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Recursive methods of determination of 4-point blocks in $N = 1$ superconformal field theories

by

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Contents

Introduction

Two dimensional superconformal field theories with one supersymmetry ($N = 1$ SCFT) are supersymmetric generalizations of conformally invariant field theories (CFT). The intense activity in the subject of theories with conformal symmetry began with the work by Polyakov [1]. It was pointed out there that the conformal theories describe statistical systems at critical points. In order to calculate the critical exponents the non-Hamiltonian approach to conformal theories, the so called-bootstrap program, was proposed in [2]. The realization of this program in two dimensional CFT was presented in the work by Belavin, Polyakov and Zamolodchikov (BPZ) [3]. The BPZ work introduced the method of constructing minimal models *i.e.* the special examples of completely solvable conformal field theories. Soon the realization of superconformal invariance in quantum field theory and the superconformal minimal models were analyzed in [4], [5], [6].

Since another two papers by Polyakov [7], [8] the role of CFT in string theory was widely recognized [9], [10]. The string scattering amplitudes can be expressed in terms of correlation functions of (super)conformal field theories. An additional motivation for studies on conformal field theories comes from the AdS/CFT correspondence that has been rapidly developing field of research in the last years [11], [12], [13].

In CFT any n-point correlation function can be expressed by 3-point structure constants and conformal blocks [3]. The conformal blocks are universal functions completely determined by conformal symmetry. The 4-point conformal block is defined as a power series in projective invariant z. It is a function of central charge c, intermediate weight Δ and four external conformal weights Δ_i. Coefficients of the z-expansion are defined by the Gram matrices in Verma modules and the 3-point conformal block. The properties of these objects are well studied, nevertheless, the calculation of the block from definition is not effective and its general form is not known. There are however recursive methods of determination of the 4-point conformal block developed by Al. Zamolodchikov [14, 15, 16]. They are based on the fact that 4-point block can be expressed as a sum over poles in the central charge (or intermediate weight) and a term non singular in c (or in Δ, respectively) [14]. The residues are proportional to the block itself, which leads to recursion relations.

The first recursion relation, *i.e.* the z-recurrence, is related to the block's expansion as the sum over the poles in the central charge. In order to close the recurrence the term regular in c is necessary. It is given by the $c \to \infty$ limit of the 4-point block [14].

The second, more efficient method of determining 4-point block, *i.e.* the elliptic recurrence, can be derived by analyzing large Δ asymptotic of the block. According to Zamolodchikov's works [16], the first two terms in the large Δ expansion of the 4-point block can be read off from its classical asymptotic. A multiplicative factor related to these two terms can be separated from the 4-point block, which leads to a definition of the so-called elliptic block. The regular in Δ term of the elliptic block does not depend on external weights and central charge. Thus it can be determined from an explicit analytic formula of the block derived in a certain model [15], [17]. Since each elliptic block's residue in Δ is proportional to the elliptic block itself, knowing the regular term, one gets the second closed recursion relation for the coefficients.

The recursive methods of determination of a general 4-point conformal block allowed, for instance, for numerical consistency check of Liouville theory with 3-point structure constants proposed by Otto and Dorn [18] and by A. and Al. Zamolodchikov [19]. The methods were also used in the study of the $c \to 1$ limit of minimal models [20] or in obtaining new results in the classical geometry of hyperbolic surfaces [21, 22]. In a more general context of arbitrary CFT model, with the help of the recursive methods one can numerically calculate any 4-point function once the structure constants of the model are known.

The present thesis is devoted to the problem of definition and calculation of 4-point superconformal blocks in $N = 1$ SCFT. It is based on the results published in [23], [24], [25], [26]. In the first chapter we recall the derivation of the two recursive methods of determination of the 4-point conformal block [14, 15, 16] in detail. The ideas presented in this chapter are a basis for supersymmetric generalization. Let us note that we formulate the original Zamolodchikov's results in the language of chiral 3-forms. It turns out that the technique of identifying 3-point blocks as suitably normalized chiral 3-forms can be effectively extended to the supersymmetric case. It leads to a successful definition of all types of 4-point superconformal blocks.

In $N = 1$ superconformal field theories there exist two types of fields: the Neveu-Schwarz (NS) fields local with respect to fermionic current $S(z)$ and the Ramond fields "half-local" with respect to $S(z)$. We discuss in the first place the superconformal blocks corresponding to correlation functions of NS fields [23].

The superconformal Ward identities determine correlation functions up to two independent types of structure constants. In NS sector of SCFT there is however an important simplification: each given 3-point function of arbitrary NS fields is proportional to just one out of two structure constants. This implies similar as in non supersymmetric case definition and properties of 3-point NS blocks. Since the algebra in supersymmetric case is more general than in CFT, there are 2 types of 3-point NS blocks and 4 types of 4-point NS blocks. For each type of the superconformal blocks there is one even and one odd supersymmetric block. All the 3-point and the 4-point NS blocks are defined in the second chapter. Analyzing their properties one can check that it is possible to derive the recursive relations for 4-point NS blocks. As in the bosonic case, the 4-point superconformal blocks can be represented as a sum over poles in the central charge (or intermediate weight) and a term nonsingular in c (or in Δ, respectively). The residues of a given type of block are proportional to even and odd blocks of the same type. The term regular in c can be calculated as the $c \to \infty$ limit of the block. With the NS superconformal blocks correctly defined it is not difficult to obtain the generalization of the first recursive method for determining the 4-point blocks.

The derivation of the elliptic recursion is more complicated [24]. Analyzing supersymmetric Liouville theory one has to investigate the classical limit of the superconformal blocks. It turns out that the asymptotical behavior of all types of 4-point NS blocks is given by one universal block. It is the same classical block which is present in the limit of 4-point conformal block. In the classical limit the contribution from fermions is noticeable just in coefficients proportional to exponent of the classical block. Using the relation between large Δ asymptotic and the classical block one can calculate the first two terms in large Δ expansions of the NS blocks. The multiplicative factors related with these two terms describe the c and Δ_i dependence of the non singular in Δ parts of the superconformal blocks. Dividing the blocks by the multiplicative factors one can define superconformal elliptic blocks with terms regular in Δ which are independent of central charge and external weights. In order to compute the regular terms and complete the elliptic recursion one needs an explicit example of superconformal blocks with an arbitrary intermediate weight. In the last chapter we propose a model where such blocks can be calculated.

Before that, in the third chapter, the problem of 4-point superconformal blocks in Ramond sector is discussed [26]. We restrict our interest to the class of SCFT models where the Ramond fields have a common parity (for the left and the right sector) [4], [6]. We present in detail the case of the 4-point blocks corresponding to correlation functions of Ramond fields factorized on NS states. The other types of Ramond blocks are briefly discussed in the end of this chapter.

As in the NS sector, the supersymmetric Ward identities allow to reduce any correlator containing Ramond fields up to two independent structure constants. In this case, however, an arbitrary 3-point function is always given by a sum of two terms, each proportional to a different structure constant. The Ward identities have a more complex form because the correlation functions of the fermionic current $S(z)$, two Ramond fields and one NS field is double valued. Additional complication comes from the fact that the Ramond field operators correspond to states from irreducible representation of the tensor product $\mathcal{R} \otimes \bar{\mathcal{R}}$ of the left $\mathcal{R}$ and the right $\bar{\mathcal{R}}$ Ramond algebras extended by the common parity operator. Thus it is not obvious how one should express the 3-point correlation functions of Ramond fields by the 3-point blocks which are chiral objects with definite left (or right) parity. Nevertheless, using the technique of identifying the 3-point blocks as suitably normalized chiral 3-forms, it is possible to define 4-point Ramond blocks and to analyze their properties. As in the NS sector, there are four even and four odd 4-point Ramond blocks. The elliptic recurrence for the Ramond blocks can be investigated by the same method as in the case of NS blocks.

In the last chapter we recall Zamolodchikovs' calculation of a 4-point block in the $c = 1$

scalar theory extended by Ramond states of the free scalar current [15], [17]. Then, by adding free fermion current we obtain the supersymmetric generalization of this model [25]. The explicit formulae for the $c = \frac{3}{2}$ blocks are necessary in order to find the closed elliptic recursion relations for the general 4-point superconformal blocks. Additionally, we can use these formulae to check if our constructions of the 4-point blocks and the recursion relations are correct.

The recursive representations of the 4-point superconformal blocks in the $N = 1$ SCFT discussed in the present thesis yield approximate (with arbitrary accuracy), analytic expressions for general 4-point superconformal blocks. Some of the results were already used for numerical verification of the consistency of the $N = 1$ supersymmetric Liouville theory in the NS sector [27], [28]. A consistency check of the Ramond sector of $N = 1$ supersymmetric Liouville theory is not yet done.

Chapter 1

Conformal block in CFT

1.1 Basic definitions and notation

1.1.1 Operator Product Expansion

We shall consider two dimensional conformal field theories (CFT) defined on a complex plane. Within the BPZ formulation [3] the basic dynamical assumption comes under the name of operator product expansion. It can be formulated as follows

In an arbitrary correlation function the product of any two local operators can be expressed as a series of local operators

$$\varphi_i(z_2,\bar{z}_2)\varphi_j(z_1,\bar{z}_1) = \sum_k C^k_{ij}(z_2-z_1,\bar{z}_2-\bar{z}_1)\varphi_k(z_1,\bar{z}_1), \tag{1.1}$$

where the coefficients $C_{kij}(z_2-z_1,\bar{z}_2-\bar{z}_1)$ are c-number functions.

This is a strong version of the Wilson operator product expansion. It allows to express any correlator in terms of $C_{kij}(z_2-z_1,\bar{z}_2-\bar{z}_1)$ functions. There are two kinds of restrictions imposed on these functions. The first group follows form the conformal symmetry. It determines for instance the z and the $\bar{z}$ dependence of $C_{kij}(z_2-z_1,\bar{z}_2-\bar{z}_1)$. The second is a consequence of the operator product expansion. Since the left hand side of (1.1) is associative, the OPE coefficients should respect associativity as well. This leads to non linear equations imposing strong constraints on $C_{kij}(z_2-z_1,\bar{z}_2-\bar{z}_1)$. The idea to construct a CFT model by solving the symmetry and the associativity requirements is called the conformal bootstrap program. It was proposed by Polyakov [2] and to large extend realized by BPZ [3].

1.1.2 Conformal symmetry and Ward identities

In two dimensions there exists an infinite parameter family of local conformal transformations

$$\mathbb{C} \supset U \ni z \to f(z) \in \mathbb{C}$$

leaving the standard metric $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$ unchanged up to a scale factor. Such maps are analytic functions $w = f(z)$ so that

$$\mathrm{d}w\mathrm{d}\bar{w} = |f'(z)|^2\mathrm{d}z\mathrm{d}\bar{z}.$$

The conformal transformations that map the compactified complex plane $\mathbb{C}^* \equiv \mathbb{C}\cup\{\infty\}$ onto itself are called global conformal transformations or projective transformations and form a group isomorphic to $PSL(2,\mathbb{C})$. Projective transformations can be parameterized as

$$f(z) = \frac{az+b}{cz+d}\,, \qquad ad - bc = 1$$

where a, b, c, d are complex numbers. In CFT one implements both global and local conformal symmetries [1].

In the Lagrangian formulation the scaling invariance of a theory leads to a traceless energy-momentum tensor. This condition together with the standard continuity equation imply that the non vanishing components of energy-momentum tensor on the complex plane are holomorphic and antiholomorphic functions:

$$T^{zz} \equiv T = T(z), \qquad \bar{T}^{\bar{z}\bar{z}} \equiv \bar{T} = \bar{T}(\bar{z}).$$

Since the energy momentum tensor is a generator of local coordinate transformation, one can assume that in a general CFT model:

There exist an holomorphic $T(z)$ and an antiholomorphic $\bar{T}(\bar{z})$ fields which are generators of conformal symmetry:

$$\delta_{\epsilon,\bar{\epsilon}}\varphi(w,\bar{w}) = \frac{1}{2\pi i}\oint_w \mathrm{d}z\ \epsilon(z)T(z)\varphi(w,\bar{w}) + \frac{1}{2\pi i}\oint_{\bar{w}} \mathrm{d}\bar{z}\ \bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\varphi(w,\bar{w}). \tag{1.2}$$

where $\delta_{\epsilon,\bar{\epsilon}}\varphi(w,\bar{w})$ is the variation of a local field $\varphi(w,\bar{w})$ with respect to infinitesimal conformal transformation $z \to z+\epsilon(z), \bar{z} \to \bar{z}+\bar{\epsilon}(\bar{z})$.

We assume that the algebra of local fields contains *primary fields* which under conformal transformation $z \to w(z)$ change in a particularly simple way:

$$\phi'_{\Delta,\bar{\Delta}}(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-\Delta}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{\Delta}}\phi_{\Delta,\bar{\Delta}}(z,\bar{z}), \tag{1.3}$$

where the parameters $\Delta, \bar{\Delta}$ are called the holomorphic (left) and the antiholomorphic (right) conformal weight. From this definition it follows that the variation of the primary field with respect to the infinitesimal transformations has the form:

$$\delta_{\epsilon,\bar{\epsilon}}\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) = \left(\Delta\partial\epsilon(w) + \bar{\Delta}\bar{\partial}\bar{\epsilon}(\bar{w}) + \epsilon(w)\partial + \bar{\epsilon}(\bar{w})\bar{\partial}\right)\phi_{\Delta,\bar{\Delta}}(w,\bar{w}).$$

The equation (1.2) implies:

$$\begin{aligned} T(z)\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) &= \frac{\Delta}{(z-w)^2}\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) + reg. \qquad (1.4)\\ \overline{T}(\bar{z})\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) &= \frac{\bar{\Delta}}{(\bar{z}-\bar{w})^2}\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi_{\Delta,\bar{\Delta}}(w,\bar{w}) + reg. \end{aligned}$$

[1]For an extensive introduction to CFT one can consult for example [29], [30], [31], [10]

These are the local Ward identities for the primary field with the conformal weights Δ, $\bar{\Delta}$.

In the known Lagrangian models the transformation law for generator $T(z)$ takes the form:

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2} \left[T(z) - \frac{c}{12}\{w;z\}\right],$$

where $\{w;z\}$ is the Schwarz derivative:

$$\{w;z\} = \frac{(\mathrm{d}^3w/\mathrm{d}z^3)}{(\mathrm{d}w/\mathrm{d}z)} - \frac{3}{2}\left(\frac{\mathrm{d}^2w/\mathrm{d}z^2}{\mathrm{d}w/\mathrm{d}z}\right)^2.$$

The second term is proportional to the central charge c which is the parameter of the theory. Note that for global conformal transformations the Schwarz derivative is zero.

The transformation law and the equation (1.2) for generators $T(z), \overline{T}(\bar{z})$ lead to the local conformal Ward identities for the generator $T(z)$:

$$\begin{aligned} T(z)T(0) &= \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + reg. \\ \bar{T}(z)\bar{T}(0) &= \frac{c}{2\bar{z}^4} + \frac{2}{\bar{z}^2}\bar{T}(0) + \frac{1}{\bar{z}}\bar{\partial}\bar{T}(0) + reg. \\ \overline{T}(z)T(0) &= reg. \end{aligned} \tag{1.5}$$

We assume this form of the local conformal Ward identities in a general CFT.

The operators

$$L_n = \frac{1}{2\pi i}\oint_0 \mathrm{d}z\ z^{n+1}T(z), \qquad \bar{L}_n = \frac{1}{2\pi i}\oint_0 \mathrm{d}\bar{z}\ \bar{z}^{n+1}\bar{T}(\bar{z}) \tag{1.6}$$

form two copies of the Virasoro algebra:

$$\begin{aligned} [L_n, L_m] &= (n-m)\ L_{n+m} + \frac{c}{12}n(n^2-1)\ \delta_{n+m,0} \\ [L_n, \bar{L}_m] &= 0 \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\ \bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\ \delta_{n+m,0} \end{aligned} \tag{1.7}$$

We will call L_n and $\bar{L}_n$ as the left and the right Virasoro generators, respectively.

1.1.3 Verma module

The state $|\nu_\Delta\rangle$ is called *the highest weight state with weight* Δ if it satisfies the following conditions:

$$L_m\,|\nu_\Delta\rangle = 0, \qquad L_0\,|\nu_\Delta\rangle = \Delta\,|\nu_\Delta\rangle\,, \qquad m > 0. \tag{1.8}$$

A *descendant state* is defined as a state created by an action of operators L_{-M} on the highest weight state. The descendant states form vector space $\mathcal{V}^n_{\Delta,c}$ with the basis:

$$|\nu_{\Delta,M}\rangle = L_{-M}\ |\nu_\Delta\rangle\ \equiv\ L_{-m_j}\ldots L_{-m_1}\,|\nu_\Delta\rangle\ , \tag{1.9}$$

where $M = \{m_1, m_2, \ldots, m_j\} \subset \mathbb{N}$ is arbitrary ordered set of indices $m_j \leq \ldots \leq m_2 \leq m_1$, such that $|M| = m_1 + \ldots + m_j = n$. Each $\mathcal{V}^n_{\Delta,c}$ is an eigenspace of L_0 with the eigenvalue $\Delta + n$. The direct sum of such spaces over all levels of excitations n composes *Verma module* $\mathcal{V}_{\Delta,c}$ *i.e.* the highest weight representation of the Virasoro algebra:

$$\mathcal{V}_{\Delta,c} = \bigoplus_{n\in\mathbb{N}\cup\{0\}} \mathcal{V}^n_{\Delta,c} \qquad \mathcal{V}^0_{\Delta,c} = \mathbb{C}\,\nu_\Delta\,,$$

As a scalar product on $\mathcal{V}_{\Delta,c}$ we can choose a symmetric bilinear form $\langle .\,|.\rangle_{c,\Delta}$ on $\mathcal{V}_{\Delta,c}$ such that

$$\langle \nu_\Delta\,|\nu_\Delta\rangle = 1, \quad L_n^\dagger = L_{-n}.$$

The operator L_0 is hermitean with respect to $\langle .\,|.\rangle_{c,\Delta}$ what ensures reality of conformal weights.

1.1.4 Gram matrix

The Gram matrix is a matrix of $\langle .\,,.\rangle_{c,\Delta}$ for each subspace $\mathcal{V}^n_{\Delta,c}$ calculated in the basis (1.9):

$$\left[B^n_{c,\Delta}\right]_{M,N} = \langle \nu_{\Delta,M}, \nu_{\Delta,N}\rangle_{c,\Delta}. \tag{1.10}$$

It has the following properties [32], [33]:

1. *The determinant of Gram matrix matrix is given by Kac theorem*

$$\det B^n_{c,\Delta} = C \prod_{1\leqslant rs\leqslant n} (\Delta - \Delta_{rs})^{p(n-rs)} \tag{1.11}$$

where C does not depend of Δ, c and Δ_{rs} weight has the following form:

$$\begin{aligned} \Delta_{rs}(c) &= -\frac{rs-1}{2} + \frac{r^2-1}{4}\beta^2 + \frac{s^2-1}{4}\frac{1}{\beta^2}\,, \\ \beta &= \frac{1}{\sqrt{24}}\left(\sqrt{1-c} + \sqrt{25-c}\right), \end{aligned} \tag{1.12}$$

As a function of c Kac determinant vanishes at

$$c = c_{rs}(\Delta) \equiv 1 - 6\left(\beta_{rs}(\Delta) - \frac{1}{\beta_{rs}(\Delta)}\right)^2, \tag{1.13}$$

where $r, s \in \mathbb{Z}, \quad r \geq 2,\ s \geq 1, \quad 1 \leq rs \leq n$, and

$$\beta^2_{rs}(\Delta) = \frac{1}{r^2-1}\left(rs - 1 + 2\,\Delta + \sqrt{(r-s)^2 + 4\,(r\,s-1)\,\Delta + 4\,\Delta^2}\right)$$

The multiplicity of each zero in (1.11) is given by: $p(n-rs) = \dim \mathcal{V}^{n-rs}_{c,\Delta}$.

2. *The Gram matrix is nonsingular if and only if $\mathcal{V}^n_{\Delta,c}$ does not contain singular vectors of degrees $rs \leq n$.*

The singular vector χ_{rs} is a descendant state from $\mathcal{V}^{rs}_{c,\Delta_{rs}}$ which is in the same time the highest weight state satisfying condition (1.8) with $L_0\chi_{rs} = (\Delta_{rs} + rs)\chi_{rs}$. It generates a singular subspace $\mathcal{V}^{n-rs}_{c,\Delta_{rs}+rs} \subset \mathcal{V}^n_{c,\Delta_{rs}}$, which consists of vectors ξ orthogonal to any vector $\zeta \in \mathcal{V}^n_{c,\Delta_{rs}}$: $\langle \xi\,, \zeta\rangle_{c,\Delta_{rs}} = 0$.

3. *The only singularities of inverse Gram matrix* $\left[B^n_{c,\Delta}\right]^{M,N}$ *are poles of first order at* $\Delta = \Delta_{rs}$.

   If the Verma module $\mathcal{V}_{c,\Delta_{rs}+rs}$ is not reducible for all $rs \le n$, then the multiplicity of each zero of the Kac determinant coincides with the dimension of the singular subspace $\mathcal{V}^{n-rs}_{c,\Delta_{rs}+rs}$ and the following lemma applies:

   Let $A(\delta)$ be a family of linear operators acting in n-dimensional space V and let $A(\delta)$ be a polynomial function of δ. If the order of the zero of $det A(\delta)$ at $\delta = 0$ equals the dimension of the null space of $A(0)$, then in an arbitrary basis each matrix elements of $A(\delta)^{-1}$ has at most a simple pole at $\delta = 0$.

   The same pole structure is true for the inverse Gram matrix as a function of the central charge c.

1.1.5 The space of states

We assume that there exist a unique vacuum state $|0\rangle$ *i.e.* the highest weight state invariant with respect to the global conformal transformations generated by L_{-1}, L_0, L_1.

Let us consider a state generated by the primary field $\phi_{\Delta,\bar{\Delta}}$ acting on the vacuum:

$$\lim_{z,\bar{z}\to 0} \phi_{\Delta,\bar{\Delta}}(z,\bar{z})\,|0\rangle \;=\; \left|\Delta,\bar{\Delta}\right\rangle . \tag{1.14}$$

The bra state $\left\langle\Delta,\bar{\Delta}\right|$ is defined as

$$\left\langle\Delta,\bar{\Delta}\right| = \lim_{z,\bar{z}\to 0} \left[\phi_{\Delta,\bar{\Delta}}(z,\bar{z})|0\rangle\right]^{\dagger} \equiv \langle 0|\phi_{\Delta,\bar{\Delta}}\left(\infty,\infty\right), \tag{1.15}$$

where the hermitean conjugated primary field is defined as follows

$$\left[\phi_{\Delta,\bar{\Delta}}(z,\bar{z})\right]^{\dagger} = \bar{z}^{-2\Delta} z^{-2\bar{\Delta}} \phi_{\Delta,\bar{\Delta}}\left(\frac{1}{\bar{z}},\frac{1}{z}\right).$$

Such a definition of the conjugated field can be justified by considering the continuation to the Minkowski space cylinder [30]. The time reversal $\sigma^0 \to -\sigma^0$ on the cylinder by the map $z = \mathrm{e}^{\sigma^0+i\sigma^1}$, $\bar{z} = \mathrm{e}^{\sigma^0-i\sigma^1}$ becames $z \to \frac{1}{\bar{z}}$. The additional $z,\bar{z}$ dependent factors are necessary to ensure the proper transformation properties of the conjugated field with respect to the conformal group.

Any two point correlation function of primary fields is determined by the global conformal transformations up to a constant:

$$\langle 0|\,\phi_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2)\phi_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)\,|0\rangle = D_{21}\delta_{\Delta_1,\Delta_2}\delta_{\bar{\Delta}_1,\bar{\Delta}_2}(z_2-z_1)^{-2\Delta_1}(\bar{z}_2-\bar{z}_1)^{-2\bar{\Delta}_1}.$$

We impose normalization of primary fields, what leads to the condition for the 2-point correlators: $D_{21} \equiv 1$. Thus the states defined by (1.14),(1.15) are normalized:

$$\left\langle\Delta,\bar{\Delta}\right|\Delta,\bar{\Delta}\rangle = 1.$$

From the definition of the Virasoro generators (1.6) and OPE of $T(z)$ and $\overline{T}(\bar{z})$ with the primary field $\phi_{\Delta,\bar{\Delta}}$ (1.4), it follows that the state $\left|\Delta,\bar{\Delta}\right\rangle$ is a highest weight state with respect to the left and to the right Virasoro algebras:

$$\left|\Delta,\bar{\Delta}\right\rangle = \left|\nu_\Delta \otimes \bar{\nu}_{\bar{\Delta}}\right\rangle,$$

what implies for the bra state: $\left\langle\Delta,\bar{\Delta}\right| = \left|\nu_\Delta \otimes \bar{\nu}_{\bar{\Delta}}\right\rangle^\dagger$.

The states created by the action of the Virasoro generators on $\left|\Delta,\bar{\Delta}\right\rangle$ form the tensor product of Verma modules $\mathcal{V}_{c,\Delta}$ and $\overline{\mathcal{V}}_{\bar{\Delta},c}$. We assume that all the states in CFT are of this type. *The space of states in a conformal field theory is a sum of the tensor products of the left and the right Verma modules:*

$$\mathcal{H}_c = \bigoplus_{(\Delta,\bar{\Delta})} \mathcal{V}_{\Delta,c} \otimes \bar{\mathcal{V}}_{\bar{\Delta},c}.$$

$(\Delta,\bar{\Delta})$ are the pairs of conformal weights of the corresponding primary fields present in the theory. The set of conformal weights $(\Delta,\bar{\Delta})$ is called the spectrum of primary fields.

1.1.6 Field operators

The last assumption that will be made concerns the states-fields correspondence and can be formulated in the following way:

In CFT there is one to one correspondence between the states from the space of states $\mathcal{H}_c$ and the field operators from the space of fields.

Each primary field $\phi_{\Delta,\bar{\Delta}}$ is related to the state $\left|\Delta,\bar{\Delta}\right\rangle$ (1.14). The fields corresponding to the states $\xi_\Delta \otimes \bar{\xi}_{\bar{\Delta}} \in \mathcal{V}_{\Delta,c} \otimes \bar{\mathcal{V}}_{\bar{\Delta},c}$ are called *descendant fields*:

$$\lim_{z,\bar{z}\to 0} \varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z})\left|0\right\rangle = \left|\xi_\Delta \otimes \bar{\xi}_{\bar{\Delta}}\right\rangle.$$

The action of the Virasoro generators L_{-m} on states extends by the correspondence to the action on the fields and has the form:

$$\mathcal{L}_{-m}\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}) \equiv \varphi_{\Delta,\bar{\Delta}}(L_{-m}\xi,\bar{\xi}|z,\bar{z}) = \oint_z \mathrm{d}w\, \frac{T(w)}{(w-z)^{m-1}}\, \varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}). \tag{1.16}$$

This relation is the definition of the descendant field. The descendant fields together with appropriate primary field $\phi_{\Delta,\bar{\Delta}} = \varphi_{\Delta,\bar{\Delta}}(\nu,\bar{\nu}|z,\bar{z})$ constitute a conformal family $[\phi_{\Delta,\bar{\Delta}}]$. Any field in the space of fields belongs to some conformal family $[\phi_{\Delta,\bar{\Delta}}]$ with $(\Delta,\bar{\Delta})$ from the spectrum of primary fields. For example, the identity operator is the primary field with both left and right conformal weights equal zero. $T(z)$, $\overline{T}(\bar{z})$ are descendants of the identity operator, with weights $(2,0)$ and (0,2) respectively.

1.1.7 Correlation functions

Let us consider arbitrary correlation function containing a descendant field corresponding to a state $L_{-M}\left|\Delta,\bar{\Delta}\right\rangle$. Using the definition (1.16) one can write the descendant as a multiple integral of the form:

$$\mathcal{L}_{-m_n}\dots\mathcal{L}_{-m_1}\phi_{\Delta,\bar{\Delta}}(\nu,\bar{\nu}|z,\bar{z}) = \oint_{\{w_i,z\}}\dots\oint_z \frac{\mathrm{d}w_n}{2\pi i}\dots\frac{\mathrm{d}w_1}{2\pi i}\frac{T(w_n)}{(w_n-z)^{m_n-1}}\dots\frac{T(w_1)}{(w_1-z)^{m_1-1}}\,\phi_{\Delta,\bar{\Delta}}(\nu,\bar{\nu}|z,\bar{z})$$

The integral around location of this field can be written as a sum of integrals with contours around locations of the other fields in the correlator. By such a contour deformation one can express the action of $\mathcal{L}_{-m}$ on one field by a linear combination of $\mathcal{L}_n$ with $n \geq -1$ acting on the other fields. Using this method one can derive the relations between correlation functions *i.e.* the conformal Ward identities. The commutator of the operators acting on a field is given by the formula following from (1.16) and OPE (1.5):

$$[\mathcal{L}_n,\mathcal{L}_{-m}]\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}) = (n+m)\mathcal{L}_{n-m}\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}) + \frac{c}{12}\,n(n^2-1)\delta_{n,m}\,\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}). \tag{1.17}$$

One can see that due to the contour deformation procedure it is possible to obtain the correlator of fields from lower levels of excitation. Using Ward identities one can express any correlation function of descendants as a linear differential operator acting on the correlator of primary field.

Moreover, thanks to the basic dynamical assumption of CFT (1.1), the n-point correlators can be reduced to 3-point correlation functions. The global conformal transformations $\mathrm{SL}(2,\mathbb{C})$ determine the $z_i, \bar{z}_i$ dependence of the 3-point functions [30]:

$$\langle 0|\,\phi_3(z_3,\bar{z}_3)\phi_2(z_2,\bar{z}_2)\phi_1(z_1,\bar{z}_1)\,|0\rangle = C_{321}\prod_{p>q}(z_p-z_q)^{-\Delta_{pq}}(\bar{z}_p-\bar{z}_q)^{-\bar{\Delta}_{pq}} \tag{1.18}$$

where $\Delta_{32} = \Delta_3+\Delta_2-\Delta_1$, *ect.* The structure constants C_{321} are 3-point correlation functions of primary fields in the standard locations $0, 1, \infty$:

$$C_{321} \equiv \langle 0|\,\phi_3(\infty,\infty)\phi_2(1,1)\phi_1(0,0)\,|0\rangle = \left\langle \nu_{\Delta_3}\otimes\bar{\nu}_{\bar{\Delta}_3}\right|\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar{\nu}_{\bar{\Delta}_1}\right\rangle.$$

Notice that thanks to the correspondence between fields and states (1.14), (1.15), the correlation function can be written as a matrix element between two primary states.

In general one can say that when the structure constants C_{321} in a given theory are known then due to conformal symmetry it is possible to calculate any correlation function in this theory. In practice however it is very non-trivial problem to determine the correlation functions even in the case of four primary fields.

The global conformal transformations enable to fix three locations of the fields in a correlator [30]:

$$\langle 0|\,\phi_4(z_4,\bar{z}_4)\phi_3(z_3,\bar{z}_3)\phi_2(z_2,\bar{z}_2)\phi_1(z_1,\bar{z}_1)\,|0\rangle = \prod_{i>j}(z_i-z_j)^{-(\Delta_i+\Delta_j)+\frac{\Delta}{3}}(\bar{z}_i-\bar{z}_j)^{-(\bar{\Delta}_i+\bar{\Delta}_j)+\frac{\bar{\Delta}}{3}}\,\langle 0|\,\phi_4(\infty,\infty)\phi_3(1,1)\phi_2(z,\bar{z})\phi_1(0,0)\,|0\rangle$$

where $\Delta = \sum_{i=1}^4 \Delta_i$, $\bar\Delta = \sum_{i=1}^4 \bar\Delta_i$ and $z, \bar z$ are the projective invariants

$$z = \frac{z_{21}z_{43}}{z_{31}z_{42}}, \quad \bar z = \frac{\bar z_{21}\bar z_{43}}{\bar z_{31}\bar z_{42}}, \qquad \text{where} \quad z_{ij} = z_i - z_j, \bar z_{ij} = \bar z_i - \bar z_j. \tag{1.19}$$

Such a 4-point function reduces to the structure constants and functions completely determined by the symmetry, which are called 4-point conformal blocks:

$$\langle 0|\, \phi_4(\infty,\infty)\phi_3(1,1)\phi_2(z,\bar z)\phi_1(0,0)\,|0\rangle = \sum_p C_{43p}C_{p21}\left|\mathcal{F}_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2.$$

In order to present exact definition of 4-point conformal block and Zamolodchikov's recursive methods of determining these objects we have to analyze properties of 3-point correlation functions in more detail.

1.2 The 3-point block

1.2.1 Ward identities for the 3-point correlation function

We shall apply the contour deformation procedure discussed in the previous section to the 3-point correlation function of descendant fields. For $m > 1$ we have:

$$\begin{aligned}
&\left\langle \xi_3,\bar\xi_3\right|\varphi_2(L_{-m}\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle \\
&\quad= \oint_z \frac{dw}{2\pi i}(w-z)^{1-m}\langle\, \varphi_3(\xi_3,\bar\xi_3|\infty,\infty)\,T(w)\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\,\varphi_1(\xi_1,\bar\xi_1|0,0)\,\rangle \qquad (1.20)\\
&\quad= \oint_\infty \frac{dw}{2\pi i}\sum_{n=0}^\infty \binom{1-m}{n}(-z)^n w^{1-m-n}\,\langle\, \varphi_3(\xi_3,\bar\xi_3|\infty,\infty)\,T(w)\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\,\varphi_1(\xi_1,\bar\xi_1|0,0)\,\rangle\\
&\quad- \oint_0 \frac{dw}{2\pi i}\sum_{n=0}^\infty \binom{1-m}{n}(-z)^{1-m-n} w^{n}\,\langle\, \varphi_3(\xi_3,\bar\xi_3|\infty,\infty)\,\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\,T(w)\varphi_1(\xi_1,\bar\xi_1|0,0)\,\rangle
\end{aligned}$$

Using definition (1.16) we can write:

$$T(w)\varphi(\xi,\bar\xi|0,0) = \sum_{n\in\mathbb{Z}} w^{n-2}\varphi(L_{-n}\xi,\bar\xi|0,0).$$

Inserting this OPE into integrals above one gets the Ward identity:

$$\begin{aligned}
\left\langle \xi_3,\bar\xi_3\right|\varphi_2(L_{-m}\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle &= \sum_{n=0}^\infty \binom{m-2+n}{n} z^n \left\langle L_{m+n}\xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle\\
&+ (-1)^m\sum_{n=0}^\infty \binom{m-2+n}{n} z^{-m+1-n}\left\langle \xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|L_{n-1}\xi_1,\bar\xi_1\right\rangle
\end{aligned}$$

In the same way one can derive the other Ward identities for 3-point correlation functions:

$$\begin{aligned}
\left\langle \xi_3,\bar\xi_3\right|\varphi_2(L_{m}\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle &= \sum_{n=0}^{m+1} \binom{m+1}{n}(-z)^n\Big(\left\langle L_{n-m}\xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle\\
&- \left\langle \xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|L_{m-n}\xi_1,\bar\xi_1\right\rangle \qquad m > -1,
\end{aligned}$$

and

$$\langle L_{-n}\xi_3, \bar{\xi}_3 | \varphi_{\Delta_2,\bar{\Delta}_2}(\xi_2, \bar{\xi}_2|z,\bar{z})| \xi_1, \bar{\xi}_1 \rangle = \langle \xi_3, \bar{\xi}_3 | \varphi_{\Delta_2,\bar{\Delta}_2}(\xi_2, \bar{\xi}_2|z,\bar{z})| L_n\xi_1, \bar{\xi}_1 \rangle + \sum_{m=-1}^{l(n)} \begin{pmatrix} n+1 \\ m+1 \end{pmatrix} z^{n-m} \langle \xi_3, \bar{\xi}_3 | \varphi_{\Delta_2,\bar{\Delta}_2}(L_m\xi_2, \bar{\xi}_2|z,\bar{z})| \xi_1, \bar{\xi}_1 \rangle$$

where $l(n) = n$ for $n+1 \geqslant 0$, and $l(n) = \infty$ for $n+1 < 0$. Additionally, since the L_{-1} operator is the generator of translations we have:

$$\left\langle \xi_3, \bar{\xi}_3 \right| \varphi_2(L_{-1}\xi_2, \bar{\xi}_2|z,\bar{z}) \left| \xi_1, \bar{\xi}_1 \right\rangle = \partial_z \left\langle \xi_3, \bar{\xi}_3 \right| \varphi_2(\xi_2, \bar{\xi}_2|z,\bar{z}) \left| \xi_1, \bar{\xi}_1 \right\rangle$$

Using these relations it is possible to take off a creation operator ($\mathcal{L}_{-n}$) from one field and change it to some combination of anihilation operators ($\mathcal{L}_n$) and $\mathcal{L}_0$,$\mathcal{L}_{-1}$ acting on the other fields. With the help of commutation relations (1.7) or (1.17), step by step one can get rid of all the left Virasoro creation operators.

The analogous Ward identities containing right Virasoro generators can be derived as well. Since $L_{-m}, \bar{L}_{-m}$ commute (1.7), one can take off left and right operators independently. Thus the Ward identities allow to reduce the 3-point function of descendant fields to the structure constant and the functions determined by the symmetry. The latter functions factorize into holomorphic and antiholomorphic part.

1.2.2 Definition of the 3-point block

We shell define the chiral trilinear map on Verma modules:

$$\varrho(\xi_3, \xi_2, \xi_1|z) : \mathcal{V}_{\Delta_3} \times \mathcal{V}_{\Delta_2} \times \mathcal{V}_{\Delta_1} \mapsto \mathbb{C} , \tag{1.21}$$

such that 3-point function could be written in terms of it:

$$\langle \xi_3, \bar{\xi}_3 | \varphi_{\Delta_2,\bar{\Delta}_2}(\xi_2, \bar{\xi}_2|z,\bar{z})| \xi_1, \bar{\xi}_1 \rangle = \varrho(\xi_3, \xi_2, \xi_1|z)\varrho(\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_1|\bar{z}).$$

From the Ward identities for 3-point correlation function we can derive conditions that the form (1.21) has to obey [34]:

$$\varrho(\xi_3, L_{-1}\xi_2, \xi_1|z) = \partial_z \varrho(\xi_3, \xi_2, \xi_1|z), \tag{1.22}$$

$$\varrho(\xi_3, L_n\xi_2, \xi_1|z) = \sum_{m=0}^{n+1} \binom{n+1}{m} (-z)^m \Big(\varrho(L_{m-n}\xi_3, \xi_2, \xi_1|z) - \varrho(\xi_3, \xi_2, L_{n-m}\xi_1|z) \Big), \quad n > -1, \tag{1.23}$$

$$\varrho(\xi_3, L_{-n}\xi_2, \xi_1|z) = \sum_{m=0}^{\infty} \binom{n-2+m}{n-2} z^m \varrho(L_{n+m}\xi_3, \xi_2, \xi_1|z) + (-1)^n \sum_{m=0}^{\infty} \binom{n-2+m}{n-2} z^{-n+1-m} \varrho(\xi_3, \xi_2, L_{m-1}\xi_1|z), \quad n > 1, \tag{1.24}$$

$$\varrho(L_{-n}\xi_3,\xi_2,\xi_1|z) = \varrho(\xi_3,\xi_2,L_n\xi_1|z) + \sum_{m=-1}^{l(n)} \left(\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix}\right) z^{n-m}\varrho(\xi_3,L_m\xi_2,\xi_1|z), \tag{1.25}$$

where $l(m)=m$ for $m+1\geqslant 0$, and $l(m)=\infty$ for $m+1<0$.

The form $\varrho(\xi_3,\xi_2,\xi_1|z)$ is almost completely determined by the constraints above. In particular, for L_0-eingenstates, $L_0\,\xi_i=\Delta_i(\xi_i)\xi_i,\ i=1,2,3$, the z dependence is fixed:

$$\varrho(\xi_3,\xi_2,\xi_1|z) = z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\,\varrho(\xi_3,\xi_2,\xi_1|1). \tag{1.26}$$

For any descendants states ξ_i one can use formulae (1.22)-(1.25) to express $\varrho(\xi_3,\xi_2,\xi_1|z)$ in terms of one constant $\varrho(\nu_3,\nu_2,\nu_1|1)$, where ν_3,ν_2,ν_1 are primary states in modules $\mathcal{V}_{\Delta_3},\mathcal{V}_{\Delta_2},\mathcal{V}_{\Delta_1}$ respectively.

The 3-point block is defined as normalized 3-form $\rho(\xi_3,\xi_2,\xi_1|z)$:

$$\varrho(\xi_3,\xi_2,\xi_1|z)\equiv\rho(\xi_3,\xi_2,\xi_1|z)\varrho(\nu_3,\nu_2,\nu_1|1). \tag{1.27}$$

The normalization condition simply means:

$$\rho(\nu_3,\nu_2,\nu_1|1) = 1.$$

The 3-point correlation function can thus be written in terms of the 3-point blocks:

$$\langle\,\xi_3,\bar\xi_3\,|\varphi_{\Delta_2,\bar\Delta_2}(\xi_2,\bar\xi_2|z,\bar z)|\,\xi_1,\bar\xi_1\,\rangle = \rho(\xi_3,\xi_2,\xi_1|z)\,\rho(\bar\xi_3,\bar\xi_2,\bar\xi_1|\bar z)\,C_{321} \tag{1.28}$$

where the structure constant:

$$C_{321} = \varrho(\nu_3,\nu_2,\nu_1|1)\,\varrho(\bar\nu_3,\bar\nu_2,\bar\nu_1|1).$$

1.2.3 Chiral vertex operator

The 3-point block is a chiral object in terms of which the correlation function of three fields can be expressed. Let us define now a chiral object that corresponds to individual field.

For any state $\xi_2\in\mathcal{V}_{\Delta_2}$ we define the chiral vertex operator [2]

$$V(\xi_2|z)\,:\,\mathcal{V}_{\Delta_1}\to\mathcal{V}_{\Delta_3}$$

through its matrix elements:

$$\langle\xi_3|\,V(\xi_2|z)\,|\xi_1\rangle\equiv\rho(\xi_3,\xi_2,\xi_1|z).$$

The relation between a field and the vertex operators has the form:

$$\varphi_{\Delta_2,\bar\Delta_2}(\xi_2,\bar\xi_2|z,\bar z) = \bigoplus_{\Delta_3,\Delta_1} C_{321}V(\xi_2|z)\otimes V(\bar\xi_2|\bar z).$$

[2]The basic facts on the vertex operators can be found in [35], [36], [37] [38]

In what follows we will focus on 3-point correlation functions with two primary fields and one descendant located either at zero or at infinity. Thus it is sufficient to consider vertex operator corresponding to highest weight state $V(\nu_2|z)$, for which commutation relations with Virasoro generators are given by (cf. (1.4), (1.25)):

$$[\mathrm{L}_m, V(\nu_2|z)] \;=\; z^m\,(z\partial_z + (m+1)\Delta_2)\,V(\nu_2|z). \tag{1.29}$$

The relations above determine completely the form of the 3-point block with one descendant state from level $f = |M| = |m_1 + \ldots + m_j|$:

$$\rho(L_{-M}\nu_3, \nu_2, \nu_1|z) = \langle L_{-M}\nu_3|\,V(\nu_2|z)\,|\nu_1\rangle = z^{\Delta_3+|M|-\Delta_2-\Delta_1}\gamma_{\Delta_3}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_M$$

where

$$\gamma_\Delta\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_M \stackrel{\text{def}}{=} (\Delta-\Delta_1+m_1\Delta_2)\,(\Delta-\Delta_1+m_2\Delta_2+m_1)\cdots\left(\Delta-\Delta_1+m_j\Delta_2+\sum_{l=1}^{j-1}m_l\right). \tag{1.30}$$

Similarly, the commutation relations (1.29) allow to find the 3-point block with descendant state $|L_{-M}\nu_1\rangle$:

$$\rho(\nu_3, \nu_2, L_{-M}\nu_1|z) = z^{\Delta_3-\Delta_2-\Delta_1-|M|}\gamma_{\Delta_1}\begin{bmatrix}\Delta_2\\ \Delta_3\end{bmatrix}_M,$$

what gives

$$\rho(\nu_3, \nu_2, L_{-M}\nu_1|1) = \rho(L_{-M}\nu_1, \nu_2, \nu_3|1).$$

As a function of each conformal weight the 3-point block $\rho(L_{-M}\nu_3, \nu_2, \nu_1|1)$ is thus a polynomial of maximal degree equal to the number of creation operators j.

Let us stress one more important property of the 3-point block *i.e.* the factorization. From the commutation relation (1.29) one can see that shifting each creation operator L_{-m} from left side of the vertex to the right gives one multiplicative factor. This factor does not depend on the action of the other operators on the state ν_3, it depends just on the level of descendant state. Therefore we can stop process of removing creation operators from descendant state in any moment $\xi_3 \in \mathcal{V}^n_{\Delta_3}$ and as a result we will obtain the same polynomial as if the initial state on right was primary one with shifted weight $\Delta_3 + n$:

$$\varrho(L_{-M}\xi_3, \nu_2, \nu_1|z) \;=\; \rho(L_{-M}\nu_{\Delta_3+n}, \nu_2, \nu_1|z)\;\varrho(\xi_3, \nu_2, \nu_1|1). \tag{1.31}$$

1.2.4 Fusion rules and fusion polynomials

The null vector χ_{rs} appears in Verma module $\mathcal{V}_{\Delta_{rs}}$ for degenerate weight Δ_{rs} given by a location of a zero of Kac determinant (1.11). Such a vector is orthogonal to any state, in particular it has zero norm. The field which corresponds to the null state is called the *zero field*. Any correlation function which includes this field vanishes.

Let us consider the 3-point function with a zero field $\varphi(\chi_{rs}, \bar{\xi}|z, \bar{z})$. Since the zero field is a descendant of degenerate primary field $\phi(\nu_{rs}, \bar{\nu}|z, \bar{z})$, the 3-point function can be expressed by the following 3-point blocks:

$$\langle\,\varphi(\chi_{rs}, \bar{\xi}_3|\infty, \infty)\,\phi(\nu_2, \bar{\nu}_2|1, 1)\phi(\nu_1, \bar{\nu}_1|0, 0)\,\rangle \;=\; \rho(\xi_{rs}, \nu_2, \nu_1|1)\rho(\bar{\xi}_3, \bar{\nu}_2, \bar{\nu}_1|1)\,C_{(rs)21},$$

where the structure constant

$$C_{(rs)21} = \langle\, \phi(\nu_{rs},\bar\nu_3|\infty,\infty)\,\phi(\nu_2,\bar\nu_2|1,1)\phi(\nu_1,\bar\nu_1|0,0)\,\rangle.$$

Vanishing of the correlator with zero field implies that for a given set of weights either the 3-point block or the structure constant has to be zero.

The Null Vector Decoupling Theorem [39] states that the 3-point block with degenerate field has zero only when the weights

$$\Delta_i = -\frac{1}{4}\left(\beta-\frac{1}{\beta}\right)^2 + \frac{\alpha_i^2}{4}$$

satisfy the fusion rule:

$$\alpha_2 \pm \alpha_1 \;=\; (1-r+2k)\beta - (1-s+2l)\frac{1}{\beta}, \tag{1.32}$$

with integers from the set $0 \le k \le r-1$, $0 \le l \le s-1$.

The theorem above can be justified with the help of Feigin-Fuchs construction [39], [29]. It shows that the structure constants $C_{(rs)21}$ in a free scalar theory with a background charge $Q = (b+\frac{1}{b})$ are indeed non-zero each time the fusion rules (1.32) are fulfilled. Within this approach primary fields are represented by exponential operators $\phi_a(z) = \mathrm{e}^{2a\varphi(z)}$ with conformal weights $\Delta_a = a(Q-a)$. The n-point correlators of the exponential operators gain the factor $\mathrm{e}^{2\lambda(a_1+\ldots a_n)}$ under the transformation $\varphi(z)\to\varphi(z)+\lambda$. Variation of the action in the free scalar theory with a background charge upon the shift of $\varphi(z)$ is $\delta S = \lambda Q$. This implies the constraint on the correlation function called the charge conservation condition:

$$\sum_{i=1}^{n} 2a_i = Q.$$

One can modify a correlation function by changing its total charge without changing its conformal properties. It can be done by inserting into the correlator the so called screening operators with zero conformal weight:

$$Q_b = \oint dz\, \mathrm{e}^{2b\varphi(z)}, \qquad Q_{\frac{1}{b}} = \oint dz\, \mathrm{e}^{\frac{2}{b}\varphi(z)}.$$

Consider the 3-point function with degenerate field $\phi_{rs} = \mathrm{e}^{2a_{rs}\varphi(z)}$ and screening operators:

$$C_{(rs)21} \;=\; \left\langle \mathrm{e}^{2a_{rs}\varphi(z)}\mathrm{e}^{2a_2\varphi(z)}\mathrm{e}^{2a_1\varphi(z)}Q_b^k\, Q_{\frac{1}{b}}^l \right\rangle, \qquad k,l\in\mathbb{N}\;. \tag{1.33}$$

The correlator does not vanish if the charge conservation rule is satisfied. This condition is in agreement with the fusion rule (1.32), where $a_{rs} = -\frac{b}{2}r - \frac{1}{2b}s$, $a_i = \frac{Q+i\alpha_i}{2}$ and $b = i\beta$. Since the block is a model independent function, it should vanish each time the structure constant (1.33) is non zero.

The *Null Vector Decoupling Theorem* leads to the definition of the fusion polynomial:

$$P_c^{rs}\!\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right] \;=\; \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\frac{\alpha_2+\alpha_1+p\beta-q\beta^{-1}}{2}\right)\left(\frac{\alpha_2-\alpha_1+p\beta-q\beta^{-1}}{2}\right) \tag{1.34}$$

where $p = r - 1 - 2k,\ q = s - 1 - 2l$. $P_c^{rs}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]$ is a polynomial of degree rs in the variable $\Delta_2 - \Delta_1$ and of degree $[\frac{rs}{2}]$ in $\Delta_2 + \Delta_1$. Coefficient of highest power of $\Delta_2 - \Delta_1$ is equal 1. The function $\rho(\xi_{rs}, \nu_2, \nu_1|1)$ has the same properties (1.30) thus the 3-point block with null vector is equal to fusion polynomial:

$$\rho(\chi_{rs}, \nu_2, \nu_1|1) \quad = \quad P_c^{rs}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]. \tag{1.35}$$

1.3 The 4-point block

1.3.1 Definition

The 4-point functions reduce to structure constants and 4-point conformal blocks - chiral objects completely determined by the Ward identities. We will define the 4-point conformal blocks in terms of the 3-point blocks introduced in the last section.

Let us consider the 4-point correlation function with identity operator inserted between two fields:

$$\langle 0|\,\phi_4(\infty,\infty)\phi_3(1,1)\phi_2(z,\bar z)\phi_1(0,0)\,|0\rangle = \langle \nu_4\otimes\bar\nu_4|\,\phi_3(1,1)\,\mathbf{1}\,\phi_2(z,\bar z)\,|\nu_1\otimes\bar\nu_1\rangle =$$

$$\sum_p \sum_{n=|M|=|N|} \langle \nu_4\otimes\bar\nu_4|\,\phi_3(1,1)|\nu_{p,M}\otimes\bar\nu_{p,\bar M}\rangle \left[B^n_{c,\Delta_p}\right]^{MN} \left[\bar B^n_{c,\bar\Delta_p}\right]^{\bar M\bar N} \langle \nu_{p,N}\otimes\bar\nu_{p,\bar N}|\phi_2(z,\bar z)\,|\nu_1\otimes\bar\nu_1\rangle$$

where the form of $\mathbf{1}$ follows from definition of Gram matrix (1.10):

$$\mathbf{1} = \sum_p \sum_{n=|M|=|N|} |\nu_{p,M}\rangle \left[B^n_{c,\Delta_p}\right]^{MN} \langle \nu_{p,N}|$$

and p numbers conformal weights in the spectrum of primary fields. Expressing 3-point correlation functions by 3-point blocks (1.28) one gets:

$$\begin{aligned}\langle \nu_4\otimes\bar\nu_4|\,\phi_3(1,1)\phi_2(z,\bar z)\,|\nu_1\otimes\bar\nu_1\rangle \quad &= \sum_p \sum_{n=|M|=|N|} C_{43p}\, C_{p21}\, \rho(\nu_4,\nu_3,\nu_{p,M}|1) \left[B^n_{c,\Delta_p}\right]^{MN} \rho(\nu_{p,N},\nu_2,\nu_1|z)\\ &\qquad \times \rho(\bar\nu_4,\bar\nu_3,\bar\nu_{p,\bar M}|1) \left[\bar B^n_{c,\bar\Delta_p}\right]^{\bar M\bar N} \rho(\bar\nu_{p,\bar N},\bar\nu_2,\bar\nu_1|\bar z)\\ &= \sum_p C_{43p}\, C_{p21} \left|\mathcal{F}_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2\end{aligned}$$

The 4-point conformal block is defined as the following series:

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \quad = \quad z^{\Delta-\Delta_2-\Delta_1}\left(1 + \sum_{n\in\mathbb{N}} z^n F^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\right). \tag{1.36}$$

with the coefficients:

$$F^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] = \sum_{n=|M|=|N|} \rho(\nu_4,\nu_3,\nu_{p,M}|1) \left[B^n_{c,\Delta_p}\right]^{MN} \rho(\nu_{p,N},\nu_2,\nu_1|1) \tag{1.37}$$

There are two important assumptions concerning analytical properties of the conformal block. First, it is believed that the radius of convergence of the series in z is equal 1. The exact proof does not exist, but all known examples of blocks calculated in some special cases confirm this hypothesis. The second assumption, also supported by some explicit formulae for analytical continuations, is that the only singularities of block as a function of z are the branching points at $0, 1, \infty$.

This means that conformal block is a single-valued analytical function on the universal covering of the sphere with 3 punctures at $0, 1, \infty$. Let us remind the definition of elliptic nome:

$$q(z) = \mathrm{e}^{i\pi\tau}, \qquad \tau(z) = i\frac{K(1-z)}{K(z)},$$

where $K(z)$ is complete elliptic integral of the first kind. The inverse of elliptic nome

$$z(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)}$$

is a universal covering of 3-punctured sphere by the Poincare disc $\mathbb{D}$. Thus the conformal block is a single-valued analytic function of q. The elliptic nome will naturally appear in the context of the so called elliptic block, which as power series in q is supposed to converge for $|q| < 1$.

The conformal block is also an analytical function of four external weights Δ_i, internal weight Δ and central charge c. Its coefficients depend on external weights entirely through the 3-point blocks which are polynomials in all weights (1.30). Due to inverse Gram matrix contribution, as functions of the intermediate weight Δ and the central charge c the 4-point blocks' coefficients are rational functions.

Even though the block is completely determined by the conformal symmetry, its exact form is in general not known. There exist a set of the so called minimal models for which the block was computed. These models are parameterized by $c = 1 - \frac{6}{(n+2)(n+3)}$ and have discreet and finite spectrum consisting of degenerate primary fields exclusively [3].

In general, one could try to compute the block from the definition, but the computation of the inverse Gram matrix is problematical. Thus a method of an approximate determination of conformal block is needed. The problem was solved by Zamolodchkov [14] who presented at first a recursion relation for block's coefficients of the expansion in z (1.37). His next two works [15], [16] were devoted to the second, more effective method based on recursion relation for coefficients of the block expanded in terms of elliptic nome q.

1.3.2 Residua

In derivation of the recursion relations for the 4-point conformal block the properties of the 3-point blocks and inverse Gram matrix play crucial role. From the third property of Gram matrix and Mittag-Leffler theorem it follows that blocks' coefficients can be expressed as a sum over simple poles in Δ and a term non-singular in Δ :

$$F^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; \mathrm{h}^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] + \sum_{1\leq rs\leq n} \frac{\mathcal{R}^n_{c,\,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]}{\Delta-\Delta_{rs}(c)}\,, \tag{1.38}$$

The same is true for the central charge dependence:

$$F^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; \mathrm{f}^n_{\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] + \sum_{1< rs\leq n} \frac{\widetilde{\mathcal{R}}^n_{\Delta,\,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]}{c-c_{rs}(\Delta)}\,. \tag{1.39}$$

The residues in both cases $\Delta_{rs}(c)$ and $c_{rs}(\Delta)$ are related (1.12),(1.13):

$$\begin{aligned}\widetilde{\mathcal{R}}^n_{\Delta,\,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= -\frac{\partial c_{rs}(\Delta)}{\partial\Delta}\,\mathcal{R}^n_{c_{rs}(\Delta),\,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right], \\ \frac{\partial c_{rs}(\Delta)}{\partial\Delta} &= 4\,\frac{c_{rs}(\Delta)-1}{(r^2-1)\,\beta^4_{rs}(\Delta)-(s^2-1)}.\end{aligned} \tag{1.40}$$

The structure of the residues is essential for the recurrence relations for the blocks' coefficients. We shall present now the basic steps of calculation of the residuum at Δ_{rs}.

First let us notice that a pole in degenerate weight Δ_{rs} is connected with the presence of a null vector $\chi_{rs}\in\mathcal{V}^{rs}_{\Delta_{rs}}$ generating the submodule $\mathcal{V}_{\Delta_{rs}+rs}\subset\mathcal{V}_{\Delta_{rs}}$. Hence, among the states from $\mathcal{V}^n_{\Delta_{rs}}$ $(n>rs)$, there are null vector descendants belonging to $\mathcal{V}^{n-rs}_{\Delta_{rs}+rs}$. This fact motivates the specific choice of the basis for states at level $n>rs$ in Verma module with arbitrary weight Δ.

Let χ^M_{rs} be the coefficients of the null vector χ_{rs} in the basis $L_{-M}\nu_{\Delta_{rs}}$:

$$\chi_{rs}=\sum_M \chi^M_{rs}L_{-M}\nu_{\Delta_{rs}}.$$

Consider the states at level $n>rs$ which can be written in terms of χ^M_{rs} coefficients::

$$L_{-N}\chi^\Delta_{rs}\in\mathcal{V}^n_\Delta,\qquad\text{where}\qquad \chi^\Delta_{rs}=\sum_M\chi^M_{rs}L_{-M}\nu_\Delta,\quad |N|=n-rs,$$

so that the null vector appears in the limit: $\chi_{rs}=\lim_{\Delta\to\Delta_{rs}}\chi^\Delta_{rs}$. The set of these states can be always extended to a full basis in $\mathcal{V}^n_\Delta$.

Working in that basis one gets the following result for the residue:

$$\begin{aligned}\mathcal{R}^n_{c,\,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \lim_{\Delta\to\Delta_{rs}}(\Delta-\Delta_{rs}(c))F^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \\ &= A_{rs}(c)\sum_{n=|M|=|N|}\rho(\nu_4,\nu_3,L_{-M}\chi_{rs}|1)\;\left[B^{n-rs}_{c,\Delta_{rs}+rs}\right]^{M,N}\rho(L_{-N}\chi_{rs},\nu_2,\nu_1|1),\end{aligned}$$

with

$$A_{rs}(c)\;=\;\lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\chi^\Delta_{rs}|\chi^\Delta_{rs}\rangle}{\Delta-\Delta_{rs}(c)}\right)^{-1}.$$

The factorization property of 3-point block (1.31) gives:

$$\rho(L_{-N}\chi_{rs},\nu_2,\nu_1|1) \;=\; \rho(L_{-M}\nu_{\Delta_{rs}+rs},\nu_2,\nu_1|1)\;\rho(\chi_{rs},\nu_2,\nu_1|1).$$

One can notice that the block $\rho(L_{-M}\nu_{\Delta_{rs}+rs},\nu_2,\nu_1|1)$, analogical one $\rho(\nu_4,\nu_3,L_{-N}\nu_{\Delta_{rs}+rs}|1)$ and inverse Gram matrix corresponding to weight $\Delta_{rs}+rs$ together give the 4-point block coefficient:

$$\sum_{n=|M|=|N|}\rho(L_{-M}\nu_{\Delta_{rs}+rs},\nu_2,\nu_1|1)\left[B^{n-rs}_{c,\Delta_{rs}+rs}\right]^{M,N}\rho(\nu_4,\nu_3,L_{-N}\nu_{\Delta_{rs}+rs}|1) \;=\; F^{n-rs}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]$$

The remaining two 3-point blocks, with the singular vector χ_{rs} as one of the arguments, are given by the fusion polynomials (1.34), (1.35). Thus the final result for residuum is the following:

$$\mathcal{R}^{n}_{c,rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; A_{rs}(c)\,P^{rs}_{c}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_{c}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]F^{n-rs}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \tag{1.41}$$

The exact form of the coefficient $A_{rs}(c)$ was proposed in [14] and derived afterwards in [40] by Al. Zamolodchikov:

$$A_{rs}(c) \;=\; \frac{1}{2}\prod_{m=1-r}^{r}\prod_{n=1-s}^{s}\left(p\beta-\frac{q}{\beta}\right)^{-1},\qquad (m,n)\neq(0,0),(r,s).$$

The formula for residuum (1.41) inserted into equation (1.38) gives the recursion relation for block's coefficients:

$$F^{n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; \mathrm{h}^{n}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]+\sum_{1\leq rs\leq n}\frac{A_{rs}(c)\,P^{rs}_{c}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_{c}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta-\Delta_{rs}(c)}F^{n-rs}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]. \tag{1.42}$$

One can sum up all the block's coefficients to obtain the relation for the 4-point block (1.36):

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; \mathrm{h}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)+\sum_{1\leq rs\leq n}\frac{A_{rs}(c)\,P^{rs}_{c}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_{c}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta-\Delta_{rs}(c)}\mathcal{F}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \tag{1.43}$$

Analogical relations hold for sum over the poles in c (1.39) with residua given by (1.40) and (1.41).

1.3.3 Term regular in c

In order to complete the recursion relations for block's coefficients it is necessary to derive an exact form of the regular terms in (1.38), (1.39). Since these functions do not have poles in intermediate weight (or central charge) they can be determined from the behavior of the 4-point blocks for large Δ or c respectively.

In the case of c-dependence, nonsingular term is simply a limit of the block for $c\to\infty$. The block's coefficients depend on c only through inverse Gram matrix $\left[B^{n}_{c,\Delta}\right]^{M,N}$. The Kac determinant is a polynomial of the order $dim\mathcal{V}^{n-rs}_{c,\Delta}$ in c. A minor of the Gram matrix can be a polynomial in c of the order not greater than the order of Kac determinant. Hence the elements of inverse Gram matrix are given by non positive power of c. In fact there is only one matrix element that does not vanish in limit the $c \to \infty$.

From the Virasoro algebra (1.7) it follows that the central charge appears in Gram matrix due to commutators of the type $[L_n, L_{-n}]$ for $n \neq 1$. Hence the diagonal elements are polynomials of maximal degree in c. For $n = 1$ the diagonal element does not depend on c, the same is true for all elements from the row and column that include this element. Thus the only element in inverse Gram matrix which does not depend on c and hence does not vanish in the limit $c \to \infty$ is the diagonal one:

$$\lim_{c\to\infty} \left[B^n_{c,\Delta}\right]^{\mathbb{I}\,\mathbb{I}} = \frac{1}{\langle \nu_\Delta | \, L_1^n L_{-1}^n \, |\nu_\Delta\rangle} = \frac{1}{n!(2\Delta)_n},$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol.

The 3-point block in that case is given by (1.30) with all $m_i = 1$:

$$\rho\left(\mathrm{L}^n_{-1}\nu, \nu_2, \nu_1|1\right) \quad = \quad (\Delta + \Delta_2 - \Delta_1)_n.$$

The regular term in block's coefficient expansion (1.39), given by the $c \to \infty$ limit of (1.37), has thus the following form:

$$\mathrm{f}^n_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \quad = \quad \frac{1}{n!}\frac{(\Delta + \Delta_3 - \Delta_4)_n(\Delta + \Delta_2 - \Delta_1)_n}{(2\Delta)_n}.$$

Since all the functions defining coefficients of the 4-point block depend on the intermediate weight, the derivation of the term regular in Δ is more complicated. The large Δ behavior of the 4-point block was worked out by Al. Zamolodchikov [14, 15, 16]. We shall present the basic steps of this derivation in the next section.

1.4 Elliptic recurrence for 4-point block

The essential observation for deriving the large Δ asymptotic of 4-point block made by Al. Zamolodchikov [14, 15, 16] is the following: to write down the block's asymptotic it is necessary to study the classical limit of the block. The first two terms of the expansion of classical block in terms of large classical intermediate weight δ fully determine the dependence on external weights and central charge of the first two terms in the $\frac{1}{\Delta}$ expansion of conformal quantum block. It was also stressed that the classical limit of the block can be investigated by analyzing the Liouville theory.

1.4.1 Classical block

The main assumption concerning classical limit of quantum 4-point conformal block reads that the limit exist. By analyzing the asymptotical behavior of correlation functions in the

Liouville theory it was possible to find heuristic arguments indicating that the classical limit of conformal block has a form of exponential function of the classical block [16]. We will present here the basic points of reasoning leading to the exact definition of the classical block.

First, let us remind the action in the Liouville theory [34]:

$$\mathcal{S}_{\rm LFT} = \frac{1}{2\pi}\int\left(|\partial\phi|^2 + 4\pi^2 b^2\mu^2 \mathrm{e}^{2b\phi}\right) d^2z,$$

where the scale parameter μ is called the cosmological constant and b is the dimensionless coupling constant. This definition assumes a trivial background metric $g_{ab} = \delta_{ab}$. The Liouville theory on a sphere can be given in terms of the flat action above, but the additional special boundary condition has to be satisfied by the Liouville field :

$$\phi(z,\bar{z}) = -Q\log(z\bar{z}) + \mathcal{O}(1) \quad \text{at} \quad |z| \to \infty.$$

This constraint is equivalent to taking away all the curvature to the infinity. The background charge

$$Q = b + \frac{1}{b}$$

determines the central charge of the theory

$$c = 1 + 6Q^2. \tag{1.44}$$

Modes of the energy-momentum tensor

$$T(z) = -(\partial\phi)^2 + Q\partial^2\phi, \qquad \bar{T}(\bar{z}) = -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2\phi$$

satisfy the Virasoro algebra with the cental charge given by (1.44).

Spectrum of the Liouville theory consists of an infinite family of Verma modules [41]

$$\mathcal{H} = \int_{\mathbb{S}}^{\oplus} \mathrm{d}a\ \mathcal{V}_a \otimes \bar{\mathcal{V}}_a, \qquad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+.$$

The primary fields are represented as exponents $\mathsf{V}_a = \mathrm{e}^{2a\phi}$ with conformal dimensions $\Delta_a = \bar{\Delta}_a = a(Q-a)$.

Within the path-integral approach the n-point correlation function of the exponential fields is defined as the functional integral:

$$\langle V_n(z_n)\ldots V_1(z_1)\rangle = \int D\phi\ V_{a_n}(z_n)\ldots V_{a_1}(x_1)\ e^{-S_{LFT}[\phi]}. \tag{1.45}$$

Depending on the value of conformal weight there are two types of operators: "light" fields with $a \sim b$ and fields with "heavy" weights:

$$a = \tfrac{Q}{2}(1-\lambda) \ \ , \ \ ba \to \tfrac{1-\lambda}{2} \ \ , \ \ b^2\Delta \to \delta = \tfrac{1-\lambda^2}{4}.$$

where δ is called classical weight. In the classical limit *i.e.* $b \to 0, 2\pi\mu b^2 \to m = const$, only the presence of the heavy fields in the correlator has influence on the classical solution of the field equations.

In the case of 4-point correlation function of "heavy" fields the classical limit

$$\langle V_{a_4} V_{a_3} V_{a_2} V_{a_1} \rangle \sim \mathrm{e}^{-\frac{1}{b^2} S_{\mathrm{cl}}[\delta_4,\delta_3,\delta_2,\delta_1]}$$

is determined by the action $S_{\mathrm{cl}}[\delta_4, \delta_3, \delta_2, \delta_1]$ given by

$$S[\phi] = \frac{1}{2\pi} \int \left(|\partial\phi|^2 + m^2 \mathrm{e}^{2\phi} \right) d^2 z, \tag{1.46}$$

calculated on the classical configuration φ satisfying the Liouville equation

$$\partial\bar{\partial}\varphi - m^2 \mathrm{e}^{2\phi} = \sum_1^4 \frac{1-\lambda_i}{4} \delta(z - z_i).$$

On the other hand, before taking the classical limit, we can express 4-point function in terms of 4-point blocks and structure constants:

$$\langle V_{a_4} V_{a_3} V_{a_2} V_{a_1} \rangle = \int\limits_{\frac{Q}{2}+i\mathbb{R}_+} \frac{da}{2\pi i} C_{43a} C_{a21} \left| \mathcal{F}_{\Delta_a} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (z) \right|^2. \tag{1.47}$$

The asymptotic behavior of structure constants also follows from (1.45):

$$C_{a21} \quad \sim \quad \mathrm{e}^{-\frac{1}{b^2} S_{\mathrm{cl}}[\delta,\delta_2,\delta_1]}$$

with $S_{\mathrm{cl}}[\delta, \delta_2, \delta_1]$ as the 3-point classical Liouville action.

Now compare the classical limit of 4-point function projected on one conformal family Δ_a:

$$\langle V_{a_4} V_{a_3} \, |_{\Delta_a} \, V_{a_2} V_{a_1} \rangle \sim \mathrm{e}^{-\frac{1}{b^2} S_{\mathrm{cl}}[\delta_4,\delta_3,\delta_2,\delta_1|\delta_a]}$$

with the limit of (1.47) for the same weight. The $Q \to \infty$ asymptotic of the quantum block is thus given by:

$$\mathcal{F}_{1+6Q^2,\Delta} \left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix} \right] (z) \quad \sim \quad \mathrm{e}^{Q^2 f_\delta \left[\begin{smallmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{smallmatrix} \right] (z)} \tag{1.48}$$

where $f_\delta \left[\begin{smallmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{smallmatrix} \right] (z)$ is the classical block [19], [16] satisfying the relation:

$$S_{\mathrm{cl}}[\delta_4, \delta_3, \delta_2, \delta_1 | \delta] = S_{\mathrm{cl}}[\delta_4, \delta_3, \delta] + S_{\mathrm{cl}}[\delta, \delta_2, \delta_1] - f_\delta \left[\begin{smallmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{smallmatrix} \right] (z) - \bar{f}_\delta \left[\begin{smallmatrix} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{smallmatrix} \right] (\bar{z}).$$

1.4.2 $\frac{1}{\delta}$ expansion of classical block

It would be extremely hard problem to calculate the classical block in general. Fortunately, as it will be discussed in the next subsection, in order to find the recursive relation for block's coefficient (1.38) only the first two terms of $\frac{1}{\delta}$ expansion of the classical block are needed. The solution of this problem was presented by Zamolodchikov in the work [16]. It is based on a Fuchsian equation in which the classical block is present as an accessory parameter. Knowing a condition for monodromy properties of the Fuchsian equation's solutions one can find appropriate value of the accessory parameter *i.e.* to calculate the classical block. We will remind below all the basic steps of this calculation.

Consider 5-point correlation function of primary fields with one degenerate field $V_5 = V_{-\frac{b}{2}}$:

$$\langle V_4 V_3 V_5 V_2 V_1 \rangle \equiv \Big\langle V_4(\infty)\, V_3(1,1)\, V_{-\frac{b}{2}}(z,\bar{z})\, V_2(x,\bar{x})\, V_1(0,0) \Big\rangle \tag{1.49}$$

In the degenerate family $[V]_{-\frac{b}{2}}$ there is a zero field of the form:

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2\Delta_{-\frac{b}{2}}+1)}\mathcal{L}_{-1}^2\right) V_{-\frac{b}{2}} = 0, \qquad \Delta_{-\frac{b}{2}} = \Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}b^2.$$

On the one hand the correlation function which includes the zero field has to vanish. On the other hand one can use the fact that zero field is a descendant. The contour deformation calculations applied to the definition of descendant field (1.16) together with Ward identities (1.4) lead to the differential equation for the correlator (1.49):

$$\begin{aligned} &\left\{\partial_z^2 \;+\; b^2\left[\frac{\Delta_4-\Delta_3-\Delta_2-\Delta_1}{z(z-1)} + \frac{\Delta_3}{(z-1)^2} + \frac{\Delta_2}{(z-x)^2} + \frac{\Delta_1}{z^2}\right]\right\}\langle V_4 V_3 V_5 V_2 V_1\rangle \\ &\quad +\; b^2 \frac{x(x-1)}{z(z-1)(z-x)}\frac{\partial}{\partial x}\langle V_4 V_3 V_5 V_2 V_1 \rangle = 0. \end{aligned} \tag{1.50}$$

In the classical limit operator $V_{-\frac{b}{2}}$ is a light field and it does not contribute to classical dynamics. Thus for a given intermediate conformal family Δ_a the projected 5-point function behave as:

$$\Big\langle V_4(\infty)\, V_3(1,1)\, V_{-\frac{b}{2}}(z,\bar{z})\,|_{\Delta_a}\, V_2(x,\bar{x})\, V_1(0,0) \Big\rangle \sim \psi(z)\; \mathrm{e}^{\frac{1}{b^2} f_{\delta_a}\left[\substack{\delta_3\ \delta_2 \\ \delta_4\ \delta_1}\right](x)}, \tag{1.51}$$

where $f_{\delta}\left[\substack{\delta_3\ \delta_2 \\ \delta_4\ \delta_1}\right](x)$ is the classical conformal block (1.48). Substituting this limit into differential equation (1.50) one gets the Fuchsian equation [16]:

$$\begin{aligned} \frac{d^2\psi(z)}{dz^2} + \left(\frac{\delta_4-\delta_3-\delta_2-\delta_1}{z(z-1)} + \frac{\delta_1}{z^2} + \frac{\delta_2}{(z-x)^2} + \frac{\delta_3}{(z-1)^2}\right)\psi(z) & \\ + \frac{x(x-1)\mathcal{C}(x)}{z(z-x)(z-1)}\psi(z) &= 0, \end{aligned} \tag{1.52}$$

with the accessory parameter $\mathcal{C}(x)$:

$$\mathcal{C}(x) \;=\; \frac{\partial}{\partial x} f_{\delta}\left[\substack{\delta_3\ \delta_2 \\ \delta_4\ \delta_1}\right](x). \tag{1.53}$$

The functions on either side of (1.51) should have the same monodromy properties along the contour encircling the points 0 and x. First, let us notice that the monodromy properties of the 5-point correlator (1.51) along a curve encircling both 0 and x are the same as the monodromy properties of the 4-point function $\left\langle V_4(\infty)\, V_3(1,1)\, V_{-\frac{b}{2}}(z,\bar{z})\, V_a(0,0)\right\rangle$ for a curve encircling 0. The z dependence of this correlator for $z \to 0$ can be read off from the OPE of degenerate field with primary operator V_a:

$$\begin{aligned} V_{-\frac{b}{2}}(z,\bar{z})V_a(0,0) &= C_{(a_+,-\frac{b}{2},a)}(z\bar{z})^{\frac{bQ}{2}(1+\lambda)}\, V_{a_+}(0,0) \\ &+ C_{(a_-,-\frac{b}{2},a)}(z\bar{z})^{\frac{bQ}{2}(1-\lambda)}\, V_{a_-}(0,0) + \text{descendants}, \end{aligned}$$

The families $a_\pm = a \pm -\frac{b}{2}$ appearing in the OPE are determined by the fusion rules (1.32), where $a = \frac{Q(1-\lambda)}{2}$.

Hence, in the space of solutions of (1.52) there exist a basis $\psi_\pm(z)$ such that functions analytically continued in z along the path encircling the points 0 and x satisfy the condition:

$$\psi_\pm\left(\mathrm{e}^{2\pi i}z\right) = -\mathrm{e}^{\pm i\pi\lambda}\,\psi_\pm(z)\,. \tag{1.54}$$

This corresponds to the monodromy matrix with trace equal $-2cos(\pi\lambda)$ which is invariant with respect to the choice of the basis of solutions.

The idea which allow to determine the classical block is the following: adjust $\mathcal{C}$ in such a way that the equation (1.50) admits solutions with the monodromy around 0 and x given by (1.54).

The technical details of the calculation leading to the result for the first two terms of $\frac{1}{\delta}$ expansion of classical block are given in Appendix A. The classical block has the following form (A.10):

$$\begin{aligned} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= i\pi\tau\left(\delta-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{3}{4}-\delta_1-\delta_2-\delta_3-\delta_4\right)\ln K^4(x) \\ &+ \left(\frac{1}{4}-\delta_2-\delta_3\right)\ln(1-x)+\left(\frac{1}{4}-\delta_1-\ \delta_2\right)\ln(x)+\mathcal{O}\left(\frac{1}{\delta}\right). \end{aligned} \tag{1.55}$$

1.4.3 Large Δ asymptotic of conformal block from the classical block

In this subsection we will present Zamolodchikov's reasoning leading to statement that the first two terms of the $\frac{1}{\delta}$ expansion of classical block fully determine the dependence on external weights and central charge of the first two terms in the $\frac{1}{\Delta}$ expansion of conformal quantum block.

Let us denote:

$$\mathcal{G}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = \ln \mathcal{F}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z).$$

Since the conformal block (1.36) can be written as a series in z, the function $\mathcal{G}_\Delta$ also admit an expansion of the form:

$$\mathcal{G}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = (\Delta-\Delta_2-\Delta_1)\ln z+\sum_{i=0}^{\infty}G_n z^n,$$

where G_n, as the blocks coefficients (1.37), are rational functions of Δ, c, Δ_i:

$$G_n = \frac{P_n(\Delta,\Delta_i,c)}{Q_n(\Delta,c)}.$$

The functions $P_n(\Delta, \Delta_i, c)$, $Q_n(\Delta, c)$ are polynomials in their arguments. The assumption concerning existence of the classical block as the classical limit of conformal block implies that the maximal homogeneous power of $P_n(\Delta, \Delta_i, c)$ is greater by 1 than the maximal homogeneous power of $Q_n(\Delta, c)$.

The $\frac{1}{\Delta}$ expansion of G_n can be easily computed if one write the first terms of polynomials explicitly:

$$\begin{aligned} G_n &= \frac{P_n(\Delta,\Delta_i,c)}{Q_n(\Delta,c)} = \frac{A_n\Delta^{N_n+1} + \Delta^{N_n}\left(\sum\limits_{i=1}^{4} B_n^i\Delta_i + C_n c + D_n\right) + \ldots}{a_n\Delta^{N_n} + \Delta^{N_n-1}(b_n c + d_n) + \ldots} \\ &= \left[A_n\Delta + \left(\sum_{i=1}^{4} B_n^i\Delta_i + C_n c + D_n\right) + \ldots\right]\left[\frac{1}{a_n} - \frac{1}{a_n^2}\left(\frac{1}{\Delta}(b_n c + d_n) + \frac{1}{\Delta^2}\ldots\right) + \ldots\right] \\ &= \frac{A_n}{a_n}\Delta + \sum_{i=1}^{4}\frac{B_n^i}{a_n}\Delta_i + \frac{(C_n a_n - A_n b_n)}{a_n^2}c + \frac{D_n}{a_n} - \frac{A_n d_n}{a_n^2} + \mathcal{O}\left(\frac{1}{\Delta}\right). \end{aligned} \tag{1.56}$$

The coefficient a_n is non-zero due to the properties of Kac determinant (1.11) and inverse Gram matrix. We want to compare this formula with $\frac{1}{\delta}$ expansion of classical block.

Let us write the classical block also as a power series in z:

$$f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z) = (\delta - \delta_2 - \delta_1)\ln z + \sum_{n=1}^{\infty} z^n\, f_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]; \qquad \text{where} \quad \lim_{b\to 0}\frac{P_n(\Delta,\Delta_i,c)}{Q_n(\Delta,c)} = \tfrac{1}{b^2} f_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right].$$

Next define the polynomials of maximal homogeneous degree:

$$\begin{aligned} P_n^{N_n+1}(\Delta,\Delta_i,c) &= \Delta^{N_n}\left(A_n\Delta + \sum_{i=1}^{4} B_n^i\Delta_i + C_n c\right) \\ &+ \Delta^{N_n-1}\left(\sum_{i,j=1}^{4} X_n^{ij}\Delta_i\Delta_j + \sum_{i=1}^{4} Y_n^i\Delta_i c + Z_n c^2\right) + \Delta^{N_n-2}\Big(\ldots \end{aligned}$$

and

$$Q_n^{N_n}(\Delta,c) = a_n\Delta^{N_n} + b_n\Delta^{N_n-1}c + c_n\Delta^{N_n-2}c^2 + \ldots$$

Notice that $P_n^{N_n+1}$ and $Q_n^{N_n}$ do not include the coefficients D_n and d_n from (1.56) since the latter are proportional to Δ^{N_n} or Δ^{N_n-1}, respectively.

The classical limit of G_n is determined by the polynomials above:

$$\begin{aligned} \lim_{b\to0} b^2\frac{P_n(\Delta,\Delta_i,c)}{Q_n(\Delta,c)} &= \lim_{b\to0} b^2\,\frac{\left(P_n^{N_n+1}(\Delta,\Delta_i,c) + P_n^{N_n}(\Delta,\Delta_i,c) + \ldots\right)}{\left(Q_n^{N_n}(\Delta,c) + Q_n^{N_n-1}(\Delta,c) + \ldots\right)} \\ &= \lim_{b\to0} b^2\,\frac{\left(\frac{1}{b^2}\,P_n^{N_n+1}(\delta,\delta_i,b^2c) + P_n^{N_n}(\delta,\delta_i,b^2c) + \ldots\right)}{\left(Q_n^{N_n}(\delta,b^2c) + b^2\,Q_n^{N_n-1}(\delta,b^2c) + \ldots\right)} \\ &= \lim_{b\to0}\frac{1}{Q_n^{N_n}(\delta,\frac{c}{b^2})}\left(P_n^{N_n+1}(\delta,\delta_i,b^2c) + b^2\,P_n^{N_n}(\delta,\delta_i,b^2c) + \ldots\right)\left(1 - b^2\,\frac{Q_n^{N_n-1}(\delta,b^2c)}{Q_n^{N_n}(\delta,b^2c)} + \ldots\right) \\ &= \frac{P_n^{N_n+1}(\delta,\delta_i,b^2c)}{Q_n^{N_n}(\delta,b^2c)} \end{aligned}$$

Thus coefficients of classical block take the form:

$$f_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] = \frac{P_n^{N_n+1}(\delta,\delta_i,b^2c)}{Q_n^{N_n}(\delta,b^2c)} = \frac{\delta^{N_n}\left(A_n\delta + \sum\limits_{i=1}^{4} B_n^i\delta_i + 6C_n\right) + \delta^{N_n-1}\Big(\ldots}{a_n\delta^{N_n} + 6b_n\delta^{N_n-1} + 36c_n\delta^{N_n-2} + \ldots},$$

where $b^2c = b^2(1+6Q^2) \to 6$ while $b \to 0$. Expanding this function in powers of $\frac{1}{\delta}$ in the same way as G_n in powers of $\frac{1}{\Delta}$ in (1.56) one gets:

$$f_\delta^n\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right] = \frac{A_n}{a_n}\delta + \sum_{i=1}^{4}\frac{B_n^i}{a_n}\delta_i + \frac{6\left(C_na_n - A_nb_n\right)}{a_n^2} + \mathcal{O}\left(\frac{1}{\delta}\right).$$

Comparing expression above with (1.56) we can see that the first two terms in the expansion of classical block determine the coefficients proportional to Δ, Δ_i and c in $\frac{1}{\Delta}$ expansion of the function G_n.

Using formula (1.55) for classical block we can finally identify the function $\mathcal{G}$ as:

$$\begin{aligned}\mathcal{G}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = i\pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z) \\ + \left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \Delta_2\right)\ln(z) + f(z) + \mathcal{O}\left(\frac{1}{\Delta}\right),\end{aligned} \tag{1.57}$$

where $f(z)$ corresponds to parameters D_n, d_n from (1.56) and cannot be determined from the classical block. On the other hand, this function is independent from Δ_i and c and thus it can be derived from analytical expression for block calculated in some specific model. The model which was considered by Zamolodchikov is the so-called Ashkin-Teller model, the $c = 1$ scalar free theory extended by the Ramond sector [15], [17]. The conformal block calculated for external weights $\Delta_0 = \frac{1}{16}$ has the following form:

$$\mathcal{F}_\Delta\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = (16q)^\Delta\ \left[z\,(1-z)\right]^{-\frac{1}{8}}\,\theta_3^{-1}(q), \tag{1.58}$$

We will show how to compute this block in the chapter 4.1. Notice, that the asymptotic of the explicit block is in agreement with (1.57).

1.4.4 Elliptic block

The aim of the last two subsections was to derive large Δ asymptotic of conformal block so that the regular term in recursion relation (1.43) could be determined. From the large Δ asymptotic (1.57) we can read off the Δ_i and c dependence of the term non-singular in Δ. Excluding from the conformal block the multiplicative factor which takes over all the Δ_i and c dependence of the non-singular term, we define the elliptic block:

$$\begin{aligned}\mathcal{F}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) &= (16q)^{\Delta-\frac{c-1}{24}}\ z^{\frac{c-1}{24}-\Delta_1-\Delta_2}\ (1-z)^{\frac{c-1}{24}-\Delta_2-\Delta_3} \\ &\times\ \theta_3^{\frac{c-1}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\ \mathcal{H}_\Delta\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q),\end{aligned} \tag{1.59}$$

It has the same analytic structure as the conformal block

$$\mathcal{H}_{\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) = g(q) + \sum_{m,n} \frac{h_{mn}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q)}{\Delta - \Delta_{mn}}, \tag{1.60}$$

but the regular in Δ term $g(q)$ does not depend on external weights Δ_i and central charge c any more. It is related to $f(z)$ in (1.57):

$$\mathrm{e}^{f(z)} = (16q)^{\frac{1}{24}}\,[z(1-z)]^{-\frac{1}{24}}\,\theta_3^{-\frac{1}{2}}(q)\,g(q).$$

Hence $g(q)$ can be identified with a help of the explicit formula for the block calculated in Ashkin-Teller model with $c = 1$ (1.58). By comparison with definition (1.59) one can notice that the elliptic block in this case is simply equal to regular term $\mathcal{H}_{\Delta}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) = 1$, what gives $g(q) = 1$.

The residua of elliptic block (1.60) can be easily derived by inserting the definition of the elliptic block (1.59) into relation for the conformal block (1.43):

$$\mathcal{H}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) \;=\; 1 + \sum_{1\leq rs\leq n} \frac{A_{rs}(c)\,P_c^{rs}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P_c^{rs}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{rs}(c)}\,\mathcal{H}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q).$$

Finally, let us write elliptic block as a power series in the nome:

$$\mathcal{H}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) \;=\; \sum_{n=0}^{\infty}(16q)^n H^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]$$

The elliptic block's coefficients satisfy the elliptic recursion relation:

$$H^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] = g_n \;+ \sum_{r,s>0} \frac{A_{rs}(c)\,P_c^{rs}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P_c^{rs}\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{rs}}\,H^{n-rs}_{c,\Delta_{rs}+rs}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \tag{1.61}$$

where $g_n = \delta_{n,0}$ due to nonsingular term $g(q) = \sum_{n=0}^{\infty}(16q)^n g_n = 1$.

This recursion relation in practice is more useful than the z-recurrence (1.42). The inverse of elliptic nome $z(q)$ is a universal covering of 3-punctured sphere by the Poincare disc $\mathbb{D}$. If the q-series defining elliptic block converges for $|q| < 1$ it converges on the whole covering and thus it gives there a uniform approximation.

The elliptic recurrence (1.61) allows for approximate, analytic determination of the general 4-point conformal block. It was applied to numerical consistency check of Liouville theory with 3-point functions proposed by by Otto and Dorn [18] and by A. and Al. Zamolodchikov [19]. It was also used in study of the $c \to 1$ limit of minimal models [20] or in obtaining new results in the classical geometry of hyperbolic surfaces [21, 22].

Chapter 2

Conformal blocks in NS sector of $N = 1$ SCFT

2.1 Definitions

2.1.1 $N = 1$ superconformal symmetry

We shall analyze now two dimensional conformal field theories with $N = 1$ supersymmetry (SCFT). Our aim is to define the 4-point superconformal blocks and to find their recursive representations. It can be done by a proper extension of the Zamolodchikov's reasonings reminded in the previous chapter.

First we will suitably generalize the main assumptions and definitions concerning CFT. The basic dynamical assumption *i.e.* the operator product expansion remains the same (1.1) and reads:

In an arbitrary correlation function the product of any two local operators can be expressed as a series of local operators

$$\varphi_i(z_2,\bar{z}_2)\varphi_j(z_1,\bar{z}_1) = \sum_k C^k_{ij}(z_2 - z_1, \bar{z}_2 - \bar{z}_1)\varphi_k(z_1,\bar{z}_1), \tag{2.1}$$

where the coefficients $C_{kij}(z_2 - z_1, \bar{z}_2 - \bar{z}_1)$ are c-number functions.

Additionally, we assume that in a general SCFT model:

There exist an holomorphic field $S(z)$ and an antiholomorphic counterpart $\bar{S}(\bar{z})$, which together with fields $T(z), \bar{T}(\bar{z})$ (1.2) generate superconformal symmetry. $S(z), \bar{S}(\bar{z})$ have conformal weights $(\Delta, \bar{\Delta})$ equal to $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$, respectively.

The local Ward identities for the holomorphic (left) generators have the form:

$$\begin{aligned} T(z)T(0) &= \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + reg. \\ T(z)S(0) &= \frac{3}{2z^2}S(0) + \frac{1}{z}\partial S(0) + reg. \\ S(z)S(0) &= \frac{2c}{3z^3} + \frac{2}{z}T(0) + reg. \end{aligned} \tag{2.2}$$

The local Ward identities for the antiholomorphic (right) generators are given by analogous formulae. Additionally:

$$\overline{T}(\bar{z})T(0) \;=\; reg.\,, \qquad \overline{T}(\bar{z})S(0) = reg.\,, \qquad \bar{S}(\bar{z})S(0) = reg.$$

Since the conformal weight of fermionic operators $S(z)$ and $\bar{S}(\bar{z})$ is half-integer, the correlation functions containing each of them can be double-valued. Thus the space of fields splits into two parts: the *Neveu-Schwarz subspace* of fields $\varphi_{\rm NS}(z_i,\bar{z}_i)$ local with respect to $S(z)$ and the *Ramond subspace* of fields $R(z_i,\bar{z}_i)$ 'half-local' with respect to $S(z)$. The 'half-locality' of Ramond fields means that any correlation function

$$\langle S(z)R(z_i,\bar{z}_i)\ldots\rangle$$

changes the sing upon analytic continuation in z around the point $z=z_i$.

The locality properties of NS fields and Ramond fields imply that their OPEs with $S(z)$ can be written in terms of integer or half-integer powers of z respectively:

$$S(z)\varphi_{\rm NS}(0,0) \;=\; \sum_{k\in\mathbb{Z}+\frac12} z^{k-\frac32}S_{-k}\varphi_{\rm NS}(0,0), \tag{2.3}$$

$$S(z)R(0,0) \;=\; \sum_{m\in\mathbb{Z}} z^{m-\frac32}S_{-m}R(0,0). \tag{2.4}$$

It follows from the local Ward identities (2.2) that the operators

$$\begin{aligned} S_{-k}\varphi_{\rm NS}(0,0) &= \oint_0 \frac{{\rm d}z}{2\pi i}\, z^{-k+\frac12}S(z)\varphi_{\rm NS}(0,0), \quad \bar{S}_{-k}\varphi_{\rm NS}(0,0) = \oint_0 \frac{{\rm d}\bar{z}}{2\pi i}\, \bar{z}^{-k+\frac12}\bar{S}(\bar{z})\varphi_{\rm NS}(0,0),\\ S_{-m}R(0,0) &= \oint_0 \frac{{\rm d}z}{2\pi i}\, z^{-m+\frac12}S(z)R(0,0), \qquad \bar{S}_{-m}R(0,0) = \oint_0 \frac{{\rm d}\bar{z}}{2\pi i}\, \bar{z}^{-m+\frac12}\bar{S}(\bar{z})R(0,0), \end{aligned}$$

together with the Virasoro generators (1.7):

$$L_n = \frac{1}{2\pi i}\oint_0 {\rm d}z\; z^{n+1}T(z), \qquad \bar{L}_n = \frac{1}{2\pi i}\oint_0 {\rm d}\bar{z}\; \bar{z}^{n+1}\overline{T}(\bar{z})$$

form two copies of the Neveu-Schwarz-Ramond algebra:

$$\begin{aligned} [L_m,L_n] &= (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n},\\ [L_m,S_p] &= \frac{m-2p}{2}S_{m+p},\\ \{S_p,S_q\} &= 2L_{p+q} + \frac{c}{3}\left(p^2-\frac14\right)\delta_{p+q},\\ [L_n,\bar{L}_m] &= 0\,, \qquad [L_n,\bar{S}_p]=0\,, \qquad \left\{S_p,\bar{S}_q\right\}=0, \end{aligned} \tag{2.5}$$

where p,q are integer or half-integer when S_p,S_q act on NS field or Ramond field, respectively.

In the current chapter we will discuss only Neveu-Schwarz fields, thus for clarity we will skip the fields' indices NS. The analysis of the Ramond sector of SCFT will be presented in the next chapter.

2.1.2 Primary fields in the NS sector

As in the CFT case, we assume that the algebra of local fields contain primary fields which under conformal transformation change in a particularly simple way (1.3). Additionally, we assume that there exist *superprimary fields* $\phi_{\Delta,\bar{\Delta}}$ defined as such primary fields that satisfy

$$\begin{aligned} L_n\phi_{\Delta,\bar{\Delta}}(0,0) &= S_k\phi_{\Delta,\bar{\Delta}}(0,0) \;=\; 0, \qquad n,k>0, \\ L_0\phi_{\Delta,\bar{\Delta}}(0,0) &= \Delta\phi_{\Delta,\bar{\Delta}}(0,0) \end{aligned} \tag{2.6}$$

and similarly for the "right" generators $\bar{L}_n$ and $\bar{S}_k$.

There are three other primary fields related to the superprimary field:

$$\psi_{\Delta,\bar{\Delta}} \;=\; \left[S_{-1/2},\varphi_{\Delta,\bar{\Delta}}\right], \quad \bar{\psi}_{\Delta,\bar{\Delta}} \;=\; \left[\bar{S}_{-1/2},\varphi_{\Delta,\bar{\Delta}}\right], \quad \widetilde{\phi}_{\Delta,\bar{\Delta}} \;=\; \left\{S_{-1/2},\left[\bar{S}_{-1/2},\phi_{\Delta,\bar{\Delta}}\right]\right\} \tag{2.7}$$

The values of primary fields' conformal weights follows from the algebra (2.5):

$$\psi_{\Delta,\bar{\Delta}}\;:\;\left(\Delta+\frac{1}{2},\bar{\Delta}\right); \qquad \bar{\psi}_{\Delta,\bar{\Delta}}\;:\;\left(\Delta,\bar{\Delta}+\frac{1}{2}\right); \qquad \tilde{\phi}_{\Delta,\bar{\Delta}}\;:\;\left(\Delta+\frac{1}{2},\bar{\Delta}+\frac{1}{2}\right).$$

The local Ward identities for all four types of primary fields containing generator $T(z)$ are given by (1.4) with appropriate conformal weights. The local Ward identities containing generator $S(z)$ follow from (2.3), (2.5) and read:

$$\begin{aligned} S(z)\phi(w,\bar{w}) &= \frac{1}{z-w}\,\psi_{\Delta}(w,\bar{w}) + reg. \\ S(z)\psi(w,\bar{w}) &= \frac{2\Delta}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi_{\Delta}(w,\bar{w}) + reg. \end{aligned} \tag{2.8}$$

$$\begin{aligned} S(z)\bar{\psi}(w,\bar{w}) &= \frac{1}{z-w}\,\tilde{\phi}_{\Delta}(w,\bar{w}) + reg. \\ S(z)\tilde{\phi}(w,\bar{w}) &= \frac{2\Delta}{(z-w)^2}\,\bar{\psi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\bar{\psi}_{\Delta}(w,\bar{w}) + reg. \end{aligned} \tag{2.9}$$

2.1.3 NS supermodule

The *highest weight state* with respect to the NS algebra (2.5) is defined by the following conditions:

$$\mathrm{L}_m\left|\nu_\Delta\right\rangle = \mathrm{S}_k\left|\nu_\Delta\right\rangle = 0, \qquad \mathrm{L}_0\left|\nu_\Delta\right\rangle = \Delta\left|\nu_\Delta\right\rangle, \qquad m\in\mathbb{N},\; k\in\mathbb{N}-\frac{1}{2}. \tag{2.10}$$

We will denote

$$|{*\nu}\rangle \equiv S_{-\frac{1}{2}}\left|\nu_\Delta\right\rangle. \tag{2.11}$$

It is not the highest weight state with respect to NS algebra, but it is still the highest weight state with respect to the Virasoro algebra (1.8):

$$\mathrm{L}_m\left|*\nu_\Delta\right\rangle = 0, \qquad \mathrm{L}_0\left|*\nu_\Delta\right\rangle = \left(\Delta+\frac{1}{2}\right)\left|*\nu_\Delta\right\rangle, \qquad m\in\mathbb{N}.$$

The *descendant states* are the states created by an action of generators L_{-m} and S_{-k} on the highest weight state $|\nu_\Delta\rangle$. They form the vector space $\mathcal{V}^f_{c,\Delta}$ with the basis:

$$|\nu_{\Delta,KM}\rangle = S_{-K}\,L_{-M}\;|\nu_\Delta\rangle \;\equiv\; S_{-k_i}\ldots S_{-k_1}L_{-m_j}\ldots L_{-m_1}\,|\nu_\Delta\rangle\;. \tag{2.12}$$

$K=\{k_1,k_2,\ldots,k_i\}\subset\mathbb{N}-\frac{1}{2}$ and $M=\{m_1,m_2,\ldots,m_j\}\subset\mathbb{N}$ are arbitrary ordered sets of indices $k_i<\ldots<k_2<k_1,\ m_j\leq\ldots\leq m_2\leq m_1$, such that $|K|+|M|=k_1+\ldots+k_i+m_1+\ldots+m_j=f$. Each $\mathcal{V}^f_{c,\Delta}$ is an eigenspace of L_0 with the eigenvalue $\Delta+f$.

The direct sum of the spaces $\mathcal{V}^f_{c,\Delta}$ composes *superconformal NS module* of the highest weight Δ and central charge c:

$$\mathcal{V}_{c,\Delta}\;=\bigoplus_{f\in\frac{1}{2}\mathbb{N}\cup\{0\}}\mathcal{V}^f_{c,\Delta}\,,\qquad \mathcal{V}^0_{c,\Delta}=\mathbb{C}\,\nu_\Delta\,.$$

The *parity operator* is defined as follows:

$$(-1)^F=(-1)^{2(L_0-\Delta)}.$$

Any state $\xi_\Delta\in\mathcal{V}^f_{c,\Delta}$ as an eigenstate of L_0 has definite parity: $(-1)^{2f}$. Thus the NS module has a natural $\mathbb{Z}_2$ grading:

$$\mathcal{V}_{c,\Delta}\;=\;\mathcal{V}^+_{c,\Delta}\oplus\mathcal{V}^-_{c,\Delta}\,,\qquad \mathcal{V}^+_{c,\Delta}\;=\bigoplus_{m\in\mathbb{N}\cup\{0\}}\mathcal{V}^m_{c,\Delta}\,,\qquad \mathcal{V}^-_{c,\Delta}\;=\bigoplus_{k\in\mathbb{N}-\frac{1}{2}}\mathcal{V}^k_{c,\Delta}\,.$$

The scalar product is defined as a symmetric bilinear form $\langle.\,,.\rangle_{c,\Delta}$ on NS module $\mathcal{V}_{\Delta,c}$ such that

$$\langle\nu_\Delta\,,\,\nu_\Delta\rangle=1,\quad L_n^\dagger=L_{-n},\quad S_k^\dagger=S_{-k}.$$

The supersymmetric equivalent of Gram matrix is the matrix calculated in basis (2.12):

$$\left[B^n_{c,\Delta}\right]_{KM,LN}=\;\langle\nu_{\Delta,KM}\,,\,\nu_{\Delta,LN}\rangle_{c,\Delta}.$$

The Kac determinant has the form [32]:

$$\det B^f_{c,\Delta}\;=\;C\prod_{1\leqslant rs\leqslant 2f}(\Delta-\Delta_{rs})^{P_{NS}(f-\frac{rs}{2})} \tag{2.13}$$

where C depends only on the level f, the sum $r+s$ must be *even* and

$$\begin{aligned}\Delta_{rs}(c)\;&=\;-\frac{rs-1}{4}+\frac{r^2-1}{8}\beta^2+\frac{s^2-1}{8}\frac{1}{\beta^2}\,,\\ \beta\;&=\;\frac{1}{2\sqrt{2}}\left(\sqrt{1-\hat{c}}+\sqrt{9-\hat{c}}\right),\qquad \hat{c}=\frac{2}{3}c.\end{aligned} \tag{2.14}$$

As a function of central charge c the Kac determinant has zero at

$$c\;=\;c_{rs}(\Delta)\;\equiv\;\frac{3}{2}-3\left(\beta_{rs}(\Delta)-\frac{1}{\beta_{rs}(\Delta)}\right)^2, \tag{2.15}$$

where $1 < rs \leq 2n$, $1 < r$, $\;r + s \in 2\mathbb{N}$, and

$$\beta_{rs}^2(\Delta) \;=\; \frac{1}{r^2-1}\left(4\Delta + rs - 1 + \sqrt{16\Delta^2 + 8(rs-1)\Delta + (r-s)^2}\right).$$

The multiplicity of each zero is given by $P_{NS}(f) = \dim \mathcal{V}^f_{c,\Delta}$. It can be read from the formula:

$$\sum_{f=0}^{\infty} P_{NS}(f) q^f \;=\; \prod_{n=1}^{\infty} \frac{1+q^{n-\frac{1}{2}}}{1-q^n}.$$

The Gram matrix $B^f_{c,\Delta}$ is nonsingular if and only if the supermodule $\mathcal{V}_{c,\Delta}$ does not contain singular vectors of degree $\frac{1}{2}, 1, \ldots, f$. Let us remind that singular vector is a descendant state which is at the same time the highest weight state. The singular vectors appear in degenerate NS modules $\mathcal{V}_{c,\Delta_{rs}}$ at the level $\frac{rs}{2}$: $L_0\chi_{rs} = (\Delta_{rs} + \frac{rs}{2})\chi_{rs}$.

All the main properties of $B^f_{c,\Delta}$ matrix and Kac determinant are similar to the non-supersymmetric case. In particular, the inverse Gram matrix as a function of weight Δ (or central charge c) has simple poles at degenerate weights Δ_{rs} (or c_{rs}, respectively).

2.1.4 The space of states

We will assume that there exist a unique NS vacuum state $|0\rangle$ *i.e.* the highest weight state invariant with respect to the global superconformal transformations generated by $L_{\pm 1}, L_0, S_{\pm\frac{1}{2}}$ and their right counterparts.

Consider the state created by the superprimary field acting on the vacuum:

$$\lim_{z,\bar{z}\to 0} \phi_{\Delta,\bar{\Delta}}(z,\bar{z})\,|0\rangle \;=\; \left|\Delta,\bar{\Delta}\right\rangle.$$

One can show, by similar reasoning as in the bosonic case (1.14), that $\left|\Delta,\bar{\Delta}\right\rangle$ can be normalized to one. From the definition of superprimary field (2.6) it is clear that this state is a highest weight state with respect to the left and to the right NS algebras, so that

$$\left|\Delta,\bar{\Delta}\right\rangle = \left|\nu_\Delta \otimes \bar{\nu}_{\bar{\Delta}}\right\rangle.$$

The states created by the action of the NS generators on $\left|\Delta,\bar{\Delta}\right\rangle$ form the tensor product of NS supermodules $\mathcal{V}_{c,\Delta}$ and $\bar{\mathcal{V}}_{\bar{\Delta},c}$. We assume the following:

The space of states in the NS sector of superconformal field theory is a sum of the tensor products of the left and the right NS supermodules over the spectrum of NS superprimary fields:

$$\mathcal{H}_{NS} = \bigoplus_{(\Delta,\bar{\Delta})} \mathcal{V}_{\Delta,c} \otimes \bar{\mathcal{V}}_{\bar{\Delta},c}.$$

2.1.5 Field operators

In the SCFT case the assumption concerning the CFT states-fields correspondence is generalized in the following way:

In the NS sector of SCFT there is one to one correspondence between the states from the space of states $\mathcal{H}_{NS}$ and the NS field operators from the space of local operators.

Each superprimary field corresponds to a highest weight state with respect to left and right NS algebra:

$$\lim_{z,\bar{z}\to 0} \phi_{\Delta,\bar{\Delta}}(z,\bar{z})\,|0\rangle \;=\; |\nu_\Delta \otimes \bar{\nu}_{\bar{\Delta}}\rangle$$

The action of the NS generators on the states $\xi_\Delta \otimes \bar{\xi}_{\bar{\Delta}} \in \mathcal{V}_{\Delta,c} \otimes \bar{\mathcal{V}}_{\bar{\Delta},c}$ extends by the correspondence to the action on the fields. The *descendant fields* are defined by the relations:

$$\begin{aligned}
\mathcal{L}_{-m}\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}) \equiv \varphi_{\Delta,\bar{\Delta}}(L_{-m}\xi,\bar{\xi}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{1-m}T(w)\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}), \quad m\in\mathbb{N},\\
\mathcal{S}_{-k}\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}) \equiv \varphi_{\Delta,\bar{\Delta}}(S_{-k}\xi,\bar{\xi}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}-k}S(w)\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z}), \quad k\in\mathbb{N}-\tfrac{1}{2}.
\end{aligned} \tag{2.16}$$

so that

$$\lim_{z,\bar{z}\to 0} \varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z})\,|0\rangle \;=\; \left|\xi_\Delta \otimes \bar{\xi}_{\bar{\Delta}}\right\rangle .$$

The *parity of a field* $\varphi_{\Delta,\bar{\Delta}}(\xi,\bar{\xi}|z,\bar{z})$ is given by the product of chiral parities of the corresponding states $\xi \in \mathcal{V}^f_{c,\Delta}, \bar{\xi} \in \bar{\mathcal{V}}^{\bar{f}}_{c,\bar{\Delta}}$: $(-1)^{2f+2\bar{f}}$.

The superprimary field together with the descendant fields constitute a superconformal family. Any NS field belongs to some superconformal family with conformal weights from the spectrum of superprimary fields. In particular, the fields $T(z), \overline{T}(\bar{z}), S(z), \bar{S}(\bar{z})$ belong to the superconformal family with identity operator as a superprimary field.

The three types of primary fields introduced before (2.7) are the "lowest" descendants of the superprimary field. They correspond to the highest weight states with respect to the left and right Virasoro algebras (2.11):

$$\begin{aligned}
\psi_{\Delta,\bar{\Delta}}(z,\bar{z}) &= \varphi_{\Delta,\bar{\Delta}}(*\nu,\bar{\nu}|z,\bar{z}),\\
\bar{\psi}_{\Delta,\bar{\Delta}}(z,\bar{z}) &= \varphi_{\Delta,\bar{\Delta}}(\nu,*\bar{\nu}|z,\bar{z}),\\
\widetilde{\phi}_{\Delta,\bar{\Delta}}(z,\bar{z}) &= \varphi_{\Delta,\bar{\Delta}}(*\nu,*\bar{\nu}|z,\bar{z}).
\end{aligned} \tag{2.17}$$

2.1.6 Ward identities for correlation functions

First, let us impose the positive parity condition for correlation functions in the following form:

The correlation function can be non zero only if the total parity of all fields in the correlator is positive.

Consider now the n-point correlator of descendant fields defined by integrals of the form (2.16). Using the contour deformation method described in the first chapter, one can derive the Ward identities for correlation functions. The superconformal Ward identities allow to write any correlator of descendants in terms of functions of superprimary fields and primary fields (2.7).

In the case of a 3-point function of descendant fields the Ward identities containing Virasoro generators L_m have the same form as in the bosonic case (1.20). The relations containing generators S_k are given by:

$$\begin{aligned}
\left\langle \xi_3,\bar\xi_3\right|\varphi_2(S_{-k}\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle &= \sum_{n=0}^{\infty}\binom{k-\frac{3}{2}+n}{n} z^n \left\langle S_{k+n}\xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle \\
&+\epsilon\,(-1)^{k+\frac12}\sum_{n=0}^{\infty}\binom{k-\frac{3}{2}+n}{n} z^{-k+\frac12-n}\left\langle \xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|S_{n-\frac12}\xi_1,\bar\xi_1\right\rangle, \qquad k>\frac12,\\
\left\langle \xi_3,\bar\xi_3\right|\varphi_2(S_{k}\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle &= \sum_{n=0}^{k+\frac12}\binom{k+\frac12}{n}(-z)^n\Big(\left\langle S_{n-k}\xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle \\
&- \epsilon\,\left\langle \xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|S_{k-n}\xi_1,\bar\xi_1\right\rangle\Big), \qquad k\geq\frac12,\\
\left\langle S_{-k}\xi_3,\bar\xi_3\right|\varphi_2(\xi_2,\bar\xi_2|z,\bar z)\left|\xi_1,\bar\xi_1\right\rangle &= \epsilon\,\langle\, \xi_3,\bar\xi_3\,|\varphi_{\Delta_2,\bar\Delta_2}(\xi_2,\bar\xi_2|z,\bar z)|\,S_k\xi_1,\bar\xi_1\,\rangle \\
&+\sum_{m=-1}^{l(k-\frac12)}\binom{k+\frac12}{m+1} z^{k-\frac12-m}\langle\, \xi_3,\bar\xi_3\,|\varphi_{\Delta_2,\bar\Delta_2}(S_{m+\frac12}\xi_2,\bar\xi_2|z,\bar z)|\,\xi_1,\bar\xi_1\,\rangle.
\end{aligned} \tag{2.18}$$

Here ϵ denotes parity of the field $\varphi_2(\xi_2,\bar\xi_2|z,\bar z)$, $l(n)=n$ for $n+1\geqslant 0$, and $l(n)=\infty$ for $n+1<0$. Analogous identities hold for the antiholomorphic current $\bar S(\bar z)$.

Let us notice that above relations, in contrast to the Ward identities in non supersymmetric CFT, do not allow to reduce a 3-point correlation function to one structure constant. There are two independent structure constants, appearing in two different cases. The first case occurs when the total number of operators S_k acting on the fields is even. Then the correlator is proportional to structure constant C_{321}:

$$C_{321} \;=\; \langle 0|\,\phi_3(\infty,\infty)\phi_2(1,1)\phi_1(0,0)\,|0\rangle \;=\; \left\langle \nu_{\Delta_3}\otimes\bar\nu_{\bar\Delta_3}\right|\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle. \tag{2.19}$$

Otherwise, if the total number of operators S_k is odd, the function will reduce to a correlator with one operator $S_{-\frac12}$ acting on one of the fields (see the last identity in (2.18)). In order to ensure positive parity of the initial correlator, the number of operators $\bar S_k$ has to be odd as well. Thus in this case the initial function will be proportional to the structure $\widetilde C_{321}$ defined in terms of the primary field $\widetilde\phi(z,\bar z)=\varphi(S_{-\frac12}\nu,\bar S_{-\frac12}\bar\nu|z,\bar z)$:

$$\widetilde C_{321} \;=\; \langle 0|\,\phi_3(\infty,\infty)\widetilde\phi_2(1,1)\phi_1(0,0)\,|0\rangle \;=\; \left\langle \nu_{\Delta_3}\otimes\bar\nu_{\bar\Delta_3}\right|\widetilde\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle. \tag{2.20}$$

The fact that in SCFT there exist two independent structure constants can be seen also from the local Ward identities (2.8), (2.9). Take one of the primary fields. The action of $S(z)$ on this field is given in terms of a different primary field. The double action of $S(z)$ on the primary field can be expressed in terms of the same primary field.

In the case of an arbitrary 4-point function, the Ward identities allow to write the correlator in terms of eight independent correlators of primary fields:

$$\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\phi_3(z,\bar z)\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, \qquad \left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\psi_3(z,\bar z)\psi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle,$$

$$\begin{aligned}
&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\widetilde\phi_3(z,\bar z)\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, &&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\bar\psi_3(z,\bar z)\psi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, \\
&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\phi_3(z,\bar z)\widetilde\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, &&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\psi_3(z,\bar z)\bar\psi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, \\
&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\widetilde\phi_3(z,\bar z)\widetilde\phi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle, &&\left\langle \nu_{\Delta_4}\otimes\bar\nu_{\bar\Delta_4}\right|\bar\psi_3(z,\bar z)\bar\psi_2(1,1)\left|\nu_{\Delta_1}\otimes\bar\nu_{\bar\Delta_1}\right\rangle.
\end{aligned} \tag{2.21}$$

The global superconformal transformations (generated by $L_0, S_{\pm\frac12}, L_{\pm1}$) allow to choose two fields as superprimary ones and to fix the locations of the fields in the standard way $(0,0),(1,1),(\infty,\infty)$ with $(z,\bar z)$ given by (1.19).

2.2 The 3-point block

2.2.1 Definition of the 3-point block

Let us define on NS modules a chiral trilinear map:

$$\varrho(\xi_3,\xi_2,\xi_1|z) : \mathcal{V}_{\Delta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{V}_{\Delta_1} \;\mapsto\; \mathbb{C}\ .$$

In order to ensure that an arbitrary 3-point function could be written in terms of it:

$$\langle\, \xi_3,\bar\xi_3\,|\varphi_{\Delta_2,\bar\Delta_2}(\xi_2,\bar\xi_2|z,\bar z)|\,\xi_1,\bar\xi_1\,\rangle \;=\; \varrho(\xi_3,\xi_2,\xi_1|z)\varrho(\bar\xi_3,\bar\xi_2,\bar\xi_1|\bar z), \tag{2.22}$$

we impose the the following conditions:

$$\begin{aligned}
\varrho(\xi_3,S_k\xi_2,\xi_1|z) \;&=\; \sum_{m=0}^{k+\frac12}\binom{k+\frac12}{m}(-z)^m\,\big(\varrho(S_{m-k}\xi_3,\xi_2,\xi_1|z) \\
&\qquad -\,(-1)^{2(N(\xi_1)+N(\xi_3))}\;\varrho(\xi_3,\xi_2,S_{k-m}\xi_1|z)\big)\,,\qquad k\geqslant -\tfrac12,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
\varrho(\xi_3,S_{-k}\xi_2,\xi_1|z) \;&=\; \sum_{m=0}^{\infty}\binom{k-\frac32+m}{m}z^m\varrho(S_{k+m}\xi_3,\xi_2,\xi_1|z) \\
-\,(-1)^{2(N(\xi_1)+N(\xi_3))+k+\frac12}&\sum_{m=0}^{\infty}\binom{k-\frac32+m}{m}z^{-k-m+\frac12}\varrho(\xi_3,\xi_2,S_{m-\frac12}\xi_1|z),\qquad k>\tfrac12.
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
\varrho(S_{-k}\xi_3,\xi_2,\xi_1|z) \;&=\; (-1)^{2(N(\xi_1)+N(\xi_3))+1}\varrho(\xi_3,\xi_2,S_k\xi_1|z) \\
&+\;\sum_{m=-1}^{l(k-\frac12)}\binom{k+\frac12}{m+1}z^{k-\frac12-m}\varrho(\xi_3,S_{m+\frac12}\xi_2,\xi_1|z)
\end{aligned} \tag{2.25}$$

and corresponding relations with Virasoro generators of the form (1.22)-(1.25), which we will repeat for completeness:

$$\varrho(\xi_3,L_{-1}\xi_2,\xi_1|z) \;=\; \partial_z\varrho(\xi_3,\xi_2,\xi_1|z), \tag{2.26}$$

$$\begin{aligned}
\varrho(\xi_3,L_n\xi_2,\xi_1|z) \;&=\; \sum_{m=0}^{n+1}\binom{n+1}{m}(-z)^m\Big(\varrho(L_{m-n}\xi_3,\xi_2,\xi_1|z) \\
&\qquad -\,\varrho(\xi_3,\xi_2,L_{n-m}\xi_1|z)\Big),\qquad n>-1,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}\varrho(\xi_3, L_{-n}\xi_2, \xi_1|z) &= \sum_{m=0}^{\infty} \binom{n-2+m}{n-2} z^m \varrho(L_{n+m}\xi_3, \xi_2, \xi_1|z) \\ &+ (-1)^n \sum_{m=0}^{\infty} \binom{n-2+m}{n-2} z^{-n+1-m} \varrho(\xi_3, \xi_2, L_{m-1}\xi_1|z), \quad n > 1,\end{aligned} \tag{2.28}$$

$$\begin{aligned}\varrho(L_{-n}\xi_3, \xi_2, \xi_1|z) &= \varrho(\xi_3, \xi_2, L_n\xi_1|z) \\ &+ \sum_{m=-1}^{l(n)} \binom{n+1}{m+1} z^{n-m} \varrho(\xi_3, L_m\xi_2, \xi_1|z).\end{aligned} \tag{2.29}$$

In the formulae (2.23)-(2.25) the symbol $N(\xi_i)$ denotes a level of excitation of the state ξ_i.

The 3-form $\varrho(\xi_3, \xi_2, \xi_1|z)$ is determined by the conditions above up to two independent constants:

$$\begin{aligned}&\varrho(\nu_3, \nu_2, \nu_1|1), \\ &\varrho(\nu_3, *\nu_2, \nu_1|1) = \varrho(*\nu_3, \nu_2, \nu_1|1) = \varrho(\nu_3, \nu_2, *\nu_1|1).\end{aligned} \tag{2.30}$$

More precisely, if the total parity of all the states ξ_i is positive, then the 3-form is proportional to $\varrho(\nu_3, \nu_2, \nu_1|1)$. Otherwise, the 3-form is proportional to the second constant $\varrho(\nu_3, *\nu_2, \nu_1|1)$.

The *3-point block* $\rho(\xi_3, \xi_2, \xi_1|z)$ is defined as a normalized 3-form, *i.e.* a function proportional to one of the constants:

$$\varrho(\xi_3, \xi_2, \xi_1|z) = \rho(\xi_3, \xi_2, \xi_1|z) \times \begin{cases} \varrho(\nu_3, \nu_2, \nu_1|1) \\ \varrho(\nu_3, *\nu_2, \nu_1|1) \end{cases} \tag{2.31}$$

The upper line corresponds to the case of positive total parity of all states ξ_i, while the lower line corresponds to the case of negative total parity. The normalization condition for the block gives:

$$\rho(\nu_3, \nu_2, \nu_1|1) = \rho(\nu_3, *\nu_2, \nu_1|1) = 1.$$

The z dependence of the block is determined by the Ward identities containing Virasoro generators and has the same form as the 3-point block in CFT (1.26):

$$\rho(\xi_3, \xi_2, \xi_1|z) = z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)} \rho(\xi_3, \xi_2, \xi_1),$$

where

$$\rho(\xi_3, \xi_2, \xi_1) \equiv \rho(\xi_3, \xi_2, \xi_1|1).$$

Any 3-point correlation function can be written in terms of the 3-point blocks and structure constants. Inserting equation (2.31) into (2.22) one gets the following formula:

$$\begin{aligned}\langle\, \xi_3, \bar{\xi}_3 \,| \varphi_{\Delta_2, \bar{\Delta}_2}(\xi_2, \bar{\xi}_2|z, \bar{z}) |\, \xi_1, \bar{\xi}_1 \,\rangle &= z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)} \bar{z}^{\bar{\Delta}_3(\bar{\xi}_3)-\bar{\Delta}_2(\bar{\xi}_2)-\bar{\Delta}_1(\bar{\xi}_1)} \\ &\times \rho(\xi_3, \xi_2, \xi_1|1)\rho(\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_1|1) \times \begin{cases} C_{321} \\ \widetilde{C}_{321} \end{cases}\end{aligned} \tag{2.32}$$

The structure constants are specific combinations of the constants (2.30):

$$\begin{aligned} C_{321} &= \left\langle \nu_{\Delta_3} \otimes \bar{\nu}_{\bar{\Delta}_3} \right| \phi_2(1,1) \left| \nu_{\Delta_1} \otimes \bar{\nu}_{\bar{\Delta}_1} \right\rangle = \varrho(\nu_3, \nu_2, \nu_1|1)\, \varrho(\bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1|1)\,, \\ \widetilde{C}_{321} &= \left\langle \nu_{\Delta_3} \otimes \bar{\nu}_{\bar{\Delta}_3} \right| \widetilde{\phi}_2(1,1) \left| \nu_{\Delta_1} \otimes \bar{\nu}_{\bar{\Delta}_1} \right\rangle = \varrho(\nu_3, *\nu_2, \nu_1|1)\, \varrho(\bar{\nu}_3, *\bar{\nu}_2, \bar{\nu}_1|1)\,. \end{aligned} \tag{2.33}$$

Other combinations of constants (2.30) do not occur because of the positive parity condition imposed on the correlation functions.

2.2.2 Chiral vertex operator

Let us define a generalized chiral vertex operator. For a given state ξ_2 it is a linear map

$$V(\xi_2|z) : \mathcal{V}_{\Delta_1} \;\mapsto\; \mathcal{V}_{\Delta_3}\,,$$

such that its matrix element between arbitrary states $\xi_3 \in \mathcal{V}_{\Delta_3}, \xi_1 \in \mathcal{V}_{\Delta_1}$ is given by the 3-point block:

$$\langle\, \xi_3 \,| V(\xi_2|z) |\, \xi_1 \,\rangle \;=\; \rho(\xi_3, \xi_2, \xi_1|z).$$

The chiral vertex does not have definite parity and can be decomposed into even (parity preserving) and odd (parity reversing) part:

$$V(\xi_2|z) = V^{\mathrm{even}}(\xi_2|z) + V^{\mathrm{odd}}(\xi_2|z).$$

We would like to find the relations between primary fields (2.17) and chiral vertex operators. The Ward identities for 3-point function imply decomposition of the correlator onto 3-point blocks and structure constants (2.32). When the second field is a primary one, the decomposition takes the form:

$$\langle\, \xi_3, \bar{\xi}_3 \,| \varphi_{\Delta_2,\bar{\Delta}_2}({}_{-}\nu_2, {}_{-}\bar{\nu}_2 | z, \bar{z}) |\, \xi_1, \bar{\xi}_1 \,\rangle \;= \rho(\xi_3, {}_{-}\nu_2, \xi_1|z) \rho(\bar{\xi}_3, {}_{-}\bar{\nu}_2, \bar{\xi}_1|\bar{z}) \;\times \begin{cases} C_{321} \\ \widetilde{C}_{321} \end{cases},$$

where the upper (lower) line corresponds to the case of positive (negative) total parity of all states. The notation ${}_{-}\nu_i$ stands for ν_i or $*\nu_i$. Knowing that 3-point blocks are matrix elements of chiral vertex operator it is straightforward to find the relations:

$$\begin{aligned} \phi_2(z,\bar{z}) &= \bigoplus_{\Delta_3,\Delta_1} \left(C_{321} V^{\mathrm{even}}(\nu_2|z) \otimes V^{\mathrm{even}}(\bar{\nu}_2|\bar{z}) - \tilde{C}_{321} V^{\mathrm{odd}}(\nu_2|z) \otimes V^{\mathrm{odd}}(\bar{\nu}_2|\bar{z}) \right), \\ \psi_2(z,\bar{z}) &= \bigoplus_{\Delta_3,\Delta_1} \left(C_{321} V^{\mathrm{odd}}(*\nu_2|z) \otimes V^{\mathrm{even}}(\bar{\nu}_2|\bar{z}) - \tilde{C}_{321} V^{\mathrm{even}}(*\nu_2|z) \otimes V^{\mathrm{odd}}(\bar{\nu}_2|\bar{z}) \right), \\ \bar{\psi}_2(z,\bar{z}) &= \bigoplus_{\Delta_3,\Delta_1} \left(C_{321} V^{\mathrm{even}}(\nu_2|z) \otimes V^{\mathrm{odd}}(*\bar{\nu}_2|\bar{z}) + \tilde{C}_{321} V^{\mathrm{odd}}(\nu_2|z) \otimes V^{\mathrm{even}}(*\bar{\nu}_2|\bar{z}) \right), \\ \tilde{\phi}_2(z,\bar{z}) &= \bigoplus_{\Delta_3,\Delta_1} \left(C_{321} V^{\mathrm{odd}}(*\nu_2|z) \otimes V^{\mathrm{odd}}(*\bar{\nu}_2|\bar{z}) + \tilde{C}_{321} V^{\mathrm{even}}(*\nu_2|z) \otimes V^{\mathrm{even}}(*\bar{\nu}_2|\bar{z}) \right). \end{aligned} \tag{2.34}$$

The sign in front of each term is determined by the law of composition of tensor products of homogeneous elements:

$$(A \otimes \bar{A})(B \otimes \bar{B}) = (-1)^{\deg(\bar{A})\cdot\deg(B)} AB \otimes \bar{A}\bar{B} \ .$$

The commutation relations of $V(\nu|z), V(*\nu|z)$ with superconformal generators can be read off from the Ward identities for chiral 3-form (2.25), (2.26) and (2.29):

$$\begin{aligned} [\mathrm{L}_m, V(\nu_2|z)] &= z^m \left(z\partial_z + (m+1)\Delta_2\right) V(\nu_2|z), \\ [\mathrm{L}_m, V(*\nu_2|z)] &= z^m \left(z\partial_z + (m+1)\left(\Delta_2 + \tfrac{1}{2}\right)\right) V(*\nu_2|z), \end{aligned} \tag{2.35}$$

$$\begin{aligned} [\mathrm{S}_k, V^{\mathrm{even}}(\nu_2|z)] &= z^{k+\frac{1}{2}}\ V^{\mathrm{odd}}(*\nu_2|z), \\ \{\mathrm{S}_k, V^{\mathrm{odd}}(\nu_2|z)\} &= z^{k+\frac{1}{2}}\ V^{\mathrm{even}}(*\nu_2|z), \end{aligned} \tag{2.36}$$

$$\begin{aligned} [\mathrm{S}_k, V^{\mathrm{even}}(*\nu_2|z)] &= z^{k-\frac{1}{2}} \left(z\partial_z + \Delta_2(2k+1)\right)\ V^{\mathrm{odd}}(\nu_2|z), \\ \{\mathrm{S}_k, V^{\mathrm{odd}}(*\nu_2|z)\} &= z^{k-\frac{1}{2}} \left(z\partial_z + \Delta_2(2k+1)\right)\ V^{\mathrm{even}}(\nu_2|z). \end{aligned} \tag{2.37}$$

2.2.3 Properties of the 3-point block

We are interested in the properties of the 3-point block for which one of the external states is a descendant state. These objects will appear in the decomposition of the 4-point correlation functions (2.21) into chiral blocks and structure constants.

The matrix elements of a chiral vertex operators between one descendant and one highest weight state can be calculated with the help of the commutators (2.35) - (2.37):

$$\rho(\nu_{3,KM}, \nu_2, \nu_1|z) = z^{\Delta_3+|K|+|M|-\Delta_2-\Delta_1} \begin{cases} \eta^{\mathrm{o}}_{\Delta_3+|M|}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \gamma_{\Delta_3}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_M, \\[2ex] \eta^{\mathrm{e}}_{\Delta_3+|M|}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \gamma_{\Delta_3}\begin{bmatrix}\Delta_2+\frac{1}{2}\\ \Delta_1\end{bmatrix}_M, \end{cases} \tag{2.38}$$

$$\rho(\nu_{3,KM}, *\nu_2, \nu_1|z) = z^{\Delta_3+|K|+|M|-\Delta_2-\Delta_1-\frac{1}{2}} \begin{cases} \eta^{\mathrm{e}}_{\Delta_3+|M|}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \gamma_{\Delta_3}\begin{bmatrix}\Delta_2+\frac{1}{2}\\ \Delta_1\end{bmatrix}_M, \\[2ex] \eta^{\mathrm{o}}_{\Delta_3+|M|}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \gamma_{\Delta_3}\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_M, \end{cases} \tag{2.39}$$

with the upper lines corresponding to $|K| \in \mathbb{N}$, the lower lines to $|K| \in \mathbb{N} - \frac{1}{2}$ and

$$\gamma_\Delta\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_M \stackrel{\mathrm{def}}{=} \left(\Delta - \Delta_1 + m_1\Delta_2\right)\left(\Delta - \Delta_1 + m_2\Delta_2 + m_1\right)\cdots\left(\Delta - \Delta_1 + m_j\Delta_2 + \sum_{l=1}^{j-1} m_l\right),$$

$$\eta^{\mathrm{o}}_\Delta\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \stackrel{\mathrm{def}}{=} \left(\Delta - \Delta_1 + 2k_1\Delta_2\right)\left(\Delta - \Delta_1 + 2k_3\Delta_2 + k_1 + k_2\right)\ldots\left(\Delta - \Delta_1 + 2k_p\Delta_2 + \sum_{l=1}^{p-1} k_l\right),$$

$$\eta^{\mathrm{e}}_\Delta\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}_K \stackrel{\mathrm{def}}{=} \left(\Delta - \Delta_1 + 2k_2\Delta_2 + k_1\right)\left(\Delta - \Delta_1 + 2k_4\Delta_2 + \sum_{l=1}^{3} k_l\right)..\left(\Delta - \Delta_1 + 2k_{p'}\Delta_2 + \sum_{l=1}^{p'-1} k_l\right).$$

Denote the number of operators S_k acting on the state ν_3 by i. Then p is the largest odd integer not grater than i, and p' is the largest even integer not grater than i. Notice that the function $\gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_M$ obtained from the action of Virasoro generators is the same as in the bosonic case (1.30). Thus as a function of conformal weights Δ, Δ_i the 3-point block is a polynomial of maximal degree $j+p$ or $j+p'$, depending on the case (2.38),(2.39).

Applying the commutation rules (2.35) - (2.37) to the matrix elements of chiral vertex operators $\rho(\nu_3, \nu_2, \nu_{1,KM})$ one can derive the relations:

$$
\begin{aligned}
\rho(\nu_3, \nu_2, \nu_{1,KM}) &= \rho(\nu_{1,KM}, \nu_2, \nu_3),\\
\rho(\nu_3, *\nu_2, \nu_{1,KM}) &= \rho(\nu_{1,KM}, *\nu_2, \nu_3),
\end{aligned}
\tag{2.40}
$$

for $|K| \in \mathbb{N} \cup \{0\}$, and

$$
\begin{aligned}
\rho(\nu_3, \nu_2, \nu_{1,KM}) &= \rho(\nu_{1,KM}, \nu_2, \nu_3),\\
\rho(\nu_3, *\nu_2, \nu_{1,KM}) &= -\,\rho(\nu_{1,KM}, *\nu_2, \nu_3),
\end{aligned}
\tag{2.41}
$$

for $|K| \in \mathbb{N} - \frac{1}{2}$.

Another important feature of the 3-point block is the factorization property. It follows from the commutation relations (2.35) - (2.37) that for an arbitrary state $\xi_3 \in \mathcal{V}^f_{\Delta_3}$ one has:

$$
\begin{aligned}
\varrho(\mathrm{S}_{-K}\mathrm{L}_{-M}\,\xi_3, \nu_2, \nu_1|z) &= z^{\Delta_3+f+|K|+|M|-\Delta_2-\Delta_1}\times\\
&\rho(\mathrm{S}_{-K}\mathrm{L}_{-M}\nu_{\Delta_3+f}, \nu_2, \nu_1)\times\begin{cases}\varrho(\xi_3, \nu_2, \nu_1|1),\\ \varrho(\xi_3, *\nu_2, \nu_1|1),\end{cases}\\
\varrho(\mathrm{S}_{-K}\mathrm{L}_{-M}\,\xi_3, *\nu_2, \nu_1|z) &= z^{\Delta_3+f+|K|+|M|-\Delta_2-\Delta_1-\frac{1}{2}}\times\\
&\rho(\mathrm{S}_{-K}\mathrm{L}_{-M}\nu_{\Delta_3+f}, *\nu_2, \nu_1)\times\begin{cases}\varrho(\xi_3, *\nu_2, \nu_1|1),\\ \varrho(\xi_3, \nu_2, \nu_1|1),\end{cases}
\end{aligned}
\tag{2.42}
$$

where upper lines correspond to $|K| \in \mathbb{N} \cup \{0\}$, and the lower lines to $|K| \in \mathbb{N} - \frac{1}{2}$. The explicit dependence of constants $\varrho(\xi_3, \_\nu_2, \nu_1|1)$ will be helpful in analyzing the fusion rules.

2.2.4 Fusion rules and fusion polynomials

Fusion rules specify when the 3-point structure constants with one degenerate field ϕ_{rs} can be non-zero. Each time the values of conformal weights Δ_1, Δ_2 fulfill a fusion rule, the 3-point block $\rho(\chi_{rs}, \_\nu_2, \nu_1)$ vanishes. This mechanism ensures that 3-point correlation function with zero field is always zero: for a given set of weights either the structure constant or the 3-point block has to vanish.

In the case of superconformal theory there are two kinds of structure constants $C_{(rs)21}, \widetilde{C}_{(rs)21}$. To write down fusion rules for them, one can use the Feigin-Fuchs construction [5]. In this special formulation of a free superscalar theory the screening charges are given by:

$$Q_b = \oint dz\, \psi(z) e^{b\phi(z)}, \qquad Q_{\frac{1}{b}} = \oint dz\, \psi(z) e^{\frac{1}{b}\phi(z)},$$

where $\psi(z)$ is the free fermionic current. The screening charges in the right sector are constructed in the same way in terms of antiholomorphic ferminionic current $\bar{\psi}(\bar{z})$. The screening charges, once inserted into a correlation function, modify its total charge and parity, leaving other properties unchanged. Because positive total parity of the structure constants has to be preserved, only even number of screening charges can be added into the correlators. Thus there are two possible choices: one can add either even number of left screening charges or odd number of left and odd number of right screening charges.

In free superscalar theory there is additional condition concerning parity. Not only total parity of the correlation function should be positive, the left and the right chiral parities have to be positive as well. Thus the structure constant $C_{(rs)21}$ composed from superprimary fields, which are even in both holomorphic and antiholomorphic sector, is represented by correlator with even number of left screening charges:

$$C_{(\alpha_{rs},\delta),(\alpha_2,0),(\alpha_1,0)} \;=\; \left\langle \phi_{rs}\phi_2\phi_1\, Q_b^k\, Q_{\frac{1}{b}}^l \right\rangle \;, \quad k+l \in 2\mathbb{N} \;, \quad \delta = -\tfrac{1}{2\sqrt{2}}(\tfrac{1}{b}+b)\;;$$

It does not vanish if and only if charge conservation condition is satisfied. For even number of screening charges this condition (with $b = i\beta$) is equivalent to the *even fusion rule*:

$$\alpha_2 \pm \alpha_1 \;=\; (1-r+2k)\beta - (1-s+2l)\frac{1}{\beta}, \qquad k+l \in 2\mathbb{N}\cup\{0\}, \tag{2.43}$$

which has to be fulfilled by weights $\Delta_i = -\frac{1}{8}\left(\beta - \frac{1}{\beta}\right)^2 + \frac{\alpha_i^2}{8}$. The parameters k, l are integers in the range $0 \le k \le r-1,\ 0 \le l \le s-1$.

The constant $\widetilde{C}_{(rs)21}$ including one field $\tilde{\phi}$, which has negative both left and right chiral parities, is represented by the correlator with an odd number of insertions of left screening charges:

$$\widetilde{C}_{(\alpha_{rs},\delta),(\alpha_2,0),(\alpha_1,0)} \;=\; \left\langle \phi_{rs}\tilde{\phi}_2\phi_1\, Q_b^k\, Q_{\frac{1}{b}}^l\, \bar{Q}_b \right\rangle \;, \quad k+l \in 2\mathbb{N}+1 \;, \quad \delta = \tfrac{1}{2\sqrt{2}}(\tfrac{1}{b}-b).$$

Charge conservation implies that this structure constant does not vanish if and only if the *odd fusion rule*:

$$\alpha_2 \pm \alpha_1 \;=\; (1-r+2k)\beta - (1-s+2l)\frac{1}{\beta}, \qquad k+l \in 2\mathbb{N}-1, \tag{2.44}$$

is satisfied.

Discussed conditions for structure constants together with the definition of the 3-form as chiral part of correlation function:

$$\begin{aligned} C_{(rs)21} &= \varrho(\nu_{rs},\nu_2,\nu_1|1)\,\varrho(\bar{\nu}_3,\bar{\nu}_2,\bar{\nu}_1|1)\,, \\ \widetilde{C}_{(rs)21} &= \varrho(\nu_{rs},*\nu_2,\nu_1|1)\,\varrho(\bar{\nu}_3,*\bar{\nu}_2,\bar{\nu}_1|1)\,. \end{aligned}$$

lead to the following conclusions:

1. $\varrho(\nu_{rs},\nu_2,\nu_1|1)\neq 0$ if and only if *even fusion rule* is satisfied,

2. $\varrho(\nu_{rs},*\nu_2,\nu_1|1)\neq 0$ if and only if *odd fusion rule* is satisfied.

Consider now the 3-point function with a zero field corresponding to the null vector χ_{rs}:

$$\langle\,\chi_{rs},\bar{\xi}_3\,|\varphi_{\Delta_2,\bar{\Delta}_2}(\_\nu_2,\bar{\nu}_2|z,\bar{z})|\,\nu_1,\bar{\nu}_1\,\rangle \;=\; \varrho(\chi_{rs},\_\nu_2,\nu_1|1)\varrho(\bar{\xi}_3,\bar{\nu}_2,\bar{\nu}_1|1).$$

Vanishing of this function implies that the 3-form depending of χ_{rs} has to be zero:

$$\varrho(\chi_{rs},\_\nu_2,\nu_1|1) \;=\; 0,$$

From factorization property (2.42), which in this case takes the form:

$$\varrho(\chi_{rs},\nu_2,\nu_1|1) \;=\; \rho(\chi_{rs},\nu_2,\nu_1)\times\begin{cases}\varrho(\nu_{rs},\nu_2,\nu_1|1), & \text{for } \frac{rs}{2}\in\mathbb{N},\\ \varrho(\nu_{rs},*\nu_2,\nu_1|1), & \text{for } \frac{rs}{2}\in\mathbb{N}-\frac{1}{2},\end{cases}$$

$$\varrho(\chi_{rs},*\nu_2,\nu_1|1)=\rho(\chi_{rs},*\nu_2,\nu_1)\times\begin{cases}\varrho(\nu_{rs},*\nu_2,\nu_1|1), & \text{for } \frac{rs}{2}\in\mathbb{N},\\ \varrho(\nu_{rs},\nu_2,\nu_1|1), & \text{for } \frac{rs}{2}\in\mathbb{N}-\frac{1}{2},\end{cases}$$

one can see that the 3-point block has to vanish

$$\rho(\chi_{rs},\_\nu_2,\nu_1) \;=\; 0$$

each time the appropriate fusion rule is fulfilled and the corresponding constant is non-zero.

Let us now define the fusion polynomials. The first one is a function which vanishes if one of even fusion rules is satisfied:

$$P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix} \;=\; \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\frac{\alpha_2-\alpha_1+p\beta-q\beta^{-1}}{2\sqrt{2}}\right)\left(\frac{\alpha_2+\alpha_1+p\beta-q\beta^{-1}}{2\sqrt{2}}\right) \tag{2.45}$$

where p,q are related to previously used variables: $p=r-1-2k,\;q=s-1-2l;\;k+l\in 2\mathbb{N}$ and thus $p+q-(r+s)\in 4\mathbb{Z}+2$. The second fusion polynomial has zero each time the odd fusion rule holds:

$$P^{rs}_c\begin{bmatrix}*\Delta_2\\ \Delta_1\end{bmatrix} \;=\; \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\frac{\alpha_2-\alpha_1+p\beta-q\beta^{-1}}{2\sqrt{2}}\right)\left(\frac{\alpha_2+\alpha_1+p\beta-q\beta^{-1}}{2\sqrt{2}}\right) \tag{2.46}$$

with $p+q-(r+s)\in 4\mathbb{Z}$ (corresponding to $k+l\in 2\mathbb{N}-1$).

One can check that $P^{rs}_c\begin{bmatrix}\Delta_2\\ \Delta_1\end{bmatrix}$ is a polynomial of degree $\left[\frac{rs+1}{2}\right]$ in the variable $\Delta_2-\Delta_1$ and of degree $\left[\frac{rs+1}{4}\right]$ in $\Delta_2+\Delta_1$. $P^{rs}_c\begin{bmatrix}*\Delta_2\\ \Delta_1\end{bmatrix}$ is a polynomial of degree $\left[\frac{rs}{2}\right]$ in the variable $\Delta_2-\Delta_1$ and of degree $\left[\frac{rs}{4}\right]$ in $\Delta_2+\Delta_1$. The coefficients of highest powers of $\Delta_2-\Delta_1$ in both polynomials are equal 1. All these properties uniquely determine the polynomials P^{rs}_c. They are also common for the 3-point blocks (2.38), (2.39) with appropriate choice of arguments $\rho\,(\chi_{rs}, \_\nu_2, \nu_1)$. Thus we have the following equalities:

$$\rho\,(\chi_{rs}, \nu_2, \nu_1) \;=\; \begin{cases} P^{rs}_c \left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right] & \text{for } \frac{rs}{2} \in \mathbb{N}, \\ P^{rs}_c \left[\begin{smallmatrix}*\Delta_2\\ \Delta_1\end{smallmatrix}\right] & \text{for } \frac{rs}{2} \in \mathbb{N} - \frac{1}{2}, \end{cases}$$
$$\rho\,(\chi_{rs}, *\nu_2, \nu_1) \;=\; \begin{cases} P^{rs}_c \left[\begin{smallmatrix}*\Delta_2\\ \Delta_1\end{smallmatrix}\right] & \text{for } \frac{rs}{2} \in \mathbb{N}, \\ P^{rs}_c \left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right] & \text{for } \frac{rs}{2} \in \mathbb{N} - \frac{1}{2}, \end{cases} \tag{2.47}$$

which will be helpful in deriving recursive methods of determining of 4-point superconformal blocks.

2.3 4-point NS superconformal blocks

2.3.1 Definition

Let us consider a correlation function of four superprimary fields:

$$\begin{aligned}\langle 0|\, \phi_4(\infty,\infty)\phi_3(1,1)\phi_2(z,\bar z)\phi_1(0,0)\,|0\rangle &= \langle \nu_4 \otimes \bar\nu_4|\, \phi_3(1,1)\,\mathbf{1}\,\phi_2(z,\bar z)\,|\nu_1 \otimes \bar\nu_1\rangle \\ &= \sum_p \sum_{f=|K|+|M|=|L|+|N|} \langle \nu_4 \otimes \bar\nu_4|\, \phi_3(1,1)|\nu_{p,KM} \otimes \bar\nu_{p,\bar K\bar M}\rangle \left[B^f_{c,\Delta_p}\right]^{KM,LN} \\ &\qquad \times \left[\bar B^f_{c,\bar\Delta_p}\right]^{\bar K\bar M,\bar L\bar N} \langle \nu_{p,LN} \otimes \bar\nu_{p,\bar L\bar N}|\phi_2(z,\bar z)\,|\nu_1 \otimes \bar\nu_1\rangle\end{aligned}$$

Using relation (2.32) following from Ward identities, we can write the 3-point functions in terms of 3-point blocks with one descendant state from level f. Notice that the sum over f decompose into two parts: a sum over even states ($f \in \mathbb{N}$) and a sum over odd states ($f \in \mathbb{N} - \frac{1}{2}$).

$$\begin{aligned}&\langle \nu_4 \otimes \bar\nu_4|\, \phi_3(1,1)\phi_2(z,\bar z)\,|\nu_1 \otimes \bar\nu_1\rangle \\ &\quad = \sum_p \sum_{f\in\mathbb{N}} C_{43p}\; C_{p21}\,\rho(\nu_4,\nu_3,\nu_{p,KM}) \left[B^f_{c,\Delta_p}\right]^{KM,LN} \rho(\nu_{p,LN},\nu_2,\nu_1|z) \\ &\qquad\qquad \times \rho(\bar\nu_4,\bar\nu_3,\bar\nu_{p,\bar K\bar M}) \left[\bar B^f_{c,\bar\Delta_p}\right]^{\bar K\bar M,\bar L\bar N} \rho(\bar\nu_{p,\bar L\bar N},\bar\nu_2,\bar\nu_1|\bar z) \\ &\quad + \sum_p \sum_{f\in\mathbb{N}-\frac{1}{2}} \widetilde C_{43p}\; \widetilde C_{p21}\;\rho(\nu_4,\nu_3,\nu_{p,KM}) \left[B^f_{c,\Delta_p}\right]^{KM,LN} \rho(\nu_{p,LN},\nu_2,\nu_1|z) \\ &\qquad\qquad \times \rho(\bar\nu_4,\bar\nu_3,\bar\nu_{p,\bar K\bar M}) \left[\bar B^f_{c,\bar\Delta_p}\right]^{\bar K\bar M,\bar L\bar N} \rho(\bar\nu_{p,\bar L\bar N},\bar\nu_2,\bar\nu_1|\bar z) \\ &\quad = \sum_p C_{43p}\; C_{p21} \left|\mathcal{F}^1_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2 - \widetilde C_{43p}\; \widetilde C_{p21} \left|\mathcal{F}^{\frac{1}{2}}_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2.\end{aligned}$$

The example above illustrates how the 4-point superconformal blocks should be defined. There are four types of NS superconformal blocks corresponding to correlation functions of different primary fields (2.17), (2.21). For each type there is one even block:

$$\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \;=\; z^{\Delta-\_\Delta_2-\Delta_1}\left(1+\sum_{m\in\mathbb{N}} z^m F^{m}_{c,\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\right),$$

and one odd block:

$$\mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \;=\; z^{\Delta-\_\Delta_2-\Delta_1}\sum_{k\in\mathbb{N}-\frac{1}{2}} z^k F^{k}_{c,\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right],$$

where $\_\Delta_i$ stands for Δ_i or $*\Delta_i$, and $z^{\Delta-*\Delta_2-\Delta_1} = z^{\Delta-\Delta_2-\Delta_1-\frac{1}{2}}$. The coefficients are defined by 3-point blocks and inverse Gram matrix:

$$F^{f}_{c,\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \;=\; \sum_{|K|+|M|=|L|+|N|=f} \rho(\nu_4, \_\nu_3, \nu_{\Delta,KM})\,\left[B^{f}_{c,\Delta}\right]^{KM,LN}\rho(\nu_{\Delta,LN}, \_\nu_2, \nu_1), \tag{2.48}$$

For example:

$$\begin{aligned}
\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \;&=\; z^{\Delta-\Delta_2-\Delta_1-\frac{1}{2}}\left(1+\sum_{m\in\mathbb{N}} z^m F^{m}_{c,\Delta}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]\right),\\
F^{m}_{c,\Delta}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \;&=\; \sum \rho^{\Delta_4\ \Delta_3\ \Delta}_{\infty\ \ 1\ \ 0}(\nu_4, \nu_3, \nu_{\Delta,KM})\,\left[B^{\ m}_{c,\Delta}\right]^{KM,LN}\rho^{\Delta\ \Delta_2\ \Delta_1}_{\infty\ \ 1\ \ 0}(\nu_{\Delta,LN}, *\nu_2, \nu_1).
\end{aligned}$$

The formulae (2.40), (2.41) imply simple relations:

$$\begin{aligned}
\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \;&=\; z^{-\Delta_2-\Delta_1-\frac{1}{2}+\Delta_4+\Delta_3}\;\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix} *\Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{smallmatrix}\right](z),\\
\mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \;&=\; -z^{-\Delta_2-\Delta_1-\frac{1}{2}+\Delta_4+\Delta_3}\;\mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} *\Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{smallmatrix}\right](z),
\end{aligned}$$

reducing the number of blocks to 6 independent functions.

Since the 3-point blocks are matrix elements of the chiral vertex operators, one can say that the even (odd) blocks are defined in terms of matrix elements of even (odd) parts of the vertex operators $V(\_\nu_3), V(\_\nu_2)$. Using the formula expressing primary fields in terms of vertex operators (2.34) one can easily write the decomposition of any 4-point function of primaries (for simplicity we write the expressions in the diagonal case $\Delta_i = \bar{\Delta}_i$):

$$\begin{aligned}
\langle\nu_4\otimes\bar{\nu}_4|\,\phi_3(1,1)\tilde{\phi}_2(z,\bar{z})\,|\nu_1\otimes\bar{\nu}_1\rangle \;&=\\
&\sum_p\left(C_{43p}\tilde{C}_{p21}\left|\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z)\right|^2+\tilde{C}_{43p}C_{p21}\left|\mathcal{F}^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z)\right|^2\right),\\
\langle\nu_4\otimes\bar{\nu}_4|\,\tilde{\phi}_3(1,1)\phi_2(z,\bar{z})\,|\nu_1\otimes\bar{\nu}_1\rangle \;&=\\
&\sum_p\left(\tilde{C}_{43p}C_{p21}\left|\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} *\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z)\right|^2+C_{43p}\tilde{C}_{p21}\left|\mathcal{F}^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix} *\Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z)\right|^2\right),
\end{aligned}$$

$$\langle \nu_4 \otimes \bar\nu_4 | \, \tilde\phi_3(1,1)\tilde\phi_2(z,\bar z) \, |\nu_1 \otimes \bar\nu_1\rangle = \sum_p \left(\tilde C_{43p}\tilde C_{p21} \left| \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \right|^2 - C_{43p}C_{p21} \left| \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \right|^2 \right),$$

$$\langle \nu_4 \otimes \bar\nu_4 | \, \psi_3(1,1)\psi_2(z,\bar z) \, |\nu_1 \otimes \bar\nu_1\rangle = \sum_p \Bigg(\tilde C_{43p}\tilde C_{p21} \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) + C_{43p}C_{p21} \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) \Bigg),$$

$$\langle \nu_4 \otimes \bar\nu_4 | \, \bar\psi_3(1,1)\bar\psi_2(z,\bar z) \, |\nu_1 \otimes \bar\nu_1\rangle = \sum_p \Bigg(C_{43p}C_{p21} \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) + \tilde C_{43p}\tilde C_{p21} \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) \Bigg),$$

$$\langle \nu_4 \otimes \bar\nu_4 | \, \psi_3(1,1)\bar\psi_2(z,\bar z) \, |\nu_1 \otimes \bar\nu_1\rangle = \sum_p \Bigg(-\tilde C_{43p}C_{p21} \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) + C_{43p}\tilde C_{p21} \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) \Bigg),$$

$$\langle \nu_4 \otimes \bar\nu_4 | \, \bar\psi_3(1,1)\psi_2(z,\bar z) \, |\nu_1 \otimes \bar\nu_1\rangle = \sum_p \Bigg(-C_{43p}\tilde C_{p21} \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) + \tilde C_{43p}C_{p21} \mathcal{F}^{\frac12}_{\Delta_p}\left[\begin{smallmatrix} \Delta_3 & {*\Delta_2} \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix} {*\Delta_3} & \Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](\bar z) \Bigg).$$

2.3.2 Recurrence relations for the NS blocks

Practically, it is not possible to calculate the 4-point blocks' coefficients from the definition. But, as in the non-supersymmetric case, due to properties of the 3-point blocks and inverse Gram matrix, one can derive the recurrence relations for the coefficients. As we demonstrated, the 3-point blocks are polynomial functions of the weights Δ,Δ_i (2.38), (2.39) and the inverse Gram matrix is a rational function of intermediate weight Δ and central charge c. More precisely, the inverse Gram matrix has poles of first order at degenerate weights Δ_{rs} (2.14) (or at degenerate c_{rs}). Therefore the blocks' coefficients can be expressed as a sum over simple poles and a regular term:

$$F^{f}_{c,\Delta}\left[\begin{smallmatrix} \text{-}\Delta_3 & \text{-}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] = \mathrm{h}^{f}_{c,\Delta}\left[\begin{smallmatrix} \text{-}\Delta_3 & \text{-}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] + \sum_{\substack{1<rs\leq 2f \\ r+s\in 2\mathbb{N}}} \frac{\mathcal{R}^{f}_{c,\,rs}\left[\begin{smallmatrix} \text{-}\Delta_3 & \text{-}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]}{\Delta - \Delta_{rs}(c)}\,, \tag{2.49}$$

or as a sum over the poles in c :

$$F^{f}_{c,\Delta}\left[\begin{smallmatrix} {}_{\_}\Delta_3 & {}_{\_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \;=\; \mathrm{f}^{f}_{\Delta}\left[\begin{smallmatrix} {}_{\_}\Delta_3 & {}_{\_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] + \sum_{\substack{1<rs\leq 2f,\; r>1 \\ r+s\in 2\mathbb{N}}} \frac{\mathcal{R}^{f}_{\Delta,\,rs}\left[\begin{smallmatrix} {}_{\_}\Delta_3 & {}_{\_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right]}{c-c_{rs}(\Delta)}\,, \tag{2.50}$$

where $c_{rs}(\Delta)$ is given by (2.15). The residues in both cases are related by:

$$\begin{aligned}
\mathcal{R}^{f}_{\Delta,\,rs}\left[\begin{smallmatrix} {}_{_}\Delta_3 & {}_{_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \;&=\; -\frac{\partial c_{rs}(\Delta)}{\partial \Delta}\,\mathcal{R}^{f}_{c_{rs}(\Delta),\,rs}\left[\begin{smallmatrix} {}_{_}\Delta_3 & {}_{_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right], \\
\frac{\partial c_{rs}(\Delta)}{\partial \Delta} \;&=\; \frac{8c_{rs}(\Delta)-12}{(r^2-1)\,\beta^4_{rs}(\Delta)-(s^2-1)}.
\end{aligned} \tag{2.51}$$

We expect that each residuum is proportional to other block's coefficient. In order to check our guess it is convenient to choose a specific basis in $\mathcal{V}^{f}_{\Delta}$ (for $f>\frac{rs}{2}$). We can do this because the definition of 4-point block is independent of the choice of basis. Let χ^{KM}_{rs} be the coefficients of the null vector χ_{rs} in the basis $S_{-K}L_{-M}\nu_{\Delta_{rs}}$,

$$\chi_{rs} = \sum_{K,M} \chi^{KM}_{rs} S_{-K}L_{-M}\,\nu_{\Delta_{rs}}\,.$$

We normalize χ_{rs} in such a way that for $rs\in 2\mathbb{N}$ the coefficient at $(L_{-1})^{\frac{rs}{2}}\,\nu_{\Delta_{rs}}$ is equal 1, and for $rs\in 2\mathbb{N}-1$ the coefficient at $S_{-\frac{1}{2}}(L_{-1})^{\frac{rs-1}{2}}\,\nu_{\Delta_{rs}}$ is equal 1. To check that these two coefficients can not vanish one can compare powers of $\Delta_2-\Delta_1$ in 3-point blocks depending on χ_{rs} and fusion polynomials (2.47).
Consider the states

$$S_{-I}L_{-N}\,\chi^{\Delta}_{rs}\in\mathcal{V}^{f}_{\Delta}\,, \tag{2.52}$$

where

$$\chi^{\Delta}_{rs} = \sum_{K,M}\chi^{KM}_{rs}S_{-K}L_{-M}\,\nu_{\Delta}\,,\quad |I|+|N| = f-\tfrac{rs}{2}\,,$$

so that $\chi_{rs}=\lim_{\Delta\to\Delta_{rs}}\chi^{\Delta}_{rs}$. The set of these states can be always completed to a full basis in $\mathcal{V}^{f}_{\Delta}$. Working in such a basis and using the properties of the Gram matrix $B^{f}_{c,\Delta}$ and its inverse one obtains

$$\begin{aligned}
&\mathcal{R}^{f}_{c,\,rs}\left[\begin{smallmatrix} {}_{_}\Delta_3 & {}_{_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] = \lim_{\Delta\to\Delta_{rs}}\left(\Delta-\Delta_{rs}(c)\right)\,F^{f}_{c,\Delta}\left[\begin{smallmatrix} {}_{_}\Delta_3 & {}_{_}\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right] \\
&= A_{rs}(c)\sum_{|K|+|M|=|L|+|N|=f-\frac{rs}{2}}\rho(\nu_4,{}_{_}\nu_3,S_{-K}L_{-M}\chi_{rs})\,\left[B^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\right]^{KM,LN}\rho(S_{-L}L_{-N}\chi_{rs},{}_{_}\nu_2,\nu_1),
\end{aligned}$$

with

$$A_{rs}(c) \;=\; \lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\chi^{\Delta}_{rs}|\chi^{\Delta}_{rs}\rangle}{\Delta-\Delta_{rs}(c)}\right)^{-1}. \tag{2.53}$$

The exact form of the coefficient above was proposed by A. Belavin and Al. Zamolodchikov [42]:

$$A_{rs}(c) \;=\; \frac{1}{2}(-1)^{rs-1}\prod_{m=1-r}^{r}\prod_{n=1-s}^{s}\left(\frac{1}{\sqrt{2}}\left(p\beta-\frac{q}{\beta}\right)\right)^{-1}, \qquad m+n\in 2\mathbb{Z},\ (m,n)\neq(0,0),(r,s).$$

The block's coefficient appears in the residuum due to the factorization property of 3-point blocks (2.42), which in that case have the form:

$$\rho(S_{-L}L_{-N}\chi_{rs},\nu_2,\nu_1) \;=\; \rho(S_{-L}L_{-N}\nu_{\Delta_{rs}+\frac{rs}{2}},\nu_2,\nu_1)\;\times\;\begin{cases}\rho(\chi_{rs},\nu_2,\nu_1),\\[2mm] \rho(\chi_{rs},*\nu_2,\nu_1),\end{cases}$$

$$\rho(S_{-L}L_{-N}\chi_{rs},*\nu_2,\nu_1) = \rho(S_{-L}L_{-N}\nu_{\Delta_{rs}+\frac{rs}{2}},*\nu_2,\nu_1)\;\times\;\begin{cases}\rho(\chi_{rs},*\nu_2,\nu_1),\\[2mm] \rho(\chi_{rs},\nu_2,\nu_1),\end{cases}$$

where upper lines correspond to $f-\frac{rs}{2}\in\mathbb{N}\cup\{0\}$, and the lower lines to $f-\frac{rs}{2}\in\mathbb{N}-\frac{1}{2}$. Using the reflection properties (2.40), (2.41) one can write analogical formulae for 3-point block $\rho(\nu_4,{}_{-}\nu_3,S_{-K}L_{-M}\chi_{rs})$. The 3-point blocks depending of descendants of $\nu_{\Delta_{rs}+\frac{rs}{2}}$ together with inverse Gram matrix give the block's coefficient:

$$\sum_{|K|+|M|=|L|+|N|=f-\frac{rs}{2}}\rho(S_{-K}L_{-M}\nu_{\Delta_{rs}+\frac{rs}{2}},{}_{-}\nu_2,\nu_1)\left[B^{n-rs}_{c,\Delta_{rs}+rs}\right]^{KM,LN}\rho(\nu_4,{}_{-}\nu_3,S_{-L}L_{-N}\nu_{\Delta_{rs}+\frac{rs}{2}})$$
$$= F^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\left[\begin{smallmatrix}{}_{-}\Delta_3 & {}_{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right].$$

The remaining 3-point blocks $\rho(\chi_{rs},{}_{-}\nu_2,\nu_1)$ (2.47) are given by the fusion polynomials $P^{rs}_c\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]$ or $P^{rs}_c\left[\begin{smallmatrix}*\Delta_2\\ \Delta_1\end{smallmatrix}\right]$, depending on a type of block and the case of $\frac{rs}{2}$ being integer or half integer. Thus the final result for the residue has the form:

$$\mathcal{R}^{m}_{c,\,rs}\left[\begin{smallmatrix}{}_{-}\Delta_3 & {}_{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; A_{rs}(c)\,S_{rs}({}_{-}\Delta_3)\,P^{rs}_c\left[\begin{smallmatrix}{}_{-}\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}{}_{-}\Delta_2\\ \Delta_1\end{smallmatrix}\right]F^{m-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\left[\begin{smallmatrix}{}_{-}\Delta_3 & {}_{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \tag{2.54}$$

for $m\in\mathbb{N}\cup\{0\}$ and

$$\mathcal{R}^{k}_{c,\,rs}\left[\begin{smallmatrix}{}_{-}\Delta_3 & {}_{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \;=\; A_{rs}(c)\,S_{rs}({}_{-}\Delta_3)\,P^{rs}_c\left[\begin{smallmatrix}\widetilde{{}_{-}\Delta_3}\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}\widetilde{{}_{-}\Delta_2}\\ \Delta_1\end{smallmatrix}\right]F^{k-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\left[\begin{smallmatrix}{}_{-}\Delta_3 & {}_{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] \tag{2.55}$$

for $k\in\mathbb{N}-\frac{1}{2}$, where $\widetilde{\Delta}=*\Delta$, $\widetilde{*\Delta}=\Delta$, and due to minus sign in reflection relation (2.41):

$$S_{rs}(\Delta)=1\quad,\quad S_{rs}(*\Delta)=(-1)^{rs}.$$

The residue at c_{rs} in (2.50) is given by the formulae above and relation (2.51).

In order to complete the recursion relations for blocks' coefficients (2.49),(2.50) we need an exact form of the regular terms. In the case of c-dependence, the regular terms can be calculated from the $c\to\infty$ limit of the blocks' coefficients. The coefficients depend on c only through inverse Gram matrix, which elements are given by a non positive power of c.

By means of NS algebra one can check that Gram matrix depends on central charge due to commutators of type $[L_n, L_{-n}]$ for $n \neq 1$ and $[S_k, S_{-k}]$ for $k \neq \frac{1}{2}$. Elements of Gram matrix which correspond to the states

$$
\begin{aligned}
|\nu_{0\mathbb{I}}\rangle &= L_{-1}^{n}\,|\nu_\Delta\rangle \qquad && \in \mathcal{V}_{c,\Delta}^{f} \qquad \text{if} \qquad f = n,\\
|\nu_{1\mathbb{I}}\rangle &= S_{-\frac{1}{2}} L_{-1}^{n}\,|\nu_\Delta\rangle && \in \mathcal{V}_{c,\Delta}^{f} \qquad \text{if} \qquad f = n + \frac{1}{2}
\end{aligned}
$$

are independent of central charge. The minor for the diagonal element $\langle \nu_{0\mathbb{I}}\,|\nu_{0\mathbb{I}}\rangle$ or $\langle \nu_{1\mathbb{I}}\,|\nu_{1\mathbb{I}}\rangle$ is thus a polynomial in c of the same order as the Kac determinant. Therefore, for $c \to \infty$, the only non vanishing element of inverse Gram matrix is:

$$
\begin{aligned}
\lim_{c\to\infty}\left[B_{c,\Delta}^{f}\right]^{0\mathbb{I}\,0\mathbb{I}} &= \frac{1}{\langle \nu_\Delta|\, L_1^n L_{-1}^n\,|\nu_\Delta\rangle} = \frac{1}{n!(2\Delta)_n} \quad \text{if} \quad f = n,\\
\text{or} \quad \lim_{c\to\infty}\left[B_{c,\Delta}^{f}\right]^{1\mathbb{I}\,1\mathbb{I}} &= \frac{1}{\langle \nu_\Delta|\, L_1^n L_{-1}^n\,|\nu_\Delta\rangle} = \frac{1}{n!(2\Delta)_n} \quad \text{if} \quad f = n + \frac{1}{2},
\end{aligned}
$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. In the $c \to \infty$ limit the 4-point blocks' coefficients are given by the formulae above and the 3-point blocks calculated with help of equations (2.38),(2.39):

$$
\begin{aligned}
\rho\left(\mathrm{L}_{-1}^{n}\nu, \nu_2, \nu_1\right) &= (\Delta + \Delta_2 - \Delta_1)_n,\\
\rho\left(S_{-\frac{1}{2}}\mathrm{L}_{-1}^{n}\nu, \nu_2, \nu_1\right) &= \left(\Delta + \Delta_2 - \Delta_1 + \frac{1}{2}\right)_n,\\
\rho\left(\mathrm{L}_{-1}^{n}\nu, *\nu_2, \nu_1\right) &= \left(\Delta + \Delta_2 - \Delta_1 + \frac{1}{2}\right)_n,\\
\rho\left(S_{-\frac{1}{2}}\mathrm{L}_{-1}^{n}\nu, *\nu_2, \nu_1\right) &= -\rho\left(\nu_1, *\nu_2, S_{-\frac{1}{2}}\mathrm{L}_{-1}^{n}\nu\right) = (\Delta + \Delta_2 - \Delta_1)_{n+1}.
\end{aligned}
$$

Thus the terms regular in c have the following form:

$$
\begin{aligned}
\mathrm{f}_\Delta^{n}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \frac{1}{n!}\frac{(\Delta + \Delta_3 - \Delta_4)_n(\Delta + \Delta_2 - \Delta_1)_n}{(2\Delta)_n},\\
\mathrm{f}_\Delta^{n+\frac{1}{2}}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \frac{1}{n!}\frac{\left(\Delta + \Delta_3 - \Delta_4 + \frac{1}{2}\right)_n\left(\Delta + \Delta_2 - \Delta_1 + \frac{1}{2}\right)_n}{(2\Delta)_{n+1}},\\
\mathrm{f}_\Delta^{n}\left[\begin{smallmatrix}*\Delta_3 & *\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= \frac{1}{n!}\frac{\left(\Delta + \Delta_3 - \Delta_4 + \frac{1}{2}\right)_n\left(\Delta + \Delta_2 - \Delta_1 + \frac{1}{2}\right)_n}{(2\Delta)_n},\\
\mathrm{f}_\Delta^{n+\frac{1}{2}}\left[\begin{smallmatrix}*\Delta_3 & *\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] &= -\frac{1}{n!}\frac{(\Delta + \Delta_3 - \Delta_4)_{n+1}(\Delta + \Delta_2 - \Delta_1)_{n+1}}{(2\Delta)_{n+1}},
\end{aligned}
\tag{2.56}
$$

and so on.

Having calculated the regular terms we can finally write the close recursion relations for the coefficients in the z-expansion of the NS 4-point superconformal blocks. Remembering that the residuum $\mathcal{R}_{\Delta,rs}^{f}$ at c_{rs} is given by (2.51) and (2.54), (2.55) it is convenient to introduce simplified notation:

$$
A_{rs}(\Delta) = -\frac{\partial c_{rs}(\Delta)}{\partial \Delta}\, A_{rs}\Big(c_{rs}(\Delta)\Big), \qquad P_\Delta^{rs}\left[\begin{smallmatrix}\Delta_a\\ \Delta_b\end{smallmatrix}\right] = P_{c_{rs}(\Delta)}^{rs}\left[\begin{smallmatrix}\Delta_a\\ \Delta_b\end{smallmatrix}\right].
$$

Then, the recursion relations take the form:

$$F^{m}_{c,\Delta}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} = \mathrm{f}^{m}_{\Delta}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} + \sum_{\substack{1<rs\leq 2m,\ r>1 \\ r+s\in 2\mathbb{N}}} \frac{A_{rs}(\Delta)}{c-c_{rs}(\Delta)}\, P^{rs}_{\Delta}\begin{bmatrix} \_\Delta_3 \\ \Delta_4 \end{bmatrix} P^{rs}_{\Delta}\begin{bmatrix} \_\Delta_2 \\ \Delta_1 \end{bmatrix} F^{m-\frac{rs}{2}}_{c_{rs},\Delta+\frac{rs}{2}}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}, \tag{2.57}$$

where $m \in \mathbb{N}$ and

$$F^{k}_{c,\Delta}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} = \mathrm{f}^{k}_{\Delta}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix} + \sum_{\substack{1<rs\leq 2k,\ r>1 \\ r+s\in 2\mathbb{N}}} \frac{S_{rs}(\_\Delta_3)A_{rs}(\Delta)}{c-c_{rs}(\Delta)}\, P^{rs}_{\Delta}\begin{bmatrix} \widetilde{\_\Delta_3} \\ \Delta_4 \end{bmatrix} P^{rs}_{\Delta}\begin{bmatrix} \widetilde{\_\Delta_2} \\ \Delta_1 \end{bmatrix} F^{k-\frac{rs}{2}}_{c_{rs},\Delta+\frac{rs}{2}}\begin{bmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}, \tag{2.58}$$

where $k \in \mathbb{N} - \frac{1}{2}$. Let us note that for each type of the 4-point block one gets independent recursion formulae mixing coefficients of the even and the odd blocks of the same type.

2.4 Elliptic recurrence for NS 4-point blocks

In the last subsection we have presented a derivation of closed recursion relations for the coefficients in the z-expansion of the NS 4-point superconformal blocks. These formulae are based on the fact that blocks' coefficients can be expressed as a sum over the simple poles in central charge c (2.50) with regular term which is given by the $c \to \infty$ limit of blocks' coefficients.

One can investigate another set of recursion relations for the coefficients of 4-point blocks, namely the elliptic recurrence. The blocks' coefficients can be written in terms of a sum over the simple poles in intermediate weight Δ (2.49), with residues given by equations (2.54), (2.55). To complete the recursion relations one needs to know the terms regular in Δ. These functions can be determined from large Δ behavior of the blocks' coefficients but, as in the bosonic case, it will be more complicate than the calculation of the terms regular in c.

We will start from checking if one can repeat Al. Zamolodchikov's reasoning concerning conformal block. In the bosonic case the first two terms of the expansion of classical block in terms of large classical intermediate weight δ fully determine the Δ_i and c dependence of the first two terms in the $\frac{1}{\Delta}$ expansion of conformal quantum block. Analyzing supersymmetric Liouville theory we will investigate the classical limit of the supersymmetric blocks.

2.4.1 Classical limit of the superconformal blocks

The $N = 1$ supersymmetric Liouville theory is defined by the action [43]:

$$\mathcal{S}_{\rm SLFT} \int d^2z \left(\frac{1}{2\pi}\,|\partial\phi|^2 + \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + 2i\mu b^2\bar{\psi}\psi \mathrm{e}^{b\phi} + 2\pi b^2\mu^2\mathrm{e}^{2b\phi}\right) \tag{2.59}$$

with b a dimensionless coupling constant and μ the scale parameter. The background charge $Q = b + \frac{1}{b}$ determines the central charge of the theory $c = \frac{3}{2} + 3Q^2$. Let us notice that parameterizations of the central charge is different from the one in the non supersymmetric Liouville theory (1.44).

The super-primary field V_a and all its descendant Virasoro primaries are represented by exponentials:

$$\begin{aligned} V_a &= \mathrm{e}^{a\phi}, \\ \Lambda_a &= \left[S_{-1/2}, V_a\right] - ia\psi\mathrm{e}^{a\phi} \\ \bar{\Lambda}_a &= \left[\bar{S}_{-1/2}, V_a\right] - ia\bar{\psi}\mathrm{e}^{a\phi}, \\ \widetilde{V}_a &= \left\{S_{-1/2}, \left[\bar{S}_{-1/2}, V_a\right]\right\} a^2\psi\bar{\psi}\mathrm{e}^{a\phi} - 2i\pi\mu ba\mathrm{e}^{(a+b)\phi} \end{aligned}$$

The exponent V_a has conformal dimension $\Delta_a = \bar{\Delta}_a = \frac{a(Q-a)}{2}$.

Within the path-integral approach the correlation functions are represented by functional integrals, for example:

$$\langle V_{a_4} V_{a_3} \tilde{V}_{a_2} V_{a_1} \rangle = \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{-\mathcal{S}_{\mathrm{SLFT}}[\phi,\psi]}\mathrm{e}^{a_4\phi}\mathrm{e}^{a_3\phi}\left(a_2^2\psi\bar{\psi}\mathrm{e}^{a_2\phi} - 2i\pi\mu b a_2 \mathrm{e}^{(a_2+b)\phi}\right)\mathrm{e}^{a_1\phi}. \tag{2.60}$$

In order to analyze the classical limit ($b \to 0$, $2\pi\mu b^2 \to m = const$) of this correlator one can integrate fermions out. The integration is gaussian and the operator $\mathrm{e}^{b\phi}$ is light, thus one may expect that in the case of heavy weights

$$a = \tfrac{Q}{2}(1-\lambda) \quad , \quad ba \to \tfrac{1-\lambda}{2} \quad , \quad 2b^2\Delta \to \delta = \tfrac{1-\lambda^2}{4},$$

the 4-point function (2.60) has the following asymptotical behavior:

$$\langle V_{a_4} V_{a_3} \tilde{V}_{a_2} V_{a_1} \rangle \sim \tfrac{1}{b^2}\mathrm{e}^{-\frac{1}{2b^2}S_{\mathrm{cl}}[\delta_4,\delta_3,\delta_2,\delta_1]},$$

where $S_{\mathrm{cl}}[\delta_4,\delta_3,\delta_2,\delta_1]$ is the bosonic Liouville action (1.46). On the other hand, before taking the classical limit one can express the 4-point function (2.60) by superconformal blocks:

$$\langle V_{a_4} V_{a_3} \tilde{V}_{a_2} V_{a_1} \rangle = \int\limits_{\frac{Q}{2}+i\mathbb{R}_+} \frac{da}{2\pi i}\left(C_{43a}\tilde{C}_{a21}\left|\mathcal{F}^1_\Delta\left[\begin{smallmatrix}\Delta_3 & {*}\Delta_2 \\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2 + \tilde{C}_{43a}C_{a21}\left|\mathcal{F}^{\frac{1}{2}}_\Delta\left[\begin{smallmatrix}\Delta_3 & {*}\Delta_2 \\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)\right|^2\right) \tag{2.61}$$

where C and $\tilde{C}$ are the Liouville structure constants :

$$C_{a21} = \langle V_a(\infty,\infty)V_{a_2}(1,1)V_{a_1}(0,0)\rangle \quad , \quad \tilde{C}_{a21} = \langle V_a(\infty,\infty)\tilde{V}_{a_2}(1,1)V_{a_1}(0,0)\rangle$$

The asymptotic behavior of the 3-point functions, as in the case of 4-point functions, can be read from the path integral representation:

$$\begin{aligned} C_{a21} &\sim \mathrm{e}^{-\frac{1}{2b^2}S_{\mathrm{cl}}[\delta,\delta_2,\delta_1]} \\ \tilde{C}_{a21} &\sim \tfrac{1}{b^2}\mathrm{e}^{-\frac{1}{2b^2}S_{\mathrm{cl}}[\delta,\delta_2,\delta_1]} \end{aligned} \tag{2.62}$$

where $S_{\mathrm{cl}}[\delta,\delta_2,\delta_1]$ is the 3-point classical bosonic Liouville action.

To find out what is the behavior of the 4-point blocks one should compare asymptotics of functions on both sides of (2.61) for a fixed intermediate weight Δ. The classical limit of the correlator projected on the even (or odd) subspace $\mathcal{V}_\Delta \otimes \mathcal{V}_{\bar\Delta}$ reads:

$$\begin{aligned} \langle V_{a_4} V_{a_3} \,|^{\rm even}_{\Delta}\, \tilde V_{a_2} V_{a_1}\rangle &\sim \tfrac{1}{b^2}\,{\rm e}^{-\frac{1}{2b^2} S_{\rm cl}[\delta_4,\delta_3,\delta_2,\delta_1|\delta]}, \\ \langle V_{a_4} V_{a_3} \,|^{\rm odd}_{\Delta}\, \tilde V_{a_2} V_{a_1}\rangle &\sim \tfrac{1}{b^2}\,{\rm e}^{-\frac{1}{2b^2} S_{\rm cl}[\delta_4,\delta_3,\delta_2,\delta_1|\delta]} \end{aligned} \qquad (2.63)$$

where the "Δ-projected" classical action is given by

$$S_{\rm cl}[\delta_4,\delta_3,\delta_2,\delta_1|\delta] = S_{\rm cl}[\delta_4,\delta_3,\delta] + S_{\rm cl}[\delta,\delta_2,\delta_1] - f_\delta\!\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\!(z) - \bar f_\delta\!\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\!(\bar z).$$

The classical block $f_\delta\!\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\!(x)$ is defined in terms of the $\tilde Q \to \infty$ limit of the quantum conformal block in the Virasoro $c = 1 + 6\tilde Q^2$ CFT (1.48):

$$\mathcal{F}_{1+6\tilde Q^2,\Delta}\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) \sim \exp\left(\tilde Q^2 f_\delta\!\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right]\!(z)\right).$$

The right hand side of (2.61) for a fixed subspace of intermediate Δ states gives the blocks and structure constants. The classical limit of structure constants (2.62) and the projected correlators (2.63) imply the behavior of the blocks:

$$\mathcal{F}^1_\Delta\!\left[\begin{smallmatrix}\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) \sim {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}\ , \quad \mathcal{F}^{\frac12}_\Delta\!\left[\begin{smallmatrix}\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) \sim {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}$$

Using representations analogous to (2.61) and the same reasoning for other 4-point correlators of primary fields $V_a, \Lambda_a, \bar\Lambda_a, \tilde V_a$, one gets

$$\begin{aligned} \mathcal{F}^1_\Delta\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) &\sim {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}\ , & \mathcal{F}^{\frac12}_\Delta\!\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) &\sim b^2 {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)} \\ \mathcal{F}^1_\Delta\!\left[\begin{smallmatrix}{}^*\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) &\sim {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}\ , & \mathcal{F}^{\frac12}_\Delta\!\left[\begin{smallmatrix}{}^*\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\!(z) &\sim \tfrac{1}{b^2} {\rm e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)} \end{aligned} \qquad (2.64)$$

2.4.2 Classical block

The path-integral arguments lead to the conclusion that the asymptotical behavior of all types of 4-point superconformal blocks is given by one universal block. Moreover, this block is the classical block defined in CFT as the classical limit of quantum 4-point conformal block (1.48).

In the following subsection we would like to carry out a check confirming that the function appearing in the classical limit of superconformal blocks is indeed equal to the classical block defined in the bosonic case. In the CFT the classical block can be investigated with the help of a null vector decoupling equation (1.50). In the classical limit the equation becomes a Fuch type equation with the classical block present as an accessory parameter. In order to calculate the classical block the following problem can be formulated: adjust the accessory parameter in such a way that the equation has solutions with given monodromy around 0 and x. We will show that in the supersymmetric case it is possible to find a null vector decoupling equation, which in the classical limit turns out to be the same Fuchs type equation with identical monodromy properties as the one in the conformal case. This will be an additional argument supporting the correctness of the path integral reasoning presented in the last subsection.

Let us start by introducing the zero field V_0 corresponding to a null vector $\chi_{3,1}$:

$$V_0(z_5, \bar{z}_5) \equiv \left(L_{-1}S_{-\frac{1}{2}} + b^2 S_{-\frac{3}{2}}\right) V_{-b}(z_5, \bar{z}_5).$$

Any correlation function containing the zero field vanishes, what implies that a correlator including degenerate field $V_{-b}(z_5, \bar{z}_5)$ satisfies some differential equation.

Consider four 5-point correlators of primary fields $V_i = V_{a_i}(z_i, \bar{z}_i)$ or $\Lambda_i = \Lambda_{a_i}(z_i, \bar{z}_i)$ and the zero field V_0: $\langle V_4\Lambda_3V_0V_2V_1\rangle$, $\langle V_4V_3V_0\Lambda_2V_1\rangle$, $\langle V_4V_3V_0V_2\Lambda_1\rangle$, $\langle V_4\Lambda_3V_0\Lambda_2\Lambda_1\rangle$. Since V_0 has negative parity, in order to ensure positive total parity of the correlation function, odd number out of the remaining fields has to have negative parity as well. One can apply to these correlators the local superconformal Ward identities (2.8). As a result, in the limit $z_4 \to \infty$, the following set of equation for 5-point functions of primary fields and degenerate field $V_{a_5} = V_{-b}$ arises:

$$\begin{aligned}
&\left[\partial_5^2 + b^2\left(\frac{1}{z_{53}}\partial_3 + \frac{1}{z_{52}}\partial_2 + \frac{1}{z_{51}}\partial_1 + \frac{2\Delta_1}{z_{51}^2}\right)\right]\langle V_4V_3V_5V_2V_1\rangle \\
&= \left(\partial_{z_5} - \frac{b^2}{z_{52}}\right)\langle V_4V_3V_5\Lambda_2V_1\rangle - \left(\partial_{z_5} - \frac{b^2}{z_{53}}\right)\langle V_4\Lambda_3V_5V_2V_1\rangle \\
&+ b^2\left(\frac{1}{z_{53}} - \frac{1}{z_{52}}\right)\langle V_4\Lambda_3V_5\Lambda_2V_1\rangle, \\
&b^2\left[\left(\frac{1}{z_{15}} + \frac{1}{z_{52}}\right)\partial_2 + \frac{2\Delta_2}{z_{52}^2}\right]\langle V_4V_3V_5V_2V_1\rangle \\
&= b^2\left(\frac{1}{z_{35}} + \frac{1}{z_{51}}\right)\langle V_4\Lambda_3V_5\Lambda_2V_1\rangle - \left(\partial_{z_5} + \frac{b^2}{z_{15}}\right)\langle V_4V_3V_5\Lambda_2V_1\rangle, \\
&b^2\left[\left(\frac{1}{z_{15}} + \frac{1}{z_{53}}\right)\partial_3 + \frac{2\Delta_3}{z_{53}^2}\right]\langle V_4V_3V_5V_2V_1\rangle \\
&= -b^2\left(\frac{1}{z_{25}} + \frac{1}{z_{51}}\right)\langle V_4\Lambda_3V_5\Lambda_2V_1\rangle + \left(\partial_{z_5} + \frac{b^2}{z_{15}}\right)\langle V_4\Lambda_3V_5V_2V_1\rangle.
\end{aligned} \tag{2.65}$$

This is a system of three equations for four independent functions and it is impossible to write a closed relation for any correlator. Fortunately, such a relation can be derived in the classical limit. Adding the second equation to the first one and subtracting from the result the third equation we obtain:

$$\begin{aligned}
&\left[\partial_{z_5}^2 + b^2\left(\frac{1}{z_{51}}\partial_1 + \left(\frac{1}{z_{15}} + \frac{2}{z_{52}}\right)\partial_2 + \left(\frac{1}{z_{15}} + \frac{2}{z_{53}}\right)\partial_3 + \frac{2\Delta_1}{z_{51}^2} + \frac{2\Delta_2}{z_{52}^2} + \frac{2\Delta_3}{z_{53}^2}\right)\right]\langle V_4V_3V_5V_2V_1\rangle \\
&= b^2\left(\frac{1}{z_{51}} - \frac{1}{z_{52}}\right)\langle V_4V_3V_5\Lambda_2V_1\rangle - b^2\left(\frac{1}{z_{51}} - \frac{1}{z_{53}}\right)\langle V_4\Lambda_3V_5V_2V_1\rangle
\end{aligned} \tag{2.66}$$

The operator V_{-b} is a “light” field, thus the correlation function $\langle V_4V_3V_5V_2V_1\rangle$ in the classical limit has the form:

$$\chi(z_5)\,\mathrm{e}^{-\frac{1}{2b^2}S_{\rm cl}[\delta_4,\delta_3,\delta_2,\delta_1]}.$$

The other two correlators have similar behavior because each one includes the "light" field $\Lambda_{-b} \sim b\mathrm{e}^{-b\phi}$ and a "heavy" field $\Lambda_i \sim \frac{1}{b}\mathrm{e}^{a_i\phi}$. Now we can take the limit $b \to 0$ of relation (2.66), noticing that for $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ of order b^{-2}:

$$\partial_1,\ \partial_2,\ \partial_3 = \mathcal{O}\left(b^{-2}\right), \qquad \Delta_5,\ \partial_{z_5} = \mathcal{O}\left(1\right).$$

Keeping only the leading terms we get the closed equation for the classical limit of $\langle V_4V_3V_5V_2V_1\rangle$. In the standard locations $z_1 = 0, z_3 = 1,\ z_5 = z,\ z_2 = x$, it takes the form:

$$\begin{aligned}\Bigg\{\partial_z^2 \ &+\ 2b^2\left[\frac{\Delta_4-\Delta_3-\Delta_2-\Delta_1}{z(z-1)}+\frac{\Delta_3}{(z-1)^2}+\frac{\Delta_2}{(z-x)^2}+\frac{\Delta_1}{z^2}\right]\Bigg\}\langle V_4V_3V_5V_2V_1\rangle \\ &+\ 2b^2\frac{x(x-1)}{z(z-1)(z-x)}\frac{\partial}{\partial x}\langle V_4V_3V_5V_2V_1\rangle = 0. \end{aligned} \tag{2.67}$$

The classical limit of the 5-point correlation function (containing one "light" field) projected on even subspace $\mathcal{V}_\Delta\otimes\overline{\mathcal{V}}_\Delta$ is given by:

$$\Big\langle V_4\left(\infty\right)V_3\left(1,1\right)V_{-b}\left(z,\bar z\right)|_{\Delta}^{\mathrm{even}}\ V_2\left(x,\bar x\right)V_1\left(0,0\right)\Big\rangle \sim \chi_\Delta(z)\ \mathrm{e}^{\frac{1}{2b^2}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)}, \tag{2.68}$$

where according to path integral arguments $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$ is the classical conformal block. Defining an accessory parameter $\mathcal{C}(x)$:

$$\mathcal{C}(x) \quad = \quad \frac{\partial}{\partial x}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) \tag{2.69}$$

we get from (2.67) a Fuchsian equation:

$$\begin{aligned}\frac{d^2\chi_\Delta(z)}{dz^2}+\left(\frac{\delta_4-\delta_3-\delta_2-\delta_1}{z(z-1)}+\frac{\delta_1}{z^2}+\frac{\delta_2}{(z-x)^2}+\frac{\delta_3}{(z-1)^2}\right)\chi_\Delta(z)& \\ +\frac{x(x-1)\mathcal{C}(x)}{z(z-x)(z-1)}\chi_\Delta(z) \quad &= \quad 0.\end{aligned} \tag{2.70}$$

This is the same equation as the one obtained in the bosonic case (1.52). Now let us check what are the monodromy properties of the solutions $\chi_\Delta(z)$ along the path encircling the points 0 and x. Similarly as in the conformal case, the monodromy properties of the 5-point correlator (2.68) along a curve encircling both 0 and x are the same as the monodromy properties of the 4-point function of superprimary fields $\Big\langle V_4\left(\infty\right)V_3\left(1,1\right)V_{-b}\left(z,\bar z\right)V_a\left(0,0\right)\Big\rangle$ for a curve encircling 0. The OPE of degenerate field with operator V_a contains the information about z-dependence of the function:

$$\begin{aligned}V_{-b}(z,\bar z)V_a(0,0) = C_{(a_+,-b,a)}(z\bar z)^{\frac{bQ}{2}(1+\lambda)}\,V_{a_+}(0,0)+C_{(a_-,-b,a)}(z\bar z)^{\frac{bQ}{2}(1-\lambda)}\,V_{a_-}(0,0)\\ +\tilde C_{(a,-b,a)}(z\bar z)^{1+b^2}\,\frac{1}{(2\Delta_{a_s})^2}\,\tilde V_a(0,0)\ +\ \text{descendants},\end{aligned}$$

where $a_\pm = a \pm b$ and $a = \frac{Q}{2}(1-\lambda)$. The conformal families present in the OPE above are dictated by the fusion rules (2.43), (2.44). In the limit $b \to 0$ the third term is sub-leading with respect to the first two.

Thus the monodromy condition for the function $\chi(z)$ reads:

$$\chi^{\pm}_{\Delta}\left(\mathrm{e}^{2\pi i}z\right) \;=\; -\mathrm{e}^{\pm i\pi\lambda}\,\chi^{\pm}_{\Delta}(z)\,, \tag{2.71}$$

where $\chi^{\pm}_{\Delta}$ is a basis in the space of solutions of (2.70). This constraint for $\chi^{\pm}_{\Delta}$ analytically continued in z along the contour encircling the points 0 and x has the same form as the corresponding monodromy condition (1.54) in the bosonic case.

We have shown that the equation (2.70) together with the monodromy condition (2.71) are the same as in the non supersymmetric theory. Therefore the block which is given by the accessory parameter (2.69) has to be equal to the classical block which was calculated from the accessory parameter (1.53).

2.4.3 Large Δ asymptotic of superconformal blocks from the classical block

In non supersymmetric CFT, according to Zamolodchikov reasoning [15] reminded in the chapter (1.4.3), the first two terms in the δ-expansion of the classical block determine the first two terms of large Δ-expansion of quantum 4-point conformal block.

The conclusion which follows from the path integral arguments is that the asymptotical behavior of 4-point superconformal blocks is given by the same classical block. Moreover, each type of even superconformal block in the limit $b \to 0$ behaves in analogous way as a conformal block:

$$\mathcal{F}^{1}_{c,\Delta}\left[\begin{smallmatrix}\text{-}\Delta_3 & \text{-}\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;\sim\; \mathrm{e}^{\frac{1}{2b^2}f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \tag{2.72}$$

Thus the reasoning leading to the relation between the Δ-expansion of even NS blocks and δ-expansion of the classical block is the same as in the CFT case. The same arguments are true also in the case of one type of odd superconformal block:

$$\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & *\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \sim \mathrm{e}^{\frac{1}{2b^2}f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}.$$

The other two types of odd blocks have slightly modified asymptotics:

$$\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;\sim\; b^2\mathrm{e}^{\frac{1}{2b^2}f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \qquad \mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}*\Delta_3 & *\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \sim \frac{1}{b^2}\mathrm{e}^{\frac{1}{2b^2}f_{\delta}\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}. \tag{2.73}$$

The differences in the asymptotical behavior of odd 4-point blocks are caused by the certain property of 3-point superconformal block. Consider a 3-point block with given external state $\nu_{\Delta,KM}$ from non integer level $f \in \mathbb{N} - \frac{1}{2}$:

- $\rho(\nu_4, \nu_3, \nu_{\Delta,KM})$ is a polynomial in Δ of maximal order $f - \frac{1}{2}$,
- $\rho(\nu_4, *\nu_3, \nu_{\Delta,KM})$ is a polynomial in Δ of maximal order greater by 1: $f + \frac{1}{2}$.

From definition (2.48), it is thus clear that different types of odd 4-point blocks' coefficients (without star, with one or two stars) are proportional to polynomials with various value of maximal order ($2f-1, 2f$ or $2f+1$ respectively). Notice, that inverse Gram matrix is the same in each case.

From path integral arguments it follows that the coefficients of the block with one star $\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)$ have in numerator such a power of Δ which ensure existence of the classical block as a proper classical limit of the block. Therefore:

- the leading power in the *denominator* of each coefficient of $\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)$ is greater by 1 than it should be to give a classical block in the classical limit,
- the leading power in the *numerator* of each coefficient of $\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}{}^*\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)$ is greater by 1 than it should be to give a classical block in the classical limit.

Notice, that the power series defining odd blocks, unlike in the case of even blocks, do not contain zeroth order term. Hence, one can collect common for each term factor and put it in front of the series.

Consider first the block without stars. Since Kac determinant contains factor Δ connected with degenerate weight $\Delta_{11}=0$, each element of inverse Gram matrix is proportional to Δ^{-1}. Thus each block's coefficient is proportional to Δ^{-1} and one can put this factor in front of the series. Consequently, the function defined as a logarithm of a block

$$\mathcal{G}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}{}^-\Delta_3 & {}^-\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) = \ln \mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}{}^-\Delta_3 & {}^-\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)$$

in the current case admits the following power series expansion:

$$\mathcal{G}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; (\Delta-\Delta_2-\Delta_1)\ln z - \ln\Delta + \sum_{i=0}^{\infty} G_n z^n.$$

For the block with one star there is no need to change the maximal power of Δ in denominator:

$$\mathcal{G}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; (\Delta-\Delta_2-\Delta_1)\ln z + \sum_{i=0}^{\infty} G^*_n z^n.$$

The coefficients of the block $\mathcal{F}^{\frac{1}{2}}_{c,\Delta}\left[\begin{smallmatrix}{}^*\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z)$ have maximal power of Δ in numerator greater by 1 than it should be to give simply a classical block as a classical limit. One can multiply each term of coefficients by Δ^{-1}, leaving Δ in front of the series. It will ensure existence of the proper classical limit (2.73) of the block and give:

$$\mathcal{G}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}{}^*\Delta_3 & {}^*\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](z) \;=\; (\Delta-\Delta_2-\Delta_1)\ln z + \ln\Delta + \sum_{i=0}^{\infty} G^{**}_n z^n$$

Now all coefficients G_n, G^*_n, G^{**}_n are rational functions of Δ, Δ_i, c having polynomials of correct order in numerator and denominator. Using Zamolodchikov's technique one can find the relation between the $\frac{1}{\Delta}$ expansion of G_n, G^*_n, G^{**}_n and the $\frac{1}{\delta}$ expansion of the classical block.

Finally, the analogous relation as in the CFT case (1.57) can be obtained:

$$\mathcal{G}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) = P_{\_\_}(\Delta) + i\pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z)$$
$$+\left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \ \Delta_2\right)\ln(z) + f^{\frac{1}{2}}_{\_\_}(z) + \mathcal{O}\left(\frac{1}{\Delta}\right), \quad (2.74)$$

where

$$P(\Delta) = -\ln\Delta, \qquad P_{*}(\Delta) = 0, \qquad P_{**}(\Delta) = \ln\Delta,$$

and $f^{\frac{1}{2}}_{\_\_}(z)$ are functions of z specific for each type of block and independent of Δ_i and c. The exact form of these functions can be derived from analytic expressions for NS superconformal blocks in a specific model. In the chapter (4.2) we will present the method of computing NS blocks with external weights $\Delta_i = \frac{1}{8}$ in the $c = \frac{3}{2}$ superscalar free theory extended by the Ramond states both in the bosonic and fermionic sector.

2.4.4 Elliptic blocks

The main difficulty concerning derivation of the recursion relation for blocks' coefficients expressed as a sum over the poles in Δ (2.49) is the problem of calculation of the term regular in Δ. The large Δ asymptotic (2.74) (and corresponding one for even blocks given by (1.57)) show how the first two terms in $\frac{1}{\Delta}$ expansion of the superconformal blocks depend on the external weights Δ_i and central charge c. This information is sufficient to define elliptic blocks in such a way that, for each type of the block, the term regular in Δ will be independent of Δ_i and c. To this end we will write explicitly the multiplicative factor which takes over all the Δ_i and c dependence of the term non-singular in Δ:

$$\begin{aligned}\mathcal{F}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](z) &= (16q)^{\Delta - \frac{c-3/2}{24}}\, z^{\frac{c-3/2}{24} - \Delta_1 - \_\Delta_2}\,(1-z)^{\frac{c-3/2}{24} - \_\Delta_2 - \_\Delta_3} \\ &\times\ \theta_3^{\frac{c-3/2}{2} - 4(\Delta_1 + \_\Delta_2 + \_\Delta_3 + \Delta_4)}\,\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](q),\end{aligned} \quad (2.75)$$

where $q = \mathrm{e}^{i\pi\tau}$ is the elliptic nome. The additional in comparison with asymptotics terms are independent of the external weights and central charge. Their forms, different for different types of blocks, are suggested by the explicit expressions of superconformal blocks in $c = \frac{3}{2}$ model (4.51) – (4.57).

The elliptic blocks $\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](q)$ have the same analytic structure as the superconformal ones:

$$\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](q) = g^{1,\frac{1}{2}}_{\_\_}(q) + \sum_{m,n} \frac{h^{1,\frac{1}{2}}_{mn}\left[\begin{smallmatrix} \_\Delta_3 & \_\Delta_2 \\ \Delta_4 & \Delta_1 \end{smallmatrix}\right](q)}{\Delta - \Delta_{mn}}.$$

The regular in Δ functions $g^{1,\frac{1}{2}}_{\_\_}(q)$ are related to the functions $f^{1,\frac{1}{2}}_{\_\_}(z)$ in (2.74):

$$\mathrm{e}^{f^{1,\frac{1}{2}}_{\_\_}(z)} = (16q)^{\frac{1}{16}}(1-z)^{-\frac{1}{16}}\theta_3(q)^{-\frac{3}{4}} g^{1,\frac{1}{2}}_{\_\_}(q).$$

They do not depend on the external weights Δ_i and central charge c any more. Thus their analytical form can be extracted with the help of $c = \frac{3}{2}$ elliptic blocks (4.58), (4.59) with external weights $\Delta_i = \Delta_0 = \frac{1}{8}$:

$$\begin{aligned}
\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= \theta_3(q^2), & \mathcal{H}^{\frac{1}{2}}_\Delta\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= \tfrac{1}{\Delta}\theta_2(q^2),\\
\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= \theta_3(q^2), & \mathcal{H}^{\frac{1}{2}}_\Delta\left[\begin{smallmatrix}\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= \theta_2(q^2),
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^1_\Delta\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= \theta_3(q^2)\left(1 - \tfrac{q}{\Delta}\theta_3^{-1}\tfrac{\partial}{\partial q}\theta_3(q) + \tfrac{\theta_2^4(q)}{4\Delta}\right),\\
\mathcal{H}^{\frac{1}{2}}_\Delta\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) &= -\theta_2(q^2)\Delta\left(1 - \tfrac{q}{\Delta}\theta_3^{-1}\tfrac{\partial}{\partial q}\theta_3(q) + \tfrac{\theta_2^4(q)}{4\Delta}\right).
\end{aligned}$$

The regular in Δ parts determine the $g^{1,\frac{1}{2}}_{\_\_}$ functions:

$$\begin{aligned}
g^1(q) &= \theta_3(q^2), & g^{\frac{1}{2}}(q) &= 0,\\
g^1_*(q) &= \theta_3(q^2), & g^{\frac{1}{2}}_*(q) &= \theta_2(q^2),\\
g^1_{**}(q) &= \theta_3(q^2), & g^{\frac{1}{2}}_{**}(q) &= -\theta_2(q^2)\Delta\left(1 - \frac{q}{\Delta}\theta_3^{-1}\frac{\partial}{\partial q}\theta_3(q) + \frac{\theta_2^4(q)}{4\Delta}\right)
\end{aligned} \tag{2.76}$$

The residua of elliptic blocks $h^{1,\frac{1}{2}}_{mn}\left[\begin{smallmatrix}\_\Delta_3 & \_\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q)$ are given by the corresponding residua of superconformal blocks (2.49), (2.54), (2.55):

$$\begin{aligned}
\mathcal{H}^{1,\frac{1}{2}}_\Delta\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) &= g^{1,\frac{1}{2}}_{__}(q)\\
&+ \sum_{\substack{m,n>0\\ m,n\in 2\mathbb{N}}} (16q)^{\frac{mn}{2}} \frac{A_{rs}(c)P^{rs}_c\left[\begin{smallmatrix}_\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}_\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{mn}} \mathcal{H}^{1,\frac{1}{2}}_{\Delta_{mn}+\frac{mn}{2}}\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q)\\
&+ \sum_{\substack{m,n>0\\ m,n\in 2\mathbb{N}+1}} (16q)^{\frac{mn}{2}} \frac{S_{rs}(_\Delta_3)A_{rs}(c)P^{rs}_c\left[\begin{smallmatrix}\widetilde{_\Delta_3}\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}\widetilde{_\Delta_2}\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{mn}} \mathcal{H}^{\frac{1}{2},1}_{\Delta_{mn}+\frac{mn}{2}}\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q).
\end{aligned} \tag{2.77}$$

The coefficients of elliptic blocks written as series in nome are defined as follows:

$$\mathcal{H}^{1,\frac{1}{2}}_\Delta\left[\begin{smallmatrix}\_\Delta_3 & \_\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](q) = \sum_f (16q)^f H^f_{c,\Delta}\left[\begin{smallmatrix}\_\Delta_3 & \_\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]$$

where for even blocks $f \in \mathbb{N}\cup 0$ and for odd blocks $f \in \mathbb{N} - \frac{1}{2}$. The coefficients satisfy the elliptic recursion relations:

$$\begin{aligned}
H^f_{c,\Delta}\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right] = g^f_{__} &+ \sum_{\substack{r,s>0\\ r,s\in 2\mathbb{N}}} \frac{A_{rs}(c)\,P^{rs}_c\left[\begin{smallmatrix}_\Delta_3\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}_\Delta_2\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{rs}} H^{f-\frac{rs}{2}}_{\Delta_{rs}+\frac{rs}{2}}\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right]\\
&+ \sum_{\substack{r,s>0\\ r,s\in 2\mathbb{N}+1}} \frac{S_{rs}(_\Delta_3)A_{rs}(c)P^{rs}_c\left[\begin{smallmatrix}\widetilde{_\Delta_3}\\ \Delta_4\end{smallmatrix}\right]P^{rs}_c\left[\begin{smallmatrix}\widetilde{_\Delta_2}\\ \Delta_1\end{smallmatrix}\right]}{\Delta - \Delta_{rs}} H^{f-\frac{rs}{2}}_{\Delta_{rs}+\frac{rs}{2}}\left[\begin{smallmatrix}_\Delta_3 & _\Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right].
\end{aligned} \tag{2.78}$$

with nonsingular terms

$$g^{1}_{--}(q) = \sum_{m\in\mathbb{N}} (16q)^m g^m_{--}, \qquad g^{\frac{1}{2}}_{--}(q) = \sum_{k\in\mathbb{N}-\frac{1}{2}} (16q)^k g^k_{--},$$

given by (2.76).

2.5 Conclusions on the recursive representations of the NS blocks

In the current chapter we have presented the results concerning 4-point superconformal blocks corresponding to correlation function of NS fields [23], [24]. The 4-point blocks are defined as z-power series with coefficients given in terms of 3-point superconformal blocks and inverse NS Gram matrix. Analyzing properties of these objects one can derive the recursion relations for the block coefficients. It is also possible to work out the elliptic recurrence representation of the NS superconformal blocks. First, investigating classical limit of correlators in $N = 1$ supersymmetric Liouville theory one finds out that the exponential part of the classical limit of all 4-point superconformal blocks is given by the classical conformal block. Next, analysis of large Δ asymptotic of the superconformal blocks leads to the relations between first two terms of $\frac{1}{\Delta}$ expansions of the superconformal blocks and known first two terms of $\frac{1}{\delta}$ expansion of the classical block. These relations suggest a definition of NS elliptic blocks ensuring that the (additive) term regular in Δ for each elliptic block is independent of external weights Δ_i and central charge c. In order to compute these terms and close the elliptic recursion relations one thus needs to calculate explicit formula for each type of superconformal block with arbitrary fixed external weights in some specific model. We use the $c = \frac{3}{2}$ superconformal blocks with $\Delta_i = \frac{1}{8}$. As a result we obtain the elliptic recursion relations for all types of the NS superconformal blocks.

One should note, that the definition and z-recurrence representation of 4-point blocks related to correlator of NS superprimary fields were worked out independently by V.A. Belavin [44]. For this type of block (*i.e.* block without stars) the elliptic recursion relations were conjectured in [27] and applied to numerical consistency check of $N = 1$ supersymmetric Liouville theory. Further numerical verification of $N = 1$ supersymmetric Liouville theory was given in [28] where the elliptic recurrence representation of the block with one star was proposed. The complete derivation of the elliptic recursion relations however was missing until [24]. Our results (2.76), (2.77) confirm the relations previously conjectured in [27], [28] and provide new recurrence representation of the blocks with two stars. One can notice that the regular in Δ part of the odd elliptic block with two stars $g^{\frac{1}{2}}_{**}(q)$ (2.76) is much more complicated that the corresponding functions in the other cases and it would be difficult to guess its form numerically.

Chapter 3

Conformal blocks in the Ramond sector of $N = 1$ SCFT

3.1 Definitions

3.1.1 NS and Ramond sectors in N=1 SCFT

In the section 2.1.1 we have presented the main assumptions concerning $N = 1$ SCFT. Let us remind that the space of fields in SCFT decomposes onto two parts: Neveu-Schwarz sector with $\varphi_{\rm NS}(z_i, \bar{z}_i)$ local with respect to $S(z)$ and Ramond sector with $R(z_i, \bar{z}_i)$ defined as 'half-local' with respect to $S(z)$. The locality properties of the fields, together with the basic dynamical assumption (2.1), impose some constraints on OPEs of the fields. Namely, OPE of two Ramond fields should be expressed as a series of NS operators, while OPE of one Ramond field and one NS field should be given in terms of Ramond operators. This means that the NS operators φ and the Ramond operators R have the following block structure:

$$\varphi = \left[\begin{array}{c|c} \varphi_{\rm NN} & 0 \\ \hline 0 & \varphi_{\rm RR} \end{array}\right] \quad , \quad R = \left[\begin{array}{c|c} 0 & R_{\rm NR} \\ \hline R_{\rm RN} & 0 \end{array}\right] \tag{3.1}$$

with respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_{\rm NS} \oplus \mathcal{H}_{\rm R}$ of the space of states.

In the previous chapter the matrix elements of operators $\varphi_{\rm NN}$ acting between NS states were considered. In the current chapter we will analyze matrix elements of Ramond operators between Neveu-Schwarz and Ramond states. The case of NS fields $\varphi_{\rm RR}$ acting between Ramond states will be briefly discussed in the end of this chapter.

We assume that in the set of Ramond fields there exist *Ramond primary fields* $R^{\pm}_{\Delta,\bar{\Delta}}(w, \bar{w})$[1] with conformal weights $\Delta, \bar{\Delta}$ and parity $\pm$. It is defined by the following local Ward identities (1.4):

$$T(z)R^{\pm}_{\Delta,\bar{\Delta}}(w, \bar{w}) \quad \sim \quad \frac{\Delta}{(z-w)^2} R^{\pm}_{\Delta,\bar{\Delta}}(w, \bar{w}) + \frac{1}{z-w} \partial R^{\pm}_{\Delta,\bar{\Delta}}(w, \bar{w}) + reg.$$

[1] Following [6] we chose the "symmetric" convention for $\pm$ components of the Ramond fields.

$$\bar{T}(\bar{z})R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) \sim \frac{\bar{\Delta}}{(\bar{z}-\bar{w})^2}R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) + reg. \tag{3.2}$$

and

$$\begin{aligned} S(z)R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{i\beta e^{\mp i\frac{\pi}{4}}}{(z-w)^{\frac{3}{2}}}R^{\mp}_{\Delta,\bar{\Delta}}(w,\bar{w}), \\ \bar{S}(\bar{z})R^{\pm}_{\Delta,\bar{\Delta}}(w,\bar{w}) &\sim \frac{-i\bar{\beta} e^{\pm i\frac{\pi}{4}}}{(\bar{z}-\bar{w})^{\frac{3}{2}}}R^{\mp}_{\Delta,\bar{\Delta}}(w,\bar{w}), \end{aligned} \tag{3.3}$$

where $\beta, \bar{\beta}$ are related to the conformal weights by

$$\Delta = \frac{c}{24} - \beta^2 \quad , \quad \bar{\Delta} = \frac{c}{24} - \bar{\beta}^2.$$

3.1.2 R supermodule

We shall consider the left Ramond algebra (2.5) extended by the fermion parity operator $(-1)^{F_L}$:

$$[(-1)^F_L, L_m] = \{(-1)^{F_L}, S_n\} = 0 \quad , \quad m,n \in \mathbb{Z}. \tag{3.4}$$

The *highest weight state* w^+_Δ with respect to the extended Ramond algebra is defined by the following conditions:

$$L_0 w^+_\Delta = \Delta w^+_\Delta\,, \qquad (-1)^{F_L} w^+_\Delta = w^+_\Delta\,, \qquad L_m w^+_\Delta = S_n w^+_\Delta = 0, \qquad m,n \in \mathbb{N}, \tag{3.5}$$

where $\mathbb{N}$ is the set of positive integers. The state generated by S_0 acting on w^+_Δ has the same conformal weight as the highest weight state and it is anihilated by L_m, S_n with $m,n > 0$ as well. But due to the condition (3.4) such a state is an eigenvector of the parity operator $(-1)^{F_L}$ to eigenvalue -1.

The *descendant states* are defined as states created by an action of generators L_{-m} and S_{-n} on the highest weight state w^+_Δ. They form the vector space $\mathcal{W}^f_\Delta$ with the basis:

$$w_{\Delta,KM} = S_{-K}L_{-M}w^+_\Delta \equiv S_{-k_i}\ldots S_{-k_1}L_{-m_j}\ldots L_{-m_1}w^+_\Delta\,, \tag{3.6}$$

where $K = \{k_1, k_2, \ldots, k_i\} \subset \mathbb{N} \cup \{0\}$ and $M = \{m_1, m_2, \ldots, m_j\} \subset \mathbb{N}$ are arbitrary ordered sets of indices

$$k_i < \ldots < k_2 < k_1, \qquad m_j \leq \ldots \leq m_2 \leq m_1,$$

such that $|K|+|M| = k_1+\ldots+k_i+m_1+\ldots+m_j = f$. Let us denote the number of operators in the set S_{-K} by $\sharp K$. The parity of the state is given by: $(-1)^{F_L}w_{\Delta,KM} = (-1)^{\sharp K}w_{\Delta,KM}$.

The *R supermodule* of the highest weight Δ and the central charge c is defined as the $\mathbb{Z}$-graded representation of the extended Ramond algebra determined on the space

$$\mathcal{W}_\Delta = \bigoplus_{f\in\mathbb{N}\cup\{0\}} \mathcal{W}^f_\Delta\,,$$

by the relations (2.5) and (3.5). In order to simplify the notation we omit the subscript c at $\mathcal{W}$. Each $\mathcal{W}_\Delta^f$ is an eigenspace of L_0 with the eigenvalue $\Delta + f$. The space $\mathcal{W}_\Delta$ has a natural $\mathbb{Z}_2$-grading:

$$\mathcal{W}_\Delta = \mathcal{W}_\Delta^+ \oplus \mathcal{W}_\Delta^-, \quad \mathcal{W}_\Delta^+ = \bigoplus_{f\in\mathbb{N}\cup\{0\}} \mathcal{W}_\Delta^{f+}, \quad \mathcal{W}_\Delta^- = \bigoplus_{f\in\mathbb{N}\cup\{0\}} \mathcal{W}_\Delta^{f-},$$

where $\mathcal{W}_\Delta^{f\pm}$ are common eigenspaces of the operators $(-1)^{F_L}$ and L_0. Note that the subspaces $\mathcal{W}_\Delta^{0+}, \mathcal{W}_\Delta^{0-}$ are 1-dimensional except the case $\Delta = \frac{c}{24}$ where $\mathcal{W}_\Delta^{0-} = \{0\}$.

As a scalar product we introduce a symmetric bilinear form $\langle .\,, .\rangle_{c,\Delta}$ on $\mathcal{W}_\Delta$ such that

$$\langle w_\Delta, w_\Delta\rangle = 1, \quad \langle w_\Delta, S_0\, w_\Delta\rangle = 0, \quad (L_m)^\dagger = L_{-m},\ (S_n)^\dagger = S_{-n}.$$

It is block-diagonal with respect to the L_0- and $(-1)^{F_L}$-grading.

The Ramond equivalent of Gram matrix is the matrix of $\langle .\,,.\rangle_{c,\Delta}$ on $\mathcal{W}_\Delta^{f\pm}$ calculated in the basis (3.6):

$$\left[B_{c,\Delta}^{f\pm}\right]_{KM,LN} = \langle w_{\Delta,KM}^\pm, w_{\Delta,LN}^\pm\rangle_{c,\Delta}.$$

It is nonsingular if and only if the R supermodule $\mathcal{W}_\Delta$ does not contain singular vectors of degrees $0, 1, 2, \ldots, f$. A singular vector in Ramond sector is defined as a state $\chi \in \mathcal{W}_\Delta^f$ of degree f satisfying the highest weight conditions (3.5) with $L_0\chi = (\Delta + f)\chi$. It generates its own R supermodule $\mathcal{W}_{\Delta+f}$ which is a submodule of $\mathcal{W}_\Delta$.

The determinant of $B_{c,\Delta}^{f\pm}$ is given by the formula conjectured by Friedan,Qiu, Shenker [4] and proven by Meurman and Rocha-Caridi [45]. For level zero it reads

$$\det B_{c,\Delta}^{0+} = 1 \quad , \quad \det B_{c,\Delta}^{0-} = \Delta - \frac{c}{24},$$

and for higher levels

$$\det B_{c,\Delta}^{f\pm} = \left(\Delta - \frac{c}{24}\right)^{\frac{P_R(f)}{2}} \prod_{1\leqslant rs\leqslant 2f} (\Delta - \Delta_{rs})^{P_R(f-\frac{rs}{2})}, \tag{3.7}$$

where $r, s \in \mathbb{N}$, the sum $r + s$ must be *odd* and

$$\Delta_{rs}(c) = \frac{1}{16} - \frac{rs-1}{4} + \frac{1-r^2}{8}b^2 + \frac{1-s^2}{8}\frac{1}{b^2}, \qquad c = \frac{3}{2} + 3\left(b + \frac{1}{b}\right)^2. \tag{3.8}$$

The multiplicity of each zero is given by $P_R(f) = \dim \mathcal{W}_\Delta^f$ and can be read off from the relation

$$\sum_{f=0}^\infty P_R(f) q^f = \prod_{n=1}^\infty \frac{1+q^n}{1-q^n}.$$

3.1.3 The space of states

Let us consider a Ramond state generated by a Ramond primary field acting on the NS vacuum introduced in (2.1.4):

$$\lim_{z,\bar{z}\to 0} R^{\pm}_{\Delta,\bar{\Delta}}(z,\bar{z})\,|0\rangle \;=\; \left|w^{\pm}_{\Delta,\bar{\Delta}}\right\rangle .$$

Due to the local Ward identities (3.2),(3.3) imposed on $R^{\pm}_{\Delta,\bar{\Delta}}$, the state $w^{\pm}_{\Delta,\bar{\Delta}}$ is anihilated by the left and the right generators with positive indices. Thus it can be expressed in terms of the highest weight states with respect to left and right extended Ramond algebras (3.4).

The tensor product $\mathcal{W}_{\Delta} \otimes \bar{\mathcal{W}}_{\bar{\Delta}}$ of the left and the right R supermodules is defined as a graded tensor product of representations of $\mathbb{Z}_2$-graded algebras. This provides a representation of the direct sum $R \oplus \bar{R}$ of left and right Ramond algebras extended by the left $(-1)^{F_L}$ and the right $(-1)^{F_R}$ parity operators. Since the Ramond fields have definite one total parity instead of two chiral parities, we are interested in the extension of $R \otimes \bar{R}$ by the common parity operator

$$(-1)^F = (-1)^{F_L}(-1)^{F_R}$$

and the corresponding $\mathbb{Z}_2$-grading. For $\Delta, \bar{\Delta} \neq \frac{c}{24}$ an appropriate representation can be obtained restricting the action of $R \otimes \bar{R}$ and $(-1)^F$ to the invariant subspace $\mathcal{W}_{\Delta,\bar{\Delta}} \subset \mathcal{W}_{\Delta} \otimes \bar{\mathcal{W}}_{\bar{\Delta}}$ generated by the vectors

$$\begin{aligned} w^{+}_{\Delta,\bar{\Delta}} &= \tfrac{1}{\sqrt{2}}\left(w^{+}_{\Delta} \otimes w^{+}_{\bar{\Delta}} - i\, w^{-}_{\Delta} \otimes w^{-}_{\bar{\Delta}}\right) , \\ w^{-}_{\Delta,\bar{\Delta}} &= \tfrac{1}{\sqrt{2}}\left(w^{+}_{\Delta} \otimes w^{-}_{\bar{\Delta}} \;+\; w^{-}_{\Delta} \otimes w^{+}_{\bar{\Delta}}\right) . \end{aligned} \tag{3.9}$$

where $w^{-}_{\Delta} = \frac{e^{i\frac{\pi}{4}}}{i\beta} S_0 w^{+}_{\Delta}$, $w^{-}_{\bar{\Delta}} = \frac{e^{-i\frac{\pi}{4}}}{i\bar{\beta}} \bar{S}_0 w^{+}_{\bar{\Delta}}$. We shall call it the "small representation".

The choice of basis (3.9) in the zero level subspace $\mathcal{W}^{0}_{\Delta,\bar{\Delta}}$ is consistent with the definition of the corresponding Ramond primary fields (3.2):

$$S_0 w^{\pm}_{\Delta,\bar{\Delta}} \;=\; i\beta e^{\mp i\frac{\pi}{4}} w^{\mp}_{\Delta,\bar{\Delta}} \quad , \quad \bar{S}_0 w^{\pm}_{\Delta,\bar{\Delta}} \;=\; -i\bar{\beta} e^{\pm i\frac{\pi}{4}} w^{\mp}_{\Delta,\bar{\Delta}} . \tag{3.10}$$

The descendant states in the "small representation" have the form:

$$S_{-L}L_{-N}\bar{S}_{-K}\bar{L}_{-M} w^{+}_{\Delta,\bar{\Delta}} = \frac{1}{\sqrt{2}}\left(S_{-L}L_{-N} w^{+}_{\Delta} \otimes \bar{S}_{-K}\bar{L}_{-M} w^{+}_{\bar{\Delta}} - (-1)^{\sharp K} i\, S_{-L}L_{-N} w^{-}_{\Delta} \otimes \bar{S}_{-K}\bar{L}_{-M} w^{-}_{\bar{\Delta}}\right)$$

We assume that the space of Ramond states $\mathcal{H}_R$ is a sum of the "small representations" over the weights from the spectrum of Ramond primary fields.

3.1.4 Field operators

We assume that the correspondence between states and field operators in SCFT is true in the Ramond sector as well: *There is one to one correspondence between the states from the space of Ramond states and the field operators from the space of Ramond fields.*

The primary fields $R^{\pm}_{\Delta,\bar{\Delta}}$ create from the vacuum $w^{\pm}_{\Delta,\bar{\Delta}}$ states. We expect that descendant fields generate the descendant states:

$$\lim_{z,\bar{z}\to 0} R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z})\,|0\rangle \equiv S_{-K}L_{-M}\bar{S}_{-L}\bar{L}_{-N}w^{+}_{\Delta,\bar{\Delta}}, \tag{3.11}$$

where $\eta,\bar{\eta}$ denote the states from chiral R supermodules $\mathcal{W}_\Delta,\bar{\mathcal{W}}_{\bar{\Delta}}$:

$$\eta = S_{-K}L_{-M}w^{+}_{\Delta}, \qquad \bar{\eta} = \bar{S}_{-L}\bar{L}_{-N}w^{+}_{\bar{\Delta}}.$$

Let us define the descendant fields as operators satisfying the following relations

$$\begin{aligned}
\mathcal{L}_{-m}R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}) \equiv R_{\Delta,\bar{\Delta}}(L_{-m}\eta,\bar{\eta}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{1-m}T(w)R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}), \quad m\in\mathbb{N},\\
\mathcal{S}_{-k}R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}) \equiv R_{\Delta,\bar{\Delta}}(S_{-k}\eta,\bar{\eta}|z,\bar{z}) &= \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}-k}S(w)R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z}), \quad k\in\mathbb{N},
\end{aligned} \tag{3.12}$$

together with the condition (3.11). It follows from (3.11) that the parity of the descendant is given by $(\sharp K + \sharp\bar{L})$, which is the number of $\mathcal{S}_{-k_i}$ and $\bar{\mathcal{S}}_{-l_i}$ operators acting on the even primary field $R^{+}_{\Delta,\bar{\Delta}}(z,\bar{z})$. Due to the action of $\mathcal{S}_0,\bar{\mathcal{S}}_0$, each Ramond field with a given parity has its counterpart with the same conformal weights but the opposite parity. The primary field $R^{+}_{\Delta,\bar{\Delta}}(z,\bar{z})$ with $\beta=\bar{\beta}=0$ is the only exception and does not have corresponding field with negative parity (3.3).

All primary fields with their descendants $R_{\Delta,\bar{\Delta}}(\eta,\bar{\eta}|z,\bar{z})$ form the space of Ramond fields.

Let us note one more important feature of Ramond primary fields. We assume that $R^{+}_{\Delta,\bar{\Delta}}(z,\bar{z})$ is hermitian. Then the hermicity of $R^{-}_{\Delta,\bar{\Delta}}(z,\bar{z})$ follows from the definition of $\mathcal{S}_0R^{+}$ (3.12) and we have:

$$R^{\pm}_{\Delta,\bar{\Delta}}(z,\bar{z})^{\dagger} = \bar{z}^{-2\Delta}z^{-2\bar{\Delta}}R^{\pm}_{\Delta,\bar{\Delta}}\left(\tfrac{1}{\bar{z}},\tfrac{1}{z}\right) \quad ,$$

Thus the off-diagonal blocks of the primary Ramond field (3.1) are related to each other:

$$\overline{\langle R(\infty)R^{\pm}_{\rm RN}(z,\bar{z})\varphi(0)\rangle} = \bar{z}^{-2\Delta}z^{-2\bar{\Delta}}\langle\varphi(\infty)R^{\pm}_{\rm NR}(\tfrac{1}{\bar{z}},\tfrac{1}{z})R(0)\rangle \tag{3.13}$$

and it is sufficient to analyze the matrix elements of the operators $R^{\pm}_{\rm NR}$.

3.1.5 Ward identities for 3-point correlation functions

Let us consider a correlation function

$$\Big\langle\varphi_3(\infty,\infty)S(w)R^{\epsilon}_2(z,\bar{z})R^{\epsilon'}_1(0,0)\Big\rangle,$$

where $R^{\epsilon_i}_i(z,\bar{z}) = R_i(\eta,\bar{\eta}|z,\bar{z})$ are two arbitrary excited Ramond fields of definite parities ϵ,ϵ' and $\varphi_3(\infty,\infty)=\varphi_3(\xi,\bar{\xi}|\infty,\infty)$ is an arbitrary NS excited field. Due to the locality properties of Ramond fields such a correlator is a double valued function of w. Thus in order to derive Ward identities for 3-point correlators one cannot use the the standard contour deformation method. Another approach based on considering a single valued function

$$\sqrt{w(w-z)}\,\Big\langle\varphi_3(\infty,\infty)S(w)R^{\epsilon}_2(z,\bar{z})R^{\epsilon'}_1(0,0)\Big\rangle$$

was proposed by [4] (see also [46] for more detailed analysis). Contour integrals of this function around a location of each field can be expressed in terms of fields excitations. For example, the integral around the second Ramond field location is given by:

$$\oint_z \frac{dw}{2\pi i}\,(w-z)^{-n+\frac{1}{2}}w^{\frac{1}{2}}\left\langle \varphi_3(\infty,\infty)S(w)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle$$
$$=\oint_z \frac{dw}{2\pi i}\,\sum_p \binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}(w-z)^{p-n+\frac{1}{2}}\left\langle \varphi_3(\infty,\infty)S(w)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle =$$
$$=\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\left\langle \varphi_3(\infty,\infty)\mathcal{S}_{p-n}R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle$$

The deformation of the contour of integral above leads to the formula:

$$\begin{aligned}
&\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\left\langle \varphi_3(\infty,\infty)\mathcal{S}_{p-n}R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle\\
&\qquad= \oint_{|w|>|z|}\frac{dw}{2\pi i}\,(w-z)^{-n+\frac{1}{2}}w^{\frac{1}{2}}\left\langle \varphi_3(\infty,\infty)S(w)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle\\
&\qquad\quad-\epsilon\oint_{|w|<|z|}\frac{dw}{2\pi i}\,(w-z)^{-n+\frac{1}{2}}w^{\frac{1}{2}}\left\langle \varphi_3(\infty,\infty)R_2^{\epsilon}(z,\bar z)S(w)R_1^{\epsilon'}(0,0)\right\rangle\\
&\qquad= \sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-z)^p\left\langle \mathcal{S}_{p+n-\frac{1}{2}}\varphi_3(\infty,\infty)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle\\
&\qquad\quad-i\epsilon\sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-1)^{n+p}z^{-n+\frac{1}{2}-p}\left\langle \varphi_3(\infty,\infty)R_2^{\epsilon}(z,\bar z)\mathcal{S}_pR_1^{\epsilon'}(0,0)\right\rangle,
\end{aligned}\tag{3.14}$$

where the relation $S(w)R^{\epsilon}(z,\bar z)=\epsilon\,R^{\epsilon}(z,\bar z)S(w)$ was used.

Similarly, considering an integral of the function

$$w^n\sqrt{w(w-z)}\left\langle \varphi_3(\infty,\infty)S(w)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle$$

one can derive the second Ward identity:

$$\begin{aligned}
&\sum_p\binom{\frac{1}{2}}{p}(-z)^p\left\langle \mathcal{S}_{p-n-\frac{1}{2}}\varphi_3(\infty,\infty)R_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle\\
&\qquad=\sum_{p=0}^{\infty}\binom{n+\frac{1}{2}}{p}\; z^{n+\frac{1}{2}-p}\left\langle \varphi_3(\infty,\infty)\,\mathcal{S}_pR_2^{\epsilon}(z,\bar z)R_1^{\epsilon'}(0,0)\right\rangle=\\
&\qquad\quad+i\epsilon\sum_p\binom{\frac{1}{2}}{p}(-1)^p\; z^{\frac{1}{2}-p}\left\langle \varphi_3(\infty,\infty)R_2^{\epsilon}(z,\bar z)\mathcal{S}_{n+p}R_1^{\epsilon'}(0,0)\right\rangle
\end{aligned}\tag{3.15}$$

Corresponding relations hold also in the anti-holomorphic sector. The Ward identities with Virasoro generators have the same form as in (1.20).

Notice that there are some features of the Ward identities (3.14),(3.15) which make them more complicated than the corresponding relations in NS sector.

Firstly the action of a creation operator $\mathcal{S}_{-k}$ on a field is given not only by the $\mathcal{S}_0$ or $\mathcal{S}_{-\frac{1}{2}}$ and anihilation operators acting on the other fields, but also by the lower creation operators $\mathcal{S}_{-k+p}$ acting on the same field.

Secondly, any 3-point correlation function can be reduced to a combination of two out of eight correlators of primary fields:

$$\begin{aligned} C^{\pm}_{321} &= \Big\langle \phi_3(\infty,\infty) R^{\pm}_2(1,1) R^{\pm}_1(0,0) \Big\rangle, \\ \widetilde{C}^{\pm}_{321} &= \Big\langle \widetilde{\phi}_3(\infty,\infty) R^{\pm}_2(1,1) R^{\pm}_1(0,0) \Big\rangle, \\ D^{\pm}_{321} &= \Big\langle \psi_3(\infty,\infty) R^{\pm}_2(1,1) R^{\mp}_1(0,0) \Big\rangle, \\ \bar{D}^{\pm}_{321} &= \Big\langle \bar{\psi}_3(\infty,\infty) R^{\pm}_2(1,1) R^{\mp}_1(0,0) \Big\rangle. \end{aligned} \tag{3.16}$$

With the help of formula (3.15) for $n = 0$ and the corresponding antiholomorphic one:

$$\begin{aligned} &\Big\langle S_{-\frac{1}{2}} \varphi_3(\infty,\infty) R^{\epsilon}_2(1,1) R^{\epsilon'}_1(0,0) \Big\rangle \\ &\quad = \Big\langle \phi_3(\infty,\infty) S_0 R^{\epsilon}_2(1,1) R^{\epsilon'}_1(0,0) \Big\rangle + i\epsilon \Big\langle \phi_3(\infty,\infty) R^{\epsilon}_2(1,1) S_0 R^{\epsilon'}_1(0,0) \Big\rangle \\ &\Big\langle \bar{S}_{-\frac{1}{2}} \varphi_3(\infty,\infty) R^{\epsilon}_2(1,1) R^{\epsilon'}_1(0,0) \Big\rangle \\ &\quad = \Big\langle \varphi_3(\infty,\infty) \bar{S}_0 R^{\epsilon}_2(1,1) R^{\epsilon'}_1(0,0) \Big\rangle - i\epsilon \Big\langle \varphi_3(\infty,\infty) R^{\epsilon}_2(1,1) \bar{S}_0 R^{\epsilon'}_1(0,0) \Big\rangle \end{aligned}$$

we can find the relations expressing all structure constants in terms of two independent constants:

$$\begin{aligned} \widetilde{C}^{\pm}_{321} &= \mp i \left[(\bar{\beta}_1\beta_1 + \bar{\beta}_2\beta_2) C^{\pm}_{321} - (\bar{\beta}_1\beta_2 + \bar{\beta}_2\beta_1) C^{\mp}_{321} \right] \\ D^{\pm}_{321} &= i\mathrm{e}^{\pm i\frac{\pi}{4}} \left[\beta_2 C^{\mp}_{321} + \beta_1 C^{\pm}_{321} \right] \\ \bar{D}^{\pm}_{321} &= -i\mathrm{e}^{\mp i\frac{\pi}{4}} \left[\bar{\beta}_2 C^{\mp}_{321} + \bar{\beta}_1 C^{\pm}_{321} \right] \end{aligned} \tag{3.17}$$

3.2 The 3-point Ramond block

3.2.1 Ramond field vs. chiral vertex operators

We would like to define a chiral 3-form in such a way that any 3-point function with Ramond field $R^{\pm}_{NR}$ could be written in terms of it. Since the Ramond fields correspond to states from "small representation" $\mathcal{W}_{\Delta,\bar{\Delta}} \subset \mathcal{W}_{\Delta} \otimes \bar{\mathcal{W}}_{\bar{\Delta}}$ (3.9), the relation between a correlator of Ramond fields and the chiral form defined on $\mathcal{W}_{\Delta}$ and NS module is not straightforward.

Let us start by defining the form which is anti-linear in the left argument and linear in the central and the right ones:

$$\varrho_{NR}(\xi_3, \eta_2, \eta_1) \; : \; \mathcal{V}_{\Delta_3} \times \mathcal{W}_{\Delta_2} \times \mathcal{W}_{\Delta_1} \; \mapsto \; \mathbb{C}$$

and satisfies the following relations:

$$
\begin{aligned}
&\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\; \varrho_{NR}(\xi_3, S_{p-n}\eta_2, \eta_1|z) \\
&\quad = \sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-z)^p\; \varrho_{NR}(S_{p+n-\frac{1}{2}}\xi_3, \eta_2, \eta_1|z) \\
&\qquad -i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{-n+\frac{1}{2}}{p}(-1)^{n+p} z^{\frac{1}{2}-n-p}\varrho_{NR}(\xi_3, \eta_2, S_p\eta_1|z) \\
&\sum_{p=0}^{\infty}\binom{n+\frac{1}{2}}{p}\; z^{n+\frac{1}{2}-p}\; \varrho_{NR}(\xi_3, S_p\eta_2, \eta_1|z) \\
&\quad = \sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-z)^p\; \varrho_{NR}(S_{p-n-\frac{1}{2}}\xi_3, \eta_2, \eta_1|z) \\
&\qquad -i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-1)^p\; z^{\frac{1}{2}-p}\varrho_{NR}(\xi_3, \eta_2, S_{n+p}\eta_1|z),
\end{aligned}
\tag{3.18}
$$

and the relations with Virasoro generators of the form (2.26)-(2.29). An even or an odd number $|\xi_3|, |\eta_1|$ denote even or odd parities of $\xi_3 \in \mathcal{V}_{\Delta_3}$, $\eta_1 \in \mathcal{W}_{\Delta_1}$, respectively. We will call these formulae Ward identities for the non normalized form ϱ_{NR}.

The form is determined by these relations up to four independent constants. We define the forms $\rho^{ij}_{NR}\,; i,j = \pm$ as coefficients in front of these constants:

$$
\begin{aligned}
\varrho_{NR}(\xi_3, \eta_2, \eta_1|z) &= \rho^{++}_{NR}(\xi_3, \eta_2, \eta_1|z)\varrho_{NR}(\nu_3, w_2^+, w_1^+|1) \\
&+ \rho^{+-}_{NR}(\xi_3, \eta_2, \eta_1|z)\varrho_{NR}(\nu_3, w_2^+, w_1^-|1) \\
&+ \rho^{-+}_{NR}(\xi_3, \eta_2, \eta_1|z)\varrho_{NR}(\nu_3, w_2^-, w_1^+|1) \\
&+ \rho^{--}_{NR}(\xi_3, \eta_2, \eta_1|z)\varrho_{NR}(\nu_3, w_2^-, w_1^-|1)\;.
\end{aligned}
\tag{3.19}
$$

The Ward identities for ϱ_{NR} containing the Virasoro generators determine the z dependence of all the coefficients:

$$
\rho^{ij}_{NR}(\xi_3, \eta_2, \eta_1|z) = z^{\Delta_3(\xi_3)-\Delta_2(\eta_2)-\Delta_1(\eta_1)}\rho^{ij}_{NR}(\xi_3, \eta_2, \eta_1)
$$

where

$$
\rho^{ij}_{NR}(\xi_3, \eta_2, \eta_1) \equiv \rho^{ij}_{NR}(\xi_3, \eta_2, \eta_1|1).
$$

Our aim is to express any 3-point function containing primary $R^{\pm}_{\rm NR}$ fields and satisfying Ward identities (3.14),(3.15) in terms of the 3-form defined by the formulae (3.18). In the first step we shall find a relation between two independent structure constants $C^{\pm}$ (3.17) and eight chiral constants ϱ_{NR}, $\bar{\varrho}_{NR}$. The correlators $C^{\pm}$ are matrix elements of Ramond primary fields between states $|\nu\otimes\bar{\nu}\rangle$ and $w^{\pm}_{\Delta,\bar{\Delta}}$. The constants ϱ_{NR} are matrix elements of non normalized chiral vertex operators. These operators are defined in the following way:

$$
\begin{aligned}
\langle\xi_3|V^{\pm}_{\rm NRe}(z)|\eta_1\rangle &= \varrho_{NR}(\xi_3, w^{\pm}, \eta_1|z) \quad , \quad |\xi_3|, |\eta_1| \quad \text{equal parities} \\
\langle\xi_3|V^{\pm}_{\rm NRo}(z)|\eta_1\rangle &= \varrho_{NR}(\xi_3, w^{\pm}, \eta_1|z) \quad , \quad |\xi_3|, |\eta_1| \quad \text{theopposite parities}
\end{aligned}
$$

Thus, in order to find the relation between $C^{\pm}$ and constants ϱ_{NR}, $\bar{\varrho}_{NR}$ one should express the fields $R^{\pm}_{\rm NR}$ by non normalized chiral vertex operators. The construction of the states $w^{\pm}_{\Delta,\bar{\Delta}}$ in the "small representation" (3.9) suggest the following form of the Ramond fields

$$
\begin{aligned}
R^{+}_{\rm NR} &= AV^{+}_{\rm NRe}\otimes\bar{V}^{+}_{\rm NRe}+BV^{+}_{\rm NRo}\otimes\bar{V}^{+}_{\rm NRo}+iBV^{-}_{\rm NRe}\otimes\bar{V}^{-}_{\rm NRe}-iAV^{-}_{\rm NRo}\otimes\bar{V}^{-}_{\rm NRo} \\
R^{-}_{\rm NR} &= AV^{+}_{\rm NRe}\otimes\bar{V}^{-}_{\rm NRo}-BV^{+}_{\rm NRo}\otimes\bar{V}^{-}_{\rm NRe}+BV^{-}_{\rm NRe}\otimes\bar{V}^{+}_{\rm NRo}+AV^{-}_{\rm NRo}\otimes\bar{V}^{+}_{\rm NRe}.
\end{aligned} \tag{3.20}
$$

The coefficients are fixed up to A and B by Ward identities for matrix elements of the fields $R^{\pm}_{\rm NR}$ (3.17). The relations above imply that the structure constants are given by the following formulae:

$$
\begin{aligned}
C^{+} &= A\,\varrho_{NR}(\nu,w^{+},w^{+};1)\,\bar{\varrho}_{NR}(\nu,w^{+},w^{+};1)+iB\,\varrho_{NR}(\nu,w^{-},w^{+};1)\,\bar{\varrho}_{NR}(\nu,\bar{w}^{-},w^{+};1) \\
&+ iB\,\varrho_{NR}(\nu,w^{+},w^{-};1)\,\bar{\varrho}_{NR}(\nu,w^{+},\bar{w}^{-};1)+A\,\varrho_{NR}(\nu,w^{-},w^{-};1)\,\bar{\varrho}_{NR}(\nu,w^{-},w^{-};1) \\
C^{-} &= A\,\varrho_{NR}(\nu,w^{+},w^{+};1)\,\bar{\varrho}_{NR}(\nu,w^{-},w^{-};1)+B\,\varrho_{NR}(\nu,w^{-},w^{+})\,\bar{\varrho}_{NR}(\nu,w^{+},w^{-};1) \\
&- B\,\varrho_{NR}(\nu,w^{+},w^{-};1)\,\bar{\varrho}_{NR}(\nu,w^{-},w^{+};1)+A\,\varrho_{NR}(\nu,w^{-},w^{-};1)\,\bar{\varrho}_{NR}(\nu,w^{+},w^{+};1)\ .
\end{aligned} \tag{3.21}
$$

Knowing formulae (3.20) one can consider arbitrary matrix elements of primary Ramond fields. In order to compute how the 3-point correlators reduce to the structure constants given by (3.21), some properties of the forms $\rho^{ij}_{NR}; i,j=\pm$ will be useful. One can derive these properties by analyzing Ward identities for non normalized 3-form (3.18). Since the Ward identities are complicated, the derivations are laborious even in the case of relations concerning basic features of the 3-forms.

First, one can check that for a given $S_{-I}\nu, S_{-J}w^{+}_{1}$ the 3-form ϱ_{NR} is proportional to two out of four constants:

$$
\begin{aligned}
&\rho^{\pm\pm}_{NR}(S_{-I}\nu,w^{+}_{2},S_{-J}w^{+}_{1})=0, \qquad &&\text{if}\quad (2|I|+\sharp J)\in 2\mathbb{N}+1 \\
&\rho^{\pm\mp}_{NR}(S_{-I}\nu,w^{+}_{2},S_{-J}w^{+}_{1})=0, \qquad &&\text{if}\quad (2|I|+\sharp J)\in 2\mathbb{N}
\end{aligned}
$$

Secondly, the formulae (3.18) hardly depend on the parity of the second state $w^{\pm}_{2}$. More precisely, only the action of the last S_0 on $w^{\pm}_{2}$ is relevant:

$$
\begin{aligned}
\varrho_{NR}(S_{-I}\nu,w^{+}_{2},S_{-J}w^{+}_{1}|1) &= f_1\,\varrho_{NR}(\nu_3,w^{+}_{2},w^{+}_{1}|1)+f_2\,\varrho_{NR}(\nu_3,S_0w^{+}_{2},S_0w^{+}_{1}|1) \\
\varrho_{NR}(S_{-I}\nu,w^{-}_{2},S_{-J}w^{+}_{1}|1) &= f_1\,\varrho_{NR}(\nu_3,w^{-}_{2},w^{+}_{1}|1)+f_2\,\varrho_{NR}(\nu_3,S_0w^{-}_{2},S_0w^{+}_{1}|1),
\end{aligned} \tag{3.22}
$$

where $(2|I|+\sharp J)\in 2\mathbb{N}$. Since f_1, f_2 are results of the action of S_0^2, L_{-1}, L_0, they are functions of conformal weights or β_i^2, independent of the parity of $w^{\pm}_{2}$ or the sign of β_i. Using the definition (3.19) one can read off the relations:

$$
\begin{aligned}
\rho^{++}_{NR}(S_{-I}\nu,w^{+}_{2},S_{-J}w^{+}_{1}) &= f_1=\rho^{-+}_{NR}(S_{-I}\nu,w^{-}_{2},S_{-J}w^{+}_{1}) \\
\rho^{--}_{NR}(S_{-I}\nu,w^{+}_{2},S_{-J}w^{+}_{1}) &= -i\beta_1\beta_2 f_2=-i\,\rho^{+-}_{NR}(S_{-I}\nu,w^{-}_{2},S_{-J}w^{+}_{1})
\end{aligned} \tag{3.23}
$$

and

$$\begin{aligned}\rho^{++}_{NR}(S_{-I}\nu, w^{+}_{-\beta_2}, S_{-J}w^{+}_1) &= \rho^{++}_{NR}(S_{-I}\nu, w^{+}_{\beta_2}, S_{-J}w^{+}_1)\\ \rho^{--}_{NR}(S_{-I}\nu, w^{+}_{-\beta_2}, S_{-J}w^{+}_1) &= -\rho^{--}_{NR}(S_{-I}\nu, w^{+}_{\beta_2}, S_{-J}w^{+}_1)\end{aligned} \tag{3.24}$$

Similarly, for $(2|I| + \sharp J) \in 2\mathbb{N}+1$ one gets:

$$\begin{aligned}\rho^{-+}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1) &= -i\,\rho^{++}_{NR}(S_{-I}\nu, w^{-}_{2}, S_{-J}w^{+}_1)\\ \rho^{+-}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1) &= \rho^{--}_{NR}(S_{-I}\nu, w^{-}_{2}, S_{-J}w^{+}_1)\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\rho^{+-}_{NR}(S_{-I}\nu, w^{+}_{-\beta_2}, S_{-J}w^{+}_1) &= \rho^{+-}_{NR}(S_{-I}\nu, w^{+}_{\beta_2}, S_{-J}w^{+}_1)\\ \rho^{-+}_{NR}(S_{-I}\nu, w^{+}_{-\beta_2}, S_{-J}w^{+}_1) &= -\rho^{-+}_{NR}(S_{-I}\nu, w^{+}_{\beta_2}, S_{-J}w^{+}_1)\end{aligned} \tag{3.26}$$

Moreover, analyzing the Ward identities (3.18) one can investigate how the 3-form depends on the parity of $w^{\pm}_1$:

$$\begin{aligned}\rho^{+-}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{-}_1) &= (-1)^{|J|}\,\rho^{++}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1)\\ \rho^{-+}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{-}_1) &= -i\,(-1)^{|J|}\,\rho^{--}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1)\\ \rho^{++}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{-}_1) &= -i\,(-1)^{|J|}\,\rho^{+-}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1)\\ \rho^{--}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{-}_1) &= (-1)^{|J|}\,\rho^{-+}_{NR}(S_{-I}\nu, w^{+}_{2}, S_{-J}w^{+}_1).\end{aligned} \tag{3.27}$$

Using the definition (3.19) and relations (3.23), (3.25), (3.27) one can check that all matrix elements of the Ramond field $R^{\pm}_{\rm NR}$ depend on the arbitrary constants only via combinations (3.21). The 3-point functions reduce to the structure constants in the following way:

$$\begin{aligned}&\left\langle S_{-I}\bar S_{-\bar I}\,\nu\otimes\bar\nu\right|\; R^{+}_{2}\;\left|S_{-J}\bar S_{-\bar J}\,w^{+}_{\Delta_1,\bar\Delta_1}\right\rangle\\ &\quad= C^{(+)}\rho^{(+)}_{\rm NRe}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(+)}_{\rm NRe}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\quad+ C^{(-)}\rho^{(-)}_{\rm NRe}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(-)}_{\rm NRe}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\left\langle S_{-I}\bar S_{-\bar I}\,\nu\otimes\bar\nu\right|\; R^{-}_{2}\;\left|S_{-J}\bar S_{-\bar J}\,w^{+}_{\Delta_1,\bar\Delta_1}\right\rangle\\ &\quad= (-1)^{|J|}C^{(+)}\rho^{(+)}_{\rm NRe}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(+)}_{\rm NRo}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\quad- (-1)^{|J|}C^{(-)}\rho^{(-)}_{\rm NRe}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(-)}_{\rm NRo}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\end{aligned} \tag{3.28}$$

for $S_{-I}\,\nu, S_{-\bar J}\,w^{+}_1$ of the same parity, and

$$\begin{aligned}&\left\langle S_{-I}\bar S_{-\bar I}\,\nu\otimes\bar\nu\right|\; R^{+}_{2}\;\left|S_{-J}\bar S_{-\bar J}\,w^{+}_{\Delta_1,\bar\Delta_1}\right\rangle\\ &\quad= -i(-1)^{|J|}C^{(+)}\rho^{(+)}_{\rm NRo}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(+)}_{\rm NRo}(\bar S_{-\bar I}\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\qquad -i\,(-1)^{|J|}C^{(-)}\rho^{(-)}_{\rm NRo}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(-)}_{\rm NRo}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\left\langle S_{-I}\bar S_{-\bar I}\,\nu\otimes\bar\nu\right|\; R^{-}_{2}\;\left|S_{-J}\bar S_{-\bar J}\,w^{+}_{\Delta_1,\bar\Delta_1}\right\rangle\\ &\quad= C^{(+)}\rho^{(+)}_{\rm NRo}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(+)}_{\rm NRe}(\bar S_{-\bar I}\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\\ &\quad- C^{(-)}\rho^{(-)}_{\rm NRo}(S_{-I}\,\nu, w^{+}_{2}, S_{-J}\,w^{+}_1)\,\bar\rho^{(-)}_{\rm NRe}(\bar S_{-\bar I}\,\nu, w^{+}_{2}, \bar S_{-\bar J}\,w^{+}_1)\end{aligned} \tag{3.29}$$

for $S_{-I}\,\nu, S_{-\bar{J}}\,w_1^+$ of the opposite parity. The functions proportional to the constants

$$C^{(\pm)} = \frac{C^+ \pm C^-}{2}$$

have the form:

$$\begin{aligned} \rho^{(\pm)}_{\rm NRe} &= \rho^{++}_{NR} \pm \rho^{--}_{NR} \quad , \quad \bar\rho^{(\pm)}_{\rm NRe} = \bar\rho^{++}_{NR} \pm \bar\rho^{--}_{NR} \\ \rho^{(\pm)}_{\rm NRo} &= \rho^{+-}_{NR} \pm i\rho^{-+}_{NR} \quad , \quad \bar\rho^{(\pm)}_{\rm NRo} = \bar\rho^{+-}_{NR} \mp i\bar\rho^{-+}_{NR} \end{aligned} \tag{3.30}$$

We will call them normalized 3-point Ramond blocks. Introducing normalized chiral vertex operators,

$$\begin{aligned} \langle\xi_3|V^{(\pm)}_{\rm NRe}(w^\pm|z)|\eta_1\rangle &= \rho^{(\pm)}_{\rm NRe}(\xi_3, w^\pm, \eta_1|z) \quad , \quad |\xi_3|, |\eta_1| \quad \text{equal parities} \\ \langle\xi_3|V^{(\pm)}_{\rm NRo}(w^\pm|z)|\eta_1\rangle &= \rho^{(\pm)}_{\rm NRo}(\xi_3, w^\pm, \eta_1|z) \quad , \quad |\xi_3|, |\eta_1| \quad \text{theopposite parities} \end{aligned} \tag{3.31}$$

one gets from the calculations of 3-point functions

$$\begin{aligned} R^+_{\rm NR}(w^+, \bar w^+|z,\bar z) &= C^{(+)}\left(V^{(+)}_{\rm NRe}(w^+|z)\otimes\bar V^{(+)}_{\rm NRe}(\bar w^+|\bar z) - i\,V^{(+)}_{\rm NRo}(w^+|z)\otimes\bar V^{(+)}_{\rm NRo}(\bar w^+|\bar z)\right) \\ &+ C^{(-)}\left(V^{(-)}_{\rm NRe}(w^+|z)\otimes\bar V^{(-)}_{\rm NRe}(\bar w^+|\bar z) - i\,V^{(-)}_{\rm NRo}(w^+|z)\otimes\bar V^{(-)}_{\rm NRo}(\bar w^+|\bar z)\right) \\ R^-_{\rm NR}(w^+, \bar w^+|z,\bar z) &= C^{(+)}\left(V^{(+)}_{\rm NRe}(w^+|z)\otimes\bar V^{(+)}_{\rm NRo}(\bar w^+|\bar z) + V^{(+)}_{\rm NRo}(w^+|z)\otimes\bar V^{(+)}_{\rm NRe}(\bar w^+|\bar z)\right) \\ &- C^{(-)}\left(V^{(-)}_{\rm NRe}(w^+|z)\otimes\bar V^{(-)}_{\rm NRo}(\bar w^+|\bar z) + V^{(-)}_{\rm NRo}(w^+|z)\otimes\bar V^{(-)}_{\rm NRe}(\bar w^+|\bar z)\right). \end{aligned} \tag{3.32}$$

The Ramond fields in the R-NS sector can be directly obtained from the hermicity condition $\left(R^\pm_{\rm NR}\right)^\dagger = R^\pm_{\rm RN}$ (3.13).

Let us observe that

$$w^+_\Delta\otimes w^+_{\bar\Delta} + i\,w^-_\Delta\otimes w^-_{\bar\Delta}\ ,\ w^+_\Delta\otimes w^-_{\bar\Delta} - w^-_\Delta\otimes w^+_{\bar\Delta}\ \in\ \ker R^\pm_{\rm NR}$$

hence the states from "small representation" form an invariant subspace of the full Ramond fields $R^\pm$.

The matrix elements of the normalized vertex operators *i.e.* the 3-point blocks are the proper objects in terms of which the 4-point Ramond blocks should be defined. Let us emphasize the basic fact concerning the Ramond 3-point blocks. The definition (3.30) implies that all properties of the blocks are given by the corresponding properties of the 3-forms and can be derived solely from the Ward identities (3.18). In particular, relations (3.23), (3.25) lead to:

$$\begin{aligned} \rho^{(\pm)}_{\rm NRe}(S_{-I}\nu, w_2^-, S_J w_1^+) &= \pm\rho^{(\pm)}_{\rm NRo}(S_{-I}\nu, w_2^+, S_J w_1^+) \\ \rho^{(\pm)}_{\rm NRo}(S_{-I}\nu, w_2^-, S_J w_1^+) &= \pm i\rho^{(\pm)}_{\rm NRe}(S_{-I}\nu, w_2^+, S_J w_1^+)\ . \end{aligned}$$

These imply for chiral vertex operators (3.31):

$$V^{(\pm)}_{\rm NRe}(w^-|z) = \pm V^{(\pm)}_{\rm NRo}(w^+|z) \qquad V^{(\pm)}_{\rm NRo}(w^-|z) = \pm i\,V^{(\pm)}_{\rm NRe}(w^+|z).$$

Taking into account the graded tensor product structure:

$$\langle \xi_3 \otimes \bar{\xi}_3 | V^{(\pm)}_{\mathrm{NRp}}(w_2^{\pm}) \otimes \bar{V}^{(\pm)}_{\mathrm{NR\bar{p}}}(\bar{w}_2^{\pm}) | \eta_1 \otimes \bar{\eta}_1 \rangle \;=\; (-1)^{|p||\bar{w}_2^{\pm}| + |\bar{p}||\eta_1|} \varrho^{(\pm)}_{\mathrm{NRp}}(\xi_3, w_2^{\pm}, \eta_1) \bar{\varrho}^{(\pm)}_{\mathrm{NR\bar{p}}}(\bar{\xi}_3, \bar{w}_2^{\pm}, \bar{\eta}_1)$$

where $\mathrm{p}, \bar{\mathrm{p}} = \mathrm{e}, \mathrm{o}$ and $|\mathrm{e}| = 0, |\mathrm{o}| = 1$, one gets

$$\begin{aligned} -iR^{+}_{\mathrm{NS}}(w^-, \bar{w}^- | z, \bar{z}) &= R^{+}_{\mathrm{NS}}(w^+, \bar{w}^+ | z, \bar{z}) \\ R^{+}_{\mathrm{NS}}(w^+, \bar{w}^- | z, \bar{z}) &= R^{+}_{\mathrm{NS}}(w^-, \bar{w}^+ | z, \bar{z}) \;=\; R^{-}_{\mathrm{NS}}(w^+, \bar{w}^+ | z, \bar{z}) \;. \end{aligned}$$

Moreover, due to relations (3.24), (3.26) the 3-point blocks depend on the sign of β_2 in a very simple way:

$$\begin{aligned} \rho^{(\pm)}_{\mathrm{NRe}}(S_{-I}\nu, w^+_{-\beta_2}, S_J w^+_1) &= \pm \rho^{(\mp)}_{\mathrm{NRe}}(S_{-I}\nu, w^+_{\beta_2}, S_J w^+_1) \\ \rho^{(\pm)}_{\mathrm{NRo}}(S_{-I}\nu, w^+_{-\beta_2}, S_J w^+_1) &= \pm \rho^{(\mp)}_{\mathrm{NRo}}(S_{-I}\nu, w^+_{\beta_2}, S_J w^+_1) \;. \end{aligned} \tag{3.33}$$

This leads to the following relation for the Ramond primary fields with the same conformal weights Δ_β but opposite sign in front of β:

$$R^{\epsilon}_{-\beta} = \epsilon R^{\epsilon}_{\beta} \;. \tag{3.34}$$

3.2.2 Fusion rules and fusion polynomials

Consider the three point correlation functions with degenerate NS field ϕ_{rs} within the Feigin-Fuchs construction [5]. In this approach the Ramond fields are represented by vertex operators in the free superscalar Hilbert space

$$R^{+}_{\beta,\bar{\beta}}(z, \bar{z}) = \mathrm{e}^{a\phi(z) + \bar{a}\bar{\phi}(\bar{z}) + i\frac{\pi}{4}} \sigma^+(z, \bar{z}) \quad , \quad R^{-}_{\beta,\bar{\beta}}(z, \bar{z}) = \mathrm{e}^{a\phi(z) + \bar{a}\bar{\phi}(\bar{z}) - i\frac{\pi}{4}} \sigma^-(z, \bar{z}) \tag{3.35}$$

where $a = \frac{Q}{2} - \sqrt{2}\beta$ and $\sigma^{\pm}$ are the twist operators of the fermionic sector:

$$\psi(z)\sigma^{\pm}(z, \bar{z}) \sim \frac{1}{\sqrt{2(z-w)}}\, \sigma^{\mp}(z, \bar{z}) \;. \tag{3.36}$$

The left chiral screening charges are given by:

$$Q_b = \oint dz\, \psi(z) \mathrm{e}^{b\phi(z)}, \qquad Q_{\frac{1}{b}} = \oint dz\, \psi(z) \mathrm{e}^{\frac{1}{b}\phi(z)},$$

and the same construction holds in the right sector. The Feigin-Fuchs representation of three point functions with various number of left screening charges has the form:

$$\begin{aligned} C^{\epsilon}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} &= \left\langle \phi_{rs} R^{\epsilon}_{\beta_2} R^{\epsilon}_{\beta_1} Q^k_b Q^l_{\frac{1}{b}} \right\rangle \;, \qquad k + l \in 2\mathbb{N} \;, \qquad \delta = -\tfrac{1}{2\sqrt{2}}(\tfrac{1}{b} + b) \;; \\ C^{\epsilon}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} &= \left\langle \phi_{rs} R^{\epsilon}_{\beta_2} R^{\epsilon}_{\beta_1} Q^k_b Q^l_{\frac{1}{b}} \bar{Q}_b \right\rangle \;, \quad k + l \in 2\mathbb{N} + 1 \;, \quad \delta = \tfrac{1}{2\sqrt{2}}(\tfrac{1}{b} - b). \end{aligned}$$

The charge conservation implies that the structure constants above are non-zero if and only if the even fusion rules (for $k + l \in 2\mathbb{N} \cup \{0\}$):

$$\beta_1 + \beta_2 \;=\; \frac{1}{2\sqrt{2}}(1 - r + 2k)b + \frac{1}{2\sqrt{2}}(1 - s + 2l)\frac{1}{b}, \tag{3.37}$$

or the odd fusion rules (for $k+l \in 2\mathbb{N}-1$):

$$\beta_1+\beta_2 \;=\; \frac{1}{2\sqrt{2}}(1-r+2k)b+\frac{1}{2\sqrt{2}}(1-s+2l)\frac{1}{b} \tag{3.38}$$

are satisfied (k, l are integers in the range $0 \le k \le r-1,\ 0 \le l \le s-1$).

Moreover, additional relations between structure constants can be obtained. In the Feigin-Fuchs representation one can show that for any even integer $n \in 2\mathbb{N}$:

$$\begin{aligned} \big\langle \psi(w_1)\ldots\psi(w_n)\sigma^-(1,1)\sigma^-(0,0)\big\rangle &= -\big\langle \psi(w_1)\ldots\psi(w_n))\sigma^+(1,1)\sigma^+(0,0)\big\rangle \\ \big\langle \psi(w_1)\ldots\psi(w_{n-1})\,\bar\psi(\bar w)\sigma^-(1,1)\sigma^-(0,0)\big\rangle &= \big\langle \psi(w_1)\ldots\psi(w_{n-1})\,\bar\psi(\bar w)\sigma^+(1,1)\sigma^+(0,0)\big\rangle \end{aligned}$$

If the even fusion rules (3.37) are satisfied this implies

$$C^{+}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} = -C^{-}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)}.$$

what gives $C^{(+)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} = 0$ and $C^{(-)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} \neq 0$.

Consider now a 3-point function with zero field corresponding to the null vector χ_{rs}. It follows from equations (3.28),(3.29) that, if the even fusion rules (3.37) are satisfied and $C^{(-)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} \neq 0$, then:

$$\rho^{(-)}_{\rm NRe,o}(\chi_{rs}, w_2^+, w_1^+) = 0$$

Similarly, for the odd fusion rules (3.38) one gets $C^{(+)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)} \neq 0$ and

$$\rho^{(+)}_{\rm NRe,o}(\chi_{rs}, w_2^+, w_1^+) = 0\ .$$

An additional information on zeros of the forms in question can be derived from the formula

$$C^{(\pm)}_{(\alpha_{rs},\delta),(-\beta_2,0),(\beta_1,0)} = C^{(\mp)}_{(\alpha_{rs},\delta),(\beta_2,0),(\beta_1,0)}$$

which is a simple consequence of (3.34). The form $\rho^{(+)}_{\rm NRe,o}(\chi_{rs}, w_2^+, w_1^+)$ has to vanish for the even fusion rules (3.37) and $\rho^{(+)}_{\rm NRe,o}(\chi_{rs}, w_2^+, w_1^+)$ for the odd fusion rules (3.38) with the opposite sign in front of β_2 in both cases.

The discussion above suggests the following definition of the fusion polynomials in the Ramond sector:

$$P^{rs}_{c}\left[{}^{\pm\beta_2}_{\;\beta_1}\right] \;=\; \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\beta_1\mp\beta_2+\frac{pb+qb^{-1}}{2\sqrt{2}}\right)\prod_{p'=1-r}^{r-1}\prod_{q'=1-s}^{s-1}\left(\beta_1\pm\beta_2+\frac{p'b+q'b^{-1}}{2\sqrt{2}}\right) \tag{3.39}$$

where p, q, p', q' run with the step 2 and satisfy the conditions: $p+q-(r+s) \in 4\mathbb{Z}+2$ and $p'+q'-(r+s) \in 4\mathbb{Z}$, corresponding to even and odd fusion rules, respectively. One easily checks that for $rs \in 2\mathbb{N}$, $P^{rs}_{c}\left[{}^{\pm\beta_2}_{\;\beta_1}\right]$ are polynomials of degree $\frac{rs}{2}$ in $(\Delta_2-\Delta_1)$, and for $rs \in 2\mathbb{N}-1$ – of degree $\frac{rs-1}{2}$ in $(\Delta_2-\Delta_1)$ with the additional factor $(\beta_1\mp\beta_2)$. On the other hand using the definition of the 3-point block (3.30) together with the Ward identities for non normalized 3-form one can calculate:

$$\begin{aligned}\rho^{(\pm)}_{\rm NRe}(L_{-1}^n\nu, w_2^+, w_1^+) &= (\Delta+\Delta_2-\Delta_1)_n\\ \rho^{(\pm)}_{\rm NRo}(S_{-\frac12}L_{-1}^n\nu, w_2^+, w_1^+) &= {\rm e}^{-i\frac{\pi}{4}}(\beta_1\mp\beta_2)(\Delta+\Delta_2-\Delta_1)_n\end{aligned}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. Taking into account the normalization condition for χ_{rs} one thus finally obtains:

$$\begin{aligned}\rho^{(\pm)}_{\rm NRe}(\chi_{rs}, w_2^+, w_1^+) &= P^{rs}_c\left[{}^{\pm\beta_2}_{\beta_1}\right] \qquad \text{for} \quad rs\in 2\mathbb{N}\ ,\\ \rho^{(\pm)}_{\rm NRo}(\chi_{rs}, w_2^+, w_1^+) &= {\rm e}^{-i\frac{\pi}{4}}\, P^{rs}_c\left[{}^{\pm\beta_2}_{\beta_1}\right] \quad \text{for} \quad rs\in 2\mathbb{N}-1\ .\end{aligned} \tag{3.40}$$

3.3 Ramond 4-point blocks

3.3.1 Definition

We shall restrict ourselves to 4-point blocks corresponding to correlation functions of four Ramond fields factorized on NS states. The formulae (3.32) expressing Ramond primary fields $R^{\pm}_{\rm NR}, R^{\pm}_{\rm RN}$ in terms of normalized chiral vertex operators together with the properties of 3-point blocks (3.33) suggests the following definition of Ramond 4-point blocks:

$$F^f_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right] = \sum_{|K|+|M|=|L|+|N|=f} \overline{\rho^{(\pm)}_{|f|}(\nu_{\Delta,KM}, w^+_{\beta_3}, w^+_4)}\ \left[B^f_{c,\Delta}\right]^{KM,LN} \rho^{(\pm)}_{|f|}(\nu_{\Delta,LN}, w^+_{\beta_2}, w^+_1). \tag{3.41}$$

where $|f| = {\rm e}$ for $f\in\mathbb{N}$, $|f| = {\rm o}$ for $f\in\mathbb{N}-\frac12$, $\nu_{\Delta,KM}$ is the standard basis in the NS Verma module $\mathcal{V}_{c,\Delta}$ (2.12), and $\left[B^f_{c,\Delta}\right]^{KM,LN}$ denotes the inverse NS Gram matrix. One has four even:

$$\mathcal{F}^1_{\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right](z) = z^{\Delta-\Delta_2-\Delta_1}\left(1+\sum_{m\in\mathbb{N}} z^m F^m_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right]\right), \tag{3.42}$$

and four odd,

$$\mathcal{F}^{\frac12}_{\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right](z) = z^{\Delta+\frac12-\Delta_2-\Delta_1}\sum_{k\in\mathbb{N}-\frac12} z^{k-\frac12} F^k_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right], \tag{3.43}$$

conformal blocks.

It follows from the definition of the blocks' coefficients $F^f_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right]$ that they are polynomials in β_i, and rational functions of the intermediate weight Δ and the central charge c. Due to the properties of inverse NS Gram matrix (2.1.3), the coefficients can be expressed as a sum over the poles in Δ :

$$F^f_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right] = {\rm h}^f_{c,\Delta}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right] + \sum_{\substack{1<rs\leq 2f\\ r+s\in 2\mathbb{N}}} \frac{\mathcal{R}^f_{c,\,rs}\left[{}^{\pm\beta_3}_{\ \beta_4}\,{}^{\pm\beta_2}_{\ \beta_1}\right]}{\Delta-\Delta_{rs}(c)}\,, \tag{3.44}$$

with $\Delta_{rs}(c)$ given by Kac determinant formula (2.13).

The calculation of the residue at Δ_{rs} is essentially the same as in the NS case. With a suitable choice of basis in $\mathcal{V}_{c,\Delta}$ (2.52) one gets

$$\mathcal{R}^f_{c,\,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} = A_{rs}(c) \times \tag{3.45}$$
$$\sum \overline{\rho^{(\pm)}_{|f|}(S_{-K}L_{-M}\chi_{rs}, w_3^+, w_4^+)}\,\left[B^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\right]^{KM,LN}\rho^{(\pm)}_{|f|}(S_{-L}L_{-N}\chi_{rs}, w_2^+, w_1^+),$$

with coefficients $A_{rs}(c)$ given by (2.53).

Analyzing Ward identities for the 3-form (3.15) one can check that the factorization property of the forms $\rho^{(\pm)}_{\rm NR}$ holds on singular vectors. In the case $\frac{rs}{2} \in \mathbb{N}$ one gets

$$\begin{aligned}
\rho^{(\pm)}_{\rm NRe}(S_{-I}\chi_{rs}, w_2^+, w_1^+) &= \rho^{(\pm)}_{\rm NRe}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}}, w_2^+, w_1^+)\,\rho^{(\pm)}_{\rm NRe}(\chi_{rs}, w_2^+, w_1^+)\ ,\ |I| \in \mathbb{N}\ ,\\
\rho^{(\pm)}_{\rm NRo}(S_{-K}\chi_{rs}, w_2^+, w_1^+) &= \rho^{(\pm)}_{\rm NRo}(S_{-K}\nu_{\Delta_{rs}+\frac{rs}{2}}, w_2^+, w_1^+)\,\rho^{(\pm)}_{\rm NRe}(\chi_{rs}, w_2^+, w_1^+)\ ,\ |K| \in \mathbb{N}+\tfrac{1}{2}\ ,
\end{aligned}$$

while in the case $\frac{rs}{2} \in \mathbb{N}+\frac{1}{2}$

$$\begin{aligned}
\rho^{(\pm)}_{\rm NRo}(S_{-I}\chi_{rs}, w_2^+, w_1^+) &= \rho^{(\mp)}_{\rm NRe}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}}, w_2^+, w_1^+)\,\rho^{(\pm)}_{\rm NRo}(\chi_{rs}, w_2^+, w_1^+)\ ,\ |I| \in \mathbb{N}\ ,\\
\rho^{(\pm)}_{\rm NRe}(S_{-K}\chi_{rs}, w_2^+, w_1^+) &= i\,\rho^{(\mp)}_{\rm NRo}(S_{-I}\nu_{\Delta_{rs}+\frac{rs}{2}}, w_2^+, w_1^+)\,\rho^{(\pm)}_{\rm NRo}(\chi_{rs}, w_2^+, w_1^+)\ ,\ |K| \in \mathbb{N}+\tfrac{1}{2}\ .
\end{aligned}$$

Using the factorization properties one can obtain formulae for the residue:

$$\mathcal{R}^f_{c,\,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} = A_{rs}(c)\,\overline{\rho^{(\pm)}_{\rm RNe}(\chi_{rs}, w_3^+, w_4^+)}\rho^{(\pm)}_{\rm NRe}(\chi_{rs}, w_2^+, w_1^+)F^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} \tag{3.46}$$

for $\frac{rs}{2} \in \mathbb{N}\cup\{0\}$, and

$$\mathcal{R}^f_{c,\,rs}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix} = A_{rs}(c)\,\overline{\rho^{(\pm)}_{\rm RNo}(\chi_{rs}, w_3^+, w_4^+)}\rho^{(\pm)}_{\rm NRo}(\chi_{rs}, w_2^+, w_1^+)F^{f-\frac{rs}{2}}_{c,\Delta_{rs}+\frac{rs}{2}}\begin{bmatrix}\mp\beta_3 & \mp\beta_2\\ \beta_4 & \beta_1\end{bmatrix} \tag{3.47}$$

for $\frac{rs}{2} \in \mathbb{N}-\frac{1}{2}$. The 3-point blocks $\rho^{(\pm)}_{\rm NRp}(\chi_{rs}, w_2^+, w_1^+)$ are given by the fusion polynomials (3.40).

In order to derive the recursion relation for blocks' coefficients one has to calculate the regular in Δ term in (3.44). As in the case of NS sector, it can be achieved following Zamolodchikov's derivation [16].

Notice that the coefficients of Ramond 4-point blocks can be expressed as a sum over simple poles in c_{rs} as well. For these coefficients however, the regular terms in c are not given by the limit of the blocks in $c \to \infty$. The 3-point blocks are polynomials in β_i rather than in conformal weights. This implies that, unlike NS 3-point blocks, Ramond 3-point blocks depend of conformal weights *and* central charge. Thus the reasoning leading to determination of the terms regular in c (2.56) does not work in case of Ramond 4-point blocks.

3.3.2 Large Δ asymptotic of Ramond blocks

The first step in the derivation of the elliptic recurrence is to find the large Δ asymptotic of 4-point block. As it was reminded in the section 1.4, Zamolodchkov's reasoning is based on the observation that the full Δ_i, c dependence of the first two terms in the large Δ expansion of the conformal block can be read off from the first two terms of the $\frac{1}{\delta}$ expansion of the classical block. The basic assumption of this approach concerns the existence of the classical limit of conformal blocks. Analyzing the $N = 1$ super-Liouville correlation functions represented by functional integrals one can justify the the existence of the classical limit of Ramond blocks. The action in the $N = 1$ super-Liouville is defined as in (2.59). Within the functional approach the Ramond fields are represented by vertex operators (3.35). Since the twist fields are light the fermionic sector does not contribute to the classical limit at all.

The path integral arguments allow to investigate the asymptotical behavior of 4-point and 3-point correlation functions. Considering the 4-point correlators projected on the even or the odd subspace of intermediate NS states one may get information concerning individual blocks. This leads to the assumption that in the limit

$$b \to 0 \quad , \quad i\,b\beta_i \to p_i \quad , \quad b^2\Delta_i \to \delta_i = p_i^2$$

the asymptotic behavior of Ramond blocks reads

$$\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) \;\sim\; r_1 \mathrm{e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \qquad \mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) \;\sim\; r_{\frac{1}{2}} \mathrm{e}^{\frac{1}{2b^2} f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](z)}, \tag{3.48}$$

where $f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$ is the classical conformal block and coefficients $r_1, r_{\frac{1}{2}}$ are independent of b. In order to determine the δ dependence of $r_1, r_{\frac{1}{2}}$ one has to check the leading powers of Δ in the 3-point blocks defining coefficients of 4-point blocks. Analyzing Ward identities (3.18) and using reasoning similar as the one leading to relation (3.22), one finds the maximal powers of Δ in the forms $\rho^{\pm\pm}_{NR}, \rho^{\pm\mp}_{NR}$:

$$\begin{aligned}
\rho^{+-}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\;\propto\; \beta_1 \Delta^{|K|-\frac{1}{2}} + \dots\,, \qquad & \rho^{++}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\propto \Delta^{|K|} + \dots\,,\\
\rho^{-+}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\;\propto\; \beta_2 \Delta^{|K|-\frac{1}{2}} + \dots\,, \qquad & \rho^{--}_{NR}(S_{-K}\nu, w_2^+, w_1^+) &\propto \beta_1\beta_2 \Delta^{|K|-1} + \dots\,.
\end{aligned}$$

This implies for the 3-point blocks:

$$\rho^{(\pm)}_{NRe}(S_{-K}\nu, w_2^+, w_1^+) \propto \Delta^{|K|} + \dots\,, \qquad \rho^{(\pm)}_{NRo}(S_{-K}\nu, w_2^+, w_1^+) \propto (\beta_1 \mp \beta_2)\Delta^{|K|-\frac{1}{2}} + \dots\,.$$

It follows from the Δ dependence of the inverse NS Gram matrix and from the relations above that the coefficients $r_1, r_{\frac{1}{2}}$ are indeed independent of b. Moreover, even 4-point Ramond blocks have the same asymptotical behavior as even NS blocks (2.72):

$$r_1 \sim \text{const}$$

as functions of δ. The coefficients of odd Ramond blocks, as in the case of odd NS block without stars (2.73), have the leading power of Δ in the numerator smaller by 1 in comparison to the coefficients of even blocks. Since the power series defining odd blocks do not contain zeroth order term, one can put the factor $\frac{(\beta_1 \mp \beta_2)}{\Delta}$ in front of these series. This implies that

$$r_{\frac{1}{2}} \sim \frac{1}{\delta}\,.$$

Once the classical limits of the blocks are known one can again follow Zamolodchikov's derivation in order to find large Δ behavior of the blocks. The even blocks have the same asymptotics as non supersymmetric and even NS blocks (1.57), while the odd blocks behave similarly as odd NS block without stars (2.74):

$$\ln \mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) = \pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z)$$
$$+\left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \Delta_2\right)\ln(z) + f^{\pm\pm}(z) + \mathcal{O}\left(\frac{1}{\Delta}\right),$$
$$\ln \mathcal{F}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) = -\ln\Delta + \pi\tau\left(\Delta - \frac{c}{24}\right) + \left(\frac{c}{8} - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_4\right)\ln K^2(z)$$
$$+\left(\frac{c}{24} - \Delta_2 - \Delta_3\right)\ln(1-z) + \left(\frac{c}{24} - \Delta_1 - \Delta_2\right)\ln(z) + \mathcal{O}\left(\frac{1}{\Delta}\right),$$

where $f^{\pm\pm}(z)$ are functions of z specific for each type of block and independent of Δ_i and c.

3.3.3 Elliptic recurrence

The large Δ asymptotic suggests the following form of superconformal blocks:

$$\begin{aligned}\mathcal{F}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) &= (16q)^{\Delta-\frac{c-3/2}{24}}\, z^{\frac{c-3/2}{24}-\Delta_1-\Delta_2}\,(1-z)^{\frac{c-3/2}{24}-\Delta_2-\Delta_3}\\ &\times\ \theta_3^{\frac{c-3/2}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\,\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q).\end{aligned} \tag{3.49}$$

The elliptic blocks $\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z)$ have the same analytic structure as superconformal ones:

$$\begin{aligned}\mathcal{H}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q) &= g^{\pm\pm}(q) + \sum_{m,n}\frac{h^{1}_{mn}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)}{\Delta-\Delta_{mn}},\\ \mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q) &= \sum_{m,n}\frac{h^{\frac{1}{2}}_{mn}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)}{\Delta-\Delta_{mn}}.\end{aligned}$$

In the case of odd elliptic blocks the regular in Δ terms are zero due to $(-\ln\Delta)$ in large asymptotics. The functions $g^{\pm\pm}(q)$, non singular in Δ, depend on the block type and are independent of β_i and the central charge c. Thus they can be determined from the analytical formula of the $\hat{c}=1$ 4-point block with $\Delta_i = \Delta_0 = \frac{1}{16}$, $\beta_0 = 0$. This block can be calculated using the techniques of the chiral superscalar model, which will be discussed in the next chapter (4.69):

$$\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](q) = (16q)^{\Delta}\,[z(1-z)]^{-\frac{1}{8}}\,\theta_3(q)^{-1} \tag{3.50}$$

The $b\to i$, $\beta_i\to\beta_0$ limit of each type of general even block is regular for generic values of Δ and yields

$$\lim_{\beta\to 0}\lim_{b\to i}\mathcal{F}^{1}_{\Delta}\left[\begin{smallmatrix}\pm\beta & \pm\beta\\ \beta & \beta\end{smallmatrix}\right](q) = (16q)^{\Delta}\,[z(1-z)]^{-\frac{1}{8}}\,\theta_3(q)^{-1}g^{\pm\pm}(q)\ .$$

Comparing this with (3.50) one gets

$$g^{\pm\pm}(q) = 1\,.$$

The residua of elliptic blocks $h^{1}_{mn}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)$ are given by the corresponding residua of superconformal blocks (3.46),(3.47). The final form of the elliptic recurrence in the Ramond sector thus reads:

$$\begin{aligned}\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q) = g^{1,\frac{1}{2}} &+ \sum_{\substack{m,n>0\\ m,n\in 2\mathbb{N}}} (16q)^{\frac{mn}{2}} \frac{A_{mn}(c)\overline{P^{mn}_{c}\left[\begin{smallmatrix}\pm\beta_3\\ \beta_4\end{smallmatrix}\right]}P^{mn}_{c}\left[\begin{smallmatrix}\pm\beta_2\\ \beta_1\end{smallmatrix}\right]}{\Delta-\Delta_{mn}}\,\mathcal{H}^{1,\frac{1}{2}}_{\Delta_{mn}+\frac{mn}{2}}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)\\ &+ \sum_{\substack{m,n>0\\ m,n\in 2\mathbb{N}+1}} (16q)^{\frac{mn}{2}} \frac{A_{mn}(c)\overline{P^{mn}_{c}\left[\begin{smallmatrix}\pm\beta_3\\ \beta_4\end{smallmatrix}\right]}P^{mn}_{c}\left[\begin{smallmatrix}\pm\beta_2\\ \beta_1\end{smallmatrix}\right]}{\Delta-\Delta_{mn}}\,\mathcal{H}^{\frac{1}{2},1}_{\Delta_{mn}+\frac{mn}{2}}\left[\begin{smallmatrix}\mp\beta_3 & \mp\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q),\end{aligned} \tag{3.51}$$

where $g^1 = 1, g^{\frac{1}{2}} = 0$.

3.4 Remarks concerning other types of 4-point blocks

In the current chapter the matrix elements of Ramond primary fields between arbitrary Ramond and NS states have been considered. We have expressed the 3-point functions by 3-point blocks $\rho^{(\pm)}_{\rm NRe,o}(S_{-I}\nu, w_2^+, S_J w_1^+)$ (3.30) and structure constants $C^{(\pm)}$ (3.28),(3.29). This allowed to write Ramond primary fields in terms of normalized chiral vertex operators (3.32). The matrix elements of the vertex operators *i.e.* the 3-point blocks are basic objects needed for the definition of 4-point blocks. We have defined the 4-point blocks corresponding to correlation functions of four primary Ramond fields factorized on NS states. The coefficients of these blocks are given by 3-point blocks $\rho^{(\pm)}_{\rm NRe,o}(\nu_{KM}, w_2^+, w_1^+)$ and inverse NS Gram matrix. Due to properties of the inverse NS Gram matrix and factorization property of the 3-point blocks it was possible to derive the recursive relations for the 4-point blocks.

Let us note, that one can define three more types of 4-point blocks. They appear in the following types of correlation functions of two Ramond fields and two NS fields:

$$\langle \phi_4\phi_3 R_2 R_1\rangle\,, \qquad \langle R_4\phi_3\phi_2 R_1\rangle\,, \qquad \langle \phi_4 R_3\phi_2 R_1\rangle\,. \tag{3.52}$$

The first correlator factorized on NS states corresponds to a 4-point block which should be defined by NS 3-point blocks $\rho(\nu_4, {}_{-}\nu_3, \nu_{\Delta,KM})$ (2.32), Ramond 3-point blocks $\rho^{(\pm)}_{\rm NRe,o}(\nu_{KM}, w_2^+, w_1^+)$ and inverse NS Gram matrix. The properties of all these objects have been already discussed in this dissertation. The recurrence relations for the coefficients of 4-point blocks of this type can be derived by repeating the steps presented in subsection 3.3.

The other types of 4-point blocks correspond to the second and the third correlator in (3.52) factorized on Ramond states. These blocks can be defined in terms of Ramond equivalent of inverse Gram matrix, 3-point blocks $\rho^{(\pm)}_{\rm NRe,o}(\nu_4, w_3^+, w_{LM}^+)$ and one more type of

3-point blocks corresponding to matrix elements of NS field ϕ_{RR} between Ramond states. The latter type of 3-point block can be constructed in a similar way as the two discussed already types of 3-point blocks. The starting point would be the following definition of 3-form:

$$\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(\ldots;z) \;:\; \mathcal{W}_{\Delta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{W}_{\Delta_1} \;\mapsto\; \mathbb{C}$$

satisfying the relations

$$\begin{aligned}
\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(S_{-n}\eta_3,\xi_2,\eta_1;z) &= (-1)^{|\eta_1|+|\eta_3|+1}\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(\eta_3,\xi_2,S_n\eta_1;z)\\
&+ \sum_{k=-\frac{1}{2}}^{\infty}\binom{n+\frac{1}{2}}{k+\frac{1}{2}}z^{n-k}\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(\eta_3,S_k\xi_2,\eta_1;z),
\end{aligned}$$

$$\begin{aligned}
\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\;\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(\eta_3,S_{p-k}\xi_2,\eta_1;z) &=\\
&\sum_{p=0}^{\infty}\binom{\frac{1}{2}-k}{p}(-z)^p\;\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(S_{p+k-\frac{1}{2}}\eta_3,\xi_2,\eta_1;z)\\
&-(-1)^{|\eta_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}-k}{p}(-z)^{\frac{1}{2}-k-p}\varrho_{RR}^{\Delta_3\Delta_2\Delta_1}(\eta_3,\xi_2,S_p\eta_1;z).
\end{aligned}$$

In the case of one excited external state (if S_{-K} does not contain operator S_0) the 3-form is determined up to one of 8 constants:

$$\varrho(S_{-K}w_3^{\pm},{}_{-}\nu_2,w_1^{\pm}|z) \;=\; \rho(S_{-K}w_3,{}_{-}\nu_2,w_1|z)\times\begin{cases}\varrho(w_3^{\pm},{}_{-}\nu_2,w_1^{\pm}|1)\\ \varrho(w_3^{\pm},\widetilde{{}_{-}\nu_2},w_1^{\pm}|1)\end{cases}$$

where $\widetilde{\nu}=*\nu,\widetilde{*\nu}=\nu$. The upper (lower) line corresponds to even (odd) number of operators S_{-k_i}. The 8 constants are related with each other in such a way that 4 of them are independent. The coefficient of proportionality $\rho(S_{-K}w_3,{}_{-}\nu_2,w_1|z)$ is the proper 3-point block. Using the relations defining 3-form one can derive the basic properties of the 3-point block, in particular the factorization property. One should also find the fusion polynomials corresponding to 3-point blocks with a null vector. Afterwards, the definition of 4-point blocks will be straightforward.

Let us note that the 4-point correlators factorize on Ramond states from the space $\mathcal{W}_{\Delta}\otimes\bar{\mathcal{W}}_{\bar{\Delta}}$. This should not cause a problem since it is possible to write primary fields ϕ_{RR}, similar as $R^{\pm}_{\rm NR}$, in terms of chiral vertex operators for which the "small representation" $\mathcal{W}_{\Delta,\bar{\Delta}}$ is an invariant subspace.

The elliptic recurrence representation of the 4-point blocks corresponding to the factorization on Ramond states can be derived by using the techniques presented in this thesis. The residua of the blocks' coefficients will be proportional to other blocks' coefficient, new fusion polynomials corresponding to a null Ramond vector ω_{rs} and the following factor

$$A^R_{rs}(c) \;=\; \lim_{\Delta\to\Delta_{rs}}\left(\frac{\langle\omega^{\Delta}_{rs}|\omega^{\Delta}_{rs}\rangle}{\Delta-\Delta_{rs}(c)}\right)^{-1},$$

where $\omega^{\Delta}_{rs} \in \mathcal{W}_{\Delta,c}$. The form of this factor has been already proposed on the basis of higher equations of motion in the Ramond sector of $N = 1$ supersymmetric Liouville theory by A. Belavin and Al. Zamolodchikov [42]. The terms regular in Δ can be investigated from the large Δ asymptotic of the 4-point blocks by analogous reasoning as the one presented in section (3.3).

Concluding, it seems that by applying the methods presented in the current chapter it should be possible to derive the recursive methods of determining all types of 4-point block appearing in $N = 1$ SCFT.

Chapter 4

Superconformal blocks in $c = \frac{3}{2}$ SCFT

In order to derive the elliptic recursive method of determining 4-point block we need an explicit formula for the block with an arbitrary intermediate weight and any, specific c and Δ_i. In [15], [17] Zamolodchikov and Apikyan worked out an exact formula for the 4-point conformal block corresponding to correlation function of fields with $\Delta_0 = \frac{1}{16}$, associated with the continuous limit of spin operators in the Ashkin-Teller model. They considered the $c = 1$ scalar theory extended by Ramond states of the free scalar current. In such a theory two types of field operators are present. The fields corresponding to NS states of free scalar current are given by the exponential operators $V_p(z, \bar{z}) = \mathrm{e}^{2ip\varphi(z) - 2i\bar{p}\bar{\varphi}(\bar{z})}$, where $\varphi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$ is the free scalar field. Because of the charge conservation condition, only one conformal family appears in the OPE of two exponential operators. Thus by considering the correlation functions of these fields it is impossible to derive a formula for the 4-point block with arbitrary intermediate weight.

The second type of field operators present in the theory are fields corresponding to Ramond states of the free scalar current. Any 4-point correlation function of these fields factorizes on NS current states with arbitrary conformal weight. Using additional to conformal Ward identities relations following from current algebra one can find a differential equation for the 4-point function of fields corresponding to so-called Ramond vacuum state. As a result an explicit analytic formula for the 4-point block can be derived.

In the first section we recall the original derivation presented in [15], [47] in detail. In the second section we generalize the model by adding free fermion current. We consider the correlation functions of the NS superconformal fields corresponding to Ramond-Ramond states of scalar and fermion currents respectively. The 4-point correlation function of these fields factorize on NS-NS states with arbitrary conformal weight. Applying similar reasoning as the one proposed by Zamolodchikovs it is possible to derive the explicit formulae for each type of NS 4-point superconformal blocks in the $c = \frac{3}{2}$ model.

In order to calculate Ramond 4-point superconformal blocks one should consider correlation functions of fields corresponding to Ramond-NS states of scalar and fermion currents respectively. These correlators also factorize on NS-NS states with arbitrary conformal weight. Examples of exact analytic formulae of the Ramond superconformal blocks are presented in the third section.

4.1 The 4-point conformal block in $c = 1$ CFT

4.1.1 NS and Ramond states of scalar current

The chiral scalar (bosonic) current $j(z)$ is defined by the following OPE:

$$j(z)j(z') \sim \frac{1}{2(z-z')^2} \tag{4.1}$$

and is related with free scalar field $j(z) = i\partial_z\varphi(z,\bar{z})$. We shall consider two types of states: the Neveu-Schwarz (NS) states for which modes of the bosonic current obey the following algebra:

$$j(z)\,|\xi\rangle_{NS} = \sum_{n\in\mathbb{N}} z^{-n-1} j_n\,|\xi\rangle_{NS}\,, \qquad [j_n, j_m] = \frac{m}{2}\delta_{n+m}, \tag{4.2}$$

and the Ramond (R) states defined by:

$$j(z)\,|\xi\rangle_R = \sum_{k\in\mathbb{N}+\frac{1}{2}} z^{-k-1} j_k\,|\xi\rangle_R\,, \qquad [j_k, j_l] = \frac{k}{2}\delta_{k+l}. \tag{4.3}$$

The highest weight representation of the algebra (4.2) with the highest weight state

$$j_0\,|p\rangle_{NS} = p\,|p\rangle_{NS}\,, \qquad j_n\,|p\rangle_{NS} = 0, \qquad n \in \mathbb{N},$$

constitute the NS current module $\mathcal{B}_p^{NS}$. The highest weights representation of the algebra (4.3) with the highest weight state

$$j_k\,|\sigma_0\rangle = 0,\ \ k \in \mathbb{N} - \frac{1}{2}$$

form the R current module $\mathcal{B}^R$. The space of states is a direct sum of the current modules:

$$\mathcal{B} = \left(\bigoplus_p \mathcal{B}_p^{NS}\right) \oplus \mathcal{B}^R,$$

where p runs through the spectrum of NS highest weight states.

The holomorphic component of energy-momentum tensor of the theory is constructed from the bosonic current in accordance with the Sugawara-Sommerfeld formula ([47] and included references):

$$T(z) =: j(z)j(z):$$

One can check that $T(z)$ satisfies the local Ward identities (1.5) with central charge $c = 1$. The OPE of $T(z)$ and scalar current implies that $j(z)$ has conformal weight $\Delta = 1$.

The Virasoro generators L_n (1.7) are related to the current modes. In NS sector:

$$L_0 = j_0^2 + 2\sum_{n\in\mathbb{N}} j_{-n}j_n, \qquad L_m = \sum_{n\in\mathbb{Z}} j_{m-n}j_n, \qquad m \neq 0 \tag{4.4}$$

and in R sector:

$$L_0 = \frac{1}{16} + 2\sum_{k\in\mathbb{N}-\frac{1}{2}} j_{-k}j_k, \qquad L_m = \sum_{k\in\mathbb{Z}+\frac{1}{2}} j_{m-k}j_k, \qquad m \neq 0. \tag{4.5}$$

It follows from (4.4) that the NS current module $\mathcal{B}_p^{NS}$ is a Verma module $\mathcal{V}_{\Delta_p}$ with conformal weight $\Delta_p = p^2$. The conformal vacuum is the highest weight state from NS current module with $p = 0$.

Let us consider now the R current sector. The highest weight state, the so called Ramond vacuum state $|\sigma_0\rangle$, is also the highest weight state with respect to Virasoro algebra. Its conformal weight is equal $\Delta_0 = \frac{1}{16}$ (4.5). The current module builded on this state is a direct sum of irreducible Verma modules with conformal weights $\Delta_n = \frac{(2n+1)^2}{16}$. On each level $n(n+1)/4$ in R current module the new Virasoro highest weight state with weight Δ_n appears. In particular, for $n = 1$ one has:

$$|\sigma_1\rangle = 2\, j_{-\frac{1}{2}}\, |\sigma_0\rangle\,.$$

Using the states-operators correspondence in CFT one can define the primary fields associated with the Virasoro highest weight states:

$$\lim_{z,\bar z\to 0}\sigma_0(z,\bar z)\,|0\rangle = |\sigma_0\otimes\bar\sigma_0\rangle\,, \qquad \lim_{z,\bar z\to 0}\sigma_1(z,\bar z)\,|0\rangle = |\sigma_1\otimes\bar\sigma_0\rangle\,, \qquad \lim_{z,\bar z\to 0}\phi_p(z,\bar z)\,|0\rangle = |p\otimes 0\rangle\,.$$

Since we are interested in z dependence of fields and correlation functions, we will always choose the corresponding states in the right sector as the Ramond vacuum or the NS vacuum. For the sake of brevity we will ignore the $\bar z$ dependence and write simply $\sigma_0(z), \sigma_1(z)$ and $\phi_p(z)$.

The OPEs of current $j(z)$ with the first two primary fields can be read off from the Ramond algebra (4.3):

$$\begin{aligned} j(\xi)\sigma_0(z) &\sim \frac{1}{2}(\xi-z)^{-\frac{1}{2}}\sigma_1(z) \\ j(\xi)\sigma_1(z) &\sim \frac{1}{2}(\xi-z)^{-\frac{3}{2}}\sigma_0(z) + 2(\xi-z)^{-\frac{1}{2}}\partial_z\sigma_0(z), \end{aligned} \tag{4.6}$$

while the OPE with NS field is follows from (4.2):

$$j(\xi)\phi_p(z) \sim \frac{p}{(\xi-z)}\,\phi_p(z). \tag{4.7}$$

In the second relation in (4.6) the following Knizhnik-Zamolodchikov (KZ) equation implied by (4.5) was used:

$$L_{-1}\sigma_0(z) - j_{-\frac{1}{2}}^2\sigma_0(z) = 0, \qquad L_{-1}\sigma_0(z) = \partial_z\sigma_0(z)$$

4.1.2 Relations for the correlation functions

With the help of the OPEs of bosonic current with primary fields (4.6), (4.7) it is possible to derive the set of relations for 4-point correlation functions of $\sigma_0(z), \sigma_1(z)$.

First, let us consider the following correlation function containing scalar current:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\sigma_0(z_2)\sigma_0(z_1)\rangle.$$

Because of the square root-like singularities in OPE of $j(\xi)$ with Ramond fields, the correlator multiplied by the coefficient:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\sigma_0(z_2)\sigma_0(z_1)\rangle \sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)}$$

is a single valued, holomorphic function of ξ. Since any correlator of $j(\xi)$ without operator insertions at infinity falls like ξ^{-2} for large ξ, the function above has to be a constant. Thus the correlator is given by:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\sigma_0(z_2)\sigma_0(z_1)\rangle \quad = \quad \frac{A(z_4, z_3, z_2, z_1)}{\sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)}}.$$

I order to determine the ξ-independent function one can expand the r.h.s of this relation around $\xi = z_2$. Comparing the result with the OPE $\langle j(\xi)\sigma_0(z_2)$ (4.6) one gets:

$$A(z_4, z_3, z_2, z_1) = \frac{1}{2}\sqrt{z_{21}z_{23}z_{24}}\,\langle\sigma_0(z_4)\sigma_0(z_3)\sigma_1(z_2)\sigma_0(z_1)\rangle.$$

Consider now the correlators projected on NS states:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle = \frac{1}{2}\sqrt{z_{21}z_{23}z_{24}}\frac{\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_1(z_2)\sigma_0(z_1)\rangle}{\sqrt{(\xi - z_4)(\xi - z_3)(\xi - z_2)(\xi - z_1)}}.$$

Integrating the l.h.s of this equation along contour $\mathcal{C}_{[z_2,z_1]}$ enclosing points z_1 and z_2 one gets:

$$\begin{aligned}\oint\limits_{\mathcal{C}_{[z_2,z_1]}}\frac{d\xi}{2\pi i}\,\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle \quad &= \quad \langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,j_0\,\sigma_0(z_2)\sigma_0(z_1)\rangle \\ &= \quad p\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle\end{aligned}$$

The left hand site is given by:

$$\mathcal{K}(z_i) \quad = \quad \oint\limits_{\mathcal{C}_{[z_2,z_1]}}\frac{d\xi}{2\pi i}\,\frac{1}{\sqrt{(\xi - z_1)(\xi - z_2)(\xi - z_3)(\xi - z_4)}} \quad = \quad \frac{2K(z)}{\pi\sqrt{z_{31}z_{42}}}\,, \tag{4.8}$$

where $K(z)$ is the complete elliptic integral of the first kind:

$$K(z) \quad = \quad \int\limits_0^1\frac{dt}{\sqrt{(1 - t^2)\,(1 - t^2 z)}}, \qquad z \quad = \quad \frac{z_{21}z_{43}}{z_{31}z_{42}}.$$

Thus the first relation for the 4-point correlation functions reads:

$$p\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle = \frac{1}{2}\,\mathcal{K}(z_i)\,\sqrt{z_{21}z_{23}z_{24}}\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_1(z_2)\sigma_0(z_1)\rangle \tag{4.9}$$

It also follows from the OPEs (4.6) that the correlator multiplied by the square root factor:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\sigma_1(z_2)\sigma_0(z_1)\rangle\sqrt{(\xi-z_4)(\xi-z_3)(\xi-z_2)(\xi-z_1)},$$

as a function of ξ, is holomorphic on $\mathbb{C}\setminus\{z_2\}$ and has a simple pole at $\xi = z_2$. The correlator thus reads:

$$\langle j(\xi)\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_1(z_2)\sigma_0(z_1)\rangle = \frac{1}{\sqrt{(\xi-z_4)(\xi-z_3)(\xi-z_2)(\xi-z_1)}}\left(\frac{B(z_i)}{\xi-z_2}+C(z_i)\right) \tag{4.10}$$

Comparing the expansion of the r.h.s of (4.10) around $\xi = z_2$ with the OPE (4.6) one can determine the coefficients:

$$\begin{aligned} B(z_i) &= \frac{1}{2}\sqrt{z_{21}z_{23}z_{24}}\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle, \\ C(z_i) &= 2\sqrt{z_{21}z_{23}z_{24}}\left[\partial_{z_2}+\frac{1}{8}\left(\frac{1}{z_{21}}+\frac{1}{z_{23}}+\frac{1}{z_{24}}\right)\right]\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle. \end{aligned}$$

Inserting these functions into (4.10) and integrating along contour $\mathcal{C}_{[z_2,z_1]}$ the second equation for correlation functions is obtained:

$$\begin{aligned} p\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_1(z_2)\sigma_0(z_1)\rangle &= 2B(z_i)\,\partial_{z_2}\mathcal{K}(z_i)+C(z_i)\,\mathcal{K}(z_i) \\ &= 2\Big(z_{21}z_{32}z_{42}\Big)^{\frac{3}{8}}\mathcal{K}^{\frac{1}{2}}(z_i)\;\partial_{z_2}\left[\Big(z_{21}z_{32}z_{42}\Big)^{\frac{1}{8}}\mathcal{K}^{\frac{1}{2}}(z_i)\,\langle\sigma_0(z_4)\sigma_0(z_3)\,|_p\,\sigma_0(z_2)\sigma_0(z_1)\rangle\right] \end{aligned} \tag{4.11}$$

For the 4-point functions of fields in standard locations the equations (4.9) and (4.11) read:

$$\begin{aligned} \langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_0(0)\rangle &= \frac{1}{2}\left(\frac{2K(z)}{\pi}\right)^{\frac{3}{2}}\sqrt{z(1-z)}\;\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1(z)\sigma_0(0)\rangle, \\ \langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1(z)\sigma_0(0)\rangle &= \frac{2}{\Delta_p}\left(\frac{2K(z)}{\pi}\right)^{\frac{1}{2}}[z(1-z)]^{\frac{7}{8}} \\ &\quad\times\;\partial_z\left[\left(\frac{2K(z)}{\pi}\right)^{\frac{1}{2}}[z(1-z)]^{\frac{1}{8}}\;\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_0(0)\rangle\right]. \end{aligned}$$

In order to find an equation for the 4-point block one should write the 4-point correlation function in terms of conformal blocks:

$$\begin{aligned} \langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_0(0)\rangle &= \sum_{|N|=|M|=n}\rho(\sigma_0,\sigma_0,L_{-M}p|1)\left[B^n_{c,\Delta_p}\right]^{MN}\rho(L_{-N}p,\sigma_0,\sigma_0|z)\times C_p \\ &= C_p\,\mathcal{F}_{c,\Delta_p}\!\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z), \end{aligned} \tag{4.12}$$

where C_p is a z-independent constant. Then, the set of equations for correlators is equivalent to the following relation:

$$\Delta_p G_p(z) = \left(\frac{2K(z)}{\pi}\right)^2[z(1-z)]\,\partial_z G_p(z),$$

where

$$G_p(z) = \left(\frac{2K(z)}{\pi}\right)^{\frac{1}{2}} [z(1-z)]^{\frac{1}{8}} \mathcal{F}_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$$

Using the relations for elliptic theta function:

$$\theta_3^2(q) = \left(\frac{2K(z)}{\pi}\right), \qquad \theta_3^4(q)\,[z(1-z)]\,\partial_z = q\partial_q$$

one finds: $G_p(z) = D\,q^{\Delta_p}$. The q-independent constant D can be fixed using normalization condition for the 4-point block (1.36). Finally, one gets the following expression for the 4-point block:

$$\mathcal{F}_{c,\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = (16q)^{\Delta_p}\,[z(1-z)]^{-\frac{1}{8}}\,\theta_3^{-1}(q).$$

4.2 NS superconformal blocks related to the R-R states in the $c = \frac{3}{2}$ SCFT

4.2.1 Holomorphic currents

We shall consider the supersymmetric generalization of the model presented in the previous section. In addition to the free bosonic current we will introduce the free fermionic current.

First, let us rewrite the definition of the bosonic current $j(z)$ (with conformal weights $\Delta = 1$, $\bar{\Delta} = 0$):

$$j(z)j(z') \sim \frac{1}{(z-z')^2}.$$

In comparison to definition (4.1) the normalization of the current has changed. This implies also a different normalization of modes of the currents:

$$j(z)\,|\xi\rangle_{NS} = \sum_{n\in\mathbb{N}} z^{-n-1} j_n\,|\xi\rangle_{NS}\,, \qquad [j_n, j_m] = m\delta_{n+m}, \tag{4.13}$$

$$j(z)\,|\xi\rangle_{R} = \sum_{k\in\mathbb{N}+\frac{1}{2}} z^{-k-1} j_k\,|\xi\rangle_{R}\,, \qquad [j_k, j_l] = k\delta_{k+l}. \tag{4.14}$$

As before, the space of states $\mathcal{B}$ is a direct sum

$$\mathcal{B} = \left(\bigoplus_p \mathcal{B}_p^{NS}\right) \oplus \mathcal{B}^R$$

where $\mathcal{B}_p^{NS}$ are the NS current modules defined as a highest weight representations of the algebra (4.13) with the highest weight state

$$j_0\,|p\rangle_{NS} = p\,|p\rangle_{NS}\,, \qquad j_n\,|p\rangle_{NS} = 0, \qquad n\in\mathbb{N}, \tag{4.15}$$

and $\mathcal{B}^R$ is the R current module defined as a highest weight representation of the algebra (4.14) with the highest weight state

$$j_k\,|0\rangle_R = 0, \qquad k\in\mathbb{N}-\tfrac{1}{2}. \tag{4.16}$$

The fermionic current (with conformal weights $\Delta = \frac{1}{2}, \bar{\Delta} = 0$) is defined by the OPE

$$\psi(z)\psi(z') \sim \frac{1}{z - z'} \,. \tag{4.17}$$

We will consider the fermionic NS states and fermionic Ramond states:

$$\psi(z)\,|\zeta\rangle_{NS} = \sum_{k\in\mathbb{N}+\frac{1}{2}} z^{-k-\frac{1}{2}}\psi_k\,|\zeta\rangle_{NS}\,, \qquad \{\psi_k, \psi_l\} = \delta_{k+l}, \tag{4.18}$$

$$\psi(z)\,|\zeta\rangle_{R} = \sum_{n\in\mathbb{N}} z^{-n-\frac{1}{2}}\psi_n\,|\zeta\rangle_{R}\,, \qquad \{\psi_n, \psi_m\} = \delta_{n+m}. \tag{4.19}$$

The highest weight representation of the algebra (4.18) with the highest weight state

$$\psi_k\,|0\rangle_{NS} = 0, \qquad k \in \mathbb{N} - \frac{1}{2}$$

constitute the fermionic NS current module $\mathcal{F}^{NS}$. The fermionic R current module $\mathcal{F}^{R}$ is built on the highest weight state $|+\rangle$ with respect to algebra (4.19) defined by the relations:

$$\psi_0\,|+\rangle_R = \tfrac{1}{\sqrt{2}}\,|-\rangle_R\,, \qquad \psi_n\,|+\rangle_R = 0, \qquad n \in \mathbb{N}.$$

The space of states $\mathcal{F}$ is a direct sum of the fermionic NS current module $\mathcal{F}^{NS}$ and the fermionic R current module $\mathcal{F}^{R}$. The tensor product $\mathcal{B}\otimes\mathcal{F}$ decomposes into the direct sum

$$\mathcal{B}\otimes\mathcal{F} = \left[\left(\bigoplus_p \mathcal{B}_p^{NS}\otimes\mathcal{F}^{NS}\right)\oplus\mathcal{B}^{R}\otimes\mathcal{F}^{R}\right]\oplus\left[\left(\bigoplus_p \mathcal{B}_p^{NS}\otimes\mathcal{F}^{R}\right)\oplus\mathcal{B}^{R}\otimes\mathcal{F}^{NS}\right] \tag{4.20}$$

of highest weight supercurrent modules. The Sugawara construction

$$\begin{aligned} T(z) &= \frac{1}{2}:j(z)j(z): - \frac{1}{2}:\psi(z)\partial\psi(z):\,, \\ S(z) &= j(z)\psi(z), \end{aligned}$$

defines on the first summand a free field representation of the NS superconformal algebra with the central charge $c = \frac{3}{2}$. In this sector

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad S(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^{-k-\frac{3}{2}}S_k,$$

where

$$\begin{aligned} L_0 &= \frac{1}{2}j_0^2 + \sum_{n\in\mathbb{N}} j_{-n}j_n + \sum_{k\in\mathbb{N}-\frac{1}{2}} (k+\tfrac{1}{2})\psi_{-k}\psi_k, \\ L_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} :j_{m-n}j_n: + \frac{1}{4}\sum_{k\in\mathbb{Z}+\frac{1}{2}} (2k-m)\,:\psi_{m-k}\psi_k:\,, \\ S_k &= \sum_{n\in\mathbb{Z}} j_n\psi_{k-n}, \end{aligned}$$

on the subspace $\bigoplus_p \mathcal{B}_p^{NS} \otimes \mathcal{F}^{NS}$ and

$$\begin{aligned} L_0 &= \sum_{k \in \mathbb{N}-\frac{1}{2}} j_{-k} j_k + \sum_{n \in \mathbb{N}} (n + \frac{1}{2}) \psi_{-n} \psi_n + \frac{1}{8}, \\ L_m &= \frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} : j_{m-k} j_k : + \frac{1}{4} \sum_{n \in \mathbb{Z}} (2n - m) : \psi_{m-n} \psi_n : , \\ S_k &= \sum_{n \in \mathbb{Z}} \psi_n j_{k-n} \end{aligned} \tag{4.21}$$

on $\mathcal{B}^R \otimes \mathcal{F}^R$.

Let us note, that the states from the second term in the direct sum (4.20) correspond to superconformal Ramond fields. In the next section we shall analyze correlation functions of fields associated with the states from $\mathcal{B}^R \otimes \mathcal{F}^{NS}$.

The NS-NS supercurrent module $\mathcal{B}_p^{NS} \otimes \mathcal{F}^{NS}$ is a superconformal NS module $\mathcal{V}_{\Delta_p}$ with the conformal weight $\Delta_p = \frac{p^2}{2}$. The highest weight state with respect to NS superconformal algebra is given by:

$$\nu_p \equiv |p\rangle_{NS} \otimes |0\rangle_{NS} .$$

By the one to one state-operator correspondence in SCFT the state ν_p is associated with the superprimary field $\phi_p(z)$:

$$\lim_{z \to 0} \phi_p(z) \; |0\rangle = |p\rangle_{NS} \otimes |0\rangle_{NS} ,$$

where $|0\rangle = |0\rangle_{NS} \otimes |0\rangle_{NS} \in \mathcal{B}_0^{NS} \otimes \mathcal{F}^{NS}$ is the "true" vacuum invariant with respect to superconformal transformations. As in the first section, for the sake of brevity, we will ignore the $\bar{z}$ dependence of the fields keeping in mind that as the corresponding state in right sector one can choose the vacuum.

In the R-R supercurrent module $\mathcal{B}^R \otimes \mathcal{F}^R$ there exists the highest weight state which corresponds to superprimary field with conformal weight $\Delta_0 = \frac{1}{8}$:

$$\chi_0^+(0) \; |0\rangle = |0\rangle_R \otimes |+\rangle_R \equiv \chi_0^+ .$$

There is one more superprimary field with the same conformal weight $\Delta_0 = \frac{1}{8}$ but with opposite parity. It corresponds to the state:

$$\chi_0^-(0) \; |0\rangle = \sqrt{2} \psi_0 \chi_0^+(0) \; |0\rangle = |0\rangle_R \otimes |-\rangle_R \equiv \chi_0^- .$$

In supercurrent modules the parity of the fields is defined by the number of fermionic excitations. One can check that on each $\frac{n(n+1)}{2}$ level, among descendants of χ_0^+, there are two states corresponding to superprimary fields with equal weights and opposite parity. Since all $c = \frac{3}{2}$, $\Delta_n = \frac{1}{2}\left(n + \frac{1}{2}\right)^2$ superconformal NS modules are not degenerate the superconformal content of the R-R module can be read off from the ratio

$$\frac{\chi_{RR}(t)}{\chi_c(t)} \;=\; 2 \sum_{n=0}^{\infty} t^{\frac{n(n+1)}{2}}$$

where $\chi_{RR}(t)$ is the character of $\mathcal{B}^R \otimes \mathcal{F}^R$

$$\chi_{RR}(t) = 2t^{\frac{1}{8}} \prod_{k=1}^{\infty} \frac{1+t^k}{1-t^{\frac{2k-1}{2}}}$$

and

$$\chi_c(t) = t^{\frac{1}{8}} \prod_{k=1}^{\infty} \frac{1+t^{\frac{2k-1}{2}}}{1-t^k}$$

is the character of the superconformal NS module.

The R-R module is hence a direct sum of irreducible NS superconformal Verma modules with conformal weights

$$\Delta_n = \frac{1}{2}\left(n+\frac{1}{2}\right)^2, \qquad n = 0, 1, \ldots,$$

each weight appearing twice in the sum. We shall denote the corresponding superprimary fields by $\chi_n^{\pm}(z)$, in particular $\chi_0^{\pm}(z)$ and

$$\chi_1^{\pm}(z) \;=\; \frac{1}{2}\left(j_{-\frac{1}{2}}^2 - \psi_{-1}\psi_0\right)\chi_0^{\pm}(z). \tag{4.22}$$

We define these fields in such a way that in each case $|0\rangle_R \otimes |+\rangle_R$ is the corresponding state in the right sector.

4.2.2 Relations for the correlation functions

Using the technique presented in the first section (4.1.2) of this chapter one can derive the set of equations for the 4-point correlation functions of R-R fields.

First, consider the correlator with arbitrary pattern of upper signs ensuring positive total parity of the function:

$$\left\langle j(\xi)\chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle.$$

The OPEs of the bosonic current and the superprimary field $\chi_0^{\pm}$ can be derived from the algebra (4.14) and (4.16):

$$j(\xi)\chi_0^{\pm}(z) \;\sim\; \frac{1}{\sqrt{\xi - z}}\, j_{-\frac{1}{2}}\chi_0^{\pm}(z). \tag{4.23}$$

Repeating the steps leading to the equation (4.9) one gets:

$$\left\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\, j_0\, \chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle \;=\; \sqrt{z_{21}z_{23}z_{24}}\,\mathcal{K}(z_i)\left\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3) j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle, \tag{4.24}$$

where $\mathcal{K}(z_i)$ is related with the elliptic integral of the first kind (4.8).

Next, in order to derive a relation for the correlator containing $j_{-\frac{1}{2}}\chi_0^{\pm}(z)$, the OPE of bosonic current and the field $j_{-\frac{1}{2}}\chi_0^{\pm}(z)$ is necessary. It follows from the algebra of modes j_k and ψ_m and relations (4.21), (4.22) that the OPE reads:

$$j(\xi)\, j_{-\frac{1}{2}}\chi_0^{\pm}(z) \;\sim\; \frac{1}{2(\xi - z)^{\frac{3}{2}}}\,\chi_0^{\pm}(z) + \frac{1}{\sqrt{\xi - z}}\, L_{-1}\chi_0^{\pm}(z) + \frac{1}{\sqrt{\xi - z}}\,\chi_1^{\pm}(z), \tag{4.25}$$

where the KZ equation was used:

$$L_{-1}\chi_0^{\pm}(z) \;=\; \frac{1}{2}\left(j^2_{-\frac{1}{2}} + \psi_{-1}\psi_0\right)\chi_0^{\pm}(z). \tag{4.26}$$

With the help of analogous reasoning as the one leading to (4.11), using the OPE above instead of (4.6), one can derive the second relation for the 4-point functions:

$$\begin{aligned}
&\left\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\, j_0\, j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle \\
&= \left(z_{21}z_{32}z_{42}\right)^{\frac{1}{4}}\frac{\partial}{\partial z_2}\left[\left(z_{21}z_{32}z_{42}\right)^{\frac{1}{4}}\mathcal{K}(z_i)\left\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle\right] \\
&+ \left(z_{21}z_{32}z_{42}\right)^{\frac{1}{2}}\mathcal{K}(z_i)\left\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_1^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle.
\end{aligned} \tag{4.27}$$

Since the relations (4.24) and (4.27) do not form a closed set of equations we need to find some formulae relating the correlator containing one field $\chi_1^{\pm}$ with the correlation function of four $\chi_0^{\pm}$. This can be done with the help of the OPEs of the fermionic current with superprimary fields:

$$\begin{aligned}
\sqrt{2}\psi(\xi)\,\chi_0^{\pm}(z) &\sim \frac{1}{\sqrt{\xi - z}}\,\chi_0^{\mp}(z), \\
\sqrt{2}\psi(\xi)\,\chi_1^{\pm}(z) &\sim -\frac{1}{2(\xi - z)^{\frac{3}{2}}}\,\chi_0^{\mp}(z) + \frac{1}{\sqrt{\xi - z}}\,L_{-1}\chi_0^{\mp}(z).
\end{aligned} \tag{4.28}$$

These OPEs can be derived from the algebra of modes ψ_m together with the relations (4.21) and (4.22). It follows from (4.28) that

$$\frac{\left\langle \sqrt{2}\psi(\xi)\,\chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_m^{\pm}(z_2)\chi_0^{\mp}(z_1)\right\rangle}{\sqrt{(\xi - z_1)(\xi - z_2)(\xi - z_3)(\xi - z_4)}}$$

is an analytic function of ξ. It has poles at the locations z_i, but at infinity it vanish faster than ξ^{-1}. Thus the sum of its residues must vanish. This and (4.28) imply:

$$\begin{aligned}
0 \;=\; &\frac{\left\langle \chi_0^{-}(z_4)\chi_0^{+}(z_3)\chi_0^{+}(z_2)\chi_0^{-}(z_1)\right\rangle}{\sqrt{z_{41}z_{42}z_{43}}} + \frac{\left\langle \chi_0^{+}(z_4)\chi_0^{-}(z_3)\chi_0^{+}(z_2)\chi_0^{-}(z_1)\right\rangle}{\sqrt{z_{31}z_{32}z_{34}}} \\
+ \;&\frac{\left\langle \chi_0^{+}(z_4)\chi_0^{+}(z_3)\chi_0^{-}(z_2)\chi_0^{-}(z_1)\right\rangle}{\sqrt{z_{21}z_{23}z_{24}}} + \frac{\left\langle \chi_0^{+}(z_4)\chi_0^{+}(z_3)\chi_0^{+}(z_2)\chi_0^{+}(z_1)\right\rangle}{\sqrt{z_{12}z_{13}z_{14}}}
\end{aligned} \tag{4.29}$$

for $m = 0$ and

$$\begin{aligned}
&-\left(z_{21}z_{23}z_{24}\right)^{-\frac{3}{4}}\frac{\partial}{\partial z_2}\left[\left(z_{21}z_{23}z_{24}\right)^{\frac{1}{4}}\left\langle \chi_0^{+}(z_4)\chi_0^{+}(z_3)\chi_0^{-}(z_2)\chi_0^{-}(z_1)\right\rangle\right] = \\
&= \left(z_{41}z_{42}z_{43}\right)^{-\frac{1}{2}}\left\langle \chi_0^{-}(z_4)\chi_0^{+}(z_3)\chi_1^{+}(z_2)\chi_0^{-}(z_1)\right\rangle \\
&+ \left(z_{31}z_{32}z_{34}\right)^{-\frac{1}{2}}\left\langle \chi_0^{+}(z_4)\chi_0^{-}(z_3)\chi_1^{+}(z_2)\chi_0^{-}(z_1)\right\rangle \\
&+ \left(z_{12}z_{13}z_{14}\right)^{-\frac{1}{2}}\left\langle \chi_0^{+}(z_4)\chi_0^{+}(z_3)\chi_1^{+}(z_2)\chi_0^{+}(z_1)\right\rangle
\end{aligned} \tag{4.30}$$

for $m = 1$.

One more set of relations for the correlator with two $j_{-\frac{1}{2}}\chi_0^{\pm}$ fields will be needed. It can be derived using OPEs (4.23), (4.25):

$$\begin{aligned}
&\Big\langle \chi_0^{\pm}(z_4)\, j_{-\frac{1}{2}}\chi_0^{\pm}(z_3)\, j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \qquad (4.31)\\
&\quad = \oint\limits_{z_3} \frac{d\xi}{2\pi i}\, \frac{1}{\sqrt{\xi - z_3}} \Big\langle j(\xi)\, \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\, j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \\
&\quad = \sqrt{\frac{z_{21}z_{23}z_{24}}{z_{31}z_{32}z_{34}}} \left[\frac{\partial}{\partial z_2} \big\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\big\rangle + \big\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_1^{\pm}(z_2)\chi_0^{\pm}(z_1)\big\rangle \right.\\
&\quad + \left. \frac{1}{4}\left(\frac{1}{z_{21}} + \frac{1}{z_{32}} + \frac{1}{z_{24}} \right) \big\langle \chi_0^{\pm}(z_4)\chi_0^{\pm}(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\big\rangle \right]
\end{aligned}$$

The relations (4.24), (4.27), (4.29) and (4.30) form the closed set of equations for the three types of 4-point correlation functions. In the next step one should write the correlators in terms of 4-point blocks and structure constants so that a set of equations for the blocks could be obtained. Before that, however, we shall derive relations for the 3-point functions which allow to express the structure constants through one independent constant $\big\langle \phi_p(\infty)\chi_0^{+}(1)\chi_0^{+}(0)\big\rangle$.

4.2.3 3-point blocks

The 3-point correlation functions of fields corresponding to states from supercurrent modules can be written in terms of a 3-form:

$$\langle\, \phi_3(\xi_3,\bar{\xi}_3|\infty,\infty)\phi_2(\xi_2,\bar{\xi}_2|z,\bar{z})\phi_1(\xi_1,\bar{\xi}_1|0,0)\,\rangle \;= \eta(\xi_3,\xi_2,\xi_1|z)\,\eta(\bar{\xi}_3,\bar{\xi}_2,\bar{\xi}_1|\bar{z}). \qquad (4.32)$$

The 3-form, defined on the one NS-NS and two R-R supercurrent modules:

$$\eta(\xi,\zeta,\zeta'|z) \qquad \xi \in \mathcal{B}_p^{NS}\otimes\mathcal{F}^{NS}, \qquad \zeta,\zeta' \in \mathcal{B}^{R}\otimes\mathcal{F}^{R},$$

is a nontrivial extension of the NS superconformal 3-form (2.22). It is determined by Ward identities for currents $j(z), \psi(z)$ up to one constant

$$\eta(\nu_p,\chi_0^{+},\chi_0^{+}) \equiv \eta(\nu_p,\chi_0^{+},\chi_0^{+}|1).$$

Since in the free superscalar theory the left and the right fermionic parities are independently preserved, the 3-form identically vanishes if total parity of all arguments is odd.

If the states ζ,ζ' belong to definite superconformal Verma modules the form η satisfies Ward identities for the non-normalized superconformal 3-form (2.23)- (2.29). For instance, for even vectors

$$\nu_{p,KM} \;=\; S_{-K}L_{-M}\nu_p \;\equiv\; S_{-k_i}\ldots S_{-k_1}L_{-m_j}\ldots L_{-m_1}\nu_p\,, \quad |K| \in \mathbb{N}\cup\{0\},$$

one has:

$$
\begin{aligned}
\eta(\nu_{p,KM}, \chi_m^{\pm}, \chi_n^{\pm}|z) &= \rho(\nu_{p,KM}, \chi_m, \chi_n|z)\, \eta(\nu_p, \chi_m^{\pm}, \chi_n^{\pm}), \\
\eta(\nu_{p,KM}, S_{-\frac{1}{2}}\chi_m^{\mp}, \chi_n^{\pm}|z) &= \rho(\nu_{p,KM}, *\chi_m, \chi_n|z)\, \eta(\nu_p, S_{-\frac{1}{2}}\chi_m^{\mp}, \chi_n^{\pm}),
\end{aligned}
\tag{4.33}
$$

and for odd ones ($|K| \in \mathbb{N} - \frac{1}{2}$):

$$
\begin{aligned}
\eta(\nu_{p,KM}, \chi_m^{\mp}, \chi_n^{\pm}|z) &= \rho(\nu_{p,KM}, \chi_m, \chi_n|z)\, \eta(\nu_p, S_{-\frac{1}{2}}\chi_m^{\mp}, \chi_n^{\pm}), \\
\eta(\nu_{p,KM}, S_{-\frac{1}{2}}\chi_m^{\pm}, \chi_n^{\pm}|z) &= \rho(\nu_{p,KM}, *\chi_m, \chi_n|z)\, \eta(\nu_p, \chi_m^{\pm}, \chi_n^{\pm}).
\end{aligned}
\tag{4.34}
$$

The form ρ in the formulae above is the normalized 3-point superconformal block (1.27) and χ_m stands for the highest weight state in the superconformal Verma module with the central charge $c = \frac{3}{2}$ and the conformal weight $\Delta_m = \frac{1}{2}\left(m + \frac{1}{2}\right)^2$.

In what follows we shall derive the formulae expressing 3-forms

$$
\eta(\nu_p, \chi_m^{\pm}, \chi_0^{\pm}), \quad \eta(\nu_p, S_{-\frac{1}{2}}\chi_m^{\mp}, \chi_0^{\pm}), \qquad m = 0, 1
$$

by the one independent constant $\eta(\nu_p, \chi_0^{+}, \chi_0^{+})$. We will consider the correlation functions of the corresponding fields and the bosonic or fermionic current.

First, take the function

$$
f(\xi) = \frac{1}{\sqrt{(\xi - z_2)(\xi - z_1)}} \Big\langle \psi(\xi)\phi_p(z_3)\chi_0^{-}(z_2)\chi_0^{+}(z_1) \Big\rangle.
$$

It follows from the OPE-s:

$$
\begin{aligned}
\psi(\xi)\chi_0^{\pm}(z) &\sim \frac{1}{\sqrt{\xi - z}}\psi_0\chi_0^{\pm}(z) = \frac{1}{\sqrt{2(\xi - z)}}\chi_0^{\mp}(z), \\
\psi(\xi)\phi_p(z) &\sim 1,
\end{aligned}
$$

that $f(\xi)$ is analytic in the complex ξ plane, have simple poles at $\xi = z_2,\ \xi = z_1$, and falls off at infinity faster than ξ^{-1}. We thus have

$$
\begin{aligned}
0 &= \oint_{z_3} \frac{d\xi}{2\pi i}\, f(\xi) = -\oint_{z_2} \frac{d\xi}{2\pi i}\, f(\xi) + \oint_{z_1} \frac{d\xi}{2\pi i}\, f(\xi) \\
&= -\frac{1}{\sqrt{2z_{21}}}\Big\langle \phi_p(z_3)\chi_0^{+}(z_2)\chi_0^{+}(z_1) \Big\rangle + \frac{1}{\sqrt{2z_{12}}}\Big\langle \phi_p(z_3)\chi_0^{-}(z_2)\chi_0^{-}(z_1) \Big\rangle,
\end{aligned}
$$

so that

$$
\Big\langle \phi_p(z_3)\chi_0^{-}(z_2)\chi_0^{-}(z_1) \Big\rangle = \frac{\sqrt{z_{12}}}{\sqrt{z_{21}}}\Big\langle \phi_p(z_3)\chi_0^{+}(z_2)\chi_0^{+}(z_1) \Big\rangle = i\Big\langle \phi_p(z_3)\chi_0^{+}(z_2)\chi_0^{+}(z_1) \Big\rangle.
$$

Since the antiholomorphic 3-form present in the decomposition of the 3-point functions

$$
\Big\langle \phi_p(\infty)\chi_m^{\pm}(1)\chi_n^{\pm}(0) \Big\rangle = \eta(\nu_p, \chi_m^{\pm}, \chi_n^{\pm})\bar{\eta}(0, \bar{\chi}_0^{+}, \bar{\chi}_0^{+})
$$

is always the same, equations for the 3-point functions imply relations for the 3-forms. In particular,

$$\eta(\nu_p, \chi_0^-, \chi_0^-) \;=\; i\,\eta(\nu_p, \chi_0^+, \chi_0^+). \tag{4.35}$$

Here and below we adopt the convention that for $j < l$:

$$z_{jl} \;=\; \mathrm{e}^{i\pi} z_{lj}.$$

Next, integrating around $\xi = z_3$ the identity

$$\Big\langle j(\xi)\phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \;=\; \frac{z_{23}\sqrt{z_{21}}}{(\xi - z_3)\sqrt{(\xi - z_2)(\xi - z_1)}}\Big\langle \phi_p(z_3) j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle$$

we get

$$p\Big\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \;=\; \sqrt{\frac{z_{21}z_{32}}{z_{31}}}\,\Big\langle \phi_p(z_3) j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle,$$

what gives

$$\eta(\nu_p, j_{-\frac{1}{2}}\chi_0^{\pm}, \chi_0^{\pm}) \;=\; p\,\eta(\nu_p, \chi_0^{\pm}, \chi_0^{\pm}). \tag{4.36}$$

Analogous computation gives

$$\eta(\chi_0^{\pm}, j_{-\frac{1}{2}}\chi_0^{\pm}, \nu_p) \;=\; -ip\,\eta(\chi_0^{\pm}\,\chi_0^{\pm}, \nu_p).$$

In the case of correlation function containing the field χ_1 or $S_{-\frac{1}{2}}\chi_1$ the calculations are similar but involve more steps. Using the OPE (4.25) in the form

$$j(\xi) j_{-\frac{1}{2}}\chi_0^{\pm}(z) \;\sim\; \frac{1}{2(\xi - z)^{\frac{3}{2}}}\,\chi_0^{\pm}(z) + \frac{1}{\sqrt{\xi - z}}\, j_{-\frac{1}{2}}^2\chi_0^{\pm}(z)$$

we get

$$\Big\langle j(\xi)\phi_p(z_3) j_{-\frac{1}{2}}\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \;=\; \left(\frac{a}{\xi - z_2} + b\right)\frac{1}{(\xi - z_3)\sqrt{(\xi - z_1)(\xi - z_2)}}, \tag{4.37}$$

with

$$a \;=\; \frac{1}{2} z_{23}\sqrt{z_{21}}\Big\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle,$$

$$b \;=\; z_{23}\sqrt{z_{21}}\left[\Big\langle \phi_p(z_3) j_{-\frac{1}{2}}^2\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle + \frac{1}{4}\left(\frac{1}{z_{21}} + \frac{2}{z_{23}}\right)\Big\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle\right].$$

Integrating (4.37) around $\xi = z_3$ we derive a relation

$$\Big\langle \phi_p(z_3) j_{-\frac{1}{2}}^2\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \;=\; \left(\frac{2\Delta_p - \frac{1}{4}}{z_{21}} + \frac{2\Delta_p}{z_{32}}\right)\Big\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \tag{4.38}$$

From the definition of $\chi_1(z)$ (4.22) and the KZ equation (4.26) we get

$$\Big\langle \phi_p(z_3)\chi_1^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle \;=\; \Big\langle \phi_p(z_3) j_{-\frac{1}{2}}^2\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle - \frac{1}{2}\partial_{z_2}\Big\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\Big\rangle$$

The z-dependence of the 3-point function is given by (1.18):

$$\left\langle \phi_p(z_3)\chi_0^{\pm}(z_2)\chi_0^{\pm}(z_1)\right\rangle = z_{32}^{-\Delta_p} z_{31}^{-\Delta_p} z_{21}^{\Delta_p-\frac{1}{4}} \left\langle \phi_p(\infty)\chi_0^{\pm}(1)\chi_0^{\pm}(0)\right\rangle,$$

thus

$$\eta(\nu_p, \chi_1^{\pm}, \chi_0^{\pm}) = \Delta_p\, \eta(\nu_p, \chi_0^{\pm}, \chi_0^{\pm}). \tag{4.39}$$

Finally, similar calculation with the help of the relation

$$S_{-\frac{1}{2}}\chi_1^{\pm} = \left(3j^3_{-\frac{1}{2}}\psi_0 + 2S_{-\frac{3}{2}} - 5L_{-1}S_{-\frac{1}{2}}\right)\chi_0^{\pm},$$

gives

$$\eta(\nu_p, S_{-\frac{1}{2}}\chi_1^{+}, \chi_0^{-}) = \frac{ip}{\sqrt{2}}\left(\Delta_p - \frac{1}{2}\right)\eta(\nu_p, \chi_0^{+}, \chi_0^{+}). \tag{4.40}$$

4.2.4 4-point NS superconformal blocks

We shall express the 4-point correlation functions in terms of 4-point superconformal blocks and one independent structure constants. Then, on the basis of relations (4.24)-(4.30), one can derive a set of equations for the blocks. Any 4-point function of R-R fields factorizes on NS-NS states, for example:

$$\begin{aligned}&\left\langle \chi_0^{+}(\infty)\chi_0^{+}(1)\, j_0\, \chi_0^{+}(z)\chi_0^{+}(0)\right\rangle \\ &= \sum_p \sum_{K,M,L,N} \eta(\chi_0^{+}, \chi_0^{+}, \nu_{p,KM}) B^{KM,LN} \eta(j_0\nu_{p,LN}, \chi_0^{+}, \chi_0^{+}|z)\, \bar{D}\end{aligned}$$

where due to the left parity conservation the sum runs over even states $|K|, |L| \in \mathbb{N}\cup\{0\}$. $\bar{D}$ denotes some $\bar{z}$-dependent function which is the same in the case of each discussed 4-point correlator. Taking into account properties of the 3-form (4.33) and definitions of the superconformal blocks (2.48) one gets

$$\begin{aligned}&\left\langle \chi_0^{+}(\infty)\chi_0^{+}(1)\, j_0\, \chi_0^{+}(z)\chi_0^{+}(0)\right\rangle \qquad (4.41)\\ &= \sum_p \sum_{K,M,L,N} p\,\eta(\chi_0^{+}, \chi_0^{+}, \nu_p)\, \eta(\nu_p, \chi_0^{+}, \chi_0^{+})\ \rho(\chi_0, \chi_0, \nu_{p,KM})\, B^{KM,LN}\ \rho(j_0\nu_{p,LN}, \chi_0, \chi_0|z)\, \bar{D} \\ &= \sum_p p\, C_p F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)\end{aligned}$$

where $C_p \equiv \eta(\chi_0^{+}, \chi_0^{+}, \nu_p)\eta(\nu_p, \chi_0^{+}, \chi_0^{+})\,\bar{D}$. The correlation function of fields with different parities $\left\langle \chi_0^{+}(\infty)\chi_0^{-}(1)\, j_0\, \chi_0^{+}(z)\chi_0^{-}(0)\right\rangle$ factorizes on odd states. Using formulae (4.34), the following relation

$$S_{-\frac{1}{2}}\chi_0^{\mp} = j_{-\frac{1}{2}}\chi_0^{\pm}\sqrt{2} \tag{4.42}$$

and (4.35), (4.36) one has in this case

$$\begin{aligned}\left\langle \chi_0^+(\infty)\chi_0^-(1)\, j_0\, \chi_0^+(z)\chi_0^-(0)\right\rangle &= \sum_p p\, \eta(\chi_0^+, S_{-\frac{1}{2}}\chi_0^-, \nu_p)\eta(\nu_p, S_{-\frac{1}{2}}\chi_0^+, \chi_0^-) \\ \times & \sum_{K,M,L,N} \rho(\chi_0, \chi_0, \nu_{p,KM}) B^{KM,LN} \rho(j_0\nu_{p,LN}, \chi_0, \chi_0|z)\, \bar{D} \\ &= \sum_p p\, \Delta_p\, C_p\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)\end{aligned}$$

It follows from the relation (4.42) that the correlators with $j_{-\frac{1}{2}}\chi_0^{\pm}$ decompose onto superconformal blocks with one star:

$$\left\langle \chi_0^+(\infty)\chi_0^+(1)\, j_{-\frac{1}{2}}\, \chi_0^+(z)\chi_0^+(0)\right\rangle = \sum_p p\, C_p\, F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) \tag{4.43}$$

and

$$\left\langle \chi_0^+(\infty)\chi_0^-(1)\, j_{-\frac{1}{2}}\, \chi_0^+(z)\chi_0^-(0)\right\rangle = \sum_p p\, C_p\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$$

Analogously, using properties of the 3-form η (4.33), (4.34) and relations (4.35) - (4.40) one can write all the 4-point functions discussed in section (4.2.2) in terms of the 4-point superconformal blocks and one constant C_p.

Inserting the relations expressing correlators in terms of 4-point blocks (4.41), (4.43) into equation (4.24) one gets the following equation for superconformal blocks:

$$F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = \frac{2K(z)}{\pi}\sqrt{z(1-z)}\, F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z). \tag{4.44}$$

For the odd blocks corresponding to the correlators of fields with different parities the equation (4.24) implies:

$$\Delta_p\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = \frac{2K(z)}{\pi}\sqrt{z(1-z)}\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z). \tag{4.45}$$

Similarly, the relation (4.27) leads to the equations:

$$\begin{aligned}2\Delta_p\, F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) &= [z(1-z)]^{\frac{1}{4}}\frac{\partial}{\partial z}\left[\frac{2K(z)}{\pi}[z(1-z)]^{\frac{1}{4}} F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)\right] \\ &+ \Delta_p\frac{2K(z)}{\pi}[z(1-z)]^{\frac{1}{2}} F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z),\end{aligned} \tag{4.46}$$

$$\begin{aligned}2\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & {*\Delta_0} \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) &= [z(1-z)]^{\frac{1}{4}}\frac{\partial}{\partial z}\left[\frac{2K(z)}{\pi}[z(1-z)]^{\frac{1}{4}} F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)\right] \\ &+ \left(\Delta_p - \tfrac{1}{2}\right)\frac{2K(z)}{\pi}[z(1-z)]^{\frac{1}{2}} F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z).\end{aligned} \tag{4.47}$$

The next two equations can be obtained from (4.29) and (4.30), respectively:

$$\Delta_p\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = (1-\sqrt{1-z})\, z^{-\frac{1}{2}} F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) \tag{4.48}$$

$$\begin{aligned}&[z(1-z)]^{-\frac{1}{4}}\frac{\partial}{\partial z}\left[[z(1-z)]^{\frac{1}{4}} F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)\right] = \\ &= \Delta_p\left(\Delta_p - \tfrac{1}{2}\right)\sqrt{z}\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) + \Delta_p\frac{2K(z)}{\pi}\sqrt{1-z}\, F^{1}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1 \\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z).\end{aligned} \tag{4.49}$$

Formulae (4.44) – (4.48) allow to express the functions $F^f_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$, $F^f_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$ and $F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$ in terms of $F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$. Using (4.49) we then arrive at the equation

$$\frac{dG_p(z)}{dz} = \left[\frac{\Delta_p}{z(1-z)}\left(\frac{2K(z)}{\pi}\right)^{-2} - \frac{1-\sqrt{1-z}}{4z\sqrt{1-z}}\right] G_p(z), \tag{4.50}$$

where

$$G_p(z) = [z(1-z)]^{\frac{1}{4}} \left(\frac{2K(z)}{\pi}\right)^{\frac{1}{2}} F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z).$$

Integrating (4.50) we get

$$F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = (16q)^{\Delta_p} \left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}} [z(1-z)]^{-\frac{1}{4}} \left(\frac{2K(z)}{\pi}\right)^{-\frac{1}{2}}.$$

Using relations:

$$\frac{2K(z)}{\pi} = \theta_3^2(q), \qquad \left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}} \theta_3(q) = \theta_3(q^2), \qquad \left(\frac{1-\sqrt{1-z}}{2}\right)^{\frac{1}{2}} \theta_3(q) = \theta_2(q^2)$$

where theta functions are defined in the standard way:

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}$$

one finally obtains:

$$F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{1}{4}} (16q)^{\Delta_p} \theta_3^{-2}(q)\, \theta_3(q^2), \tag{4.51}$$

$$F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{1}{4}} \frac{(16q)^{\Delta_p}}{\Delta_p} \theta_3^{-2}(q)\, \theta_2(q^2), \tag{4.52}$$

$$F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \theta_3^{-4}(q)\, \theta_3(q^2), \tag{4.53}$$

$$F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{3}{4}} (16q)^{\Delta_p} \theta_3^{-4}(q)\, \theta_2(q^2), \tag{4.54}$$

$$F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{5}{4}} (16q)^{\Delta_p} \theta_3^{-6}(q) \left(\theta_3(q^2) - \frac{q}{\Delta_p}\frac{\partial}{\partial q}\theta_3(q^2)\right), \tag{4.55}$$

$$F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = [z(1-z)]^{-\frac{5}{4}} \frac{(16q)^{\Delta_p}}{\Delta_p - \frac{1}{2}} \theta_3^{-6}(q) \left(\theta_2(q^2) - \frac{q}{\Delta_p}\frac{\partial}{\partial q}\theta_2(q^2)\right). \tag{4.56}$$

Equations for functions $F^f_{\Delta_p}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z)$ can be obtained from (4.31) using the relations (4.36), (4.39), (4.40):

$$2\Delta_p F^1_{\Delta_p}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = \sqrt{z}\left(\frac{\partial}{\partial z} + \frac{1}{4z(1-z)}\right) F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) + \sqrt{z}\,\Delta_p F^1_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z),$$

$$\frac{2}{\Delta_p} F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = \sqrt{z}\left(\frac{\partial}{\partial z} + \frac{1}{4z(1-z)}\right) F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) + \sqrt{z}\,(\Delta_p - \frac{1}{2})\, F^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\Delta_0 & \Delta_1\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z).$$

From results (4.51), (4.55) and (4.52), (4.56) one gets, respectively:

$$
\begin{aligned}
F^{1}_{\Delta_p}\left[{}^{*\Delta_0\,*\Delta_0}_{\;\Delta_0\;\;\Delta_0}\right](z) &= z^{-\frac{3}{4}}(1-z)^{-\frac{5}{4}}\,(16q)^{\Delta_p}\,\frac{\theta_3(q^2)}{\theta_3^6(q)}\left(1-\frac{q}{\Delta_p}\,\theta_3^{-1}(q)\,\frac{\partial\theta_3(q)}{\partial q}+\frac{\theta_2^4(q)}{4\Delta_p}\right),\\
F^{\frac{1}{2}}_{\Delta_p}\left[{}^{*\Delta_0\,*\Delta_0}_{\;\Delta_0\;\;\Delta_0}\right](z) &= -z^{-\frac{3}{4}}(1-z)^{-\frac{5}{4}}\,(16q)^{\Delta_p}\,\frac{\theta_2(q^2)}{\theta_3^6(q)}\,\Delta_p\left(1-\frac{q}{\Delta_p}\,\theta_3^{-1}(q)\,\frac{\partial\theta_3(q)}{\partial q}+\frac{\theta_2^4(q)}{4\Delta_p}\right).
\end{aligned}
\tag{4.57}
$$

Explicit expressions for the conformal blocks (4.51) – (4.57) were used in the derivation of the elliptic recurrence representation of the NS blocks (2.4.4).

Using the definition of NS elliptic 4-point blocks (2.75) one can read from the formulae (4.51) – (4.57) the form of $c = \frac{3}{2}$ elliptic blocks with weights $\Delta_0 = \frac{1}{8}$:

$$
\begin{aligned}
\mathcal{H}^{1}_{\Delta}\left[{}^{\Delta_0\,\Delta_0}_{\Delta_0\,\Delta_0}\right](q) &= \theta_3(q^2), & \mathcal{H}^{\frac{1}{2}}_{\Delta}\left[{}^{\Delta_0\,\Delta_0}_{\Delta_0\,\Delta_0}\right](q) &= \tfrac{1}{\Delta}\theta_2(q^2),\\
\mathcal{H}^{1}_{\Delta}\left[{}^{\Delta_0\,*\Delta_0}_{\Delta_0\;\;\Delta_0}\right](q) &= \theta_3(q^2), & \mathcal{H}^{\frac{1}{2}}_{\Delta}\left[{}^{\Delta_0\,*\Delta_0}_{\Delta_0\;\;\Delta_0}\right](q) &= \theta_2(q^2),
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
\mathcal{H}^{1}_{\Delta}\left[{}^{*\Delta_0\,*\Delta_0}_{\;\Delta_0\;\;\Delta_0}\right](q) &= \theta_3(q^2)\left(1-\tfrac{q}{\Delta}\theta_3^{-1}\tfrac{\partial}{\partial q}\theta_3(q)+\tfrac{\theta_2^4(q)}{4\Delta}\right),\\
\mathcal{H}^{\frac{1}{2}}_{\Delta}\left[{}^{*\Delta_0\,*\Delta_0}_{\;\Delta_0\;\;\Delta_0}\right](q) &= -\theta_2(q^2)\Delta\left(1-\tfrac{q}{\Delta}\theta_3^{-1}\tfrac{\partial}{\partial q}\theta_3(q)+\tfrac{\theta_2^4(q)}{4\Delta}\right).
\end{aligned}
\tag{4.59}
$$

4.3 Ramond superconformal blocks related to the R-NS states in the $c = \frac{3}{2}$ SCFT

We shall present the derivation of explicit formulae for several examples of Ramond elliptic blocks in $c = \frac{3}{2}$ SCFT model introduced in the previous section.

The generators of superconformal symmetry are defined in terms of scalar and fermion currents:

$$
\begin{aligned}
T(z) &= \frac{1}{2}:j(z)j(z): -\frac{1}{2}:\psi(z)\partial\psi(z):,\\
S(z) &= j(z)\psi(z).
\end{aligned}
$$

This relation defines on the subspace $\mathcal{B}^R \otimes \mathcal{F}^{NS}$ (4.20) a free field representation of the Ramond superconformal algebra. In this sector

$$
T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad S(z) = \sum_{m\in\mathbb{Z}} z^{-m-\frac{3}{2}}S_m,
$$

where

$$
\begin{aligned}
L_0 &= \sum_{k\in\mathbb{N}-\frac{1}{2}} j_{-k}j_k + \sum_{l\in\mathbb{N}-\frac{1}{2}} (l+\frac{1}{2})\psi_{-l}\psi_l + \frac{1}{16},\\
L_m &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} :j_{m-k}j_k: +\frac{1}{4}\sum_{l\in\mathbb{Z}+\frac{1}{2}} (2l-m):\psi_{m-l}\psi_l:,\\
S_n &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_k j_{n-k}.
\end{aligned}
\tag{4.60}
$$

We are interested in the R-NS states because the 4-point correlation functions of corresponding Ramond fields factorizes on NS-NS states with arbitrary conformal weight. This allows to calculate 4-point superconformal blocks as functions of intermediate weight.

The highest weight state with respect to supercurrent algebra in this sector is given by:

$$\sigma_0 \equiv |0\rangle_R \otimes |0\rangle_{NS} .$$

This is at the same time the highest weight state with respect to Ramond superconformal algebra. Since it is anihilated by the S_0 operator, σ_0 does not have the counterpart with negative parity. Its conformal weight is equal $\Delta_0 = \frac{1}{16}$, $\beta = 0$ and it is the so called superconformal Ramond vacuum. Its first descendants in the supercurrent module are also superconformal highest weight states:

$$\sigma_1^+ = j_{-\frac{1}{2}}\sigma_0 \qquad \sigma_1^- = \frac{1}{\sqrt{2}}\mathrm{e}^{i\frac{\pi}{4}}\psi_{-\frac{1}{2}}\sigma_0 \tag{4.61}$$

with $\Delta_1 = \frac{9}{16}$ and $\beta_1 = \frac{1}{\sqrt{2}}$. We will denote the corresponding Ramond primary fields by $\sigma_0(z), \sigma_1^{\pm}(z)$. We ignore the $\bar{z}$ dependence of these fields, in all cases choosing the $\bar{\sigma}_0$ as the corresponding state in the right sector.

From the current algebras (4.14), (4.18) and relation (4.60) it follows that the OPEs of $j(z)$ with fields σ_0, σ_1 have the same form as (4.6):

$$\begin{aligned} j(\xi)\sigma_0(z) &\sim \frac{1}{2}(\xi - z)^{-\frac{1}{2}}\sigma_1(z) \\ j(\xi)\sigma_1(z) &\sim \frac{1}{2}(\xi - z)^{-\frac{3}{2}}\sigma_0(z) + 2(\xi - z)^{-\frac{1}{2}}\partial_z\sigma_0(z), \end{aligned} \tag{4.62}$$

Thus the 4-point function of σ_0 superconformal fields is given by the same formula as the one derived by Zamolodchikovs:

$$\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_0(0)\rangle = (16)^{\Delta_p}\,[z(1-z)]^{-\frac{1}{8}}\,\theta_3^{-1}(q)\,C_p \equiv G_0(z), \tag{4.63}$$

where C_p is a z independent constant.

Using the technique presented in the previous sections one can derive the following relations between 4-point correlation functions:

$$\left\langle\sigma_1^+(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_1^+(0)\right\rangle = 2i\,z^{\frac{3}{8}}(1-z)^{\frac{9}{8}}\partial_z\left[\left(\frac{z}{1-z}\right)^{\frac{1}{8}}G_0(z)\right], \tag{4.64}$$

$$\left\langle\sigma_1^-(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_1^-(0)\right\rangle = \frac{i}{2}G_0(z), \tag{4.65}$$

$$\left\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1^+(z)\sigma_1^+(0)\right\rangle = -2iz^{\frac{1}{8}}(1-z)^{\frac{3}{8}}\partial_z\left[\left(\frac{1-z}{z}\right)^{\frac{1}{8}}G_0(z)\right], \tag{4.66}$$

$$\left\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1^-(z)\sigma_1^-(0)\right\rangle = \frac{i}{2z}G_0(z). \tag{4.67}$$

Notice that the formulae (4.65) and (4.67) follow directly from the definition of the fermionic current (4.17).

Let us note that the fields $\sigma_n^{\pm}(z)$ corresponds to the states $\sigma_n^{\pm} \otimes \bar{\sigma}_0$ from $\mathcal{W}_{\Delta_n} \otimes \bar{\mathcal{W}}_{\bar{\Delta}_0}$. They are not the same type of Ramond fields $R^{\pm}_{\rm NR}$ corresponding to the states from the "small representation" $\mathcal{W}_{\Delta,\bar{\Delta}}$ which were discussed in the third chapter. Thus the relations expressing $R^{\pm}_{\rm NR}$ in terms of normalized chiral vertex operators (3.32) do not apply to $\sigma_n^{\pm}(z)$ fields. In consequence one can not write straightforwardly the decomposition of the 4-point correlators in (4.63)-(4.67) onto the 4-point Ramond blocks defined by (3.42),(3.43).

In order to compute the 4-point Ramond blocks we have to find a way to express all the 4-point correlators in terms of chiral 3-point blocks (3.30). To this end we introduce an extension of superconformal 3-form (anti-linear in left argument and linear in the central and rights ones):

$$\eta(\xi, \varsigma, \varsigma'|z) \qquad \xi \in \mathcal{B}_p^{NS} \otimes \mathcal{F}^{NS}, \qquad \varsigma, \varsigma' \in \mathcal{B}^{R} \otimes \mathcal{F}^{NS},$$

It is determined by the Ward identities for currents $j(z), \psi(z)$ up to one constant

$$\eta(\nu_p, \sigma_0, \sigma_0|1) \equiv \eta(\nu_p, \sigma_0, \sigma_0).$$

In the case of states belonging to superconformal modules $\mathcal{V}_{\Delta_p}, \mathcal{W}_{\Delta_i}$, the 3-form satisfies the superconformal Ward identities (3.18). It can thus be written in terms of the forms $\rho^{ij}_{NR}; i, j = \pm$ (3.19). In particular:

$$\begin{aligned}
\eta(\nu_{p,KM}, \sigma_m^{\pm}, \sigma_n^{\pm}|z) &= \rho^{++}_{NR}(\nu_{p,KM}, \sigma_m^{\pm}, \sigma_n^{\pm}|z)\eta(\nu_p, \sigma_m^{+}, \sigma_n^{+}) \\
&+ \rho^{--}_{NR}(\nu_{p,KM}, \sigma_m^{\pm}, \sigma_n^{\pm}|z)\eta(\nu_p, \sigma_m^{-}, \sigma_n^{-}).
\end{aligned}$$

Notice, that forms $\eta(\nu_p, \sigma_m^{+}, \sigma_n^{-})$ and $\eta(\nu_p, \sigma_m^{-}, \sigma_n^{+})$ vanish because the fermionic total parity of all arguments has to be even.

Using the properties of ρ^{ij}_{NR} (3.23),(3.25), (3.27) and the definition of 3-point block (3.30) one can derive some relations for η:

$$\begin{aligned}
\eta(\nu_{p,KM}, \sigma_0, \sigma_0|z) &= \rho^{++}_{NR}(\nu_{p,KM}, \sigma_0, \sigma_0|z)\, \eta(\nu_p, \sigma_0, \sigma_0) \\
&= \rho^{(\pm)}_{NRe}(\nu_{p,KM}, \sigma_0, \sigma_0|z)\, \eta(\nu_p, \sigma_0, \sigma_0), \\
\eta(\nu_{p,KM}, \sigma_0, \sigma_1^{+}|z) &= \rho^{++}_{NR}(\nu_{p,KM}, \sigma_0, \sigma_1^{+}|z)\, \eta(\nu_p, \sigma_0, \sigma_1^{+}) \\
&= \rho^{(\pm)}_{NRe}(\nu_{p,KM}, \sigma_0, \sigma_1^{+}|z)\, \eta(\nu_p, \sigma_0, \sigma_1^{+}), \\
\eta(\nu_{p,KM}, \sigma_0, \sigma_1^{-}|z) &= -i\rho^{+-}_{NR}(\nu_{p,KM}, \sigma_0, \sigma_1^{+}|z)\, \eta(\nu_p, \sigma_0, \sigma_1^{+}) \\
&= -i\rho^{(\pm)}_{NRo}(\nu_{p,KM}, \sigma_0, \sigma_1^{+}|z)\, \eta(\nu_p, \sigma_0, \sigma_1^{+}).
\end{aligned} \tag{4.68}$$

With the help of these relations one can write the following correlators in terms of 4-point blocks:

$$\begin{aligned}
\langle \sigma_0(\infty)\sigma_0(1)\,|_p\, \sigma_0(z)\sigma_0(0)\rangle &= \sum_{K,M,L,N} \eta(\sigma_0, \sigma_0, \nu_{p,KM}) B^{KM,LN} \eta(\nu_{p,LN}, \sigma_0, \sigma_0|z)\, \bar{D}_p \\
&= \mathcal{F}^{1}_{\Delta_p}\begin{bmatrix}\beta_0\, \beta_0 \\ \beta_0\, \beta_0\end{bmatrix}(z)\, C_p, \\
\left\langle \sigma_1^{+}(\infty)\sigma_0(1)\,|_p\, \sigma_0(z)\sigma_1^{+}(0)\right\rangle &= \sum_{K,M,L,N} \eta(\sigma_1^{+}, \sigma_0, \nu_{p,KM}) B^{KM,LN} \eta(\nu_{p,LN}, \sigma_0, \sigma_1^{+}|z)\, \bar{D}_p
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](z)\,(-ip^2)\,C_p,\\
\left\langle\sigma_1^-(\infty)\sigma_0(1)\,|_p\,\sigma_0(z)\sigma_1^-(0)\right\rangle &= \sum_{K,M,L,N}\eta(\sigma_1^-,\sigma_0,\nu_{p,KM})B^{KM,LN}\eta(\nu_{p,LN},\sigma_0,\sigma_1^-|z)\,\bar{D}_p\\
&= \mathcal{F}^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](z)\,(-ip^2)\,C_p,
\end{aligned}
$$

where $C_p = \big(\eta(\sigma_0,\sigma_0,\nu_p)\eta(\nu_p,\sigma_0,\sigma_0)\bar{D}_p\big)$ is a z-independent constant. The relations

$$
\eta(\sigma_1^+,\sigma_0,\nu_p) = -p\,\eta(\sigma_0,\sigma_0,\nu_p), \qquad \eta(\nu_p,\sigma_0,\sigma_1^+) = ip\,\eta(\nu_p,\sigma_0,\sigma_0)
$$

follow from OPEs of bosonic current $j(z)$ with fields $\phi_p, \sigma_{0,1}$ and can be derived in the similar way as (4.36).

Inserting the correlators above into the equations (4.63)-(4.65) one can compute the 4-point blocks:

$$
\begin{aligned}
\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](z) &= (16)^{\Delta_p}\,[z(1-z)]^{-\frac{1}{8}}\,\theta_3^{-1}(q), \qquad (4.69)\\
\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](z) &= \frac{1}{\Delta_p}\,z^{\frac{3}{8}}(1-z)^{\frac{9}{8}}\partial_z\left[\left(\frac{z}{1-z}\right)^{\frac{1}{8}}\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](z)\right],\\
\mathcal{F}^{\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](z) &= \frac{1}{4\Delta_p}\,\mathcal{F}^{1}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](z).
\end{aligned}
$$

These are examples of the simplest 4-point Ramond blocks in $c = \frac{3}{2}$ model. The superconformal Ramond vacuum σ_0 does not have an odd counterpart ($\beta_0 = 0$), thus the two types of 3-point blocks are equal $\rho^{(+)}_{NRe}(\nu_{p,KM},\sigma_0,\sigma_n^{\pm}|z) = \rho^{(-)}_{NRe}(\nu_{p,KM},\sigma_0,\sigma_n^{\pm}|z)$ and there exist only one out of four types of even (or odd) 4-point block. In order to derive formulae for the blocks $\mathcal{F}^{1,\frac{1}{2}}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \pm\beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z)$ with different sign in front of non zero β_1 one can use equations (4.66), (4.67).

The correlation functions written in terms of the 3-form η read:

$$
\begin{aligned}
\left\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1^+(z)\sigma_1^+(0)\right\rangle &= \sum_{K,M,L,N}\rho^{++}(\sigma_0,\sigma_0,\nu_{p,KM})\,\eta(\sigma_0,\sigma_0,\nu_p)\,B^{KM,LN}\\
&\times\big(\rho^{++}(\nu_{p,KM},\sigma_1^+,\sigma_1^+|z)\eta(\nu_p,\sigma_1^+,\sigma_1^+) + \rho^{--}(\nu_{p,KM},\sigma_1^+,\sigma_1^+|z)\,\eta(\nu_p,\sigma_1^-,\sigma_1^-)\big)\\
&= -2i(\Delta-\frac{1}{4})\,C_p\,F^{++}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) + \frac{i}{2}\,C_p\,F^{+-}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) \equiv f_1\,C_p,\\
\left\langle\sigma_0(\infty)\sigma_0(1)\,|_p\,\sigma_1^-(z)\sigma_1^-(0)\right\rangle &= \sum_{K,M,L,N}\rho^{++}(\sigma_0,\sigma_0,\nu_{p,KM})\,\eta(\sigma_0,\sigma_0,\nu_p)\,B^{KM,LN} \qquad (4.70)\\
&\times\big(\rho^{--}(\nu_{p,KM},\sigma_1^+,\sigma_1^+|z)\eta(\nu_p,\sigma_1^+,\sigma_1^+) + \rho^{++}(\nu_{p,KM},\sigma_1^+,\sigma_1^+|z)\,\eta(\nu_p,\sigma_1^-,\sigma_1^-)\big)\\
&= -2i(\Delta-\frac{1}{4})\,C_p\,F^{+-}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) + \frac{i}{2}\,C_p\,F^{++}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) \equiv f_2\,C_p,
\end{aligned}
$$

where the following relations between constants in the supercurrent module were used:

$$
\eta(\sigma_1^+,\sigma_1^+,\nu_p) = -2i(\Delta-\frac{1}{4})\,\eta(\sigma_0,\sigma_0,\nu_p), \qquad \eta(\nu_p,\sigma_1^-,\sigma_1^-) = \frac{i}{2}\,\eta(\nu_p,\sigma_0,\sigma_0).
$$

The functions

$$F^{+\pm}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) = \sum_{K,M,L,N} \rho^{++}(\sigma_0,\sigma_0,\nu_{p,KM})\, B^{KM,LN} \rho^{\pm\pm}(\nu_{p,KM},\sigma_1^+,\sigma_1^+|z)$$

are related to the 4-point superconformal blocks (3.30), (3.41):

$$\begin{aligned}
\mathcal{F}^1_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) &= F^{++}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) + F^{+-}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z)\\
\mathcal{F}^1_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) &= F^{++}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) - F^{+-}_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z).
\end{aligned}$$

Solving the set of equations for the blocks (4.70) one obtains:

$$\mathcal{F}^1_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) = \frac{1}{\Delta-\frac{1}{2}}\frac{i}{2}(f_1+f_2), \qquad \mathcal{F}^1_{\Delta_p}\left[\begin{smallmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](z) = \frac{1}{\Delta}\frac{i}{2}(f_1-f_2), \tag{4.71}$$

where the functions f_1, f_2 are given by the relations (4.66), (4.67).

Applying the definition of elliptic blocks in Ramond sector of $N = 1$ SCFT (3.49):

$$\begin{aligned}
\mathcal{F}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](z) &= (16q)^{\Delta-\frac{c-3/2}{24}}\, z^{\frac{c-3/2}{24}-\Delta_1-\Delta_2}\,(1-z)^{\frac{c-3/2}{24}-\Delta_2-\Delta_3}\\
&\times\ \theta_3^{\frac{c-3/2}{2}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\,\mathcal{H}^{1,\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{smallmatrix}\right](q)
\end{aligned}$$

to the blocks (4.69), (4.71) one can calculate the following formulae for the elliptic blocks:

$$\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](q) = 1, \qquad \mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](q) = 0, \tag{4.72}$$

$$\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q) = 1 + \frac{1}{\Delta}\left(\frac{\theta_2^4(q)}{4} - \theta_3^{-1}(q)\, q\frac{\partial}{\partial q}\theta_3(q)\right), \tag{4.73}$$

$$\mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q) = \frac{1}{4\Delta}\,\theta_2^2(q)\,\theta_3^2(q), \tag{4.74}$$

$$\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) = 1 + \frac{1}{\Delta-\frac{1}{2}}\left\{\frac{1}{2} + \frac{\theta_2^4(q)}{4} - \frac{1}{2}\theta_3^2(q^2)\theta_3^2(q) - \theta_3^{-1}(q)\, q\frac{\partial}{\partial q}\theta_3(q)\right\}, \tag{4.75}$$

$$\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) = 1 + \frac{1}{\Delta}\left\{\frac{\theta_2^4(q)}{4} - \frac{1}{2}\theta_2^2(q^2)\theta_3^2(q) - \theta_3^{-1}(q)\, q\frac{\partial}{\partial q}\theta_3(q)\right\}, \tag{4.76}$$

$$\mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) = \mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) = 0. \tag{4.77}$$

These formulae can be seen as a consistency check of the construction of Ramond 4-point blocks presented in the third chapter. As it was suggested by the path-integral arguments (3.48), all the odd blocks do not have the regular in Δ term. The block $\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_0 & \beta_0\end{smallmatrix}\right](q)$ was used to fix the regular in Δ term in the even blocks. One can notice that all the even blocks indeed have the regular terms equal to 1.

4.4 Elliptic blocks in $c = \frac{3}{2}$ model vs. elliptic recursive relations

One can ask if it is possible to make a consistency check of the elliptic recursion relations using the elliptic blocks calculated in $c = \frac{3}{2}$ model. In principle, inserting some explicit analytical formulae for elliptic blocks into the recursion relations one should be able to check if the residua in (2.77), (3.51) are correct *i.e.* do not lead to contradictions. However, in the case of the 4-point blocks in $c = \frac{3}{2}$ model, the derived recursion relations can be incorrect. They are valid for the theories with c such that NS supermodules $\mathcal{V}_{\Delta_{rs}(c)+\frac{rs}{2}}$ are not reducible (cf. the third property of Gram matrix (1.1.4)).

In the $c = \frac{3}{2}$ model the NS supermodules $\mathcal{V}_{\Delta_{rs}+\frac{rs}{2}}$ are reducible. It follows from Kac theorem (2.14) that weights $\Delta_{rs} + \frac{rs}{2}$ are degenerate:

$$\Delta_{r,s}(c = 3/2) = \frac{(r-s)^2}{8}, \qquad \Delta_{r,s} + \frac{rs}{2} = \Delta_{r+2s,s}.$$

One can notice that all calculated $c = \frac{3}{2}$ NS and Ramond blocks, as any other blocks for generic c, have simple poles in Δ, but the residua can be given by different formulae than the ones derived in the previous chapters: (2.54), (2.55), (3.46), (3.47). Let us consider the following coefficients in the residues:

$$\begin{aligned} R^{r,s}_{NS}\left[{}^{\text{-}\Delta_3\,\text{-}\Delta_2}_{\;\Delta_4\;\;\Delta_1}\right] &= A_{rs}(c)\, P^{rs}_c\left[{}^{\text{-}\Delta_3}_{\;\Delta_4}\right] P^{rs}_c\left[{}^{\text{-}\Delta_2}_{\;\Delta_1}\right] \\ R^{r,r}_{R}\left[{}^{\pm\beta_3\,\pm\beta_2}_{\;\;\beta_4\;\;\;\beta_1}\right] &= A_{rs}(c)\overline{P^{rs}_c\left[{}^{\pm\beta_3}_{\;\beta_4}\right]}P^{rs}_c\left[{}^{\pm\beta_2}_{\;\beta_1}\right] \end{aligned}$$

where $A_{rs}(c)$ is given by (2.53):

$$A_{rs}(c) \;=\; \frac{1}{2}\prod_{m=1-r}^{r}\prod_{n=1-s}^{s}\left(\frac{1}{\sqrt{2}}\left(pb - \frac{q}{b}\right)\right)^{-1}, \qquad m+n \in 2\mathbb{Z},\ (m,n) \neq (0,0),(r,s)$$

and the fusion polynomials (2.45),(2.46) (with $\Delta_a = \frac{a(Q-a)}{2}$) or (3.39):

$$\begin{aligned} P^{rs}_c\left[{}^{\Delta_2}_{\Delta_1}\right] &= \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\frac{2a_1 - 2a_2 - pb - qb^{-1}}{2\sqrt{2}}\right)\left(\frac{2a_1 + 2a_2 + pb + qb^{-1}}{2\sqrt{2}}\right), \\ P^{rs}_c\left[{}^{*\Delta_2}_{\;\Delta_1}\right] &= \prod_{p'=1-r}^{r-1}\prod_{q'=1-s}^{s-1}\left(\frac{2a_1 - 2a_2 - p'b - q'b^{-1}}{2\sqrt{2}}\right)\left(\frac{2a_1 + 2a_2 + p'b + q'b^{-1}}{2\sqrt{2}}\right), \\ P^{rs}_c\left[{}^{\pm\beta_2}_{\;\beta_1}\right] &= \prod_{p=1-r}^{r-1}\prod_{q=1-s}^{s-1}\left(\beta_1 \mp \beta_2 + \frac{pb+qb^{-1}}{2\sqrt{2}}\right)\prod_{p'=1-r}^{r-1}\prod_{q'=1-s}^{s-1}\left(\beta_1 \pm \beta_2 + \frac{p'b+q'b^{-1}}{2\sqrt{2}}\right), \end{aligned}$$

where p, q, p', q' run with the step 2 and satisfy the conditions: $p + q - (r + s) \in 4\mathbb{Z} + 2$ and $p' + q' - (r + s) \in 4\mathbb{Z}$. Analyzing the $c \to 1, (b \to i)$ limit of these coefficients for $a_i = \frac{i}{2}$,$\beta_0 = 0, \beta_1 = \frac{1}{\sqrt{2}}$, one can calculate that almost all the residues vanish. In the case of the limit of NS residues with $\Delta_0 = \frac{1}{8}$, $(a = \frac{i}{2})$, the non zero terms occur for $r = s$ and they read:

$$(16)^{\frac{r^2}{2}}\, R^{r,r}_{NS}\left[{}^{*\Delta_0\,*\Delta_0}_{\;\Delta_0\;\;\Delta_0}\right] = \begin{cases} -r^2\left[\frac{(r-3)!!}{(r-2)!!}\right]^2 & \text{if} \quad r \in 2\mathbb{N}, \\ 2\left[\frac{(r-1)!!}{(r-2)!!}\right]^2 & \text{if} \quad r \in 2\mathbb{N}+1. \end{cases} \tag{4.78}$$

In the Ramond case the non vanishing limits of residues corresponding to calculated blocks with $\beta_0 = 0, \beta_1 = \frac{1}{\sqrt{2}}$ are given by:

$$\begin{aligned}
(16)^{\frac{r^2}{2}} \; R_R^{r,r}\begin{bmatrix}\beta_0 & \beta_0\\ \beta_1 & \beta_1\end{bmatrix} &= (-1)^{r+1}\, r^2 \left[\frac{(r-3)!!}{(r-2)!!}\right]^2 \\
(16)^{\frac{r^2}{2}} \; R_R^{r,r}\begin{bmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{bmatrix} &= -r^2 \left[\frac{(r-3)!!}{(r-2)!!}\right]^2 \qquad \text{if} \qquad r \in 2\mathbb{N}, \\
(16)^{\frac{r(r+2)}{2}} \; R_R^{r,r+2}\begin{bmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{bmatrix} &= -\frac{r(r+2)}{2} \left[\frac{(r-3)!!(r-1)!!}{(r-2)!!r!!}\right] \qquad \text{if} \qquad r \in 2\mathbb{N}+1.
\end{aligned} \tag{4.79}$$

The non vanishing residues correspond to poles in degenerate weights:

$$\Delta_{r,r} = 0, \qquad \Delta_{r,r+2} = \frac{1}{2}.$$

It is clear that the structure of poles suggested by the limits of residues is in complete agrement with the form of calculated elliptic NS and Ramond blocks:

- both even and odd NS blocks with two stars have non zero terms proportional to $\frac{1}{\Delta}$,
- Ramond blocks $\mathcal{H}_\Delta^{1,\frac{1}{2}}\begin{bmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{bmatrix}(q)$ and $\mathcal{H}_\Delta^{1}\begin{bmatrix}\beta_0 & -\beta_1\\ \beta_0 & \beta_1\end{bmatrix}(q)$ have poles in $\Delta_{r,r} = 0$,
- the block $\mathcal{H}_\Delta^{1}\begin{bmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{bmatrix}(q)$ has pole in $\Delta_{r,r+2} = \frac{1}{2}$.

Moreover, we have checked that the all calculated elliptic blocks do satisfy the recursion relations (2.77), (3.51) , but with the residues (4.78),(4.79) modified in such a way that all the square brackets [...] are equal 1 (see Appendix B). Since the unwanted coefficients [...] are similar in both NS and Ramond cases, one can guess that the residues in the case of $c = \frac{3}{2}$ blocks should be given by some modification of the coefficient

$$A_{rs}(c) \;=\; \lim_{\Delta\to\Delta_{rs}} \left(\frac{\langle\chi_{rs}^{\Delta}|\chi_{rs}^{\Delta}\rangle}{\Delta - \Delta_{rs}(c)}\right)^{-1}.$$

It is still open question how to derive the exact formulae for an arbitrary residue of NS or Ramond 4-point blocks in the special case $c = \frac{3}{2}$.

Conclusion

The present thesis is aimed as a comprehensive view of the problem of definition and determination of the 4-point superconformal blocks in $N = 1$ superconformal field theories. We have presented a detailed discussion of the superconformal blocks corresponding to 4-point correlation functions factorized on NS fields in both Neveu-Schwarz and Ramond sectors of $N = 1$ SCFT.

The superconformal blocks are nontrivial generalizations of the conformal block. The problem of construction and determination of the superconformal blocks is very interesting since its solution needs new ideas and suitable extensions of the methods applicable to ordinary CFT.

The most efficient way of introducing the 4-point superconformal blocks is to define them in terms of 3-point superconformal blocks. Properties of the 4-point blocks, then, can be analyzed through the symmetry constraints imposed on the 3-point blocks. The problem of determination of 3-point blocks, however, easier than in the case of 4-point blocks, is not straightforward either. We have acted in the following scheme. The first step is to derive the Ward identities for 3-point correlation functions of arbitrary fields. On the basis of these identities one can postulate a set of relations defining chiral 3-form. Next, one should find a relation between structure constants and the chiral 3-forms. This allows for a definition of 3-point superconformal blocks as suitably normalized chiral 3-forms.

In the NS sector, with the help of Ward identities, one can reduce any 3-point correlator to one out of two independent structure constants. The relation between structure constants and normalization constants for the 3-form is clear. Thus, the definition of 3-point NS blocks as normalized 3-form is not troublesome.

The definition of Ramond 3-point blocks is more problematic. Due to "half-locality" of Ramond fields the Ward identities in Ramond sector have a complicated form. An arbitrary correlator of one NS and two Ramond fields can be reduced to a sum of two terms, each term proportional to one out of eight different structure constants. Even though only two structure constants are independent, the choice of a correct basis of the constants is not obvious. The second complication arises from the fact that Ramond field operators correspond to states from the "small representation", *i.e.* irreducible representation of the tensor product of the chiral Ramond algebras extended by the common parity operator. It is thus difficult to find a relation between the structure constants and the chiral 3-forms. The solution of these problems is to write the Ramond primary field in terms of non normalized chiral vertex operators. Considering its matrix elements one can find how the eight independent 3-form normalization constants reduce to two independent combinations of structure constants. Then the the definition of the 3-point block as normalized 3-form is possible.

The way of defining the 3-point superconformal blocks as normalized 3-form may seem unnecessarily complicated. It occurred, however, that it is just the most efficient and universal method of identifying all types of 3-point superconformal blocks in $N = 1$ SCFT, including the problematic 3-point Ramond blocks. Moreover, it allows to investigate the basic properties (e.g. factorization, c and Δ dependence) of the 3-point superconformal blocks from the relations defining the 3-forms. These properties and the features of inverse (NS or Ramond) Gram matrix are crucial in derivation of the recursive representations for the 4-point superconformal blocks.

Generalizing the reasonings known from the ordinary CFT, we have derived the z-recurrence for 4-point NS blocks (2.57), (2.58) and the elliptic recurrence for the NS and Ramond blocks (2.78), (3.51). In order to close the elliptic recursion relations the explicit formulae for each type of 4-point NS blocks and one type of 4-point Ramond block with arbitrary intermediate weight were necessary. We have calculated such blocks in the $c = \frac{3}{2}$ model, the supersymmetric generalization of the $c = 1$ scalar theory extended by Ramond states of the free scalar current [15]. The superconformal blocks in the $c = \frac{3}{2}$ model can be seen as a confirmation that the general constructions of the 4-point superconformal blocks and reasonings leading to the recursion relations are correct.

The recursive methods for the 4-point superconformal blocks in $N = 1$ SCFT presented in the thesis yield approximate (with arbitrary accuracy), analytic expressions for general 4-point superconformal blocks. Using these methods one can numerically calculate any 4-point function once the structure constants of the model are known. The methods can be used, for instance, for the consistency check of the $N = 1$ super Liouville theory with structure constants calculated by Poghosian [43]. The verification of consistency of the NS sector of $N = 1$ super Liouville theory was already done by A.Belavin, V.Belavin, A.Neveu and Al.Zamolodchikov [27, 28]. The check of Ramond sector of $N = 1$ super Liouville theory is still an open question. Moreover, the presented results can be helpful in the analysis of a common $c \to \frac{3}{2}$ limit of super Liouville theory and superconformal minimal models [48]. Finally, we believe that it possible to extend the developed techniques of constructing 4-point superconformal blocks and determining their recursive representations to the case of superconformal field theories with $N = 2$ supersymmetry.

Appendix A

$\frac{1}{\delta}$ expansion of classical block

We shall present some technical details of the elliptic Ansatz used in [15, 16] in order to calculate the first two terms of the $\frac{1}{\delta}$ expansion of classical block.

Consider the equation

$$\frac{d^2\psi(z)}{dz^2} + U(z)\psi(z) + \frac{x(x-1)\mathcal{C}(x)}{z(z-x)(z-1)}\psi(z) \quad = \quad 0 \tag{A.1}$$

with potential

$$U(z)\frac{1}{4}\left(\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2-\lambda_4^2-2}{z(z-1)} + \frac{1-\lambda_1^2}{z^2} + \frac{1-\lambda_2^2}{(z-x)^2} + \frac{1-\lambda_3^2}{(z-1)^2}\right).$$

We want to find $\mathcal{C}(x)$ such that equation (A.1) admits a pair of solutions $\psi^{\pm}(z)$ satisfying the monodromy condition:

$$\psi_{\pm}\left(\mathrm{e}^{2\pi i}z\right) = -\mathrm{e}^{\pm i\pi\lambda}\psi_{\pm}(z), \tag{A.2}$$

where $\psi_{\pm}\left(\mathrm{e}^{2\pi i}z\right)$ denotes a function analytically continued in z along the contour encircling the points 0 and x.

Following [16] we perform an elliptic change of variables:

$$\xi(z) = \frac{1}{2}\int^{\frac{z}{x}} \frac{dt}{\sqrt{t(1-t)(1-xt)}}, \qquad \psi(\xi) = \left(\frac{dz(\xi)}{d\xi}\right)^{-\frac{1}{2}}\psi\Big(z(\xi)\Big). \tag{A.3}$$

This gives

$$\begin{aligned}
\frac{d}{dz}\psi(z) &= -\frac{1}{2}\xi''\left(\xi'\right)^{-\frac{3}{2}}\psi(\xi) + \left(\xi'\right)^{+\frac{1}{2}}\left.\frac{d\psi(\xi)}{d\xi}\right|_{\xi=\xi(z)}, \\
\frac{d^2}{dz^2}\psi(z) &= -\frac{1}{2}\left(\xi'\right)^{-\frac{1}{2}}\{\xi(z),z\}\,\psi(\xi) + \left(\xi'\right)^{+\frac{3}{2}}\left.\frac{d^2\psi(\xi)}{d\xi^2}\right|_{\xi=\xi(z)},
\end{aligned} \tag{A.4}$$

where $\{\xi(z),z\}$ is the Schwarzian derivative of the map (A.3):

$$\{\xi(z),z\} = \frac{3}{8}\left[\frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{1}{(z-1)^2}\right] - \frac{1}{4}\left[\frac{1}{z(z-x)} + \frac{1}{z(z-1)} + \frac{1}{(z-x)(z-1)}\right].$$

Using (A.3) and (A.4) we can rewrite equation (A.1) in the form of a Schroedinger equation:

$$\frac{d^2\psi(\xi)}{d\xi^2} + U(\xi)\psi(\xi) + 4x(x-1)\mathcal{C}(x)\psi(\xi) = 0, \tag{A.5}$$

with the double periodic in ξ potential

$$\begin{aligned} U(\xi) &= \Big(\xi'(z)\Big)^{-2}\left[U(z) - \frac{1}{2}\{\xi(z), z\}\right]\Bigg|_{z=z(\xi)} \\ &= \left(\frac{1}{4}-\lambda_1^2\right)\left(\frac{x}{z(\xi)}-1\right) + \left(\frac{1}{4}-\lambda_2^2\right)\left[\frac{x(x-1)}{z(\xi)-x}+2x-1\right] \\ &+ \left(\frac{1}{4}-\lambda_3^2\right)\left(1-\frac{1-x}{1-z(\xi)}\right) + \left(\frac{1}{4}-\lambda_4^2\right)(z(\xi)-x)+x-\frac{1}{2}. \end{aligned} \tag{A.6}$$

After analytical continuation of the function $\xi(z)$ along the contour encircling the points 0 and x one gets:

$$\xi\left(e^{2\pi i}z\right) = \xi(z) + \frac{1}{2}\oint\limits_{[0,1]} \frac{dt}{\sqrt{t(1-t)(1-xt)}} = \xi(z) + \int\limits_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}} = \xi(z) + 2K(x).$$

where $K(x)$ is the complete elliptic integral of the first kind:

$$K(x) \equiv \int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}} = \frac{1}{2}\int\limits_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}}.$$

The monodromy condition (A.2) for $\psi(\xi)$ as a function of ξ thus takes the form

$$\psi_\pm\Big(\xi + 2K(x)\Big) = e^{\pm i\pi\lambda}\psi_\pm(\xi). \tag{A.7}$$

We will solve the equation (A.5) in the limit $\lambda \gg 1$ using a standard perturbative method. Assuming

$$U(\xi) = o(\mathcal{C}(x)), \tag{A.8}$$

the solutions in the leading order are in the form of plane waves:

$$\psi_\pm^{(0)}(\xi) = e^{\pm ip\xi}, \qquad p^2 = 4x(x-1)\mathcal{C}(x).$$

On the other hand the monodromy condition (A.7) implies:

$$e^{\pm 2ipK(x)} = e^{\pm i\pi\lambda} \qquad \Rightarrow \qquad p = -\frac{\pi\lambda}{2K(x)},$$

what also proves the consistency of our assumption (A.8). Hence, in the leading order one gets:

$$\mathcal{C}^{(0)}(x) = \frac{\pi^2\lambda^2}{16x(x-1)K^2(x)}.$$

In order to find the first correction let us define:

$$\psi_{\pm}(\xi) = \psi_{\pm}^{(0)}(\xi) + \psi_{\pm}^{(1)}(\xi), \qquad \mathcal{C}(x) = \mathcal{C}^{(0)}(x) + \mathcal{C}^{(1)}(x).$$

Inserting the expansions above to (A.5) we obtain:

$$\frac{d^2\psi_+^{(1)}(\xi)}{d\xi^2} + U(\xi)\psi_+^{(0)}(\xi) + 4x(x-1)\mathcal{C}^{(0)}(x)\,\psi_+^{(1)}(\xi) + 4x(x-1)\mathcal{C}^{(1)}(x)\,\psi_+^{(0)}(\xi) = 0. \quad \text{(A.9)}$$

One can multiply both sides by $\psi_-^{(0)}(\xi)$ and integrate over ξ in $[\xi_0, \xi_0 + 2K]$. Since

$$\psi_-^{(0)}(\xi)\frac{d\psi_+^{(1)}(\xi)}{d\xi}\Bigg|_{\xi_0}^{\xi_0+2K(x)} = \frac{d\psi_-^{(0)}(\xi)}{d\xi}\psi_+^{(1)}(\xi)\Bigg|_{\xi_0}^{\xi_0+2K(x)} = 0,$$

and

$$\frac{d^2\psi_-^{(0)}(\xi)}{d\xi^2} + 4x(x-1)\mathcal{C}^{(0)}(x)\psi_-^{(0)}(\xi) = 0,$$

it follows from (A.9) that the first correction has the form:

$$\mathcal{C}^{(1)}(x) = \frac{-1}{8x(x-1)K(x)}\int_{\xi_0}^{\xi_0+2K(x)} d\xi\ U(\xi) = \frac{-1}{16x(x-1)K(x)}\int_{[0,x]} \frac{U(\xi(z))dz}{\sqrt{z(1-z)(x-z)}}.$$

Using (A.6) one can rewrite the integral (A.9) in the following way:

$$\int_{[0,x]} \frac{U(\xi(z))\,dz}{\sqrt{z(1-z)(x-z)}} = \Big\{ \left(1-4\lambda_1^2\right)(I_1 - K(x)) + \left(1-4\lambda_2^2\right)(I_2 + (2x-1)K(x))$$

$$+ \left(1-4\lambda_3^2\right)(K(x) - I_3) + \left(1-4\lambda_4^2\right)(I_4 - xK(x)) + 4\left(x - \frac{1}{2}\right)K(x)\Big\}$$

Integrating, one gets:

$$\begin{aligned}
I_1 &= \frac{1}{4}\int_{[0,x]} \frac{xdz}{z\sqrt{z(1-z)(x-z)}} = K(x) - E(x),\\
I_2 &= \frac{1}{4}\int_{[0,x]} \frac{x(1-x)dz}{(z-x)\sqrt{z(1-z)(x-z)}} = (1-x)K(x) - E(x),\\
I_3 &= \frac{1}{4}\int_{[0,x]} \frac{(1-x)dz}{(1-z)\sqrt{z(1-z)(x-z)}} = E(x),\\
I_4 &= \frac{1}{4}\int_{[0,x]} \frac{zdz}{z\sqrt{(1-z)(x-z)}} = K(x) - E(x),
\end{aligned}$$

where $E(x)$ is the complete elliptic integral of the second kind:

$$E(x) \equiv \int_0^1 \frac{(1-xt^2)\,dt}{\sqrt{(1-t^2)(1-xt^2)}} = \frac{1}{2}\int_0^1 \frac{(1-xt)\,dt}{\sqrt{t(1-t)(1-xt)}}.$$

This leads to the formula for the correction to the accessory parameter:

$$\begin{aligned}\mathcal{C}^{(1)}(x) &= \frac{-1}{16x(x-1)\,K(x)}\int_{[0,x]}\frac{U(\xi(z))\,dz}{\sqrt{z(1-z)(x-z)}}\\ &= \frac{-1}{4x(x-1)}\left\{\frac{E(x)}{K(x)}\left(-1+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2\right)+x\left(1-\lambda_2^2+\lambda_4^2\right)-\left(\lambda_3^2+\lambda_4^2\right)\right\}\end{aligned}$$

Since $\mathcal{C}=\partial_x f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x)$ one can calculate the classical block:

$$\begin{aligned}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) = \int\frac{dx}{4x(x-1)}&\left\{\frac{(\pi\lambda)^2}{4K^2(x)}+\frac{E(x)}{K(x)}\left(1-\lambda_1^2-\lambda_2^2-\lambda_3^2-\lambda_4^2\right)\right.\\ &\left.-x\left(1-\lambda_2^2+\lambda_4^2\right)+\lambda_3^2+\lambda_4^2\right\}+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\end{aligned}$$

Using properties of elliptic integrals:

$$\begin{aligned}\int\frac{dx}{x(x-1)}\frac{1}{4K^2(x)} &= \frac{1}{\pi}\frac{K(1-x)}{K(x)}=\frac{\tau}{i\pi}\\ \int\frac{dx}{x(x-1)}\frac{E(x)}{K(x)} &= -\frac{1}{2}\ln K^4(x)-\ln x\end{aligned}$$

we get

$$\begin{aligned}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= \frac{1}{4}\left\{-i\pi\tau\lambda^2-\frac{1}{2}\left(1-\lambda_1^2-\lambda_2^2-\lambda_3^2-\lambda_4^2\right)\ln K^4(x)\right.\\ &\left.-(1-\lambda_2^2-\lambda_3^2)\ln(1-x)-(1-\lambda_1^2-\lambda_2^2)\ln(x)\right\}+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\end{aligned}$$

In terms of $\delta=\frac{1-\lambda^2}{4}$, $\delta_i=\frac{1-\lambda_i^2}{4}$ the classical block reads:

$$\begin{aligned}f_\delta\left[\begin{smallmatrix}\delta_3 & \delta_2\\ \delta_4 & \delta_1\end{smallmatrix}\right](x) &= i\pi\tau\left(\delta-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{3}{4}-\delta_1-\delta_2-\delta_3-\delta_4\right)\ln K^4(x)\\ &+\left(\frac{1}{4}-\delta_2-\delta_3\right)\ln(1-x)+\left(\frac{1}{4}-\delta_1-\ \delta_2\right)\ln(x)+\mathcal{O}\left(\frac{1}{\delta}\right).\end{aligned}\tag{A.10}$$

Let us note, that the classical block is given by this formula up to x-independent integration constant which can be fixed by the normalization condition of the 4-point conformal block $\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ \Delta_4 & \Delta_1\end{smallmatrix}\right](x)$ (1.36).

Appendix B

Check of recursion relations for calculated $c = \frac{3}{2}$ elliptic blocks

In this appendix we will show that NS and Ramond elliptic blocks in $c = \frac{3}{2}$ model calculated in the forth chapter satisfy recursion relations (2.77), (3.51) with modified residues. The modified non vanishing residua of NS blocks with $\Delta_0 = \frac{1}{8}$ are given by:

$$(16)^{\frac{r^2}{2}} \; R^{r,r}_{NS} \begin{bmatrix} *\Delta_0 & *\Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix} = \begin{cases} -r^2 & \text{if} \quad r \in 2\mathbb{N}, \\ 2 & \text{if} \quad r \in 2\mathbb{N}+1. \end{cases} \tag{A.1}$$

and in the case of Ramond blocks with $\beta_0 = 0, \beta_1 = \frac{1}{\sqrt{2}}$:

$$\begin{aligned} (16)^{\frac{r^2}{2}} \; R^{r,r}_{R} \begin{bmatrix} \beta_0 & \beta_0 \\ \beta_1 & \beta_1 \end{bmatrix} &= (-1)^{r+1}\, r^2 \\ (16)^{\frac{r^2}{2}} \; R^{r,r}_{R} \begin{bmatrix} \beta_0 & -\beta_1 \\ \beta_0 & \beta_1 \end{bmatrix} &= -r^2 \qquad \text{if} \quad r \in 2\mathbb{N}, \\ (16)^{\frac{r(r+2)}{2}} \; R^{r,r+2}_{R} \begin{bmatrix} \beta_0 & \beta_1 \\ \beta_0 & \beta_1 \end{bmatrix} &= -\tfrac{r(r+2)}{2} \qquad \text{if} \quad r \in 2\mathbb{N}+1. \end{aligned} \tag{A.2}$$

The non vanishing residues correspond to poles in degenerate weights:

$$\Delta_{r,r} = 0, \qquad \Delta_{r,r+2} = \frac{1}{2}.$$

First let us consider NS elliptic blocks (4.58), (4.59). Four of them trivially satisfy the recursion relations (2.77) because they are equal to the regular in Δ terms:

$$\begin{aligned} &\mathcal{H}^1_\Delta \begin{bmatrix} \Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(q) = g^1_\Delta \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(q), \qquad \mathcal{H}^1_\Delta \begin{bmatrix} \Delta_0 & *\Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(q) = g^1_\Delta \begin{bmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(q), \\ &\mathcal{H}^{\frac{1}{2}}_\Delta \begin{bmatrix} \Delta_0 & *\Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(q) = g^{\frac{1}{2}}_\Delta \begin{bmatrix} \Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(q), \qquad \mathcal{H}^{\frac{1}{2}}_\Delta \begin{bmatrix} *\Delta_0 & *\Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(z) = g^{\frac{1}{2}}_\Delta \begin{bmatrix} *\Delta_3 & *\Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}(q), \end{aligned}$$

In the other cases the formula (A.1) becomes helpful:

$$\mathcal{H}^{\frac{1}{2}}_\Delta \begin{bmatrix} \Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(q) \;=\; \sum_{r \in 2\mathbb{N}} (16q)^{\frac{r^2}{2}} \frac{\mathcal{R}^{\frac{1}{2}}_{c,mm} \begin{bmatrix} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{bmatrix}}{\Delta} \mathcal{H}^{\frac{1}{2}}_{\frac{m^2}{2}} \begin{bmatrix} \Delta_0 & \Delta_0 \\ \Delta_0 & \Delta_0 \end{bmatrix}(q)$$

$$
\begin{aligned}
&+ \sum_{r\in 2\mathbb{N}+1} (16q)^{\frac{r^2}{2}} \frac{\mathcal{R}^{\frac{1}{2}}_{c,mm}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right]}{\Delta} \mathcal{H}^1_{\frac{r^2}{2}}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) \\
&= \frac{1}{\Delta}\sum_{r\in 2\mathbb{N}} q^{\frac{r^2}{2}}\,(-m^2)\,\mathcal{H}^{\frac{1}{2}}_{\frac{m^2}{2}}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) + \frac{2}{\Delta}\sum_{r\in 2\mathbb{N}+1} q^{\frac{r^2}{2}}\,\theta_3(q^2)
\end{aligned}
$$

The definitions the theta functions

$$
\theta_2(q^2) = \sum_{n=-\infty}^{\infty} q^{\frac{(2n+1)^2}{2}} = 2\sum_{n=0}^{\infty} q^{\frac{(2n+1)^2}{2}}, \qquad \theta_3(q^2) = \sum_{n=-\infty}^{\infty} q^{2n^2} = 1 + 2\sum_{n=1}^{\infty} q^{2n^2}
$$

imply:

$$
\sum_{r\in 2\mathbb{N}} q^{\frac{r^2}{2}} = \frac{1}{2}\left(\theta_3(q^2) - 1\right), \qquad \sum_{r\in 2\mathbb{N}+1} q^{\frac{r^2}{2}} = \frac{1}{2}\left(\theta_2(q^2)\right). \tag{A.3}
$$

Substituting $\mathcal{H}^{\frac{1}{2}}_{\frac{r^2}{2}}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) = \frac{2}{r^2}\,\theta_2(q^2)$ one thus gets:

$$
\begin{aligned}
\mathcal{H}^{\frac{1}{2}}_{\Delta}\left[\begin{smallmatrix}\Delta_0 & \Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](z) &= \frac{-2}{\Delta}\sum_{r\in 2\mathbb{N}} q^{\frac{r^2}{2}}\theta_2(q^2) + \frac{2}{\Delta}\sum_{r\in 2\mathbb{N}+1} q^{\frac{r^2}{2}}\theta_3(q^2) \\
&= \frac{-1}{\Delta}\,(\theta_3(q^2) - 1)\,\theta_2(q^2) + \frac{1}{\Delta}\theta_2(q^2)\,\theta_3(q^2) = \frac{1}{\Delta}\,\theta_2(q^2).
\end{aligned}
$$

The last block $\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q)$ in (4.59) also satisfies the recursion relation:

$$
\begin{aligned}
&\mathcal{H}^1_{\Delta}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) = \theta_3(q^2) + \sum_{r\in 2\mathbb{N}}\left(\frac{-r^2}{\Delta}\right) q^{\frac{r^2}{2}}\,\mathcal{H}^1_{\frac{r^2}{2}}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) \\
&+ \sum_{r\in 2\mathbb{N}+1}\left(\frac{-2}{\Delta}\right) q^{\frac{r^2}{2}}\,\mathcal{H}^{\frac{1}{2}}_{\frac{r^2}{2}}\left[\begin{smallmatrix}*\Delta_0 & *\Delta_0\\ \Delta_0 & \Delta_0\end{smallmatrix}\right](q) \\
&= \theta_3(q^2) - \frac{2}{\Delta}\left(\sum_{r\in 2\mathbb{N}} q^{\frac{r^2}{2}}\,\theta_3(q^2) - \sum_{r\in 2\mathbb{N}+1} q^{\frac{r^2}{2}}\,\theta_2(q^2)\right)\left(-q\,\theta_3^{-1}\frac{\partial}{\partial q}\theta_3(q) + \frac{\theta_2^4(q)}{4}\right) \\
&-\frac{2}{\Delta}\sum_{r\in 2\mathbb{N}} \frac{r^2}{2}\, q^{\frac{r^2}{2}}\,\theta_3(q^2) + \frac{2}{\Delta}\sum_{r\in 2\mathbb{N}+1} \frac{r^2}{2}\, q^{\frac{r^2}{2}}\,\theta_2(q^2) \\
&= \theta_3(q^2)\left(1 - \frac{q}{\Delta}\theta_3^{-1}\frac{\partial}{\partial q}\theta_3(q) + \frac{\theta_2^4(q)}{4}\right) \\
&+\frac{1}{\Delta}\left(q\theta_3^{-1}\frac{\partial}{\partial q}\theta_3(q) - \frac{\theta_2^4(q)}{4} + \frac{q}{2}\frac{\partial}{\partial q}\right)\left(\theta_3^2(q^2) - \theta_2^2(q^2)\right)
\end{aligned} \tag{A.4}
$$

From the identities

$$
\theta_3(q^2) = \theta_3(q)\left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}}, \qquad \theta_2(q^2) = \theta_3(q)\left(\frac{1-\sqrt{1-z}}{2}\right)^{\frac{1}{2}} \tag{A.5}
$$

it follows that

$$
\theta_3(q^2) - \theta_2(q^2) = \sqrt{1-z}\,\theta_3(q).
$$

Since

$$q\frac{\partial}{\partial q} = z(1-z)\,\theta_3^4(q)\frac{\partial}{\partial q}, \qquad \text{and} \qquad z = \frac{\theta_2^4(q)}{\theta_3^4(q)} \tag{A.6}$$

we have

$$\frac{q}{2}\frac{\partial}{\partial q}\left(\sqrt{1-z}\theta_3(q)\right) = \left(q\theta_3^{-1}\frac{\partial}{\partial q}\theta_3(q) - \frac{\theta_2^4(q)}{4}\right),$$

what implies that the last line in (A.4) is zero.

Four of the Ramond elliptic blocks (4.73) - (4.76) nontrivially satisfy recursion relations with residues of the form (A.2). For example, the block in (4.73):

$$\begin{aligned}
\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q) &= 1 + \sum_{r\in 2\mathbb{N}}\left(\frac{-r^2}{\Delta}\right)q^{\frac{r^2}{2}}\,\mathcal{H}^1_{\frac{r^2}{2}}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q) + \sum_{r\in 2\mathbb{N}+1}\left(\frac{r^2}{\Delta}\right)q^{\frac{r^2}{2}}\,\mathcal{H}^{\frac{1}{2}}_{\frac{r^2}{2}}\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q)\\
&= 1 - \frac{1}{\Delta}\left(q\frac{\partial}{\partial q}\theta_3(q^2) + \left(\theta_3(q^2)-1\right)\left(\frac{1}{4}\theta_2^4(q) - \theta_3^{-1}(q)q\frac{\partial}{\partial q}\theta_3(q)\right)\right) + \frac{1}{4\Delta}\sqrt{z}\theta_2(q^2)\theta_3^4(q)\\
&= \mathcal{H}^1_\Delta\left[\begin{smallmatrix}\pm\beta_0 & \pm\beta_0\\ \beta_1 & \beta_1\end{smallmatrix}\right](q) - \frac{1}{\Delta}\theta_3^4(q)\left\{-\frac{1}{8}z(1-z)^{\frac{1}{2}}\theta_3^2(q)\theta_3^{-1}(q^2) + \frac{1}{4}\theta_2(q^2)(z-\sqrt{z})\right\},
\end{aligned} \tag{A.7}$$

where (A.3), (A.5), (A.6) and relation

$$q\frac{\partial}{\partial q}\theta_3(q^2) = \theta_3^{-1}(q)q\frac{\partial}{\partial q}\theta_3(q) - \frac{1}{8}z(1-z)^{\frac{1}{2}}\theta_3^6(q)\theta_3^{-1}(q^2)$$

were used. One can check, once more applying (A.5), that the bracket in the last line in (A.7) vanishes.

The block $\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q)$ (4.75) with poles at $\Delta = \frac{1}{2}$, satisfies the recursion relation with residues from the last line in (A.2):

$$\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) = 1 + \frac{1}{\Delta-\frac{1}{2}}\sum_{r\in 2\mathbb{N}}\left(-\frac{r(r+2)}{2}\right)q^{\frac{r^2}{2}}\,\mathcal{H}^1_{\frac{r(r+2)}{2}}\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q).$$

The series in nome can be written in terms of theta function:

$$\sum_{r\in 2\mathbb{N}} q^{\frac{r(r+2)}{2}} = \frac{1}{2}q^{-\frac{1}{2}}\,\theta_2(q^2) - 1, \qquad \sum_{r\in 2\mathbb{N}}\frac{r(r+2)}{2}\,q^{\frac{r(r+2)}{2}} = \frac{q}{2}\frac{\partial}{\partial q}\left(q^{-\frac{1}{2}}\theta_2(q^2)\right),$$

what implies

$$\begin{aligned}
\mathcal{H}^1_\Delta\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) &= 1 - \frac{1}{\Delta-\frac{1}{2}}\left\{\frac{q}{2}\frac{\partial}{\partial q}\left(q^{-\frac{1}{2}}\,\theta_2(q^2)\right)\right.\\
&\left.+\left(\frac{1}{2}q^{-\frac{1}{2}}\,\theta_2(q^2)-1\right)\left(\frac{1}{2}+\frac{1}{4}\theta_2^4(q) - \frac{1}{2}\theta_3^2(q^2)\theta_3^2(q) - \theta_3^{-1}(q)q\frac{\partial}{\partial q}\theta_3(q)\right)\right\} \qquad \text{(A.8)}\\
&= \mathcal{H}^1_\Delta\left[\begin{smallmatrix}\beta_0 & \beta_1\\ \beta_0 & \beta_1\end{smallmatrix}\right](q) - \frac{1}{\Delta-\frac{1}{2}}\theta_3^4(q)\left\{\frac{1}{8}z(1-z)^{\frac{1}{2}}\theta_3^2(q)\theta_2^{-1}(q^2) - \frac{1}{4}\theta_2(q^2)(1-z-\sqrt{1-z})\right\},
\end{aligned}$$

where we used the relations (A.5), (A.6) and

$$q\frac{\partial}{\partial q}\left(q^{-\frac{1}{2}}\theta_2(q^2)\right) = -\frac{1}{2}q^{-\frac{1}{2}}\theta_2(q^2) + q^{-\frac{1}{2}}\left(\theta_2(q^2)\theta_3^{-1}(q)q\frac{\partial}{\partial q}\theta_3(q) + \frac{1}{8}z(1-z)^{\frac{1}{2}}\theta_3^6(q)\theta_2^{-1}(q^2)\right).$$

With the help of (A.5), similar like in (A.7), one can calculate that the bracket in the last line in (A.8) vanishes. By analogous calculations one can check that the other Ramond elliptic blocks (4.74) and (4.76) also satisfy recursive relations (3.51) with residues given by (A.2).

Bibliography

[1] A. M. Polyakov, *Conformal symmetry of critical fluctuations*, *JETP Lett.* **12** (1970) 381.

[2] A. M. Polyakov, *Non-Hamiltonian approach to conformal quantum field theory*, *Sov. Phys. JETP* **39** (1974) 10.

[3] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B241** (1984) 333–380.

[4] D. Friedan, Z. a. Qiu and S. H. Shenker, *Superconformal Invariance In Two-Dimensions And The Tricritical Ising Model*, *Phys. Lett.* **B 151** (1985) 37.

[5] M. Bershadsky, V. Knizhnik and M. G. Teitelman, *Superconformal Symmetry In Two-Dimensions*, *Phys. Lett.* **B 151** (1985) 31.

[6] A. Zamolodchikov and R. Pogosian, *Operator algebra in two-dimensional superconformal field theory*, *Sov. J. Nucl. Phys.* **47** (1988) 929.

[7] A. M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B103** (1981) 207–210.

[8] A. M. Polyakov, *Quantum geometry of fermionic strings*, *Phys. Lett.* **B103** (1981) 213.

[9] M. Green, J. Schwarz and E. Witten, *Superstring theory.* Cambridge University Press, 1987.

[10] J. Polchinski, *String Theory.* Cambridge University Press, 1998.

[11] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386 [`hep-th/9905111`].

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114 [`hep-th/9802109`].

[13] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, `hep-th/0201253`.

[14] A. B. Zamolodchikov, *Conformal symmetry in two-dimensions: an explicit recurrence formula for the conformal partial wave amplitude*, *Commun. Math. Phys.* **96** (1984) 419–422.

[15] A. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, *Sov. Phys. JETP* **63** (1986) 1061.

[16] A. Zamolodchikov, *Conformal symmetry in two-dimensional space: recursion representation of conformal block*, *Theor.Math.Phys.* **73** (1987) 1088.

[17] S. A. Apikyan and A. B. Zamolodchikov, *Conformal Blocks, Related To Conformally Invariant Ramond States Of A Free Scalar Field*, *Sov. Phys. JETP* **65** (1987) 19.

[18] H. Dorn and H. J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B429** (1994) 375–388 [hep-th/9403141].

[19] A. B. Zamolodchikov and A. B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B477** (1996) 577–605 [hep-th/9506136].

[20] I. Runkel and G. M. T. Watts, *A non-rational CFT with $c = 1$ as a limit of minimal models*, *JHEP* **09** (2001) 006 [hep-th/0107118].

[21] L. Hadasz, Z. Jaskolski and M. Piatek, *Classical geometry from the quantum Liouville theory*, *Nucl. Phys.* **B724** (2005) 529–554 [hep-th/0504204].

[22] L. Hadasz and Z. Jaskolski, *Liouville theory and uniformization of four-punctured sphere*, *J. Math. Phys.* **47** (2006) 082304 [hep-th/0604187].

[23] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursion representation of the Neveu-Schwarz superconformal block*, *JHEP* **03** (2007) 032 [hep-th/0611266].

[24] L. Hadasz, Z. Jaskolski and P. Suchanek, *Elliptic recurrence representation of the N=1 Neveu- Schwarz blocks*, *Nucl. Phys.* **B798** (2008) 363–378 [0711.1619].

[25] L. Hadasz, Z. Jaskolski and P. Suchanek, *Conformal blocks related to the R-R states in the $\hat{c} = 1$ SCFT*, *Phys. Rev.* **D77** (2008) 026012 [0711.1618].

[26] L. Hadasz, Z. Jaskolski and P. Suchanek, *Elliptic recurrence representation of the $N = 1$ superconformal blocks in the Ramond sector*, *JHEP* **11** (2008) 060 [0810.1203].

[27] A. Belavin, V. Belavin, A. Neveu and A. Zamolodchikov, *Bootstrap in supersymmetric Liouville field theory. I: NS sector*, *Nucl. Phys.* **B784** (2007) 202–233 [hep-th/0703084].

[28] V. A. Belavin, *On the $N = 1$ super Liouville four-point functions*, *Nucl. Phys.* **B798** (2008) 423–442 [0705.1983].

[29] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory.* Springer, New York, 1997.

[30] P. H. Ginsparg, *Applied conformal field theory*, `hep-th/9108028`.

[31] S. Ketov, *Conformal Field Theory.* World Scientific Publishing Co. Pte. Ltd., 1995.

[32] V. G. Kac, *Contravariant Form for Infinite Dimensional Lie Algebras and Superalgebras*, . In *Austin 1978, Proceedings, Group Theoretical Methods In Physics*, Berlin 1979, 441-445.

[33] B. L. Feigin and D. B. Fuchs, *Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra*, *Funkts. Anal. Prilozh.* **16** (1982) 47–63.

[34] J. Teschner, *Liouville theory revisited*, *Class. Quant. Grav.* **18** (2001) R153–R222 [`hep-th/0104158`].

[35] G. W. Moore and N. Seiberg, *Polynomial Equations For Rational Conformal Field Theories*, *Phys. Lett.* **B 212** (1988) 451.

[36] G. W. Moore and N. Seiberg, *Classical And Quantum Conformal Field Theory*, *Commun. Math. Phys.* **123** (1989) 177.

[37] Frenkel, E. and Ben-Zvi, D., *Vertex algebras and algebraic curves.* AMS Bookstore, 2004.

[38] V. G. Kac, *Infinite-dimensional Lie algebras*, *Progress in Math. Phys.* **44** (1984).

[39] B. L. Feigin and D. B. Fuchs, *Representations of Virasoro algebra* . In: Representations of Lie group and related topics, eds. A.M. Vershik, D.P. Zhelobenko (Gordon and Breach, London 1990).

[40] A. Zamolodchikov, *Higher equations of motion in Liouville field theory*, *Int. J. Mod. Phys.* **A19S2** (2004) 510–523 [`hep-th/0312279`].

[41] T. L. Curtright and C. B. Thorn, *Conformally Invariant Quantization of the Liouville Theory*, *Phys. Rev. Lett.* **48** (1982) 1309.

[42] A. Belavin and A. Zamolodchikov, *Higher equations of motion in $N = 1$ SUSY Liouville field theory*, *JETP Lett.* **84** (2006) 418–424 [`hep-th/0610316`].

[43] R. H. Poghosian, *Structure constants in the $N = 1$ super-Liouville field theory*, *Nucl. Phys.* **B496** (1997) 451–464 [`hep-th/9607120`].

[44] V. A. Belavin, *$N = 1$ SUSY conformal block recursive relations*, `hep-th/0611295`.

[45] A. Meurman and A. Rocha-Caridi, *Highest weight representation of the Neveu-Schwarz and Ramond algebras*, *Commun. Math. Phys.* **107** (1986) 263.

[46] M. R. Gaberdiel, *Fusion of twisted representations*, *Int. J. Mod. Phys.* **A12** (1997) 5183–5208 [hep-th/9607036].

[47] A. B. Zamolodchikov and A. B. Zamolodchikov, *Conformal field theory and 2-D critical phenomena. 5a. Conformal theory of critical Ashkin-Teller model*, . ITEP-90-91.

[48] S. Fredenhagen and D. Wellig, *A common limit of super Liouville theory and minimal models*, *JHEP* **09** (2007) 098 [0706.1650].