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Beyond the Standard Model with Noncommutative Geometry

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Oświadczenie

Ja niżej podpisany Thomas E. Williams (nr indeksu: 1127347) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „Beyond the Standard Model with Noncommutative Geometry" jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem dr. hab. Leszka Hadasza. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

Kraków, dnia ....

podpis doktoranta

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Abstract

The purpose of this thesis is threefold: firstly to introduce the reader to the basic principles of noncommutative geometry which are requisite for the construction of models of physics in this formalism. Secondly, in order to lend context to the models constructed in this thesis, two possible extensions of the Standard Model are briefly introduced. Namely, the Pati-Salam grand unified theory and supersymmetry, both of which have deservedly received significant attention over the years. And thirdly, to present in some detail several works to which I have contributed, and which advances the development and study of these models in the context of noncommutative geometry.

Streszczenie

Niniejsza praca ma trzy cele: po pierwsze, zapoznać czytelnika z podstawowymi zasadami geometrii nieprzemiennej, niezbędnymi do budowy modeli fizyki w tym formalizmie. Po drugie, aby nadać kontekst modelom skonstruowanym w tej pracy, pokrótce przedstawiono dwa możliwe rozszerzenia Modelu Standardowego: teorię wielkiej unifikacji Pati-Salama oraz supersymetrię, które przez lata przyciągały znaczące zainteresowanie. Trzeci cel to przedstawić szczegółowo szereg prac, do których wniosłem swój wkład, a które posuwają naprzód badania tych teorii w kontekście geometrii nieprzemiennej.

Contents

Chapter 1

Introduction

Let's begin with an introductory overview of the primary topics to be discussed in this work, and the essential language and tools with which the reader should be equipped before proceeding to subsequent chapters.

1.1 Noncommutative geometry

Noncommutative geometry (NCG) is a relatively new branch of mathematics, the bulk of which was worked out in the latter half of the previous century by French mathematician Alain Connes [1]. In its essence, noncommutative geometry tells us how to abstract all pertinent information about a classical differentiable manifold to the level of operators and abstract algebras of functions defined over its coordinates. Furthermore, Connes goes on to formulate a set of conditions for which the converse is also achieved. Via his celebrated *reconstruction theorem*, one may recover the geometric information about the underlying manifold from a set of purely algebraically defined quantities.

In the formalisms of operator theory and abstract algebra there is no need to restrict to the study of commutative algebras only, thus by considering noncommutative algebras, a broader class of "geometric spaces" is studied, for which a classical geometric description is impossible. Such spaces are frequently referred to as *noncommutative geometries* or *noncommutative manifolds*. As it turns out, it is in this class of noncommutative spaces (or more correctly a restricted subclass known as *almost-commutative manifolds* or *AC-manifolds*) in which Connes found a unified theory of the complete Standard Model (SM) of particle physics coupled with classical (Einsteinian) gravity [2].

Due to the enormity and mathematical depth of the subject, the reader is referred to the existing literature for a more comprehensive introduction [3, 4] and instead herein is given only a rapid overview of the elements which will be necessary for what follows.

As previously stated, due to its success in producing the SM, the recipes in this thesis will, in each case, follow the *AC-geometry* approach with conditions suitably relaxed to accommodate a particular extension of the Standard Model. With this in mind, recall that the AC-geometry approach begins with a *total space* of the form

$$M \times F, \tag{1.1}$$

where M is a compact Reimannian spin manifold and F is some finite, discrete topological space.

The next step is to pass to a set of algebraic structures which equivalently describes this space, known as a *spectral triple*. This is done in 2 steps. First, to M and F, associate the spectral triples

$$\mathcal{M} \equiv (\mathcal{A}_M, \mathcal{H}_M, \mathcal{D}_M) \quad \text{and} \quad \mathcal{F} \equiv (\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F), \tag{1.2}$$

where $\mathcal{M}$ consists of a unital, associative $*$-algebra, $\mathcal{A}_M$, faithfully represented on a Hilbert space of bounded operators, $\mathcal{H}_M$, and a Hermetian operator, $\mathcal{D}_M : \mathcal{H}_M \to \mathcal{H}_M$, often taken to be a *Dirac operator*, with compact resolvent, and such that $[\mathcal{D}_M, a]$ is bounded for any $a \in \mathcal{A}_M$, and where $\mathcal{F}$ consists of a finite dimensional, unital, associative $*$-algebra, $\mathcal{A}_F$, faithfully represented on a finite dimesional Hilbert space, $\mathcal{H}_F$, upon which acts a symmetric operator $\mathcal{D}_F$. Then the spectral triple encoding the geometric structure of the total space is given by the tensor product

$$\mathcal{M} \otimes \mathcal{F} \equiv (\mathcal{A}, \mathcal{H}, \mathcal{D}) \equiv (\mathcal{A}_M \otimes \mathcal{A}_{\mathcal{F}}, \mathcal{H}_M \otimes \mathcal{H}_F, \mathcal{D}_M \otimes \mathcal{D}_F), \tag{1.3}$$

where $\mathcal{D} \equiv \mathcal{D}_M \otimes \mathcal{D}_F \equiv \mathcal{D}_M \otimes \mathbf{1}_F + \gamma_M \otimes \mathcal{D}_F$ is called the Dirac operator of the AC-manifold.

A spectral triple is said to be *even* if the Hilbert space is equipped with a $\mathbb{Z}_2$-grading (an operator $\gamma : \mathcal{H} \to \mathcal{H}$, such that $\gamma^2 = 1$) which satisfies $[\gamma, a] = 0$ and $\{\gamma, \mathcal{D}\} = 0$. Also, a spectral triple is called *real*, if the Hilbert space admits a *real structure*, that is, an anti-unitary operator, $J : \mathcal{H} \to \mathcal{H}$ such that $J^2 = \epsilon$, $J\mathcal{D} = \epsilon' \mathcal{D} F$, and in the case that the spectral triple is even, $J\gamma = \epsilon'' \gamma J$, where ϵ, ϵ', and ϵ'' are each ± 1 and together determine the *KO-dimension* of the spectral triple. Moreover, it is required that the *commutant property* (or 0^{th}-order condition) and the *1st-order condition* hold,

$$[a, b^\circ] = 0, \quad \text{and} \quad [[\mathcal{D}, a], b^\circ] = 0, \tag{1.4}$$

where $a, b \in \mathcal{A}$, and $b^\circ \equiv J b^* J^{-1}$ implements a right action of $\mathcal{A}$ on $\mathcal{H}$.

Finally then, the object of central importance to this story, namely a real, even spectral "triple"

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}; \gamma, J), \tag{1.5}$$

where $\gamma \equiv \gamma_M \otimes \gamma_F$, and $J \equiv J_M \otimes J_F$, may be written down (generally).

In noncommutative geometry, the *gauge fields* arise by considering *Morita (self-)equivalence* of the algebra, meanwhile the *gauge group* implements unitary equivalence of spectral triples (which is itself an instantiation of Morita equivalence). Briefly, the above real, even spectral triple is equivalent, up to Morita self-equivalence, to

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}_A; \gamma, J), \tag{1.6}$$

where $\mathcal{D}_A = \mathcal{D} + A + \epsilon' JAJ^{-1}$ is the *inner fluctuated* Dirac operator and $A \in \Omega^1_{\mathcal{D}}(\mathcal{A}) \equiv \{\sum_i a_i[\mathcal{D}, b_i] : a_i, b_i \in \mathcal{A}\}$ are the *gauge fields*, or *inner fluctuations* of the Dirac operator, $\mathcal{D}$. Meanwhile,

$$(\mathcal{A}, \mathcal{H}, U\mathcal{D}U^*; \gamma, J), \tag{1.7}$$

is a unitarily equivalent triple obtained by an element of the gauge group $U = uJuJ^{-1}$ where u is a unitary element of $\mathcal{A}$. Ultimately, $U\mathcal{D}U^* = \mathcal{D}_A$ for $A = u[\mathcal{D}, u^*]$.

Physics then emerges from the noncommutative formalism from the *action functional*,

$$S = S_b + S_f \equiv \mathrm{Tr}\left(f\left(\frac{\mathcal{D}_A}{\Lambda}\right)\right) + \langle \xi, \mathcal{D}_A \xi \rangle \,, \tag{1.8}$$

where $\xi \in \mathcal{H}$ and f is some sufficiently well behaved function. This action consists of the *spectral action*, S_b, responsible for the bosonic terms, and the *fermionic action*, S_f, taking care of fermionic particle content. The former is usually evaluated by heat kernal methods and is spectral in the sense that it counts eigenvalues of the fluctuated Dirac operator up to some predetermined cut-off, Λ.

1.2 Pati-Salam and L-R symmetric models

The Standard Model of Particle Physics is the preeminent theory of fundamental constituents and their interactions. It has been experimentally tested and verified to be accurate with a very high degree of precision. Nevertheless, there are several open questions which, to date, the Standard Model has left unresolved. Among them, several are related to the masses of neutrinos and the seesaw mechanism [5], and others to baryon asymmetry [6]. Nowadays, there is also a tremendous set of cosmological data [7] which suggest the existence of dark matter. Several attempts to explain the aforementioned questions have already been proposed. Due to the proven success of the Standard Model, most of them are extensions thereof and are known as theories which go *Beyond the Standard Model*. An interesting one, which has already been under consideration for several years and intensively studied by several physicists, is the model introduced by J.C. Pati and A. Salam [8].

The *Pati-Salam model* is a Yang-Mills-type model based on the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(4)$ gauge group. It extends the usual Standard Model by e.g. introducing leptoquark symmetry and left-right symmetry. It was also considered to explain the origin of parity symmetry breaking [9], [10].

Models which describe theories of Particle Physics are traditionally constructed in the *Lagrangian formalism*, i.e. the form of the action is postulated based upon the desired symmetries of the resulting model. A possible geometrical explanation for the structure of such theories is provided by spectral geometry. The Standard Model has been studied for several years in the framework of almost-commutative geometry (see e.g. [11],[12] and [2]) but some puzzles remain unsolved, not only for the product spectral triple, but also for its finite part. Recently, in [13], it was proposed to consider the finite spectral triple for the Standard Model as the shadow of some pseudo-Riemannian triple in such a way that the pseudo-Riemannian structure leads to the existence of some nontrivial grading on the Riemannian triple.

The Pati-Salam model has been considered as a noncommutative geometry by several authors, see e.g. [14],[15],[16],[17] and [18]. As a survey of the historical development of these methods and their applications in Particle Physics, [19] is also recommended. Reduced versions of the Pati-Salam models, i.e. the Left-Right Symmetric Models, were also considered in the framework of noncommutative geometry, firstly as potentially interesting examples for the Connes-Lott scheme of spectral geometry, but then also from the point of view of possible physical applications — see e.g. [20],[21],[22] and [23]. However, since some of the fundamental axioms of noncommutative geometry were not satisfied, such models were not satisfactory. Later on, due to the trend of relaxing some of the axioms, e.g. the first-order condition, and further development of the spectral theory in their absence (see e.g. [24]), the family of Pati-Salam models was analyzed.

1.3 Supersymmetry

Simply stated, supersymmetry (SUSY) is a proposed symmetry of nature that relates to each boson (a particle with integer spin) an associated partner particle with half-integer spin (a fermion), and vice versa. Although initially posited as a meson/baryon symmetry in the theory of hadrons [25], it was reincarnated several years later as a global spacetime symmetry in the context of quantum field theories (QFTs). It is perhaps in the work of Julius Wess and Bruno Zumino [26], that supersymmetry really came of age. Their work provided the first example of a four-dimensional, supersymmetric quantum field theory with interactions.

While the defining principle of supersymmetry is concisely stated, its simplicity is disproportionate to its value. With the additional principle of supersymmetry, many of the curiosities and apparent inconsistencies of the SM are readily explained. The unexpectedly low mass of the Higgs particle, the hierarchy problem, and the non-unification of the gauge coupling constants at high energies, to name a few. Additionally, SUSY provides a candidate for the particle(s) responsible for dark matter, the only possible "workaround" for the Coleman-Madula theorem, and a ray of hope for (potentially) physically relevant string theories which without SUSY, would be out of business [27, 28].

A particularly pleasing formulation of SUSY arises from the notion of *superspace*. One may argue that this should be the natural starting point for combining SUSY with NCG since it elegantly encodes the SUSY transformations as geometric translations of its coordinates [29, 30]. Essentially, a superspace is obtained by adding one or more anticommuting coordinates to an ordinary manifold. The component fields of a multiplet of particles and their superpartners are encoded in a single object called a *superfield*, which is a function of the superspace coordinates. Then, infinitesimal translations of the superspace coordinates correspond with global supersymmetry transformations, transforming bosonic component fields into fermionic ones, and vice versa. Moreover, for a superfield that has been written in an expansion of its anticommuting coordinates, the variation of the highest order term, with respect to SUSY transformations, is always a total derivative, making it a natural choice for building the action of a supersymmetric model of physics.

1.4 Summary of results

The following chapters present the details of several projects to which I have contributed and whose aim is to advance the development and study of physics beyond the Standard Model by using mathematical techniques drawn from and inspired by noncommutative geometry [31, 32, 33].

In Chapter 2 we analyze the finite part of the pseudo-Riemannian spectral triple for the Pati-Salam model and determine the possible pseudo-Riemannian structures which may exist for the various permissible gradings of the input algebra. Given that this triple should “restrict" to a Riemannian triple with an additional grading, we argue that the allowed pseudo-Riemannian structures restrict the family of Pati-Salam models to the Left-Right symmetric ones, i.e. No-go for lepto-quarks. Furthermore, the induced grading on the Riemannian triple greatly reduces the freedom of admissible Dirac operators. These restrictions are identified, and we note that this class of Dirac operators still allows for the existence of $SU(2)$-doublets of right-handed particles. Thus, still allowing for physics beyond the Standard Model.

In Chapter 3, with an eye toward its application as the Dirac operator for a superspace based spectral triple, we factorize the ordinary Dirac operator on Minkowski space and show that a particular solution of its action on a restricted space of superfields is a massless spinor superfield together with a Maxwell gauge field. We then examine and discuss the implications of this factorization on the total-space Dirac operator of a spectral triple in the usual AC-geometry approach to noncommutative model building.

And finally, in Chapter 4, a possible framework for incorporating SUSY in its superspace formulation into the AC-geometry approach to noncommutative geometry is proposed. Working in an example superspace and together with a 2-point finite space, the input data of the “spectral triple" is examined, i.e. the unital, associative Grassmann algebra, its representation on a (super) Hilbert space, gradings, real structure (charge conjugation), and a candidate Dirac operator. Also, the fermionic action for a simply chosen inner product and the dimensionally meaningful term of the spectral action are both calculated.

Chapter 2

Pseudo-Riemmanian finite spaces for PS models

In this chapter, the pseudo-Riemannian structure of the finite triple for the Pati-Salam model is considered, in the sense introduced in [13]. Further, the finite spectral triples for such models are analyzed, possible pseudo-Riemannian structures are discussed and are shown to be related to the grading that distinguishes leptons from quarks, and it is argued that the existence of this pseudo-Riemannian structure restricts the family of Pati-Salam models to the Left-Right symmetric ones. The pulished version of these results can be found in [31]

2.1 Finite spectral triples for Pati-Salam models

In this section we consider the finite spectral triple for the family of Pati-Salam models. We discuss algebras and their commutants, different choices of chiral structures and possible Dirac operators.

2.1.1 Spectral data

The algebra for the Pati-Salam model is of the form

$$\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}), \tag{2.1}$$

where $\mathbb{H}_L$ and $\mathbb{H}_R$ respectively denote left and right chiral copies of the quaternion algebra. Consider $F = M_4(\mathbb{C})$ with the inner product $\langle v, w\rangle = \mathrm{Tr}(v^* w)$ (where v^* denotes the Hermitian conjugate of $v \in F$). The elements, v in F can be presented in the following form

$$v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix}. \tag{2.2}$$

Let $\mathcal{H} = F \oplus F^*$ be the Hilbert space for the model we consider here, where F^* is the dual representation to F. Following [34] we can identify

$$\mathrm{End}_{\mathbb{C}}(\mathcal{H}) \cong M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C}), \tag{2.3}$$

where the matrix algebra is represented on $\mathcal{H}$ as

$$\widetilde{\pi}\left(\alpha \otimes 1_2 \otimes \beta\right) \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \alpha v \beta^t \\ \alpha w \beta^t \end{bmatrix}, \tag{2.4}$$

and

$$\widetilde{\pi}\left(1_4 \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes 1_4\right) \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} av + bw \\ cv + dw \end{bmatrix}, \tag{2.5}$$

for all $\alpha, \beta, v, w \in M_4(\mathbb{C})$ and $a, b, c, d \in \mathbb{C}$.

Therefore, we have to represent all operators acting on $\mathcal{H}$ as elements of this tensor product space, and also find the form of the representation $\pi : \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H})$ in this language.

Let e_{ij} be the matrix that has 1 in the entry (i, j) and zero otherwise. Then the grading γ has the following matrix representation

$$\gamma = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + 1_4 \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix}. \tag{2.6}$$

There is also another possible choice of grading [35], for which left-handed leptons have the same parity as right-handed quarks, and vice versa for the opposite chirality:

$$\gamma_\star = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} -1 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}. \tag{2.7}$$

With respect to the decomposition $\mathcal{H} = F \oplus F^*$, let J be the real structure, i.e.

$$J \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w^* \\ v^* \end{bmatrix}. \tag{2.8}$$

It is used to define the opposite representation [34]. For $\xi = \widetilde{\pi}\left(\alpha \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \beta\right)$ we take $\xi^\circ = J\xi^* J^{-1} = \widetilde{\pi}\left(\beta^t \otimes \begin{bmatrix} d & b \\ c & a \end{bmatrix} \otimes \alpha^t\right)$. From now on we will omit the $\widetilde{\pi}$ symbol for the representation of an endomorphism.

The elements of the algebra $\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \otimes M_4(\mathbb{C})$ are represented on $\mathcal{H}$ as

$$\pi(q_1, q_2, m) = \begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + m \otimes e_{22} \otimes 1_4, \tag{2.9}$$

where $q_1 \in \mathbb{H}_R$, $q_2 \in \mathbb{H}_L$ and $m \in M_4(\mathbb{C})$.

Notice that $\gamma_\star$ does not commute with this representation of the algebra $\mathcal{A}$ unless the symmetry following from $M_4(\mathbb{C})$ is broken into $\mathbb{C} \oplus M_3(\mathbb{C})$.

Therefore, here we are considering two algebras. The first one being $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ which we refer to as corresponding to an unreduced Pati-Salam model, and the second one $\mathbb{H}_R \oplus \mathbb{H}_L \oplus \mathbb{C} \oplus M_3(\mathbb{C})$, which we will call reduced.

Since the Dirac operator $D \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$, it is of the form

$$D = \sum_{\substack{i,j=1,2 \\ 1 \le k \le K}} D^k_{1ij} \otimes e_{ij} \otimes D^k_{2ij}, \tag{2.10}$$

with $D^k_{1ij}, D^k_{2ij} \in M_4(\mathbb{C})$, for some natural number K. From now on, summations will be understood to be over the entire range of all indices unless explicitly stated otherwise.

2.1.2 Commutants

We now consider the commutants of several algebras related to the unreduced and reduced Pati-Salam models. These results will be crucial to the discussion in section 2.2.3.

Notice first, that for any (real or complex) matrix algebra $\mathcal{A}$ contained in $M_N(\mathbb{C})$ for some N, the commutant $\mathcal{A}'$ is the same as $\mathcal{A}'_{\mathbb{C}}$, where $\mathcal{A}_{\mathbb{C}}$ denotes the complexification of $\mathcal{A}$.

By a straightforward computation we can check that the commutant of the algebra of elements $\begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix}$, with $q_1, q_2 \in \mathbb{H}$ is the algebra with elements $\begin{bmatrix} \alpha 1_2 & \\ & \beta 1_2 \end{bmatrix}$, where $\alpha, \beta \in \mathbb{C}$. We denote this algebra by $\mathcal{C}_1$.

In a similar manner, the commutant of the algebra of elements $\begin{bmatrix} \lambda & \\ & n \end{bmatrix}$, with $\lambda \in \mathbb{C}$, $n \in M_3(\mathbb{C})$ is the algebra with elements $\begin{bmatrix} \alpha & \\ & \beta 1_3 \end{bmatrix}$, where $\alpha, \beta \in \mathbb{C}$. We denote this algebra by $\mathcal{C}_2$.

Furthermore, notice that $M_4(\mathbb{C})' \cong \mathbb{C}$. Therefore, we can describe the commutants of the Pati-Salam algebra for the unreduced (i.e. with $\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$) case, and the reduced one (i.e. with $\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus \mathbb{C} \oplus M_3(\mathbb{C})$).

Proposition 2.1.1. *The commutant* $(\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}))'$ *in* $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ *is the algebra generated by elements of the form*

$$A_1 \otimes e_{11} \otimes E_1 + 1_4 \otimes e_{22} \otimes E_2, \tag{2.11}$$

where $A_1 \in \mathcal{C}_1$ *and* $E_1, E_2 \in M_4(\mathbb{C})$.

Proof. Any element of the considered algebra may be represented as

$$\pi(q_1, q_2, m) = \begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + m \otimes e_{22} \otimes 1_4, \tag{2.12}$$

with $q_1, q_2 \in \mathbb{H}$ and $m \in M_4(\mathbb{C})$. It is enough to find which elements of the form

$$A_1 \otimes e_{11} + A_2 \otimes e_{12} + A_3 \otimes e_{21} + A_4 \otimes e_{22} \in M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \tag{2.13}$$

commute with $\begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix} \otimes e_{11} + m \otimes e_{22}$ for all q_1, q_2 and m.

The only possible solutions are with $A_1 \in \mathcal{C}_1$, $A_4 \sim 1_4$ and $A_2 = A_3 = 0$. □

In a perfectly similar way, we get the following

Proposition 2.1.2. *The commutant* $(\mathbb{H}_R \oplus \mathbb{H}_L \oplus \mathbb{C} \oplus M_3(\mathbb{C}))'$ *in* $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ *is the algebra generated by elements of the form*

$$A_1 \otimes e_{11} \otimes E_1 + A_2 \otimes e_{22} \otimes E_2, \tag{2.14}$$

where $A_1 \in \mathcal{C}_1$, $A_2 \in \mathcal{C}_2$ *and* $E_1, E_2 \in M_4(\mathbb{C})$.

2.1.3 Dirac operators and reality

Let us now consider self-adjoint Dirac operators that commute with the real structure J. Because $DJ = JD$, we have $D = JDJ^{-1} = (D^*)^\circ$, but since $D = D^*$, the necessary condition that has to be satisfied is $D = D^\circ$. Notice that this is a weaker condition than the original two together. Therefore, we will consider them separately.

Observe first, that the Dirac operator $D = \sum D_{ijklrs} e_{kl} \otimes e_{ij} \otimes e_{rs}$ is self-adjoint if and only if $D_{ijklrs} = \bar{D}_{jilksr}$ for all indices.

Suppose now that $DJ = JD$, i.e. $D = JDJ^{-1}$. We may write the Dirac operator as

$$D = \underbrace{D_{11} + D_{12}}_{D_0} + \underbrace{D_{22} + D_{21}}_{D_1}, \tag{2.15}$$

where

$$D_{ij} = \sum_k D^k_{1ij} \otimes e_{ij} \otimes D^k_{2ij}, \tag{2.16}$$

as previously described in (2.10). Notice that for an operator X of the form $A \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes B$ we have

$$JXJ^{-1} = \overline{B} \otimes \begin{pmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{pmatrix} \otimes \overline{A}, \tag{2.17}$$

so it follows that $D_1 = JD_0J^{-1}$. Moreover, as a result we immediately get the following

Proposition 2.1.3. *The Dirac operator $D = \sum D_{ijklrs} e_{kl} \otimes e_{ij} \otimes e_{rs}$ commutes with J if and only if*

$$D_{11klrs} = \bar{D}_{22rskl}, \qquad \textit{and} \qquad D_{12klrs} = \bar{D}_{21rskl}, \tag{2.18}$$

for all k, l, r, s.

Let A be an operator such that $AJ = \alpha JA$ for some $\alpha = \pm 1$. Moreover, suppose that $A = A_{11} + A_{22}$, where

$$A_{ij} = \sum_k A^k_{1ij} \otimes e_{ij} \otimes A^k_{2ij}, \tag{2.19}$$

for some $A^k_{lij} \in M_4(\mathbb{C})$. Then we have the following

Lemma 2.1.4. *$[A, D] = 0$ if and only if $[A, D_0] = 0$. And analogously for anticommutators.*

Proof. Observe that $[A, D] = [A, D_0] + \alpha J[A, D_0]J$, since $J^2 = \mathrm{id}$. Furthermore, $[A, D_0]$ only contains terms with $\cdots \otimes e_{11} \otimes \cdots$ and $\cdots \otimes e_{12} \otimes \cdots$, while $J[A, D_0]J$ only contains terms with $\cdots \otimes e_{21} \otimes \cdots$ and $\cdots \otimes e_{22} \otimes \cdots$. $\square$

2.1.4 Dirac operators for γ

Since we are interested in Dirac operators that commute with the real structure J we can write $D = D_0 + JD_0J^{-1}$. We first consider all Dirac operators D that anticommute with γ, given by (2.6), as a grading. It is enough to restrict ourselves to the D_0 part.

We have the following

Proposition 2.1.5. *$D = D_0 + JD_0J^{-1}$ anticommutes with γ if and only if D_0 is of the form*

$$\begin{aligned} D_0 = \sum_k \Bigg\{ &\begin{bmatrix} 0_2 & X_k \\ Y_k & 0_2 \end{bmatrix} \otimes e_{11} \otimes A_k + \begin{bmatrix} P_k & Q_k \\ 0_2 & 0_2 \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} Z_k & 0_2 \\ T_k & 0_2 \end{bmatrix} + \\ &+ \begin{bmatrix} 0_2 & 0_2 \\ U_k & V_k \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0_2 & W_k \\ 0_2 & S_k \end{bmatrix} \Bigg\}, \end{aligned} \tag{2.20}$$

where $X_k, Y_k, Z_k, T_k, U_k, V_k, W_k, S_k, P_k, Q_k \in M_2(\mathbb{C})$ and $A_k \in M_4(\mathbb{C})$.

Proof. Notice that

$$\gamma = \sum_{n=1,2} (e_{nn} \otimes e_{11} \otimes e_{mm} - e_{mm} \otimes e_{22} \otimes e_{nn}) + \sum_{n=3,4} (e_{mm} \otimes e_{22} \otimes e_{nn} - e_{nn} \otimes e_{11} \otimes e_{mm}). \tag{2.21}$$

Let us write D as $D = \sum \hat{D}_{ijklrs}$, where $\hat{D}_{ijklrs} = D_{ijklrs} e_{kl} \otimes e_{ij} \otimes e_{rs}$. Simple computation shows that

$$D\gamma = \sum_{n=1,2} \left(\hat{D}_{i1knrm} - \hat{D}_{i2kmrn} \right) + \sum_{n=3,4} \left(\hat{D}_{i2kmrn} - \hat{D}_{i1knrm} \right), \tag{2.22}$$

and

$$\gamma D = \sum_{n=1,2} \left(\hat{D}_{1jnlms} - \hat{D}_{2jmlns} \right) + \sum_{n=3,4} \left(\hat{D}_{2jmlns} - \hat{D}_{1jnlms} \right), \tag{2.23}$$

hence by direct inspection we see that the Dirac operator has to be of the claimed form. $\square$

2.1.5 Dirac operators for $\gamma_\star$

Again, we are considering all Dirac operators which commute with J, and therefore of the form $D = D_0 + JD_0J^{-1}$, but now which anticommute with $\gamma_\star$, given by (2.7), as a grading. This time we have,

Proposition 2.1.6. *$D = D_0 + JD_0J^{-1}$ anticommutes with $\gamma_\star$ if and only if D_0 is of the form*

$$\begin{aligned} D_0 = \sum_k \Bigg\{ & \begin{bmatrix} X_k & \\ & Y_k \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} & \alpha_{1k} & \alpha_{2k} & \alpha_{3k} \\ \beta_{1k} & & & \\ \beta_{2k} & & & \\ \beta_{3k} & & & \end{bmatrix} + \begin{bmatrix} & Z_k \\ T_k & \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \gamma_k & \\ & C_k \end{bmatrix} + \\ & + \begin{bmatrix} \delta_{1k} & \\ \delta_{2k} & \\ & P_k \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} \sigma_{1k} & \sigma_{2k} & \\ & & E_k \end{bmatrix} + \begin{bmatrix} & F_k \\ \mu_{1k} & \\ \mu_{2k} & \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} & \nu_{1k} & \nu_{2k} \\ G_k & & \end{bmatrix} \Bigg\}, \end{aligned} \tag{2.24}$$

where $X_k, Y_k, Z_k, T_k \in M_2(\mathbb{C})$, $E_k, G_k \in M_{3\times 2}(\mathbb{C})$, $P_k, F_k \in M_{2\times 3}(\mathbb{C})$, $C_k \in M_3(\mathbb{C})$, and $\alpha_{lk}, \beta_{lk}, \gamma_k, \delta_{lk}, \sigma_{lk}, \mu_{lk}, \nu_{lk} \in \mathbb{C}$.

Proof. As before, let us first write the grading in a more convenient form,

$$\begin{aligned} \gamma_\star = & \sum_{n=1,2} (e_{nn} \otimes e_{11} \otimes e_{11} - e_{11} \otimes e_{22} \otimes e_{nn}) + \sum_{\substack{n=3,4 \\ m=2,3,4}} (e_{nn} \otimes e_{11} \otimes e_{mm} - e_{mm} \otimes e_{22} \otimes e_{nn}) + \\ & + \sum_{n=3,4} (e_{11} \otimes e_{22} \otimes e_{nn} - e_{nn} \otimes e_{11} \otimes e_{11}) + \sum_{\substack{m=1,2 \\ n=2,3,4}} (e_{nn} \otimes e_{22} \otimes e_{mm} - e_{mm} \otimes e_{11} \otimes e_{nn}). \end{aligned} \tag{2.25}$$

For $D = \sum \hat{D}_{ijklrs}$ notice that

$$\begin{aligned} D\gamma_\star = & \sum_{n=1,2} \left(\hat{D}_{i1knr1} - \hat{D}_{i2k1rn} \right) + \sum_{\substack{n=3,4 \\ m=2,3,4}} \left(\hat{D}_{i1knrm} - \hat{D}_{i2kmrn} \right) + \\ & + \sum_{n=3,4} \left(\hat{D}_{i2k1rn} - \hat{D}_{i1knr1} \right) + \sum_{\substack{m=1,2 \\ n=2,3,4}} \left(\hat{D}_{i2knrm} - \hat{D}_{i1kmrn} \right), \end{aligned} \tag{2.26}$$

and likewise

$$\begin{aligned}\gamma_\star D = \sum_{n=1,2}\left(\hat{D}_{1jnl1s}-\hat{D}_{2j1lns}\right) + \sum_{\substack{n=3,4\\ m=2,3,4}}\left(\hat{D}_{1jnlms}-\hat{D}_{2jmlns}\right)+ \\ + \sum_{n=3,4}\left(\hat{D}_{2j1lns}-\hat{D}_{1jnl1s}\right) + \sum_{\substack{m=1,2\\ n=2,3,4}}\left(\hat{D}_{2jnlms}-\hat{D}_{1jmlns}\right).\end{aligned} \tag{2.27}$$

Therefore, a straightforward comparison shows that D anticommutes with $\gamma_\star$ if and only if it is of the claimed form. □

2.2 Pseudo-Riemannian Structures

Let us recall that a pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$, of signature (p, q), is a system consisting of an algebra $\mathcal{A}$, Hilbert space $\mathcal{H}$, Dirac operator $\mathcal{D}$, $\mathbb{Z}/2\mathbb{Z}$-grading γ, real structure J and an additional grading $\beta \in \mathrm{End}(\mathcal{H})$ such that $\beta^* = \beta$, $\beta^2 = 1$ and which commutes with the representation of $\mathcal{A}$ and defines a Krein structure on the Hilbert space. These objects are supposed to satisfy several conditions that are collected in [13], section II. For our purposes it is enough to recall that β has to satisfy $\beta\gamma = (-1)^p\gamma\beta$ and $\beta J = (-1)^{\frac{p(p-1)}{2}}\epsilon^p J\beta$, where $\mathcal{D}J = \epsilon J\mathcal{D}$ and $\epsilon = \pm 1$ depending on the KO-dimension of the triple.

Furthermore, we assume that $\mathcal{D}$ is β-selfadjoint, i.e. $\mathcal{D}^* = (-1)^p\beta\mathcal{D}\beta$. We say that the triple is time-oriented if β can be presented as the image of a certain Hochschild p-cycle.

Out of the pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$ one can construct its Riemannian restriction, i.e. a triple $(\mathcal{A}, \mathcal{H}, D_+, \gamma, J, \beta)$ with $\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^*)$ which is a self-adjoint operator and $\beta\mathcal{D}_+ = (-1)^p\mathcal{D}_+\beta$. This spectral triple is of the same KO-dimension as the one we started with.

As was noticed in [13] the spectral triple for the Standard Model can be treated as a Riemannian restriction of some pseudo-Riemannian triple, with an additional grading originating from the time-orientation on the pseudo-Riemannian level. The existence of such a grading results in a restriction on the number of possible Dirac operators, compatible with the other elements of the triple.

Here we are looking for similar effects in the case of Pati-Salam models. From now on we will denote $\mathcal{D}_+$ by D, and since the spectral triple for the Pati-Salam models has to be of KO-dimension 6, we take the signature to be $(0, 2)$. We are looking for all possible βs and (self-adjoint) Dirac operators D such that $\beta D = D\beta$.

Therefore we are looking for β of the form

$$\beta = \pi(q_1, q_2, m)J\pi(q_1', q_2', m')^*J^{-1}, \tag{2.28}$$

with $q_1, q_2, q_1', q_2' \in \mathbb{H}$ and $m, m' \in M_4(\mathbb{C})$ for the unreduced case, and

$$\beta = \pi(q_1, q_2, \lambda, n)J\pi(q_1', q_2', \lambda', n')^*J^{-1}, \tag{2.29}$$

with $q_1, q_2, q_1', q_2' \in \mathbb{H}$, $\lambda, \lambda' \in \mathbb{C}$ and $n, n' \in M_3(\mathbb{C})$ in the reduced case.

For simplicity we have assumed here that β is a 0-cycle containing only one summand. We postpone the discussion about the more general situation until section 2.2.3.

Moreover, we require that

$$\beta\gamma = \gamma\beta, \qquad \beta J = J\beta. \tag{2.30}$$

2.2.1 The unreduced Pati-Salam model

Let us start with this case first. Then β can be represented as

$$\beta = \begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix} \otimes e_{11} \otimes m'^t + m \otimes e_{22} \otimes \begin{bmatrix} q_1'^t & \\ & q_2'^t \end{bmatrix}. \tag{2.31}$$

Since β is a 0-cycle, it commutes with the grading by construction. Notice the fact that β commutes both with the algebra and the opposite algebra (since $\beta J = J\beta$ and the order zero condition holds) fixes all matrices q_1, q_2, q_1', q_2' and m, m' to be proportional to the identity, and moreover, it enforces $m = m'^t$ and $q_i = q_i'^t$ for $i = 1, 2$. The condition $\beta^2 = 1$ fixes all these proportionality factors to be a sign.

Therefore, the only possible pseudo-Riemannian structures are

$$\beta = \pi(\eta_1 1_2, \eta_2 1_2, \eta_3 1_4) J \pi(\eta_1 1_2, \eta_2 1_2, \eta_3 1_4) J^{-1} \tag{2.32}$$

with $\eta_1, \eta_2, \eta_3 = \pm 1$.

So, up to the trivial rescaling by a factor of -1 there are only two such possible operators:

$$\pi(1_2, 1_2, 1_4) J \pi(1_2, 1_2, 1_4) J^{-1}, \quad \text{and} \quad \pi(1_2, -1_2, 1_4) J \pi(1_2, -1_2, 1_4) J^{-1}. \tag{2.33}$$

The first of these is the identity operator, so it commutes with everything. This is the trivial case in which we are not interested.

Compatible Dirac operators for the unreduced Pati-Salam model

We are looking for all possible generic Dirac operators D (not necessarily anticommuting with a grading) such that $D\beta = \beta D$. Moreover, we already assume that D commutes with J, so it is of the form $D_0 + JD_0J^{-1}$. For such a D, we get the following

Proposition 2.2.1. *The Dirac operator $D = D_0 + JD_0J^{-1}$ commutes with*

$$\begin{aligned} \beta &= \pi(1_2, -1_2, 1_4) J \pi(1_2, -1_2, 1_4) J^{-1} = \\ &= \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + 1_4 \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}, \end{aligned} \tag{2.34}$$

if and only if D_0 is of the following form

$$
\begin{aligned}
D_0 = \sum_k \Bigg\{ & \begin{bmatrix} \widetilde{X}_k & 0_2 \\ 0_2 & \widetilde{Y}_k \end{bmatrix} \otimes e_{11} \otimes \widetilde{A}_k + \begin{bmatrix} \widetilde{P}_k & \widetilde{Q}_k \\ 0_2 & 0_2 \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} \widetilde{Z}_k & 0_2 \\ \widetilde{T}_k & 0_2 \end{bmatrix} + \\
& + \begin{bmatrix} 0_2 & 0_2 \\ \widetilde{U}_k & \widetilde{V}_k \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0_2 & \widetilde{W}_k \\ 0_2 & \widetilde{S}_k \end{bmatrix} \Bigg\},
\end{aligned} \tag{2.35}
$$

where $\widetilde{X}_k, \widetilde{Y}_k, \widetilde{P}_k, \widetilde{Q}_k, \widetilde{Z}_k, \widetilde{T}_k, \widetilde{U}_k, \widetilde{V}_k, \widetilde{W}_k, \widetilde{S}_k \in M_2(\mathbb{C})$ and $\widetilde{A}_k \in M_4(\mathbb{C})$.

Proof. Following a similar argument as that used to prove Propositions 2.1.5 and 2.1.6, but with β replacing γ or $\gamma_\star$, we first write

$$
\begin{aligned}
\beta = & \sum_{n=1,2} e_{nn} \otimes e_{11} \otimes e_{mm} - \sum_{n=3,4} e_{nn} \otimes e_{11} \otimes e_{mm} + \\
& + \sum_{m=1,2} e_{nn} \otimes e_{22} \otimes e_{mm} - \sum_{m=3,4} e_{nn} \otimes e_{22} \otimes e_{mm}.
\end{aligned} \tag{2.36}
$$

Noticing that for $D = \sum \hat{D}_{ijklrs}$ we have

$$
D\beta = \sum_{n=1,2} \hat{D}_{i1knrm} - \sum_{n=3,4} \hat{D}_{i1knrm} + \sum_{m=1,2} \hat{D}_{i2knrm} - \sum_{m=3,4} \hat{D}_{i2knrm}, \tag{2.37}
$$

and similarly

$$
\beta D = \sum_{n=1,2} \hat{D}_{1jnlms} - \sum_{n=3,4} \hat{D}_{1jnlms} + \sum_{m=1,2} \hat{D}_{2jnlms} - \sum_{m=3,4} \hat{D}_{2jnlms}. \tag{2.38}
$$

The result follows from a straightforward comparison of these expressions. □

Physical consequences of the unreduced Pati-Salam model with γ

Since D anticommutes with γ, we see that the only possibility for such a D to commute with the nontrivial β discussed above is that terms of the form $\cdots \otimes e_{11} \otimes \cdots$ and $\cdots \otimes e_{22} \otimes \cdots$ must vanish.

Here we are looking for possible extensions of the Standard Model, for which the Dirac operator contains terms of the form

$$
\begin{bmatrix} & M_l \\ M_l^\dagger & \end{bmatrix} \otimes e_{11} \otimes e_{11} \quad \text{and} \quad \begin{bmatrix} & M_q \\ M_q^\dagger & \end{bmatrix} \otimes e_{11} \otimes (1_4 - e_{11}), \tag{2.39}
$$

which encode the Yukawa parameters for leptons and quarks. Therefore, physically acceptable extensions of the Standard Model, i.e. those that can be reduced to the Standard Model after imposing additional conditions on terms of the Dirac operator, also have to contain these terms. Therefore, no Pati-Salam model with the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$, grading γ, and with the pseudo-Riemannian structure β is physically acceptable.

2.2.2 The reduced Pati-Salam model

In this case

$$\beta = \begin{bmatrix} q_1 & \\ & q_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \lambda' & \\ & n'^t \end{bmatrix} + \begin{bmatrix} \lambda & \\ & n \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} q_1'^t & \\ & q_2'^t \end{bmatrix}. \tag{2.40}$$

As before, since β is a 0-cycle, it commutes with the grading, and commutation with the algebra, the fact that $\beta^2 = 1$ and that it commutes with J fixes all matrices $q_i = q_i' = \pm 1_2$, for $i = 1, 2$, $\lambda = \lambda' = \pm 1$ and $n = n' = \pm 1_3$. Therefore, up to trivial rescaling we have the following four cases

$$\begin{gathered} \pi(1_2, 1_2, 1, 1_3) J \pi(1_2, 1_2, 1, 1_3) J^{-1}, \quad \pi(1_2, -1_2, 1, 1_3) J \pi(1_2, -1_2, 1, 1_3) J^{-1}, \\ \pi(1_2, 1_2, 1, -1_3) J \pi(1_2, 1_2, 1, -1_3) J^{-1}, \quad \pi(-1_2, 1_2, 1, -1_3) J \pi(-1_2, 1_2, 1, -1_3) J^{-1}. \end{gathered} \tag{2.41}$$

Only three of them are nontrivial, and the case with $\beta = \pi(1_2, -1_2, 1, 1_3) J \pi(1_2, -1_2, 1, 1_3) J^{-1}$ is exactly the same as the one discussed in subsection 2.2.1.

Compatible Dirac operators for the reduced Pati-Salam model

Now, we will discuss restrictions on a generic Dirac operator (not necessarily anticommuting with a grading) which follow from commutation with the nontrivial βs allowed in the case of the reduced Pati-Salam model. As before, we assume that D commutes with the real structure so that it is of the form $D_0 + JD_0J^{-1}$. For the two remaining cases in (2.41) we get the following results. Firstly, we have

Proposition 2.2.2. *The Dirac operator $D = D_0 + JD_0J^{-1}$ commutes with*

$$\begin{aligned} \beta &= \pi(-1_2, 1_2, 1, -1_3) J \pi(-1_2, 1_2, 1, -1_3) J^{-1} = \\ &= \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix} \end{aligned} \tag{2.42}$$

if and only if D_0 is of the form

$$\begin{aligned} D_0 = \sum_k \Bigg\{ & \begin{bmatrix} \widetilde{X}_k & \\ & \widetilde{Y}_k \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \widetilde{\gamma}_k & \\ & \widetilde{C}_k \end{bmatrix} + \begin{bmatrix} & \widetilde{Z}_k \\ \widetilde{T}_k & \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} & \widetilde{\alpha}_{1k} & \widetilde{\alpha}_{2k} & \widetilde{\alpha}_{3k} \\ \widetilde{\beta}_{1k} & & & \\ \widetilde{\beta}_{2k} & & & \\ \widetilde{\beta}_{3k} & & & \end{bmatrix} + \\ & + \begin{bmatrix} \widetilde{\delta}_{1k} & \\ \widetilde{\delta}_{2k} & \\ & \widetilde{P}_k \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} \widetilde{\sigma}_{1k} & \widetilde{\sigma}_{2k} & \\ & & \widetilde{E}_k \end{bmatrix} + \begin{bmatrix} & \widetilde{F}_k \\ \widetilde{\mu}_{1k} & \\ \widetilde{\mu}_{2k} & \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} & \widetilde{\nu}_{1k} & \widetilde{\nu}_{2k} \\ \widetilde{G}_k & & \end{bmatrix} \Bigg\}, \end{aligned} \tag{2.43}$$

where $\widetilde{X}_k, \widetilde{Y}_k, \widetilde{Z}_k, \widetilde{T}_k \in M_2(\mathbb{C})$, $\widetilde{E}_k, \widetilde{G}_k \in M_{3\times 2}(\mathbb{C})$, $\widetilde{P}_k, \widetilde{F}_k \in M_{2\times 3}(\mathbb{C})$, $\widetilde{C}_k \in M_3(\mathbb{C})$, and $\widetilde{\alpha}_{lk}, \widetilde{\beta}_{lk}, \widetilde{\gamma}_k, \widetilde{\delta}_{lk}, \widetilde{\sigma}_{lk}, \widetilde{\mu}_{lk}, \widetilde{\nu}_{lk} \in \mathbb{C}$.

Proof. It follows from a by now familiar computation that, for $D = \sum \hat{D}_{ijklrs}$,

$$\begin{aligned} D\beta = &- \sum_{n=1,2} \hat{D}_{i1knr1} + \sum_{\substack{n=1,2\\m=2,3,4}} \hat{D}_{i1knrm} + \sum_{n=3,4} \hat{D}_{i1knr1} - \sum_{\substack{n=3,4\\m=2,3,4}} \hat{D}_{i1knrm} - \\ &- \sum_{m=1,2} \hat{D}_{i2k1rm} + \sum_{m=3,4} \hat{D}_{i2k1rm} + \sum_{\substack{n=2,3,4\\m=1,2}} \hat{D}_{i2knrm} - \sum_{\substack{n=2,3,4\\m=3,4}} \hat{D}_{i2knrm}, \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} \beta D = &- \sum_{n=1,2} \hat{D}_{1jnl1s} + \sum_{\substack{n=1,2\\m=2,3,4}} \hat{D}_{1jnlms} + \sum_{n=3,4} \hat{D}_{1jnl1s} - \sum_{\substack{n=3,4\\m=2,3,4}} \hat{D}_{1jnlms} - \\ &- \sum_{m=1,2} \hat{D}_{2j1lms} + \sum_{m=3,4} \hat{D}_{2j1lms} + \sum_{\substack{n=2,3,4\\m=1,2}} \hat{D}_{2jnlms} - \sum_{\substack{n=2,3,4\\m=3,4}} \hat{D}_{2jnlms}. \end{aligned} \tag{2.45}$$

Comparing these two expressions we get the claimed result. □

Similarly,

Proposition 2.2.3. *The Dirac operator $D = D_0 + JD_0J^{-1}$ commutes with*

$$\begin{aligned} &\beta = \pi(1_2, 1_2, 1, -1_3)J\pi(1_2, 1_2, 1, -1_3)J^{-1} = \\ &= \begin{bmatrix} 1_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & 1_2 \end{bmatrix} \end{aligned} \tag{2.46}$$

if and only if D_0 is of the form

$$\begin{aligned} D_0 = \sum_k \Bigg\{ &\widetilde{A}_k \otimes e_{11} \otimes \begin{bmatrix} \widetilde{\gamma}_k & \\ & \widetilde{C}_k \end{bmatrix} + \begin{bmatrix} \widetilde{\delta}_{1k} & \\ \widetilde{\delta}_{2k} & 0_{4\times 3} \\ \widetilde{\delta}_{3k} & \\ \widetilde{\delta}_{4k} & \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} \widetilde{\sigma}_{1k} & \widetilde{\sigma}_{2k} & \widetilde{\sigma}_{3k} & \widetilde{\sigma}_{4k} \\ & & 0_{3\times 4} & \end{bmatrix} + \\ &+ \begin{bmatrix} 0_{4\times 1} & \widetilde{F}_k \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0_{1\times 4} \\ \widetilde{G}_k \end{bmatrix} \Bigg\}, \end{aligned} \tag{2.47}$$

where $\widetilde{A}_k \in M_4(\mathbb{C})$, $\widetilde{G}_k \in M_{3\times 4}(\mathbb{C})$, $\widetilde{F}_k \in M_{4\times 3}(\mathbb{C})$, $\widetilde{C}_k \in M_3(\mathbb{C})$, and $\widetilde{\gamma}_k, \widetilde{\delta}_{lk}, \widetilde{\sigma}_{lk} \in \mathbb{C}$.

Proof. In a similar manner to the previous cases we compute

$$D\beta = \sum \left(\hat{D}_{i1klr1} + \hat{D}_{i2k1rs}\right) - \sum_{m=2,3,4} \left(\hat{D}_{i1klrm} + \hat{D}_{i2knrs}\right), \tag{2.48}$$

and

$$\beta D = \sum \left(\hat{D}_{1jkl1s} + \hat{D}_{2j1lrs}\right) - \sum_{m=2,3,4} \left(\hat{D}_{1jklms} + \hat{D}_{2jnlrs}\right). \tag{2.49}$$

The result follows from a straightforward comparison of these terms. □

Physical consequences of the reduced Pati-Salam model with $\gamma_\star$

Notice that since it is required of the Dirac operator to anticommute with the grading $\gamma_\star$, and moreover any physically interesting model should be an extension of the Standard Model, we conclude that the only possibility is therefore the β from Proposition 2.2.3, in which case the freedom of possible Dirac operators $D = D_0 + JD_0J^{-1}$ is reduced to those with D_0 of the form

$$
\begin{aligned}
D_0 = \sum_k \Bigg\{ &\begin{bmatrix} & Z_k \\ T_k & \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \gamma_k & \\ & C_k \end{bmatrix} + \\
&+ \begin{bmatrix} \delta_{1k} & \\ \delta_{2k} & 0_{4\times 3} \\ 0 & \\ 0 & \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} \sigma_{1k} & \sigma_{2k} & 0 & 0 \\ & 0_{3\times 4} & & \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} & 0_{2\times 3} \\ 0_{2\times 1} & E_{1k} \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0_{1\times 2} & 0_{1\times 2} \\ 0_{3\times 2} & F_{1k} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & \\ 0 & 0_{4\times 3} \\ \delta_{3k} & \\ \delta_{4k} & \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0 & 0 & \sigma_{3k} & \sigma_{4k} \\ & 0_{3\times 4} & & \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} & E_{2k} \\ 0_{2\times 1} & 0_{2\times 3} \end{bmatrix} \otimes e_{12} \otimes \begin{bmatrix} 0_{1\times 2} & 0_{1\times 2} \\ F_{2k} & 0_{3\times 2} \end{bmatrix} \Bigg\},
\end{aligned}
\tag{2.50}
$$

where $T_k, Z_k \in M_2(\mathbb{C})$, $C_k \in M_3(\mathbb{C})$, $E_{lk} \in M_{2\times 3}(\mathbb{C})$, $F_{lk} \in M_{3\times 2}(\mathbb{C})$ and $\gamma_k, \sigma_{lk}, \delta_{lk} \in \mathbb{C}$.

We can treat this model as an extension of the Standard Model with modified chiralities, i.e. in which left-handed (resp. right-handed) leptons have the same parity as right-handed (resp. left-handed) quarks. Therefore, the only compatible extension beyond the Standard Model and contained within the family of Pati-Salam models which have $\gamma_\star$ as a grading, and possesses a pseudo-Riemannian structure in the sense of the existence of a one-term 0-cycle β, is precisely the reduced Pati-Salam model with exactly the same pseudo-Riemannian structure which was uniquely possible in the case of the Standard Model [13]. Since $\gamma_\star$ explicitly breaks the SU(4)-symmetry into $\mathrm{U}(1)\times\mathrm{U}(3)$, it is not surprising that the resulting class of models also has this property. Nevertheless, it is worth noting that there is exactly one (up to an irrelevant global sign in β) such possibility, and moreover that it is consistent with the one that is known to be the only possibility in the case of the Standard Model. Furthermore, we observe that the pseudo-Riemannian structure still allows for the existence of SU(2)-doublets of right-handed particles. This is an interesting feature. Notice also, that there are further restrictions on the entries of the compatible Dirac operators which follow from the assumption of self-adjointness. These restrictions are summarized at the beginning of Section 2.1.3.

Physical consequences of the reduced Pati-Salam model with γ

Since the grading γ is compatible with the unreduced Pati-Salam model, it is also compatible with the reduced model. Therefore we can consider Dirac operators that anticommute with

γ and commute with β for this case. For the nontrivial βs we see that only one of them, i.e. β from Proposition 2.2.3, is compatible with the requirement of being an extension of the Standard Model. For this choice we have the following

Proposition 2.2.4. *The Dirac operator $D = D_0 + JD_0J^{-1}$ that commutes with β and anticommutes with γ has to be of the same form as in* (2.50)*, i.e. exactly the same form as in the case with $\gamma_\star$.*

Moreover, notice that the only β which is admissible in this case is exactly the same one that prevented the existence of leptoquarks in the Standard Model [13]. Therefore, the only possible extension of the Standard Model contained within the family of Pati-Salam models, which takes into account the pseudo-Riemannian structure for finite triples in the sense defined in [13], has to be of the reduced form. That is, the SU(4)-symmetry is broken into a U(1) $\times$ U(3)-symmetry. Therefore, instead of the full $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(4)$ Pati-Salam gauge group we must reduce to the case with $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1) \times \mathrm{U}(3)$. In this extension the right particles are doublets under the SU(2)-symmetry, and leptons are separated from quarks, i.e. they are not the fourth color, so there are no leptoquarks. In a similar manner to the previous case, there are further restrictions on the entries of the compatible Dirac operators which follow from the assumption of self-adjointness — see Section 2.1.3.

2.2.3 Generic β-structures

Here we are looking for all possible βs that are 0-cycles and which satisfy all required conditions but we do not assume these operators to consist of only one term.

The unreduced Pati-Salam model

Since we require that β commutes with the representation of the unreduced Pati-Salam algebra it follows from Proposition 2.1.1 that

$$\beta = \begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes E_1 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes E_2 + 1_4 \otimes e_{22} \otimes F, \tag{2.51}$$

where $E_1, E_2, F \in M_4(\mathbb{C})$.

Since $\beta^2 = 1$, we get

$$F^2 = 1_4, \qquad \begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes E_1^2 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes E_2^2 = 1_{16}, \tag{2.52}$$

hence $E_1^2 = E_2^2 = 1_4$. Since β commutes with J, this implies that

$$\bar{F} \otimes 1_4 = \begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes E_1 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes E_2. \tag{2.53}$$

Let us write $F = \begin{bmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{bmatrix}$, then (2.53) is equivalent to the following set of conditions

$$\begin{bmatrix} \overline{F_{11}} & \overline{F_{12}} \\ \overline{F_{21}} & \overline{F_{22}} \end{bmatrix} \otimes 1_4 = \begin{bmatrix} 1_2 \otimes E_1 & \\ & 1_2 \otimes E_2 \end{bmatrix} \tag{2.54}$$

Therefore $\overline{F_{12}} = \overline{F_{21}} = 0_2$, $\overline{F_{11}} \otimes 1_4 = 1_2 \otimes E_1$ and $\overline{F_{22}} \otimes 1_4 = 1_2 \otimes E_2$, so

$$E_1 = \eta_1 1_4, \qquad \overline{F_{11}} = \eta_1 1_2, \qquad E_2 = \eta_2 1_4, \qquad \overline{F_{22}} = \eta_2 1_2, \tag{2.55}$$

for some nonzero complex numbers η_1 and η_2. Notice that since $\beta^* = \beta$ and $\beta^2 = 1$ the zero solutions are not allowed, and moreover we deduce that both $\eta_1 = \pm 1$ and $\eta_2 = \pm 1$. Therefore,

$$\beta = \begin{bmatrix} \eta_1 1_2 & \\ & \eta_2 1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + 1_4 \otimes e_{22} \otimes \begin{bmatrix} \eta_1 1_2 & \\ & \eta_2 1_2 \end{bmatrix}, \tag{2.56}$$

for some η_1, η_2 being ± 1.

Notice that all such βs are 0-cycles, and there are (up to a trivial rescaling) only two possibilities:

$$\pi(1_2, 1_2, 1_4) J \pi(1_2, 1_2, 1_4) J^{-1}, \quad \pi(1_2, -1_2, 1_4) J \pi(1_2, -1_2, 1_4) J^{-1}. \tag{2.57}$$

These are exactly the same as under the assumption of β being only a one-term 0-cycle, i.e. the conclusion of section 2.2.1 remains valid for more general βs.

The reduced Pati-Salam model

In a similar manner, since we require that β commutes with the representation of the reduced Pati-Salam algebra, it follows from Proposition 2.1.2 that

$$\beta = \begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes E_1 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes E_2 + \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes F_1 + \begin{bmatrix} 0 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes F_2, \tag{2.58}$$

where $E_1, E_2, F_1, F_2 \in M_4(\mathbb{C})$. Since $\beta^2 = 1$, we infer that

$$E_1^2 = E_2^2 = 1_4 = F_1^2 = F_2^2. \tag{2.59}$$

Now, from the condition $\beta J = J\beta$, repeating the previously used argument, we end up with the following form of β:

$$\begin{aligned} \beta = &\begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \eta_1 & \\ & \eta_2 1_3 \end{bmatrix} + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} \eta_3 & \\ & \eta_4 1_3 \end{bmatrix} + \\ &+ \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} \eta_1 1_2 & \\ & \eta_3 1_2 \end{bmatrix} + \begin{bmatrix} 0 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} \eta_2 1_2 & \\ & \eta_4 1_2 \end{bmatrix}, \end{aligned} \tag{2.60}$$

where $\eta_i = \pm 1$ for $i = 1, ..., 4$. There are only eight independent (i.e. up to a global sign) possibilities. They are listed below:

$$\beta_1 = \mathrm{id}, \tag{2.61}$$

$$\begin{aligned}\beta_2 = &\begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \\ &+ \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes 1_4 + \begin{bmatrix} 0 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix},\end{aligned} \tag{2.62}$$

$$\beta_3 = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + 1_4 \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}, \tag{2.63}$$

$$\begin{aligned}\beta_4 = &\begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} -1 & \\ & 1_3 \end{bmatrix} + \\ &+ \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix} + \begin{bmatrix} 0 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes 1_4,\end{aligned} \tag{2.64}$$

$$\begin{aligned}\beta_5 = &\begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 0_2 & \\ & 1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + \\ &+ \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes 1_4 + \begin{bmatrix} 0 & \\ & 1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix},\end{aligned} \tag{2.65}$$

$$\beta_6 = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}, \tag{2.66}$$

$$\beta_7 = 1_4 \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} \otimes e_{22} \otimes 1_4, \tag{2.67}$$

$$\begin{aligned}\beta_8 = &\begin{bmatrix} 1_2 & \\ & 0_2 \end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix} + \begin{bmatrix} 0_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1_4 + \\ &+ \begin{bmatrix} 1 & \\ & 0_3 \end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} + \begin{bmatrix} 0 & \\ & -1_3 \end{bmatrix} \otimes e_{22} \otimes 1_4.\end{aligned} \tag{2.68}$$

All of the above are 0-cycles (more precisely: $\beta = \pi(1_2, 0_2, \eta_1, \eta_2 1_3) J \pi(1_2, 0_2, \eta_1, \eta_2 1_3) J^{-1} + \pi(0_2, 1_2, \eta_3, \eta_4 1_3) J \pi(0_2, 1_2, \eta_3, \eta_4 1_3) J^{-1}$), but only four of them are images of all non-zero elements of the algebra: β_1, β_3, β_6 and β_7. These are exactly the same cases we had in the case of the single term 0-cycles. Moreover, an analogous computation to before shows that βs which are not of the one term type do not allow for physically acceptable Dirac operators in the sense explained in Section 2.2.1. The reason is that the restrictions on the $\cdots \otimes e_{11} \otimes \cdots$ and $\cdots \otimes e_{22} \otimes \cdots$ terms that follow from the fact that the coexistance of β and γ (or $\gamma_\star$) requires that such terms contain matrices that are simultaneously block-diagonal and anti-diagonal. To sum up, the conclusion in Section 2.2.2 remains valid when more general βs are considered.

The Standard Model

Let us now discuss the generic case for the Standard Model. In [13] the one term case was discussed. Now, mirroring the above computation we can get the following family of possible βs:

$$\begin{aligned}\beta = &\begin{bmatrix}1 & \\ & 0_3\end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix}\eta_1 & \\ & \eta_2 1_3\end{bmatrix} + \begin{bmatrix}0 & & \\ & 1 & \\ & & 0_2\end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix}\eta_3 & \\ & \eta_4 1_3\end{bmatrix} + \\ &+ \begin{bmatrix}0_2 & \\ & 1_2\end{bmatrix} \otimes e_{11} \otimes \begin{bmatrix}\eta_5 & \\ & \eta_6 1_3\end{bmatrix} + \begin{bmatrix}1 & \\ & 0_3\end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix}\eta_1 & & \\ & \eta_3 & \\ & & \eta_5 1_2\end{bmatrix} + \\ &+ \begin{bmatrix}0 & \\ & 1_3\end{bmatrix} \otimes e_{22} \otimes \begin{bmatrix}\eta_2 & & \\ & \eta_4 & \\ & & \eta_6 1_2\end{bmatrix},\end{aligned} \tag{2.69}$$

where $\eta_i = \pm 1$, for $i = 1, ..., 6$. This straightforward check shows that the only case in which β is a 0-cycle and allows for an extension of the Standard Model (in the previously discussed sense) is of the one term type and is exactly the same β that prevented the existence of leptoquarks in [13]. Therefore, the validity of the conclusion therein is maintained when generalizing to allow, from the outset, for more general βs.

2.3 Conclusions

We have discussed a role which may be played by the existence of pseudo-Riemannian structures for the finite spectral triples associated with the family of Pati-Salam models. We have shown that their existence reduces the algebra to $\mathbb{H}_R \oplus \mathbb{H}_L \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. Despite the fact that the existence of the additional grading as the shadow of a pseudo-Riemannian structure does not determine the Dirac operator uniquely, we have a huge reduction of the possible choices.

We would like to stress that due to such a reduction, the family of Left-Right Symmetric (LRS) models appears to be the interesting one. This class of models was broadly considered both from theoretical and phenomenological points of view - see e.g. [5], [9],[10], [36], [37], [38] and [39]. In such models the gauge group is

$$\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(3) \times \mathrm{U}(1)_{B-L}. \tag{2.70}$$

The chiral fermions consist of three families of quarks and leptons, and are given by

$$q_L = \left(1, 2, 3, \frac{1}{3}\right), \qquad q_R = \left(2, 1, 3, \frac{1}{3}\right), \qquad l_L = (1, 2, 1, -1)\,, \qquad l_R = (2, 1, 1, -1)\,, \tag{2.71}$$

where the parameters denote the quantum numbers under $SU(2)_R$, $SU(2)_L$, SU(3) and $U(1)_{B-L}$ gauge groups, respectively [5].

The charge of a particle in such a model is defined as $Q = I_{3L} + I_{3R} + \frac{B-L}{2}$, where I_3 is the third component of an SU(2)-isospin.

The Left-Right Symmetric models were also considered from the point of view of noncommutative geometry, initially as possibly interesting examples for the Connes-Lott scheme, but later on also as possible extensions of the Standard Model — see e.g. [20],[21],[22] and [23]. The main interest was in the determination of whether, in this framework, such models provide a mechanism to explain the origin of parity symmetry breaking. In [20] it was argued that in the almost-commutative Yang-Mills-Higgs models, parity cannot be spontaneously broken. This followed from the requirement that Poincaré duality must be satisfied.

The family of reduced Pati-Salam models generalizes both the Left-Right Symmetric Models and also the Chiral Electromagnetism Model [20]. The latter contains the $U(1)_R \times U(1)_L$ gauge group instead of the SU(2) ones. This theory played the role of a toy model for the application of the Connes-Lott scheme to LRS theories.

Chapter 3

A Superspace Dirac Operator in NCG from the "factorization" of the Ordinary Dirac Operator

The previous chapter was focused on aspects of Pati-Salam and L-R symmetric models in the context of NCG. But this is merely one possible extension of the Standard Model which has the potential to explain several of the unresolved questions in modern physics and make new, testable physical predictions. Another possible class of extensions to the Standard Model, namely supersymmetric models, will be the motivating topic of the next chapters.

In 1928 P.A.M. Dirac reported his now-famous procedure for deriving an equation governing the quantum mechanical properties for particles with half-integer spin [40]. The process he pioneered may be essentially described as taking the "square root" of the Klein-Gordon equation.

The natural question, whether this process is iterable, was posed and solved by the use of superspace coordinates and their (first-order) derivatives [41]. A series of papers followed, studying the free and interacting forms of the resulting equations acting on (super)spaces of superfields [42, 43, 44, 45, 46].

As we have already seen in the previous chapters, the mathematical tools and methods of noncommutative geometry are well suited to developing models of theoretical physics. It bears repeating that it is a certain subclass of noncommutative geometries, known as almost commutative (AC) geometries, which is practically tailor-made for the description of such physical models. This adaptation was pioneered in [47], but again, for the working physicist, we also recommend the presentation in [4].

As will be presented in the next chapter, I have recently proposed an alternative procedure for the construction of physical models which exhibit supersymmetry (arising from an underlying superspace), and within the framework of noncommutative geometry (up to and including a spectral action), [33]. Of central importance to this framework is the notion of a suitable Dirac operator. Given that the natural geometric setting for supersymmetry may well be superspace [27, 28], we expect that any Dirac operator which is claimed to govern the dynamics of particles in a supersymmetric model, should, in an essential way, take into account superspace coordinates and their derivatives.

One possibility could be to construct a superspace Dirac operator associated with the underlying superspace spin bundle. This would be a sort of "inside-out" approach where the fundamental space under consideration is a superspace exhibiting supersymmetry through infinitesimal global translations of its coordinates. Considered in this way, supersymmetry is an explicit, unavoidable property of the model.

Alternatively, inspired by the procedure outlined in [41], one may consider an "outside-in" approach. This time, the basic ingredients are those of the usual AC-geometry approach for obtaining physical models from NCG, i.e. the underlying space is an ordinary Riemannian spin manifold and the Dirac operator is the spin connection acting fiberwise on square-integrable sections of the spin bundle. Supersymmetry and the gauge fields then emerge when considering the action of the "square root" of the (possibly unfluctuated!) total space Dirac operator on a restricted space of spinor superfields.

We now proceed to construct such an operator via the latter, "outside-in" approach. The content of this chapter is based on the paper [32].

3.1 Factorization of the Dirac operator

3.1.1 Minkowski space – the Szwed approach

Using two-component spinor notation (ofttimes referred to as Van der Waerden notation) and the chiral representation for the Dirac matrices (for the conventions see [48]), one can write the Dirac equation in four dimensional Minkowski space as

$$-\begin{pmatrix} i\bar{\sigma}^{\mu\,\dot{\alpha}\beta}\partial_\mu & m\delta^{\dot{\alpha}}_{\dot{\beta}} \\ m\delta^{\beta}_{\alpha} & i\sigma^{\mu}_{\alpha\dot{\beta}}\partial_\mu \end{pmatrix}\begin{pmatrix} \psi_\beta \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \equiv \mathcal{D}\begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} = 0. \tag{3.1}$$

Taking a "square root" of the Dirac operator corresponds to the construction of an operator, A, which satisfies

$$A^\dagger A = \mathcal{D}. \tag{3.2}$$

If one requires A to be a local operator and to contain space-time derivatives, then, since there is no second order derivative in the Dirac operator, one is compelled to assume that the coefficients of ∂_μ in A are nilpotent. Therefore one is lead to consider the operator A as acting on a superspace with the coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.

There are several first order differential operators which can be defined on this space. In particular, the spinorial ones,

$$\begin{aligned} D_\alpha &= \partial/\partial\theta^\alpha + i\sigma^\mu{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \\ \bar D_{\dot\alpha} &= -\partial/\partial\bar\theta^{\dot\alpha} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu. \end{aligned} \tag{3.3}$$

satisfy an algebra with relations given by

$$\begin{aligned} \{D_\alpha, D_\beta\} &= \left\{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\right\} = 0, \\ \left\{D_\alpha, \bar D_{\dot\beta}\right\} &= -2i\sigma^\mu{}_{\alpha\dot\beta}\partial_\mu. \end{aligned} \tag{3.4}$$

If we now define 2×2 matrices

$$A_{\beta\alpha} = \begin{pmatrix} D^\beta & -\bar D_{\dot\beta} \\ \bar D^{\dot\alpha} & D_\alpha \end{pmatrix}, \tag{3.5}$$

then

$$(A_{\alpha\beta})^\dagger A_{\beta\alpha} = \begin{pmatrix} \{D^\beta, \bar D^{\dot\alpha}\} & \bar D_{\dot\beta}\bar D^{\dot\alpha} + D^\beta D_\alpha \\ \bar D_{\dot\beta}\bar D^{\dot\alpha} + D^\beta D_\alpha & \{D_\alpha, \bar D_{\dot\beta}\} \end{pmatrix}. \tag{3.6}$$

In particular

$$(A_{\alpha\alpha})^\dagger A_{\alpha\alpha} = -2\begin{pmatrix} i\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu & M \\ M & i\sigma^\mu{}_{\alpha\dot\alpha}\partial_\mu \end{pmatrix} \tag{3.7}$$

with

$$M = -\frac{1}{4}\left(\bar D\bar D + DD\right) \equiv -\frac{1}{4}\left(\bar D_{\dot\alpha}\bar D^{\dot\alpha} + D^\alpha D_\alpha\right). \tag{3.8}$$

The equality (3.7) was the motivation in [41, 44] for postulating the following set of equations as a “square root” of the Dirac equation:

$$D^\alpha\psi_\alpha - \bar D_{\dot\alpha}\bar\chi^{\dot\alpha} = 0, \qquad \bar D^{\dot\alpha}\psi_\alpha + D_\alpha\bar\chi^{\dot\alpha} = 0, \tag{3.9}$$

in which the spinors ψ_α and $\bar\chi^{\dot\alpha}$ are considered to be functions of the superspace coordinates $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$, and are subject to the additional constraint

$$\left(DD + \bar D\bar D\right)\psi_\alpha + 4m\psi_\alpha = \left(DD + \bar D\bar D\right)\bar\chi^{\dot\alpha} + 4m\bar\chi^{\dot\alpha} = 0. \tag{3.10}$$

The solution set of these equations turned out to be nonempty and interesting. In particular, a simple case in which $\psi_\alpha = \chi_\alpha$ corresponds to the Maxwell superfield [44].

3.1.2 4d Euclidean space

It is essential to the noncommutative methods, which we intend to employ in section 3, that the "total-space" Dirac operator is Hermitian. Therefore we proceed in a Riemannian signature and, for simplicity, choose to work in 4-dimensional Euclidean space.

In particular, in this setting the Lorentz transformations are the 4-dimensional rotations characterized by the symmetry group $SO(4)$. Their spin representation is given by the universal covering Lie group, $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ and the corresponding Clifford algebra is isomorphic to the Lie algebra of infinitesimal generators, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This may be understood by straightforward extension of a similar calculation in 3 dimensions which can be followed in appendix D.

After defining

$$\sigma^m \equiv (i\tau_1, i\tau_2, i\tau_3, \mathbf{1}_2) \quad \text{and} \quad \tilde{\sigma}^m \equiv (-i\tau_1, -i\tau_2, -i\tau_3, \mathbf{1}_2), \tag{3.11}$$

where τ_i are the Pauli matrices, it is immediate to check that the Hermitian matrices

$$\gamma_{\mathrm{E}}^m \equiv \begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix} \tag{3.12}$$

generate the Clifford algebra of 4-dimensional Euclidean space,

$$\{\gamma_{\mathrm{E}}^m, \gamma_{\mathrm{E}}^n\} = 2\delta^{mn}\mathbf{1}_4. \tag{3.13}$$

Furthermore, this algebra possesses a natural grading induced by the operator

$$\gamma_{\mathrm{E}}^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \tag{3.14}$$

The Euclidean Dirac operator has the form

$$\mathcal{D} = i\gamma_E^m \partial_m + m\mathbf{1}_4 = \begin{pmatrix} m\mathbf{1}_2 & i\sigma^m\partial_m \\ i\tilde{\sigma}^m\partial_m & m\mathbf{1}_2 \end{pmatrix} \tag{3.15}$$

and acts on a bispinor

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\chi} \end{pmatrix}. \tag{3.16}$$

As for the spinorial indices, we declare

$$\begin{gathered} \psi = (\psi_\alpha), \quad \tilde{\chi} = \left(\tilde{\chi}^{\dot{\alpha}}\right), \\ \tilde{\sigma}^m = \left(\tilde{\sigma}^{m\dot{\alpha}\alpha}\right), \quad \sigma^m = (\sigma^m_{\alpha\dot{\alpha}}) \end{gathered} \tag{3.17}$$

which allows us to present the Dirac equation as

$$\begin{gathered} i\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m\psi_\alpha + m\tilde{\chi}^{\dot{\alpha}} = 0, \\ i\sigma^m_{\alpha\dot{\alpha}}\partial_m\tilde{\chi}^{\dot{\alpha}} + m\psi_\alpha = 0. \end{gathered} \tag{3.18}$$

Unlike the Minkowski case, the spinors ψ and $\tilde{\chi}$ transform independently under the action of Spin(4). Indeed, if we parameterize a matrix $L \in \mathrm{SO}(4)$ as $L = \exp\omega$ (with $\omega_{mn} = -\omega_{nm}$) then

$$\psi'_\alpha(x) = M_\alpha{}^\beta \psi_\beta\big(L^{-1}x\big), \qquad \tilde{\chi}'^{\dot\alpha} = W^{\dot\alpha}{}_{\dot\beta}\tilde{\chi}^{\dot\beta}\big(L^{-1}x\big) \tag{3.19}$$

where

$$M(L) = \exp\left(\tfrac{1}{8}\omega_{mn}(\sigma^m\tilde{\sigma}^n - \sigma^n\tilde{\sigma}^m)\right), \quad W(L) = \exp\left(\tfrac{1}{8}\omega_{mn}(\tilde{\sigma}^m\sigma^n - \tilde{\sigma}^n\sigma^m)\right), \tag{3.20}$$

are distinct operators. i.e., $M(L)$ depends on ω_{mn} only through a combination

$$\sum_{j=1}^{3}\Big(\sum_{k=1}^{3}\sum_{l=1}^{k-1}\epsilon_{jkl}\omega_{kl} + \omega_{j4}\Big)\tau_j \tag{3.21}$$

while $W(L)$ depends on ω_{mn} through a combination

$$\sum_{j=1}^{3}\Big(\sum_{k=1}^{3}\sum_{l=1}^{k-1}\epsilon_{jkl}\omega_{kl} - \omega_{j4}\Big)\tau_j. \tag{3.22}$$

In order to construct a relevant superspace, we introduce two constant (anticommuting) spinors ξ_α and $\tilde{\zeta}^{\dot\alpha}$. By construction, under the action of Spin(4) we have

$$\xi_\alpha \to M_\alpha{}^\beta\xi_\beta, \qquad \tilde{\zeta}^{\dot\alpha} \to W^{\dot\alpha}{}_{\dot\beta}\tilde{\zeta}^{\dot\beta} \tag{3.23}$$

and thus ξ_α and $\tilde{\zeta}^{\dot\alpha}$ are necessarily complex, i.e. we may treat ξ_α and $\overline{\xi}^\beta = (\xi_\beta)^\dagger$, as well as $\tilde{\zeta}^{\dot\alpha}$ and $\overline{\tilde{\zeta}}_{\dot\beta} = \left(\tilde{\zeta}^{\dot\beta}\right)^\dagger$, as independent Grasmann variables.

For the Levi-Civita tensor we adapt the convention $\varepsilon_{12} = \varepsilon_{\dot1\dot2} = \varepsilon^{21} = \varepsilon^{\dot2\dot1} = 1$. In effect

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha}$$

and

$$\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^m_{\beta\dot\beta} = \tilde{\sigma}^{m\dot\alpha\alpha}. \tag{3.24}$$

Let us now define the spinorial derivatives

$$D^\alpha = \frac{\partial}{\partial\xi_\alpha} + i\,\overline{\tilde{\zeta}}_{\dot\beta}\,\tilde{\sigma}^{m\dot\beta\alpha}\,\partial_m, \qquad \tilde{D}_{\dot\alpha} = \frac{\partial}{\partial\tilde{\zeta}^{\dot\alpha}} + i\,\bar{\xi}^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m, \tag{3.25}$$

and consequently

$$\overline{D}_\alpha = \frac{\partial}{\partial\bar{\xi}^\alpha} + i\,\sigma^m_{\alpha\dot\beta}\,\tilde{\zeta}^{\dot\beta}\,\partial_m, \qquad \overline{\tilde{D}}^{\dot\alpha} = \frac{\partial}{\partial\overline{\tilde{\zeta}}_{\dot\alpha}} + i\,\tilde{\sigma}^{m\dot\alpha\beta}\,\xi_\beta\,\partial_m. \tag{3.26}$$

They satisfy an algebra

$$\left\{D^\alpha, \overline{\tilde{D}}^{\dot\alpha}\right\} = 2i\,\tilde\sigma^{m\dot\alpha\alpha}\,\partial_m, \qquad \left\{\overline{D}_\alpha, \tilde{D}_{\dot\alpha}\right\} = 2i\,\sigma^m_{\alpha\dot\alpha}\,\partial_m, \tag{3.27}$$

with all the remaining anticommutators vanishing. Moreover, if we define

$$Q^\alpha = \frac{\partial}{\partial\xi_\alpha} - i\,\overline{\tilde\zeta}_{\dot\beta}\,\tilde\sigma^{m\dot\beta\alpha}\,\partial_m, \qquad \tilde{Q}_{\dot\alpha} = \frac{\partial}{\partial\tilde\zeta^{\dot\alpha}} - i\,\bar\xi^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m, \tag{3.28}$$

and

$$\overline{Q}_\alpha = \frac{\partial}{\partial\bar\xi^\alpha} - i\,\sigma^m_{\alpha\dot\beta}\,\tilde\zeta^{\dot\beta}\,\partial_m, \qquad \overline{\tilde{Q}}^{\dot\alpha} = \frac{\partial}{\partial\overline{\tilde\zeta}_{\dot\alpha}} - i\,\tilde\sigma^{m\dot\alpha\beta}\,\xi_\beta\,\partial_m, \tag{3.29}$$

then it is immediate to check that all of the anticommutators involving one of the operators (3.25) or (3.26), and one of the operators (3.28) or (3.29), vanish. In effect, all equations formulated in terms of derivatives (3.25) and (3.26) are invariant under the (supersymmetry) transformations generated by (3.28) and (3.29).

We next promote ψ_α and $\tilde\chi^{\dot\alpha}$ to spinor valued functions on the Euclidian superspace with coordinates $(x, \xi_\alpha, \bar\xi^\alpha, \tilde\zeta^{\dot\alpha}, \overline{\tilde\zeta}_{\dot\alpha})$ and, guided by (3.9), subject them to the following set of equations:

$$D^\alpha\psi_\alpha + \tilde{D}_{\dot\alpha}\tilde\chi^{\dot\alpha} = 0 \tag{3.30}$$

and

$$\overline{\tilde{D}}^{\dot\alpha}\psi_\alpha + \overline{D}_\alpha\tilde\chi^{\dot\alpha} = 0. \tag{3.31}$$

In (3.30) the indices are summed over (so that the l.h.s. is a scalar), while (3.31) is a vanishing condition for a certain tensor, and thus also has an invariant meaning.

From (3.27), (3.30) and (3.31) we get

$$i\tilde\sigma^{m\dot\alpha\alpha}\partial_m\psi_\alpha + \tilde{M}^{\dot\alpha}{}_{\dot\beta}\tilde\chi^{\dot\beta} = 0, \tag{3.32}$$

$$i\sigma^m_{\alpha\dot\alpha}\partial_m\tilde\chi^{\dot\alpha} + M_\alpha{}^\beta\psi_\beta = 0. \tag{3.33}$$

where

$$M_\alpha{}^\beta = \frac{1}{2}\left(\delta^\beta_\alpha\,\tilde{D}_{\dot\alpha}\overline{\tilde{D}}^{\dot\alpha} + \overline{D}_\alpha D^\beta\right), \qquad \tilde{M}^{\dot\alpha}{}_{\dot\beta} = \frac{1}{2}\left(\delta^{\dot\alpha}_{\dot\beta}\,D^\alpha\overline{D}_\alpha + \overline{\tilde{D}}^{\dot\alpha}\tilde{D}_{\dot\beta}\right). \tag{3.34}$$

We conclude that (3.30) and (3.31) imply Dirac equations for the (super) spinors ψ_α and $\tilde\chi^{\dot\alpha}$ on the subspace of superfields satisfying

$$M_\alpha{}^\beta\psi_\beta = m\psi_\alpha, \qquad \tilde{M}^{\dot\alpha}{}_{\dot\beta}\tilde\chi^{\dot\beta} = m\tilde\chi^{\dot\alpha}. \tag{3.35}$$

To see that there exist nontrivial solutions of the set of equations (3.32), (3.33) and (3.35) we consider a simple case

$$\tilde\chi^{\dot\alpha} = \frac{\partial\psi_\alpha}{\partial\bar\xi^\beta} = \frac{\partial\psi_\alpha}{\partial\tilde\zeta^{\dot\beta}} = 0. \tag{3.36}$$

Equation

$$\tilde{\bar{D}}^{\dot\alpha}\psi_\alpha = 0 \tag{3.37}$$

then implies that ψ_α depends on $\bar{\tilde{\zeta}}_{\dot\alpha}$ only through a combination of the form

$$y^m = x^m - i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}\xi_\alpha. \tag{3.38}$$

If we take

$$\psi_\alpha(x, \xi_\alpha, \tilde\zeta^{\dot\alpha}) = \lambda_\alpha(y) + F_{mn}(y)(\sigma^m\tilde\sigma^n)_\alpha{}^\beta\xi_\beta, \tag{3.39}$$

then, since

$$D^\alpha y^m = 2i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}, \tag{3.40}$$

we get

$$D^\alpha\psi_\alpha = \mathrm{Tr}(\sigma^m\tilde\sigma^n)\,F_{mn}(y) + 2i\bar{\tilde{\zeta}}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}\partial_m\lambda_\alpha(y) + 2i\bar{\tilde{\zeta}}_{\dot\alpha}\xi_\beta(\tilde\sigma^p\sigma^m\tilde\sigma^n)^{\dot\alpha\beta}\partial_p F_{mn}(y). \tag{3.41}$$

Vanishing of the second term on the r.h.s. of formula (3.41) implies that λ_α satisfies the massless Dirac equation,

$$i\tilde\sigma^{m\dot\alpha\alpha}\partial_m\lambda_\alpha = 0, \tag{3.42}$$

meanwhile, vanishing of the first term implies that the tensor F_{mn} is antisymmetric, and consequently the identity

$$\tilde\sigma^p\sigma^m\tilde\sigma^n = \epsilon^{pmnr}\,\tilde\sigma^r + \delta^{mp}\tilde\sigma^n + \delta^{mn}\tilde\sigma^p - \delta^{np}\tilde\sigma^m, \qquad \epsilon^{1234} = 1, \tag{3.43}$$

applied to the last term, gives

$$\epsilon^{rpmn}\partial_p F_{mn} = 0, \qquad \partial^m F_{mn} = 0. \tag{3.44}$$

We conclude that a particular solution of the postulated set of equations is a spinor superfield with component fields consisting of a massless spinor field and a Maxwell gauge field.

Since the matrices (3.20) are unitary with unit determinant, the spinors $\xi^\alpha \equiv \epsilon^{\alpha\beta}\xi_\beta$ and $\bar\xi^\alpha$ (as well as $\tilde\zeta_{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\tilde\zeta^{\dot\beta}$ and $\bar{\tilde{\zeta}}_{\dot\alpha}$) transform in the same way under Spin(4). We can therefore construct spinorial derivatives

$$D^\alpha = \frac{\partial}{\partial\xi_\alpha} + i\,\tilde\zeta_{\dot\beta}\,\tilde\sigma^{m\dot\beta\alpha}\,\partial_m, \qquad \tilde D_{\dot\alpha} = \frac{\partial}{\partial\tilde\zeta^{\dot\alpha}} + i\,\xi^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m, \tag{3.45}$$

and corresponding supercharges

$$Q^\alpha = \frac{\partial}{\partial\xi_\alpha} - i\,\tilde\zeta_{\dot\beta}\,\tilde\sigma^{m\dot\beta\alpha}\,\partial_m, \qquad \tilde Q_{\dot\alpha} = \frac{\partial}{\partial\tilde\zeta^{\dot\alpha}} - i\,\xi^\beta\,\sigma^m_{\beta\dot\alpha}\,\partial_m, \tag{3.46}$$

without invoking conjugated Grasmann variables. Then the set of equations:

$$D^{\alpha}\psi_{\alpha} + \tilde{D}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}} = 0, \tag{3.47}$$

and

$$\tilde{D}^{\dot{\alpha}}\psi_{\alpha} + D_{\alpha}\tilde{\chi}^{\dot{\alpha}} = 0, \tag{3.48}$$

where $D_{\alpha} = D^{\beta}\epsilon_{\beta\alpha}$ and $\tilde{D}^{\dot{\alpha}} = \tilde{D}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}$, imposed on "analytic", spinorial superfields

$$\psi_{\alpha} = \psi_{\alpha}(x, \xi, \tilde{\zeta}), \qquad \tilde{\chi}^{\dot{\alpha}} = \tilde{\chi}^{\dot{\alpha}}(x, \xi, \tilde{\zeta}), \tag{3.49}$$

is invariant with respect to both Spin(4) and supersymmetric transformations (generated by (3.46)) and implies the Dirac equation (3.18) on a subspace satisfying the "mass" constraints

$$\begin{aligned} &\frac{1}{4}\left(\delta^{\beta}_{\alpha}\epsilon^{\dot{\gamma}\dot{\alpha}}\left[\tilde{D}_{\dot{\alpha}}, \tilde{D}_{\dot{\gamma}}\right] + \epsilon_{\gamma\alpha}\left[D^{\gamma}, D^{\beta}\right]\right)\psi_{\beta} = m\psi_{\alpha}, \\ &\frac{1}{4}\left(\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{\gamma\alpha}\left[D^{\alpha}, D^{\gamma}\right] + \epsilon^{\dot{\gamma}\dot{\alpha}}\left[\tilde{D}_{\dot{\gamma}}, \tilde{D}_{\dot{\beta}}\right]\right)\tilde{\chi}^{\dot{\beta}} = m\tilde{\chi}^{\dot{\alpha}}. \end{aligned} \tag{3.50}$$

Nontrivial solutions of (3.47), (3.48) and (3.50) with $m = 0$ can be found (even if by "brute force", i.e. expanding ψ_{α} and $\tilde{\chi}^{\dot{\alpha}}$ in a series of non-vanishing powers of ξ and $\tilde{\zeta}$ and then working out and solving the resulting differential equations for the coefficient functions). Notice that necessarily both ψ_{α} and $\tilde{\chi}^{\dot{\alpha}}$ are nonzero. Indeed, for $\tilde{\chi}^{\dot{\alpha}} = 0$ equations (3.47) and (3.48) imply

$$D^{\alpha}\psi_{\alpha} = 0, \qquad \tilde{D}_{\dot{\beta}}\psi_{\alpha} = 0, \tag{3.51}$$

which is inconsistent since the anticommutator $\{D^{\alpha}, \tilde{D}_{\dot{\beta}}\}$ does not vanish.

3.2 Almost-commutative geometry

As was already introduced and utilized in the preceding chapters, we again turn to the AC-manifold subclass of noncommutative geometries to examine the implications of taking such a factorized form of the Dirac operator in the framework of NCG. Starting from a "total-space" spectral triple of the form

$$(\mathcal{A} \otimes \mathcal{A}_F, \mathcal{H} \otimes \mathcal{H}_F, D_{AC}), \tag{3.52}$$

and in the context of our 4-dimensional Euclidean space, we have a total space Dirac operator of the form

$$D_{AC} = \mathcal{D} \otimes \mathbf{1}_N + \gamma^5_E \otimes D_F, \tag{3.53}$$

where $\mathcal{D}$ is the Euclidean Dirac operator defined in (3.15), γ_E^5 is of the form given in (3.14), and D_F is a finite Dirac operator on $\mathbb{C}^N$, i.e. a Hermitian $N \times N$ matrix. Therefore, D_{AC} can be explicitly written as a $4N \times 4N$ matrix, acting on bispinors of the form

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\chi} \end{pmatrix}, \tag{3.54}$$

where

$$\psi = (\psi_{i\alpha}), \quad \tilde{\chi} = \left(\tilde{\chi}_i^{\dot{\alpha}}\right), \quad i = 1, \ldots, N, \tag{3.55}$$

and the Dirac equation can be written in the form

$$\begin{aligned} &i\,\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m\psi_{i\alpha} + m\tilde{\chi}_i^{\dot{\alpha}} + (D_F)_{ij}\,\tilde{\chi}_j^{\dot{\alpha}} = 0, \\ &i\,\sigma^m_{\alpha\dot{\alpha}}\partial_m\tilde{\chi}_i^{\dot{\alpha}} + m\psi_{i\alpha} - (D_F)_{ij}\,\psi_{j\alpha} = 0. \end{aligned} \tag{3.56}$$

Now, consider the algebra

$$\begin{aligned} &\left\{D_i^\alpha, D_j^\beta\right\} = 2\epsilon^{\alpha\beta}Z_{ij}, \qquad && Z_{ij} = -Z_{ji}, \\ &\left\{\tilde{D}_{i\dot{\alpha}}, \tilde{D}_{j\dot{\beta}}\right\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{Z}_{ij}, \qquad && \tilde{Z}_{ij} = -\tilde{Z}_{ji}, \end{aligned} \tag{3.57}$$

together with

$$\left\{D_i^\alpha, \tilde{D}_j^{\dot{\alpha}}\right\} = 2i\,\delta_{ij}\tilde{\sigma}^{m\dot{\alpha}\alpha}\,\partial_m, \qquad \left\{D_{i\alpha}, \tilde{D}_{j\dot{\alpha}}\right\} = 2i\,\delta_{ij}\sigma^m_{\alpha\dot{\alpha}}\,\partial_m, \tag{3.58}$$

where $D_{i\alpha} = D_i^\beta\epsilon_{\beta\alpha}$ and $\tilde{D}_j^{\dot{\alpha}} = \tilde{D}_{j\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}$. It can be realized as an algebra of differential operators on a superspace with coordinates $(x^m, \xi_{i\alpha}, \tilde{\zeta}_i^{\dot{\alpha}})$:

$$\begin{aligned} D_i^\alpha &= \frac{\partial}{\partial\xi_{i\alpha}} + i\tilde{\zeta}^{i\dot{\alpha}}\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m + Z_{ij}\xi_j^\alpha, \\ \tilde{D}_{i\dot{\alpha}} &= \frac{\partial}{\partial\tilde{\zeta}_j^{\dot{\alpha}}} + i\xi_i^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m + \tilde{Z}_{ij}\tilde{\zeta}_{j\dot{\alpha}}. \end{aligned} \tag{3.59}$$

The corresponding supercharges, anticommuting with derivatives (3.59), have the form:

$$\begin{aligned} Q_i^\alpha &= \frac{\partial}{\partial\xi_{i\alpha}} - i\tilde{\zeta}^{i\dot{\alpha}}\tilde{\sigma}^{m\dot{\alpha}\alpha}\partial_m - Z_{ij}\xi_j^\alpha, \\ \tilde{Q}_{i\dot{\alpha}} &= \frac{\partial}{\partial\tilde{\zeta}_j^{\dot{\alpha}}} - i\xi_i^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m - \tilde{Z}_{ij}\tilde{\zeta}_{j\dot{\alpha}}. \end{aligned} \tag{3.60}$$

If we postulate equations of the form

$$\begin{aligned} D_i^\alpha\psi_{j\alpha} + \tilde{D}_{j\dot{\alpha}}\tilde{\chi}_i^{\dot{\alpha}} &= 0, \\ \tilde{D}_i^{\dot{\beta}}\psi_{i\alpha} + D_{i\alpha}\tilde{\chi}_i^{\dot{\beta}} &= 0, \end{aligned} \tag{3.61}$$

then, using (3.58), we can conclude that solutions of (3.61) satisfy the Dirac equation, (3.56), provided that the "mass" conditions

$$
\begin{aligned}
(m\delta_{ij} + (D_F)_{ij})\,\tilde{\chi}_j^{\dot{\alpha}} &= \frac{1}{2}\left(\delta^{\dot{\alpha}}_{\dot{\beta}} D_i^{\alpha} D_{j\alpha} + \delta_{ij}\tilde{D}_k^{\dot{\alpha}}\tilde{D}_{k\dot{\beta}}\right)\tilde{\chi}_j^{\dot{\beta}}, \\
(m\delta_{ij} - (D_F)_{ij})\,\psi_{j\alpha} &= \frac{1}{2}\left(\delta^{\beta}_{\alpha}\tilde{D}_{i\dot{\alpha}}\tilde{D}_j^{\dot{\alpha}} + \delta_{ij} D_{k\alpha} D_k^{\beta}\right)\psi_{j\beta},
\end{aligned}
\tag{3.62}
$$

are satisfied. With the help of (3.57), equations (3.62) can be alternatively presented as

$$
(m\delta_{ij} + (D_F)_{ij} - Z_{ij})\,\tilde{\chi}_j^{\dot{\alpha}} = \frac{1}{4}\left(\delta^{\dot{\alpha}}_{\dot{\beta}}\epsilon_{\beta\alpha}\left[D_i^{\alpha}, D_j^{\beta}\right] + \delta_{ij}\epsilon^{\dot{\gamma}\dot{\alpha}}\left[\tilde{D}_{k\dot{\gamma}}, \tilde{D}_{k\dot{\beta}}\right]\right)\tilde{\chi}_j^{\dot{\beta}}, \tag{3.63}
$$

and

$$
\left(m\delta_{ij} - (D_F)_{ij} - \tilde{Z}_{ij}\right)\psi_{j\alpha} = \frac{1}{4}\left(\delta^{\beta}_{\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}\left[\tilde{D}_{i\dot{\alpha}}, \tilde{D}_{j\dot{\beta}}\right] + \delta_{ij}\epsilon_{\gamma\alpha}\left[D_k^{\gamma}, D_k^{\beta}\right]\right)\psi_{j\beta}. \tag{3.64}
$$

The simplest solutions of these equations (and, most likely, the only that is consistent with (3.61), although the general proof of this claim is still missing) correspond to a situation in which both the l.h.s. and the r.h.s. of (3.63) and (3.64) vanish. This implies that the constructed framework, which reconciles non-commutative geometry with supersymmetry in a simple setting, requires the finite part of the Dirac operator (3.53) to be antisymmetric and expressible through central charges of the algebra (3.57, 3.58) as

$$
(D_F)_{ij} = Z_{ij} = -\tilde{Z}_{ij}. \tag{3.65}
$$

It is worth recalling that in the usual development via the AC-geometry approach to noncommutative geometry, the finite spectral triple only contains data about the fermionic particle content of the model. The bosonic particle content of the theory, or gauge fields, are then given by the inner fluctuations which arise through consideration of Morita equivalences of the algebra. The Morita (self-)equivalent total-space spectral triple is then comprised of the algebra, Hilbert space, and the "fluctuated" Dirac operator taking into account the gauge fields. While we have seen that gauge fields arise naturally through the "factorization" procedure which we have herein described, one could also consider the implications of factorizing the fluctuated Dirac operator.

Chapter 4

Superspace SUSY in NCG with spectral action

In the first two sections of this chapter, the approach and results of two noteworthy examples from the literature will be briefly summarized. Each typifies a particular approach to building supersymmetric models in the context of noncommutative geometry. The first is predicated upon obtaining a supersymmetry invariant spectral action through appropriate modification of the particle content. Meanwhile, the second approach slightly relaxes the axioms of noncommutative geometry to accommodate a supermanifold as the base space for the data of the spectral triple. This second approach avoids passing to a Riemannian signature by using algebraic techniques, rather than spectral ones, to build an action. The rest of the chapter is based on the the work published in [33], and presents a proposal for incorporating a superspace formulation of the principle of supersymmetry into the formalism of noncommutative geometry with a spectral action.

4.1 Spectral models of SUSY in NCG

With the exception of [49], the work by Wim Beenakker, Thijs van den Broek, and Walter D. van Sujlekom, *Supersymmetry and Noncommutative Geometry* [50], seems to be the only previous attempt to reconcile NCG with SUSY which makes use of the *spectral action*. Starting from the requirement that the resulting spectral action functional be supersymmetric, Beeneker et al. provide a classification of all supersymmetric AC-geometries whose particle content this ensures. Meanwhile avoiding mention of *superfields* or *supermanifolds*.

They accomplish this by employing the so-called *Krajewski diagrams* which were introduced as a tool for categorizing finite spectral triples, [51], but which they cleverly note, can also be used to compute the values of the traces of the powers of the finite Dirac operator which appear in the action functional. The key observation which allows for the construction these terms being: It is only the continuous, closed loops with n edges in a Krajewski diagram that contribute to the trace of the n^{th} power of D_F.

With this knowledge in hand, they then proceed to outline and follow a procedure for identifying the irreducible "building blocks" of potentially supersymmetric models. It is interesting to note that the action corresponding to a particular building block may not, itself, be supersymmetric, but rather, the building blocks are constructively supersymmetric. That is, they are defined in such a way that a total action may become supersymmetric again by introducing a proper combination of additional building blocks. In the end, five such "irreducible building blocks" are identified.

Since not all possible combinations of the five building blocks should result in a supersymmetric action, a strategy is developed to determine a list of requirements that must be fulfilled for it to be so. In particular, an off-shell version of a particular on-shell spectral action is written down using the auxiliary fields and undetermined coefficients, and then constraints are determined from the requirement that the fermionic action, as well as the off-shell action, be supersymmetric. These constraints are then used as requirements for the coefficients of the on-shell spectral action to be supersymmetric, thus ensuring that the noncommutative spectral action is the on-shell counterpart of a supersymmetric off-shell action.

By following this procedure as it is applied to several constructively supersymmetric examples built from the five irreducible building blocks previously identified, a list of requirements is given, effectively providing a litmus test for potentially supersymmetric models arising from the class of almost commutative geometries. It is observed that such a restrictive set of conditions severely limits the number of supersymmetric models which can be constructed in this way, meanwhile instantly elevating the status of any model which can successfully navigate this prohibitively demanding gauntlet.

Finally, after a brief detour to explore the concept of soft supersymmetry breaking in the spectral action, it is shown that an AC-geometry version of the minimally supersymmetric Standard Model fails to have a spectral action which is supersymmetric according to the previously identified criteria. This is a sort of No-go theorem that is specific to the MSSM, and in no way precludes the existence of other, AC-geometry-based models of supersymmetry.

4.2 Superspace models of SUSY in NCG

Several attempts have been made to combine SUSY with NCG that do relax the definition of AC-manifold sufficiently to allow an algebra of superfields over a supermanifold to be the primary object of study, but with the exception [49], they do not employ the spectral action [52, 53, 54, 55].

A particularly noteworthy example of this type of formulation can be found in the work of W. Kalau and M. Waltze [54]. Starting from the usual notion of a spectral triple, they proceed to define the input parameters of a spectral triple which determines the (noncommutative) geometry of a supermanifold.

Namely, the unital associative algebra which they choose is given by the usual algebra of superfields, enlarged by taking spinor doublets as generating elements. Likewise, the supersymmetric generators and involution operator are naturally extended. This algebra is then faithfully represented on an appropriate (super) Hilbert space, and an inner product is defined in terms of the Berezin integral, projecting the product superfield onto the highest order component of its expansion in terms of Grassmann variables. Finally, they propose that the correct Dirac operator to employ should be one which is constructed from the superspace covariant derivatives related with the supersymmetry generators, and which takes into account the commutation relations of the "Clifford algebra" also generated by them. They claim that such an operator has a natural interpretation as the "square root" of the ordinary Dirac operator.

In order to avoid the necessity of invoking the noncommutative analogue of the integral, the so-called Dixmier trace, which is not well-defined for non-Euclidean spaces, Kalau and Waltze turn to the development of a generalized differential algebra and corresponding super-Clifford algebra in order that they may use the result to construct the necessary elements for a Yang-Mills theory, namely a covariant derivative and a curvature tensor. The benefit of this approach is that this construction is signature agnostic and thus their results hold for pseudo-Riemannian spaces.

4.3 SUSY on a superspace

The superspace $\mathbb{R}^{3|2}$ is a coordinate space described by three commuting (bosonic) coordinates, say x^0, x^1, x^2, and two independent, anticommuting (fermionic) coordinates, say θ^1 and θ^2 which are assumed to form a spinor of the 3 dimensional Lorentz group. In the superspace construction, the global supersymmetry transformations correspond to translations of the superspace coordinates of the form

$$\delta\theta^\alpha \equiv \epsilon^\alpha \quad \text{and} \quad \delta x^m \equiv \theta_\alpha (\gamma^m)^\alpha{}_\beta \epsilon^\beta, \tag{4.1}$$

in accordance with the transformation properties of θ^α and x^m under Lorentz transformations. Component fields of a *supermultiplet* (an *irreducible representation* of the *supersymmetry algebra*) are combined into a function of the superspace coordinates called a *superfield*,

$$S(x,\theta) = f(x) + g_\beta(x)\theta^\beta + h(x)\theta\theta, \tag{4.2}$$

for some component functions f, g, h. Here the convention adopted is

$$\theta\theta \equiv \theta^2\theta^1 = \frac{1}{2}\varepsilon_{\alpha\beta}\theta^\beta\theta^\alpha = \frac{1}{2}\theta_\alpha\theta^\alpha, \quad \text{where} \quad \varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = -\varepsilon^{21} = 1. \tag{4.3}$$

Comparing the two expressions for the infintesimal variation of S under supersymmetry transformations:

$$\delta S(x,\theta) = \delta f(x) + \delta g_\beta(x)\theta^\beta + \delta h(x)\theta\theta \tag{4.4}$$

and

$$\delta S(x,\theta) = S(x+\delta x, \theta+\delta\theta) - S(x,\theta), \tag{4.5}$$

it is seen, in particular, that

$$\delta h(x) = \partial_m \left(g_\beta (\gamma^m \epsilon)^\beta \right). \tag{4.6}$$

The fact that the $\theta\theta$ component of a superfield transforms as a total derivative implies that the integral,

$$\int h \, d^3x \equiv \int S_{\theta\theta} \, d^3x, \tag{4.7}$$

is invariant under supersymmetry transformations.

The infinitesimal supersymmetry variation of a superfield can be expressed through a first order differential operator Q_β. Indeed, writing

$$\delta S = [\epsilon^\beta Q_\beta, S], \tag{4.8}$$

implies

$$Q_\beta = -\partial_\beta + (\theta\gamma^m)_\beta \partial_m. \tag{4.9}$$

It is also useful to define the *superspace covariant* derivative (or *super-covariant* derivative),

$$D_\alpha = \partial_\alpha + (\theta\gamma^m)_\alpha \partial_m, \tag{4.10}$$

which anticommutes with the generator of supersymmetry transformations,

$$\{D_\alpha, Q_\beta\} \equiv D_\alpha Q_\beta + D_\beta Q_\alpha = 0, \tag{4.11}$$

and squares to a proportion of the spinorial derivative,

$$\{D^\alpha, D_\beta\} = -2(\gamma^m)^\alpha{}_\beta \partial_m. \tag{4.12}$$

The present construction will utilize a *spinor superfield*,

$$\Psi^\alpha(x,\theta) = \psi^\alpha(x) + F^\alpha{}_\beta(x)\theta^\beta + \chi^\alpha(x)\theta\theta, \tag{4.13}$$

which under Lorentz transformations changes as $\Psi'(x,\theta) = S(L)\Psi(L^{-1}x, S(L)^{-1}\theta)$. The infinitesimal supersymmetry transformations of the component fields of $\Psi(x,\theta)$ read

$$\begin{aligned} \delta\psi^\alpha &= F^\alpha{}_\beta \epsilon^\beta, \\ \delta F^\alpha{}_\beta &= \partial_m \psi^\alpha (\gamma^m \epsilon)^\rho \varepsilon_{\rho\beta} - \chi^\alpha \epsilon_\beta, \\ \text{and} \quad \delta\chi^\alpha &= \partial_m F^\alpha{}_\beta (\gamma^m \epsilon)^\beta. \end{aligned}$$

The components of the spinor superfield do not form an irreducible representation of the supersymmetry transformation. They may be constrained by requiring $D_\alpha \Psi^\alpha = 0$. This is compatible with the supersymmetry transformation of Ψ since, thanks to (4.11),

$$D_\alpha \Psi^\alpha = 0 \quad \Rightarrow \quad D_\alpha \delta \Psi^\alpha = D_\alpha \epsilon_\beta Q^\beta \Psi^\alpha = \epsilon_\beta Q^\beta D_\alpha \Psi^\alpha = 0. \tag{4.14}$$

Explicitly,

$$\begin{aligned} D_\alpha \Psi^\alpha &= \partial_\alpha \Psi^\alpha + \theta_\beta (\gamma^m)^\beta{}_\alpha \partial_m \Psi^\alpha \\ &= F^\alpha{}_\alpha - \theta_\beta \chi^\beta + \theta_\beta (\gamma^m)^\beta{}_\alpha \partial_m \Psi^\alpha + (\gamma^m)^\beta{}_\alpha \partial_m F^\alpha{}_\beta \theta\theta, \end{aligned}$$

so that the components of the *chiral spinor superfields*, $\tilde{\Psi}^\alpha$, defined by the relation $D_\alpha \tilde{\Psi}^\alpha = 0$, satisfy

$$\begin{aligned} & \mathrm{Tr} F \equiv F^\alpha{}_\alpha = 0, \\ & \chi^\beta = (\gamma^m)^\beta{}_\alpha \partial_m \psi^\alpha, \\ \text{and} \quad & \mathrm{Tr}\left[\gamma^a \partial_a F\right] \equiv (\gamma^m)^\beta{}_\alpha \partial_m F^\alpha{}_\beta = 0, \end{aligned}$$

and transform according to the rule

$$\begin{aligned} \delta \psi^\alpha =& F^\alpha{}_\beta \epsilon^\beta, \\ \text{and} \quad \delta F^\alpha{}_\beta =& \partial_m \psi^\alpha (\gamma^m \epsilon)^\rho \varepsilon_{\rho\beta} - (\gamma^m)^\alpha{}_\rho \partial_m \psi^\rho(x) \epsilon_\beta. \end{aligned}$$

For now, take the Dirac operator to be the usual spinorial derivative on 3d Minkowski spacetime,

$$\mathcal{D}_M \equiv D = i\gamma^m \partial_m, \tag{4.15}$$

and it will act on the Hilbert space of chiral spinor superfields $\tilde{\Psi}(x, \theta)$ over $\mathbb{R}^{3|2}$. The chiral restricted fermionic action is then taken to be

$$\left\langle \tilde{\Psi}, \mathcal{D}_M \tilde{\Psi} \right\rangle \equiv \left(\tilde{\Psi}, \mathcal{D}_M \tilde{\Psi} \right). \tag{4.16}$$

The coefficient of the term which has the highest order in the Grassmann variables, once passed to the action integral, is by construction invariant under a supersymmetry transformation, and it is calculated to be

$$\left\langle \tilde{\Psi}, \mathcal{D}_M \tilde{\Psi} \right\rangle_{\theta\theta} = \langle \psi, i\gamma^m \partial_m \chi \rangle + \langle F_1 + F_2, i\gamma^m \partial_m (F_1 - F_2) \rangle + \langle \chi, i\gamma^m \partial_m \psi \rangle. \tag{4.17}$$

4.4 The spectral triple

4.4.1 The Grassmann algebra

The *Grassmann algebra*, $\Lambda_\infty(\mathbb{C})$, (hereafter abbreviated as Λ_∞), is the unital, associative algebra generated by a countably infinite set of anti-commuting variables ξ^i, that is,

$$\xi^i \xi^j + \xi^j \xi^i = 0, \quad \text{for all} \quad i, j \in \mathbb{N}. \tag{4.18}$$

Each element $g \in \Lambda_\infty$ may be written as the sum of its *body* and *soul*, $g = g_B + g_S \in \Lambda_\infty^B \oplus \Lambda_\infty^S$, where

$$g_S = \sum_{k=1}^{\infty} \frac{1}{k!} c_{i_1 i_2 \dots i_k} \xi^{i_1} \xi^{i_2} \dots \xi^{i_k}, \quad \text{and} \quad g_B,\, c_{i_1 i_2 \dots i_k} \in \mathbb{C}. \tag{4.19}$$

Alternatively, Λ_∞ may be decomposed into the direct sum of an *even* subalgebra and an *odd* subset, $\Lambda_\infty = \Lambda_\infty^e \oplus \Lambda_\infty^o$, where Λ_∞^e consists of Λ_∞^B and elements of Λ_∞^S with an even number of generating elements, ξ^i, and likewise Λ_∞^o consists of elements of Λ_∞^S with an odd number of generating elements. The preceding is an example of a $\mathbb{Z}_2$-*grading.* An object which carries such a grading is often referred to (especially in physics literature) as a super-object. i.e. A $\mathbb{Z}_2$-graded algebra is a superalgebra.

There are several possible involutive maps on Λ_∞ which make it a $*$-*algebra*, i.e. for any $g, h \in \Lambda_\infty$, $(gh)^* = h^* g^*$ and $(g^*)^* = g$. For now, define $* : \Lambda_\infty \to \Lambda_\infty$ to be $g \mapsto g^* = g_B^* + g_S^*$, where g_B^* is ordinary complex conjugation, and

$$g_S^* = \sum_{k=1}^{\infty} \frac{1}{k!} c^*_{i_1 i_2 \dots i_k} \xi^{i_k} \xi^{i_{k-1}} \dots \xi^{i_1} = \sum_{k=1}^{\infty} \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} c^*_{i_1 i_2 \dots i_k} \xi^{i_1} \xi^{i_2} \dots \xi^{i_k}. \tag{4.20}$$

The group of unitary elements of the Grassmann algebra is $\mathcal{U}(\Lambda_\infty) = \{u \in \Lambda_\infty \mid uu^* = u^*u = 1\}$, and since for $u \in \mathcal{U}(\Lambda_\infty)$, $u_B \neq 0$, Grassmann unitaries are *logarithmic*, that is, they are expressible as $u = \mathrm{e}^{ig}$ for some *real Grassmann number* g, i.e. $g \in \Lambda_\infty$ satisfying $g = g^*$.

For further discussion about the super Hilbert space structure of the Grassmann algebra please refer to appendix A.

4.4.2 3d Minkowski spacetime

In order to present the following ideas in a simple setting we choose to work in a three-dimensional space with metric signature $(p, q) = (1, 2)$, e.g. $\eta = \mathrm{diag}(1, -1, -1)$. In this case the universal cover of the Lorentz group is the group $\mathrm{SL}(2, \mathbb{R})$ and the Dirac matrices (i.e. generators of a matrix representation of the even graded *Clifford algebra* $\mathrm{Cl}^e_{1,2}(\mathbb{R})$) may be expressed via Pauli matrices, as

$$\gamma^0 = \sigma^2, \gamma^1 = i\sigma^3, \gamma^2 = i\sigma^1. \tag{4.21}$$

The spin representation of Lorentz transformations, (i.e. L such that $L^{\mathrm{T}} \eta L = \eta$), is constructed in the standard way,

$$S(L) = \exp\left\{ \frac{1}{4} \sum_{a<b} \xi_{ab} [\gamma^a, \gamma^b] \right\}, \quad \text{where} \quad \xi_{ab} = -\xi_{ba}, \tag{4.22}$$

and satisfies

$$S^{-1}(L)\gamma^{m'}S(L) = L^{m'}_{\ m}\gamma^m, \tag{4.23}$$

which guarantees covariance with the *spinorial derivative* $D \equiv i\gamma^m\partial_m$. Explicitly, D acts on *spinors* as $D\psi^\alpha(x) = i(\gamma^m)^\alpha_{\ \beta}\partial_m\psi^\beta(x)$, which transform under Lorentz transformations as $\psi^{\alpha'}(x) = S(L)^{\alpha'}_{\ \alpha}\psi^\alpha(L^{-1}x)$, so covariance means $D'\psi'(x') = S(L)D\psi(x)$.

For a Lorentz invariant, hermitian inner product, take $(\xi,\psi) \equiv i\bar{\xi}\psi$, where $\bar{\xi} \equiv \xi^\dagger\gamma^0$. D is hermitian with respect to this product, i.e. $(D\chi,\psi) = (\chi, D\psi)$, and moreover, the complex conjugation operator, C, which acts by $C\psi = \psi^*$ is an anti-unitary operator, i.e. $(C\chi, C\psi) = (\chi,\psi)^*$.

For more details of this calculation please refer to appendix B.

4.4.3 N-point superspace and the distance function

Take the (unital, associative) $*$-(super)algebra $\Lambda(F)$ of Grassmann number (Λ_∞) valued functions over a finite topological space F consisting of N distinct points and endowed with the discrete topology. Let this algebra be equipped with pointwise linear multiplication, addition, and with involution as defined in (4.20), i.e. for any $f, g \in \Lambda(F)$ and $\lambda \in \mathbb{C}$,

$$(f+g)(x) = f(x) + g(x), \tag{4.24a}$$

$$(\lambda f)(x) = \lambda f(x), \tag{4.24b}$$

$$(fg)(x) = f(x)g(x). \tag{4.24c}$$

Notice that for the case of a finite discrete space F, the map

$$\Lambda(F) \ni \varphi \mapsto (\varphi(1), \varphi(2), \ldots, \varphi(N)) \in \Lambda^N \equiv \Lambda \oplus \Lambda \oplus \cdots \text{N-copies} \cdots \oplus \Lambda, \tag{4.25}$$

is a $*$-algebra isomorphism, $\Lambda(F) \simeq \Lambda^N$. The above copies of the Grassmann algebra may conveniently arranged as entries along the main diagonal of an $N \times N$ matrix

$$\begin{pmatrix} \varphi(1) & 0 & \cdots & 0 \\ 0 & \varphi(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi(N) \end{pmatrix}, \tag{4.26}$$

so that pointwise multiplication and addition are simply matrix multiplication and addition, respectively.

Now, if F is endowed with a metric d_{ij}, then there exists a representation, π, of $\Lambda(F)$ on a finite dimensional Hilbert space, and a bounded symmetric operator, D, such that

$$d_{ij} = \sup_{f\in\Lambda^N} \{|f(i) - f(j)| : ||[\mathsf{D}, \pi(f)]|| \leq 1\}. \tag{4.27}$$

This claim follows from the equality

$$||[D,\pi(f)]|| = \max_{k\neq l}\left\{\frac{1}{d_{kl}}|\phi(k)-\phi(l)|\right\}, \tag{4.28}$$

which is proved by an induction argument following [4] Thm 2.18. pp 19-20. Therefore, it also makes sense in the present context to speak of the Dirac operator as a fundamental object which determines the geometry of a superspace.

Henceforth, F is taken to be a 2-point discrete space.

4.4.4 The total space "spectral triple"

All the ingredients are now available to construct the spectral triple for the total space which is to be presently considered. The base space spectral triple is characterized by the algebraic data

$$\mathcal{M}^{3|2} \equiv \left(\mathcal{A}_M = \Lambda^e_\infty, \mathcal{H}_M, \mathcal{D}_M = i\gamma^m\partial_m; \gamma_M \equiv \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, J_M \equiv \begin{pmatrix}G & 0\\ 0 & G\end{pmatrix}\right), \tag{4.29}$$

where $\mathcal{H}_M$ is a (super) Hilbert space of spinor superfields, (4.13), where γ_M implements the $\mathbb{Z}_2$-grading of the Grassmann algebra, and where G denotes the Grassmann involution operator, (4.20). And the finite space spectral triple is

$$\mathcal{F}_F \equiv \left((\Lambda^e_\infty)^2, (\Lambda^o_\infty)^2, \mathcal{D}_F = 0, \gamma_F = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, J_F = \begin{pmatrix}0 & G\\ G & 0\end{pmatrix}\right), \tag{4.30}$$

where the form of J_F presented here is one of several possible, to be discussed in the subsequent section, and $\mathcal{D}_F = 0$ follows from the spectral triple for a 2-point finite space with 2 dimensional Hilbert space representation being equipped with such a real structure, J_F. The resulting triple for the total space is then

$$\mathcal{M}^{3|2}\otimes\mathcal{F}_F \equiv \left(\mathcal{A} = (\Lambda^e_\infty)^2, \mathcal{H} = (\Psi(x,\theta))^2, \mathcal{D} = \mathcal{D}_M\otimes\mathbf{1}_F; \gamma = \gamma_M\otimes\gamma_F, J = J_M\otimes J_F\right). \tag{4.31}$$

It should be stressed that the tensor product used here, and in all that follows is the *graded tensor product* over the Grassmann algebra rather than the usual one.

4.5 Inner fluctuations and the spectral action

4.5.1 Fluctuating the Dirac operator

The Dirac operator for the total space, $\mathcal{D} = \mathcal{D}_M\otimes\mathbf{1}_F$, where $\mathcal{D}_M = i\gamma^m\partial_m$, may be written in a matrix form for a 2 point finite space geometry, as

$$\mathcal{D} = i\gamma^m\begin{pmatrix}\partial_m & 0\\ 0 & \partial_m\end{pmatrix}. \tag{4.32}$$

To calculate $\mathcal{D}_A = \mathcal{D} + A + JAJ^{-1}$ the form of $A \in \Omega^1_{\mathcal{D}}(\mathcal{A}) \equiv \{a[\mathcal{D}, b] : a, b \in \mathcal{A} = (\Lambda^e_\infty)^2\}$ is needed. So, taking

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \tag{4.33}$$

gives

$$A = a[\mathcal{D}, b] = \begin{pmatrix} ia_1\gamma^m\partial_m b_1 & 0 \\ 0 & ia_2\gamma^m\partial_m b_2 \end{pmatrix}. \tag{4.34}$$

Now, the anti-unitarity of J_F implies that it must be of the form $J_F = U \circ G$, where U is a unitary operator on $\mathcal{H}_F$, i.e. a representation of unitary elements, u_i, of the algebra $\mathcal{A}_F$. In the case of a Grassmann number valued algebra over a 2 point finite space geometry, the restrictions $u_1u_2^* = u_2u_1^* = -1$ and $u_1u_1^* = u_2u_2^* = -1$ which arise from the possibility that $J_F^2 = -\mathbf{1}_F$, cannot be satisfied except trivially, and thus such choices of J_F are excluded in the present situation.

However, using $J_F = \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix}$,

$$JAJ^{-1} = \begin{pmatrix} ia_2^*\gamma^m\partial_m b_2^* & 0 \\ 0 & ia_1^*\gamma^m\partial_m b_1^* \end{pmatrix}. \tag{4.35}$$

Then, the fact that $A + JAJ^{-1}$ is traceless, (this follows from the hermiticity of A), implies that

$$(a_i\partial_m b_i)^* = -a_i\partial_m b_i, \tag{4.36}$$

and so,

$$A + JAJ^{-1} = \begin{pmatrix} i\gamma^m(a_1\partial_m b_1 - a_2\partial_m b_2) & 0 \\ 0 & -i\gamma^m(a_1\partial_m b_1 - a_2\partial_m b_2) \end{pmatrix}. \tag{4.37}$$

Thus, the fluctuated Dirac operator for this choice of J_F is

$$\mathcal{D}_A = i\gamma^m \begin{pmatrix} \partial_m & 0 \\ 0 & \partial_m \end{pmatrix} + \begin{pmatrix} \gamma^m A_m & 0 \\ 0 & -\gamma^m A_m \end{pmatrix} \tag{4.38a}$$

$$= \mathcal{D} + \gamma^m A_m \otimes \gamma_F \quad \text{where} \quad A_m = i(a_1\partial_m b_1 - a_2\partial_m b_2). \tag{4.38b}$$

Similarly,

$$JAJ^{-1} = \begin{pmatrix} ia_1^*\gamma^m\partial_m b_1^* & 0 \\ 0 & ia_2^*\gamma^m\partial_m b_2^* \end{pmatrix}, \tag{4.39}$$

is obtained by using $J_F = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$. This time, the previously invoked trace free condition results in a trivial fluctuation, i.e. $\mathcal{D}_A = \mathcal{D}$. Instead, let $a_1\mathcal{D}_M b_1 = -a_2\mathcal{D}_M b_2$ and

$(a_1 \mathcal{D}_M b_1)^* = -(a_1 \mathcal{D}_M b_1)^*$ so that $A + JAJ^{-1}$ is again traceless as required. Changing labels so that $a_2 = -a_1$ and $b_2 = b_1$,

$$A + JAJ^{-1} = \begin{pmatrix} i\gamma^m(a_1\partial_m b_1 - a_1^*\partial_m b_1^*) & 0 \\ 0 & -i\gamma^m(a_1\partial_m b_1 - a_1^*\partial_m b_1^*) \end{pmatrix}. \tag{4.40}$$

As before then, the fluctuated Dirac operator for this choice of J_F may be written as

$$\mathcal{D}_A = i\gamma^m \begin{pmatrix} \partial_m & 0 \\ 0 & \partial_m \end{pmatrix} + \begin{pmatrix} \gamma^m A_m & 0 \\ 0 & -\gamma^m A_m \end{pmatrix} \tag{4.41a}$$

$$= \mathcal{D} + \gamma^m A_m \otimes \gamma_F \quad \text{where this time} \quad A_m = i(a_1\partial_m b_1 - a_1^*\partial_m b_1^*). \tag{4.41b}$$

4.5.2 The gauge group and chiral superfield covariance

Considering the finite space, $\mathcal{F}_F$, associated with the 2-point discrete topological space F, take $\mathcal{U}(\mathcal{A}_F)$ to be the unitary elements of $\mathcal{A}_F$, i.e. $u \in \mathcal{U}(\mathcal{A}_F)$ which have the form

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{ig^{(1)}} & 0 \\ 0 & \mathrm{e}^{ig^{(2)}} \end{pmatrix}, \tag{4.42}$$

where $g^{(1)}$ and $g^{(2)}$ are real, even, Grassmann elements, i.e. $g^{(i)} = (g^{(i)})^*$ and $g^{(i)}a = ag^{(i)}$ for any $a \in \mathcal{A}_F$.

Now recall the adjoint map Ad: $\mathcal{U}(\mathcal{A}_F) \ni u \mapsto U_F \equiv \pi(u)J_F\pi(u)J_F^* \in \mathrm{End}(\mathcal{H}_F)$, and note that for brevity the representation symbol π will be implicit when no danger of confusion is present. Then, for $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{H}_F$ it is readily checked that:

$$U_F^* U_F = U_F U_F^* = \mathbf{1}_F, \tag{4.43a}$$

$$U_F^* \gamma U_F = \gamma, \tag{4.43b}$$

$$U_F^* J_F U_F = J_F, \tag{4.43c}$$

$$U_F h = \begin{pmatrix} u_1 h_1 u_2^* \\ u_2 h_2 u_1^* \end{pmatrix}. \tag{4.43d}$$

Given the last property above, computing $\mathrm{Ker(Ad)} = \{u \in \mathcal{U}(A_F) : U_F h = h \quad \text{for all} \quad h \in H_F\}$ yields the conditions $g^{(1)} = g^{(2)} \equiv g_e$, i.e. an element of the kernel has the form,

$$\mathrm{Ker(Ad)} \in \begin{pmatrix} \mathrm{e}^{ig_e} & 0 \\ 0 & \mathrm{e}^{ig_e} \end{pmatrix}. \tag{4.44}$$

Now, the gauge group of $\mathcal{A}$ is defined to be

$$\mathcal{G}(\mathcal{M}^{3|2} \otimes \mathcal{F}_F) \equiv \{U = uJuJ^* | u \in \mathcal{U}(\mathcal{A})\}, \tag{4.45}$$

but $\mathcal{G}(\mathcal{A}_M)$ is trivial, and since $\mathrm{Ker}(\mathrm{Ad}) = \mathcal{U}(\mathcal{A}_F)_{J_F} \equiv \{u \in \mathcal{A}_F : uJ_F = J_F u^*\}$, the gauge group of the finite space is given by

$$\mathcal{G}(\mathcal{F}_F) = \mathcal{U}(\mathcal{A}_F)/\mathrm{Ker}(\mathrm{Ad}). \tag{4.46}$$

It is immediate to calculate that an element $\underline{u} \in \mathcal{G}(\mathcal{F}_F)$ is of the form

$$\underline{u} = \begin{pmatrix} e^{\frac{i}{2}g} & 0 \\ 0 & e^{-\frac{i}{2}g} \end{pmatrix}, \tag{4.47}$$

where $g \equiv g^{(1)} - g^{(2)}$, and, if $\underline{U} = \underline{u}J_F\underline{u}J_F^*$, then

$$\underline{U}h = \begin{pmatrix} \mathrm{e}^{ig}h_1 \\ \mathrm{e}^{-ig}h_2 \end{pmatrix}. \tag{4.48}$$

It is interesting to note that the chiral restriction imposed on a superspinor is not consistent with gauge covariance. Indeed, the compatibility condition

$$\mathrm{e}^{ig}D_\alpha\tilde{\Psi}^\alpha = D_\alpha \mathrm{e}^{ig}\tilde{\Psi}^\alpha, \tag{4.49}$$

is satisfied if and only if $D_\alpha g = 0$. In the case of a real, even superfield g the latter condition yields, after a short calculation, that g has to be a real, constant element of Λ_∞.

4.5.3 The fermionic action

Since there is no concern with regards to the so-called fermion doubling problem which is encountered when one reproduces the standard model by the techniques of NCG, here the fermionic action is taken in it's original form:

$$\langle \xi, \mathcal{D}_A \xi \rangle, \tag{4.50}$$

for $\xi \in \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F$. Such elements have the form

$$\xi = \Psi(x,\theta) \otimes h = \Psi_+ \otimes e + \Psi_- \otimes \bar{e}, \tag{4.51a}$$

$$= \begin{pmatrix} \Psi \otimes h_1 & 0 \\ 0 & \Psi \otimes h_2 \end{pmatrix} = \begin{pmatrix} \Psi_+ & 0 \\ 0 & \Psi_- \end{pmatrix}, \tag{4.51b}$$

where $\{e, \bar{e}\}$ is an orthonormal basis for $\mathcal{H}_F$, such that $e \in \mathcal{H}_F^+$ and $\bar{e} \in \mathcal{H}_F^-$, (i.e. $\gamma_F e = e$ and $\gamma_F \bar{e} = -\bar{e}$), and such that $J_F e = \bar{e}$ and $J_F \bar{e} = e$. Also, recall that each $\Psi_\pm \in \mathcal{H}_M$ is a super-spinor with the form

$$\Lambda^o_\infty \ni \Psi^\alpha_\pm(x,\theta) = \psi^\alpha_\pm(x) + F^\alpha_{\pm\,\beta}(x)\theta^\beta + \chi^\alpha_\pm(x)\theta\theta, \tag{4.52}$$

which means that $\psi^\alpha_\pm(x), \chi^\alpha_\pm(x) \in \Lambda^o_\infty$ and $F^\alpha_{\pm\,\beta}(x) \in \Lambda^e_\infty$.

Given the fluctuated Dirac operator given in (4.38a) and (4.38b), the fermionic action is calculated to be

$$\langle \xi, \mathcal{D}_A \xi \rangle = \langle \xi, (\mathcal{D}_M \otimes \mathbf{1}_F)\xi \rangle + \langle \xi, (\gamma^m A_m \otimes \gamma_F)\xi \rangle \,. \tag{4.53}$$

For the first term,

$$\begin{aligned}
\langle \xi, (\mathcal{D}_M \otimes \mathbf{1}_F)\xi \rangle &= \langle \Psi_+, \mathcal{D}_M \Psi_+ \rangle \langle e, e \rangle + \langle \Psi_-, \mathcal{D}_M \Psi_- \rangle \langle \bar{e}, \bar{e} \rangle && \text{(4.54a)} \\
&= \langle \Psi_+, \mathcal{D}_M \Psi_+ \rangle + \langle \Psi_-, \mathcal{D}_M \Psi_- \rangle && \text{(4.54b)} \\
&\equiv \langle \Psi_\pm, \mathcal{D}_M \Psi_\pm \rangle \,, && \text{(4.54c)}
\end{aligned}$$

and similarly the second term,

$$\begin{aligned}
\langle \xi, (\gamma^m A_m \otimes \gamma_F)\xi \rangle &= \langle \Psi_+, \gamma^m A_m \Psi_+ \rangle + \langle \Psi_-, \gamma^m A_m \Psi_- \rangle && \text{(4.55a)} \\
&\equiv \langle \Psi_\pm, \gamma^m A_m \Psi_\pm \rangle \,. && \text{(4.55b)}
\end{aligned}$$

As before, supersymmetry invariance of the action under a supersymmetry transformation is guaranteed for terms which are of highest order in the Grassmann variables. But since $A_m = i(a_1 \partial_m b_1 - a_2 \partial_m b_2)$ where $a_i, b_i \in \Lambda^e_\infty$ for $i = 1, 2$, A_m is itself represented by an even superfield on $\mathbb{R}^{3|2}$ and can be written in the form

$$A_m = \mathsf{A}_m + \lambda_{m,\alpha} \theta^\alpha + \mathsf{B}_m \theta\theta, \tag{4.56}$$

for some independent fields A_m, $\lambda_{m,\alpha}$, and B_m.

In these terms

$$\begin{aligned}
\langle \xi, (\gamma^m A_m \otimes \gamma_F)\xi \rangle_{\theta\theta} = {}& \langle \psi_\pm, \gamma^m \mathsf{B}_m \psi_\pm \rangle + \left\langle \psi_\pm, \gamma^m \lambda_{m,[2} F_{\pm\, 1]} \right\rangle + \langle \psi_\pm, \gamma^m \mathsf{A}_m \chi_\pm \rangle \\
&+ \left\langle F_{\pm\,[1}, \gamma^m \lambda_{m,2]} \psi_\pm \right\rangle + \left\langle F_{\pm\,[2}, \gamma^m \mathsf{A}_m F_{\pm\, 1]} \right\rangle + \langle \chi_\pm, \gamma^m \mathsf{A}_m \psi_\pm \rangle \,.
\end{aligned} \tag{4.57}$$

And finally, we may write down the complete supersymmetry invariant fermionic action

$$\begin{aligned}
\langle \xi, \mathcal{D}_A \xi \rangle_{\theta\theta} = {}& \langle \psi_\pm, \mathcal{D}_M \chi_\pm \rangle + \left\langle F_{\pm\,[2}, \mathcal{D}_M F_{\pm\, 1]} \right\rangle + \langle \chi_\pm, \mathcal{D}_M \psi_\pm \rangle \\
&+ \langle \psi_\pm, \gamma^m \mathsf{B}_m \psi_\pm \rangle + \left\langle \psi_\pm, \gamma^m \lambda_{m,[2} F_{\pm\, 1]} \right\rangle + \langle \psi_\pm, \gamma^m \mathsf{A}_m \chi_\pm \rangle \\
&+ \left\langle F_{\pm\,[1}, \gamma^m \lambda_{m,2]} \psi_\pm \right\rangle + \left\langle F_{\pm\,[2}, \gamma^m \mathsf{A}_m F_{\pm\, 1]} \right\rangle + \langle \chi_\pm, \gamma^m \mathsf{A}_m \psi_\pm \rangle \,.
\end{aligned} \tag{4.58}$$

4.5.4 The spectral action

Within the Connes approach, the dynamics of a gauge field is encoded in the *spectral action*,

$$S_A = \mathrm{Tr} f(\mathcal{D}_A), \tag{4.59}$$

where f is a smooth, rapidly vanishing function whose moments determine the parameters (e.g. coupling constants) of the discussed model. For a detailed calculation of the heat kernal expansion in a simple case please refer to appendix C.

However, in the present context of a 3 dimensional Minkowski space, heat kernel methods are problematic. Firstly, as was discussed at the beginning of this chapter, for spaces with non-Euclidean signature the Dixmier trace is not defined. But, even if we passed to a Euclidean signature, e.g. by means of a Wick rotation, in the case of a 3-dimensional space, the dimensionally meaningful term is the one containing the trace of the third power of the fluctuated Dirac operator. Therefore, in the present situation of a flat (super)space, it is enough to note that the calculation of S_A essentially trivializes, and boils down to calculating the trace of the third power of the fluctuated Dirac operator.

Since

$$(D_A)^3 = \frac{i}{2}\left(-\partial^2\partial_m + (A\cdot A)\partial_m + 2A_m(A\cdot\partial) + \partial_m(A\cdot A) + A_m(\partial A)\right)\gamma^m\otimes\mathbf{1}_F \quad (4.60a)$$

$$-\frac{1}{2}\left(2A_m\partial^2 + 2(A\cdot\partial)\partial_m + (\partial A)\partial_m + 2\partial_m(A\cdot\partial) + \partial_m(\partial A) - 2(A\cdot A)A_m\right)\gamma^m\otimes\gamma_F \quad (4.60b)$$

$$-\frac{1}{2}\partial_p F_{mn}\left(\gamma^p\left[\gamma^m,\gamma^n\right]\right)\otimes\gamma_F + \frac{i}{2}A_pF_{mn}\left(\gamma^p\left[\gamma^m,\gamma^n\right]\right)\otimes\mathbf{1}_F, \quad (4.60c)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$, we get

$$S_A \sim \int \epsilon^{pmn}A_pF_{mn}d^3x. \quad (4.61)$$

The SUSY invariant action in the gauge sector is thus of the form

$$(S_A)_{\theta\theta} = \frac{1}{g}\int \epsilon^{pmn}\left(\mathsf{A}_p(\partial_m\mathsf{B}_n - \partial_n\mathsf{B}_m) + \mathsf{B}_p(\partial_m\mathsf{A}_n - \partial_n\mathsf{A}_m) - \lambda_{p,\alpha}(\partial_m\lambda_n^\alpha - \partial_n\lambda_m^\alpha)\right), \quad (4.62a)$$

where g is a coupling constant. This action is again automatically invariant under the SUSY transformation of $\mathsf{A}_m, \lambda_{m,\alpha}$ and B_m defined by

$$\delta A_m(x,\theta) = \delta\mathsf{A}_m + \delta\lambda_{m,\alpha}\theta^\alpha + \delta\mathsf{B}_m\theta\theta = A_m(x+\delta x,\theta+\delta\theta) - A_m(x,\theta), \quad (4.63)$$

where δx and $\delta\theta$ are of the form (4.1).

4.6 Conclusions

The preceding sections propose and exemplify a strategy for the incorporation of a superspace formulation of the principle of supersymmetry into the formalism of noncommutative geometry, up to and including the spectral action. This has been done in as simple a setting as possible, not solely for computational convenience, but as well, so as to avoid obfuscation of the guiding principles and machinery of the noncommutative method.

In fact, the perspicacious reader will have undoubtedly (and rightly) noted that there is nothing truly noncommutative in the example which is presently investigated. Through consideration of a less trivial finite space (e.g. *Supermatrix* algebras), one may introduce noncommutativity into the picture and expect the resulting theory to have a richer structure (e.g. non-abelian gauge fields and a Higgs sector analogue).

It is also worth noting that some care was taken to avoid mentioning the KO-dimension of the spaces used in the above construction. The reason being that it is not clear what should be the analogous notion in the context of this proposed program which includes anti-commuting variables in an essential way.

On a related note, the Dirac operator being considered here is a somehow naive choice as it only contains derivatives of the commuting coordinates. A consequence of this choice, it should be emphasized, is that the resulting field theory is nothing like a physically relevant one. For example, electrodynamics is the usual result of the AC-geometry approach when a 2-point finite space is considered. But by choosing the spinorial derivative as Dirac operator, absent are terms of the form $\psi^\alpha D\psi^\alpha$. It is expected that an honest superspace Dirac operator built from supercovariant derivatives will further extend the richness, and it is hoped, the physical relevance, of any theory developed according to this strategy. One possible recipe for the construction of such an operator was the topic of the previous chapter.

Appendices

Appendix A

The Grassmann algebra as a super Hilbert space

We are in search of a module over a finitely generated Grassmann algebra with a Grassmann number valued inner product.

A.1 Λ_n

Take the *finitely generated Grassmann algebra*, Λ_n to be the unital, associative algebra (over $\mathbb{C}$) generated by $1 \in \mathbb{C}$ and a finite set of anti-commuting symbols $\{\xi^{i_1}, \xi^{i_2}, \ldots, \xi^{i_n}\}$ where the set $(i_1, i_2, \ldots, i_n) \in M_n := \{(m_1, m_2, \ldots, m_k) : 1 \leq k \leq n, m_i \in \{1, 2, \ldots, n\}, 1 \leq m_1 < \cdots < m_k \leq n\}$ is a particular choice of generating basis.

Recall that each element $g \in \Lambda_n$ may be written as the sum of it's *body* and *soul*, $g = g_B + g_S \in \Lambda_n^B \oplus \Lambda_n^S$, where

$$g_S = \sum_{k=1}^{n} \frac{1}{k!} g_{i_1 i_2 \ldots i_k} \xi^{i_1} \xi^{i_2} \ldots \xi^{i_k}, \quad \text{and} \quad g_B,\ g_{i_1 i_2 \ldots i_k} \in \mathbb{C}.$$

There are several possible definitions of an involutive map on Λ_n which would make Λ_n a $*$-*algebra*, i.e. for any $g, h \in \Lambda_n$, $(gh)^* = h^* g^*$ and $(g^*)^* = g$. For now we'll adopt $* : \Lambda_n \to \Lambda_n$ with $g \mapsto g^* = g_B^* + g_S^*$, where g_B^* is ordinary complex congugation, and

$$g_S^* = \sum_{k=1}^{n} \frac{1}{k!} g^*_{i_1 i_2 \ldots i_k} \xi^{i_k} \xi^{i_{k-1}} \ldots \xi^{i_1} = \sum_{k=1}^{n} \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} g^*_{i_1 i_2 \ldots i_k} \xi^{i_1} \xi^{i_2} \ldots \xi^{i_k}.$$

Another useful way of thinking about Λ_n is as a direct sum of complex vector spaces,

$$\Lambda_n = \bigoplus_{r=0}^{n} V_r \quad \text{where} \quad V_r = \mathrm{span}_{\mathbb{C}}\{\xi^{i_1}, \xi^{i_2}, \ldots, \xi^{i_r}\}.$$

Thus, each $g \in \Lambda_n$ may be written as $g = \sum_{r=0}^{n} g_r$ where each $g_r \in V_r$.

There is an isomorphism $\star : \Lambda_n \to \Lambda_n$ defined on elements $\xi^{i_1}\xi^{i_2}\ldots\xi^{i_d}$ by

$$\star\left[\xi^{i_1}\xi^{i_2}\ldots\xi^{i_d}\right] := \xi^{j_1}\xi^{j_2}\ldots\xi^{j_{n-d}},$$

where $(j_1, \ldots, j_{n-d})$ is chosen so that $(i_1, \ldots, i_d, j_1, \ldots, j_n)$ is an even permutation of $(1, 2, \ldots, n)$. $\star$ extends to all of Λ_n by requiring conjugate linearity, i.e $\star[\alpha g] = \alpha^* \star [g]$ for $\alpha \in \mathbb{C}$ and $g \in \Lambda_n$, and that $\star$ be a real linear transformation.

A.2 Norms

A simple and natural choice could be to take the *Rogers norm*,

$$|g|_1 := |g_b| + \sum_{\substack{(m_1,\ldots,m_k)\in M_n \\ k=1}}^{n} |g_{i_1 i_2 \ldots i_k}|$$

which is submultiplicative, i.e. $|gh|_1 \leq |g|_1|h|_1$ for all $g, h \in \Lambda_n$, which means that Λ_n with norm $|\cdot|_1$ is a (complex) Banach space called the *Rogers algebra* and denoted $\Lambda_n(1)$. But since this norm depends implicitly on the choice of generating basis, and given that Grassmann number valued quantities are not physically observable, there may be no universally preferred basis for generating a Grassmann algebra.

A norm which is trivially independent of the generators is the *body norm*,

$$||g||_B := |g_B|$$

but this norm essentially erases any pertinent information (physical or otherwise) which may be contained in the soul of the Grassmann number.

An alternative norm, possibly due to Whitney, and well described by Federer, called the *mass norm* is constructed starting from the vector space norm, $||\cdot||_r$ on V_r given by

$$||g_r||_r := \inf\left\{\sum_{(m_1,\ldots,m_r)\in M_n} |g_{i_1 i_2 \ldots i_r}|\right\},$$

where the infimum is over all possible generating basis choices. $||\cdot||_r$ is submultiplicative across vector spaces in the following way.

$$||p_r q_s||_{r+s} \leq ||p_r||_r ||q_s||_s, \quad \text{for all} \quad p_r \in W_r \quad \text{and} \quad q_s \in V_s.$$

So the *mass norm*, defined to be

$$||g|| := \sum_{r=0}^{n} ||g_r||_r, \quad \text{for all} \quad g \in \Lambda_n,$$

is, by construction, independent of the choice of generating basis, and is submultiplicative by the inherited submultiplicativity of the vector space norm. Explicitly,

$$\begin{aligned}
||pq|| &= \sum_r ||(pq)||_r \\
&\leq \sum_r \sum_{k\leq r} ||p_{r-k}q_k||_r \\
&\leq \sum_r \sum_{k\leq r} ||p_{r-k}||_{r-k}||q_k||_k \\
&\leq \sum_r \sum_k ||p_r||_r||q_k||_k \\
&= ||p||||q||.
\end{aligned}$$

One may now notice that $||g||_B \leq ||g||$. (Note that $g_B \equiv g_0$).

A.3 Hilbert Λ_n-Modules

A *pre-Hilbert Λ_n-module* is a $\mathbb{Z}_2$-graded (left/right/bi) Λ_n-module, call it E and denote the grading by $E = E^0 \oplus E^1$ into even and odd subsets, respectively, together with a Λ_n-valued inner product, $\langle \cdot, \cdot \rangle : E \times E \to \Lambda_n$, which, for $e, e_i \in E$ and $\alpha \in \mathbb{C}$ satisfies

1. $\langle e_1 + e_2, e_3 + e_4 \rangle = \langle e_1, e_3 \rangle + \langle e_1, e_4 \rangle + \langle e_2, e_3 \rangle + \langle e_2, e_4 \rangle$,
2. $\langle e_1, \alpha e_2 \rangle = \alpha \langle e_1, e_2 \rangle = \langle \alpha^* e_1, e_2 \rangle$,
3. $\langle e_1, e_2 \rangle_B = \langle e_2, e_1 \rangle_B^* = \overline{\langle e_2, e_1 \rangle}_B$,
4. $\langle e, e \rangle_B \geq 0 \quad \text{for all} \quad e \in E, \quad \text{and} \quad \langle e, e \rangle = 0 \quad \text{if and only if} \quad e = 0.$

Conditions 1 and 2 constitute sesquilinearity. We note that these are the conditions of Rudolph, who points out that the sesqui-Λ_n-linearity (i.e. an additional property requiring, in the right Λ_n-module case, $\langle e_1, e_2 g \rangle = \langle e_1, e_2 \rangle g$ for $g \in \Lambda_n$) taken by DeWitt and others is too restrictive for his purposes working in the functional Schrödinger representation of spinor quantum field theory. Since it is not yet clear what we will be required for us, we start with the most general situation. Condition 3 simply says the body is Hermetian, meanwhile condition 4 says the inner product is definite and has positive body. Again there are some generalizations here compared with the original assumptions of DeWitt, whose utility in our present situation we may debate. It's perhaps worth noting that van Sujlekom includes such a sesqui-Λ_n-linearity requirement in his definition of a Hilbert bimodule (2.9 in his book).

Now, we may use the inner product to define a norm on our (pre)-Hilbert Λ_n-module E in terms of whichever norm we choose on Λ_n: $||e||^2 := ||\langle e, e \rangle||_{\Lambda_n}$.

Example 1: Arising from the Rogers norm,

$$||e||_1 = ||\langle e, e\rangle|_1,$$

and Example 2: arising form the mass norm,

$$||e||^2 = ||\langle e, e\rangle||.$$

A pre-Hilbert Λ_n-module E with norm $||\cdot||_E$ is a *Hilbert* Λ_n*-module* if it is complete with respect to its norm. In our present case of finitely generated Grassmann algebras this is trivially true and all pre-Hilbert Λ_n-modules are Hilbert Λ_n-modules.

This would be a good place to discuss Hilbert Λ_n-module morphisms and Morita equivalences. But it is a work in progress and for brevity we press on.

A.4 Super (pre)-Hilbert Space

A pre-Hilbert Λ_n-module, H is a *super pre-Hilbert* Λ_n*-module* if the inner product on H is continuous. (i.e. continuous in the norm topology; if there exists a $c > 0$ such that $||\langle e_1, e_2\rangle||_{\Lambda_n} \leq c||e_1||_E||e_2||_E$). If H is complete with respect to the norm then H is a *super Hilbert space.*

Example: The Grassmann algebra Λ_n equipped with the mass norm $||\cdot||$ becomes a super Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle p, q\rangle := \star\left[p \star [q]\right].$$

Reference material for the content of this appendix can be found in [56, 30, 29, 57, 58, 4]

Appendix B

3D Minkowski signature

If we take as our metric $\eta = \mathrm{diag}(1,-1,-1)$, and L preserving η, i.e. satisfying $L^{\mathrm{T}}\eta L = \eta$, then we may write $L = e^A$ where A satisfies $(\eta A)^T = -\eta A$. Any such A may be written as a combination

$$A = \xi_{01}K_{01} + \xi_{02}K_{02} + \xi_{12}K_{12}$$

where

$$K_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad K_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We may verify the algebra relations

$$[K_{01}, K_{02}] = K_{12}, \quad [K_{01}, K_{12}] = K_{02}, \quad \text{and} \quad [K_{12}, K_{02}] = K_{01}.$$

Also, by defining $K_{01} = K_{10}, K_{02} = K_{20}$, and $K_{12} = -K_{21}$, together with $\xi_{10} = \xi_{01}, \xi_{02} = \xi_{20}$, and $\xi_{12} = -\xi_{21}$, then the above algebra relations may be written more compactly, as

$$[K_{ab}, K_{bc}] = K_{ac}, \quad \text{and also} \quad A = \frac{1}{2}\sum_{a,b=0}^{2} \xi_{ab}K_{ab}.$$

Now, the spin representation of the Lorentz transformations in the Minkowski signature case will be

$$S(L) = \exp\left\{\frac{1}{2}\sum_{a<b} \xi_{ab}\tau_{ab}\right\}$$

satisfying

$$S^{-1}(L)\gamma^{a'}S(L) = L^{a'}{}_{a}\gamma^{a}$$

for some matrices τ_{ab}, and for γ^a chosen such that they generate a matrix representation of $\mathrm{Cl}_{1,2}(\mathbb{R})$. Thus we take

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3, \quad \text{and} \quad \gamma^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1,$$

and as the label suggests, σ_1, σ_2, and σ_3 are the usual Pauli matrices. To linear order,

$$L = 1 + A = \begin{pmatrix} 1 & \xi_{01} & \xi_{02} \\ \xi_{01} & 1 & \xi_{12} \\ \xi_{02} & -\xi_{12} & 1 \end{pmatrix},$$

and the RHS of the above condition yields

$$\delta\gamma^0 = \xi_{01}\gamma^1 + \xi_{02}\gamma^2, \quad \delta\gamma^1 = \xi_{10}\gamma^0 + \xi_{12}\gamma^2, \quad \text{and} \quad \delta\gamma^2 = \xi_{20}\gamma^0 - \xi_{12}\gamma^1.$$

Meanwhile, also to linear order,

$$S(L) = 1 + \frac{1}{2}\xi_{01}\tau_{01} + \frac{1}{2}\xi_{02}\tau_{02} + \frac{1}{2}\xi_{12}\tau_{12}, \quad \text{and} \quad S^{-1}(L) = 1 - \frac{1}{2}\xi_{01}\tau_{01} - \frac{1}{2}\xi_{02}\tau_{02} - \frac{1}{2}\xi_{12}\tau_{12}$$

so that with the same level of accuracy, the LHS gives

$$S^{-1}(L)\gamma^a S(L) = \gamma^a - \frac{1}{2}\xi_{01}[\tau_{01}, \gamma^a] - \frac{1}{2}\xi_{02}[\tau_{02}, \gamma^a] - \frac{1}{2}\xi_{12}[\tau_{12}, \gamma^a].$$

Comparing LHS and RHS we immediately see

$$[\tau_{12}, \gamma^0] = 0, \quad [\tau_{02}, \gamma^1] = 0, \quad \text{and} \quad [\tau_{01}, \gamma^2] = 0,$$

which means that $\tau_{12} = a\sigma_2$, $\tau_{02} = b\sigma_3$ and $\tau_{01} = c\sigma_1$ for some complex numbers a, b, and c. Furthermore, recalling the identity $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$ and comparing terms with like coefficients for γ^0, we get equations

$$\gamma^1 = i\sigma_3 = -\frac{1}{2}[\tau_{01}, \gamma^0] = -\frac{1}{2}[\tau_{01}, \sigma_2], \quad \text{which implies} \quad \tau_{01} = -\sigma_1 = \frac{1}{2}[\gamma^0, \gamma^1]$$

and

$$\gamma^1 = i\sigma_1 = -\frac{1}{2}[\tau_{02}, \gamma^0] = -\frac{1}{2}[\tau_{02}, \sigma_2], \quad \text{which implies} \quad \tau_{02} = \sigma_3 = \frac{1}{2}[\gamma^0, \gamma^2].$$

Similarly, comparing coefficients for γ^1 leads to

$$\gamma^2 = i\sigma_1 = -\frac{1}{2}[\tau_{12}, \gamma^1] = -\frac{1}{2}[\tau_{01}, i\sigma_3], \quad \text{which implies} \quad \tau_{12} = i\sigma_2 = \frac{1}{2}[\gamma^1, \gamma^2].$$

Perhaps unsurprisingly in retrospect, we observe that these matrices are linearly independent, real, and traceless, that is, they form a basis for the Lie algebra $\mathrm{sl}(2, \mathbb{R})$. And finally we are ready to write down the spin representation of the Lorentz transformations in three dimensions with Minkowski signature.

$$S(L) = \exp\left\{\frac{1}{4}\sum_{a<b}\xi_{ab}[\gamma^a, \gamma^b]\right\}.$$

Appendix C

The Spectral Action for d=2

In this section we will use several well known facts to obtain a perturbative expansion for the bosonic spectral action in terms of a postive even function, f, its moments $f_{d-k} := \int_0^\infty u^{(d-k)-1} f(u)du$, and a cut-off parameter Λ. To begin, the bosonic spectral action is defined to be

$$S_b := \mathsf{Tr}\left(f\left(\frac{\mathcal{D}_\omega}{\Lambda}\right)\right).$$

Given that $\mathcal{D}_\omega^2$ is a generalized Laplacian, we have the following expansion in t, known as the heat expansion

$$\mathsf{Tr}\left(\mathrm{e}^{-t\mathcal{D}_\omega^2}\right) \approx \sum_{k\geq 0} t^{\frac{k-d}{2}} a_k(\mathcal{D}_\omega^2), \quad \text{where} \quad a_k(\mathcal{D}_\omega^2) = \int_{\mathcal{M}} a_k(x, \mathcal{D}_\omega^2)\sqrt{|g|} d^d x.$$

Here d is the dimension of the manifold ($d = 2$ in this example,) the trace is taken over the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable sections of the spinor bundle, and the coefficients $a_k(x, \mathcal{D}_\omega^2)$ are the Seeley-DeWitt coefficients, the first two of which are, in our case

$$a_0\left(x, \mathcal{D}_\omega^2\right) = \frac{1}{2\pi}, \quad a_1\left(x, \mathcal{D}_\omega^2\right) = 0, \quad a_2\left(x, \mathcal{D}_\omega^2\right) = \frac{-1}{24\pi} s,$$

where s is the scalar curvature of the Levi-Civita connection, ∇.

Now, given a function $h(s)$, and its Laplace transform

$$g(v) = \int_0^\infty \mathrm{e}^{-sv} h(s)ds,$$

we can write

$$g(t\mathcal{D}_\omega^2) = \int_0^\infty \mathrm{e}^{-st\mathcal{D}_\omega^2} h(s)ds.$$

Then, taking its trace and using the heat expansion yields

$$\begin{aligned}\mathsf{Tr}\left[g(t\mathcal{D}_\omega^2)\right] &= \int_0^\infty \mathsf{Tr}\left(\mathrm{e}^{-st\mathcal{D}_\omega^2}\right) h(s)ds \\ &\approx \int_0^\infty \sum_{k\geq 0}(st)^{\frac{k-d}{2}} a_k(\mathcal{D}_\omega^2)h(s)ds \\ &= \sum_{k\geq 0} t^{\frac{k-d}{2}} a_k(\mathcal{D}_\omega^2)\int_0^\infty s^{\frac{k-d}{2}} h(s)ds.\end{aligned}$$

Notice that for t small and in the case of our example $d = 2$, terms with $k > 2$ vanish and the $k = 2$ term in this sum is just $a_2(\mathcal{D}_\omega^2)g(0)$. The remaining terms may be rewritten by performing a Mellin transform, that is, by using the definition of the Γ-function as the analytic continuation of

$$\Gamma(z) = \int_0^\infty r^{z-1}\mathrm{e}^{-r}dr$$

and letting $z = \frac{2-k}{2}$ and $r = sv$, to get

$$\Gamma\left(\frac{2-k}{2}\right) = \int_0^\infty (sv)^{\frac{2-k}{2}-1}\mathrm{e}^{-sv}d(sv) = s^{\frac{2-k}{2}}\int_0^\infty v^{\frac{2-k}{2}-1}\mathrm{e}^{-sv}dv,$$

and thus

$$s^{\frac{k-2}{2}} = \frac{1}{\Gamma\left(\frac{2-k}{2}\right)}\int_0^\infty v^{\frac{2-k}{2}-1}\mathrm{e}^{-sv}dv.$$

By substituting this into the heat expansion and performing the integral over s we obtain

$$\begin{aligned}\mathsf{Tr}\left[g(t\mathcal{D}_\omega^2)\right] &\approx a_2(\mathcal{D}_\omega^2)g(0) + \sum_{0\leq k<2} t^{\frac{k-2}{2}} a_k(\mathcal{D}_\omega^2)\frac{1}{\Gamma\left(\frac{2-k}{2}\right)}\int_0^\infty h(s)\int_0^\infty v^{\frac{2-k}{2}-1}\mathrm{e}^{-sv}dvds \\ &= a_2(\mathcal{D}_\omega^2)g(0) + \sum_{0\leq k<2} t^{\frac{k-2}{2}} a_k(\mathcal{D}_\omega^2)\frac{1}{\Gamma\left(\frac{2-k}{2}\right)}\int_0^\infty v^{\frac{2-k}{2}-1}g(v)dv.\end{aligned}$$

Choosing the function $g(v)$ so that $g(u^2) = f(u)$ and letting $v = u^2$, the integral in the series is just

$$\int_0^\infty v^{\frac{2-k}{2}-1}g(v)dv = \int_0^\infty u^{2-k-2}f(u)2udu = 2\int_0^\infty u^{2-k-1}f(u)du = 2f_{2-k}.$$

and finally, setting $t = \Lambda^{-2}$, and putting in the Seeley-DeWitt coefficients the spectral action is

$$\begin{aligned}
S_b &= \mathsf{Tr}\left[f\left(\frac{\mathcal{D}_\omega}{\Lambda}\right)\right] = \mathsf{Tr}\left[g\left(\Lambda^{-2}\mathcal{D}_\omega^2\right)\right] \\
&\approx a_2(\mathcal{D}_\omega^2)f(0) + 2\sum_{0\le k<2}\Lambda^{2-k}a_k(\mathcal{D}_\omega^2)\frac{1}{\Gamma\left(\frac{2-k}{2}\right)}f_{2-k} + \mathcal{O}(\Lambda^{-1}) \\
&= a_2(\mathcal{D}_\omega^2)f(0) + 2\Lambda^2 a_0(\mathcal{D}_\omega^2)\frac{1}{\Gamma(1)}f_2 + \mathcal{O}(\Lambda^{-1}) \\
&= \int_{\mathcal{M}}\left(\frac{f_2\Lambda^2}{\pi} - \frac{f(0)}{24\pi}s\right)\sqrt{|g|}d^2x + \mathcal{O}(\Lambda^{-1})
\end{aligned}$$

Appendix D

Riemannian signature

If we take a positive definite metric $g = \mathrm{diag}(1, 1, 1)$, and L satisfying $L^{\mathrm{T}} g L = g$, then we may write $L = e^A$ where A is skew-symmetric, (i.e. $A^{\mathrm{T}} = -A$). Any such A may be written as a combination

$$A = \xi_1 K_1 + \xi_2 K_2 + \xi_3 K_3$$

where

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

form a basis for skew-symmetric 3×3 matrices with real entries. We may calculate the algebra relations

$$[K_1, K_2] = K_3, \quad [K_3, K_1] = K_2, \quad \text{and} \quad [K_2, K_3] = K_1.$$

Or, more compactly,

$$[K_a, K_b] = \varepsilon_{abc} K_c$$

where ε_{abc} is the usual Levi-Civita symbol.

Now, the spin representation of the Lorentz transformations in the Riemannian signature case will be

$$S(L) = \exp\left\{ \frac{1}{2} \sum_{a=1}^{3} \xi_a \tau_a \right\}$$

satisfying

$$S^{-1}(L) \gamma^{a'} S(L) = L_a^{a'} \gamma^a$$

for some matrices τ_a. To linear order, the right hand side yields

$$\delta\gamma^1 = \xi_1 \gamma^2 - \xi_2 \gamma^3, \quad \delta\gamma^2 = -\xi_1 \gamma^1 + \xi_3 \gamma^3, \quad \text{and} \quad \delta\gamma^3 = \xi_2 \gamma^1 - \xi_3 \gamma^2.$$

Meanwhile, also to linear order,

$$S(L) = 1 + \xi_1\tau_1 + \xi_2\tau_2 + \xi_3\tau_3, \quad \text{and} \quad S^{-1}(L) = 1 - \xi_1\tau_1 - \xi_2\tau_2 - \xi_3\tau_3$$

so that with the same level of accuracy

$$S^{-1}(L)\gamma^a S(L) = \gamma^a - \frac{1}{2}\xi_1[\tau_1, \gamma^a] - \frac{1}{2}\xi_2[\tau 2, \gamma^2] - \frac{1}{2}\xi_3[\tau_3, \gamma^a].$$

Comparing LHS and RHS we immediately see that

$$[\tau_3, \gamma^1] = 0, \quad [\tau_2, \gamma^2] = 0, \quad \text{and} \quad [\tau_1, \gamma^3] = 0.$$

So, taking

$$\gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma^3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and recalling the identity for the Pauli matrices which says that $[\sigma_a, \sigma_b] = \varepsilon_{abc}\sigma_c$, we find by comparing terms with like coefficients in the LHS and RHS that

$$\tau_1 = i\sigma_1, \quad \tau_2 = i\sigma_2, \quad \text{and} \quad \tau_3 = i\sigma_3.$$

In retrospect this is unsurprising since Lorentz transformations in Euclidean 3-space are just rotations, and we expect the rotation group $SU(2)$ to be our spin representation; and of course the infinitesimal generators may be taken to be the standard basis for the Lie algebra $\mathfrak{su}(2)$, (i.e. $i\sigma_1, i\sigma_2, i\sigma_3$).

Chapter 5

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