enforces that D is a differential operator of the first order.

Apart from the commutative spectral triples, 0-dimensional spectral triples and the noncommutative tori there are very few genuine noncommutative examples satisfy that part of definition.

Out of the examples based on quantum spaces only the spectral triple over the standard Podles sphere [22] satisfies the reality axiom (with the order one condition). In other case cases the order-one is satisfied *almost* - that is - up to compact operators.

It is known that in many examples the metric dimension and the KR-dimension might be different (for example, for finite matrix algebras, or the standard Podles sphere). Classically, both are equal to the dimension of the manifold.

Axiom (VI. - Orientability)**.** *There exists a Hochschild cycle $c \in \mathcal{A} \otimes \mathcal{A}^0 \otimes \mathcal{A}^{\otimes n}$ such that $b(c) = 0$ and for n even $\pi_D(c) = \gamma$, for n odd $\pi_D(c) = 1$.*

Let us recall Hochschild $k-$chain is defined as an element of $C_k(M, \mathcal{A}) = M \otimes \mathcal{A}^{\otimes k}$, where M is a bimodule over $\mathcal{A}$. A boundary map is defined through:

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_k) = & ma_1 \otimes a_2 \cdots \otimes a_k + \\ & \sum_{i=1}^{k-1} (-1)^i m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k + \\ & (-1)^k a_k m \otimes a_1 \otimes \cdots \otimes a_{k-1}. \end{aligned} \tag{2.1}$$

As $b^2 = 0$ we conclude that (C_k, b) is a chain complex. The representation of c on the Hilbert space $\mathcal{H}$ is defined as:

$$\pi_D(m \otimes a_1 \otimes \cdots \otimes a_k) = m[D, a_1] \ldots [D, a_k].$$

Axiom (VII. - Poincaire duality)**.** *The additive pairing with the K-theory, $K_i(\mathcal{A})$, determined by the index map of D, is nondegenerate.*

This axiom, though satisfied classically, is, apart from the zero-dimensional examples and the noncommatative tori most difficult to verify and most difficult to satisify.

Axiom (VIII. - Irreducibility)**.** *There is no nontrivial subspace of $\mathcal{H}' \subsetneq \mathcal{H}$, such that $(\mathcal{A}, D, \mathcal{H}', J, \gamma)$ is itself a real spectral triple.*

This axiom, as we shall see, requires either some additional data or needs reformulation.

Definition 2.29. *The set $(\mathcal{A}, , D\mathcal{H}, J)$ consisting of not necessarily commutative pre$-C^*-$algebra $\mathcal{A}$, its representation on a Hilbert space $\mathcal{H}$, Dirac operator D and a real structure operator J will be called a spectral triple if it fulfils the axiom 0. and axioms from I. to VIII.*

In our work we shall be considering algebraic real spectral triples in the following sense:

Definition 2.30. *An algebraic real spectral triple, is an object consisting of a dense subalgebra $\mathcal{A}$ of a C^*-algebra, faithfully represented on a separable Hilbert space, together with D, J (and possibly γ, such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and axioms I, IV and V are satisfied.*

2.3.2 Commutative Real Spectral Triples

As we already said in this dissertation we are especially interested in the correspondence between the irreducible real spectral triples and spin structures. Let us recall that Gefland-Naimark theorem provides a one to one correspondence between the locally compact Hausdorff spaces and commutative C^*-algebras. Similar theorem which would connect spin structures and real commutative spectral triples would be the Holy Grail of noncommutative geometry - to be precise - of noncommutative geometry described in the language of spectral triples. There are two steps of this research. First we need to show that for any spin manifold with a given *spin* structure one can construct a corresponding real commutative spectral triple. Then there is a question of reconstruction procedure, i.e. one need to prove that having a real commutative spectral triple there exists a corresponding spin manifold with a uniquely determined spin structure. The second step is much more difficult. Note that as each spin or spin$^{\mathbb{C}}$ manifold is a locally compact Hausdorff space the reconstruction theorem for spectral triples would be refinement of Gelfand-Naimark result.

The problem is very complex and we shall not go into much details. We shall only sketch the main results.

Theorem 2.31 ([26], Definition 11.1, Theorem 11.1)**.** *Let $\mathcal{S}(M)$ be a spinor bundle (which is uniquely determined by the spin structure $P_{Spin}(TM)$) on a compact spin manifold without boundary. Then $(C^\infty(M), D, \Sigma(M), C)$, if M is odd-dimensional, and $(C^\infty(M), D, \Sigma(M), C, \chi)$, if M is even-dimensional, is a commutative real spectral triple, where:*

- *$C^\infty(M)$ is a pre$-C^*-$algebra of smooth complex functions over M;*
- *$\Sigma(M)$ is a Hilber space completion of smooth square summable sections of $\mathcal{S}(M)$ (see Section 2.2);*
- *D is a classical Dirac operator (see Definition 2.25);*
- *C is a charge conjugation operator over $\mathbb{C}\mathrm{liff}(Tm)$ (see Definition 2.14);*
- *if dimension of M is even then χ is a chirality operator (see Lemma 2.13).*

First let us note that in the axiomatic definition the components of a real spectral triple were tacitly assumed to be a noncommutative generalisation of classical Dirac operator, spinor bundle, charge conjugation etc. Thus the construction of commutative spectral triple for a given spin manifold is nothing else but the exemplification of its definition.

Remark 2.32. *The way backward is, as we already said, much more difficult. The most up to date result in this matter is so called reconstruction theorem proved by Connes in 2008. Consider a commutative pre*$-C^*-$*algebra* $\mathcal{A}$ *and* $(\mathcal{A}, D, \mathcal{H})$ *be a spectral triple fulfilling certain technical assumptions (for details see [17]), Connes proved, that there exists a compact smooth spin*$^{\mathbb{C}}$ *manifold* M *such that* $\mathcal{A} \simeq C^\infty(M)$. *This result justifies the concept of real spectral triple as a generalisation of spin structure.*

Chapter 3

Noncommutative Spin Structures

In this section we shall discuss further concepts connected with spectral triples. The most important is the noncommutative generalisation of spin structure. This concept involves the notions of reducibility and equivalence of spectral triples. Both notions are still quite unspecified in literature and we shall use our examples (i.e. a noncommutative three torus, a toy model discussed at the end of this section and Bieberbach manifolds in the next chapters) to discuss the possibilities. The second concept is the G-equivariant spectral triple. We shall restrict our attention only to the odd spectral triples. The even case is straightforward.

3.1 Noncommutative Spin Structure

Equivalence of spectral triples

The most intuitive concept of equivalence is the following one.

Definition 3.1. *Consider two spectral triples $(\mathcal{A}, D, \mathcal{H}, J)$ and $(\mathcal{A}, D', \mathcal{H}', J')$. We shall say that $(\mathcal{A}, D, \mathcal{H}, J)$ is unitarily equivalent to $(\mathcal{A}, D', \mathcal{H}', J')$ if there exists a unitary operator $t\colon \mathcal{H} \to \mathcal{H}'$ and σ an automorphism of $\mathcal{A}$, such that:*

$$tDt^{-1} = D', \quad tJt^{-1} = J'$$

$$\text{and} \quad t\pi(a)t^{-1} = \pi'(\sigma(a)).$$

In order to express the unitarily equivalence we shall write:

$$(\mathcal{A}, D, \mathcal{H}, J) \simeq_t (\mathcal{A}, D', \mathcal{H}', J').$$

The other notion specific for the noncommutative algebras is the following definition of internal perturbation.

Definition 3.2. *Let $(\mathcal{A}, D, \mathcal{H}, J)$ be a spectral triple, then we shall call*

$$D_A = D + A + \zeta_D J^{-1} A J$$

an internal perturbation of Dirac operator, where ζ_D depends on the dimension of the spectral triple, and

$$A = \sum_i \pi(a_i)[D, \pi(b_i)],$$

for $a_i, b_i \in \mathcal{A}$.

Although in our computation we shall use another concept of perturbation:

Definition 3.3. *Let $(\mathcal{A}, D, \mathcal{H}, J)$ be a real spectral triple. Consider a bounded operator $A \in B(\mathcal{H})$ such that:*

$$[\pi(a), A] = 0 \quad \forall a \in \mathcal{A}.$$

If $(\mathcal{A}, D + A, \mathcal{H}, J)$ is a real spectral triple, then we shall call $D' = D + A$ a bounded perturbation of Dirac operator D.

Reducibility

The most common and the strongest is the following definition of reducibility.

Definition 3.4. *The odd spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ is called reducible (in a strong sense) if there exists a subspace $\mathcal{H}' \subsetneq \mathcal{H}$ such that $\mathcal{A}$ is faithfully represented on $\mathcal{H}'$ and $(\mathcal{A}, D|_{\mathcal{H}'}, \mathcal{H}', J|_{\mathcal{H}'})$ is an odd spectral triple. Then we shall call $(\mathcal{A}, D|_{\mathcal{H}'}, \mathcal{H}', J|_{\mathcal{H}'})$ a reduction of $(\mathcal{A}, D, \mathcal{H}, J)$.*

In the literature also another two definition can be found.

Definition 3.5. *We shall say that an odd real spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ is $J-$reducible if there exists a subspace $\mathcal{H}' \subsetneq \mathcal{H}$ on which $\mathcal{A}$ is faithfully represented, $J\mathcal{H}' \subset \mathcal{H}'$ and moreover there exists an operator D' defined on a dense subspace of $\mathcal{H}'$ such that $(\mathcal{A}, D', \mathcal{H}', J|_{\mathcal{H}'})$ is a real spectral triple.*

Definition 3.6. *We shall say that an odd real spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ is $D-$reducible, if there exists a subspace $\mathcal{H}' \subsetneq \mathcal{H}$ with faithful representation of $\mathcal{A}$ such that $(\mathcal{A}, D|_{\mathcal{H}'}, \mathcal{H}')$ is a complex spectral triple.*

Moreover we shall introduce another concept of reducibility connected with the bounded perturbations.

Definition 3.7. *We say that spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ is reducible up to bounded perturbation if there exists an operator $A \in B(\mathcal{H})$ such that spectral triple $(\mathcal{A}, D + A, \mathcal{H}, J)$:*

- *is a bounded perturbation of $(\mathcal{A}, D, \mathcal{H}, J)$;*
- *is reducible in a strong sense.*

Note that we have following implications:

$$\begin{array}{ccc} \text{reducible} & \Rightarrow & \text{reducible} \\ \text{(strong sense)} & & \text{up to bounded perturbation;} \end{array}$$

$$\begin{array}{ccc} \text{reducible} & \Rightarrow & J\text{-reducible.} \\ \text{up to bounded perturbation} & & \end{array}$$

Definition 3.8. *We shall call a real spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ irreducible if it is not reducible up to bounded perturbation. The class of unitarily equivalent irreducible spectral triples over a $*-$algebra $\mathcal{A}$ shall be called a noncommutative real spin structure.*

It is easy to see the dependence of the definition of noncommutative spin structure on the assumed definition of reducibility. In the latter part of dissertation we shall show that the number of noncommuative spin structures agrees with the number of classical spin structure when the reducibility up to perturbation is chosen.

3.2 Equivariant Spectral Triples - Definition

One of the most important concepts of the tools of spectral geometry is the concept of equivariant spectral triple. It is the noncommutative counterpart to the classical notion of symmetries of manifolds, e.g. symmetry of metric tensor. The original definition elaborated by Sitarz and Paschke in [50, 43] involves operation connected with Hopf algebras, which generalise the action of classical groups, groups algebras and Lie algebras. As in this dissertation we shall not use "the noncommutative symmetries" of Hopf algebras we shall not go into detailed general definition of equivariance of spectral triples. In our case we only need to deal with the equivariance induced by the classical symmetries rephrased in the algebraic language. We shall distinguish to specific types of such symmetries - the Lie algebra type symmetry (induced by derivations) and the group action type symmetry.

Lie algebra type

Definition 3.9. *Consider a real spectral triple $(\mathcal{A}, D, \mathcal{H}J)$. Let $\mathcal{L}$ be the Lie algebra of derivations acting on $\mathcal{A}$ and denote by l the representation of $\mathcal{L}$ on $\mathcal{H}^l$ the dense subspace of Hilbert space $\mathcal{H}$. Then we shall say that $(\mathcal{A}, D, \mathcal{H}, J)$ is $\mathcal{L}$-equivariant spectral triple if for all $\lambda \in \mathcal{L}$ and $\psi \in \mathcal{H}^l$ we have:*

$$l(\lambda)D\psi = Dl(\lambda)\psi, \quad l(\lambda)J\psi = -l(\lambda)J\psi$$

and a Leibniz rule:

$$l(\lambda)\pi(a)\psi = \pi(\lambda \triangleright a)\psi + \pi(a)l(\lambda)\psi \quad \forall a \in \mathcal{A}.$$

One of the most transparent and instructive examples of Lie algebra type symmetries are the flat spectral triples over noncommutative tori. We shall present the details of this construction in the next section.

Group action type

Definition 3.10. *Consider a real spectral triple $(\mathcal{A}, D, \mathcal{H}J)$ and a group G acting by automorphisms on the algebra $\mathcal{A}$. Let ρ be the representation of group G on $\mathcal{H}^G$. i.e. the dense subspace of Hilbert space $\mathcal{H}$. Then we shall say that $(\mathcal{A}, D, \mathcal{H}, J)$ is G-equivariant spectral triple if for all $g \in G$ and $\psi \in \mathcal{H}^G$ we have:*

$$\rho(g)D\psi = D\rho(g)\psi, \quad \rho(g)J\psi = \rho(g)J\psi,$$

$$\rho(g)\pi(a)\psi = \pi(g \triangleright a)\rho(g)\psi \quad \forall a \in \mathcal{A}.$$

3.3 Noncommutative Spin Structures over the Noncommutative Torus

In this section we shall examine the equivariant spectral triples over nonocommutative three torus. We will briefly summarise the main results obtained by Sitarz in [44] and by Venselaar in [55], the most important of them is full classification of equivariant real spectral triples or simply noncommutative spin structures.

Consider a twisted group algebra $C^*(\mathbb{Z}^n, \omega_\Theta)$ with the cocycle over $\mathbb{Z}^n$:

$$\omega_\theta(p, q) = \exp(\pi i \sum_{j,k=1}^{n} \theta_{jk} p_j q_k), \quad \forall p, q \in \mathbb{Z}^3,$$

where θ_{jk} is a real antisymmetric matrix ($0 \leq \theta_{jk} < 1$). We denote the product in the twisted convolution algebra by $*_\omega$. We have

$$\delta_p *_{\omega_\theta} \delta_q = \omega_\theta(p,q)\delta_{p+q} \quad \forall p,q \in \mathbb{Z}^n.$$

Taking the canonical basis of $\mathbb{Z}^n$, $e_1, e_2, \ldots, e_n$, we then identify u_i with δ_{e_i} for each $i = 1, \ldots, n$. We will now define an abstract algebra generated by the set of n elements u_i. We have the following relations on generators:

$$(u_i)^* = u_i^{-1}, \quad u_j u_i = e^{2\pi i \theta_{ji}} u_i u_j \quad \forall i,j = 1, \ldots, n,$$

where $\theta_{ij} = -\theta_{ji}$. Moreover we define unitaries:

$$x^p \colon = e^{\pi i \sum_{i<j} \theta_{ji} p_j p_i} (u_1)^{p_1} (u_2)^{p_2} \ldots (u_n)^{p_n} \quad \forall p \in \mathbb{Z}^n.$$

For such elements we have:

$$x^p x^q = \omega_\Theta(p,q) x^{p+q},$$

where $\omega_\Theta(p,q)$ is a cocycle over $\mathbb{Z}^n$ defined above.

3.3.1 Algebra

Now we can define the algebraical noncommuatative generalisation of n-dimensional torus.

Definition 3.11. *We shall denote by $\mathcal{A}(\mathbb{T}^n_\Theta)$ the set consisting of polynomials in unitaries $u_1, u_2, \ldots, u_n$, i.e. a finite sums:*

$$\sum_{p \in I} a_p x^p,$$

where $I \subset \mathbb{Z}^n$ is finite and $a_p \in \mathbb{C}$ for all $p \in I$. Then $\mathcal{A}(\mathbb{T}^n_\Theta)$ is closed under addition and multiplication and forms a complex $*$-algebra. We shall call it an algebra of polynomials on noncommutative n-torus.

We can complete $\mathcal{A}(\mathbb{T}^n_\Theta)$ to the algebra of smooth functions on noncommutative $n-$dimensional torus, $C^\infty(\mathbb{T}^n_\Theta)$ consisting of power series:

$$x = \sum_{p \in \mathbb{Z}^n} a_p x^p,$$

where a_p belongs to a Schwartz space $\mathcal{S}(\mathbb{Z}^n)$, i.e.

$$\|(a_p)\|_k \colon = \sup_{p \in \mathbb{Z}^n} |a_p| (1 + \sum_{i=1}^n |p_i|)^k < \infty \quad \forall k \geq 0.$$

Moreover we can also consider a C^*-algebra completion in a unique norm respecting C^*-identity $\|aa^*\| = \|a\|^2$ for all $a \in C^\infty(\mathbb{T}^n_\Theta)$. We shall denote the C^*-algebra closure of $C^\infty(\mathbb{T}^n_\Theta)$ (which is at the same time a C^*-closure of $\mathcal{A}(\mathbb{T}^n_\Theta)$) by $C(\mathbb{T}^n_\Theta)$. The latter algebra is commonly called the algebra of functions over noncommuatative $n-$torus.

Remark 3.12. *If $\theta_{ij} = 0$ for all $i, j = 1, \ldots, n$ then the C^*-algebra defined in abstract approach coincides with the algebra of complex functions over a topological $n-$torus, i.e. $C(\mathbb{T}^n_\Theta) \simeq C(\mathbb{T}^n)$. It is easy to see if we define the complete set of characters:*

$$\chi_x(u_i) = e^{2\pi i x_i} \quad \forall x \in [0,1)^n.$$

Then one can easily check that densely defined:

$$t\colon C^\infty(\mathbb{T}^n_\Theta) \ni \sum_{p\in\mathbb{Z}^n} a_p \prod_{i=1,\ldots,n} (u_i)^{p_i} \to \sum_{p\in\mathbb{Z}^n} a_p e^{2\pi i (p,x)} \in C(\mathbb{R}^n/\mathbb{Z}^n) \simeq C(\mathbb{T}^n)$$

extends to an isomorphism of C^*-algebras due to Gelfand-Naimark theorem.

3.3.2 Representation

To construct the spectral triple over three torus we shall conduct most of computation on a pre$-C^*-$algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$. We shall denote three unitaries generating polynomials of $\mathcal{A}(\mathbb{T}^3_\Theta)$ by u_1, u_2, u_3. As we are dealing with spectral triple over flat (i.e. $u(1)\times u(1)\times u(1)$ equivariant) torus we need both a a representation of the Lie algebra of derivations and an equivariant representation of $\mathcal{A}(\mathbb{T}^3_\Theta)$. We set the universal enveloping algebra of Lie algebra $u(1)^3$ as generated by three selfadjoint mutually commuting derivations $\delta_1, \delta_2, \delta_3$ acting on $\mathcal{A}(\mathbb{T}^3_\Theta)$:

$$\delta_i \triangleright u_j = \begin{cases} u_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The minimal equivariant space is then an infinite direct sum: $\bigoplus_{m\in\mathbb{Z}^3} \mathcal{H}_{\epsilon+m}$, where each $\mathcal{H}_{\epsilon+m} \approxeq \mathbb{C}$ is an eigenspace of derivations:

$$l(\delta_i)e_\mu = \mu_i e_\mu,$$

for e_μ a basis vector of $\mathcal{H}_\mu$ where $\mu = m + \epsilon \in \mathbb{Z}^3 + \epsilon$.

When equipped with the inner product and completed it becomes a Hilbert space of square summable series $l^2(\mathbb{Z}^3)$. Moreover the representation π of noncommutative $3-$torus have to respect the Leibniz rule:

$$l(\delta_i)\pi(a) = \pi(\delta_i \triangleright a) + \pi(a)l(\delta_i) \quad \forall a \in \mathcal{A}(te^3_\Theta).$$

Theorem 3.13 ([55]). *All representations of $\mathcal{A}(\mathbb{T}^3_\Theta)$ on $l^2(\mathbb{Z}^3)$ respecting the Leibniz rule are unitarily equivalent to the following:*

$$\pi(x^p)e_\mu = \omega_\theta(p,\mu)e_{p+\mu},$$

where $\omega_\theta(p,\mu) = \exp(\pi i \sum_{1\le j<i\le 3}\theta_{ji}(p_j\mu_i - p_i\mu_j))$ is the cocycle of twisted group algebra $C^*(\mathbb{Z}^3,\omega)$.

3.3.3 Equivariant real spectral triples over $\mathcal{A}(\mathbb{T}^3_\Theta)$

Now we proceed to the definition of the spectral triples over noncommutative three torus. We recall that it will consist of an equivariant representation of $\mathcal{A}(\mathbb{T}^3_\Theta)$ on a Hilbert space $\mathcal{H}$, Dirac operator D and a real structure J. It occurs that those requirements are enough to define an inequivalent noncommutative spectral triples uniquely up to choice of spin structure.

Firstly we define the Hilber space $\mathcal{H} \simeq l^2(\mathbb{Z}) \otimes \mathbb{C}^2$. Now let us choose an $\epsilon \in \mathbb{R}^3$. The basis vectors of $\mathcal{H}$ are e^a_μ, i.e indexed by two numbers: $\mu \in \mathbb{Z}^3 + \epsilon$ and $a = \pm 1$. The representation of the Lie algebra of derivation and the representation of the algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$ is diagonal and to simplify notation also denoted l and π:

$$l(\delta_i)e^a_\mu = \mu_i e^a_\mu, \quad \pi(x_p)e^a_\mu = \omega(p,\mu)e^a_{p+\mu}.$$

Let $(\mathcal{A}(\mathbb{T}^3_\Theta), D, \mathcal{H}, J)$ be an equivariant real spectral triple, i.e. moreover we assume that $Dl(\delta_i) = l(\delta_i)D$ and $Jl(\delta_i) = l(\delta_i)J$. It occurs that we obtain a huge restriction on the parameters ϵ's, each $\epsilon_i = 0, \frac{1}{2}$.

Theorem 3.14 ([55]). *Let $(\mathcal{A}(\mathbb{T}^3_\Theta), D, \mathcal{H}, J)$ be a real irreducible flat spectral triple over the noncommutative three torus, then there exist three numbers ϵ_1, ϵ_2 and ϵ_3 such that each $\epsilon_i = 0, \frac{1}{2}$ and*

- *Hilbert space $\mathcal{H}$ has a countable basis consisting of vectors e^a_μ, where $a = \pm 1$ and $\mu \in \mathbb{Z}^3 + \epsilon$;*
- *the $\mathcal{A}(\mathbb{T}^3_\Theta)$ representation is unitarily equivalent to the following:*

  $$\pi(x_p)e^a_\mu = e^{\pi i \sum_{i<j}\theta_{ji}(p_j\mu_i - p_i\mu_j)}e^a_{p+\mu}.$$

- *the Dirac operator (up to rescaling) is :*

  $$De^a_\mu = aR\mu_1 e^a_\mu + (\mu_2 + \tau^{-a}\mu_3)e^{-a}_\mu,$$

  where $\tau \in U(1)$ and $R \in \mathbb{R}$;

- *the real structure:*

$$Je^a_\mu = ae^{-a}_{-\mu}.$$

There are in total 8 *inequivalent irreducible real spectral triples over noncommutative three torus. They are indexed by three parameters* $\epsilon_1, \epsilon_2, \epsilon_3$.

3.4 Spectral Triples over Quotient Spaces - Toy Model

We shall now discuss the reducibility and the group type equivariance on the second example. As a toy model we will consider the case of a circle, with a standard one-dimensional spectral triple. The algebra taken are smooth functions over the circle, its C^*-algebra completion being continuous functions $C(\mathbb{T}^1)$, while most of the computations are done on the algebra of polynomials, generated by one unitary element u and denoted $\mathcal{A}(\mathbb{T}^1)$. We also define π, its representation on the Hilbert space $\mathcal{H}$ (spanned by the eigenvectors e_μ) via:

$$\pi(u)e_\mu = e_{\mu+1},$$

where e_μ are eigenvectors of derivations with respect to the eigenvalue μ. A triple $(\mathcal{A}(\mathbb{T}^1), \mathcal{H}, d, j)$ is a real, $u(1)$-equivariant spectral triple if we define:

$$de_\mu = \mu e_\mu; \quad je_\mu = e_{-\mu}.$$

3.4.1 Reducible spectral triples

There are two inequivalent irreducible, equivariant real spectral triples over $\mathcal{A}(\mathbb{T}^1)$ - one for $\mu \in \mathbb{Z}$ and the other for $\mu \in \mathbb{Z} + \frac{1}{2}$ (see [55]). We will distinguish these two cases using parameter $\epsilon = 0, \frac{1}{2}$, such that $\mu \in \mathbb{Z} + \epsilon$ and $\mathcal{S}_\epsilon = (\mathcal{A}(\mathbb{T}^1), \mathcal{H}^\epsilon, d, j)$.

However let us consider new spectral triple $\mathcal{S}_* = (\mathcal{A}(\mathbb{T}^1), \mathcal{H}^\nu \oplus \mathcal{H}^{-\nu}, d, j)$, where $\mathcal{H}^\nu \oplus \mathcal{H}^{-\nu} = \overline{Span(e^\pm_{m\pm\nu})_{m\in\mathbb{Z}}}$ and $\nu \neq 0, \frac{1}{2}$. Representation of $\mathcal{A}(\mathbb{T}^1)$ and the action of d and j is defined as follows:

$$\pi(u)e^\pm_{m\pm\nu} = e^\pm_{m+1\pm\nu}; \quad de^\pm_{m\pm\nu} = (m \pm \nu)e^\pm_{m\pm\nu}; \quad je^\pm_{m\pm\nu} = e^\mp_{-m\mp\nu}.$$

It is easy to verify that $\mathcal{S}_*$ is indeed a real and $u(1)$-equivariant spectral triple. Moreover one can also easily check that it is not reducible in a strong sense ($d, j-$reducibility). However, we have:

Lemma 3.15. *$\mathcal{S}_*$ is reducible up to bounded perturbation of d. Moreover*

$$\mathcal{S}_* \simeq \mathcal{S}_0 \oplus \mathcal{S}_0 \simeq \mathcal{S}_{\frac{1}{2}} \oplus \mathcal{S}_{\frac{1}{2}}.$$

Proof. Firstly let us define (for $a = 0, 1$):

$$f_m^a := e^+_{m+a+\nu} + e^-_{m-\nu}, \quad g_m^a = ie^+_{m+a+\nu} - ie^-_{m-\nu}.$$

These sets of vectors span two Hilbert subspaces $\mathcal{H}_a^+ = \overline{Span(f_m^a)_{m\in\mathbb{Z}}}$ and $\mathcal{H}_a^- = \overline{Span(g_m^a)_{m\in\mathbb{Z}}}$ and since each vector from $\mathcal{H}^\nu \oplus \mathcal{H}^{-\nu}$ can be uniquely decompose into vectors from $\mathcal{H}_a^\pm$ we have:

$$\mathcal{H}^\nu \oplus \mathcal{H}^{-\nu} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- = \mathcal{H}_1^+ \oplus \mathcal{H}_1^-.$$

Spaces $H_a^\pm$ are subrepresentations of algebra $\mathcal{A}(\mathbb{T}^1)$ since $\pi(u)f_m^a = f_{m+1}^a$ and $\pi(u)g_m^a = g_{m+1}^a$. Moreover for fixed a those spaces are isomorphic, i.e. $\mathcal{H}_a^+ \simeq \mathcal{H}_a^-$. Then one can check, that for a real structure operator j we have:

$$\begin{gathered} jf_m^a = j(e^+_{m+a+\nu} + e^-_{m-\nu}) = f^a_{-m-a} \\ \text{and} \\ jg_m^a = j(ie^+_{m+a+\nu} - ie^-_{m-\nu}) = g^a_{-m-a}. \end{gathered}$$

Vectors f_m^a and g_m^a are not eigenvectors of the Dirac operator d, moreover even Hilbert spaces $\mathcal{H}_a^\pm$ are not eigenspaces neither for $a = 0$ or $a = 1$.

The only possibility to get a suitable Dirac operator is to modify d by some perturbation. We demand, firstly that new Dirac operator d_a anticommutes with j and secondly that it is still $u(1)$-equivariant. This means that we are looking for a Dirac operator of the form

$$d_a e^\pm_{m\pm\nu} = (\pm x_a + d)e^\pm_{m\pm\nu},$$

where x_a is a real number. We define:

$$d_a e^\pm_{m\pm\nu} = (\mp\nu \mp \frac{a}{2} + d)e^\pm_{m\pm\nu}.$$

Applying this to f^a and g^a we get:

$$\begin{gathered} d_a f_m^a = d_a(e^+_{m+a+\nu} + e^-_{m-\nu}) = (m + \frac{a}{2})f_m^a \\ \text{and} \\ d_a g_m^a = d_a(ie^+_{m+a+\nu} - ie^-_{m-\nu}) = (m + \frac{a}{2})g_m^a. \end{gathered}$$

This almost ends the proof. It is easy to see, that the spectral triples $(\mathcal{A}(\mathbb{T}^1), \mathcal{H}_a^+, d_a, j)$ and $(\mathcal{A}(\mathbb{T}^1), \mathcal{H}_a^-, d_a, j)$ are unitarily equivalent. If we now just rename vectors f^a and g^a via:

$$e_\mu : = f_m^a = g_m^a,$$

where $\mu = m + \frac{a}{2}$, we check that:

$$\pi(u)e_\mu = e_{\mu+1}, \quad je_\mu = e_{-\mu}, \quad d_a e_\mu = \mu e_\mu.$$

So in the end we have up to bounded perturbations of Dirac operator a unitary equivalence of spectral triples:

$$\mathcal{S}_0 \simeq (\mathcal{A}(\mathbb{T}^1), \mathcal{H}_0^+, d_a, j) \simeq (\mathcal{A}(\mathbb{T}^1), \mathcal{H}_0^-, d_a, j).$$

$$\mathcal{S}_{\frac{1}{2}} \simeq (\mathcal{A}(\mathbb{T}^1), \mathcal{H}_1^+, d_a, j) \simeq (\mathcal{A}(\mathbb{T}^1), \mathcal{H}_1^-, d_a, j).$$

and

$$\mathcal{S}_* \simeq \mathcal{S}_0 \oplus \mathcal{S}_0 \simeq \mathcal{S}_{\frac{1}{2}} \oplus \mathcal{S}_{\frac{1}{2}}.$$

Finally let us have a look at the perturbation of the Dirac operator, which gives the full (j, d, π)-reducibility. It could be explicitely checked that the perturbation is neither in $\Omega_d^1(\mathcal{A}(\mathbb{T}^1))$ nor in $j\Omega_d^1(\mathcal{A}(\mathbb{T}^1))j^{-1}$. In fact, the perturbation is in the common commutant of $\mathcal{A}(\mathbb{T}^1), \Omega_d^1(\mathcal{A}(\mathbb{T}^1)), j\mathcal{A}(\mathbb{T}^1)j^{-1}$ and $j\Omega_d^1 D(\mathcal{A}(\mathbb{T}^1))j^{-1}$. Indeed, it is a generic situation, if that set contains more than only mutiples of the identity, then one can always perturb the Dirac by an element from that set. However, such perturbation has no geometric intepretation. □

3.4.2 Spectral Triples over $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$

We will now consider spectral triples over quotient algebras in the simplest toy example one can think of: S^1. Let us firstly define the action of the group $\mathbb{Z}_N$ on the algebra $\mathcal{A}(\mathbb{T}^1)$. For h, the generator of $\mathbb{Z}_N$ we assume:

$$h \triangleright u = \lambda u,$$

where λ is the N-th root of unity. The fixed point algebra is generated by the unitary element u^N and clearly it is isomorphic to the initial algebra.

$$\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N} \simeq \mathcal{A}(\mathbb{T}^1).$$

The full spectral triple

We will consider spectral triple over fixed point algebra $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$ with Hilbert space, Dirac operator and real structure taken from the spectral triple over initial algebra and ask about its reducibility.

Let us define $\tilde{\mathcal{S}}^{\mathbb{Z}_N}: = (\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}, \mathcal{H}^\epsilon, d, j)$. To study the possible reductions of the spectral triples, we need to find the subspaces, which are preserved by the invariant subalgebra, d and j. Assuming that such subspace is spanned by the eigenvectors of the Dirac operator, we find that the minimal subspaces preserved by the invariant subalgebra are

$$\mathcal{H}_s: = \overline{Span(e_{Nm+\epsilon+s})_{m\in\mathbb{Z}}},$$

where $s = 0, 1, 2, \ldots, N-1$ so that $e_{Nm+s+\epsilon} \in \mathcal{H}^0$ or $\mathcal{H}^{\frac{1}{2}}$.

Let us see whether they are j-invariant. From the definition $je_\mu = e_{-\mu}$, so we see that:

$$j\colon \mathcal{H}_s \longrightarrow \mathcal{H}_{-s-2\epsilon},$$

and, of course, $-s$ is taken $\mod N$. Thus we obtain a condition (if we demand the j-invariance of $\mathcal{H}_s$):

$$2s = -2\epsilon \mod N.$$

If N is even it has to solutions for $\epsilon = 0$, i.e. $s = 0, \frac{N}{2}$, and does not have solution if $\epsilon = \frac{1}{2}$. If N is odd there is one solution for each ϵ: if $\epsilon = 0$ then $s = 0$ and if $\epsilon = \frac{1}{2}$ then $s = \frac{N-1}{2}$.

ϵ	N even	N odd
0	$0, \frac{N}{2}$	0
$\frac{1}{2}$	$-$	$\frac{N-1}{2}$

Table 3.1: $j-$equivariance condition

If s is one of the listed in the table (for given N and ϵ) one concludes that $\mathcal{H}_s$ is the eigenspace of both d and j. Otherwise the invariant subspace is the direct sum of $\mathcal{H}_s \oplus \mathcal{H}_{-s-2\epsilon}$. Any d and j-invariant subspace gives rise to a spectral triple over $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$. To summarise these results:

Theorem 3.16. *Spectral triple* $\tilde{\mathcal{S}}_\epsilon^{\mathbb{Z}_N}: = (\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}, \mathcal{H}^\epsilon, d, j)$ *is reducible. We have following reduction:*

- *for N even:*

$$\tilde{\mathcal{S}}_0^{\mathbb{Z}_N} \simeq \mathcal{S}_0 \oplus \mathcal{S}_{\frac{1}{2}} \oplus (\mathcal{S}_*)^{\oplus\frac{N}{2}-1} \quad \text{and} \tag{3.1}$$

$$\tilde{\mathcal{S}}_{\frac{1}{2}}^{\mathbb{Z}_N} \simeq (\mathcal{S}_*)^{\oplus\frac{N}{2}}. \tag{3.2}$$

- *for N odd:*

$$\tilde{\mathcal{S}}_0^{\mathbb{Z}_N} \simeq \mathcal{S}_0 \oplus (\mathcal{S}_*)^{\oplus \frac{N-1}{2}} \quad \text{and} \tag{3.3}$$

$$\tilde{\mathcal{S}}_{\frac{1}{2}}^{\mathbb{Z}_N} \simeq \mathcal{S}_{\frac{1}{2}} \oplus (\mathcal{S}_*)^{\oplus \frac{N-1}{2}}. \tag{3.4}$$

Note that by the Lemma 3.15 each of the $\mathcal{S}_*$ spectral triples is irreducible in the strong sense but is reducible up to perturbation of the Dirac operator, i.e. up to bounded perturbation we have $\mathcal{S}_* \simeq \mathcal{S}_0 \oplus \mathcal{S}_0 \simeq \mathcal{S}_{\frac{1}{2}} \oplus \mathcal{S}_{\frac{1}{2}}$.

The equivariant action The aim of classification of irreducible flat spectral triples over $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$ can be obtained using different path: that of looking for an equivariant action. It is easy to see that the d-invariant and $u(1)$-equivariant representation of $\mathbb{Z}_N$ on the Hilbert space, which implements the action on the algebra is (for h the generator of $\mathbb{Z}_N$):

$$\rho(h)e_\mu = \lambda^{\mu-\epsilon-s} e_\mu,$$

for some $s = 0, 1, 2, \ldots, N-1$. It is clear that the action is j-equivariant if and only if:

$$\lambda^{-\mu-\epsilon-s} = \lambda^{-\mu+\epsilon+s},$$

so that

$$2s + 2\epsilon = 0 \mod N.$$

The possible solutions are again listed in Table 3.1. Observe that each of the cases corresponds exactly to the *single* invariant subspace giving a spectral triple $\mathcal{S}_0$ or $S_{\frac{1}{2}}$ and not $\mathcal{S}_*$.

3.4.3 Summary

We see that taking spectral triples over $\mathcal{A}(\mathbb{T}^1)^{\mathbb{Z}_N}$ we should restore the situation of $\mathcal{A}(\mathbb{T}^1)$ (as in fact those two algebras are isomorphic), i.e two inequivalent irreducible flat real spectral triples which corresponds to two classically defined spin structures over circle. But using canonical definition of full $(d, j, \pi)-$reducibility in a strong sense we obtain a lot too many strongly irreducible spectral triples. To get the correct result we should use the new definition - reducibility up to bounded perturbation of the Dirac operator. We have also observed that this result coincides with the number of all possible $d-$equivariant and $j-$equivariant actions of group $\mathbb{Z}_N$ on a spectral $(\mathcal{A}(\mathbb{T}^1), \mathcal{H}, d, j)$.

Chapter 4

Noncommutative Bieberbach Manifolds

Exactly one hundred years ago, in 1912, Ludwig Bieberbach proved a theorem which states that any n-dimensional flat compact manifold is finitely covered by the n-torus, i.e. is a Bieberbach manifold (see [5, 6] for details). Moreover Bieberbach showed that any flat compact manifold is a quotient $\mathbb{T}^n/G$, where G is a finite group. It was a significant result for the topology. From this moment there was a new tool to solve problems related to flat compact manifolds M - one could rephrase the problem in terms of groups G such that: $\mathbb{T}^n \xrightarrow{G} M$. For example the aim of classification of manifolds for a given dimension could be translate into the problem of classification of possible free actions of finite groups on the n-torus, which appears to be a simpler question.

Today we shall deal with the fixed point algebras of the noncommuatative three torus $\mathcal{A}(\mathbb{T}^3_\Theta)$, which was defined in the previous chapter, under the action of finite groups. The algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$ is a noncommutative generalisation of a topological three torus, to be precise the C^*-algebra closure of $\mathcal{A}(\mathbb{T}^3_\Theta)$ for $\theta_{ij} = 0$ is isomorphic to the algebra of continuous complex valued functions over a topological $\mathbb{T}^3$. It is quite trivial to see that torus is a Bieberbach manifold. Moreover for any manifold M and group G (acting freely on M) one has a C^*-algebraical one to one correspondence:

$$\begin{array}{ccc} \text{continuous functions} & \simeq & \text{fixed point subalgebras of } C(M) \\ \text{on the quotient manifold } M/G & & \text{under the action of } G\ . \end{array}$$

We shall use this to define the noncommutative counterpart to topological three-dimensional Bieberbach manifolds. The schedule of this chapter is as follows. Firstly we shall recall the basic facts relating to the definition of topological Bieberbach manifolds. Then we shall focus on the three-dimensional case and give a classification of free actions of finite groups on three torus in a C^*-algebraic approach, i.e. we shall characterise C^*-algebras of continuous functions over Bieberbach manifolds as a fixed point algebras of $C(\mathbb{T}^3)$. Finely we shall show how one can generalise this result to the case of noncommutative torus.

4.1 Classical Bieberbach Manifolds

Let $\mathbb{R}^n$ be a real space with canonical flat metric, i.e. $d(x,y) = \sum_{i=1}^n (x_i - y_i)^2$ for all $x, y \in \mathbb{R}^n$. The orthogonal group $O(n)$ is the set consisting of $A \in M_n(\mathbb{R})$ such that $A^T A = \mathbb{1}$ with the usual multiplication of matrices. Elements of the groups $O(n)$ and $\mathbb{R}^n$ (acting as translations) are isometries of V (T is an isometry if $d(x,y) = d(T \triangleright x, T \triangleright y)$ for all $x, y \in \mathbb{R}^n$). Moreover we can define a more general isometry of this type:

$$(A,t) \triangleright x = Ax + t, \quad \forall A \in O(n), t, x \in \mathbb{R}^n$$

We can as well consider the group generated by such transformations. A Euclidean group is a crossed product group $O(n) \ltimes \mathbb{R}^n$ composed of the orthogonal group and the group of translation, such that the multiplication is defined as:

$$(A,t)(A',t') = (AA', t + At') \quad \forall A, A' \in O(n)\, t, t' \in \mathbb{R}^n.$$

We shall now proceed to the definition of Bieberbach manifolds.

Definition 4.1. *Let $\Gamma \subset O(n) \ltimes \mathbb{R}^n$ be discrete and let the action of Γ be proper discontinous. If the quotient manifold $M = \mathbb{R}^n/\Gamma$ is compact, then we shall call Γ an infinite Bieberbach group and M a Bieberbach manifold.*

Moreover another useful lemma can be stated:

Lemma 4.2. *A Bieberbach manifold $M = \mathbb{R}^n/\Gamma$ is orientable if and only if $\Gamma \subset SO(n) \ltimes \mathbb{R}^n$.*

We shall now recall the most essential of the theorems stated by Bieberbach.

Theorem 4.3 (Bieberbach, [5, 6])**.** *Let M be a flat and compact manifold of dimension n. Then there exists Γ a subgroup of $O(n) \ltimes \mathbb{R}^n$ such, that $\mathbb{R}^n/\Gamma$ is isometric to M.*

In the Definition 4.1 we have defined Bieberbach manifolds as the quotient of real space by the action of a specific subgroup of $O(n) \ltimes \mathbb{R}^n$. On the other hand by the Theorem 4.3 we see that our definition applies to all flat and compact manifolds. As any Bieberbach manifold is flat and compact we conclude that following definition is equivalent.

Definition 4.4. *A manifold M is called a Bieberbach manifold if it is flat and compact.*

The following two theorems are especially important in our case:

Theorem 4.5 (Bieberbach, [5, 6])**.** *Let Γ be an infinite Bieberbach group, then the set of pure translations in Γ defined as $T_n = \Gamma \cap \mathbb{R}^n$ is a lattice isomorphic to $\mathbb{Z}^n$.*

Theorem 4.6 (Bieberbach, [5, 6])**.** *T_n is a normal subgroup of Γ. Moreover the quotient group $G := \Gamma/\mathbb{Z}^n$ is finite. We shall call G a finite Bieberbach group.*

The proof of previous theorems implies another very interesting fact (see [45]). If we define a function $r : O(n) \ltimes \mathbb{R}^n \ni (A, a) \to A \in O(n)$, then $r(\Gamma) \simeq G$. Moreover G acting on $\mathbb{R}^n$ leaves $T_n = \Gamma \cap \mathbb{R}^n$ invariant, i.e. G acts on T_n. These facts implies the following theorem.

Theorem 4.7 (Bieberbach, [5, 6])**.** *Every Bieberbach manifold is normally covered by a flat torus $\mathbb{T}^n \xrightarrow{G} M$, and the covering map is a local isometry. Moreover $M = \mathbb{R}^n/\Gamma = \mathbb{T}^n/G$.*

Bieberbach manifolds are fully classified by their fundamental groups $\pi(M)$ and holonomy groups. Recall that holonomy group of a manifold M is discrete if and only if M is flat (see [3]).

Lemma 4.8. *Let Γ be a discrete group (not necessarily a subgroup of $\subset O(n) \ltimes \mathbb{R}^n$) which acts freely on $\mathbb{R}^n$. Let $\pi(M)$ be a fundamental group of manifold $M = \mathbb{R}^n/\Gamma$. Then M is a Bieberbach manifold if and only if $\pi(M) = \Gamma$. Moreover then $G = \Gamma/\mathbb{Z}^n$ is a holonomy of M.*

The simplest example of Bieberbach manifold is torus, which has a trivial holonomy for any dimension. Whereas the lowest dimension in which the holonomy group of an orientable Bieberbach manifold is not trivial is three. The following $GN = 1, \mathbb{Z}_2, \mathbb{Z}_3\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_6$ classifies six nonhomotopic orientable Bieberbach manifolds $\mathbb{T}^3/GN$. In the following sections we shall give a precise definitions of three-dimensional Bieberbach manifolds case by case in the C^*-algebraic approach.

4.2 Three-dimensional Bieberbach Manifolds

In this section we shall briefly recall the description of three-dimensional Bieberbach manifolds as quotients of the three-dimensional tori by the action of a finite group. We use the algebraic language, taking the algebra of the polynomial functions on the three-torus $\mathbb{T}^3$ generated by three mutually commuting unitaries U, V, W. This algebra could be then completed first to the algebra of smooth functions on the torus $C^\infty(\mathbb{T}^3)$ and later to a C^*-algebra of continuous functions $C(\mathbb{T}^3)$.

There are, in total, 10 different Bieberbach three-dimensional manifolds, six orientable (including the three-torus itself) and four nonorientable ones. This follows directly from the classification of Bieberbach groups of $\mathbb{R}^3$ out of which six do not change orientation and four change the orientation. The action of the finite groups on the unitaries U, V, W, which generate the algebra of continuous functions on the three-torus is presented in the table below and comes directly from the action of Bieberbach groups on $\mathbb{R}^3$. For each GN we obtain a fixed point algebra $C(\mathbb{T}^3)^G$ (denoted $\mathfrak{B}N$ in the orientable case and by $\mathfrak{N}N$ in the nonorientable) which is isomorphic to the C^*-algebra of complex functions over three-dimensional Bieberbach manifold. The above actions give rise to five oriented flat three-manifolds different

name	group G	action of the generators of G on U, V, W		
$\mathfrak{B}2$	$\mathbb{Z}_2$	$h \triangleright U = -U,$	$h \triangleright V = V^*,$	$h \triangleright W = W^*$
$\mathfrak{B}3$	$\mathbb{Z}_3$	$h \triangleright U = e^{\frac{2}{3}\pi i}U,$	$h \triangleright V = W,$	$h \triangleright W = V^*W^*$
$\mathfrak{B}4$	$\mathbb{Z}_4$	$h \triangleright U = iU,$	$h \triangleright V = W,$	$h \triangleright W = V^*$
$\mathfrak{B}5$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$h_1 \triangleright U = -U,$	$h_1 \triangleright V = V^*,$	$h_1 \triangleright W = W^*$
		$h_2 \triangleright U = U^*,$	$h_2 \triangleright V = -V,$	$h_2 \triangleright W = -W^*$
$\mathfrak{B}6$	$\mathbb{Z}_6$	$h \triangleright U = e^{\frac{1}{3}\pi i}U,$	$h \triangleright V = W\ ,$	$h \triangleright W = V^*W$

Table 4.1: Orientable actions of finite groups on three-torus

from the torus. The remaining four nonorientable quotients, originate from the following actions:

For full details and classifications of all free actions of finite groups on three-torus see [27, 35], note, however, that the resulting quotient manifolds are always one of the above Bieberbach manifolds. It is easy to see that $\mathfrak{N}1$ is just the Cartesian product of S^1 with the Klein bottle, whereas $\mathfrak{N}3$ and $\mathfrak{N}4$ are two distinct $\mathbb{Z}_2$ quotients of $\mathfrak{B}2$.

algebra	group G	action of the generators of G on U, V, W
$\mathfrak{N}1$	$\mathbb{Z}_2$	$h \triangleright U = -U, \quad h \triangleright V = V, \quad h \triangleright W = W^*$
$\mathfrak{N}2$	$\mathbb{Z}_2$	$h \triangleright U = -U, \quad h \triangleright V = VW, \quad h_1 \triangleright W = W^*$
$\mathfrak{N}3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$h_1 \triangleright U = -U, \quad h_1 \triangleright V = V^*, \quad h_1 \triangleright W = W^*$ $h_2 \triangleright U = U, \quad h_2 \triangleright V = -V, \quad h_2 \triangleright W = W^*$
$\mathfrak{N}4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$h_1 \triangleright U = -U, \quad h_1 \triangleright V = V^*, \quad h_1 \triangleright W = W^*$ $h_2 \triangleright U = U, \quad h_2 \triangleright V = -V, \quad h_2 \triangleright W = -W^*$

Table 4.2: Nonorientable action of finite groups on three-torus

4.2.1 Spin structures over Bieberbach manifolds

Each three-dimensional orientable Bieberbach manifold is a spin manifold and as such carries a spin structure. Full classification of spin structures over topological three-dimensional Bieberbach manifolds was done by Pfäffle in [45]. To obtain this result he used the fundamental groups. Recall that in case of Bieberbach manifolds of dimension three we have the following long exact sequence:

$$0 \to \mathbb{Z}^3 \xrightarrow{i} \Gamma_N \xrightarrow{r} G_N \to 0,$$

where Γ_N is a fundamental group (i.e. infinite Bieberbach group) of $\mathbb{T}^3/G_N$ and G_N is holonomy (i.e. finite Bieberbach group) of $\mathbb{T}^3/G_N$. The morphism r is also called a holonomy of Γ_N, while i is just the inclusion of $\mathbb{Z}^3$ in Γ_N.

Theorem 4.9 (Pfäffle, [45])**.** *Let $M = \mathbb{R}^3/\Gamma$ be a Bieberbach manifold. Then there is one to one correspondence between the spin structures on M and homomorphisms $\varepsilon\colon \Gamma \to \mathrm{Spin}(3)$ such that:*

$$r = \lambda \circ \varepsilon,$$

where r is the holonomy of Γ and λ is the double covering of $SO(3)$.

Description of fundamental groups Γ_N is quite complicated and sophisticated task, on the other hand it is barely connected with our construction of spectral triples over noncommutative Bieberbach spaces. We shall only present the number of spin structures for each manifold.

4.3 Noncommutative Bieberbach Spaces

Since a convenient description of Bieberbach manifolds is as quotients of three-torus by the action of finite groups we can ask whether starting from a noncommutative three-dimensional torus we can obtain interesting examples

name	group G	number of spin structures
$C(\mathbb{T}^3)$	1	8
$\mathfrak{B}2$	$\mathbb{Z}_2$	8
$\mathfrak{B}3$	$\mathbb{Z}_3$	2
$\mathfrak{B}4$	$\mathbb{Z}_4$	4
$\mathfrak{B}5$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{B}6$	$\mathbb{Z}_6$	2

Table 4.3: Spin structures over Bieberbach manifolds

of nontrivial flat noncommutative manifolds. Recall that the most general noncommutative three-torus $\mathcal{A}(\mathbb{T}^3_\Theta)$ can be realized as a pre$-C^*-$algebra of polynomials generated by three unitaries U, V, W with respect to commutation relations:

$$VU = e^{2\pi i\theta_{21}}UV, \quad WU = e^{2\pi i\theta_{31}}UW, \quad WV = e^{2\pi i\theta_{32}}VW.$$

The algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$, i.e. the algebra of polynomials in U, V, W, can be then completed to the algebra of "smooth functions" $C^\infty(\mathbb{T}^3_\Theta)$ and to the C^*-algebra $C(\mathbb{T}^3_\Theta)$ (for details of the construction see Section 3.3). It is worth to recall that $\mathcal{A}(\mathbb{T}^3_\Theta)$ is dense in $C(\mathbb{T}^3_\Theta)$ in the C^*-norm and as such is a pre$-C^*-$algebra for $C(\mathbb{T}^3_\Theta)$.

Next, we shall find all possible values of the matrix θ_{jk} such that the actions of the finite group G (as discussed earlier) are compatible with the commutation relation. We might define the compatibility with the action of G in the following way. We say that the action of the finite group G is compatible with the commutation relations imposed by θ_{jk} if:

$$g \triangleright (ab) = (g \triangleright a)(g \triangleright b) \quad \forall a, b \in \mathcal{A}(\mathbb{T}^3_\Theta), g \in G.$$

We have:

Lemma 4.10. *The commutation relations imposed by the* θ_{jk} *are compatible with the actions of group* G*, given in the tables 4.1 and 4.2 if* $0 \leq \theta_{jk} < 1$ *are as follows:*

As we are interested in the genuine noncommutative case, where the three-dimensional torus has at least one irrational rotation subalgebra (that is, at least one of the independent entries of the matrix θ_{jk} is irrational), we see that we might obtain only 4 nontrivial orientable noncommutative Bieberbach manifolds and two nonorientable ones.

group G	θ_{12}	θ_{13}	θ_{23}	conditions
$\mathbb{Z}_2$	$\frac{k}{2}$	$\frac{l}{2}$	θ	$k, l = 0, 1,$
$\mathbb{Z}_3$	$\frac{k}{3}$	$\frac{3-k}{3}$	θ	$k = 0, 1, 2,$
$\mathbb{Z}_4$	$\frac{k}{2}$	$\frac{k}{2}$	θ	$k = 0, 1,$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{k}{2}$	$\frac{l}{2}$	$\frac{m}{2}$	$k, l, m = 0, 1$
$\mathbb{Z}_6$	$\frac{k}{3}$	$\frac{3-k}{3}$	θ	$k = 0, 1, 2,$

Table 4.4: Values of θ_{jk} for compatible cocycles for orientable actions

group G	θ_{12}	θ_{13}	θ_{23}	conditions
$\mathbb{Z}_2$	θ	$\frac{k}{2}$	$\frac{l}{2}$	$k, l = 0, 1,$
$\mathbb{Z}_2$	θ	$\frac{k}{2}$	$\frac{l}{2}$	$k, l = 0, 1$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{k}{2}$	$\frac{l}{2}$	$\frac{m}{2}$	$k, l, m = 0, 1$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{k}{2}$	$\frac{l}{2}$	$\frac{m}{2}$	$k, l, m = 0, 1$

Table 4.5: Values of θ_{jk} for compatible cocycles for nonorientable actions

Definition 4.11. *Let $C(\mathbb{T}^3_\theta)$, be a C^*-algebra of noncommutative torus with commutation relation obtained from $\theta_{12} = \theta_{21} = 0$ and $\theta_{23} = -\theta$ for an irrational $0 < \theta < 1$. Then the generating unitaries U, V, W satisfy relations:*

$$UV = VU, \quad UW = WU, \quad WV = e^{2\pi i\theta} VW.$$

We define the algebras of noncommutative Bieberbach manifolds as the fixed point algebras of the following actions of finite groups G on $C(\mathbb{T}^3_\theta)$ (note that for $\mathfrak{N}1_\theta$ and $\mathfrak{N}2_\theta$ we need to relabel the generators: $\{U, V, W\} \to \{W, U, V\}$ so that always the V and W are from the irrational rotation subalgebra), which are combine in the table 4.6. For convenience and to match the notation of other papers we rescaled the generators V, W in the case of $\mathbb{Z}_3$ and $\mathbb{Z}_6$ actions, also, for $\mathbb{Z}_3$ we take the other generator of the $\mathbb{Z}_3$ action.

name	group G	action of the generators of G on U, V, W		
$\mathfrak{B}2_\theta$	$\mathbb{Z}_2$	$h \triangleright U = -U,$	$h \triangleright V = V^*,$	$h \triangleright W = W^*,$
$\mathfrak{B}3_\theta$	$\mathbb{Z}_3$	$h \triangleright U = e^{\frac{2}{3}\pi i} U,$	$h \triangleright V = e^{-\pi i\theta} V^* W,$	$h \triangleright W = V^*,$
$\mathfrak{B}4_\theta$	$\mathbb{Z}_4$	$h \triangleright U = iU,$	$h \triangleright V = W,$	$h \triangleright W = V^*,$
$\mathfrak{B}6_\theta$	$\mathbb{Z}_6$	$h \triangleright U = e^{\frac{1}{3}\pi i} U,$	$h \triangleright V = W\ ,$	$h \triangleright W = e^{-\pi i\theta} V^* W,$
$\mathfrak{N}1_\theta$	$\mathbb{Z}_2$	$h \triangleright U = U^*,$	$h \triangleright V = -V,$	$h \triangleright W = W,$
$\mathfrak{N}2_\theta$	$\mathbb{Z}_2$	$h \triangleright U = U^*,$	$h \triangleright V = -V,$	$h \triangleright W = WU^*,$

Table 4.6: The action of finite cyclic groups on $C(T^3_\theta)$

Lemma 4.12. *The actions of the cyclic groups* $\mathbb{Z}_N$*,* $N = 2, 3, 4, 6$ *on the noncommutative three-torus, as given in the table 4.6 is free.*

Proof. This proposition was stated in our paper (for details see [40]). It involves the concept of Hopf algebras, which we shall not introduce in this dissertation. So we shall not go into details of the proof. □

Chapter 5

$K-$theory of Noncommutative Bieberbach Manifolds

The definition of spectral triple involves the concept of finitely generated projective module. To be precise, having a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ we define $\mathcal{H}^\infty$, the subspace of $\mathcal{H}$ consisting of smooth vectors, i.e. $\mathcal{H}^\infty = \mathcal{H} \cap \bigcap_{k=1}^{\infty} \mathrm{Dom}(D^k)$. One of the axioms states that $\mathcal{H}^\infty$ is a finitely generated projective module over a pre$-C^*-$algebra $\mathcal{A}$. This leads us to a question whether we can classify, in some manner, the finitely generated projective modules over the C^*-algebra $C(\mathbb{T}^3_\Theta)^{GN}$, thus to the $K-$theory of noncommutative Bieberbach spaces. In this chapter we shall present the computation of the K-theory groups of the C^*-algebras of Bieberbach spaces obtained by the action of the cyclic group $\mathbb{Z}_N$, $N = 2, 3, 4, 6$. As $K-$theory groups of pre-C^*-algebra are the same as K_0 and K_1 of its C^*-completion, in this chapter we shall concentrate on C^*-algebras.

The layout of the chapter is as follows. First we shall discuss the Morita equivalence of Bieberbach spaces and crossed product algebras, this allows us to equate the $K-$theory of both algebras. Then we shall present possible methods of computation and show how they can be explicitly applied on the toy model of Klein bottle. Finally after choosing the most transparent of them we present the computation of $K-$theory of $C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}$.

5.1 The Morita Equivalence and Takai Duality

We shall begin with the recalling the Kishimoto-Takai theorem concerning stable isomorphism of crossed product and fixed point algebras (for more details see [31, 53]).

Theorem 5.1 (Kishimoto, Takai,[31]). *Let $(\mathcal{A}, G, \alpha)$ be C^*-dynamical system based of a compact abelian group G and let $\mathcal{A}$ be unital. For each $\hat{g} \in \hat{G}$ let $I_{\hat{g}}$ be the closed ideal of $\mathcal{A}^G = \mathcal{A}^G(1)$ generated by $\mathcal{A}^G(\hat{g})^* \mathcal{A}^G(\hat{g})$. If $I_{\hat{g}} = \mathcal{A}^G$ for all $\hat{g} \in \hat{G}$ then $\mathcal{A} \rtimes_\alpha G$ is stably isomorphic to $\mathcal{A}^G \otimes \mathbb{K}\left(l^2(G)\right)$.*

We shall now discuss the assumptions of this theorem when applied to noncommutative Bieberbach spaces. In this case the group G is cyclic finite group so the assumption of compactness is trivially fulfilled. The dual $\hat{G}$ is again $\mathbb{Z}_N$. There exists a unitary $U \in C(\mathbb{T}^3_\Theta)$ such that $\alpha_h(U) = e^{\frac{2\pi i}{N}}$ for h the generator of $\mathbb{Z}_N$, thus $U^p \in C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}(p)$ is a unitary for each $p \in \hat{G} = \mathbb{Z}_N$. This implies

$$C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}(p) = U^p\, C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}(1)$$

and moreover

$$I_p = C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}.$$

The action of cyclic group fulfils the assumptions of Takai-Kishimoto theorem thus the fixed point algebra $C(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}$ is Morita equivalent to the crossed product algebra $C(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$, $N = 2, 3, 4, 6$. In fact we can apply directly the Takai isomorphism (see [53]) to determine explicitly the isomorphism between $C(\mathbb{T}^3_\theta)^{\mathbb{Z}_n} \otimes M_N(\mathbb{C})$ and $C(\mathbb{T}^3_\theta) \rtimes \mathbb{Z}_N$, which we shall present now.

Lemma 5.2. *Consider the action of a finite group $\mathbb{Z}_N$ on $C(\mathbb{T}^3_\theta)$ determined by the action of the generator h :*

$$h \triangleright U = \lambda U, \quad h \triangleright V = \beta(V), \quad h \triangleright W = \beta(W),$$

where U is central in $C(\mathbb{T}^3_\Theta)$, $\lambda = e^{\frac{2\pi i}{N}}$ and β is an automorphism of the rotation algebra $C(\mathbb{T}^2_\Theta)$ with generators V and W (with θ not necessarily irrational). The crossed product algebra $C(\mathbb{T}^3_\theta) \rtimes_{\lambda\otimes\beta} \mathbb{Z}_N$, is generated by U, V, W and h with relations:

$$hU = \lambda U h, \;\; hV = \beta(V)h, \;\; hW = \beta(W)h, \;\; h^N = 1.$$

Then $C(\mathbb{T}^3_\theta) \rtimes_{\lambda\otimes\beta} \mathbb{Z}_N$ is canonically isomorphic to $C(\mathbb{T}^3_\theta)^{\mathbb{Z}_n} \otimes M_N(\mathbb{C})$.

Proof. Consider an element $\hat{h}$ in the crossed product algebra:

$$\hat{h} := U + \left(\frac{1}{N}\sum_{k=1}^{N} h^k\right)\left(U^{1-N} - U\right). \tag{5.1}$$

It is easy to verify that:

$$\hat{h}^N = 1, \quad h\hat{h} = \lambda \hat{h} h,$$

so h and $\hat{h}$ generate the matrix algebra $M_N(\mathbb{C})$. Since U is central both h and $\hat{h}$ commute with any element of the fixed point subalgebra $C(\mathbb{T}^3_\theta)^{\mathbb{Z}_n}$.

We shall demonstrate now the isomorphism from the lemma, which we shall denote by Ψ. First, the relation (5.1) could be inverted, yielding:

$$\Psi(U) = \hat{h} + \left(\frac{1}{N} \sum_{k=1}^{N} h^k \right) \hat{h} \left(U^N - 1 \right).$$

Since U^N is an invariant element of the algebra then $\Psi(U)$ is clearly in $C(\mathbb{T}^3_\theta)^{\mathbb{Z}_N} \otimes M_N(\mathbb{C})$. Take now arbitrary $x \in C(T^3_\theta)$. It is easy to see that x could be uniquely decomposed as a sum of elements homogeneous with respect to the action of $\mathbb{Z}_N$:

$$x = \sum_{k=0}^{N-1} x_k, \quad h \triangleright x_k = \lambda^k x_k.$$

Indeed, it is sufficient to take:

$$x_k = \frac{1}{N} \sum_{j=0}^{N-1} \bar{\lambda}^{kj} (h^j \triangleright x), \quad k = 0, \dots, N-1.$$

Then if we define:

$$\Psi(x) = \sum_{k=0}^{N-1} (x_k U^{-k}) \Psi(U)^k,$$

then the range of Ψ is clearly in $C(\mathbb{T}^3_\theta)^{\mathbb{Z}_N} \otimes M_N(\mathbb{C})$, since each of the elements $x_k U^{-k}$ is invariant and in the fixed point algebra. The verification that Ψ is an algebra morphism and is an isomorphism is trivial and we shall omit it. $\square$

Note that the isomorphism Ψ which provides the Morita equivalence in our case does not depend on the value of the parameter θ, hence for the Bieberbach manifolds (commutative and noncommutative) their K-theory groups are the same as the K-theory groups of the crossed product algebras.

A technical tool for the computations is the following lemma.

Lemma 5.3. *Let $\mathcal{A}$ be a C^*-algebra, β its automorphism and let U be a unitary implementing the action β in the crossed product $\mathcal{A} \rtimes_\beta \mathbb{Z}$. Now consider an action α of $\mathbb{Z}_N$ on $\mathcal{A} \rtimes_\beta \mathbb{Z}$, such that for h the generator of $\mathbb{Z}_N$:*

- *it is a multiplication by a root of unity ($\lambda^N = 1$) on the generator of* $\mathbb{C}\mathbb{Z}$*:*
$$h \triangleright U = \lambda U;$$
- *its restriction to the algebra $\mathcal{A}$ gives an automorphism of the latter:*
$$h \triangleright a = \alpha(a) \in \mathcal{A} \quad \forall a \in \mathcal{A},$$
such that it commutes with β, i.e. $\alpha(\beta(a)) = \beta(\alpha(a))$ for any $a \in \mathcal{A}$.

Then the algebra $(\mathcal{A} \rtimes_\beta \mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z}_N$ is isomorphic to $(\mathcal{A} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}$, where the action $\hat{\alpha}$ of $\mathbb{Z}$ for its generator U is β on $\mathcal{A}$ and a multiplication by a $\bar{\lambda}$ on the generator h of $\mathbb{C}\mathbb{Z}_N$.

Proof. With the notation like above U denotes the generator of $\mathbb{Z}$ and h the generator of $\mathbb{Z}_N$. We have:
$$U \triangleright a = UaU_* = \beta(a), \quad h \triangleright a = hah_* = \alpha(a),$$
$$h \triangleright U = hUh_* = \lambda U \quad \forall a \in \mathcal{A}.$$
The action $\hat{\alpha}$ is defined as follows:
$$U \triangleright a = UaU_* = \beta(a) = \hat{\alpha}(a), \quad \forall a \in \mathcal{A},$$
$$U \triangleright h = UhU_* = \bar{\lambda} h = \hat{\alpha}(h). \tag{5.2}$$
It is easy to see that both crossed product algebras are mutually isomorphic to each other as the defining relations are identical. □

Applying this to the case of the $C(\mathbb{T}^3_\Theta)$ and cross product by the action of $\mathbb{Z}_N$ (N=2,3,4,6) we have:

Corollary 5.4. *The algebra of the noncommutative three-torus $C(\mathbb{T}^3_\Theta)$ equals $C(\mathbb{T}^2_\Theta) \otimes C(\mathbb{T}^1)$ (in the case of torus with the automorphism induced by Bieberbach groups), which can be equivalently rewritten as a crossed product $C(\mathbb{T}^2_\Theta) \rtimes \mathbb{Z}$ with the trivial action of $\mathbb{Z}$. For $N = 2, 3, 4, 6$ the action α of $\mathbb{Z}_N$ on it is by multiplication on the generator of $\mathbb{Z}$ and leaves the algebra $C(\mathbb{T}^2_\theta)$ invariant. This action comes, in fact, from the $SL(2, \mathbb{Z})$ group of automorphisms of $C(\mathbb{T}^2_\Theta)$. Therefore by the Lemma 5.3 we have for $N = 2, 3, 4, 6$ the following isomorphism:*
$$C(\mathbb{T}^3_\theta) \rtimes_\alpha \mathbb{Z}_N \simeq (C(\mathbb{T}^2_\theta) \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}, \tag{5.3}$$
where $\hat{\alpha}(a) = a, \forall a \in C(\mathbb{T}^2_\theta)$ and $\hat{\alpha}(h) = \bar{\lambda} h$, for $h \in \mathbb{C}\mathbb{Z}_n$ (generator of $\mathbb{Z}_N$).

Remark 5.5. *We shall refer several times in our further computation to algebras which fulfils the assumptions of lemma 5.3 and have trivial action β. To fix our terminology we will say that the crossed product algebra*

$$(\mathcal{A} \otimes C(\mathbb{T}^1)) \rtimes_\alpha \mathbb{Z}_N$$

is Bieberbach-like if for h the generator of group $\mathbb{Z}_N$ we have:

- *the action of h is a multiplication by a root of unity on the generator of $C(\mathbb{T}^1)$ denoted U, i.e. $h \triangleright U = e^{\frac{2\pi i}{N}} U$;*
- *restriction of the action α to the C^*-algebra $\mathcal{A}$ gives an automorphism of the latter, i.e. $h \triangleright a = \alpha(a) \in \mathcal{A} \quad \forall a \in \mathcal{A}$.*

By the Lemma 5.3 the Bieberbach-like algebra can be equivalently rewritten as:

$$(\mathcal{A} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}.$$

5.2 The Methods of Computation

We shall now discuss three methods of computation. The first one can be applied only to the case $\mathfrak{B}2_\theta$, while the other two can be used to compute $K-$theory in each of the cases $\mathfrak{B}N_\theta$. In the next sections we shall show how to use all those methods to compute the $K-$theory of our toy model, which is Klein bottle, and after this choose the best one for the computation for Bieberbach manifolds.

5.2.1 Lance-Natsume Six Term Exact Sequence

Let us recall that $G_1 * G_2$ a free product of groups is a group consisting of "words" composed of "letters" (which are elements) of G_1 and G_2. Now having a crossed product algebra $\mathcal{A} \rtimes_\alpha (G * H)$ we define $\mathcal{A} \rtimes_\alpha G_1$ and $\mathcal{A} \rtimes_\alpha G_2$ as another crossed product algebras with the action coming from restriction of α to G_1 or G_2 respectively. In the papers of Lance and Natsume the following theorem was proven.

Theorem 5.6 ([33, 38])**.** *Let $\Gamma = G_1 * G_2$ be a free product of discrete groups G_1 and G_2 and let α denote the action of Γ on the C^*-algebra $\mathcal{A}$. Then for the crossed products there is a six term exact sequence:*

$$\begin{tikzcd}
K_0(\mathcal{A}) \arrow[r, "i_1^* \oplus -i_2^*"] & K_0(\mathcal{A} \rtimes_\alpha G_1) \oplus K_0(\mathcal{A} \rtimes_\alpha G_2) \arrow[r, "j_1^* + j_2^*"] & K_0(\mathcal{A} \rtimes_\alpha \Gamma) \arrow[d] \\
K_1(\mathcal{A} \rtimes_\alpha \Gamma) \arrow[u] & K_1(\mathcal{A} \rtimes_\alpha G_1) \oplus K_1(\mathcal{A} \rtimes_\alpha G_2) \arrow[l, "j_1^* + j_2^*"'] & K_1(\mathcal{A}), \arrow[l, "i_1^* \oplus -i_2^*"']
\end{tikzcd}$$

where the restriction of action of Γ *is also denoted* α. *The horizontal line are morphism of groups induced by the canonical inclusions of algebras:*

$$i_k\colon \mathcal{A} \to \mathcal{A} \rtimes_\alpha G_k, \quad j_k\colon \mathcal{A} \rtimes_\alpha G_k \to \mathcal{A} \rtimes_\alpha (G_1 * G_2),$$

where $k = 1, 2$.

We could use this six term exact sequence providing we know how to rewrite the Bieberbach crossed products as $\mathcal{B} \rtimes (G_1 * G_2)$ for some C^*-algebra $\mathcal{B}$ and groups G_1 and G_2. In the next section we shall show how one can do this in the case of our toy model.

5.2.2 Six Term Exact Sequence for Cyclic Group

The crucial for this subsection is the six term exact sequence obtained by Blackadar in [8]. He describe it as a byproduct of proving the Pimsner-Voiculescu sequence, while we shall use it to get some relevant results concerning a class of crossed product algebras by the finite cyclic groups.

Theorem 5.7 (Blackadar, p. 77). *Let* $\mathcal{B}$ *be a* C^*-*algebra and let* β *be its automorphism such that* $\beta^N = \mathrm{id}$. *Then there is a six term exact sequence for crossed products:*

$$\begin{tikzcd} K_1(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[r] & K_0(\mathcal{B} \rtimes_\beta \mathbb{Z}) \arrow[r, "\pi^*"] & K_0(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[d, "1-\hat{\beta}^*"] \\ K_1(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[u, "1-\hat{\beta}^*"] & K_1(\mathcal{B} \rtimes_\beta \mathbb{Z}) \arrow[l, "\pi^*"'] & K_0(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[l] \end{tikzcd}$$

where connecting morphisms are induced as follows: $\hat{\beta}^*$ *by the action of dual group on the crossed product* $\mathcal{B} \rtimes_\beta \mathbb{Z}_N$ *and* π^* *by canonical surjection of* $\mathbb{C}\mathbb{Z}$ *in* $\mathbb{C}\mathbb{Z}_N$.

We are about to state the lemma dealing with crossed products Morita equivalent to Bieberbach spaces with cyclic groups using the above theorem. Firstly let us recall a theorem which is necessary to this aim.

Theorem 5.8 (Takai-Takesaki duality). *Let* $\mathcal{A}$ *be a* C^*-*algebra and let* G *be a locally compact abelian group. Consider a crossed product* $(\mathcal{A} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ *with its canonical automorphism* $\hat{\hat{\alpha}}$. *Then the following* C^*-*dynamical systems are isomorphic:*

$$((\mathcal{A} \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}; \hat{\hat{\alpha}}) \simeq (\mathcal{A} \otimes \mathbb{K}(G); \alpha \otimes \mathrm{Ad}(\lambda)),$$

where $\mathbb{K}(G)$ *is the* C^*-*algebra of compact operators on* $l^2(G)$ *and* $\mathrm{Ad}(\lambda)$ *is the adjoint action of the left translation* λ *of* G *on* $l^2(G)$.

In the case of Bieberbach-like crossed products we have following lemma.

Lemma 5.9. *Consider a C^*-algebra $\mathcal{C}$ and its automorphism α such that $\mathcal{A} = (\mathcal{C} \otimes C(\mathbb{T}^1)) \rtimes_\alpha \mathbb{Z}_N$ is a Bieberbach-like crossed product (see Remark 5.5). Then there is a six term exact sequence:*

$$\begin{tikzcd}
K_1(\mathcal{C}) \arrow[r] & K_0(\mathcal{A}) \arrow[r, "\pi^*"] & K_0(\mathcal{C}) \arrow[d, "1-\alpha^*"] \\
K_1(\mathcal{C}) \arrow[u, "1-\alpha^*"] & K_1(\mathcal{A}) \arrow[l, "\pi^*"] & K_0(\mathcal{C}) \arrow[l]
\end{tikzcd}$$

The morphisms in the vertical lines are induced by the restricion of action α to the subalgebra $\mathcal{C}$.

Proof. Recall that by the Lemma 5.3 we have the following isomorphism of C^*-algebras:

$$(\mathcal{C} \otimes C(\mathbb{T}^1)) \rtimes_\alpha \mathbb{Z}_N = (\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}.$$

The six term exact sequence of the lemma comes from the application of Theorem 5.7 with $\mathcal{B} = \mathcal{C} \rtimes_\alpha \mathbb{Z}_N$ and $\beta = \mathrm{id} \otimes \hat{\alpha}$.

Note that in the crossed product $(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}_N$ the action $\hat{\alpha}$ is in fact the action of dual group $\mathbb{Z}_N$ and, as such, fulfils the assumptions of Takai-Takesaki duality. What we need to do to prove lemma is to replace $(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}_N$ with $\mathcal{C}$ and $\hat{\hat{\alpha}}$ with α.

To be precise, with the use of the Takai-Takesaki duality, we are free to replace $(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}_N$ with $\mathcal{C} \otimes M_N(\mathbb{C})$ in the six term exact sequence if at the same time we replace $\hat{\hat{\alpha}}$ with $\alpha \otimes \mathrm{Ad}(\lambda)$. Then by the definition of unitarily equivalence of classes in K_0 and K_1 we get $\forall a \in M_\infty(\mathcal{C}), m \in M_N(\mathbb{C}), u \in \mathrm{U}(N)$:

$$[a \otimes m] = [(1 \otimes u)(a \otimes m)(1 \otimes u^*)] = [a \otimes umu^*], \quad .$$

As the adjoin action $\mathrm{Ad}(\lambda)$ is implemented by the unitary in $U(N)$ (called h in the case of Bieberbach spaces) we conclude that:

$$(\alpha \otimes \mathrm{Ad}(\lambda))^*[a \otimes m] = [\alpha(a) \otimes umu^*] = (\alpha \otimes \mathbb{1})^*[a \otimes m]$$

$$\forall a \otimes m \in M_\infty(\mathcal{C}) \otimes M_N(\mathbb{C}).$$

This shows that in the six term exact sequence we can replace the morphisms of K-theory groups via:

$$\alpha^* \otimes \mathrm{Ad}(\lambda)^* = \alpha^* \otimes \mathbb{1}.$$

The last step comes from the fact that K-theory functor is stable, so $K_i(\mathcal{C} \otimes M_N(\mathbb{C})) = K_i(\mathcal{C})$. This ends the proof. $\square$

Corollary 5.10. *Let $\mathfrak{B}N$ be a three-dimensional noncommutative Bieberbach space. We restrict α the action of $\mathbb{Z}_N$ on $C(\mathbb{T}^2_\Theta) \otimes C(\mathbb{T}^1)$ to the noncommutative part $C(\mathbb{T}^2_\Theta)$. Then as a direct consequence of previous lemma we get the six term exact sequence:*

$$\begin{tikzcd}
K_1(C(\mathbb{T}^2_\Theta)) \arrow[r] & K_0(\mathfrak{B}N) \arrow[r, "\pi^*"] & K_0(C(\mathbb{T}^2_\Theta)) \arrow[d, "1-\alpha^*"] \\
K_1(C(\mathbb{T}^2_\Theta)) \arrow[u, "1-\alpha^*"] & K_1(\mathfrak{B}N) \arrow[l, "\pi^*"'] & K_0(C(\mathbb{T}^2_\Theta)) \arrow[l]
\end{tikzcd}$$

5.2.3 The Pimsner-Voiculescu Six Term Exact Sequence

We begin this subsection by the recalling of well known Pimsner-Voiculescu six term exact sequence.

Theorem 5.11 (Pimsner,Voiculescu,1980,[47])**.** *Let $\mathcal{B}$ be a C^*-algebra, let β be its automorphism. Then consider a crossed product $\mathcal{B} \rtimes_\beta \mathbb{Z}$, where for the generator of $\mathbb{Z}$ we have $h \triangleright a = \beta(a)$ for any $a \in \mathcal{B}$. Then the diagram:*

$$\begin{tikzcd}
K_0(\mathcal{B}) \arrow[r, "1-\beta^*"] & K_0(\mathcal{B}) \arrow[r, "i^*"] & K_0(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[d] \\
K_1(\mathcal{B} \rtimes_\beta \mathbb{Z}_N) \arrow[u] & K_1(\mathcal{B}) \arrow[l, "i^*"'] & K_1(\mathcal{B}) \arrow[l, "1-\beta^*"']
\end{tikzcd}$$

is an exact sequence.

We shall use it to obtain the next lemma:

Lemma 5.12. *Let $\mathcal{A} = (\mathcal{C} \otimes C(\mathbb{T}^1)) \rtimes_\alpha \mathbb{Z}_N$ be a Bieberbach-like crossed product (see Remark 5.5) for a C^*-algebra $\mathcal{C}$ and its automorphism α. Then there is a six term exact sequence:*

$$\begin{tikzcd}
K_0(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \arrow[r, "1-\hat{\alpha}^*"] & K_0(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \arrow[r, "i^*"] & K_0(\mathcal{A}) \arrow[d] \\
K_1(\mathcal{A}) \arrow[u] & K_1(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \arrow[l, "i^*"'] & K_1(\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \arrow[l, "1-\hat{\alpha}^*"']
\end{tikzcd} \quad ,$$

where the connecting morphisms $\hat{\alpha}^*$ are induced by the action of dual group on the crossed product $\mathcal{C} \rtimes_\alpha \mathbb{Z}_N$ and i^* are induced by the canonical inclusion.

Proof. First recall that by the Lemma 5.3 we have:

$$(\mathcal{C} \otimes C(\mathbb{T}^1)) \rtimes_\alpha \mathbb{Z}_N = (\mathcal{C} \rtimes_\alpha \mathbb{Z}_N) \rtimes_{\hat{\alpha}} \mathbb{Z}.$$

Then we obtain a six term exact sequence of the Lemma as a direct consequence of Pimsner-Voiculescu sequence with $\mathcal{B} = \mathcal{C} \rtimes_\alpha \mathbb{Z}_N$ and $\beta = \hat{\alpha}$. $\square$

Corollary 5.13. *For three dimensional noncommutative Bieberbach crossed products $C(\mathbb{T}^3_\Theta) \rtimes_\alpha \mathbb{Z}_N$ we immediately get:*

$$\begin{tikzcd} K_0(C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_N) \arrow[r, "1-\hat{\alpha}^*"] & K_0(C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_N) \arrow[r, "i^*"] & K_0(\mathfrak{B}N) \arrow[d] \\ K_1(\mathfrak{B}N) \arrow[u] & K_1(C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_N) \arrow[l, "i^*"] & K_1(C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_N) \arrow[l, "1-\hat{\alpha}^*"] \end{tikzcd}$$

5.3 A toy model - Klein Bottle

While defining classical Bieberbach manifolds we restricted our attention to the orientable case. It is due to the aim of giving description of classical spin structures and propose correspondence to the irreducible real spectral triples. Manifolds which were the object of consideration need to be spin manifolds and as such necessarily orientable. Without this restriction we find an example of nontrivial (i.e. with nontrivial holonomy) two-dimensional Bieberbach manifold, namely Klein bottle.

We will now briefly define the Klein bottle in the C^*-algebraic approach. We shall as previously use the fact that for each Bieberbach manifold torus is a covering space. Let $C(\mathbb{T}^2)$ be a C^*-algebra closure of the algebra of polynomials generated by two commuting unitaries U and V. The group $\mathbb{Z}_2$ (the holonomy of Klein bottle) acts on $C(\mathbb{T}^2)$ via:

$$h \triangleright U = \alpha_h(u) = -U, \quad h \triangleright V = \alpha_h(V) = V^*,$$

for h a nontrivial element of $\mathbb{Z}_2$.

Definition 5.14. *We shall call the fixed point algebra $C(\mathbb{T}^2)^{\mathbb{Z}_2}$ the Klein bottle and denote it $\mathfrak{K}2$.*

It is easy to see using the commutative version of Gelfand-Naimark theorem that $\mathfrak{K}2$ is isomorphic as a C^*-algebra to the complex algebra of continuous functions over a topological Klein bottle.

Lemma 5.15. *Let $\mathfrak{K}2$ be a Klein bottle. Consider the crossed product algebra $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}_2$ generated by two commuting unitaries U, V and a selfadjoint unitary h such that:*

$$hU = -Uh, \quad hV = V^*h, \quad h^2 = 1.$$

Then $\mathfrak{K}2 \otimes M_2(\mathbb{C})$ is isomorphic to $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}_2$.

Proof. We define the element:

$$\hat{h} = U + \frac{1}{2}(1+h)(U^* + U)$$

and the "inverse" relation:

$$\Psi(U) = \hat{h} + \frac{1}{2}(1+h)\hat{h}(U^2 - 1).$$

Then analogously to the Lemma 5.2 the proof that Ψ can be expanded to the isomorphism $\Psi : C(\mathbb{T}^2) \rtimes \mathbb{Z}_2 \to \mathfrak{K}2 \otimes M_2(\mathbb{C})$ comes from the direct use of Takai duality for finite groups. □

$K-$theory of Klein Bottle

We shall now demonstrate three different methods of computing the K-theory of Klein bottle. We use the fact that the algebra $\mathfrak{K}2$ is stably isomorphic to $C(\mathbb{T}^2) \rtimes \mathbb{Z}_2$ and conduct the computation on the latter algebra.

Lemma 5.16. *Let $\mathfrak{K}2$ be a Klein bottle, then;*

$$K_0(\mathfrak{K}2) = \mathbb{Z} \oplus \mathbb{Z}_2 \quad K_1(\mathfrak{K}2) = \mathbb{Z}.$$

Moreover we have $\mathfrak{K}2 \otimes M_2(\mathbb{C}) = C(\mathbb{T}^2) \rtimes \mathbb{Z}_2$ for $i = 0, 1$. Then the generators of $K_i(\mathfrak{K}2)$ written via the representatives from the crossed product algebra are:

$$[\frac{1}{2}(1+h)], \quad [\frac{1}{2}(1+Vh)]$$

for K_0 and

$$[U]$$

for K_1.

I. Lance-Natsume Six Term Exact Sequence

Firstly we shall prove the following:

Lemma 5.17. *Let $C(\mathbb{T}^1) \rtimes_{(h,f)} (\mathbb{Z}_2 * \mathbb{Z}_2)$ be the algebra generated by a unitary U and two elements h, f generating the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ such that:*

$$hUh^* = fUf^* = -U, \quad h^2 = 1, \quad f^2 = 1.$$

Then the following C^*-algebras are isomorphic:

$$C(\mathbb{T}^2) \rtimes_h \mathbb{Z}_2 \simeq C(\mathbb{T}^1) \rtimes_{(h,f)} (\mathbb{Z}_2 * \mathbb{Z}_2).$$

Proof. In the algebra $C(\mathbb{T}^2) \rtimes_h \mathbb{Z}_2$ sits the element Vh. It is a unitary and moreover:

$$(Vh)^2 = 1, \quad (Vh)U = -U(Vh).$$

The algebra $C(\mathbb{T}^2) \rtimes_h \mathbb{Z}_2$ can equivalently described as generated by the set $\{U, Vh, h\}$, but on the other hand it is easy to see that it is the same as $C(\mathbb{T}^1) \rtimes_{(h,f)} (\mathbb{Z}_2 * \mathbb{Z}_2)$ if we just rename $f := Vh$. □

Let us now apply this result to the Lance-Natsume six term exact sequence (Theorem 5.6). To make full use of it we will need to know:

- the K-theory of $C(\mathbb{T}^1)$;
- the K-theory of $C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2$ and $C(\mathbb{T}^1) \rtimes_f \mathbb{Z}_2$;
- the generators of K-theory groups;
- the morphisms induced by inclusions $i_{1,2}$ and $j_{1,2}$.

Of course the simplest of those ingredients is the K-theory of torus, which is $K_0(C(\mathbb{T}^1)) = \mathbb{Z}$ with generator $[1]$ and $K_1(C(\mathbb{T}^1)) = \mathbb{Z}$ with generator $[U]$. A bit less trivial may be the K-theory of crossed products. Firstly let us note that in fact $C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2 = C(\mathbb{T}^1) \rtimes_f \mathbb{Z}_2$, the difference lies only in the name of the generator of $\mathbb{Z}_2$.

Lemma 5.18.

$$K_0(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2) = \mathbb{Z}$$

with generator $[\frac{1}{2}(1+h)]$ *(respectively* $[\frac{1}{2}(1+f)]$*).*

$$K_1(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2) = \mathbb{Z}$$

with generator $[U]$*.*

Proof. Using once again explicit form of Takai-Kishimoto isomorphism (see Lemmas 5.15, 5.2) we obtain that $C(\mathbb{T}^1) \rtimes \mathbb{Z}_2 = C(\mathbb{T}^1)^{\mathbb{Z}_2} \otimes M_2(\mathbb{C})$. Now as $C(\mathbb{T}^1)^{\mathbb{Z}_2} \simeq C(\mathbb{T}^1)$ which $K-$theory we already know we conclude that $K_0(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2) = K_1(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2) = \mathbb{Z}$. Now to find the explicit form of the generators we need to move back through all isomorphisms. Thus we get for $K_0(C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2)$:

$$[1]_{C(\mathbb{T}^1)^{\mathbb{Z}_2}} \to [\mathrm{diag}(1,0)]_{C(\mathbb{T}^1)^{\mathbb{Z}_2} \otimes M_2(\mathbb{C})} \to [\frac{1}{2}(1+h)]_{C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2}$$

and for $K_1(C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2)$:

$$[U^2]_{C(\mathbb{T}^1)^{\mathbb{Z}_2}} \to [\mathrm{diag}(U^2,1)]_{C(\mathbb{T}^1)^{\mathbb{Z}_2} \otimes M_2(\mathbb{C})} \to [U]_{C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2}.$$

The fist arrow is just an inclusion of $C(\mathbb{T}^1)^{\mathbb{Z}_2}$ in $C(\mathbb{T}^1)^{\mathbb{Z}_2} \otimes M_2(\mathbb{C})$. The second arrow in both cases comes from the direct application of the explicit form of Takai-Kishimoto isomorphism (see [31, 53] and Lemma 5.2). The computation of the generators for $C(\mathbb{T}^1) \rtimes_f \mathbb{Z}_2$ are exactly the same so we shall omit them. □

For the generator of $K_0(C(\mathbb{T}^1) \rtimes_h \mathbb{Z}_2)$ we have

$$[\frac{1}{2}(1+h)] = [U(\frac{1}{2}(1+h))U^*] = [\frac{1}{2}(1-h)]$$

and moreovwer $\frac{1}{4}(1+h)(1-h) = 0$, so :

$$[\frac{1}{2}(1+h)] + [\frac{1}{2}(1+h)] = [1]. \tag{5.4}$$

and similarly for $K_0(C(\mathbb{T}^1) \rtimes_f \mathbb{Z}_2)$. We are now ready to explicitly determine the morphisms $i^*_{1,2}$ and $j^*_{1,2}$:

$$\begin{array}{rclrcl} i_1^*([1]) &=& 2[\frac{1}{2}(1+h)], & j_1^*([\frac{1}{2}(1+h)]) &=& [\frac{1}{2}(1+h)], \\ i_1^*([U]) &=& [U], & j_1^*([U]) &=& [U], \\ i_2^*([1]) &=& 2[\frac{1}{2}(1+f)], & j_2^*([\frac{1}{2}(1+f)]) &=& [\frac{1}{2}(1+Vh)], \\ i_2^*([U]) &=& [U], & j_2^*([U]) &=& [U]. \end{array}$$

We shall now recall all we know about $K_i(C(\mathbb{T}^2) \rtimes \mathbb{Z}_2)$, put it to the Lance-Natsume six term exact sequence and use it to compute the $K_i(\mathfrak{K}2)$. We have:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2\oplus -2"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "j_1^*+j_2^*"] & K_0(\mathfrak{K}2) \arrow[d, "0"] \\ K_1(\mathfrak{K}2) \arrow[u, "0"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[l, "j_1^*+j_2^*"'] & \mathbb{Z}, \arrow[l, "1\oplus -1"'] \end{tikzcd}$$

We compute the K-theory groups from the fact that:

$$K_0(\mathfrak{K}2) = \operatorname{coker}(i_1^* \oplus -i_2^*)_{K_0\oplus K_0},$$

$$K_1(\mathfrak{K}2) = \operatorname{coker}(i_1^* \oplus -i_2^*)_{K_1\oplus K_1},$$

where $\operatorname{coker}(i_1^* \oplus -i_2^*)_{K_i\oplus K_i}$, is the cokernel of the morphism

$$i_1^* \oplus -i_2^* : K_i(C(\mathbb{T}^1)) \to K_i(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2) \oplus K_i(C(\mathbb{T}^1) \rtimes \mathbb{Z}_2).$$

When we apply the explicit form of the morphism $i_1^* \oplus -i_2^*$ we obtain:

$$K_0(\mathfrak{K}2) = \mathbb{Z} \oplus \mathbb{Z}_2,$$

Now we can use morphisms j_k^* to determine the generators of $K_i(\mathfrak{K}2)$. As $[\frac{1}{2}(1+h)] \oplus -[\frac{1}{2}(1+f)]$ is not in the image of $i_1^* \oplus -i_2^*$ in the case of K_0 we conclude that $j_1^*([\frac{1}{2}(1+h)]) \neq j_2^*([\frac{1}{2}(1+f)])$. So we see that both $[\frac{1}{2}(1+h)]$ and $[\frac{1}{2}(1+Vh)]$ are independently generators of $K_0(\mathfrak{K}2)$. On the other hand $2[\frac{1}{2}(1+h)] \oplus -2[\frac{1}{2}(1+f)]$ is in the image of $i_1^* \oplus -i_2^*$, so there is a relation:

$$2[\frac{1}{2}(1+h)] = 2[\frac{1}{2}(1+Vh)].$$

It is not surprising, as by the Equation 5.4:

$$2[\frac{1}{2}(1+h)] = 2[\frac{1}{2}(1+f)] = [1].$$

For the $K_1(\mathfrak{K}2)$ from the fact that $[U] \oplus [U^*]$ is in the image of $i_1^* \oplus -i_2^*$ we conclude that:

$$j_1^*([U]) = j_2^*([U]),$$

and so there is only one generator denoted $[U]$.

Remark 5.19. *Using similar consideration one can show that in the case of three-dimensional $B2$ Bieberbach space one obtatains:*

$$C(\mathbb{T}^3_\Theta) \rtimes \mathbb{Z}_2 \simeq C(\mathbb{T}^2) \rtimes (\mathbb{Z}_2 * \mathbb{Z}_2).$$

This can be used to compute the $K_0(B2)$ and $K_1(B2)$. However we shall use other methods to compute it.

II. Six Term Exact Sequence for Cyclic Groups

Let $\mathfrak{K}2 = C(\mathbb{T}^2)^{\mathbb{Z}_2}$ be the Klein bottle, then the crossed product $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}_2$ is Bieberbach-like and we can write it equivalently as $(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2) \rtimes_{\hat{\alpha}} \mathbb{Z}$, where for V the generator of $C(\mathbb{T}^1)$ we have $\alpha(V) = V^*$. Then as a direct consequence of the Lemma 5.9 we get the six term exact sequence:

$$\begin{tikzcd}
K_1(C(\mathbb{T}^1)) \arrow[r] & K_0(\mathfrak{K}2) \arrow[r, "\pi"] & K_0(C(\mathbb{T}^1)) \arrow[d, "1-\alpha^*"] \\
K_1(C(\mathbb{T}^1)) \arrow[u, "1-\alpha^*"] & K_1(\mathfrak{K}2) \arrow[l, "\pi"] & K_0(C(\mathbb{T}^1)) \arrow[l]
\end{tikzcd}$$

We recall that $K_0(C(\mathbb{T}^1)) = K_1(C(\mathbb{T}^1)) = \mathbb{Z}$ and the generators of K_0 and K_1 are $[1]$ and $[U]$ respectively. Moreover the action of $\mathbb{Z}_2$ is defined on the generator of the algebra via $\alpha(V) = V^*$, thus

$$\alpha^*([1]_{K_0}) = [1]_{K_0}, \quad \alpha^*([V]_{K_1}) = -[V]_{K_1}.$$

Then the six term sequence is:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & K_0(\mathfrak{K}2) \arrow[r, "\pi"] & \mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z} \arrow[u, "2"] & K_1(\mathfrak{K}2) \arrow[l, "\pi"'] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Once again we conclude that:

$$K_0(\mathfrak{K}2) = \ker(1-\alpha^*)_{K_0} \oplus \operatorname{coker}(1-\alpha^*)_{K_1} = \mathbb{Z} \oplus \mathbb{Z}_2,$$

$$K_1(\mathfrak{K}2) = \ker(1-\alpha^*)_{K_1} = \mathbb{Z}.$$

Note that using this method we do not know how to find the explicit formula for the generators of $K-$theory groups.

III. Pimsner-Voiculescu Six Term Exact Sequence

Again recall that $\mathfrak{K}2 \otimes M_2(\mathbb{C}) = C(\mathbb{T}^2) \rtimes_\alpha \mathbb{Z}_2$ is Bieberbach-like crossed product. In the case of the Klein bottle the Pimsner-Voiculescu six term exact sequence from Lemma 5.12 yields:

$$\begin{tikzcd}
K_0(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2) \arrow[r, "1-\hat{\alpha}^*"] & K_0(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2) \arrow[r, "i^*"] & K_0(\mathfrak{K}2) \arrow[d] \\
K_1(\mathfrak{K}2) \arrow[u] & K_1(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2) \arrow[l, "i^*"'] & K_1(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2) \arrow[l, "1-\hat{\alpha}^*"']
\end{tikzcd}$$

where the action α on the generator of $C(\mathbb{T}^1)$ is $\alpha(V) = V^*$. It is easy to see that $C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2 = \mathbb{C}(\mathbb{Z}_2 * \mathbb{Z}_2)$. The K-theory of $\mathbb{C}(\mathbb{Z}_2 * \mathbb{Z}_2)$ is known (see for example [38]):

$$K_0(\mathbb{C}(\mathbb{Z}_2 * \mathbb{Z}_2)) = \mathbb{Z}^3, \quad K_1(\mathbb{C}(\mathbb{Z}_2 * \mathbb{Z}_2)) = 0.$$

The generators of K_0 (written through the representatives of $C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2$) are $[\frac{1}{2}(1+h)]$, $[\frac{1}{2}(1-h)]$ and $[\frac{1}{2}(1+hV)]$. We can now compute the endomorphism of $K_0(C(\mathbb{T}^1) \rtimes_\alpha \mathbb{Z}_2)$ induced by $\hat{\alpha}$:

$$\hat{\alpha}^*([\frac{1}{2}(1+h)]) = [\frac{1}{2}(1-h)];$$

$$\hat{\alpha}^*([\frac{1}{2}(1-h)]) = [\frac{1}{2}(1+h)];$$

$$\hat{\alpha}^*([\frac{1}{2}(1+Vh)]) = [\frac{1}{2}(1+h)] + [\frac{1}{2}(1-h)] - [\frac{1}{2}(1+Vh)].$$

Using this we get the following exact sequence:

$$\begin{tikzcd}
\mathbb{Z}^3 \arrow[r, "1-\hat{\alpha}^*"] & \mathbb{Z}^3 \arrow[r, "i^*"] & K_0(\mathfrak{K}2) \arrow[d] \\
K_1(\mathfrak{K}2) \arrow[u] & 0 \arrow[l, "i^*"'] & 0 \arrow[l, "1-\hat{\alpha}^*"']
\end{tikzcd}$$

where the endomorphism $(1-\hat{\alpha}^*)_{K_0}$ equals:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

At the end we obtain:

$$K_0(\mathfrak{K}2) = \operatorname{coker}(1-\hat{\alpha}^*)_{K_0} = \mathbb{Z} \oplus \mathbb{Z}_2,$$

$$K_1(\mathfrak{K}2) = \ker(1-\hat{\alpha}^*)_{K_0} = \mathbb{Z}.$$

Moreover we can use the morphism i^* to determine the generators of K_0. We have:

$$i^*([\frac{1}{2}(1+h)] - [\frac{1}{2}(1-h)]) = 0,$$

$$i^*(-[\frac{1}{2}(1+h)] - [\frac{1}{2}(1-h)] + 2[\frac{1}{2}(1+Vh)]) = 0.$$

Thus as the generators we can take $[\frac{1}{2}(1+h)]_{\mathfrak{K}2}$ and $[\frac{1}{2}(1+Vh)]_{\mathfrak{K}2}$ with the relation:

$$2[\frac{1}{2}(1+h)]_{\mathfrak{K}2} = 2[\frac{1}{2}(1+Vh)]_{\mathfrak{K}2}.$$

5.4 K-theory for Noncommutative Bieberbach Manifolds from the six term exact sequence of Pimsner and Voiculescu

We proceed to the computation of the $K-$theory groups for noncommutative Bieberbach spaces. Firstly we have to choose one of three methods of computation. Note that the first one, i.e. connected to Lance-Natsume six term exact sequence, enables us to carry the computation only for $\mathbb{Z}_2$. As we are interested in the computation of $K-$theory for all cyclic groups we will have to choose one of the other two. The second method does not grant us with any relevant information on the explicit formula for the generator of $K_i(\mathfrak{B}N_\theta)$. Thus the third one, which enables us to explicitly determine the generators of $K_0(\mathfrak{B}N_\theta)$, appears to be the best. We can use this method provided that we know the K-theory groups of the corresponding crossed product of noncommutative torus by the actions of the respective cyclic group $\mathbb{Z}_N$ and the exact form of the action of $\mathbb{Z}$ on the generators of these K-theory groups. Luckily in the literature on the subject we can find the necessary results .

The crossed product algebras of the noncommutative two torus by the cyclic subgroups of $SL(2,\mathbb{Z})$ have been studied recently by Echterhoff et al. as symmetric noncommutative tori and noncommutative spheres [23]. Although classically they correspond to orbifolds rather than to manifolds, we can nevertheless view the noncommutative Bieberbach algebras as circle bundles over some noncommutative spheres. One of the result of their consideration was the computation of $K_i(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N)$. They have proved that for each $N=2,3,4,6$ the group $K_0(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N)$ is a finite direct sum of the groups of integers and $K_1(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N)=0$. Thus in the Pimsner-Voiculescu six term exact sequence two groups are trivial and there is only one morphism, which we have to determine explicitly. It occurs that this method is not only the most productive but also the most transparent one. We shall now sketch briefly the schedule of investigation, while the precise computation is presented in the next subsections.

Corollary 5.20. *Using the Lemma 5.12 (see also Corollary 5.13) and by the results of Echterhoff et al. discussed above (see also the original paper [23]) we have the following six term exact sequence of $K-$theory groups for each $N=2,3,4,6$:*

$$\begin{tikzcd}
K_0(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N) \arrow[r, "1-\hat{\alpha}^*"] & K_0(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N) \arrow[r, "i^*"] & K_0(\mathfrak{B}N_\theta) \arrow[d] \\
K_1(\mathfrak{B}N_\theta) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Thus we can compute the $K-$theory of noncommutative Bieberbach spaces via:

$$K_0(\mathfrak{B}N_\theta)=\mathrm{coker}(1-\hat{\alpha}^*)_{K_0},$$
$$K_1(\mathfrak{B}N_\theta)=\ker(1-\hat{\alpha}^*)_{K_0}.$$

Moreover using morphism i^* we can determine the generators of $K_0(\mathfrak{B}N_\theta)$ if only we can express the generators of $\mathrm{coker}(1-\hat{\alpha}^*)$ *as representatives in $K_0(C(\mathbb{T}^2_\Theta)\rtimes_\alpha \mathbb{Z}_N)$.*

The only difficulty is to determine the morphism $(1-\hat{\alpha}^*)_{K_0}$. In the mentioned paper [23] by Echterhoff et al. the generators of K_0 for the crossed product $C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_N$ were characterised explicitly. The basic tool in our computation is the existence of traces on the dense subalgebra of the crossed product algebra of n oncommutative two torus by a finite cyclic group and their behaviour under the action of $\hat{\alpha}$, which we shall discuss now.

5.4.1 Traces and Twisted Traces

The origin of the additional traces is easy to understand if one recalls that in fact they come from twisted traces on the algebra of the noncommutative torus itself.

Remark 5.21. *Let $\mathcal{B}$ be an algebra and let β denote an action of a finite cyclic group $\mathbb{Z}_N$. If Φ_s is an β-invariant and β^s-twisted trace on $\mathcal{B}$, i.e. for $0 \le s < N$:*

$$\Phi_s(\beta(a)) = \Phi_s(a), \qquad \Phi_s(ab) = \Phi_s(\beta^s(b)a), \;\; \forall a, b \in \mathcal{A},$$

then Φ_s extends to a trace $\tilde{\Phi}_s$ on the crossed product algebra $\mathcal{B} \rtimes \mathbb{Z}_N$ via:

$$\tilde{\Phi}_s\left(\sum_{k=0}^{N-1} a_k h^k\right) = \Phi_s(a_{N-s}), \;\; 0 < s \le N.$$

where h is the generator of $\mathbb{Z}_N$.

The proof of the fact is a simple computation:

$$\begin{aligned}
\tilde{\Phi}_s\left(\left(\sum_{k=0}^{N-1} a_k h^k\right)\left(\sum_{j=0}^{N-1} b_j h^j\right)\right) &= \tilde{\Phi}_s\left(\sum_{k,j=0}^{N-1} a_k \beta^k(b_j) h^{k+j}\right) \\
= \sum_{k+j=N-s} \Phi_s\left(a_k \beta^k(b_j)\right) &= \sum_{k+j=N-s} \Phi_s\left(\beta^{k+s}(b_j) a_k\right) \\
= \sum_{k+j=N-s} \Phi_s\left(b_j \beta^j(a_k)\right) &= \tilde{\Phi}_s\left(\left(\sum_{k=0}^{N} b_j e^j\right)\left(\sum_{j=0}^{N} a_k e^k\right)\right).
\end{aligned}$$

In a series of papers Walters, Buck and Walters demonstrated the following crucial theorem:

Theorem 5.22. *Let $C(\mathbb{T}^2_\Theta)$ be the irrational rotation algebra. Then for $\mathbb{Z}_N, N = 2, 3, 4, 6$ with the actions (on the generators V, W) given in table (4.6) there exists a family of unbounded traces on the algebra $C(\mathbb{T}^2_\Theta) \rtimes \mathbb{Z}_N$, which together with the canonical trace τ on $C(\mathbb{T}^2_\Theta)$ provide an injective morphism from the K_0-group into $\mathbb{C}^{r(N)}$, for some $r(N)$.*

The proofs and the exact form of these traces and their value on the generators of K_0-group are to be found in [9, 10, 56, 57, 58]. We skip the presentation of details, showing as an illustration an example of the $N=2$ case. Following [56, page 592] we see that there are four unbounded traces on $C(\mathbb{T}^2_\Theta) \rtimes_\alpha \mathbb{Z}_2$: τ_{jk}, $j,k=0,1$, which are defined as follows on the basis of $C(\mathbb{T}^2_\theta) \rtimes_\alpha \mathbb{Z}_2$:

$$\tau_{jk}(V^\iota W^\kappa p^\rho) = 4e^{-\pi i\theta\iota\kappa}\delta_1^\rho \delta_j^{\bar\iota}\delta_k^{\bar\kappa}, \quad \iota,\kappa \in \mathbb{Z}, \rho = 0,1, \tag{5.5}$$

where $\bar{x} = x \mod 2$. The other cases ($N=3,4,6$) can be treated similarly.

The computation done by Walters shows that for θ rational the collection of traces provides no longer an injective map from K_0 into $\mathbb{C}^{r(N)}$. However, if one adds the nontrivial cyclic two-cocycle then it is again an injective morphism.

To have all necessary tools we only need to study the behaviour of the traces under the action of $\hat\alpha$.

Lemma 5.23. *If $\tilde\Phi_s$ is a trace on a $C(\mathbb{T}^2_\theta) \rtimes_\alpha \mathbb{Z}_N$, which comes from a α^s-twisted trace, then under the action of $\mathbb{Z}$ by $\hat\alpha$ we have:*

$$\tilde\Phi_s(\hat\alpha(a)) = e^{\frac{2\pi i s}{N}}\tilde\Phi_s(a), \;\; \forall a \in C(\mathbb{T}^2_\theta) \rtimes_\alpha \mathbb{Z}_N \tag{5.6}$$

Proof. The above property follows directly from the form of the action $\hat\alpha$ (5.2).

$$\begin{aligned}\tilde\Phi_s\left(\hat\alpha\left(\sum_{k=0}^N a_k h^k\right)\right) &= \tilde\Phi_s\left(\sum_{k=0}^N e^{-\frac{2\pi i}{N}k} a_k h e^k\right) \\ &= e^{-\frac{2\pi i}{N}(N-s)}\Phi_s(a_{N-s}) = e^{\frac{2\pi i s}{N}}\tilde\Phi_s\left(\sum_{k=0}^N a_k h^k\right).\end{aligned}$$

Observe, that, in particular, taking $s=0$ we see that the usual trace τ is $\hat\alpha$ invariant. □

5.4.2 $K-$theory of $\mathfrak{B}2_\theta$

In the case of the $\mathbb{Z}_2$ group action, the algebra $C(\mathbb{T}^2_\theta) \rtimes_\alpha \mathbb{Z}_2$ is one of the most studied and we have at our disposal all the necessary results.

Theorem 5.24 ([32], [56], [23])**.**

$$K_0(C(\mathbb{T}^2_\Theta) \rtimes \mathbb{Z}_2) = \mathbb{Z}^6, \quad K_1(C(\mathbb{T}^2_\Theta) \rtimes \mathbb{Z}_2) = 0$$

and the generators of K_0 group are:

$$[1], \;\; [e_{00}], \;\; [e_{01}], \;\; [e_{10}], \;\; [e_{11}], \;\; [\mathcal{M}_2],$$

where:

$$e_{00} = \frac{1}{2}(1+h), \;\; e_{01} = \frac{1}{2}(1+Vh), \;\; e_{10} = \frac{1}{2}(1+Wh), \;\; e_{11} = \frac{1}{2}(1+e^{i\pi\theta}VWh),$$

and $\mathcal{M}_2$ is a module, which (in the irrational case only) comes from the projection $h_{\mathcal{M}_2}$ of the form $\frac{1}{2}e_\theta(1+h)$ for a $\mathbb{Z}_2$-invariant Powers-Rieffel e_θ projection of trace θ.

To compute the explicit form of the action of the automorphism α on the above generators we use the Chern-Connes map from K_0. Using the usual trace and the unbounded traces (which we wrote explicitly) one has [56]:

generator		α twisted traces				
	τ	τ_{00}	τ_{01}	τ_{10}	τ_{11}	C
$[1]$	1	0	0	0	0	0
$[\mathcal{M}_2]$	$\frac{1}{2}\theta$	1	$-\epsilon$	ϵ	-1	1
$[e_{01}]$	$\frac{1}{2}$	2	0	0	0	0
$[e_{10}]$	$\frac{1}{2}$	0	2	0	0	0
$[e_{01}]$	$\frac{1}{2}$	0	0	2	0	0
$[e_{10}]$	$\frac{1}{2}$	0	0	0	2	0

Table 5.1: Value of traces on the generators of $K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_2)$

where ϵ is $+1$ for $0 < \theta < \frac{1}{2}$ and -1 for $\frac{1}{2} < \theta < 1$. Here, C denotes the canonical nontrivial cyclic two-cocycle over smooth sub algebra of $C(\mathbb{T}^2_\theta)$ (which naturally extends to its crossed product with $\mathbb{Z}_2$). The actual form of the cocycle is not relevant, what matters is that its pairing with generators of K_0 group is nonzero only for $\mathcal{M}$ (and it has been chosen to be 1).

Proposition 5.25.

$$K_0(\mathfrak{B}2_\theta) = \mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2, \qquad K_1(\mathfrak{B}2_\theta) = \mathbb{Z}^2.$$

Proof. First, combining (5.23) and (5.1) with the Theorem 5.22 we obtain that the induced morphism $\hat{\alpha}^*$ on the generators of K_0 is as follows:

$$\hat{\alpha}^*([e_{00}]) = [1] - [e_{00}], \qquad \hat{\alpha}^*([e_{10}]) = [1] - [e_{10}],$$
$$\hat{\alpha}^*([e_{01}]) = [1] - [e_{01}], \qquad \hat{\alpha}^*([e_{11}]) = [1] - [e_{11}],$$

and

$$\hat{\alpha}^*([\mathcal{M}_2]) = [\mathcal{M}_2] - ([e_{00}] - [e_{11}]) - \epsilon\,([e_{10}] - [e_{01}])\,, \;\; \hat{\alpha}^*([1]) = [1].$$

Using the above results we obtain the Pimsner-Voiculescu exact sequence:

$$\begin{tikzcd}
\mathbb{Z}^6 \arrow[r, "1-\hat{\alpha}^*"] & \mathbb{Z}^6 \arrow[r] & K_0(\mathfrak{B}2_\theta) \arrow[d] \\
K_1(\mathfrak{B}2_\theta) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $1-\hat{\alpha}^*$ has the form:

$$1-\hat{\alpha}^* = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 2 & 0 & -\epsilon \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Immediately we have:

$$\ker(1-\hat{\alpha}^*) = \mathbb{Z}^2, \quad \mathrm{Im}(1-\hat{\alpha}^*) = \mathbb{Z}^4,$$

and basic algebra computations give us the result that the kernel of the map and the quotient by its image, which are independent of the value of ϵ. $\square$

The generators of $K_0(\mathfrak{B}2_\theta)$ are $[\mathcal{M}_2]$, $[\frac{1}{2}(1+h)]$, $[\frac{1}{2}(1+Vh)]$ and $[\frac{1}{2}(1+Wh)]$, regarding relation:

$$2[\frac{1}{2}(1+h)] = 2[\frac{1}{2}(1+Vh)] = 2[\frac{1}{2}(1+Wh)] = [1].$$

5.4.3 $K-$theory of $\mathfrak{B}3_\theta$

Here we need to use a similar type of result as the one obtained for the $\mathbb{Z}_2$ action.

Theorem 5.26 ([9])**.** *The K-theory groups and generators of $C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_3$ are:*

$$K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_3) = \mathbb{Z}^8, \quad K_1(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_3) = 0,$$

with the generators of K_0 group:

$$[1],\ [Q_0(X)],\ [Q_0(Y)],\ [Q_0(h)],\ [Q_1(X)],\ [Q_1(Y)],\ [Q_1(h)],\ [\mathcal{M}_3],$$

where:

$$Q_j(x) = \frac{1}{3}\left(1 + e^{\frac{2\pi ji}{3}}x + e^{\frac{4\pi ji}{3}}x^2\right),$$

and

$$X = e^{\frac{1}{3}\pi i\theta} V h, \;\; Y = e^{\frac{2}{3}\pi i\theta} V^2 h,$$

with h, being the generator of $\mathbb{Z}_3$ group, $h^3 = 1$. The generator $\mathcal{M}_3$ corresponds to an exotic module related to the nontrivial $\mathbb{Z}_3-$invariant projective module over the irrational rotation algebra.

Lemma 5.27. *The action of the group $\mathbb{Z}_3$ on the above generators of K-theory is as follows:*

$$\hat{\alpha}^*([1]) = [1], \qquad \hat{\alpha}^*([\mathcal{M}_3]) = [\mathcal{M}_3]-[Q_0(h)]-[Q_0(X)]-[Q_0(Y)]+[1],$$
$$\hat{\alpha}^*([Q_1(x)]) = [Q_0(x)], \qquad \hat{\alpha}^*([Q_0(x)]) = [1] - [Q_0(x)] - [Q_1(x)],$$

for any $x = h, X, Y$.

Proof. Since the action is nontrivial only on the generator h of the crossed product algebra, all results concerning $Q_j(h), Q_j(X), Q_j(Y)$ are immediate. The only nontrivial part concerns $\mathcal{M}_3$. For this we again use the unbounded traces and the injectivity of the associated Connes-Chern character.

generator		α twisted traces			
	τ	S_{10}	S_{11}	S_{12}	C
$[1]$	1	0	0	0	0
$[\mathcal{M}_3]$	$\frac{\theta}{3}$	$\frac{-i}{3\sqrt{3}}e^{\frac{2\pi i}{3}}$	$\frac{-i}{3\sqrt{3}}e^{\frac{2\pi i}{3}}$	$\frac{-i}{3\sqrt{3}}e^{\frac{2\pi i}{3}}$	1
$[Q_0(h)]$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
$[Q_1(h)]$	$\frac{1}{3}$	$\frac{1}{3}e^{-\frac{2\pi i}{3}}$	0	0	0
$[Q_0(X)]$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0
$[Q_1(X)]$	$\frac{1}{3}$	0	0	$\frac{1}{3}e^{-\frac{2\pi i}{3}}$	0
$[Q_0(Y)]$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0
$[Q_1(Y)]$	$\frac{1}{3}$	0	$\frac{1}{3}e^{-\frac{2\pi i}{3}}$	0	0

Table 5.2: Value of traces on the generators of $K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_3)$

The values of the traces on the generators of K_0 group are taken from [9, Theorem 1.2] (with the same notation) and from them we read out the action of $\hat{\alpha}^*$. $\square$

Proposition 5.28. *The K-theory groups of $\mathfrak{B}3_\theta$ are:*

$$K_0(\mathfrak{B}3_\theta) = \mathbb{Z}^2 \oplus \mathbb{Z}_3, \qquad K_1(\mathfrak{B}3_\theta) = \mathbb{Z}^2.$$

Proof. From the Pimsner-Voiculescu (5.12) exact sequence:

$$\begin{tikzcd}
\mathbb{Z}^8 \arrow[r, "1-\hat{\alpha}^*"] & \mathbb{Z}^8 \arrow[r] & K_0(\mathfrak{B}3_\theta) \arrow[d] \\
K_1(\mathfrak{B}3_\theta) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

taking into account Lemma 5.27 we see that the matrix giving the map $1-\hat{\alpha}^*$ on the basis of $\mathbb{Z}^8$ ($[1], [Q_1(h)], [Q_0(h)], [Q_1(X)], [Q_0(X)], [Q_1(Y)], [Q_0(Y)], [\mathcal{M}_3]$) is:

$$1-\hat{\alpha}^* = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

The generators of $K_0(\mathfrak{B}3_\theta)$ are $[\frac{1}{3}(1+h+h^2)]$, $[\frac{1}{3}(1+e^{\frac{1}{3}\pi i\theta}Vh+e^{-\frac{1}{3}\pi i\theta}Wh)]$ and $[\mathcal{M}_3]$ with relation:

$$3[\frac{1}{3}(1+h+h^2)] = 3[\frac{1}{3}(1+e^{\frac{1}{3}\pi i\theta}Vh+e^{-\frac{1}{3}\pi i\theta}Wh)] = [1].$$

5.4.4 $K-$theory of $\mathfrak{B}4_\theta$

Let us begin with the following result:

Lemma 5.29 (Theorem 2.1 [57],[58])**.** *The K-groups of $\mathbb{T}^2_\theta \rtimes \mathbb{Z}_4$ are*

$$K_0(\mathbb{T}^2_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^9, \quad K_1(\mathbb{T}^2_\theta \rtimes \mathbb{Z}_4) = 0.$$

The generators are:

$$[1],\ [Q_0(h)],\ [Q_1(h)],\ [Q_2(h)],\ [Q_0(e^{\frac{\pi i\theta}{2}}Vh)],$$
$$[Q_1(e^{\frac{\pi i\theta}{2}}Vh)],\ [Q_2(e^{\frac{\pi i\theta}{2}}Vh)],\ [Q_0(Vh^2)],\ [\mathcal{M}_4],$$

where

$$Q_k(x) = \frac{1}{4}(1+(i^kx)+(i^kx)^2+(i^kx)^3),\ \ k=0,1,2,$$

and $\mathcal{M}_4$ is the nontrivial module arising from the nontrivial projective module over the noncommutative torus.

As usual we have used the values of the traces (cited with original notation):

generator		α twisted traces		α^2 twisted traces			
	τ	T_{10}	T_{11}	T_{20}	T_{21}	T_{22}	C
$[1]$	1	0	0	0	0	0	0
$[\mathcal{M}_4]$	$\frac{\theta}{4}$	$\frac{1+-i}{8}$	$\frac{1+-i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	1
$[Q_k(h)]$	$\frac{1}{4}$	$(-i)^k\frac{1}{4}$	0	$(-i)^k\frac{1}{4}$	0	0	0
$[Q_k(e^{\frac{\pi i\theta}{2}}Vh)]$	$\frac{1}{4}$	0	$(-i)^k\frac{1}{4}$	0	$(-i)^k\frac{1}{4}$	0	0
$[Q_0(Vh^2)]$	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	0

Table 5.3: Value of traces on the generators of $K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_4)$

Again, using an injective morphism coming from the Chern-Connes character from $K_0(C(\mathbb{T}^2_\Theta) \rtimes \mathbb{Z}_4)$ to $\mathbb{R}^5 \times \mathbb{C}^2$ (that does not come as a surprise as the action of $\hat{\alpha}$ is in case of some of the unbounded traces multiplication by $\pm i$) an the explicit computation of the traces [58, page 640] we obtain the following result:

Lemma 5.30. *The action of the group $\mathbb{Z}$ on the above generators of K-theory is:*

$$\hat{\alpha}^*([1]) = [1], \qquad \hat{\alpha}^*([Q_0(Vh^2)]) = [1] - [Q_0(Vh^2)],$$
$$\hat{\alpha}^*([Q_2(x)]) = [Q_1(x)], \qquad \hat{\alpha}^*([Q_1(x)]) = [Q_0(x)],$$
$$\hat{\alpha}^*([Q_0(x)]) = [1] - [Q_0(x)] - [Q_1(x)] - [Q_2(x)],$$
$$\hat{\alpha}^*([\mathcal{M}_4]) = [\mathcal{M}_4] - [Q_0(Vh^2)] - [Q_0(h)] - [Q_0(e^{\frac{\pi i\theta}{2}}Vh)] + [1],$$

for any $x = h, e^{\frac{\pi i\theta}{2}}Vh$.

Proposition 5.31. *The K-theory groups of $\mathfrak{B}4_\theta$ are:*

$$K_0(\mathfrak{B}4_\theta) = \mathbb{Z}^2 \oplus \mathbb{Z}_2, \qquad K_1(\mathfrak{B}4_\theta) = \mathbb{Z}^2,$$

Proof. From the Pimsner-Voiculescu exact sequence:

$$\begin{tikzcd}
\mathbb{Z}^9 \arrow[r, "1-\hat{\alpha}^*"] & \mathbb{Z}^9 \arrow[r] & K_1(\mathfrak{B}4_\theta) \arrow[d] \\
K_0(\mathfrak{B}4_\theta) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

using (5.30) we see that the matrix giving $1-\hat{\alpha}^*$ on the basis of $K_0(C(T^2_\theta) \rtimes \mathbb{Z}_4)$ (in this order: $[1]$,$[Q_2(h)]$, $[Q_1(h)]$, $[Q_0(h)]$, $[Q_2(e^{\frac{\pi i\theta}{2}}Vh)]$, $[Q_1(e^{\frac{\pi i\theta}{2}}Vh)]$, $[Q_0(e^{\frac{\pi i\theta}{2}}Vh)]$, $[Q_0(Vh^2)]$, $[\mathcal{M}_4]$) is:

$$1-\hat{\alpha}^* = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

The generators of $K_0(\mathfrak{B}4_\theta)$ as the representatives from $C(\mathbb{T}^3_\Theta) \rtimes_\alpha \mathbb{Z}_4$ are: $[\mathcal{M}_4]$, $[\frac{1}{4}(1+h+h^2+h^3)]$ and $[\frac{1}{4}(1+e^{\frac{\pi i\theta}{2}}Vh+e^{\pi i\theta}VWh^2+e^{-\frac{\pi i\theta}{2}}Wh)]$. Moreover there are relations:

$$2[\frac{1}{4}(1+h+h^2+h^3)] = 2[\frac{1}{4}(1+e^{\frac{\pi i\theta}{2}}Vh+e^{\pi i\theta}VWh^2+e^{-\frac{\pi i\theta}{2}}Wh)]$$

and

$$4[\frac{1}{4}(1+h+h^2+h^3)] = [1].$$

5.4.5 $K-$theory of $\mathfrak{B}6_\theta$

Similarly as in the case of cubic transform we use the results of hexic transform [9, 10].

Theorem 5.32 ([9], Theorem 1.1)**.** *The K-theory groups and generators of $C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_6$ are:*

$$K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_6) = \mathbb{Z}^{10}, \quad K_1(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_6) = 0,$$

with the generators of K_0 group:

$$[1], [\mathcal{M}_6], [Q_0(e^{\frac{\pi i}{3}}Vh^2)], [Q_2(e^{\frac{\pi i}{3}}Vh^2)], [Q_0(Vh^3)], [Q_k(h)], k = 0,1,2,3,4,$$

where:

$$Q_k(x) = \frac{1}{6}\sum_{n=0}^{5} e^{\frac{2\pi nki}{3}}x^n, \quad k = 0,1,2,3,4.$$

and the generator $\mathcal{M}_6$ is again the exotic one.

Lemma 5.33. *The action of the group $\mathbb{Z}$ through $\hat{\alpha}^*$ on the above generators of K-theory is:*

$$\hat{\alpha}^*([1]) = [1], \qquad \hat{\alpha}^*([Q_0(Vh^3)]) = [1] - [Q_0(Vh^3)],$$

$$\hat{\alpha}^*([Q_{k+1}(h)]) = [Q_k(h)], \;\; k = 1,2,3,4, \qquad \hat{\alpha}^*([Q_2(e^{\frac{\pi i}{3}}Vh^2)]) = [Q_0(e^{\frac{\pi i}{3}}Vh^2)],$$

$$\hat{\alpha}^*([Q_0(h)]) = [1] - \sum_{k=0}^{4}[Q_k(h)],$$

$$\alpha^*([\mathcal{M}_6]) = [\mathcal{M}_6] - [Q_0(h)] - [Q_0(e^{\frac{\pi i}{3}}Vh^2)] - [Q_0(Vh^3)] + [1],$$
$$\alpha^*([Q_0(e^{\frac{\pi i}{3}}Vh^2)]) = [1] - [Q_0(e^{\frac{\pi i}{3}}Vh^2)] - [Q_2(e^{\frac{\pi i}{3}}Vh^2)],$$

Proof. Again the action is immediate to read on the generators $Q_k(h)$, whereas using the property of the twisted traces, their behaviour under $\hat{\alpha}$ and the explicit table giving the values of these traces on the generators [9, Theorem 1.1]:

generator		α twisted trace	α^2 twisted traces		α^3 twisted traces		
	τ	T_{10}	T_{20}	T_{21}	T_{30}	T_{31}	C
$[1]$	1	0	0	0	0	0	0
$[\mathcal{M}_6]$	$\frac{\theta}{6}$	$\frac{1}{6}e^{\frac{\pi i}{3}}$	$\frac{-i}{6\sqrt{3}}e^{\frac{2\pi i}{3}}$	$\frac{-i\sqrt{3}}{6}e^{\frac{2\pi i}{3}}$	$\frac{1}{12}$	$\frac{1}{3}$	1
$[Q_k(h)]$	$\frac{1}{6}$	$\frac{1}{6}e^{-\frac{\pi i}{3}k}$	$\frac{1}{6}e^{-\frac{2\pi i}{3}k}$	$\frac{1}{6}e^{-\frac{2\pi i}{3}k}$	$\frac{(-1)^k}{6}$	$\frac{(-1)^k}{6}$	0
$[Q_0(e^{\frac{\pi i\theta}{3}}Vh)]$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	0	0
$[Q_2(e^{\frac{\pi i\theta}{3}}Vh)]$	$\frac{1}{3}$	0	0	$-\frac{1}{3}e^{\frac{2\pi i}{3}}$	0	0	0
$[Q_0(Vh^3)]$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	0

Table 5.4: Value of traces on the generators of $K_0(C(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}_6)$

We obtain the relations above, in particular the highly nontrivial part concerns $[\mathcal{M}_6]$. $\square$

Proposition 5.34. *The K-theory groups of $\mathfrak{B}6_\theta$ are:*

$$K_0(\mathfrak{B}6_\theta) = \mathbb{Z}^2, \qquad K_1(\mathfrak{B}6_\theta) = \mathbb{Z}^2.$$

Proof. From the Pimsner-Voiculescu exact sequence:

$$\begin{array}{ccccc}
\mathbb{Z}^{10} & \xrightarrow{1-\alpha^*} & \mathbb{Z}^{10} & \longrightarrow & K_1(\mathfrak{B}6_\theta) \\
\uparrow & & & & \downarrow \\
K_0(\mathfrak{B}6_\theta) & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

taking into account (5.33) we see that the matrix giving the map $1-\alpha^*$ on the basis of $K_0(C(\mathbb{T}^2_\theta)\rtimes\mathbb{Z}_6)$ (in the following order) ($[1]$, $[Q_4(p)]$, $[Q_3(p)]$, $[Q_2(p)]$, $[Q_1(p)]$, $[Q_0(p)]$, $[Q_2(e^{\frac{\pi i}{3}}Vp^2)]$, $[Q_0(e^{\frac{\pi i}{3}}Vp^2)]$, $[Q_0(Vp^3)]$, $[\mathcal{M}_6]$) is:

$$1-\alpha^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

□

There are two generators of $K_0(\mathfrak{B}6_\theta)$: $[\frac{1}{6}\sum_{k=0}^{5} h^k]$ and the exotic module $[\mathcal{M}_6]$.

5.5 K-theory of Classical Bieberbach Manifolds

Although the computations we have presented were for the specific case of an irrational value of θ the method works, slightly modified, for the rational case. In particular, the $K-$theory groups of $C(\mathbb{T}^2_\theta)\rtimes\mathbb{Z}_N$ and their generators are independent of θ and remain unchanged (see [23]), which follows from the fact that these are twisted group algebras and their K-theory groups depend on the homotopy class of twisting cocycle, which in this case are, of course, trivial.

Clearly, the explicit form of the generator of the nontrivial module over $C(\mathbb{T}^2_\Theta)$ very much depends on whether Θ is rational. The crucial difference between the rational and irrational case is that to have an injective morphism from the K_0 group of $C(\mathbb{T}^2_\Theta)\rtimes\mathbb{Z}_N$ into $\mathbb{R}^{r(N)}$ one needs to use the nontrivial Chern-Connes character (called second-order Chern character by Walters in [58]) coming from the cyclic two-cocycle over $C(\mathbb{T}^2_\Theta)$.

For $\mathbb{Z}_2$ the original result of Walters [56] is for the value of $\theta\in\mathbb{R}\setminus\mathbb{Q}$ but it is easy to see that the arguments are valid as well for rational θ. For $\mathbb{Z}_4$ the result in [58] is valid for all $\theta\in\mathbb{R}$, and for $\mathbb{Z}_3$ the results are combined in the papers of Buck and Walters [9] and [10], where, again, they are obtained for any value of θ, rational or irrational.

Since the nontrivial Chern-Connes character vanishes on all generators of K_0 group apart from the nontrivial one (which is called a Bott class in [23] and Fourier module by Walters in [58]). For our purpose, the crucial information is the behavior of this character under the action of $\hat{\beta}$. We have:

Lemma 5.35. *For any $N = 2, 3, 4, 6$, the Chern-Connes character induced by the cyclic 2-cocycle over $C(\mathbb{T}^2_\theta)$ is invariant under the action of $\hat{\alpha}$.*

The proof is trivial: since the action of α does not change $C(\mathbb{T}^2_\theta)$ and the trace on it as well as derivations are invariant, so must be the nontrivial Chern-Connes character. Therefore, $\hat{\alpha}^*$ of the nontrivial generator of K_0 group (which we called $\mathcal{M}_N$ for $N = 2, 3, 4, 6$) must be a sum of $\mathcal{M}_N$ with a combination of remaining generators, as is clearly the case in proof of Proposition 5.25 and Lemmas 5.27, 5.30, 5.33.

It is worth mentioning that there exists also a much more general argument, which shows that the K-theory groups of noncommutative Bieberbach manifolds are independent of θ. Using the description of the families of Bieberbach manifolds $\mathfrak{B}\mathrm{N}_\theta$ as C^*-algebra bundles one sees that combining the results of [23] and Proposition 2.2. of [24] they are a $K-$fibration. Therefore the evaluation at any point (any θ) is an isomorphism in $K-$theory and the $K-$theory groups are independent of the value of θ.

Chapter 6

Equivariant Spectral Triples over Noncommutative Bieberbach Manifolds

6.1 Spectral Triples over Bieberbachs

Each noncommutative Bieberbach algebra is a subalgebra of the noncommutative torus. Consider $(\mathcal{A}(\mathbb{T}^3_\Theta), \mathcal{H}, D, J)$ - the spectral triple over the noncommutative three torus. By restricting the representation of $\mathcal{A}(\mathbb{T}^3_\Theta)$ to the noncommutative Bieberbach space $\mathfrak{B}N_\theta$, which is a subalgebra of the latter, we obtain a spectral triple $(\mathfrak{B}N_\theta, \mathcal{H}, D, J)$, which might be, however, reducible. Now let us consider $\mathcal{H}' \subseteq \mathcal{H}$, such that $(\mathfrak{B}N_\theta, \mathcal{H}', D, J)$ is a spectral triple with the representation of $\mathfrak{B}N_\theta$, Dirac operator and real structure restricted to the subspace $\mathcal{H}'$. In such case we shall say that $(\mathfrak{B}N_\theta, \mathcal{H}', D, J)$ comes from the restriction of the $(\mathcal{A}(\mathbb{T}^3_\Theta), \mathcal{H}, D, J)$, the spectral triple over the noncommutative three torus.

In what follows we shall show that, in fact, each spectral triple over Bieberbach space is a restriction of a spectral triple over the three torus. To obtain this result we shall show that each spectral triple over Bieberbach can be lifted to a spectral triple over the noncommutative torus.

6.1.1 The Lift and the Restriction of Spectral Triples

Lemma 6.1. *Let $(\mathfrak{B}N_\theta, \mathcal{H}, D)$ be an irreducible spectral triple over a noncommutative Bieberbach manifold $\mathfrak{B}N_\theta$. Then, there exists a spectral triple over three-torus, such that $(\mathfrak{B}N_\theta, \mathcal{H}, D)$ is its restriction. Moreover the spectral triple over three torus is $\mathbb{Z}_N-$equivariant with respect to the action α.*

Proof. Using explicit form of Takai isomorphism we showed in the previous Chapter (see Section 5.1) that the crossed product algebra $\mathcal{A}(\mathbb{T}^3_\Theta) \rtimes \mathbb{Z}_N$ is isomorphic to the matrix algebra of the noncommutative Biebebarch manifold algebra. To be precise we showed that the C^*-algebra closure of those algebras are isomorphic, but the explicit form of the isomorphism found there serves just as well to prove that:

$$\mathcal{A}(\mathbb{T}^3_\Theta) \rtimes_\alpha \mathbb{Z}_N \simeq \mathfrak{B}N_\theta \otimes M_N(\mathbb{C}).$$

Since we can trivially lift the spectral triple from $\mathfrak{B}\mathrm{N}_\theta$ to $\mathfrak{B}\mathrm{N}_\theta \otimes M_N(\mathbb{C})$ using the above isomorphism we obtain a spectral triple over $\mathcal{A}(\mathbb{T}^3_\Theta) \rtimes \mathbb{Z}_N$. As $\mathcal{A}(\mathbb{T}^3_\Theta)$ is a sub algebra of $\mathcal{A}(\mathbb{T}^3_\Theta) \rtimes \mathbb{Z}_N$ we obtain, in turn, a real spectral triple over a three-dimensional noncommutative torus.

In fact, we obtain a spectral triple, which is equivariant with respect to the action of $\mathbb{Z}_N$ group. Clearly, the fact that we have a representation of the crossed product algebra is just a rephrasing of the fact that we have a $\mathbb{Z}_N$-equivariant representation of $\mathcal{A}(\mathbb{T}^3_\Theta)$. By definition, the Dirac operator lifted from the spectral triple over $\mathfrak{B}N_\theta$ commutes with h and $\hat{h}$ (the generator of the group $\mathbb{Z}_N$ and of dual group, see Section 5.1), and therefore commutes with the representation of $\mathbb{Z}_N$. Thus by this construction we have a real, $\mathbb{Z}_N$-equivariant spectral triple over $\mathcal{A}(\mathbb{T}^3_\Theta)$.

It is trivial to see that the original spectral triple over $\mathfrak{B}N_\theta$ is a reduction of the constructed spectral triple by taking the invariant subalgebra of $\mathcal{A}(\mathbb{T}^3_\Theta)$, the $\mathbb{Z}_N$-invariant subspace of $\mathcal{H}$ and the restriction of D to the latter. □

Definition 6.2. *We call the geometry (spectral triple) over the noncommutative Bieberbach manifold $\mathfrak{B}N_\theta$* **flat** *if it is a restriction of a* **flat** *spectral triple over the noncommutative three-torus, that is, the latter being equivariant with respect to the full action of $U(1) \times U(1) \times U(1)$.*

6.1.2 Equivariant Real Spectral Triples over $\mathcal{A}(\mathbb{T}^3_\Theta)$.

Let us take one of the eight canonical equivariant spectral triples over the noncommutative torus $\mathcal{A}(\mathbb{T}^3_\Theta)$ (for a definition of equivariance see [50], for classification of equivariant real spectral triples over a noncommutative two-torus see [44], for a generalization to higher dimensions see [55]).

Let us recall, that the Hilbert space $\mathcal{H}$ is spanned by two copies of $l^2(\mathbb{Z}^3)$, each of them with basis e_μ, where μ is a three-index and each μ_1, μ_2, μ_3 is either integer or half-integer depending on the choice of the spin structure. We parametrize spin structures by ϵ_i $i = 1, 2, 3$, which can take values 0 or $\frac{1}{2}$, so that $\mu_i - \epsilon_i$ is always integer [55].

For $\mathbf{k} = [k_1, k_2, k_3] \in \mathbb{Z}^3$ let us define the generic homogeneous element of the algebra of polynomials over the noncommutative torus:

$$x^{\mathbf{k}} = e^{\pi i \theta k_2 k_3} U^{k_1} V^{k_2} W^{k_3}.$$

We fix the representation of the algebra of the noncommutative torus (relevant for the construction of noncommutative Bieberbach manifolds) on $l^2(\mathbb{Z}^3)$ to be as follows:

$$\pi^{\pm}(x^{\mathbf{k}}) e^{\pm}_{\mu} = e^{\pi i \theta (k_3 \mu_2 - \mu_3 k_2)} e^{\pm}_{\mu + k}, \tag{6.1}$$

and on the Hilbert space $\mathcal{H}$ we take it diagonal:

$$\pi(x) e^{\pm}_{\mu} = \pi(x)^{\pm} e^{\pm}_{\mu}.$$

The real structure J is of the form:

$$J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix},$$

and on the basis of $\mathcal{H}$ we have:

$$J e^{\pm}_{\mu} = \pm e^{\mp}_{-\mu}, \quad \forall \mu \in \mathbb{Z}^3 + \epsilon.$$

The most general equivariant and real Dirac operator (up to rescaling) is of the form:

$$D = \begin{pmatrix} R\delta_1 & \delta_2 + \tau\delta_3 \\ \delta_2 + \tau^*\delta_3 & -R\delta_1 \end{pmatrix}, \tag{6.2}$$

where R is a real parameter and τ a complex parameter (parametrizing the conformal structure of the underlying noncommutative 2-torus).

The derivations δ_i, $i = 1, 2, 3$ act diagonally on the $l^2(\mathbb{Z}^3)$:

$$\delta_i e^{\pm}_{\mu} = \mu_i e^{\pm}_{\mu}.$$

Theorem 6.3 (see [50, 44, 55])**.** *The spectral triple, given by $\mathcal{A}_{\Theta}, \pi, J, D, \mathcal{H}$ is an $U(1)^3$-equivariant real spectral triple.*

Note that the choice of J and the Dirac operator (6.2) has still some unnecessary freedom. Indeed, changing J to $-J$ does not influence any of the axioms. Combining that with a conjugation by Pauli matrices σ^2 or σ^3 we see that we might restrict ourselves to the case $R > 0$.The difference between the choice of τ or τ^* shall be discussed at the end of this chapter.

6.2 The equivariant action of $\mathbb{Z}_N$

In this section we shall define the equivariant action of $\mathbb{Z}_N$ on the spectral triple $(\mathcal{A}(\mathbb{T}^3_\Theta), D, \mathcal{H}, J)$. To be more precise we shall find the conditions under which the spectral triple over three torus is a $\mathbb{Z}_N-$equivariant spectral triple. Of course not every action of group $\mathbb{Z}_N$ is suitable to our needs - the requirements of $\mathbb{Z}_N-$equivariance of spectral triple is rephrased as a $D-$equivariance and $J-$equivariance of the representation of $\mathbb{Z}_N$.

We shall now proceed to the part of computation devoted to determination of the representation ρ. First we introduce a shorthand general notation for the action of the group $\mathbb{Z}_N$ on the generators (and basis) of the algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$. Using the shorthand 6.1 we can write:

$$h \triangleright x^{\mathbf{p}} = e^{2\pi i \frac{p_1}{N}} x^{\mathbf{A}\mathbf{p}}, \tag{6.3}$$

where $p \in \mathbb{Z}^3$, $A = \mathrm{diag}(1, B) \in M_3(\mathbb{Z})$ and $B \in M_2(\mathbb{Z})$ is the following matrix (for each of $N = 2, 3, 4, 6$, respectively):

2	3	4	6
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

(6.4)

The Equation 6.3 comes from the direct application of the action listed in Table 4.6 to the case of elements $x^{\mathbf{p}}$. Note that although the algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$ is not commutative the action 6.3 is exactly the same as in the commutative case.

6.2.1 $D-$equivariance

We shall now present a sequence of lemmas following from the assumption of $D-$equivariance of the representation ρ.

Before we start the classification relevant for the Bieberbach manifolds, let us show another auxiliary lemma, specific for the actions of $\mathbb{Z}_N$ of the type considered.

Lemma 6.4. *Any equivariant representation of $\mathbb{Z}_N$ on $\mathcal{A}(\mathbb{T}^3_\Theta)$ must be diagonal:*

$$\rho(g) = \begin{pmatrix} \rho_+(g) & 0 \\ 0 & \rho_-(g) \end{pmatrix}.$$

and commute with $\delta_1 \otimes 1$.

Proof. First of all, observe that from the definition of equivariance the element $\sigma^1 = \frac{1}{R}U^*[D,U]$ commutes with $\rho(g)$ for any $g \in G$:

$$\rho(g)\left(U^*[D,U]\right) = \alpha_g(U^*)[D,\alpha_g(U)]\rho(g) = U^*[D,U]\rho(g),$$

since the action of g on U is by multiplication by a root of 1. Therefore the action of $\mathbb{Z}_N$ is diagonal and we can treat the two copies independently. Moreover, since $\sigma^1 D + D\sigma^1$ commutes with ρ then we obtain that $\delta_1 \otimes 1$ commutes with ρ as well. □

Let us now define the representation of group $\mathbb{Z}_N$ via the action of its generator on the basis vectors $e^{\pm}_{\mu}$. We denote by h the generator of $\mathbb{Z}_N$. As it follows from Lemma (6.4) ρ is diagonal (as it must be D-equivariant):

$$\rho(h)e^{\pm}_{\mu} = \rho^{\pm}(h)e^{\pm}_{\mu} \quad \forall \mu \in \mathbb{Z}^3 + \epsilon,$$

with $\rho^{\pm}$ being operators on $l^2(\mathbb{Z}^3)$.

Lemma 6.5. *For any $\mu \in \mathbb{Z}^3 + \epsilon$ there exist elements $\sigma^{\pm}_{\mu} \in \mathcal{A}(\mathbb{T}^3_{\Theta})$ such that for h the generator of $\mathbb{Z}_N$:*

$$\rho(h)e^{\pm}_{\mu} = \sigma^{\pm}_{\mu}e^{\pm}_{\mu}.$$

Proof. As the representation ρ is $D-$equivariant it commutes with D^2, thus:

$$D^2\rho(h)e^{\pm}_{\mu} = \rho(h)D^2e^{\pm}_{\mu} = (R^2\mu_1 + |\mu_2 + \tau\mu_3|^2)\rho(h)e^{\pm}_{\mu}.$$

From this we conclude that $\rho(h)e^{\pm}_{\mu}$ is in the eigenspace of D^2 to the eigenvalue $\|\mu\|_{D^2} = R^2\mu_1 + |\mu_2 + \tau\mu_3|^2$. Those are finite dimensional subspaces of $\mathcal{H}$ spanned by the vectors $e^{\pm}_{\kappa}$ with $\|\kappa\|_{D^2} = \|\mu\|_{D^2}$. So there exist complex numbers $\alpha^{\pm}_{\kappa}$ such that:

$$\rho(h)e^{\pm}_{\mu} = \sum_{\kappa \in \|\mu\|_{D^2}} \alpha^{\pm}_{\kappa}x^{(\kappa-\mu)}e^{\pm}_{\mu},$$

where $\kappa - \mu \in \mathbb{Z}^3$. Evidently we have that:

$$\sigma^{\pm}_{\mu} = \sum_{\kappa \in \|\mu\|_{D^2}} \alpha^{\pm}_{\kappa}x^{(\kappa-\mu)} \in \mathcal{A}(\mathbb{T}^3_{\Theta}).$$

□

Space spanned by the elements $x^{\mathbf{p}}e^{\pm}_{\epsilon}$ for $\mathbf{p} \in \mathbb{Z}^3$ is dense in $\mathcal{H}$. From the equivariance of the representation of $\mathbb{Z}_N$ and its action on the algebra for any such element we have:

$$\rho(h)x^{\mathbf{p}}e^{\pm}_{\epsilon} = (h \triangleright x^{\mathbf{p}})\rho(h)e^{\pm}_{\epsilon}.$$

Thus we obtain:

$$\sigma^{\pm}_{\mathbf{p}+\epsilon} = x^{\mathbf{Ap}}\sigma^{\pm}_{\epsilon}x^{-\mathbf{p}}. \tag{6.5}$$

Any representation ρ_σ defined with the use of elements $\sigma^{\pm}_{\mu}$ and respecting Equation 6.5 is in accordance with the action of $\mathbb{Z}_N$ on the algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$ and as such is a candidate to provide a spectral triple over a Bieberbach manifold. Still the elements $\sigma^{\pm}_{\epsilon}$ remain undetermined. We will show that if we assume that this representation is real-equivariant and D-equivariant, i.e. $\rho_\sigma(h)J = J\rho_\sigma(h)$ and $\rho_\sigma(h)D = D\rho_\sigma(h)$, then if it exists is unique. Moreover we will also determine for each N which spin structures over three torus which carry a possibility to define the equivariant action of $\mathbb{Z}_N$.

We shall start from the auxiliary lemma.

Lemma 6.6. *Let $a \in \mathcal{A}(\mathbb{T}^3_\Theta)$. If $[\delta_1, a] = 0$ and $[\delta_2 + \tau\delta_3, a] = 0$ for some $\tau \in \mathbb{C} \setminus \mathbb{R}$ then $a \in \mathbb{C}$.*

Proof. Any element $a \in \mathcal{A}(\mathbb{T}^3_\Theta)$ can be represented through its expansion in polynomials in generators U, V, W, i.e there exist a complex sequence $(a_p)_{p\in\mathbb{Z}^3}$ such that: $a = \sum_{p\in\mathbb{Z}^3} a_p x^{\mathbf{p}}$. Now:

$$[\delta_1, \sum_{p\in\mathbb{Z}^3} a_p x^{\mathbf{p}}] = \sum_{p\in\mathbb{Z}^3} a_p p_1 x^{\mathbf{p}} = 0,$$
$$[\delta_2 + \tau\delta_3, \sum_{p\in\mathbb{Z}^3} a_p x^{\mathbf{p}}] = \sum_{p\in\mathbb{Z}^3} a_p (p_2 + \tau p_3) x^{\mathbf{p}} = 0.$$

implies:

$$a_p p_1 = a_p(p_2 + \tau p_3) = 0 \quad \forall p \in \mathbb{Z}^3.$$

If only $p_2 \neq 0$ or $p_3 \neq 0$ then $p_2 + \tau p_3$ is a nonzero complex number, so of course is invertible. We obtain $a_p = 0$ if at least one of p_1, p_2 or $p_3 \neq 0$ and finely:

$$a = a_0 \in \mathbb{C}.$$

□

From D-equivairance condition we have (for $\mu \in \mathbb{Z}^3 + \epsilon$ and $a = \pm 1$):

$$D\sigma^a_\mu e^a_\mu = \rho(h)De^a_\mu = aR\mu_1\sigma^a_\mu e^a_\mu + (\mu_2 + \tau^{-a}\mu_3)\sigma^{-a}_\mu e^{-a}_\mu.$$

On the other hand we can write:

$$\sigma^a_\mu D e^a_\mu = aR\mu_1 \sigma^a_\mu e^a_\mu + (\mu_2 + \tau^{-a}\mu_3)\sigma^a_\mu e^{-a}_\mu .$$

Now we can combine the above equations to compute $[D, \sigma^a_\mu]e^a_\mu$. Since e^a_μ is a separating vector we get:

$$[\delta_1, \sigma^+_\mu] = 0, \tag{6.6}$$

$$[\delta_1, \sigma^-_\mu] = 0, \tag{6.7}$$

$$[\partial, \sigma^-_\mu] = (\mu_2 + \tau\mu_3)(\sigma^+ - \sigma^-), \tag{6.8}$$

$$[\partial^*, \sigma^+_\mu] = (\mu_2 + \tau^*\mu_3)(\sigma^- - \sigma^+). \tag{6.9}$$

The first and the second is nothing new, as we have already obtained that ρ commutes with δ_1, but the last two provides us a new information. Let us now compute the commutator of $\partial = \delta_2 + \tau\delta_3$ with $\sigma^-_\mu(\sigma^+_\mu)^*$:

$$\begin{aligned}
[\partial, \sigma^-_\mu(\sigma^+_\mu)^*] &= [\partial, \sigma^-_\mu](\sigma^+_\mu)^* - \sigma^-_\mu([\partial^*, \sigma^+_\mu])^* \\
&= (\mu_2 + \tau\mu_3)(\sigma^+_\mu - \sigma^-_\mu)(\sigma^+_\mu)^* - (\mu_2 + \tau\mu_3)\sigma^-_\mu(\sigma^-_\mu - \sigma^+_\mu)^* \\
&= (\mu_2 + \tau\mu_3)(1 - \sigma^-_\mu(\sigma^+_\mu)^* - 1 + \sigma^-_\mu(\sigma^+_\mu)^*) = 0.
\end{aligned}$$

From the Lemma 6.6 we have $\sigma^-_\mu(\sigma^+_\mu)^* \in \mathbb{C}$. We conclude that σ^+_μ and σ^-_μ are mutually proportional unitaries, i.e. $\sigma^+_\mu = \beta_\mu \sigma^-_\mu$ for $\beta_\mu \in \mathbb{C}$. From the fact that σ^a_μ can be expressed as $x^{\mathbf{Ap}}\sigma^a_\epsilon x^{-\mathbf{p}}$ for $\mu = \mathbf{p} + \epsilon$ (see Equation 6.5)we get that factors β_μ are equal for all $\mu \in \mathbb{Z}^3 + \epsilon$. We shall denoted them simply β. Thus we can rewrite the commutation relation $[\partial, \sigma^\pm]$ as follows (Equations 6.8,6.9):

$$[\partial, \sigma^+_\mu] = (\beta - 1)(\mu_2 + \tau\mu_3)\sigma^+_\mu, \tag{6.10}$$

$$[\partial^*, \sigma^+_\mu] = (\beta^* - 1)(\mu_2 + \tau^*\mu_3)\sigma^+_\mu. \tag{6.11}$$

Let us now once again use the fact that $\sigma^a_{\mathbf{p}+\epsilon} = x^{\mathbf{Ap}}\sigma^a_\epsilon x^{-\mathbf{p}}$. We get:

$$[\partial, \sigma^+_{\mathbf{p}+\epsilon}] = [\delta_2 + \tau\delta_3, x^{\mathbf{Ap}}\sigma^a_\epsilon x^{-\mathbf{p}}] \tag{6.12}$$

$$= ((\mathbf{Ap})_2 + \tau(\mathbf{Ap})_3 + (\beta - 1)(\epsilon_2 + \tau\epsilon_3) - p_2 - \tau p_3)\sigma^+_{\mathbf{p}+\epsilon}. \tag{6.13}$$

Thus we obtain a condition on $D-$equivariance:

$$(\mathbf{Ap})_2 + \tau(\mathbf{Ap})_3 = \beta(p_2 + \tau p_3) \quad \forall p_2, p_3 \in \mathbb{Z}.$$

Rewriting this in each of the cases:

$$\begin{aligned} N=2: &\qquad -p_2-\tau p_3=\beta(p_2+\tau p_3),\\ N=3: &\qquad -p_2-p_3+\tau p_2=\beta(p_2+\tau p_3),\\ N=4: &\qquad -p_2+\tau p_2=\beta(p_2+\tau p_3),\\ N=6: &\qquad -p_3+\tau(p_2+p_3)=\beta(p_2+\tau p_3). \end{aligned} \tag{6.14}$$

which are self-consistent only if:

N	β	τ	
2	-1	$\tau\in U(1)\setminus\mathbb{R}$	no restriction on τ
3	$e^{\pm\frac{2\pi i}{3}}$	$e^{\pm\frac{\pi i}{3}}$	$\tau=\sqrt{\beta}$
4	$e^{\pm\frac{\pi i}{2}}$	$e^{\pm\frac{\pi i}{2}}$	$\tau=\beta$
6	$e^{\pm\frac{\pi i}{3}}$	$e^{\pm\frac{\pi i}{3}}$	$\tau=\beta$

Table 6.1: $D-$equivariance condition

As we can see for each $N=2,3,4,6$ we have $\beta=e^{\pm\frac{2\pi i}{N}}$, thus we introduce the parameter $\sigma=\pm1$ such that $\beta=e^{\frac{2\pi i\sigma}{N}}$. When σ is chosen we at the same time fix β and τ as one can see in the table above.

We shall need one more auxiliary lemma:

Lemma 6.7. *Let $a\in\mathcal{A}(\mathbb{T}^3_\Theta)$ and suppose there exist three real numbers q_1,q_2,q_3 such that $[\delta_i,a]=q_ia$. Then $q_1,q_2,q_3\in\mathbb{Z}$ and $a\propto U^{q_1}V^{q_2}W^{q_3}$ or $a=0$.*

Proof. We begin with the observation that a is an element of pre$-C^*-$algebra generated by three unitaries and can be written as a sum of the terms of sequence: $a=\sum_{p\in\mathbb{Z}^3}a_px^{\mathbf{p}}$, where each $a_p\in\mathbb{C}$. Then applying the commutation relation with δ's we get;

$$[\delta_i,\sum_{p\in\mathbb{Z}^3}a_px^{\mathbf{p}}]-q_i\sum_{p\in\mathbb{Z}^3}a_px^{\mathbf{p}}=\sum_{p\in\mathbb{Z}^3}a_p(p_i-q_i)x^{\mathbf{p}}.$$

This provides us with the set of equations: $a_p(p_i-q_i)=0$ which have nonzero solution only if there exists $p_i=q_i$. Moreover at most one a_p is nonzero and then $a=a_px^{\mathbf{p}}$. □

We shall now give the last explicit formulation of $D-$equivariance condition. To do this we use Equations 6.10,6.11 and combine them with the values of τ and β presented in Table 6.1.

$$[\delta_1,\sigma^+_\mu]=0; \tag{6.15}$$

$$[\delta_2,\sigma^+_\mu]=q_2\sigma^+_\mu; \tag{6.16}$$

$$[\delta_3,\sigma^+_\mu]=q_3\sigma^+_\mu. \tag{6.17}$$

where the numbers q_2 and q_3 are presented in the following table for each $N = 2, 3, 4, 6$:

N	q_2	q_3
2	$-2\mu_2$	$-2\mu_3$
3	$-2\mu_2 - \mu_3$	$\mu_2 - \mu_3$
4	$-\mu_2 - \mu_3$	$\mu_2 - \mu_3$
6	$-\mu_2 - \mu_3$	μ_2

Table 6.2: $D-$equivariance condition

We are seeking a nonzero solutions of $[\delta_i, \sigma_\epsilon^+] = q_i \sigma_\epsilon^+$. By the Lemma 6.7 we conclude that numbers q_2, q_3 have to be integers. This condition provides us a restriction on the spin structures:

N	Restriction	Number of possibilities
2	$\epsilon_2 = 0, \frac{1}{2}$ and independently $\epsilon_3 = 0, \frac{1}{2}$	4
3	$\epsilon_2 = \epsilon_3 = 0$	1
4	$\epsilon_2 = \epsilon_3 = 0$ or $\epsilon_2 = \epsilon_3 = \frac{1}{2}$	2
6	$\epsilon_2 = \epsilon_3 = 0$	1

Table 6.3: Spin structure restriction

Now as the last part of the computation conducted in this section we shall determine the $D-$equivariant action of cyclic group up to the phase which shall be determined in the next part of the dissertation. From the Equations 6.15, 6.16, 6.17 we conclude that $\rho(h)e^\pm_\mu$ are eigenvectors of derivations to the eigenvalues listed in the Table 6.2. This together with the fact that the representation ρ is diagonal provides us the following equation:

$$\delta_i \rho(h) e^\pm_\mu = [\delta_i, \sigma^\pm_\mu] e^\pm_\mu + \sigma^\pm_\mu \delta_i e^\pm_\mu = (q_i + \mu_i)\rho(h) e^\pm_\mu.$$

Now we can state the following lemma.

Lemma 6.8. *Let $\mathbb{Z}_N$ be the cyclic group such that $\mathfrak{B}N_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}$. Then the $D-$equivariant and $J-$equivariant representation of $\mathbb{Z}_N$ on the spectral triple over three torus exists only if the spin structure over $\mathcal{A}(\mathbb{T}^3_\Theta)$ is one of the listed in the Table 6.3. After this restriction the action is determined up to the phase as one of the following:*

- $N = 2 :\qquad \rho(h) e^\pm_{\mu_1,\mu_2,\mu_3} \propto e^\pm_{\mu_1,-\mu_2,-\mu_3};$

- $N = 3:\qquad \rho(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} \propto e^{\pm}_{\mu_1,-\mu_2-\mu_3,\mu_2}$;
- $N = 4:\qquad \rho(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} \propto e^{\pm}_{\mu_1,-\mu_3,\mu_2}$;
- $N = 6:\qquad \rho(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} \propto e^{\pm}_{\mu_1,-\mu_3,\mu_2+\mu_3}$.

Remark 6.9. *When we compare the above Lemma 6.8 with the table 6.4 we easily see that the $D-$equivariant action is in fact, as one may have suspected, implemented by the action of matrices $\mathbf{A}$ on the indices μ. To be precise - we can now write the action of $\mathbb{Z}_N$ in a unified way for all $N = 2, 3, 4, 6$:*

$$\rho(h)e^{\pm}_{\mu} \propto e^{\pm}_{\mathbf{A}\mu}.$$

6.2.2 Complete Determination of the Equivariant Representation of $\mathbb{Z}_N$

Here we restrict our attention only to the actions for the spin structures determined in the previous section and we proceed case by case. We begin with technical lemma.

Lemma 6.10. *Let $\mathbf{A}$ be one of matrices presented in Table 6.4. Then:*

$$p_2q_3 - p_3q_2 = (\mathbf{Ap})_2(\mathbf{Aq})_3 - (\mathbf{Ap})_3(\mathbf{Aq})_2 \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^3.$$

Proof. Direct computation in each case. □

Before proceeding to the explicit computations for each case we will state an introductory lemma concerning the computation of the phases of the action ρ. The remaining part of this section is devoted to complete determination of the representation case by case and check which spin structures allows to define the $\mathbb{Z}_N-$equivariant spectral triples over $\mathcal{A}(\mathbb{T}^3_\Theta)$.

Proposition 6.11. *Let ρ be a $D-$equivariant and $J-$equivariant action of group $\mathbb{Z}_N$ on the spectral triple over three torus then:*

$$\rho(h)e^{\pm}_{\mu} = \kappa e^{\frac{2\pi i}{N}(\mu_1 \pm \frac{\sigma}{2})}e^{\pm}_{\mathbf{A}\mu},$$

where $\kappa, \sigma = \pm 1$ and $\beta = e^{\sigma\frac{2\pi i}{N}}$ in accordance with Table 6.1.

Proof. Firstly let us recall that $\rho_+ = e^{\frac{2\pi i}{N}\sigma}\rho_-$ - it fully justifies the $e^{\pm\frac{\pi i}{N}\sigma}$ factor. We will now focus to the ρ_+ action, while the ρ_- will be then determined by $e^{-\frac{2\pi i}{N}\sigma}\rho_+$. We will fix the phase for $\mu = \epsilon$ and compute the unknown factors for the other indices from the definition of the action on $\mathcal{A}(\mathbb{T}^3_\Theta)$. Let us define:

$$\rho_+(h)e^{+}_{\epsilon} = \kappa e^{\frac{2\pi i}{N}(\epsilon_1 + \frac{\sigma}{2})}e^{+}_{\mathbf{A}\epsilon}.$$

For the other $\mu = p + \epsilon$:

$$\begin{aligned}\rho_+(h)e^+_{\mathbf{p}+\epsilon} &= e^{-2\pi i\theta(p_2\epsilon_3 - p_3\epsilon_2)}\rho_+(h)x^{\mathbf{p}}e^+_\epsilon = \\ &= \kappa e^{-2\pi i\theta(p_2\epsilon_3 - p_3\epsilon_2)}e^{\frac{2\pi i}{N}p_1}e^{\frac{2\pi i}{N}(\epsilon_1+\frac{\sigma}{2})}x^{\mathbf{A}\mathbf{p}}e^+_{\mathbf{A}\epsilon} \\ &= \kappa e^{\frac{2\pi i}{N}(p_1+\epsilon_1+\frac{\sigma}{2})}e^{-2\pi i\theta(p_2\epsilon_3 - p_3\epsilon_2)}e^{2\pi i\theta((\mathbf{A}\mathbf{p})_2(\mathbf{A}\epsilon)_3-(\mathbf{A}\mathbf{p})_3(\mathbf{A}\epsilon)_2)}e^+_{\mathbf{A}\mathbf{p}+\mathbf{A}\epsilon}.\end{aligned}$$

Now by the Lemma 6.10 we have $p_2\epsilon_3 - p_3\epsilon_2 = (\mathbf{A}\mathbf{p})_2(\mathbf{A}\epsilon)_3 - (\mathbf{A}\mathbf{p})_3(\mathbf{A}\epsilon)_2$ and the equation simplifies to ($\mu = \mathbf{p} + \epsilon$):

$$\rho_+(h)e^+_\mu = \kappa e^{\frac{2\pi i}{N}(\mu_1+\frac{\sigma}{2})}e^{\pm}_{\mathbf{A}\mu}.$$

We only have to prove that $\kappa = \pm 1$. To do this we use the $J-$equivariance:

$$\rho(h)Je^+_\mu = J\rho(h)e^+_\mu = \kappa^* e^{-\frac{2\pi i}{N}(\mu_1+\frac{\sigma}{2})}e^-_{-\mathbf{A}\mu}.$$

On the other hand:

$$\rho(h)Je^+_\mu = \kappa e^{-\frac{2\pi i}{N}(\mu_1+\frac{\sigma}{2})}e^-_{-\mathbf{A}\mu}.$$

As we see $\kappa \in \mathbb{R}$ and as $|\kappa| = 1$ the only possibility is $\kappa = \pm 1$. □

Remark 6.12. *To summarise we shall briefly recall our results obtained at this point from the $D-$equivariance and $J-$equivariance of the action ρ:*

- *We know that representation is diagonal, i.e. $\rho = \rho_+ \oplus \rho_-$. Moreover we know that in each case there are two possibilities $\rho_+ = e^{\frac{2\pi i}{N}\sigma}\rho_-$, where $\sigma = \pm 1$.*
- *We have a condition connecting the parameter τ in the Dirac operator and the number σ (see Table 6.1).*
- *The $\mathbb{Z}_N-$equivariant spectral triple over three torus is possible only for spin structures listed in the Table 6.3.*
- *For those spin structures the representation is defined through:*

  $$\rho(h)e^\pm_\mu = \kappa e^{\frac{2\pi i}{N}(\mu_1\pm\frac{\sigma}{2})}e^\pm_{\mathbf{A}\mu}.$$

  At this point we still have four possibilities of different representations: $\sigma = \pm 1$ and $\kappa = \pm 1$ for each N.

We shall now proceed to explicit computations case by case.

6.2.3 Specification of the equivariant representation of $\mathbb{Z}_N$ case by case

Equivariant Representation for N = 2

Lemma 6.13. *For a given spin structure on $\mathcal{A}(\mathbb{T}^3_\Theta)$ the following is a most general real-equivariant and diagonal action of $\mathbb{Z}_2$ on $\mathcal{H}$, which implements the action on the algebra from Table 6.4:*

$$\rho(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa(-1)^{\mu_1 \pm \frac{1}{2}} e^{\pm}_{\mu_1,-\mu_2,-\mu_3}. \tag{6.18}$$

where free parameter $\kappa = \pm 1$.

Note, that this is not the canonical action inherited from the action of $\mathbb{Z}_2$ on the algebra itself (unless $\epsilon_2 = \epsilon_3 = 0$).

Proof. From direct application of Lemma 6.11 to the case $N = 2$ we have that:

$$\rho(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa(-1)^{\mu_1 \pm \frac{\sigma}{2}} e^{\pm}_{\mu_1,-\mu_2,-\mu_3}.$$

The representation is invariant under simultaneous change κ to $-\kappa$ and σ to $-\sigma$, so in fact there is only one parameter which governs the possible definition of representation. □

Lemma 6.14. *The only real equivariant and D-equivariant representation of $\mathbb{Z}_2$ exists for $\epsilon_1 = \frac{1}{2}$ choice of the spin structure.*

Proof. Above we have determined the representation of $\mathbb{Z}_2$. The only thing left which has to checked is whether $\rho(h)^2 = \mathrm{id}$. Explicit computation shows:

$$\rho e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa^2(-1)^{2\mu_1 \pm 1} e^{\pm}_{\mu_1,\mu_2,\mu_3}.$$

So we obtain $(-1)^{2\epsilon_1} = -1$ which is possible only if $\epsilon_1 = \frac{1}{2}$. □

Equivariant Representation for N = 3

Lemma 6.15. *For the spin structures $\epsilon_2 = \epsilon_3 = 0$ the following defines the most general real-equivariant action of $\mathbb{Z}_3$, which implements the action on the algebra from Table 6.4.*

$$\rho^{(\sigma)}(h)e_{\mu_1,\mu_2,\mu_3} = \kappa e^{\frac{2\pi i}{3}(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,-\mu_2-\mu_3,\mu_2}, \tag{6.19}$$

where $\sigma, \kappa = \pm 1$.

Proof. Direct application of Lemma 6.11. □

Lemma 6.16. *The $D-$equivariant and $J-$equivariant representation of $\mathbb{Z}_3$ which implements the action on the algebra $\mathcal{A}(\mathbb{T}^3_\Theta)$ presented in the Table 6.4 is one of the following:*

- *for $\epsilon_1 = 0$:*
$$\rho^{(\sigma)}(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = e^{\frac{2\pi i}{3}(\mu_1 \pm \sigma)} e^{\pm}_{\mu_1,-\mu_2-\mu_3,\mu_2}.$$

- *for $\epsilon_1 = \frac{1}{2}$:*
$$\rho^{(\sigma)}(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = -e^{\frac{2\pi i}{3}(\mu_1 \pm \sigma)} e^{\pm}_{\mu_1,-\mu_2-\mu_3,\mu_2}.$$

Proof. As in the former case we have to check whether $\rho(h)^3 = \mathrm{id}$.

$$\rho^{(\sigma)}(h)^3 e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa^3 e^{2\pi i(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa^3 (-1)^{2\epsilon_1}(-1) e^{\pm}_{\mu_1,\mu_2,\mu_3}.$$

Which is the same as $\kappa = -(-1)^{2\epsilon_1}$. To finish the proof we have to apply: $e^{\pm\frac{\pi i}{3}} = -e^{\mp\frac{2\pi i}{3}}$. □

Equivariant Representation for N = 4

Lemma 6.17. *The following defines the most general representation of $\mathbb{Z}_4$ on the Hilbert space $\mathcal{H}$:*

$$\rho^{(\sigma)}(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa e^{\frac{\pi i}{2}(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,-\mu_3,\mu_2}, \tag{6.20}$$

which implements the action of $\mathbb{Z}_4$ on the algebra and is real-equivariant.

Proof. Direct application of the previous Lemma 6.11. □

Lemma 6.18. *The only spectral triple over three torus which is $\mathbb{Z}_4-$equivariant is possible for $\epsilon_1 = \frac{1}{2}$.*

Proof. As usual we check $\rho(h)^4 = \mathrm{id}$:

$$\rho^{(\sigma)}(h)^4 e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa^4 e^{2\pi i(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,\mu_2,\mu_3}.$$

We see that there are no restriction on $\kappa = \pm 1$. On the other hand $-(-1)^{2\epsilon_1} = 1$ which is possible only for $\epsilon_1 = \frac{1}{2}$. □

Equivariant Representation for $\mathbf{N = 6}$

Lemma 6.19. *For the spin structures $\epsilon_2 = \epsilon_3 = 0$ the following defines the most general real-equivariant action of $\mathbb{Z}_6$, which implements the action on the algebra from Table 6.4.*

$$\rho^{(\sigma)}(h)e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa e^{\frac{\pi i}{3}(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,-\mu_3,\mu_2+\mu_3}. \tag{6.21}$$

Lemma 6.20. *The only $\mathbb{Z}_6-$equivariant spectral triple over three torus is possible if $\epsilon_1 = \frac{1}{2}$.*

Proof. As usual we check whether $\rho(h)^6 = \mathrm{id}$:

$$\rho^{(\sigma)}(h)^6 e^{\pm}_{\mu_1,\mu_2,\mu_3} = \kappa^6 e^{2\pi i(\mu_1 \pm \frac{\sigma}{2})} e^{\pm}_{\mu_1,\mu_2,\mu_3}.$$

We see that there are no restriction on $\kappa = \pm 1$, but $\epsilon_1 = \frac{1}{2}$ as the equation $-(-1)^{2\epsilon_1} = 1$ must be fulfilled. □

6.3 Real Flat Spectral Triples

In this section we shall classify all real spectral triples over Bieberbach manifolds, which arise from restriction of flat real spectral triples over the noncommutative torus. To simplify the notation we shall need the notion of a generalized Dirac operator on the circle, $D_{\alpha,\beta}$, which is an operator with the eigenvalues:

$$\lambda_k = \alpha k + \beta, \quad k \in \mathbb{Z}, \alpha, \beta \in \mathbb{R},$$

where $\alpha \in \mathbb{R}_+$ and $0 \leq \beta < 1$. We shall denote its spectrum by $\mathcal{S}p^1_{\alpha,\beta}$. The η invariant of this operator (see [45], Lemma 5.5) is:

$$\eta(D_{\alpha,\beta}) = \mathrm{sgn}(\beta) - \frac{2\beta}{\alpha}.$$

If we have an operator with the same spectrum, however, with a multiplicity $M > 1$, then the η invariant is M-multiple of the computed value.

Using an analogous notation, we shall denote the spectrum of the Dirac operator for a given parameter τ (see Equation 6.2 and Table 6.1) and a given spin structure over $\mathcal{A}(\mathbb{T}^3_\Theta)$ by $\mathcal{S}p^\tau_{\epsilon_1,\epsilon_2,\epsilon_3}$, with the usual multiplicities. In case the multiplicities are changed we shall introduce a factor in front.

We shall determine the $\mathbb{Z}_N$ inavriant subspaces of $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. In the previous sections in the cases $N = 2, 4, 6$ we have obtained the freedom of $\kappa = \pm 1$ (the sign in front of the representation). Of course it does not change the invariant subspaces, so for simplicity we assume that $\kappa = +1$.

6.3.1 Equivariant Real Spectral Triples over $\mathfrak{B}2_\theta$

We define the $\mathbb{Z}_2$ invariant subspaces as follows:

$$\mathcal{H}^{(\pm,j)} :\, = \{\psi \in \mathcal{H}^{\pm} | \rho(h)\psi = (-1)^j \psi\},$$

for $j = 0, 1$. It is easy to see that the representation of $\mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_2}$ restricted to such spaces is faithful.

Proposition 6.21. *The following are the real spectral triples over $\mathfrak{B}2_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_2}$:*

$$\mathcal{S}_0 = (\mathfrak{B}2_\theta, \mathcal{H}^{(+,0)} \oplus \mathcal{H}^{(-,0)}, D, J),$$
$$\mathcal{S}_1 = (\mathfrak{B}2_\theta, \mathcal{H}^{(+,1)} \oplus \mathcal{H}^{(-,1)}, D, J).$$

We have implicitly taken D and J from the three torus and restricted them to the subspaces.

Proof. By the definition of spaces $\mathcal{H}^{(\pm,j)}$ each of them carries a faithful representation of $\mathfrak{B}2_\theta$. From the $D-$equivariance of the action ρ we have (for $\psi \in \mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(-,j)}$):

$$\rho(h)D\psi = D\rho(h)\psi = (-1)^j D\psi.$$

So those subspaces are closed under Dirac operator. Similarly for the real structure:

$$\rho(h)J\psi = J\rho(h)\psi = (-1)^j J\psi.$$

□

We have eight possibilities of real spectral triples. First, the choice of the spin structures over the noncommutative torus given by ϵ_2 and ϵ_3, then there are still two spectral triples $\mathcal{S}_\nu$ where $\nu = 0, 1$. We shall now give the explicit description of vectors spanning $\mathcal{H}^{(\pm,\nu)}$. To this aim we need to distinguish two cases.

If $\epsilon_2 = \frac{1}{2}$ or $\epsilon_3 = \frac{1}{2}$

For $\nu = 0, 1$ the subspace $\mathcal{H}^{(\pm,\nu)}$, which is the $\mathbb{Z}_2-$invariant subspace of the Hilbert space of the spectral triple over the noncommutative torus is spanned by the vectors:

$$\frac{1}{\sqrt{2}} \left(e^{\pm}_{2k \mp \frac{\sigma}{2} + a, \mu_2, \mu_3} + (-1)^{a+\nu} e^{\pm}_{2k \mp \frac{\sigma}{2} + a, -\mu_2, -\mu_3} \right), \qquad (6.22)$$

with $k \in \mathbb{Z}$, $\mu_i \in \mathbb{Z} + \epsilon_i$, $a = 0, 1$.

The spectrum of the Dirac operator, when restricted to the subspace $\mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}$ consists of:

$$\mathcal{S}p^{\mathfrak{B}2_\theta}_{\epsilon_2,\epsilon_3,\nu} \ni \lambda = \pm\sqrt{R^2(2k+a+\frac{1}{2})^2 + |\mu_2+\tau\mu_3|^2}. \tag{6.23}$$

As one can see the spectrum of Dirac operator is the same for both spectral triples, i.e. $\mathcal{S}p^{\mathfrak{B}2_\theta}_{\epsilon_2,\epsilon_3,0} = \mathcal{S}p^{\mathfrak{B}2_\theta}_{\epsilon_2,\epsilon_3,1}$, there is no asymmetry in the spectrum, hence the η invariant vanishes. In fact the spectrum of this Dirac is just the same as the spectrum of the Dirac on the noncommutative torus, with the multiplicities halved, so it is $\frac{1}{2}\mathcal{S}p^{\tau}_{\frac{1}{2},\epsilon_2,\epsilon_3}$.

If $\epsilon_2 = 0$ and $\epsilon_3 = 0$

Clearly, for $\mu_2 \neq 0$ or $\mu_3 \neq 0$ the vectors 6.22 are still the invariant vectors, the spectrum of the Dirac restricted to that subspace is still given by 6.23. This part of the spectrum is, however, not the entire spectrum of the Dirac but only its part, namely:

$$\frac{1}{2}\left(\mathcal{S}p^{\tau}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^{1}_{R,\frac{1}{2}R}\right),$$

which means that we are not counting the spectrum for eigenvectors with $\mu_2 = \mu_3 = 0$. In the latter case we have the invariant subspaces spanned by the:

$$e^{\pm}_{2k\mp\frac{1}{2}+\nu,0,0}, \tag{6.24}$$

and the spectrum of the Dirac operator, restricted to that subspace consists of the following numbers:

$$\mathcal{S}p^{\mathfrak{B}2_\theta}_{0,0,\nu} \ni \lambda = \pm R(2k \mp \frac{1}{2} + \nu) \tag{6.25}$$

for $k \in \mathbb{Z}$.

It is easy to see that these spectra are:

$$\lambda_+ = R(2k+\frac{1}{2}), \quad \nu = 1, k \in \mathbb{Z},$$

which corresponds to $\mathcal{S}p^{1}_{2R,\frac{1}{2}R}$, and

$$\lambda_- = R(2k-\frac{1}{2}), \quad \nu = 0, k \in \mathbb{Z},$$

which gives $\mathcal{S}p^{1}_{2R,-\frac{1}{2}R}$. In each case the multiplicity of the spectrum is 2.

The spectra give different η invariant:

$$\eta(D_\nu^{\mathbb{Z}_2}) = \begin{cases} -1 & \text{if } \nu = 0, \\ +1 & \text{if } \nu = 1. \end{cases}.$$

So, in the end the spectrum is:

$$\frac{1}{2}\left(\mathcal{S}p^{\tau}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^{1}_{R,\frac{1}{2}R}\right) \cup 2\mathcal{S}p^{1}_{2R,\pm\frac{1}{2}R}.$$

6.3.2 Equivariant Real Spectral Triples over $\mathfrak{B}3_\theta$

Let us recall that the representation is:

$$\rho^{(\sigma)}(h)e^{\pm}_{\mu-1,\mu_2,\mu_3} = (-1)^{2\epsilon_1} e^{\frac{2\pi i}{3}(\mu_1 \pm \sigma)} e^{\pm}_{\mu_1,-\mu_2-\mu_3,\mu_2}.$$

As in the previous case we begin with the definition of $\mathbb{Z}_3-$invariant subspaces of $\mathcal{H}$, which carries the faithful representation of $\mathfrak{B}3_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_3}$:

$$\mathcal{H}^{(\pm,\sigma,j)} :\ = \{\psi \in \mathcal{H}^{\pm} | \rho^{(\sigma)}\psi = e^{\frac{2\pi i}{3}j}\psi\},$$

for $j = 0, 12$.

Proposition 6.22. *The following are the spectral triples over* $\mathfrak{B}3_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_3}$*:*

$$\mathcal{S}_{\sigma,0} = (\mathfrak{B}3_\theta, \mathcal{H}^{(+,\sigma,0)} \oplus \mathcal{H}^{(-,\sigma,0)}, D, J),$$

$$\mathcal{S}_{\sigma,1\oplus 2} = (\mathfrak{B}3_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(+,\sigma,2)} \oplus \mathcal{H}^{(-,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,2)}, D, J).$$

Where we have implicitly taken D and J as the restriction of Dirac operator and real structure from the spectral triple over three torus.

Proof. By the definition of the spaces $\mathcal{H}^{(\pm,\sigma,j)}$ each of them is a faithful projective module over $\mathfrak{B}2_\theta$. From the $D-$equivariance of the action ρ we have (for $\psi \in \mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}$):

$$\rho(h)D\psi = D\rho(h)\psi = e^{\frac{2\pi i}{3}j}D\psi.$$

So each subspace consisting of the summands $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}$ is closed under the Dirac operator. On the other hand let $\psi \in \mathcal{H}^{(\pm,\sigma,j)}$, then:

$$\rho(h)J\psi = J\rho(h)\psi = e^{-\frac{2\pi i}{3}j}J\psi \in \mathcal{H}^{(\mp,\sigma,-j)}.$$

So in the case of real structure the space closed under J must be direct sum of spaces $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,-j)}$. The two minimal possibilities of such direct sums are exactly those from the lemma. □

Remark 6.23. *The spectral triples $\mathcal{S}_{\sigma,1\oplus 2}$ are irreducible in a strong sense, i.e. there are no invariant subspaces closed under the algebra $\mathfrak{B}3_\theta$ and operators D and J. On the other hand it is reducible in both weak senses. The computation done above shows that it is $D-$reducible:*

$$\mathcal{S}_{\sigma,1\oplus 2} \simeq (\mathfrak{B}3_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,1)}, D) \oplus (\mathfrak{B}2_\theta, \mathcal{H}^{(+,\sigma,2)} \oplus \mathcal{H}^{(-,\sigma,2)}, D)$$

and $J-$reducible as well:

$$\mathcal{S}_{\sigma,1\oplus 2} \simeq (\mathfrak{B}3_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,2)}, J) \oplus (\mathfrak{B}2_\theta, \mathcal{H}^{(+,\sigma,2)} \oplus \mathcal{H}^{(-,\sigma,1)}, J).$$

As we are interested only in the case of irreducible spectral triples we shall restrict our attention only to the cases $\mathcal{S}_{+,0}$ and $\mathcal{S}_{-,0}$.

Proposition 6.24. *Up to bounded perturbation of the Dirac operator we have following unitary equivalence of spectral triples:*

$$\mathcal{S}_{\sigma,1\oplus 2} \simeq \mathcal{S}_{\sigma,0} \oplus \mathcal{S}_{\sigma,0}.$$

Proof. See Appendix 6.5. □

At this point we have four candidates for the spin structures over $\mathfrak{B}3_\theta$ - these are two spectral triples $\mathcal{S}_{+,0}$ and $\mathcal{S}_{-,0}$ respectively for the $\epsilon_1 = 0$ and $\epsilon_1 = \frac{1}{2}$. We shall now give explicit description of the vectors spanning spaces $\mathcal{H}^{(\pm,\sigma,0)}$. Here we have $\epsilon_2 = \epsilon_3 = 0$. For $\mu_2, \mu_3 \neq 0$ the invariant vectors are:

$$\frac{1}{\sqrt{3}}\Big(e^{\pm}_{3k+3\epsilon_1\mp\sigma+a,\mu_2,\mu_3} + e^{\frac{2\pi i}{3}a} e^{\pm}_{3k+3\epsilon_1\mp\sigma,-\mu_2-\mu_3,\mu_2} + e^{-\frac{2\pi i}{3}a} e^{\pm}_{3k+3\epsilon_1\mp\sigma,\mu_3,-\mu_2-\mu_3}\Big), \tag{6.26}$$

for $a = 0, 1, 2$.

The spectrum on this subspace is:

$$\mathcal{S}p^{\mathfrak{B}3_\theta}_{\epsilon_1} \ni \lambda = \pm\sqrt{R^2(3k \mp \sigma + a + 3\epsilon_1)^2 + (\mu_2)^2 + (\mu_3)^2 + \mu_2\mu_3},$$

and, as a set it is (independently of σ)

$$\frac{1}{3}\left(\mathcal{S}p^{e^{\frac{2\pi i}{3}}}_{\epsilon_1,0,0} \setminus 2\mathcal{S}p^{1}_{R,R\epsilon_1}\right).$$

In the case with $\mu_2 = \mu_3 = 0$ we have the following invariant eigenvectors:

$$e^{\pm}_{3k+3\epsilon_1\mp\sigma,0,0},$$

so that they are eigenvectors of D to the eigenvalues:

$$\mathcal{S}p_{\epsilon_1}^{\mathfrak{B}3_\theta} \ni R(3k + 3\epsilon_1 - \sigma),\ k \in \mathbb{Z}.$$

and the spectrum of the Dirac operator, restricted to that subspace is the set:

$$2\mathcal{S}p^1_{3R,(3\epsilon_1-\sigma)R}\ .$$

The $\eta(D_{\epsilon_1}^{\mathbb{Z}_3})$ invariant is, in each of the four possible cases:

$$\begin{aligned}
\sigma = +1,\ \ \epsilon_1 = \frac{1}{2}: &\qquad \eta = \frac{4}{3},\\
\sigma = +1,\ \ \epsilon_1 = 0: &\qquad \eta = -\frac{2}{3},\\
\sigma = -1,\ \ \epsilon_1 = \frac{1}{2}: &\qquad \eta = -\frac{4}{3},\\
\sigma = -1,\ \ \epsilon_1 = 0: &\qquad \eta = \frac{2}{3}.
\end{aligned}$$

It is no surprise that some of the η invariants differ by sign as, in fact, the change $\tau \to \tau^*$ corresponds to the change $D \to -D$ on the subspace considered and gives, in fact, the same geometry.

Therefore we have in the end two distinct spin structures, each projected out of different spin structure from three torus: the firs comes from $\epsilon_1 = 0$ and the second from $\epsilon_1 = \frac{1}{2}$. They are distinguishable by the η invariants of the Dirac operators. The case $\sigma = 1$ is the situation discussed in [45].

6.3.3 Equivariant Real Spectral Triples over $\mathfrak{B}4_\theta$

The case $N = 4$ is a very similar situation to that of $N = 2$. We begin as usual from defining subspaces:

$$\mathcal{H}^{(\pm,\sigma,j)}:\ = \{\psi \in \mathcal{H}^{\pm} | \rho^{(\sigma)}\psi = e^{\frac{\pi i}{2}j}\psi\},$$

for $j = 0, 1, 2, 3$.

Proposition 6.25. *The following are spectral triples over $\mathfrak{B}4_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_4}$ coming from the reduction of the $\mathbb{Z}_4-$equivariant spectral triple over three torus:*

$$\begin{aligned}
\mathcal{S}_{\sigma,0} &= (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,0)} \oplus \mathcal{H}^{(-,\sigma,0)}, D, J),\\
\mathcal{S}_{\sigma,2} &= (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,2)} \oplus \mathcal{H}^{(-,\sigma,2)}, D, J),\\
\mathcal{S}_{\sigma,1\oplus 3} &= (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(+,\sigma,3)} \oplus \mathcal{H}^{(-,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,3)}, D, J).
\end{aligned}$$

The operators D and J comes from the spectral triple over three torus and are tacitly restricted to the invariant subspaces.

Proof. As in the previous cases we show that if $\psi \in \mathcal{H}^{(\pm,\sigma,j)}$, then:

$$D\psi \in \mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}, \quad J\psi \in \mathcal{H}^{(\mp,\sigma,-j)}.$$

So the space closed under algebra $\mathfrak{B}4_\theta$, Dirac operator and real structure must be a direct sum of $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}$ and $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,-j)}$. There are exactly three minimal subspaces of this type listed in the lemma. □

Remark 6.26. *Although all three spectral triples of the lemma are irreducible only $\mathcal{S}_{\sigma,0}$ and $\mathcal{S}_{\sigma,2}$ are irreducible in a weak sense,i.e. $D-$irreducible and $J-$irreducible. The $\mathcal{S}_{\sigma,1\oplus 3}$ can be reduced to:*

$$\mathcal{S}_{\sigma,1\oplus 3} \simeq (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,1)}, D) \oplus (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,3)} \oplus \mathcal{H}^{(-,\sigma,3)}, D)$$

in the case of $D-$reducibility and:

$$\mathcal{S}_{\sigma,1\oplus 3} \simeq (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,3)}, J) \oplus (\mathfrak{B}4_\theta, \mathcal{H}^{(+,\sigma,3)} \oplus \mathcal{H}^{(-,\sigma,1)}, J)$$

in the case of real structure. We recall that we are interested only in irreducible spectral triples so from now we restrict only to the cases $\mathcal{S}_{\sigma,0}$ and $\mathcal{S}_{\sigma,2}$.

Proposition 6.27. *There is a unitary equivalence of spectral triples up to bounded perturbation of Dirac operator:*

$$\mathcal{S}_{\sigma,1\oplus 3} \simeq \mathcal{S}_{\sigma,0} \oplus \mathcal{S}_{\sigma,0} \simeq \mathcal{S}_{\sigma,2} \oplus \mathcal{S}_{\sigma,2}.$$

Proof. See Appendix 6.5. □

We will now determine the spectrum of triples $\mathcal{S}_{\sigma,\nu}$ for $\nu = 0, 2$. Here we have $\epsilon_1 = \frac{1}{2}$ and need to distinguish two cases:

If $\epsilon_2 = \epsilon_3 = \frac{1}{2}$

The invariant subspace of the Hilbert space of the spectral triple over the noncommutative torus for $\mathcal{S}_{\sigma,\nu}$, where $\nu = 0, 2$, is spanned by the following vectors:

$$\begin{aligned}\frac{1}{2}\Big(&e^{\pm}_{4k\mp\frac{\sigma}{2}+a,\mu_2,\mu_3} + e^{\frac{\pi i}{2}(a-\nu)} e^{\pm}_{4k\mp\frac{\sigma}{2}+a,-\mu_3,\mu_2} \\ &+(-1)^{a-\nu} e^{\pm}_{4k\mp\frac{\sigma}{2}+a,\mu_2,\mu_3} + e^{-\frac{\pi i}{2}(a-\nu)} e^{\pm}_{4k\mp\frac{\sigma}{2}+a,\mu_3,-\mu_2}\Big),\end{aligned} \tag{6.27}$$

for $k \in \mathbb{Z}$ and $a = 0, 1, 2, 3$.

The spectrum is symmetric, with the eigenvalues:

$$\mathcal{S}p^{\mathfrak{B}4_\theta}_{\frac{1}{2},\nu} \ni \lambda = \pm\sqrt{R^2(4k \mp \frac{\sigma}{2} + a)^2 + |\mu_2|^2 + |\mu_3|^2}, \quad k \in \mathbb{Z}, a = 0, 1, 2, 3.$$

and it is clear that it is in fact the original spectrum of the Dirac on the $\mathcal{A}(\mathbb{T}^3_\Theta)$, with $\frac{1}{4}$ of its multiplicities, i.e. $\frac{1}{4}\mathcal{S}p^i_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}$.

If $\epsilon_2 = \epsilon_3 = 0$

Again, similarly as in the $N = 2$ case if $\mu_2 \neq 0$ (note here this enforces $\mu_3 \neq 0$) we have the same part of the spectrum, on the subspace spanned by the same vectors (6.27), which are still the invariant vectors for $\mu_2 \neq 0$.

Again, this is only a part of the spectrum of the Dirac over $\mathcal{A}(\mathbb{T}^3_\Theta)$, namely:

$$\frac{1}{4}\left(\mathcal{S}p^i_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{R,\frac{1}{2}R}\right),$$

which means that we are not counting the spectrum for eigenvectors with $\mu_2 = \mu_3 = 0$ and the remaining part of the spectrum comes with the multiplicity 1 instead of 4.

In the latter case, direct computation shows that the following vectors space the invariant subspaces:

$$e^{\pm}_{4k\mp\frac{\sigma}{2}+\nu,0,0}, \tag{6.28}$$

for $\nu = 0, 2$, and the spectrum of the Dirac operator, restricted to that subspace contains all the following eigenvalues:

$$\mathcal{S}p^{\mathfrak{B}4_\theta}_{0,\nu} \ni \lambda = \pm R(4k \mp \frac{\sigma}{2} + \nu), \quad k \in \mathbb{Z}.$$

We have, in all possible cases $\sigma = \pm 1, \nu = 0, 2$ the spectra give the following $\eta(D^{\mathbb{Z}_4}_\nu)$ invariants:

$$\begin{aligned}
\sigma = +1,\ \ \nu = 0: \qquad \eta &= -\frac{3}{2},\\
\sigma = +1,\ \ \nu = 2: \qquad \eta &= \frac{1}{2},\\
\sigma = -1,\ \ \nu = 0: \qquad \eta &= \frac{3}{2},\\
\sigma = -1,\ \ \nu = 2: \qquad \eta &= -\frac{1}{2}.
\end{aligned}$$

Again, the case $\sigma = +1$ and $\sigma = -1$ are related by the map $D \to -D$, the case presented in [45] correspond to $\sigma = -1$. In the end we obtain four geometrically inequivalent spin structures two for $\epsilon_2 = \epsilon_3 = 0$ and two for $\epsilon_2 = \epsilon_3 = \frac{1}{2}$.

6.3.4 Equivariant Real Spectral Triples over $\mathfrak{B}6_\theta$

Here, one repeats most of the arguments from the $N = 3$ case. As earlier:

$$\mathcal{H}^{(\pm,\sigma,j)} :\, = \{\psi \in \mathcal{H}^\pm | \rho^{(\sigma)}\psi = e^{\frac{\pi i}{3}j}\psi\},$$

for $j = 0, 1, 2, 3, 4, 5$.

Proposition 6.28. *In the case $N = 6$ there are four spectral triples candidates for spin structures over $\mathfrak{B}6_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_6}$ coming from the restriction and then reduction of $\mathbb{Z}_6-$equivariant spectral triples over three torus. Two irreducible:*

$$\mathcal{S}_{\sigma,0} = (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,0)} \oplus \mathcal{H}^{(-,\sigma,0)}, D, J),$$
$$\mathcal{S}_{\sigma,3} = (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,3)} \oplus \mathcal{H}^{(-,\sigma,3)}, D, J)$$

and two reducible (in a sense of $D-$reducibility and $J-$reducibility):

$$\mathcal{S}_{\sigma,1\oplus 5} = (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,1)} \oplus \mathcal{H}^{(+,\sigma,5)} \oplus \mathcal{H}^{(-,\sigma,1)} \oplus \mathcal{H}^{(-,\sigma,5)}, D, J)$$

$$\mathcal{S}_{\sigma,2\oplus 4} = (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,2)} \oplus \mathcal{H}^{(+,\sigma,4)} \oplus \mathcal{H}^{(-,\sigma,2)} \oplus \mathcal{H}^{(-,\sigma,4)}, D, J).$$

Proof. Computation similar to those which were done earlier shows that the invariant subspace must be a direct sum of the spaces $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}$ and $\mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,-j)}$. Moreover the same computation shows that for $j = 1, 2$ the spectral triple $\mathcal{S}_{\sigma,j\oplus -j}$ is $D-$reducible:

$$\mathcal{S}_{\sigma,j\oplus -j} \simeq (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,j)}, D) \oplus (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,-j)} \oplus \mathcal{H}^{(-,\sigma,-j)}, D)$$

and $J-$reducible:

$$\mathcal{S}_{\sigma,j\oplus -j} \simeq (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,j)} \oplus \mathcal{H}^{(-,\sigma,-j)}, J) \oplus (\mathfrak{B}6_\theta, \mathcal{H}^{(+,\sigma,-j)} \oplus \mathcal{H}^{(-,\sigma,j)}, J).$$

□

Proposition 6.29. *There is a unitary equivalence of spectral triples up to bounded perturbation of Dirac operator:*

$$\mathcal{S}_{\sigma,1\oplus 5} \simeq \mathcal{S}_{\sigma,2\oplus 4}.$$

Moreover those spectral triples are reducible up to bounded perturbation of D to the direct sum

$$\mathcal{S}_{\sigma,0} \oplus \mathcal{S}_{\sigma,0} \simeq \mathcal{S}_{\sigma,3} \oplus \mathcal{S}_{\sigma,3}.$$

Proof. See Appendix 6.5. □

So we will deal only with $\mathcal{S}_{\sigma,\nu}$ for $\nu = 0, 3$. First of all, there exists a part of the invariant subspace of the Hilbert space where the spectrum of D is symmetric and is:

$$\frac{1}{6}\left(\mathcal{S}p^{e^{\frac{\pi i}{3}}}_{\epsilon_1,0,0} \setminus 2\mathcal{S}p^{1}_{R,R\epsilon_1}\right).$$

We do not write explicit expression for the vectors spanning this subspace, the formula is analogous to the ones derived earlier for $N = 3$.

Similarly, there exist additional invariant vectors:

$$e^{\pm}_{6k\mp\frac{\sigma}{2}+\nu,0,0}, \quad k \in \mathbb{Z},$$

where $\sigma = \pm 1$ and $\nu = 0, 3$. The eigenvalues are:

$$\mathcal{S}p^{\mathfrak{B}6_\theta}_{\nu} \ni \lambda = \pm R(6k \mp \frac{\sigma}{2} + \nu), \quad k \in \mathbb{Z}$$

and this gives the spectrum

$$2\mathcal{S}p^{1}_{6R,R(\nu-\frac{\sigma}{2})}.$$

We have, in all possible cases $\sigma = \pm 1, \nu = 0, 3$ the spectra give the following $\eta(D^{\mathbb{Z}_6}_{\nu})$ invariants:

$$\begin{aligned}
\sigma = +1, \ \ \nu = 0: \qquad \eta &= -\frac{5}{3}, \\
\sigma = +1, \ \ \nu = 3: \qquad \eta &= \frac{1}{3}, \\
\sigma = -1, \ \ \nu = 0: \qquad \eta &= \frac{5}{3}, \\
\sigma = -1, \ \ \nu = 3: \qquad \eta &= -\frac{1}{3}.
\end{aligned}$$

Again, we see that the case presented in [45] is the one $\sigma = -1$ and in the end we have two spin structures for $N = 6$.

6.4 Summary

The following table presents the number of noncommutative spin structures over the Bieberbach spaces:

Bieberbach space	Parametrisation	Number of spin structures
$\mathfrak{B}2_\theta$	$\epsilon_2, \epsilon_3 = 0, \frac{1}{2}; \nu = 0, 1$	8
$\mathfrak{B}3_\theta$	$\epsilon_1 = 0, \frac{1}{2}$	2
$\mathfrak{B}4_\theta$	$\epsilon_2 = \epsilon_3 = 0, \frac{1}{2}; \nu = 0, 2$	4
$\mathfrak{B}6_\theta$	$\nu = 0, 3$	2

Table 6.4: Noncommutative spin structures over $\mathfrak{B}N_\theta$

Note that η invariant is not zero only for the for the spin structures with $\epsilon_2 = \epsilon_3 = 0$. This gives 2 spin structures over Bieberbach spaces for each $N = 2, 3, 4, 6$. The following table shows the η invariant for those spin structures where it does not vanish:

Bieberbach space	Parametrisation	$\eta(D_\nu^{\mathbb{Z}_N})$
$\mathfrak{B}2_\theta$	$\nu = 0$	-1
	$\nu = 1$	$+1$
$\mathfrak{B}3_\theta$	$\epsilon_1 = 0$	$-\frac{2}{3}$
	$\epsilon_1 = \frac{1}{2}$	$\frac{4}{3}$
$\mathfrak{B}4_\theta$	$\nu = 0$	$\frac{3}{2}$
	$\nu = 2$	$-\frac{1}{2}$
$\mathfrak{B}6_\theta$	$\nu = 0$	$\frac{5}{3}$
	$\nu = 4$	$-\frac{1}{3}$

Table 6.5: The η invariant for Bieberbach spaces for $\epsilon_2 = \epsilon_3 = 0$

6.5 Appendix

Here we shall give the proof of Propositions 6.24,6.27,6.29. First we shall fix the notation and briefly recall basic definitions. Let $N = 3, 4, 6$ and let $\nu = 0$ for $N = 3$ and $\nu = 0, \frac{N}{2}$ for $N = 4, 6$. We have defined (for $j = 0, 1, \ldots, N-1$):

$$\mathcal{H}^{(\pm,j)} \colon = \{\psi \in \mathcal{H}^\pm | \rho(h)\psi = e^{\frac{2\pi i}{N} j}\psi\}.$$

In the original definition of representation we also allowed the freedom of parameter $\sigma = \pm 1$. Here we tacitly assume that σ is fixed through out the whole computation and to simplify notation we shall omit it. We define D_ν and J_ν as a restriction of respectively Dirac operator and real structure to the subspaces $\mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}$ and similarly for $j \neq 0, \frac{N}{2}$ $D_{j\oplus -j}$ and $J_{j\oplus -j}$ as a restriction of D and J to subspace $\mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(+,-j)} \oplus \mathcal{H}^{(-,j)} \oplus \mathcal{H}^{(-,-j)}$.

Then there are two types of real spectral triples over $\mathfrak{B}N_\theta$ listed below (see Propositions 6.22,6.25,6.28):

- irreducible ones:

$$\mathcal{S}_\nu = (\mathfrak{B}N_\theta, \mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}, D_\nu, J_\nu),$$

  where: $\nu = 0$ for $N = 3$; $\nu = 0, 2$ for $N = 4$; $\nu = 0, 3$ for $N = 6$.

- reducible in a weak sense:

$$\mathcal{S}_{j\oplus -j} = (\mathfrak{B}N_\theta, \mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(+,-j)} \oplus \mathcal{H}^{(-,j)} \oplus \mathcal{H}^{(-,-j)}, D_{j\oplus -j}, J_{j\oplus -j}),$$

  for $j \neq 0, \frac{N}{2}$.

Lemma 6.30. *Using above definitions the spectral triple:*

$$(\mathfrak{B}N_\theta, \mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(+,-j)} \oplus \mathcal{H}^{(-,j)} \oplus \mathcal{H}^{(-,-j)}, D_{j\oplus -j}, J_{j\oplus -j})$$

is up to perturbation of $D_{j\oplus -j}$ unitarily equiavalent to a direct sum of spectral triples:

$$\mathcal{S}_\nu \oplus \mathcal{S}_\nu = (\mathfrak{B}N_\theta, \mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}, D_\nu, J_\nu) \oplus (\mathfrak{B}N_\theta, \mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}, D_\nu, J_\nu),$$

if only $\nu = 0$ for $N = 3$ and $\nu = 0, \frac{N}{2}$ for $N = 4, 6$.

Proof. Firstly to fix notation let us write explicitly the operators $D_{j\oplus -j}$ and $J_{j\oplus -j}$ as a 4 by 4 matrices over $\mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(+,-j)} \oplus \mathcal{H}^{(-,j)} \oplus \mathcal{H}^{(-,-j)}$. We have:

$$D_{j\oplus -j} = \begin{pmatrix} \delta_1 & 0 & \partial & 0 \\ 0 & \delta_1 & 0 & \partial \\ \partial^* & 0 & -\delta_1 & 0 \\ 0 & \partial^* & 0 & -\delta_1 \end{pmatrix}$$

and

$$J_{j\oplus -j} = \begin{pmatrix} 0 & 0 & 0 & -J_0 \\ 0 & 0 & -J_0 & 0 \\ 0 & J_0 & 0 & 0 \\ J_0 & 0 & 0 & 0 \end{pmatrix}.$$

To simplify notation we shall write it in a block form:

$$D_{j\oplus -j} = \begin{pmatrix} \delta_1 \mathbb{1} & \partial \mathbb{1} \\ \partial \mathbb{1} & -\delta_1 \mathbb{1} \end{pmatrix}$$

and

$$J_{j\oplus -j} = \begin{pmatrix} 0 & -J_0 \sigma_2 \\ J_0 \sigma_2 & 0 \end{pmatrix},$$

where

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now let us define a unitary:

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} U^{-j+\nu} & U^{j-\nu} \\ -iU^{-j+\nu} & iU^{j-\nu} \end{pmatrix}$$

and:

$$\hat{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

Now we shall determine a $\hat{Q}-$equivalent spectral triple to $\mathcal{S}_{j\oplus -j}$. To do this we have to determine four elements: the Hilbert space $\mathcal{H}' = \mathrm{Im}(\hat{Q})$; the possibly new representation of Bieberbach space $\pi(a)' = \hat{Q}\pi(a)\hat{Q}^*$ for $a \in \mathfrak{B}N_\theta$; the real structure $J' = \hat{Q}J\hat{Q}^*$; the new Dirac operator up to bounded perturbation $D' = \hat{Q}(D+A)\hat{Q}^*$, where A is a bounded operator such that $[A, \pi(a)] = 0$ for any $a \in \mathfrak{B}N_\theta$.

Hilbert space and representation π'. Let $\psi \in \mathcal{H}^{(\pm,j)} \oplus \mathcal{H}^{(\pm,-j)}$, by the definition of subspaces and through the direct computation we have $\rho(h)Q\psi = e^{\frac{2\pi i}{N}\nu}Q\psi$. As an element Q is unitary we obtain: $Q(\mathcal{H}^{(\pm,j)} \oplus \mathcal{H}^{(\pm,-j)}) = \mathcal{H}^{(\pm,\nu)} \oplus \mathcal{H}^{(\pm,\nu)}$ and thus:

$$\hat{Q}(\mathcal{H}^{(+,j)} \oplus \mathcal{H}^{(+,-j)} \oplus \mathcal{H}^{(-,j)} \oplus \mathcal{H}^{(-,-j)}) =$$

$$= \mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)} \oplus \mathcal{H}^{(-,\nu)} = (\mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}) \otimes \mathbb{C}^2.$$

The representation π is diagonal. Moreover element U is central in $\mathcal{A}(\mathbb{T}^3_\Theta)$, so we quite trivially get that:

$$\hat{Q}\pi(a)\hat{Q}^* = \pi(a) \quad \forall a \in \mathfrak{B}N_\theta.$$

Real structure J'. By the direct computation:

$$QJ_0\sigma_2Q^* = \frac{1}{2}\begin{pmatrix} U^{-j+\nu} & U^{j-\nu} \\ -iU^{-j+\nu} & iU^{j-\nu} \end{pmatrix}\begin{pmatrix} 0 & J_0 \\ J_0 & 0 \end{pmatrix}\begin{pmatrix} U^{j-\nu} & iU^{j-\nu} \\ U^{-j+\nu} & -iU^{-j+\nu} \end{pmatrix} =$$

$$= \begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix} = J_0\mathbb{1}.$$

And for the full J:

$$J' = \hat{Q}J\hat{Q}^* = \begin{pmatrix} 0 & -J_0\mathbb{1} \\ J_0\mathbb{1} & 0 \end{pmatrix} = J_\nu \otimes \mathbb{1}.$$

The Dirac operator. Let us take $A = \mathrm{diag}(-j+\nu, j-\nu)$ which is of course bounded and commutes with diagonal representation π. Then by the direct computation for the diagonal part of $D_{j\oplus -j}$:

$$Q(\delta_1\mathbb{1} + A)Q^* =$$

$$= \frac{1}{2}\begin{pmatrix} U^{-j+\nu} & U^{j-\nu} \\ -iU^{-j+\nu} & iU^{j-\nu} \end{pmatrix}\begin{pmatrix} \delta_1 - j + \nu & 0 \\ 0 & \delta_1 + j - \nu \end{pmatrix}\begin{pmatrix} U^{j-\nu} & iU^{j-\nu} \\ U^{-j+\nu} & -iU^{-j+\nu} \end{pmatrix} =$$

$$= \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{pmatrix} = \delta_1\mathbb{1},$$

and for the off diagonal:

$$Q(\partial \mathbb{1})Q^* = \partial \mathbb{1}, \quad Q(\partial^* \mathbb{1})Q^* = \partial^* \mathbb{1}.$$

Which for the full $D_{j\oplus -j}$ is:

$$\hat{Q}(D_{j\oplus -j} + \sigma_1 \otimes A)\hat{Q}^* = \begin{pmatrix} \delta_1 \mathbb{1} & \partial \mathbb{1} \\ \partial \mathbb{1} & -\delta_1 \mathbb{1} \end{pmatrix} = D_\nu \otimes \mathbb{1}.$$

The computation conducted to this point shows the unitary $\hat{Q}-$equivalence of spectral triples:

$$\mathcal{S}_{j\oplus -j} \simeq_{\hat{Q}} (\mathfrak{B}N_\theta, (\mathcal{H}^{(+,\nu)} \oplus \mathcal{H}^{(-,\nu)}) \otimes \mathbb{C}^2, D_\nu \otimes \mathbb{1}, J_\nu \otimes \mathbb{1}).$$

It is easy to see that the latter spectral triple is reducible (in a strong sense of $D, J-$reducibility) and it could be rewritten as a direct sum of spectral triples:

$$\mathcal{S}_{j\oplus -j} \simeq_{\hat{Q}} \mathcal{S}_\nu \oplus \mathcal{S}_\nu.$$

This ends the proof. $\square$

Chapter 7

Spectrum of Dirac Operator and Spectral Action

In this chapter we shall once again focus our attention on the spectrum of Dirac operator for spectral triples classified in the previous part of dissertation. We shall use this result to compute the spectral action of Dirac operators case by case. We begin with the definition and computation of the spectral action of the Dirac operator over three torus and generalised spectrum over circle. Evidently this computation is nothing new, but one can easily see the introductory value of this consideration. Moreover this calculation will prove to be useful when we move to the case of Bieberbach manifolds.

7.1 The Spectral Action of a Flat Tori

For a given spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ the spectral action is a functional on the spectrum of the Dirac operator and depends on a real parameter Λ called the energy scale (for details see [19]). Let f be a test function, i.e. f belongs to the Schwartz space $S(\mathbb{R})$, then the spectral action is defined as:

$$\mathcal{S}(D, \Lambda) = \operatorname{tr}\big(f(\frac{D}{\Lambda})\big).$$

Usually one takes f as a smooth approximation of a cutoff function. We shall assume that the spectral action depends on D and not on D^2 (that is we do not restrict ourselves to the even functions over the spectrum), therefore there is a slight change of notation when compared for example to the [37].

Dirac Operators over Three-dimensional Torus

Let us first fix the notation to present the computation of spectral action of tori. We take the three-torus with equal lengths of three circles (i.e. when compared to the previous chapter we shall assume that in the Dirac operator the parameter $R = 1$, but still we do not fix the value of the parameter τ). By D^3_τ we denote the Dirac operator with the eigenvalues:

$$\lambda_\mu = \pm\sqrt{\mu_1^2 + |\mu_2 + \tau\mu_3|^2}, \quad \mu_i \in \mathbb{Z} + \epsilon_i, \quad \text{for } i = 1, 2, 3,$$

where ϵ_1, ϵ_2, ϵ_3 are 0 or $\frac{1}{2}$ and depend on the choice of the spin structure and $|\tau| = 1$, τ not real. The choice of $\tau = i$ corresponds to the usually assumed Dirac operator. We shall denote the spectrum of D^3_τ over three torus, counted with the multiplicities, by $\mathcal{S}p^\tau_\epsilon$ for a given spin structure determined by the choice of ϵ's. The spectral action for the torus with the Dirac D^3_τ is:

$$\mathcal{S}(D^3_\tau, \Lambda) = \sum_{\lambda \in \mathcal{S}p^\tau_\epsilon} f\left(\frac{\lambda}{\Lambda}\right).$$

Direct computation shows that:

$$\begin{aligned}
\mathcal{S}(D^3_\tau, \Lambda) &= 2\sum_{k,l,m} f\left(\pm\frac{\sqrt{(k+\epsilon_1)^2 + |l+\epsilon_2+\tau(m+\epsilon_3)|^2}}{\Lambda}\right) \\
&= \widehat{f}_e(0,0,0) + o(\Lambda^{-1}) \\
&= 2\int_{\mathbb{R}^3} f_e\left(\frac{\sqrt{x^2+|y+\tau z|^2}}{\Lambda}\right) dxdydz + o(\Lambda^{-1}) \\
&= \frac{8\pi^2}{\sin\phi}\Lambda^3 \int_0^\infty f_e(\rho)\rho^2\, d\rho + o(\Lambda^{-1}).
\end{aligned}$$

where $\tau = e^{i\phi}$, $\widehat{f}$ denotes the Fourier transform of f considered as a function of three variables:

$$\widehat{f}(k_x, k_y, k_z) = \int_{\mathbb{R}^3} f(x,y,z)e^{2\pi i(k_x x + k_y y + k_z z)}\, dxdydz,$$

and f_e denotes the even part of f. To obtain this result we have used the Poisson summation formula.

Generalised Dirac Operator

Further, we shall also need the generalised Dirac operator on the circle, taking the standard one, with eigenvalues:

$$\lambda_k = \alpha k + \beta, \quad k \in \mathbb{Z}, \alpha, \beta \in \mathbb{R},$$

we shall denote its spectrum by $\mathcal{S}p^1_{\alpha,\beta}$.

For the Dirac operator $D^1_{\alpha,\beta}$ over the circle we have:

$$\begin{aligned}\mathcal{S}(D^1_{\alpha,\beta},\Lambda) &= \sum_k f\left(\frac{\alpha k+\beta}{\Lambda}\right)\\ &= \hat{f}(0)+o(\Lambda^{-1}) = \Lambda\int_{\mathbb{R}} f\left(\frac{\alpha k+\beta}{\Lambda}\right)dx + o(\Lambda^{-1})\\ &= \frac{1}{\alpha}\Lambda\int_{\mathbb{R}} f(x)dx+o(\Lambda^{-1}).\end{aligned}$$

In the formula above $\hat{f}$ is the usual Fourier transform of f.

Remark 7.1. *Let us observe that the following identity occurs:*

$$\mathcal{S}(D^1_{1,\gamma},\Lambda) = \alpha\mathcal{S}(D^1_{\alpha,\beta},\Lambda).$$

independently of the values of α, β and γ.

7.2 The Spectra of the Dirac Operator over Bieberbach Manifolds

The spectrum of the Dirac operator on classical Bieberbach manifolds has been first calculated by Pfäffle [45]. Our result on the noncommutative Bieberbach spaces presented in the previous chapter agree with the computation carried out by Pfaäffle. We shall now briefly recapitulate it and explicitly recall the spectra of Dirac operator case by case. As the covering three-torus we shall choose the equilateral one (with lengths of three fundamental circles equal, i.e. $R=1$). Whenever we write a coefficient in front of the spectrum set we mean the same set but with the multiplicities reduced by that factor (of course, if the coefficient is $\frac{1}{n}$ this requires that the multiplicities must be divisible by n).

We shall parametrize the spins structures of the three-torus by $\{\epsilon_1,\epsilon_2,\epsilon_3\}$ being 0 or $\frac{1}{2}$, additionally the choice of invariant subspace can an add additional spin structure - this possibility is according to previous notation denoted by ν. Moreover in each case we shall take the invariant subspace with representation with $\sigma=1$ for $N=3$ and $\sigma=-1$ for $N=4,6$ (see discussion in the section 6.3). In the case of $G5=\mathbb{Z}_2\times\mathbb{Z}_2$ to define the action on $\mathcal{A}(\mathbb{T}^3_\Theta)$ we must have a commutative algebra (i.e. $\theta_{ij}=0$), which corresponds to classical manifold. Thus in this case we take the spectrum of D computed classically by Pfäffle.

We have the following spectra of the Dirac operator:

- $G2$

  Here we have $\epsilon_1 = \frac{1}{2}$ and eight possible spin structures, parametrized by choice of ϵ_2, ϵ_3 and $\nu = 0, 1$. As the Dirac operator on three torus we must take D^3_i, with eigenvalues $\pm\sqrt{(k+\frac{1}{2})^2 + (l+\epsilon_2)^2 + (m+\epsilon_3)^2}$, $k, l, m \in \mathbb{Z}$.

$$\mathcal{S}p^{\mathfrak{B}2_\theta}_{\epsilon_1,\epsilon_2,\nu} = \begin{cases} \frac{1}{2}\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},\epsilon_2,\epsilon_3} & \text{if } \epsilon_2 = \frac{1}{2} \text{ or } \epsilon_3 = \frac{1}{2}, \nu = 0, 1, \\ \frac{1}{2}\left(\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{2,-\frac{1}{2}} & \text{if } \epsilon_2 = \epsilon_3 = 0, \nu = 0 \\ \frac{1}{2}\left(\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{2,\frac{1}{2}} & \text{if } \epsilon_2 = \epsilon_3 = 0, \nu = 1. \end{cases}$$

  Observe that only in the $\epsilon_2 = \epsilon_3 = 0$ case the spectrum is not the same as for the torus.

- $G3$

  In this case only the spin structures with $\epsilon_2 = \epsilon_3 = 0$ could be projected to the quotient space. The parameter ν is fixed by the choice of the spin structure ϵ_1. As the projectable Dirac we take $D^3_{e^{\frac{2\pi i}{3}}}$.

$$\mathcal{S}p^{\mathfrak{B}3_\theta}_{\epsilon_1} = \begin{cases} \frac{1}{3}\left(\mathcal{S}p^{e^{\frac{2\pi i}{3}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{3,\frac{1}{2}} & \text{if } \epsilon_1 = \frac{1}{2}, \\ \frac{1}{3}\left(\mathcal{S}p^{e^{\frac{2\pi i}{3}}}_{0,0,0} \setminus 2\mathcal{S}p^1_{1,0}\right) \cup 2\mathcal{S}p^1_{3,-1} & \text{if } \epsilon_1 = 0. \end{cases}$$

- $G4$

  For the action of Z_4 only $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \epsilon_3$ spin structures could be projected onto the quotient, the Dirac operator which commutes with the action of the discrete group is $D^3_{e^{\frac{\pi i}{4}}}$. There are four possible spin structures and the corresponding spectra are:

$$\mathcal{S}p^{\mathfrak{B}4_\theta}_{\epsilon,\nu} = \begin{cases} \frac{1}{4}\left(\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right) & \text{if } \epsilon_2 = \epsilon_3 = \frac{1}{2}, \nu = 0, 2, \\ \frac{1}{4}\left(\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{4,\frac{1}{2}} & \text{if } \epsilon_2 = \epsilon_3 = 0, \nu = 0 \\ \frac{1}{4}\left(\mathcal{S}p^{e^{\frac{\pi i}{4}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{4,\frac{5}{2}} & \text{if } \epsilon_2 = \epsilon_3 = 0, \nu = 2. \end{cases}$$

- $G5$

  In this case the only projectable spin structure are those with $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{2}$, the projectable Dirac operator is D^3_i and the spectrum of D remains the same (apart from rescaled multiplicities) for each of four spin structures over $\mathfrak{B}5$ (result taken from [45]).

$$\mathcal{S}p^{\mathfrak{B}5} = \frac{1}{4}\mathcal{S}p^3_i.$$

- $G6$

  Here the situation is similar as in the $G3$ case and only the spin structures with $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \epsilon_3 = 0$ could be projected to the quotient space. The parameter ν is free and gives us two possible spin structures. The projectable Dirac we take $D^3_{e^{\frac{2\pi i}{3}}}$.

$$\mathcal{S}p^{\mathfrak{B}6_\theta}_\nu = \begin{cases} \frac{1}{6}\left(\mathcal{S}p^{e^{\frac{2\pi i}{3}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{6,\frac{1}{2}} & \text{if } \nu = 0. \\ \frac{1}{6}\left(\mathcal{S}p^{e^{\frac{2\pi i}{3}}}_{\frac{1}{2},0,0} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{6,\frac{7}{2}} & \text{if } \nu = 3. \end{cases}$$

7.3 The Spectral Action of the Dirac operator over Bieberbach Manifolds

In section 7.2 we have split the spectra of the Dirac operators into the sets, which corresponds to the known cases. We can explicitly calculate the difference between the spectral action on the Bieberbach manifolds and the spectral action on the three-torus, expanding then the result in Λ.

Remark 7.2. *As we have seen if $\epsilon_2 \neq 0$ or $\epsilon_3 \neq 0$ the spectrum of spectral triples over Bieberbach spaces is symmetric and equals:*

$$\mathcal{S}p^{\mathfrak{B}N_\theta} = \frac{1}{n_N}\mathcal{S}p^\tau_\epsilon,$$

where n_N is the rank of the group GN. Thus the spectral action of such Bieberbach manifolds is just $\frac{1}{n_N}\mathcal{S}(D^3_\tau, \Lambda)$.

Only for the spin structures with $\epsilon_2 = \epsilon_3 = 0$ the spectra of the Dirac operator differ from the spectrum of the three torus (apart from the trivial factor of multiplicities). From now on we shall restrict our attention only to this eight nontrivial cases.

In the nontrivial cases, i.e. for Bieberbachs $\mathfrak{B}N_\theta$ with $N = 2, 3, 4, 6$ and $\epsilon_2 = \epsilon_3 = 0$, we have the following general relation of the spectra:

$$\mathcal{S}p^{\mathfrak{B}N_\theta} = \frac{1}{N}\left(\mathcal{S}p^{\tau}_{\epsilon_1,0,0} \setminus 2\mathcal{S}p^1_{1,\epsilon_1}\right) \cup 2\mathcal{S}p^1_{N,\epsilon'},$$

where ϵ' depends on the choice of spin structure (i.e. parameter ν, see 7.2).

So the difference between the spectral action on the three-torus $\mathcal{A}(\mathbb{T}^3_\Theta)$ and on the Bieberbach $\mathfrak{B}N_\theta$ could be calculated from this difference of the spectra:

$$\mathcal{S}(\mathfrak{B}N_\theta, \Lambda) - \frac{1}{N}\mathcal{S}(D^3_\tau, \Lambda) = 2\sum_{\lambda\in\mathcal{S}p^1_{N,\epsilon'}} f\left(\frac{\lambda}{\Lambda}\right) - \frac{2}{N}\sum_{\lambda\in\mathcal{S}p^1_{1,\epsilon_1}} f\left(\frac{\lambda}{\Lambda}\right).$$

Knowing the spectra of the Dirac operator over Bieberbach manifolds when compared to the three torus we can calculate the difference of spectral actions. Firstly we shall compute the nonperturbative difference of the spectral functional. After this we shall split the test function into the even and odd part:

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)),$$

and compute those cases separately.

7.3.1 Nonperturbative Part

We can already formulate the following theorem.

Proposition 7.3. *The nonperturbative spectral action over the orientable Bieberbach manifolds with the Dirac operator projected from the equilateral Dirac operator over the three-torus is (up to an scaling and order $o(\Lambda^{-1})$) indistinguishable from the spectral action over the three-torus.*

Proof. Of course, only the cases when the spectrum differs significantly from the spectrum of the Dirac over three torus may give rise to some differences. Observe that then the difference in the spectral actions is always of the form:

$$-\frac{2}{N}\mathcal{S}(D^1_{1,\gamma}, \Lambda) + 2\mathcal{S}(D^1_{N,\beta}, \Lambda)$$

where the sign is connected to substracting or adding the component of spectral action depends on whether we substract or add the subsets of spectrum. The constants β and γ vary from case to case, N is the order of the group $\mathbb{Z}_N$ such that $\mathfrak{B}N_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{\mathbb{Z}_N}$.

From the Remark 7.1, however, we know that the resulting spectral action components will not depend on β and γ and we will obtain:

$$-\frac{2}{N}\left(\int_{\mathbb{R}} f(x)dx\right)+\frac{2}{N}\left(\int_{\mathbb{R}} f(x)dx\right)=0,$$

and hence will not contribute to the leading terms of the spectral action. □

7.3.2 Perturbative Expansion - Even Case

Consider now an even function f_e. Taking f_e to be a Laplace transform of h:

$$f_e(x)=\int_0^\infty e^{-sx}h(s)dx,$$

we can write:

$$\mathrm{tr} f_e\left(\frac{|D|}{\Lambda}\right)=\int_0^\infty \mathrm{tr} e^{-s\frac{|D|}{\Lambda}}h(s)ds.$$

First we need the technical lemma. Let $\mathcal{S}p^1_{\alpha,\beta}$ be (as denoted before) the spectrum of the rescaled Dirac over the circle. We calculate exactly the exponential $\mathrm{tr}e^{-t|D^1_{\alpha,\beta}|}$, assuming that $|\beta|<\alpha$.

$$\begin{aligned}\mathrm{tr}e^{-t|D^1_{\alpha,\beta}|}&=e^{-t|\beta|}+\sum_{k=1}^{\infty}e^{-t(\alpha k+\beta)}+\sum_{k=1}^{\infty}e^{-t(\alpha k-\beta)}\\&=e^{-t|\beta|}+\left(\frac{e^{-t\alpha}}{1-e^{-t\alpha}}\right)2\cosh(t\beta).\end{aligned}$$

Taking into account that $t=\frac{s}{\Lambda}$ we can take the Laurent expansion for large values of Λ:

$$e^{-t|\beta|}+\left(\frac{e^{-t\alpha}}{1-e^{-t\alpha}}\right)2\cosh(t\beta)\sim\frac{2}{\alpha}\frac{\Lambda}{s}+o(\frac{s}{\Lambda}). \tag{7.1}$$

We can now state:

Proposition 7.4. *The even component of the function f in the spectral action is the same up to order $o(\Lambda^{-1})$ on all three-dimensional Bieberbach manifolds (including three-torus).*

Proof. As the spectra in the difference $\mathcal{S}(\mathfrak{B}N_\theta,\Lambda)-\frac{1}{N}\mathcal{S}(D^3_\tau,\Lambda)$ are the spectra of rescaled Dirac on the circle, using the result 7.1 we see that only the Λ component in the perturbative expansion could appear. Calculating it explicitly:

$$\mathcal{S}(\mathfrak{B}N_\theta,\Lambda)-\frac{1}{N}\mathcal{S}(D^3_\tau,\Lambda)=\Lambda\left(2\frac{2}{N}-\frac{2}{N}2\right)\int_0^\infty\frac{1}{s}h(s)ds+o(\Lambda^{-1})=o(\Lambda^{-1}).$$

Therefore, irrespective of the chosen spin structure and Bieberbach manifold, even component of the function determining the spectral action gives the same result:

$$\mathcal{S}(\mathfrak{B}N_\theta,\Lambda,f_e)=\frac{1}{N}\mathcal{S}(D^3_\tau,\Lambda)+o(\Lambda^{-1}).$$

□

7.3.3 Perturbative Expansion - Odd Case

The situation discussed above is different for the odd component of f. We can always write, for an odd function f_o:

$$f_o\left(\frac{D}{\Lambda}\right)=\operatorname{sgn}(D)\phi\left(\frac{|D|}{\Lambda}\right),$$

where ϕ is an even function. Assuming that ϕ is a Laplace transform of h the odd part of the spectral action becomes:

$$\operatorname{tr} f_o\left(\frac{D}{\Lambda}\right)=\operatorname{tr}\left(\operatorname{sgn}(D)\phi\left(\frac{|D|}{\Lambda}\right)\right)=\int_0^\infty\operatorname{tr}\left(\operatorname{sgn}(D)e^{-s\frac{|D|}{\Lambda}}\right)h(s)ds.$$

For the spectra of Dirac operators, which we know, we can calculate the function under the integral: $\operatorname{tr}\left(\operatorname{sgn}(D)e^{-s\frac{|D|}{\Lambda}}\right)$ and obtain (again we denote $t=\frac{s}{\Lambda}$):

$$\begin{aligned}\operatorname{tr}\left(\operatorname{sgn}(D^1_{\alpha,\beta})e^{-t|D^1_{\alpha,\beta}|}\right)&=\operatorname{sgn}(\beta)e^{-t|\beta|}+\sum_{k=1}^\infty e^{-t(\alpha k+\beta)}-\sum_{k=1}^\infty e^{-t(\alpha k-\beta)}\\&=\operatorname{sgn}(\beta)e^{-t|\beta|}-\left(\frac{e^{-t\alpha}}{1-e^{-t\alpha}}\right)2\sinh(t\beta).\end{aligned}$$

We can expand the function for small t around 0:

$$\operatorname{sgn}(\beta)e^{-t|\beta|}-\left(\frac{e^{-t\alpha}}{1-e^{-t\alpha}}\right)2\sinh(t\beta)\sim\operatorname{sgn}(\beta)-\frac{2\beta}{\alpha}+o(t).$$

Therefore, only (up to terms of order $o(\Lambda^{-1})$) only scale invariant term can appear. We have:

Proposition 7.5. *The odd component of the function f gives rise to a difference in the spectral action on the Bieberbach manifolds in the scale invariant part of the action. The difference is proportional to the eta-invariant of the Dirac operator on the Bieberbach manifold.*

Proof. First of all, observe that for the rescaled Dirac operator on the circle $D^1_{\alpha,\beta}$ the term:

$$\mathrm{sgn}(\beta)\frac{\alpha-2|\beta|}{\alpha},$$

is the eta invariant $\eta(D^1_{\alpha,\beta})$, which measures the antisymmetry between the positive and negative parts of the spectrum of $D^1_{\alpha,\beta}$. Therefore, for any of the spin structures of the circle, the term vanishes for the standard Dirac operator (that is, $D^1_{1,\frac{1}{2}}$ or $D^1_{1,0}$, using the notation of the paper). As a consequence, the difference between the (rescaled) spectral action on the three-torus $\mathcal{A}(\mathbb{T}^3_\Theta)$ and on the Bieberbach $\mathfrak{B}N_\theta$ is (up to order $o(\Lambda^{-1})$):

$$\mathcal{S}(\mathfrak{B}N_\theta,\Lambda)-\frac{1}{N}\mathcal{S}(D^3_\tau),\Lambda)=2\eta(D^1_{N,\epsilon'_N})\phi(0),$$

where ϵ'_N depends on the chosen spin structure, and we have used that ϕ is a Laplace transform of h, so that:

$$\int_0^\infty h(s)ds=\phi(0).$$

As this is, however, the only asymmetric component of the spectrum of Dirac operator over $\mathfrak{B}N_\theta$, we have:

$$2\eta(D^1_{N,\epsilon'_N})=\eta(D^{\mathbb{Z}_N}_\nu),$$

and, finally:

$$\mathcal{S}(\mathfrak{B}N_\theta,\Lambda)=\eta(D^{\mathbb{Z}_N}_\nu)\phi(0).$$

The η invariant for Bieberbach manifolds was computed for classical Dirac operator by Pfäffle. We have redone those computation for the spectral triples in the previous chapter. The reader can find the values of η for those spin structures for which it does not vanish in the table 6.5.

We can calculate then the leading term of the spectral action arising from an odd function to be:

$$\mathcal{S}(\mathfrak{B}N_\theta,\Lambda,f_o)=\eta(D^{\mathbb{Z}_N}_\nu)\phi(0)+o(\Lambda^{-1}).$$

□

7.4 Summary

We can now express the formula for the spectral action functional for spectral triples over noncommutative Bieberbach spaces with the canonical (i.e. flat and torsion free) Dirac operator taken from the spectral triple over three torus. Let us take a test function $f(x) = f_e(x) + \mathrm{sgn}(x)\phi(x)$, where f_e and ϕ are even, then:

$$\mathcal{S}(\mathfrak{B}N_\theta, \Lambda, f) = \frac{1}{n_N}\mathcal{S}(D^3_\tau, \Lambda, f_e) + \eta(D_\nu^{\mathbb{Z}_N})\phi(0) + o(\Lambda^{-1}),$$

where n_N is the rank of group GN such that $\mathfrak{B}N_\theta = \mathcal{A}(\mathbb{T}^3_\Theta)^{GN}$, the number τ depends on the group GN (see table 6.1) and η invariant can be found in table 6.5.

As we have already noted the odd component of the spectral action is proportional to eta invariant. In fact, the result is not entirely surprising. From the general results of Bismut and Freed [7] one knows the small-t asymptotic of the following function of the Dirac operator on the odd-dimensional manifolds:

$$\mathrm{tr}\frac{D}{|D|}e^{-t|D|} = \eta(D) + \sum_{l=0}^{\infty}(A_l + B_l \log t)t^{2l+2}.$$

We shall finish this section by observing why this effect was not picked by the methods used earlier, which involved sum over the entire spectrum with the help of the Poisson summation formula.

Observe that the η invariant would appear if $\phi(0) \neq 0$. Since our function $f(x) = \mathrm{sgn}(x)\phi(|x|)$ that means that f is odd, but discontinuous at $x = 0$. Therefore, the previous considerations were valid but since were (implicitly) assuming continuity of f we could not have obtained any deviation from the spectral action over the torus.

Chapter 8

Conclusions

8.1 Results

Noncommutative Bieberbach Spaces

Classically three-dimensional Bieberbach manifolds are defined as a quotient manifolds of the real plane by the discrete subgroup of Euclidean motion $\Gamma \subset \mathbb{R}^3 \rtimes SO(3)$, which is equivalent to the quotient of a three-dimensional torus by the discrete group $G = \Gamma/\mathbb{Z}^3$. On the other hand there are more than one ways to define the noncommutative generalization of classical manifold. For example in the case of the noncommutative three torus $C(\mathbb{T}^3_\Theta)$ one can define it as a C^*-algebra closure of the abstract algebra of polynomials in three unitaries respecting certain commutation relations (which is the definition we have adopted), but there are at least two equivalent possibilities. The first method is based o a procedure called the Connes-Landi isospectral deformation of the C^*-algebra of complex valued functions over $\mathbb{T}^3$. The second uses the double crossed product $(C(\mathbb{T}^1) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$ by the group of integers acting as a rotation by the irrational angle. Our definition of noncommutative Bieberbach spaces is not similar to any of them. We have shown that the algebra $C(\mathbb{T}^3_\Theta)$ admits an action of a discrete group $\mathbb{Z}_N$ (where $N = 2, 3, 4, 6$) for a generic values of Θ. Using this we have defined noncommutative Bieberbach spaces as a fixed point subalgebras of the noncommuatative three torus. This way we have obtained another example of noncommutative spaces which serves as a testing ground for basic tools of noncommutative geometry. As our examples are not isospectral deformation of the classical Bieberbach manifolds we believe that many final result may be highly nontrivial.

K-theory of Bieberbach Spaces

Our dissertation was mainly devoted to the description of spectral triples over noncommutative Bieberbach spaces. However in the process of construction of spectral triples we posed a question of possible projective modules. This lead us to the computation of a $K-$theory of Bieberbach spaces. As we know our result is the first one in the literature of this object. During computation we have used three different six term exact sequences: the first is based on a Lance-Natsume six term exact sequence and can be used for the case $N = 2$; the second is based on a six term exact sequence for the cyclic groups obtained by Blackadar; the third one uses famous Pimsner-Voiculescu sequence and a notion of twisted traces. We showed how to compute the $K-$theory using each of these methods in a case of our toy model, i.e. Klein bottle (viewed as a nonorientable two-dimensional Bieberbach manifold). Then we chose only the most transparent and instructive method to present the explicit computation. It is worth noting that our result is that the K_0 groups for Bieberbach spaces (both commutative and noncommuative) have torsion part. As we are concerned, there are no many concrete examples of noncommutative spaces which exhibit this feature, so we believe that our result is quite interesting. It is also a noticeable fact that there exists a striking relation between the K_0 groups of the classical manifolds $\mathfrak{B}N$ ($N = 2, 3, 4, 6$) and the first homology groups of the corresponding infinite Bieberbach groups Γ_N [35], (so that $\mathfrak{B}N = \mathbb{R}^3/\Gamma_N$), namely $K_0(\mathfrak{B}N) \sim \mathbb{Z} \oplus H_1(\Gamma_N, \mathbb{Z})$, which should follow from Baum-Connes conjecture for Γ_N.

Real Flat Spectral Triples

In the third part of dissertation we presented the classification of spectral triples over noncommutative Bieberbach spaces coming from the reduction of flat spectral triples over three torus. To this aim we have used the notion of equivariant spectral triples elaborated by Sitarz and Paschke. First we classified the $\mathbb{Z}_N-$equivariant flat spectral triples over three torus. This gave us strict restriction on both: the $\mathbb{Z}_N-$equiavriant spin structures over torus and the representation of group $\mathbb{Z}_N$ on them. Using this method we obtained a vast set of spectral triples over $\mathfrak{B}N_\theta$. This have been used as a testing ground for the definitions of the reducibility. It appears that only a weaker definition, of reducibility up to bounded perturbation of the Dirac operator, gives exactly the same number of inequivalent irreducible flat spectral triples over noncommutative Bieberbach spaces (i.e. a noncommutative spin structures) as the number of classical spin structures computed for topological Bieberbach manifolds. The classification of spectral triples done in our approach, the spectrum of Dirac operators coming from them and the eta invariants fully agrees with the classification of spin structures, classical Dirac operators and their eta invariants discussed by Pfaffle. Thus the method of classifying irreducible real spectral triples appears an effective algebraic method of classification of spin structures and computation of spectra of Dirac operator.

Spectra of Dirac Operator and Spectral Action

As the last part of we computed the spectral action. We have shown that apart from the possible difference arising from the eta invariant the perturbative spectral action is exactly the same for all three-dimensional Bieberbach manifolds as for the three torus. This is not at all surprising as all terms in the perturbative expansion (for the symmetric cut-off) depend on the Riemann curvature and Bieberbach manifolds are flat. The new result is the appearance of slight modifications when the cut-off function has an asymmetric part.

8.2 Perspectives

During the computation carried out in the thesis certain observations on the three-dimensional Bieberbach manifolds were made. When treated as hypothesis they presumably give rise to more general theorems.

In classical picture one of the criterion of inequivalence between spectral triples is the difference of the eta invariants of theirs Dirac operators. The eta invariant for the spectrum of Dirac operator coming from classical consideration is known. We would like to know if for real spectral triples the eta invariant is stable under the small perturbation of the Dirac operator in the noncommutative case and (if apparently it isn't) how this perturbation looks like. However if the answer to this question is positive it would be enough to use it to distinguish inequivalent equivariant spectral triples for both commutative and noncommutative Bieberbach spaces.

During the research we have computed the spectral action for the spectral triples over noncommutative Bieberbach manifolds. It comes from the cut-off computation of the trace of Dirac operator from the spectral triples. Those manifolds are flat and three-dimensional and, as so, they are candidates for cosmological topology. It would be fruitful to discuss the consequences of computation in the field of theoretical physics. The question about the implications of the spectral action of Bieberbach manifolds on the physical cosmological models still remains without answer.

From the computation of $K_0(\mathfrak{B}N_\theta)$ we see that those groups have torsion part - an inclusion of finite discrete group. In most cases those groups are cyclic $\mathbb{Z}_n$ groups. There must exist projective modules over Bieberbach spaces which also have this feature. In noncommutative geometry a projective module over a C^*-algebra is a noncommutative counterpart to vector bundle over a classical manifold. For example for $\mathbb{T}^3/\mathbb{Z}_3$ there must exist a nontrivial vector bundle E such that $E \oplus E \oplus E$ is trivial, i.e. equals $\mathbb{T}^3/\mathbb{Z}_3 \times \mathbb{C}^3$. Then the question of possible relevant consequences for physical models arise. For example it would be instructive to investigate if it is possible to describe some $\mathbb{Z}_3-$symmetrical particle fields over Bieberbach manifolds using the sections of those vector bundles.

Recently Oliver Pfante in [46] defined the noncommutative generalisation of Chern-Simons action. Originally Chern-Simoms theory reffered to topological quantum field theory of three-dimensional manifolds. As Bieberbach manifolds which were the subject of my foregoing research are three-dimensional it is possible to use the definitions and tools elaborated by Pfante to the challenge of computation of Chern-Simons theory for Bieberbach spaces and make a step toward noncommutative counterpart of topological quantum field theory of Bieberbach manifolds.

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