Non-perturbative Aspects of Supersymmetric Plasma Physics in a Gauge Theory - String Theory Approach

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dr. hab. Romualda A. Janika

Piotr Surówka

Wydział Fizyki, Astronomii i Informatyki Stosowanej
Uniwersytetu Jagiellońskiego
Instytut Fizyki im. Mariana Smoluchowskiego
ul. Wł. Reymonta 4, 30-059, Kraków

Wydział Fizyki, Astronomii i Informatyki Stosowanej
Uniwersytetu Jagiellońskiego

Oświadczenie

Ja niżej podpisany Piotr Surówka (nr indeksu: WFAIS/D/140) doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożna przeze mnie rozprawa doktorska pt. „Non-perturbative Aspects of Supersymmetric Plasma Physics in a Gauge Theory - String Theory Approach" jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem dr. hab. Romualda A. Janika. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

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Abstract

In this Thesis we study the connection between hydrodynamics and gravity by means of the AdS/CFT correspondence. We review general construction of hydrodynamics and its applications to heavy-ion collisions. We find a novel effect due to quantum anomalies, which manifest themselves in the macroscopic hydrodynamic regime. Moreover, we find a description of hydrodynamic system with one conserved current using gauge/gravity duality. Special attention is put to the boost-invariant regime of hydrodynamics. We obtain explicit solutions of the dual system and perform a detailed regularity analysis. Finally, we investigate meson properties in the boost-invariant plasma. In particular, we study the influence of dynamical temperature and viscosity on meson spectra in the plasma.

W pracy przestudiowane zostaną związki między hydrodynamiką i grawitacją przy użyciu metod korespondencji AdS/CFT. Przedstawiona zostanie ogólna konstrukcja hydrodynamiki i jej zastosowania do zderzeń ciężkich jonów. Pokażemy istnienie nowego efektu pochodzącego od kwantowych anomalii, które manifestują się w makroskopowej hydrodynamice. Następnie opiszemy system hydrodynamiczny z jednym zachowanym prądem przy użyciu korespondencji między teorią cechowania, a teorią grawitacji. Szczególna uwaga poświęcona zostanie hydrodynamice niezmienniczej ze względu na pchnięcia Lorentza. Znajdziemy dokładne rozwiązania dla dualnego opisu, dla których przeprowadzimy szczegółową analizę regularności. Na koniec zbadamy właściwości mezonów w plazmie niezmienniczej ze względu na pchnięcia Lorentza. W szczególności, przeanalizujemy wpływ zmieniającej się temperatury i lepkości na spektrum mezonów w plazmie.

Contents

Chapter 1

Introduction

QCD has proven to be a very successful theory of strong interactions. It is a non-abelian quantum field theory. It postulates that nuclei, the constituents of atoms, are not fundamental building blocks but are actually complex structures made of 'quarks' and 'gluons'. Quarks and gluons are fundamental degrees of freedom in the theory, which can be classified by representations of the gauge group. QCD is designed to have local $SU(3)$ symmetry. Quarks are assigned to the fundamental representation, whereas gluons transform in adjoint representation of $SU(3)$. Physical hadrons are colorless, which means that they are singlets under the gauge group. QCD exhibits two very important features, asymptotic freedom and confinement of quarks.

An asymptotically free theory is a theory whose coupling strength decreases as the renormalization scale increases. In other words, interaction between quarks by exchange of gluons becomes negligible at short distances. This was proven experimentally by deep inelastic scattering, since particles with very high energies interact weakly and perturbation theory applies. On the other hand, if we are interested in the low energy physics we have to deal with very strong interactions. In between we have a transition from weakly interacting to strongly interacting field theory. QCD is a renormalizable field theory, in which the coupling constant g is a function of the energy scale,

$$g = g(\mu). \tag{1.1}$$

The renormalization scale at which the system becomes strongly interacting and the coupling becomes of order one is called Λ_{QCD}. In that regime the theory becomes very hard to solve. Therefore, there is no good explanation of the confinement problem. Confinement is a phenomenon that occurs in the low-energy regime. It implies that color charged particles cannot be isolated and directly observed. From experimental point of view this means that at low energy the particles observed in accelerators are not quarks and gluons alone, but hadrons. One of the biggest unsolved theoretical challenges is to find a spectrum of hadrons given a high-energy formulation of QCD.

There are various ways to tackle QCD analytically. Some of them rely on non-perturbative methods and some explore various possibilities of staying within high-energy regime. The most popular non-perturbative approach is based on lattice regularization

[1–3]. Euclidean space-time is represented by large but finite number of lattice sites and quantum fields as degrees of freedom on that lattice. This procedure allows to solve path integrals explicitly by performing a finite number of integrations. Although very powerful, this method has severe limitations. Real time dynamics produces complex factor in the action which makes the computation unreliable because the action oscillates. Hence, all lattice QCD computations are performed in the analytically continued Euclidean space-time. This makes lattice techniques inadequate for understanding many interesting strong interaction phenomena like showering and hadronization in high-energy collisions. Therefore, there is great need for more efficient non-perturbative tools that allow us to explore low-energy QCD. This is particularly important at the age of big scientific experiments, such as Relativistic Heavy Ion Collider (RHIC) or Large Hadron Collider (LHC). RHIC and a part of LHC are facilities dedicated to heavy-ion physics, built to produce and study the properties of new form of matter, the quark-gluon plasma (QGP). The concept of QGP, as well as some ideas how to reach it experimentally was proposed by Shuryak [4]. His arguments emerged as a consequence of idea that in a very large temperature, far above Λ_{QCD}, the color charge should not be confined but rather screened. Naïvely we expect excitations to scatter with energies of the order of the temperature, hence, to be weakly interacting and a simple perturbative approach to QGP properties describes its properties at least qualitatively. However, the experiments at RHIC actually forced us to revise our thinking about high-temperature QCD and QGP. It turned out that heavy-ion collisions can be very well described by hydrodynamics [5] (see [6–9] for a review), which is a very old effective approach to strongly coupled field theories [10]. The fact that hydrodynamic modeling of RHIC collisions works indicates that the mean free path of a particle inside the RHIC plasma must be much smaller than the size of the plasma. A short mean free path in turn suggests that the system is strongly-coupled [11, 12]. This idea is supported by lattice calculations [13, 14], as well as by the fact that perturbative techniques give wrong predictions [15, 16]. The observation that QGP might be strongly coupled does not provide us with appropriate non-perturbative field theoretic tools. However, one can make use of recent developments of string theory to get some quantitative results.

String theory was discovered in late 1960s to describe hadronic spectra and their interactions. This discovery was motivated by experiments, which showed that some hadronic states exhibited linear relation between mass squared and spin. This is a relation that can be explained if we substitute point particles by strings. However, the advances in quantum field theory which led to discovery of QCD, as well as the problems with a proper formulation of string theory, stopped the progress. In 1974 the interest in string theory was revived, Gerard 't Hooft made another important development that connected strong interactions with string theory in a completely new way. He suggested that gauge theory simplifies significantly when the number of colors N_c is large [17]. The idea was that one could take this generalized version of QCD and do an expansion in inverse powers of N_c. At first sight it seems that this large N_c theory and QCD have nothing in common. However, if we realize that typically the corrections are of order $1/N_c^2$ we see that the deviation from experiment should be around 10%. 't Hooft arguments do not say whether string theory is useful for understanding large N_c limit of QCD or how precisely find the relation between them. It took almost 25 years to find it. The solution came from unexpected direction, since in the mean time people became aware that string theory can actually describe gravity.

In 1995 Joseph Polchinski realized that string theory actually contains not only strings, but also various extended objects called D-branes [18]. The studies of D-branes and black holes in string theory led physicists to postulate the relation between a particular large N_c gauge theory and string theory [19–21]. It is often called a duality because strongly coupled gauge theory is mapped to perturbative strings. This makes the conjecture very hard to prove, since it involves solving non-perturbative field theory. However, assuming its validity we may get a lot of insight into strongly coupled regime of gauge theory, as well as into non-perturbative formulation of gravity. The dream of string theorists is to eventually find a dual description of QCD. However, it is difficult since at high energies, because of the asymptotic freedom, we need full string theory framework. At the moment the number of theories we are able to solve is limited and, unfortunately, none of them describes real physical system. Nevertheless, we can still capture some important features of real world physics. In some cases we can argue that in certain regimes there is at least qualitative agreement between theories, in other cases we can search for some universal properties of gauge theories that are accessible through gauge/gravity duality. In this Thesis we will show how to make the correspondence useful in the long-wave, low frequency limit of gauge theory - hydrodynamics.

The general framework of fluid/gravity correspondence was constructed in [22] and was motivated by works of Janik and collaborators who constructed gravity dual to Bjorken flow [23–28]. Subsequently a lot of work was done to generalize it to other dimensions [29–31], to include external forcing on the fluid [32], and to include conserved $U(1)_R$ charges [33, 34]. In that construction anomalous hydrodynamics was observed, which was later understood in [35]. Moreover, the analysis was extended to non-conformal fluids located on Dp-world-volumes [36, 37]. There was also much progress within boost-invariant setup itself. Diffusion constant was calculated in [38], meson spectra were analyzed in [39], drag force on a fundamental quark was computed in [40]. Leading α' corrections to the transport coefficients were found using the boost-invariant flow in [41, 42].

Some of the above contributions were done as a research part of this Thesis, namely a general framework for fluid/gravity duality with one conserved current was constructed in [34]. Anomalous part of that construction was later understood in [35]. Moreover, a consistent formulation of gravity dual of boost-invariant flows was found in [43], and meson spectra were calculated in [39]. This Thesis is organized as follows.

In Chapter 2 we review the heavy-ion physics and argue that hydrodynamics is a useful description in some range of time. Next, we construct a general framework of relativistic fluid dynamics. We start with perfect fluids, for which we present conservation laws. We generalize them to the case with viscous corrections. Moreover, we introduce two new transport coefficients, associated with the triangle anomalies at the quantum level of an underlying gauge theory. This effect was overlooked in the standard treatments [10, 44]. We are able to constrain these coefficients purely using hydrodynamic reasoning, as functions of temperature, chemical potential, and anomaly coefficient. Finally, we introduce the so-called boost-invariant hydrodynamics, which is useful in the central rapidity region of heavy-ion collisions.

In Chapter 3 we review some aspects of the original AdS/CFT correspondence. We introduce $\mathcal{N} = 4$ SYM and review some of its properties. Next, we give a brief introduction to string theory. We show what is the massless spectrum of superstring theory and argue that D-branes are necessary ingredient of the theory. Studying certain limit of stack of a particular kind of branes we show how the duality emerges from that picture. Then we sketch how one can relate observables on both sides of the duality. Finally, we take a path towards more realistic systems with finite temperature and show that duality is applicable there as well.

In Chapter 4 we construct a general framework for fluid/ gravity correspondence. We start with fluid with no conserved global currents, which corresponds to a gravity solution with a black hole in the center of the geometry. Including gauge field hair we generalize the construction to fluids with one conserved current. Next, we include background magnetic field, which allow us to calculate anomalous transport coefficients from gravity. These coefficients follow from Chern-Simons terms in the Einstein-Hilbert action. We find an agreement with field theory calculation, which gives an independent check of AdS/CFT correspondence and supports conclusions coming from hydrodynamics. At the end of that chapter we present an explicit solution to a particular class of fluids with boost-invariance.

In Chapter 5 we show how to introduce holographic matter in a fundamental representation. We add probe branes that do not backreact with gravity, which give rise to quarks in quenched approximation. Finally, studying small fluctuations around these brane embeddings we calculate meson spectrum in the dynamical, boost-invariant setup.

We conclude in Chapter 6 speculating about future directions of research and possible new applications of gauge/gravity duality.

Chapter 2

Relativistic heavy-ion collisions

Theoretical attempts to describe heavy-ion physics have long history. The first model based on statistical mechanics was proposed by Fermi in 1950. Soon Landau realized that hydrodynamics might be a good description of matter after the collision. In the Landau picture the colliding nucleons are significantly slowed down and then particle production occurs, mostly within the thickness of colliding nuclear matter. Subsequently, the system undergoes hydrodynamic expansion. This scheme may be a good approximation if the colliding beams don't have too much energy. Otherwise, Landau picture should be replaced. This was first suggested by Bjorken. He noted that there is an asymmetry in the particle production after the collision. The slow particles are created first, near the collision point, while fast particles emerge far from the collision point. This is known as the inside-outside cascade. The reaction volume is strongly expanded in the longitudal beam direction, which can be approximated by (1+1)-dimensional evolution. We can use Bjorken picture to follow the history of the collision process (see Fig. 2.1).

After the collision we can point out a few stages labeled by the expansion proper time defined as $\tau = (t^2 - z^2)^{1/2}$. Shortly after the collision at $0 < \tau < \tau_0$ we distinguish pre-equilibrium stage and thermalization. We do not possess a valid theoretical tool to fully describe microscopic origin of thermalization process since it involves non-abelian gauge theory. Despite difficulties, people proposed two classes of models, the so-called incoherent and coherent models, which are expected to give at least qualitative answers. Incoherent models are calculated within the framework of perturbative QCD. They propose that in the collision hard parton scatterings occur, which results in a large amount of jet production. These jets subsequently interact with each other producing equilibrated QGP [45, 46]. In coherent models the QGP follows from the formation of coherent color fields. One example is the so-called color glass condensate (CGC) [47] (see [48, 49] for a review). CGC is an effective field theory, which describes nucleus-nucleus collision as an evolution of soft classical field, created by moving partons randomly oriented in color space. Both coherent and incoherent models have their own limitations of applicability. Throughout this Thesis we will simply assume that thermalization takes place before the characteristic time $\tau_0 > \tau$, when the thermal equilibrium is reached. Then we use the relativistic hydrodynamics to describe the system evolution before it hydronizes. Eventually, at the

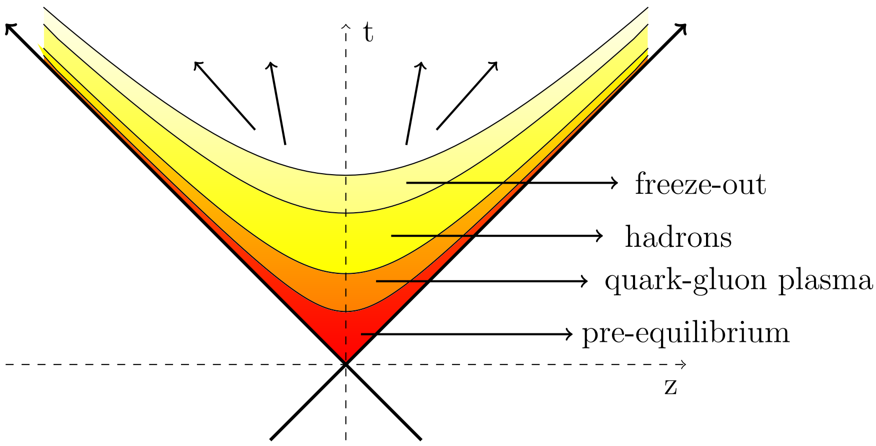

Figure 2.1: Time evolution of the collided nucleons in the Bjorken picture. Lines of constant proper times are presented as hyperbolas. The hyperbola between red and orange represents τ_0 and between yellow and pastel yellow represents the beginning of freeze-out τ_f.

proper time τ_f the system undergoes the freeze-out, when the mean free path of plasma particles becomes larger than the timescale of plasma expansion. It was argued that two kinds of freeze-out occur, the chemical, after which the number of species is constant and the kinetic equilibrium is maintained and thermal, after which the is no kinetic equilibrium. For a discussion on that see [50].

2.1 Hydrodynamics

Hydrodynamics is an effective field theory that describes fluid dynamics at scales much larger than relevant for underlying microscopic phenomena. Therefore, we can regard a fluid as a continuous medium. This medium can be divided into infinitely many infinitesimal volume elements. Each element is still treated as a macroscopic quantity that contains large number of particles. In order to have a full description of a moving fluid we will need a distribution of fluid velocity u^μ supplemented with two thermodynamic quantities. Hydrodynamics includes the dissipative effects, thus we cannot formulate it by constructing an action. Instead one postulates equations describing the field dynamics directly.

As a simple example let us take a fluid with no conserved currents. For the moment we assume that there is no dissipative effects in the medium. The only conserved quantity is the energy-momentum tensor. We expect that the equation describing dynamics of the fluid is the conservation law,

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.1}$$

In order to have a closed system of equation we have to reduce the number of independent components of $T^{\mu\nu}$. We do that assuming that our fluid is in local thermal equilibrium.

This means that the temperature in the system vary very slowly in space and time.

2.1.1 Perfect Fluid

To construct the form of energy-momentum tensor we follow the procedure of effective field theories. We expand the tensor in the powers of field derivatives. We expect that the zeroth order expansion is a perfect fluid - fluid that is specified by two quantities in the rest frame, energy density and pressure. Because we don't want to have any preferred direction in the fluid the off-diagonal part of energy-momentum tensor should vanish. Moreover, all diagonal components of the spatial part should have the same value,

$$T^{\mu\nu}_{\text{rest}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{2.2}$$

The energy-momentum tensor consist of two extensive quantities. We expect that measurable values of them should be independent of the reference frame we choose. Thus, we have to write Eq. (2.2) in a covariant way. Let us pass to a different frame,

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}_{\text{rest}} = \Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0}\varepsilon + \Lambda^{\mu}{}_{j}\Lambda^{\nu}{}_{j}P. \tag{2.3}$$

We rewrite the right-hand side of the above equation using

$$\Lambda^{\rho}{}_{j}\Lambda^{\sigma}{}_{j} = \Lambda^{\rho}{}_{0}\Lambda^{\sigma}{}_{0} - g^{\rho\sigma} = u^{\rho}u^{\sigma} - g^{\rho\sigma}, \tag{2.4}$$

which follows from the consistency condition for Lorentz transformation in General Relativity $g^{\mu\nu}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = g^{\rho\sigma}$. As a result we get the covariant form of energy-momentum tensor for the perfect fluid,

$$T^{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} - Pg^{\mu\nu}. \tag{2.5}$$

In many applications the perfect fluid approximation to physical mediums is good enough to neglect the higher order corrections. However, going beyond the zeroth order leads to the description of viscosity and chiral separation, which have a consequence of entropy production in the fluid. But before we see how the entropy production arises let us see that in the case of perfect fluid entropy is a conserved quantity. To do that we relax the condition that there is no conserved current in the medium and assume that the particle number is a conserved quantity. If the system contains a conserved current, we have an additional hydrodynamic relation,

$$\partial_{\mu}j^{\mu} = 0. \tag{2.6}$$

Again, it is the easiest to see the form of j^{μ} in the rest frame, in which at the zeroth order it is the number of particles times velocity nu^{μ}.

Now, we plug the expression for the energy-momentum tensor of the perfect fluid into Eq. (2.2),

$$\partial_{\nu}T^{\mu\nu} = u^{\mu}\partial_{\nu}\left[(\varepsilon + P)u^{\nu}\right] + (\varepsilon + P)u^{\nu}\partial_{\nu}u^{\mu} + \partial^{\mu}P. \tag{2.7}$$

In order to simplify the above equation we project it on the direction of four-velocity. Moreover, we note that the velocity of the fluid in space-time is constant $u_\mu u^\mu = -1$. Differentiating both sides gives the following identity,

$$u_\mu \partial_\nu u^\mu = 0, \tag{2.8}$$

which simplifies Eq. (2.7),

$$\partial_\nu \left[(\varepsilon + P) u^\nu \right] - u^\nu \partial_\nu P = 0. \tag{2.9}$$

Next, we use the thermodynamic relation

$$d\left(\frac{\varepsilon + P}{n}\right) = T d\left(\frac{s}{n}\right) + \frac{1}{n} dp. \tag{2.10}$$

It is valid for a particular quantity of fluid, not for a particular volume, which may contain different number of particles. Plugging Eq. (2.10) to the relation (2.7) and using the continuity equation we arrive at

$$\partial_\mu (s u^\mu) = 0. \tag{2.11}$$

The divergence of entropy current vanishes and, as expected, without dissipation fluid motion is adiabatic and reversible. However, for many physical systems perfect fluid approximation is not good enough and we need to include corrections coming from viscosity. To implement them in the above construction we have to add corrections to the energy-momentum tensor and currents.

2.1.2 Dissipative Fluid

Dissipative processes like viscosity or thermal conduction modify the equations of fluid dynamics. To see that we have to construct the form of energy-momentum tensor and the currents to the first order in the field derivatives. We denote the dissipative parts by $\tau^{\mu\nu}$, ν^μ, and σ^μ,

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}, \tag{2.12}$$

$$j^\mu = n u^\mu + \nu^\mu, \tag{2.13}$$

$$s^\mu = s u^\mu + \sigma^\mu. \tag{2.14}$$

There is one subtlety we have to be careful about, while considering first order corrections. Since we have a heat flow that results in a mass flow, the concept of velocity with respect to the mass flow of the fluid becomes ambiguous. The most frequently used reference frames are so-called Landau frame [10] and Eckart frame [51]. In Landau frame the fluid velocity is defined with reference to energy transport, while in Eckart frame with reference to charge transport. Throughout this Thesis we will use the Landau definition of velocity. It is obtained by imposing the following conditions

$$u_\mu \tau^{\mu\nu} = 0, \qquad u_\mu \nu^\mu = 0. \tag{2.15}$$

The motivation for these conditions is that we have always a freedom to choose velocity and temperature in such a way that $\tau^{0\mu}$ vanish. Moreover, the current associated with the particle number must equal the particle number density n in the proper frame. We find $\tau^{\mu\nu}$ and ν^μ from the requirement of the existence of an entropy current s^μ with non-negative derivative, $\partial_\mu s^\mu \geq 0$. We start with the relation

$$u_\nu \partial_\mu T^{\mu\nu} + \mu \partial_\mu j^\mu = 0, \tag{2.16}$$

and use the fact that if we were to obtain a thermodynamic potential with only intensive variables we would get identically zero

$$n d\mu = dp - s dT. \tag{2.17}$$

Following the procedure introduced in the previous subsection we obtain

$$\partial_\mu (s u^\mu) = -\frac{1}{T} u_\nu \partial_\mu \tau^{\mu\nu} - \frac{\mu}{T} \partial_\mu \nu^\mu. \tag{2.18}$$

Integrating by parts and using Eq. (2.15) we get the viscous contribution to the entropy current,

$$\partial_\mu (s u^\mu - \frac{\mu}{T} \nu^\mu) = \frac{1}{T} \tau^{\mu\nu} \partial_\mu u_\nu - \nu^\mu \partial_\mu \frac{\mu}{T}. \tag{2.19}$$

We can write down the most general expressions for the energy-momentum and the currents provided that there is no parity violating terms in our fluid,

$$\tau^{\mu\nu} = \eta P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) + \big(\zeta - \frac{2}{3}\eta_0\big) P^{\mu\nu} \partial \cdot u, \tag{2.20}$$

$$\nu^\mu = -\sigma T P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right), \tag{2.21}$$

$$\sigma^\mu = -\frac{\mu}{T} \nu^\mu, \tag{2.22}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and the entropy production rate is manifestly positive. However, this is not the end of the story. In principle we can add terms containing Levi-Civita symbol which contribute with both signs to the entropy current. We will show how to include them in this framework in the next section.

2.2 Hydrodynamics with triangle anomalies

Relativistic quantum field possesses a very special feature, the existence of anomalous non-conservation of axial currents due to the presence of triangle anomalies [52, 53] (see also [54]). For currents associated with global symmetries, the anomalies do not destroy current conservations, but are reflected in the three-point functions of the currents. When the theory is put in external background gauge fields coupled to the currents, some of the currents will no longer be conserved.

Dissipative hydrodynamics derived in the previous section does not contain parity violating terms. It is enough to describe many known fluids. However, if we take hydrodynamic limit of a theory containing chiral constituents such as QCD, we expect to have some ramifications of that in our description. Moreover, purely from hydrodynamic considerations it should be possible to add terms to the entropy current proportional to vorticity,

$$\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}u_\nu \partial_\lambda u_\rho. \tag{2.23}$$

This term contains one spatial derivative and it can affect viscosity and diffusion. Note, however, that vorticity does not have positive divergence and it can contribute to the entropy current with both signs. This is the reason why standard textbooks completely ignore it as not allowed by the second law of thermodynamics. In this section we will show an original result that this term is not only allowed but required if we include anomalies. Moreover, anomalous contribution will allow us to completely determine hydrodynamic coefficients associated with parity breaking terms.

Consider a relativistic fluid with U(1) anomaly. To constrain the hydrodynamic equation, we turn on a slowly-varying background gauge field A_μ coupled to the current j_μ. The strength of A_μ is of the same order as the temperature and the chemical potential, $A_\mu \sim O(p^0)$ and $F_{\mu\nu} \sim O(p)$. In presence of magnetic field the hydrodynamic equations get modified,

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda, \tag{2.24}$$

$$\partial_\mu j^\mu = C E^\mu B_\mu. \tag{2.25}$$

where electric and magnetic fields are defined in the fluid rest frame,

$$E^\mu = F^{\mu\nu}u_\nu, \tag{2.26}$$

$$B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}. \tag{2.27}$$

Because there is an external non-dynamical background field the energy-momentum tensor and current are not conserved. This is reflected by the right-hand sides of Eqs. (2.24) and (2.25). C denotes the anomaly coefficient dependent on quantum theory we consider. This coefficient is defined by the divergence of the gauge-invariant current in the presence of the external magnetic field,

$$\partial_\mu j^\mu = -\frac{1}{8}C\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{2.28}$$

The most general modification of the $U(1)$ and entropy currents in the presence of terms (2.23) and (2.27) is

$$\nu^\mu = -\sigma T P^{\mu\nu}\partial_\nu\left(\frac{\mu}{T}\right) + \sigma E^\mu + \xi\omega^\mu + \xi_B B^\mu, \tag{2.29}$$

$$s^\mu = s u^\mu - \frac{\mu}{T}\nu^\mu + D\omega^\mu + D_B B^\mu, \tag{2.30}$$

where ξ and ξ_B are two new transport coefficients associated with anomalous terms and D and D_B are two unknown functions. The entropy production contains terms with Levi-Civita symbol. This cannot be positive for general values of initial conditions. Therefore, we conclude that D and D_B are not arbitrary but they are highly constrained in such a way that non-positive divergence terms in the entropy production equation vanish. Using the following identities which relate $\partial_\mu \omega^\mu$ with ω^μ and $\partial_\mu B^\mu$ with B^μ,

$$\partial_\mu \omega^\mu = -\frac{2}{\epsilon + P}\omega^\mu(\partial_\mu P - nE_\mu), \tag{2.31}$$

$$\partial_\mu B^\mu = -2\omega \cdot E + \frac{1}{\epsilon + P}(-B \cdot \partial P + nE \cdot B), \tag{2.32}$$

one requires that the terms $\omega^\mu \partial_\mu$, $B^\mu \partial_\mu$, $\omega^\mu E_\mu$, and $\omega^\mu B_\mu$ cancel each other in Eq. (2.18). Hence, the following four relations have to be satisfied

$$\partial_\mu D - 2\frac{\partial_\mu P}{\epsilon + P} D - \xi \partial_\mu \frac{\mu}{T} = 0, \tag{2.33}$$

$$\partial_\mu D_B - \frac{\partial_\mu P}{\epsilon + P} D_B - \xi_B \partial_\mu \frac{\mu}{T} = 0, \tag{2.34}$$

$$\frac{2nD}{\epsilon + P} - 2D_B + \frac{\xi}{T} = 0, \tag{2.35}$$

$$\frac{nD_B}{\epsilon + P} + \frac{\xi_B}{T} - C\frac{\mu}{T} = 0. \tag{2.36}$$

It is convenient to change variables from μ, T to a new pair of variables, $\bar{\mu} \equiv \mu/T$ and P. From $dP = sdT + nd\mu$, integrating by parts we derive

$$\left(\frac{\partial T}{\partial P}\right)_{\bar{\mu}} = \frac{T}{\epsilon + P}, \qquad \left(\frac{\partial T}{\partial \bar{\mu}}\right)_P = -\frac{nT^2}{\epsilon + P}. \tag{2.37}$$

Writing $\partial_i D = (\partial D/\partial P)\partial_i P + (\partial D/D\bar{\mu})\partial_i \bar{\mu}$, and noting that $\partial_i P$ and $\partial_i \bar{\mu}$ can be arbitrary, as they can be considered as initial conditions on a time slice, Eq. (2.33) becomes two equations,

$$-\xi + \frac{\partial D}{\partial \bar{\mu}} = 0, \quad \frac{\partial D}{\partial P} - \frac{2}{\epsilon + P} D = 0. \tag{2.38}$$

Using Eq. (2.37), one finds that the most general solution to Eqs. (2.38) is

$$D = T^2 d(\bar{\mu}), \quad \xi = \frac{\partial}{\partial \bar{\mu}}\left(T^2 d(\bar{\mu})\right)_P, \tag{2.39}$$

where $d(\bar{\mu})$ is, for now, an arbitrary function of one variable. Equation (2.34) yields

$$D_B = T d_B(\bar{\mu}), \quad \xi_B = \frac{\partial}{\partial \bar{\mu}}\left(T d_B(\bar{\mu})\right)_P, \tag{2.40}$$

where $d_B(\bar{\mu})$ is another function of $\bar{\mu}$. From Eqs. (2.35) and (2.36) we get

$$d_B(\bar{\mu}) = \frac{1}{2}d'(\bar{\mu}), \qquad d'_B(\bar{\mu}) - C_{\text{anom}}\bar{\mu} = 0, \tag{2.41}$$

which can be integrated. We find

$$d_B(\bar{\mu}) = \frac{1}{2}C\bar{\mu}^2, \qquad d(\bar{\mu}) = \frac{1}{3}C\bar{\mu}^3. \tag{2.42}$$

So the new kinetic coefficients are

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\right), \tag{2.43}$$

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\right). \tag{2.44}$$

Let us comment on a physical significance of the above result. First of all, we have shown that there is no physical reason which disallow parity violating terms in the entropy current. Moreover, the presence of these terms leads to new hydrodynamic coefficients in the fluid. These coefficients are related to anomalies and, therefore, are independent of the coupling of underpinning quantum field theory. This independence allows to compute these coefficients from kinetic theory using perturbative methods [55]. An interesting observation is that even if the background field is turned off and there is no anomaly in the system the coefficient ξ related to vorticity knows what is the value of the anomaly coefficient.

A novel phenomenon predicted in this section may be relevant in heavy-ion collisions. The basic constituents of matter in QCD are chiral quarks. Since the microscopic theory is odd and we have very strong magnetic fields after the collision we expect that the hydrodynamic description should be anomalous. To get more physical intuition consider a volume of rotating quark matter, made of massless u and d quarks, at baryon chemical potential μ. For a moment let us neglect instanton effects, so the $U(1)_A$ current $j_5^\mu = \bar{q}\gamma^\mu\gamma^5 q$ is conserved. Because of the triangle anomaly in the three-point correlators of j_5^μ with two baryon currents, axial current will flow along the axis of rotation. This can be thought of as chiral separation, left- and right-handed quarks move with slightly different average momentum, creating an axial current. Measuring this flow through the transport coefficient ξ will be an interesting experimental challenge for the forthcoming LHC experiments. One can draw a parallel with the 'chiral magnetic effect', invoked to explain fluctuations of charge asymmetry in noncentral collisions [56, 57]. They should also affect the hydrodynamic behavior of a dense and hot neutrino gas, or of the early Universe with a large lepton chemical potential.

It is straightforward to extend the above reasoning to the case of multiple $U(1)$ commuting charges [35].

2.3 Bjorken hydrodynamics

We may represent a nucleus-nucleus collision as two discs passing through each other. Usually the z-axis is chosen to be the collision axis. If the thermalization time is short enough the transverse components of fluid velocity are almost zero. This comes from the fact that nuclei constituents, the so-called partons, collide on a very short transverse scales. There is an isotropy in the transverse plane, thus, transverse momentum averaged over a fluid element vanishes. Therefore, it is reasonable to drop off the transverse description from the fluid dynamics and consider (1+1) dimensional expansion in the (t,z) plane (see Fig. 2.2).

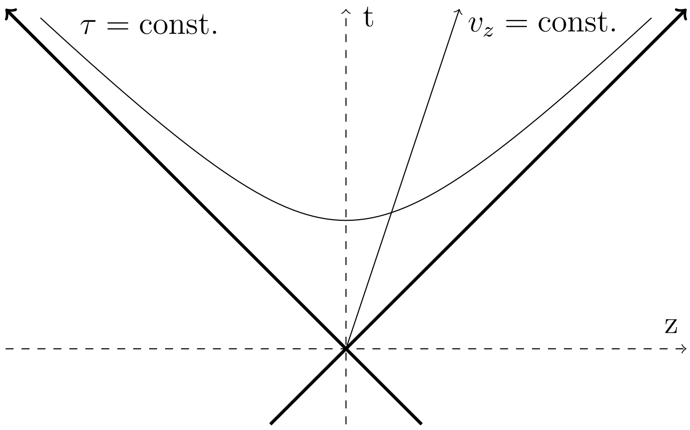

Figure 2.2: Nucleus-nucleus collision in the (z, t) plane. The thick lines represent the trajectories of the colliding nuclei. The hyperbola corresponds to a line with a constant proper time τ_0

The longitudinal motion of particles is uniform, with the velocity $v_z = z/t$, which we associate to a local fluid element. Moreover, if we perform Lorentz boost v_z remains unchanged. From that we conclude that there is a preferred set of coordinates,

$$t = \tau \cosh y, \qquad z = \tau \sinh y. \tag{2.45}$$

τ is called proper time and y is called rapidity. We can perform an inverse transformation which gives

$$\tau = \sqrt{t^2 - z^2}, \qquad y = \frac{1}{2} \ln \frac{t+z}{t-z}. \tag{2.46}$$

In these coordinates the ansatz for the local fluid velocity reads

$$u^\mu = (t/\tau, 0, 0, z/\tau) = (\cosh y, 0, 0, \sinh y). \tag{2.47}$$

This reduces Eq. (2.18) for the entropy production to a first order differential equation

$$\frac{\partial s(\tau)}{\partial \tau} = -\frac{s(\tau)}{\tau}. \tag{2.48}$$

We can easily solve the above equation to obtain an expression for the entropy

$$s(\tau) = \frac{s_{\text{initial}}}{\tau}. \tag{2.49}$$

Similarly, the Equation constraining the energy density becomes

$$\frac{\partial \varepsilon(\tau)}{\partial \tau} = -\frac{\varepsilon(\tau) + P(\tau)}{\tau}. \tag{2.50}$$

For the perfect fluid solution we impose the equation of state in the form

$$\varepsilon(\tau) = 3P(\tau). \tag{2.51}$$

This leads to the Bjorken solution of Eq. (2.50)

$$\varepsilon(\tau) = \frac{\varepsilon_{\text{initial}}}{\tau^{4/3}}. \tag{2.52}$$

For completeness of the discussion we may include the effect of dissipative corrections.

$$\tau^{00} = -\left(\frac{4}{3}\eta_0 + \zeta\right)\frac{\sinh^2 y}{\tau}, \tag{2.53}$$

$$\tau^{03} = -\left(\frac{4}{3}\eta_0 + \zeta\right)\frac{\sinh y \cosh y}{\tau}, \tag{2.54}$$

$$\tau^{33} = -\left(\frac{4}{3}\eta_0 + \zeta\right)\frac{\cosh^2 y}{\tau}, \tag{2.55}$$

$$\nu^{\mu} = 0. \tag{2.56}$$

We see that the shear and bulk viscosities always appear in combination $\frac{4}{3}\eta_0 + \zeta$ and there is no correction to the conserved current.

As a caveat we note that the first order relativistic hydrodynamics is not a causal theory. Therefore, in a series of papers Israel and Stewart constructed second order relativistic hydrodynamics, which seems to cure this undesirable feature [58, 59]. The resulting equations are hyperbolic and the signal propagation is causal. However, recent investigations show that the Israel-Stewart theory does not fully exploit symmetries and lacks some terms [60]. At the moment these effects are too small to be reliably tested. However, second order relativistic hydrodynamics can be used to study heavy-ion collisions [61], thus, an improvement of experimental methods may lead to measurements of second order transport coefficients.

We have shown that hydrodynamics provides a framework to analyze many features of heavy-ion collisions. However, we have to remember its limitations, since the initial non-equilibrium state lies outside the domain of validity. Hydrodynamics cannot tell use which value of τ we should use as a thermalization time and what is the temperature and baryon chemical potential for a given initial τ_0. Moreover, there is a set of transport coefficients which are free parameters in the theory. Therefore, they need to be adjusted to match experimental data. This is a natural procedure in physics, but it is always tempting to have some fundamental understanding of physical parameters. We pointed out that there are several microscopic models like the parton cascade models or the color glass condensate model which are currently being developed to improve the situation. However, we will not follow this path here. Instead we will employ the so-called gauge/gravity duality.

Chapter 3

AdS/CFT correspondence

Heavy-ion physics is a very rich playground to test theoretical models. As noted in the previous chapter those based on hydrodynamics work surprisingly well in certain regime after the collision. However, it is always tempting to go beyond effective description and try to understand physics from first principles. In the case of heavy-ion physics we would like to use perturbative methods of QCD. This is feasible only for a very limited number of processes, like deep inelastic scattering, due to the fact that, as recently indicated by experiments at RHIC, QGP formed after the collision is strongly coupled. This means that we cannot use methods based on the conventional expansions in coupling constant. We have to go beyond perturbative approach. However, we face many technical obstacles to overcome. As a consequence, at the moment, there are only few methods one can use. One is lattice field theory approach, in which space-time is discretised and one may answer physical questions using computers. However, because in real physical problems we have Minkowski space-time, there is complex weighting factor in the partition function. This makes many results unreliable in the lattice simulations. To say something about strongly coupled real-time field theory one usually gives up on QCD and studies a simpler theory, the so-called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) using gauge/gravity duality. At first it seems that this change is not useful for real physics. However, we may hope that there are certain universal properties of strongly coupled field theories or, at least, that one can approximate some phenomena in QCD using SYM theory.

3.1 $\mathcal{N} = 4$ Super Yang-Mills

QCD is a gauge theory with the gauge group SU(3). 't Hooft noticed that the theory simplifies if we generalize the gauge group to be SU(N_c), take N_c to be large and expand in the powers of $1/N_c$. In this limit the planar diagrams dominate the contribution to the Feynman path integral. For further simplification we assume that the field theory we consider is supersymmetric. This will lead to strong constraints on the theory, however, this step makes it also less realistic. In other words we don't know if there is any connection to QCD or how to compare physical results in both theories. Nevertheless, we may treat it as a starting point for construction of some more realistic models. We will review some basic properties of $\mathcal{N}=4$ SYM. The lagrangian is uniquely fixed by supersymmetry and have the following schematic form,

$$\mathcal{L}=\frac{1}{g_{YM}^2}\mathrm{Tr}_{N_c}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\mathcal{D}_\mu\Phi^a\mathcal{D}^\mu\Phi^a+\sum_{a,b}[\Phi^a,\Phi^b]^2+\text{fermions}\right]. \tag{3.1}$$

We set θ angle to zero. The field content of $\mathcal{N}=4$ gauge supermultiplet includes the gauge fields A_μ, four Weyl fermions and six real scalars Φ^a. Supersymmetry requires that all fields must transform in the same representation of the gauge group, namely the adjoint representation, and all must have the same mass. By gauge invariance, a mass for the gauge fields is zero, hence, the fermion and scalar fields are massless as well. Moreover, there is only one coupling constant g_{YM} which controls all interactions in the theory. Usually one combines the two parameters g_{YM} and N_c into a combination $\lambda=g_{YM}^2N_c$ known as 't Hooft coupling constant.

The bosonic part of the global symmetry group is $SO(4,2)\times SO(6)$. The first factor is the conformal group in four dimensions which includes $SO(3,1)$ group of Lorentz transformations as a subgroup. The second factor is the so-called global R-symmetry group $SO(6)\simeq SU(4)$. The bosonic part of symmetry group is supplemented with fermionic supersymmery transformations. Lagrangian (3.1) is invariant under the group $PSU(2,2|4)$. Let us focus on the bosonic subgroup $SO(2,4)$. This group in known in the literature as a conformal group in four dimensions [62]. This means that in addition to Poincaré invariance, we have scale transformations or dilatations and so-called special conformal transformations. Apart from being supersymmetric, SYM is a conformal field theory (CFT). This resembles massless QCD which is also invariant under scale transformations. However, there is a big difference here, unlike QCD SYM remains scale invariant even at the quantum level, whereas in massless QCD quantum corrections break scale invariance explicitly. This is manifested through a non-vanishing beta function. In contrast in $\mathcal{N}=4$ SYM beta function vanishes

$$\beta(g)=\mu\frac{\partial g(\mu)}{\partial\mu}=0. \tag{3.2}$$

As a consequence the dimensionless coupling g_{YM} does not run and the $SO(4,2)$ conformal invariance of the classical theory is unbroken. This gives a powerful constraint on the dynamics of the theory. The two-point correlation functions of operators of definite scaling dimension are completely fixed, and the three-point functions of such operators are fixed up to some overall constants.

$\mathcal{N}=4$ SYM has one more feature which makes it very interesting to study. As we will see later it can be reformulated as a weakly coupled string theory. Presumably this is also true for other field theories, even more realistic ones like QCD, but SYM is the first and the best understood example. Such a reformulation should give an insight in a strongly coupled regime of a field theory. To make these statements more precise let us review some basic facts about string theory. More general treatments can be found in [63–67]

3.2 String theory basics

String theory describes the dynamics of one-dimensional objects moving in a D-dimensional space-time. These objects come in two species, open strings and closed strings. The action for a relativistic string is proportional to the area that string sweeps out in space-time. This area is called world-sheet. It is convenient to parametrize the world-sheet by introducing parameters $\sigma = \sigma + 2\pi$ and τ. One can map these world-sheet coordinates to space-time coordinates $(\sigma, \tau) \mapsto X^\mu(\sigma, \tau)$ and write the action in the Nambu-Goto form,

$$S_{NG} = -T \int d^2\sigma [(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2]^{1/2}. \tag{3.3}$$

where T is a string tension. The dot denotes derivative with respect to τ and the prime denotes derivative with respect to σ. It is easily seen that Eq. (3.3) is non-linear. This fact creates substantial difficulties during quantization process. In order to circumvent them one usually rewrites the Nambu-Goto action in the so-called Polyakov form by introducing an auxiliary metric field on the world-sheet,

$$S_P = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{3.4}$$

From the condition that energy-momentum tensor vanishes one obtains a constraint on the auxiliary field $h^{\alpha\beta}$, which plugged back into Eq. (3.4) shows classical equivalence between S_P and S_{NG}. S_P is quadratic, hence, one can carry out a quantization procedure. However, we only know how to perform quantization of one excited string, which gives some spectrum later promoted to fields. At the moment we don't have a proper understanding of quantum interacting string field theory, where strings are excitations of an underlying field.

Quantization of the Polyakov action gives a spectrum which contains a negative mass state. This state is known as tachyon and its appearance in the spectrum indicates that there is unstable vacuum in the theory. In order to have a stable theory one usually requires supersymmetry. Since we have only bosonic degrees of freedom so far we need to generalize the action (3.4) by adding fermions on the world-sheet,

$$S_P = -\frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu, \tag{3.5}$$

where ρ^α, with $\alpha = 0, 1$, represent the two-dimensional Dirac matrices, which obey the Clifford algebra,

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}. \tag{3.6}$$

Because ψ^μ appear in the action quadratically we can change the sign without changing the physical content. Therefore, the full open superstring theory state spectrum breaks into two subspaces or sectors, a Ramond (R) sector which contains the states that arise using $+\psi^\mu$ for quantization, and a Neveu-Schwarz (NS) sector which contains the states that arise using $-\psi^\mu$. For closed strings we have a slight complication, since the spectrum in this case is composed multiplicatively of left and right open string modes. Hence, we have four sectors, which are usually labeled as (NS-NS), (NS-R), (R-NS), (R-R).

We demanded that the string theory spectrum is supersymmetric. However, this is not the case so far and we still have a tachyonic state in the spectrum. To cure that we have to truncate the spectrum in a way that eliminates the tachyon and leads to a supersymmetric theory. This procedure is known as GSO projection. For the oriented superstrings, in which we are interested in, we can perform the GSO projection in two ways, which lead to the so-called type IIA and type IIB string theory. In both cases the massless fields in the (NS-NS) sector are the same and include a graviton $g_{\mu\nu}$, an antisymmetric Kalb-Ramond field denoted as $B_{\mu\nu}$, and a dilaton ϕ. However, fields coming from (R-R) sector are different. In the type IIA theory the massless (R-R) bosons include a Maxwell field A_μ and a three-index antisymmetric gauge field $A_{\mu\nu\rho}$. In the type IIB theory the massless (R-R) bosons include a scalar field A, a Kalb-Ramond field $A_{\mu\nu}$, and a totally antisymmetric gauge field $A_{\mu\nu\rho\lambda}$. In a complete analogy with Maxwell electrodynamics, where a gauge field couples to a world-line of a charged particle, the existence of the multi-indexed gauge fields in the spectrum indicates that there are higher-dimensional objects in the theory. We call them D-branes.

Apart from massless excitations, string theory spectrum possesses a whole tower of massive modes. However, we will be interested in the low energy limit of the string theory or length scales that are much bigger than the fundamental string scale. In this limit massive modes decouple by acquiring infinite masses. Therefore, we don't have to include them in the analysis. Moreover, the action constructed for massless sector exactly match supergravity action. We will consider the low-energy effective description of string theory by means of supergravity.

3.3 Type IIB supergravity

Supergravity (SUGRA) is a vast field. We will review only a few basic facts. For more complete treatments see [68–71]. There is a unique theory of supergravity in eleven dimensions. One can compactify this eleven-dimensional theory in one direction obtaining ten-dimensional supergravities. These are known as type IIA or type IIB supergravity theories, which are the only maximally supersymmetric $\mathcal{N} = 2$ theories in ten dimensions. For our later purposes we will be interested mostly in type IIB theory. The field content of that theory is given in Table 3.1. We note that type IIB SUGRA has gravitini of the same chirality. Later we will take into account mostly the bosonic degrees of freedom setting gravitini and dilatini to zero.

Since the theory contains an antisymmetric field C_4 with self-dual field strength, it is difficult to construct a satisfactory action from which all equations of motion follow. However, it is feasible to find an action involving C_4 and then augment it with an additional duality condition $\tilde{F}_5 = *\tilde{F}_5$

Type IIB Supergravity particle content		
Symbol	#DOF	Field
G_{AB}	35_B	metric - graviton
$C + i\varphi$	2_B	axion - dilaton
$B_{AB} + iC_{2\,AB}$	56_B	antisymmetric rank 2
$C_{4\,ABCD}$	35_B	antisymmetric rank 4
$\psi^{1,2}_{A\alpha}$	112_F	two Majorana-Weyl gravitini
$\lambda^{1,2}_{\alpha}$	16_F	two Majorana-Weyl dilatini

Table 3.1: IIB SUGRA particle content

$$
\begin{aligned}
S_{IIB} \quad = \quad & \frac{1}{4\kappa^2}\int \sqrt{G}e^{-2\Phi}(2R + 8\partial_\mu\Phi\partial^\mu\Phi - |H_3|^2) && (3.7)\\
& -\frac{1}{4\kappa^2}\int\left[\sqrt{G}(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2) + C_4\wedge H_3\wedge F_3\right] + \text{fermions}\,.
\end{aligned}
$$

κ is ten-dimensional Newton's constatnt R is Ricci scalar and the field strengths are defined by

$$F_1 = dC\,, \quad H_3 = dB, \tag{3.8}$$

$$F_3 = dC_2\,, \quad \tilde{F}_3 = F_3 - CH_3, \tag{3.9}$$

$$F_5 = dC_4\,, \quad \tilde{F}_5 = F_5 - \tfrac{1}{2}A_2\wedge H_3 + \tfrac{1}{2}B\wedge F_3\,. \tag{3.10}$$

There is a very important class of solutions to the equations of motion coming from Eq. 3.7. These solutions are called p-branes. Their geometry is determined by the following general ansatz,

$$ds^2 = H(y)^\alpha dx^\mu dx_\mu + H(y)^\beta(dy^2 + y^2 d\Omega^2_{8-p}), \tag{3.11}$$

where α and β are constants determined from the equations of motion. x refers to coordinates on the world-volume of a brane, while y to coordinates transversal to the brane. Additionally they have a non-trivial C_{p+1}. p-branes are postulated to have natural counterparts in string theory, D-branes.

Among brane solutions of supergravity the $D3$-brane solution is of utmost importance. First of all, its world-volume has 4-dimensional Poincaré invariance with regularity at $y = 0$. Moreover, it has constant axion and dilaton fields and it is self-dual. More specifically

$$
\begin{aligned}
ds^2 &= H(y)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(y)^{1/2}(dy^2 + y^2 d\Omega^2_5),\\
&\exp(\Phi) = \text{const}, \qquad C = \text{const},\\
&B_{AB} = C_{2,AB} = 0, && (3.12)
\end{aligned}
$$

$$C_4 = H(y)^{-1} dx^0 \wedge \cdots \wedge dx^3,$$
$$H(y) = 1 + \sum_i \frac{L^4}{|\vec{y} - \vec{y}_{D3}|^4}, \qquad L^4 = 4\pi g_s N \alpha'^2.$$

Later we will argue that this solution is a basic ingredient of the gauge/gravity duality.

3.4 Branes

Strings are not the only objects in string theory. The theory also contains higher-dimensional objects called D-branes (for more exhaustive reviews see [72, 73]). They are usually classified according to the energy per unit volume - tension. We have two types of them, the solitonic Neveu-Schwarz (NS) branes, whose tension is proportional to $1/g_s^2$ and Dirichlet (D) branes, whose tension behaves like $1/g_s$. In the limit of weakly coupled strings ($g_s \to 0$) D-branes are much more dominant.

In weakly coupled type IIB string theory D-branes are viewed as hypersurfaces on which strings can end. One can see that by varying a string action. In addition to the equations of motion, there is the boundary term

$$-T \int d\tau \left(X'_\mu \delta X^\mu \mid_{\sigma=\pi} - X'_\mu \delta X^\mu \mid_{\sigma=0} \right), \tag{3.13}$$

which we demand to vanish. For closed strings the embedding functions are periodic and the boundary term vanishes identically. However, for open strings we have two possibilities. Either the component of the momentum normal to the boundary of the world-sheet vanishes

$$X'_\mu = 0 \quad \text{at} \quad \sigma = 0, \pi, \tag{3.14}$$

or the position of the string ends is fixed so that $\delta X^\mu = 0$, and

$$X^\mu \mid_{\sigma=0} = X_0^\mu = \text{const} \quad \text{and} \quad X^\mu \mid_{\sigma=\pi} = X_\pi^\mu = \text{const}. \tag{3.15}$$

The first choice is called Neumann boundary conditions, the second Dirichlet boundary conditions. Dirichlet boundary conditions break Poincaré invariance and for this reason they were not considered for many years. However, in the modern interpretation these boundary conditions indicate that strings are attached to some objects. The most natural choice is to associate them with D-branes. Moreover, the low-energy limit of them is believe to give exactly p-brane solutions which we faced in previous section. Therefore, from now on we will call them Dp-branes (see Fig. 3.1).

Dp-brane stretched in the $(X^1, \ldots, X^p)$ hyperplane, located at a point in $(X^{p+1}, \ldots, X^9)$, is then defined by including in the theory open strings with Neumann boundary conditions for $(X^0, \ldots, X^p)$ and Dirichlet boundary conditions for $(X^{p+1}, \ldots, X^9)$. The Dp-brane can be a source for various charges. It is coupled to (R-R) $(p+1)$-form potentials. In type IIA there are potentials with even p and in type IIB with odd p. Thus, we conclude that there are branes with even p in type IIA string theory and with odd p in type IIB.

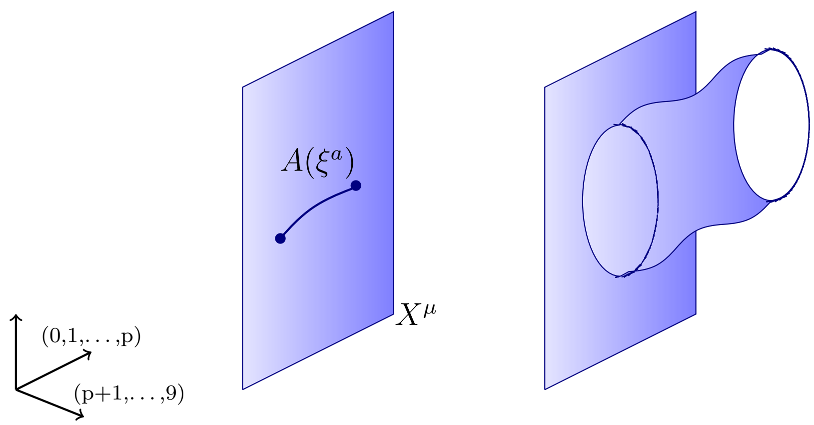

Figure 3.1: Pictorial representation of brane-string coupling. Left configuration shows an open string attached to a brane. After quantization we are left with a gauge field living on a world-volume and scalar fields denoting a position in space-time. Right configuration shows an emission of a closed string by a brane.

The action for a Dp-brane is known in the leading order in coupling constant. It is a natural generalization of the Nambu-Goto action for the string. We note that, since a D-brane is stable BPS saturated object, it has to be supersymmetric. However, we will be mostly interested in the bosonic degrees of freedom living on the world-volume, so we will skip fermionic contribution. Schematically we can write an action for a D-brane as the so-called Dirac-Born-Infeld part, which is simply an extension of two dimensional area to higher-dimensional volume plus a Wess-Zumino term which follows from the coupling to the $(p+1)$-form,

$$S_{Dp} = S_{DBI} + S_{WZ}. \tag{3.16}$$

For our purposes we can assume that the background potentials vanish so we don't need the exact form of S_{WZ}. Let us briefly discuss how to constrain the form of S_{DBI}. D-brane is a dynamical object, whose location and shape is governed through the interaction of open strings with background fields. To describe that it is convenient to introduce world-volume coordinates ξ^a, where a runs from 0 to p. If we embed the brane in some higher dimensional background metric field, we will have some map $X^\mu(\xi^a)$. We can define an induced metric on a brane or a pullback,

$$P[G_{ab}] = G_{\mu\nu}\frac{\partial X^\mu}{\partial \xi_a}\frac{\partial X^\nu}{\partial \xi^b}. \tag{3.17}$$

We emphasize that in a probe limit it is not dynamical. In addition to the metric field we expect to have a gauge field as well. This can be easily understood from the fact that we have a bunch of open strings ending on a brane. Because they have tension they tend to minimize the world-sheet area by shrinking to zero length. The low-energy description of such massless strings is a gauge field living on a brane. So the low-energy action takes the form

$$S_{Dp} = T_{Dp} \int d^{(p+1)}\xi \sqrt{-\det(P[G_{ab}] + 2\pi\alpha' F_{ab})}. \tag{3.18}$$

T_{Dp} is the tension of a Dp-brane. It has dimension of energy per unit volume

$$[T_{Dp}] = \frac{\text{mass}}{\text{length}^p} = (\text{length})^{-p-1}, \tag{3.19}$$

and in terms of fundamental parameters it can be written as

$$T_{Dp} = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}}. \tag{3.20}$$

Dp-branes have a non-zero energy per unit volume. Therfore, they are gravitating objects because, as we know from General Relativity, everything that has non-zero energy couples to gravity. In string theory the force of gravity is govern by closed strings. It means that Dp-branes can emit and absorb closed strings, as shown on Figure 3.1. The studies of gravitational physics of D-branes motivated the original statement of AdS/CFT correspondence. We will come back to that in the next section.

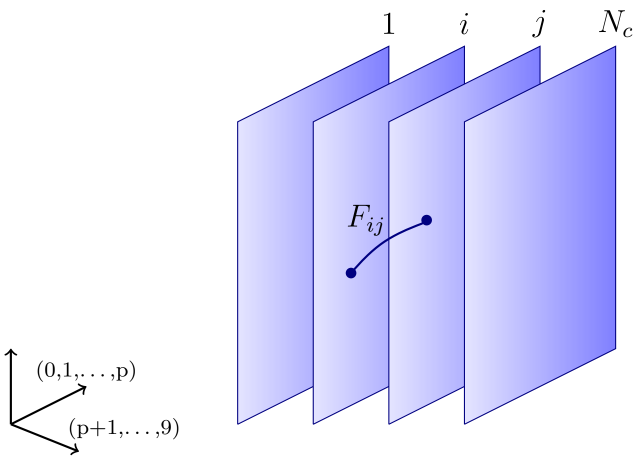

Figure 3.2: Pictorial representation of a stack of N_c Dp-branes with an attached open string.

$\alpha' \to 0$ limit of (3.18) reduces to $U(1)$ gauge symmetry in $p+1$ dimensions. In this way we may construct a $p+1$-dimensional gauge theory from string theory in 10 dimensions. One can generalize this construction to the non-abelian case by taking a stack of N_c Dp-branes placed on top of each other so that the distance between them goes to zero (see Fig. 3.2). In every point of the world-volume theory, there can be an oriented string starting from a brane and ending on some brane from the stack. From that we expect that there is a non-abelian gauge boson in every point in the theory in the adjoint representation of the $U(N_c)$ group. However, we are interested in the relative position of the branes in the stack. This implies a global $U(1)$ symmetry in a gauge theory which we can decouple leaving $SU(N_c)$ symmetry.

3.5 The Maldacena conjecture

Motivated by the D-brane physics we will show that there is a connection between quantum field theories and classical supergravity. This connection was first realized within type IIB string theory and we will review this construction in its simples form. Consider N_c parallel $D3$-branes on top of each other. The $D3$-branes are extended along a $(3+1)$ dimensional plane in $(9+1)$-dimensional space-time. String theory on this background contains two kinds of perturbative excitations, closed strings and open strings. The argument leading to the correspondence is very simple, $D3$-branes have two different low-energy descriptions, one in terms of open strings and one in terms of closed strings. Following Maldacena [19] we will conjecture that these two different descriptions are equivalent. To be more precise we write down the action for the system of interacting $D3$-branes. Schematically it takes the following form,

$$S_{\text{total}} = S_{SUGRA} + S_{DBI}, \tag{3.21}$$

where S_{SUGRA} represents the dynamics of closed string modes and S_{DBI} describes the brane system with attached open strings coupled to gravity. Expanding this effective action around flat background and taking the low-energy limit (this means we keep all dimensionless parameters fixed while taking $\alpha' \to 0$), all interaction terms vanish. We get $\mathcal{N} = 4$ SYM theory, with $g_{YM}^2 = g_s$ and free supergravity.

$\mathcal{N} = 4$ SYM is not the only low-energy limit ($\alpha' \to 0$) of $D3$-brane system. As noted previously $D3$-brane is a solution of supergravity (3.12). To see that it is convenient to shift a coordinate system such that the branes are located at the origin of the new coordinate system. Then $y_{D3} = 0$ and we introduce the distance from the branes $r = |\vec{y}|$. The metric generated by a stack of $D3$-branes may be rewritten as

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} dx^\mu dx_\mu + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2). \tag{3.22}$$

For an observer at infinity, far away from the branes, at $r \gg L$, the space-time becomes a ten-dimensional Minkowski space-time. Close to the branes, for $r \ll L$, we can neglect the '1' in the above metric

$$ds^2 = \frac{r^2}{L^2} dx^\mu dx_\mu + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2, \tag{3.23}$$

and identify the emergent geometry as a product of two spaces, five dimensional anti-de Sitter space AdS_5, with a five-dimensional sphere S^5, both with radius L. Again, we have two distinct sets of modes, those propagating in the Minkowski space and those propagating in the 'throat' region where the geometry is $AdS_5 \times S^5$. These two sets of modes decouple from each other in the low-energy limit. Far away from the throat only massless modes survive, while in the throat there is a whole tower of massive modes which cannot climb up the gravitational potential. Because we have two distinct descriptions of the same $D3$-brane system we expect that they are equivalent, meaning that four-dimensional $\mathcal{N} = 4$ SYM theory is the same as type IIB string theory propagating on $AdS_5 \times S^5$.

An important aspect of the above equivalence is that this is the so-called weak-strong coupling duality. For $\mathcal{N} = 4$ SYM we can trust the perturbative analysis if

$$\lambda = g_{YM}^2 N_c = g_s N_c \sim \frac{L^4}{\alpha'^2} \ll 1, \tag{3.24}$$

whereas the semiclassical approximation to supergravity is a good description if

$$1 \ll \frac{L^4}{\alpha'^2} \sim g_s N_c = g_{YM}^2 N_c. \tag{3.25}$$

We see that the perturbative field theory is mapped to non-perturbative gravity and vice versa. This makes the conjecture very hard to prove, but potentially very useful, because we may learn about quantum gravity by looking at the perturbative SYM and we can study non-perturbative field theory by means of classical gravity. Thus, it is difficult to have quantitative comparison between observables on both sides of the correspondence. In fact we don't know if the correspondence is valid in general for all values of N_c or only in a certain limit. In this Thesis we assume the correspondence for large 't Hooft coupling. Then the SYM is mapped to supergravity on $AdS_5 \times S^5$.

3.5.1 Symmetries

Having described heuristic arguments for the correspondence we would like to give some more detailed comparison of both sides of the correspondence and eventually give some prescriptions how to calculate physical observables in the dual language. We first focus on the comparison of symmetries between $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$. We wrote previously that the global symmetry group of the SYM theory is $PSU(2,2|4)$, which contains the bosonic part $SO(2,4) \times SO(6)$. We immediately identify the $SO(6)$ factor with the symmetry group of five dimensional sphere on the dual side. Consequently, the $SO(2,4)$ invariance is mapped to AdS_5 geometry. There is a perfect agreement, which we can easily see if we embed the anti-de Sitter space in a six-dimensional flat manifold,

$$-Y_0^2 - Y_5^2 + \sum_{i=1}^{i=4} Y_i^2 = L^2. \tag{3.26}$$

The metric element in the embedding coordinates reads

$$ds^2 = dY_0^2 + dY_5^2 - \sum_{i=1}^{i=4} dY_i^2. \tag{3.27}$$

Further insight in the symmetry properties can be acquired by a suitable choice of coordinates. One possible choice is the Poincaré coordinates (z, t, x_i), which cover one-half of the hyperboloid (3.26), related to the embedding coordinates by the following transformation,

$$Y_0 = -\frac{1}{2}z\left[1+\frac{1}{z^2}(L^2+\mathbf{x}^2-t^2)\right], \tag{3.28}$$

$$Y_i = -\frac{Lx_i}{z}, \quad (i=1,\ldots,3), \tag{3.29}$$

$$Y_4 = -\frac{Lt}{z}, \tag{3.30}$$

$$Y_5 = -\frac{1}{2}z\left[1-\frac{1}{z^2}(L^2-\mathbf{x}^2+t^2)\right]. \tag{3.31}$$

In the above transformation z plays a role of radial coordinate r from Eq. (3.23). We note that there are two common choices of radial coordinate in the literature,

$$z=\frac{L^2}{r} \quad \text{and} \quad u=\frac{r}{L^2}. \tag{3.32}$$

z has dimension of length, whereas u has dimension of energy. The line element in Poincaré coordinates reads

$$ds^2_{AdS}=\frac{L^2}{z^2}(-dt^2+d\vec{x}^2+dz^2)+L^2d\Omega_5^2. \tag{3.33}$$

Using Eq. (3.33) we can investigate the action of dilatation. On the field theory side, the beta function vanishes and the dilatation is a symmetry. On the gravity side, a dilatation should be a coordinate rescaling. We see that if we employ a transformation

$$(t,\vec{x},z)\to(ct,c\vec{x},cz), \quad c>0, \tag{3.34}$$

then the metric is preserved. Thus, we can associate dilation symmetry of $\mathcal{N}=4$ SYM with an isometry od AdS_5. Moreover, from the Eq. (3.33) we see that AdS_5 is conformal to the $z>0$ half of the five-dimensional Minkowski space. Taking the limit $z\to 0$ we reach the boundary od anti-de Sitter space. Sometimes people refer to that boundary as a place where the gauge theory is located, while string theory is in the bulk of AdS, however, we emphasize that it is more accurate to say that string theory on $AdS_5\times S^5$ is equivalent to $\mathcal{N}=4$ SYM on $\mathbf{R}^{3,1}$

3.5.2 The field/operator correspondence

We will now make the Maldacena conjecture more precise and show how it can be used to extract information about strongly coupled gauge theories [20, 21]. The $\mathcal{N}=4$ SYM is a conformal field theory. In a field theory one would like to know the correlation functions of gauge invariant operators. This is encoded in the generating functional

$$Z_{CFT}=\langle e^{\int d^4x\phi_0(x)\mathcal{O}(x)}\rangle. \tag{3.35}$$

The n-point correlation function can be computed by means of functional derivatives with respect to a source field $\phi_0(x)$,

$$\langle\mathcal{O}(x_0)\mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{\delta}{\delta\phi_0(x_0)}\frac{\delta}{\delta\phi_0(x_1)}...\frac{\delta}{\delta\phi_0(x_n)}Z_{CFT}\Big|_{\phi_0=0}. \quad (3.36)$$

To make AdS/CFT correspondence useful we have to propose a procedure for calculating such correlation functions from string theory. To do so it is actually convenient to think of conformal field theory as living on the boundary of anti-de Sitter space. We consider five-dimensional AdS space which has a four-dimensional conformal boundary. We introduce some field ϕ in AdS space a boundary value ϕ_0 and define a generating functional on the boundary by Eq. (3.35). Moreover, we have some string partition function Z_{string} supplemented with the boundary condition that at infinity ϕ approaches a given function ϕ_0. AdS/CFT correspondence identifies the string partition function Z_{string} with the generating function of correlators of $\mathcal{O}(x)$ in $\mathcal{N}=4$ super Yang-Mills living on the boundary of AdS_5 [21],

$$Z_{string} = Z_{CFT} \equiv \langle e^{\int d^4x\phi_0(x)\mathcal{O}(x)}\rangle. \quad (3.37)$$

Throughout this Thesis we will be working in the supergravity approximation, in which

$$e^{-S_{sugra}} \approx Z_{string}. \quad (3.38)$$

If classical supergravity is not an adequate approximation, then one has to include string theory corrections, or include quantum corrections. To illustrate the above ideas let us focus on the dynamics of the real scalar field φ described by the action

$$S_{sugra}(\varphi) = \frac{1}{2}\int d^4x dz\sqrt{g}[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2]. \quad (3.39)$$

To evaluate the field φ in the generating functional we have to solve the equation of motion

$$(\Box - m^2)\varphi = 0. \quad (3.40)$$

Supergravity fields encode actually two field theoretic objects, whose conformal dimension can be read off from the asymptotic behavior. For the scalar field we have

$$\varphi(z=0) \sim \varphi_0 z^{4-\Delta} + \langle\mathcal{O}\rangle z^{\Delta}. \quad (3.41)$$

The first, non-normalisable mode corresponds to a source and has conformal dimension $4-\Delta$. The normalisable part yields the corresponding VEV of conformal dimension Δ. Moreover, we get a relation between mass and conformal dimension,

$$\Delta(\Delta-4) = m^2. \quad (3.42)$$

The solution of the Eq. (3.40) can be found using bulk-to-boundary Green's function. Next, we can express the classical action S as a functional of φ_0 and calculate the two-point function using the prescription (3.36) [74, 75]. However, there is a caveat here. The quantities related by Eq. (3.37) are, in general, divergent. In order to have renormalized correlation functions we have to introduce some consistent regularization and renormalization scheme.

3.5.3 Holographic renormalization

A general renormalization scheme widely used in the context of AdS/CFT correspondence is the so-called 'holographic renormalization' [76–78] (see [79] for a review). We will recall it briefly. Suppose we have a metric that is a solution to Einstein's equation with negative cosmological constant and asymptotically approaches AdS space. Evaluation of the Einstein-Hilbert action on such solution will lead to infinity because of infinite range of integration. In holographic renormalization, we regulate the action by introducing a cut-off at $z = \epsilon$. We then add counterterms on the $z = \epsilon$ hypersurface to cancel those divergences coming from integration. A crucial step in the procedure is the choice of the coordinates that allows us to write the background metric as

$$ds^2 = g^{5D}_{MN} dx^M dx^N = \frac{g_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}. \tag{3.43}$$

Such coordinates in which the metric has the form (3.43) are called Fefferman-Graham [80] coordinates. By construction, $g_{\mu\nu}$ has a smooth limit as $z \to 0$, and can be Taylor expanded as

$$g_{\mu\nu}(z,x) = g^{(0)}_{\mu\nu}(x) + z g^{(1)}_{\mu\nu}(x) + z^2 g^{(2)}_{\mu\nu}(x) + z^4 g^{(4)}_{\mu\nu}(x) + h^{(4)}_{\mu\nu} \log(z^2) + \ldots . \tag{3.44}$$

Explicit computation shows that in pure gravity all coefficients multiplying odd powers of z vanish. $g^{(0)}_{\mu\nu}(x)$ is the physical metric of a boundary gauge theory. $g^{(2)}_{\mu\nu}(x)$ follows from the Einstein's equations. However, $g^{(4)}_{\mu\nu}$ is arbitrary. In order to generate a solution one has to give both $g^{(0)}_{\mu\nu}(x)$ and $g^{(4)}_{\mu\nu}(x)$. An important feature of this coordinate system is that $g^{(4)}_{\mu\nu}(x)$ is related to the boundary theory energy-momentum tensor [76]

$$T_{\mu\nu} = \frac{N_c^2}{2\pi^2} g^{(4)}_{\mu\nu}(x). \tag{3.45}$$

Given a boundary theory $T_{\mu\nu}$ and the metric of four-dimensional space-time we can construct its gravity dual perturbatively, since all higher order coefficients in (3.44) follow.

In other coordinate systems we can relate the energy-momentum tensor to the extrinsic curvature [81] (see also [82]) through

$$T_{\mu\nu} = -2 \lim_{r\to\infty} r^4 (K_{\mu\nu} - K\delta_{\mu\nu}), \tag{3.46}$$

where K is the trace of the extrinsic curvature.

Now, we include some matter content coupled to gravity. Near the boundary, each field has an asymptotic expansion of the form

$$\mathcal{F}(z,x) = z^{2m} \left[f^{(0)}(x) + z^2 f^{(2)}(x) + \ldots + z^{2n}(f^{(2n)}(x) + \log(z^2)\tilde{f}^{(2n)}) + \ldots \right]. \tag{3.47}$$

We interpret the boundary field $f^{(0)}(x)$ as the source for the dual operator. $f^{(2k)}(x)$, $k < n$ can be obtained iteratively in terms of $f^{(0)}(x)$ by solving field equations. These equations leave $f^{(2n)}(x)$ undetermined. The logarithmic term in (3.47) is related to conformal anomalies of the dual theory, and it is also fixed in terms of $f^{(0)}(x)$.

The most general asymptotic solution of the field equations allows us to calculate the on-shell value of the action,

$$S_{\text{reg}}[f^{(0)};\epsilon] = \int_{z^2=\epsilon} \mathrm{d}^4x\sqrt{g^{(0)}}[\epsilon^{-\nu}a^{(0)} + \epsilon^{-(\nu+1)}a^{(2)} + ... - \log\epsilon\ a^{(2\nu)} + \mathcal{O}(\epsilon^0)]. \tag{3.48}$$

where ν is a positive number that only depends on the scale dimension of the dual operator and $a^{(2k)}$ are local functions of the sources $f^{(0)}$. Last thing to do is to subtract the divergent part and take the limit $\epsilon \to 0$. The counterterms depend on the fields located on the surface $z^2 = \epsilon$, characterized by the induced metric $\gamma_{\mu\nu}/\epsilon$. The renormalized action reads

$$S_{ren}[f^{(0)}] = \lim_{\epsilon\to 0}\left(S_{reg}[f^{(0)};\epsilon] + S_{ct}\left[\mathcal{F}(x,\epsilon);\epsilon\right]\right). \tag{3.49}$$

To get the explicit form of $S_{ct}\left[\mathcal{F}(x,\epsilon);\epsilon\right]$ we have to express $f^{(0)}$ as a function of $\mathcal{F}(z,x)$, calculate $a^{(2k)}(f^{(0)})$, and determine the divergent part. We show how this method works in practice in Chapter 5, regularizing the $D7$-brane action.

3.6 Gauge/gravity duality at finite temperature

So far we have discussed a system of extremal $D3$-branes which is dual to $\mathcal{N}=4$ SYM theory at zero temperature. In order to generalize that to the case with finite temperature we need to find a gravitational solution with a scale that may correspond to temperature and some notion of entropy. This solution is known to be AdS-Schwarzchild

$$ds^2 = \frac{r^2}{L^2}(-f(r)dt^2 + dx^i dx_i) + \frac{L^2}{f(r)r^2}dr^2 + L^2 d\Omega_5^2, \tag{3.50}$$

where $f(r) = 1 - \frac{R^4}{r^4}$. The solution (3.50) describes the near-horizon geometry of near-extremal $D3$-branes. It can be depicted as a black hole located in the radial coordinate with the horizon at R. We interpret this black hole as a thermodynamical system with energy

$$E = \frac{A}{2G}, \tag{3.51}$$

and Hawking temperature

$$T_H = \frac{R}{\pi L^2}. \tag{3.52}$$

Having a field theory at finite temperature we are interested in thermodynamic quantities. Performing the Wick rotation $t \to -it_E$, the Euclidean path integral yields a thermal partition function. Furthermore, the Euclidean black hole solution is interpreted as a saddle-point in this path integral and the supergravity action evaluated for this solution is interpreted as the leading contribution to the free energy. For the $D3$-brane system we get

$$F = TS_{sugra} = -\frac{\pi^2}{8} N_c^2 T^4. \tag{3.53}$$

From Eq. (3.53) we can compute other thermodynamic variables, for example, the entropy density reads

$$s = -\frac{\partial F}{\partial T} = \frac{\pi^2}{2} N_c^2 T^3. \tag{3.54}$$

This picture becomes feasible for studying a finite temperature field theory at strong coupling. We recall that there are indications that real QGP investigated at RHIC is an example of such system. Of course our underlying theory is not QCD but $\mathcal{N} = 4$ SYM. Since these are completely different theories, we don't expect that we can formulate physical questions in terms of string theory. However, we expect that there are certain universal properties in both theories or that in some range of temperatures we can approximate real QCD plasma by a conformal $\mathcal{N} = 4$ plasma. We note that temperature introduces a scale and it breaks supersymmetry completely. Moreover, nothing prevents us from making a conjecture that there is a gravity dual for QCD itself. Studying the behavior of simpler gauge/gravity correspondences may be the first step on the road leading to QCD dual theory. At the moment we don't know precisely how to assess these similarities. Therefore, we will always keep in mind the differences, $\mathcal{N} = 4$ plasma is exactly conformal and there is no confinement/deconfinement phase transition.

Chapter 4

Fluid dynamics from gravity

In Chapter 2 we noted that at high energies an interacting QFT has an effective description in terms of fluid dynamics. This is also the case in the heavy-ion collisions. Shortly after the collision the system thermalizes and reaches thermal equilibrium. From this moment, which we denoted as τ_0 the evolution is described according to the laws of fluid dynamics. In general fluid evolution is characterized by a set of transport coefficients. In principle we can compute these coefficient from an underlying quantum field theory. For the real-world physics this is QCD. However, we noted that presumably the regime we are interested in is strongly coupled and we don't have theoretical tools to do so. We can at least study some qualitative features of strongly coupled plasma using AdS/CFT correspondence. This approach was started in the seminal papers by Policastro, Son, and Starinets [83–85] (see [86] for a review). They investigated the linearized hydrodynamics and calculated the shear viscosity over entropy density using linear response theory. This was a very important result, since in real QGP we have shear-driven elliptic flow, and the shear viscosity is the most relevant coefficient. Quite surprisingly it turned out that shear found by Policastro et al. is much smaller than any other known shear coefficient, what suggests that QGP is indeed strongly coupled. In Section 4.1 we will show how to consistently find transport coefficients in conformal fluids following [22]. Next, we generalize this procedure to more realistic fluids with one conserved current. This is an original result presented in Section 4.2. Finally, in Sections 4.3 and 4.4 we outline a construction of a gravity solution dual to boost-invariant fluids. A detailed study of regularity is presented, which is again an original result.

4.1 Gravity dual of fluid dynamics with no conserved current

In this section we will show how to establish a map between solutions of Einstein's equations and the perfect fluid solution presented in Chapter 2. As shown in Section 3.5 for every gauge invariant operator there is a dual field. We are interested in studying fluid energy-momentum tensor, hence, we need to find the corresponding gravitational field. We note that we have an idealized conformal fluid. This means that the number of transport coefficients is reduced in comparison with fluids without conformal symmetry. In particular there is no bulk viscosity, because in a conformal theory the energy-momentum tensor should be traceless

$$T^{\mu}{}_{\mu} = \varepsilon - 3P + \zeta(\partial \cdot u) = 0. \tag{4.1}$$

From that we find that $\varepsilon = 3P$ and $\zeta = 0$, since the identity (4.1) should be satisfied for an arbitrary solution.

4.1.1 Perfect fluid from gravity

We begin the discussion with the five dimensional Einstein gravity with negative cosmological constant,

$$S = \frac{1}{16\pi G_5} \int \sqrt{-g_5}\,(R - 2\Lambda)\,. \tag{4.2}$$

This is a universal subsector of type IIB supergravity. In principle we could focus on the dynamics of internal manifold, but we are interested in energy-momentum tensor dynamics so we can set all Kaluza-Klein modes on S^5 to zero. The field equations corresponding to the above action, with the radius of anti-de Sitter space set to one, are[1]

$$E_{MN} = R_{MN} - \frac{1}{2} g_{MN} R - 6 G_{MN} = 0. \tag{4.3}$$

The above equation admits AdS_5 solution, which corresponds to the vacuum state of dual field theory. Moreover, there is the so-called 'boosted black brane' solution

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br)\; u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \tag{4.4}$$

where

$$f(r) = 1 - \frac{1}{r^4}, \tag{4.5}$$

$$u^v = \frac{1}{\sqrt{1-\beta_i^2}}, \qquad u^i = \frac{\beta^i}{\sqrt{1-\beta_i^2}}. \tag{4.6}$$

This solution can be obtained by boosting the AdS-Schwarzschild solution along the spatial directions x^i. The temperature $T = \frac{1}{\pi b}$ and the velocities are constant. For brevity we have introduced the projector onto spatial dimensions,

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}. \tag{4.7}$$

We note that the parameters describing the boosted black brane solution (4.4) are precisely the hydrodynamical degrees of freedom, temperature and velocities. It posseses the holographic boundary stress tensor, which turns out to be precisely the perfect fluid (2.5).

[1] We use Latin letters $A, B \in \{r, v, x, y, z\}$ to denote the bulk indices and Greek letters $\mu, \nu \in \{v, x, y, z\}$ to denote the boundary indices.

4.1.2 Derivative expansion

We have to recall the lesson we learned in Chapter 2 about hydrodynamics and apply it to the boosted black brane. Namely we have to perturb the system away from equilibrium and perform a derivative expansion of the perturbed solution. We make the hydrodynamic variables to vary slowly along boundary directions. Thus, we want to promote the parameters b and β_i to functions of the boundary coordinates and construct the solution order by order in the boundary derivative expansion. We will measure this expansion by a formal parameter ε which we eventually set to 1. We implement the desired metric as a power series,

$$g = g^{(0)}(\beta_i, b) + \varepsilon g^{(1)}(\beta_i, b) + \varepsilon^2 g^{(2)}(\beta_i, b) + \mathcal{O}(\varepsilon^3), \tag{4.8}$$

where $g^{(0)}(\beta_i, b)$ is the metric (4.4) and $g^{(k)}(\beta_i, b)$ are yet to be determined by solving Einstein's equations (4.3). To do so we have to correct the velocity and temperature fields in a similar fashion,

$$\beta_i = \beta_i^{(0)} + \varepsilon\beta_i^{(1)} + \mathcal{O}(\varepsilon^2), \qquad b = b^{(0)} + \varepsilon b^{(1)} + \mathcal{O}(\varepsilon^2), \tag{4.9}$$

where $\beta_i^{(m)}$ and $b^{(n)}$ are functions of εx^μ. We can plug the ansatz (4.8) into Einstein's equations. However, we have to reduce the number of equations by fixing a gauge. It is convenient to work in the so-called 'background field' gauge,

$$g_{rr} = 0, \qquad g_{r\mu} \propto u_\mu, \qquad \mathrm{Tr}\left[(g^{(0)})^{-1} g^{(n)}\right] = 0 \quad \forall n > 0. \tag{4.10}$$

Before actually doing the computation let us focus for a moment on the general structure of the perturbation analysis. Suppose that we have solved the perturbation theory to the $(n-1)$th order. This means that we have determined $g^{(m)}$ for $m \leq n \leq 1$, and have determined the functions $\beta_i^{(k)}$ and $b^{(k)}$ for $k \leq n \leq 2$. From Einstein's equations we can extract the coefficient of ε^n. It has the following schematic form,

$$\mathbb{H}\left[g^{(0)}(\beta_i^{(0)}, b^{(0)})\right] g^{(n)}(x^\mu) = s_n. \tag{4.11}$$

$\mathbb{H}$ is a linear differential operator of the second order in r. It depends only on values of $\beta_i^{(0)}$ and $b^{(0)}$ at x^μ, and not on the derivatives of these functions at that point. Moreover, $\mathbb{H}$ is the same at every order of perturbation theory. The whole difficulty in the construction is to solve a homogenous second order differential equation with appropriate boundary conditions. Having done that we have to integrate it with various sources s_n, which are different at different orders of perturbation theory. It is a local expression of nth order in boundary derivatives of $\beta_i^{(0)}$ and $b^{(0)}$, as well as of $(n-k)$th order in $\beta_i^{(k)}$ and $b^{(k)}$ for all $k \leq n-1$. Finally, to make the calculation more transparent we introduce two subclasses of Eqs. (4.11). A class that determines the metric form we call dynamical equations, whereas the rest we call constraint equations.

4.1.3 Constraint equations

Equations obtained by contracting E_{MN} with the one-form normal to the boundary dr will be referred to as constraint equations. Four of them, labeled by indices associated with boundary directions have very natural physical interpretation as boundary energy-momentum conservation,

$$\partial_\mu T^{\mu\nu}_{(n-1)} = 0. \tag{4.12}$$

$T^{\mu\nu}_{(n-1)}$ is the boundary stress tensor dual the solution expanded up to $\mathcal{O}(\varepsilon^{(n-1)})$. It is a local function of temperature and velocities and their derivatives up to $(n-1)$th order.

4.1.4 Dynamical equations

Dynamical equations are those used to solve for unknown function $g^{(n)}$. We can decouple these equations into three sets. This follows from the underlying SO(3) symmetry in the spatial sections of the boundary. Thus, we decompose the solutions into SO(3) irreps

$$\begin{aligned} &\text{scalar sector}: \quad g^{(1)}_{vv}, \quad g^{(1)}_{vr}, \quad \sum_{i=1}^{3} g^{(1)}_{ii}, \\ &\text{vector sector}: \quad g^{(1)}_{vi}, \\ &\text{tensor sector}: \quad g^{(1)}_{ij}. \end{aligned} \tag{4.13}$$

Next, we determine the solutions by integration, requiring regularity of the solutions at all $r \neq 0$. This fixes the solution up to zero modes which can be absorbed in the redefinition of $\beta_i^{(0)}$ and $b^{(0)}$.

We are now in a position to investigate the technical details of the expansion. However, we would like to analyze more realistic fluids with conserved charges. Having in mind possible applications to plasma physics we note that usually not only the energy is conserved but also a baryon number.

4.2 Gravity dual of fluid dynamics with one global conserved current

The dual description of a fluid system with one conserved global charge consist of a five-dimensional metric field together with some background Maxwell field propagating in the bulk. The intuition coming from the previous discussion suggests that asymptotically AdS long-wavelength solutions of appropriate modifications of the the Einstein-Maxwell's equations are in one to one correspondence with solutions of the augmented Navier-Stokes equations described in Chapter 2. The Einstein-Maxwell's equations possess a well defined solution parameterized by the temperature, charge density and velocity. Promoting these parameters to fields, and applying the reasoning introduced in Section 4.1, we will show

how the derivative expension works in practice. We start with the five-dimensional action,

$$S = \frac{1}{16\pi G_5}\int\sqrt{-g_5}\left[R + 12 - F_{AB}F^{AB} + \frac{4\kappa}{3}\epsilon^{LABCD}A_L F_{AB}F_{CD}\right], \tag{4.14}$$

which is a consistent truncation of IIB SUGRA Lagrangian on $AdS_5\times S^5$ background with a cosmological constant $\Lambda = -6$ and the Chern-Simons parameter $\kappa = -1/(2\sqrt{3})$ [87, 88]. To keep the discussion general we will keep the κ coefficient arbitrary, and treat Eq. (4.14) as the definition of our theory. We will see that the Chern-Simons term is crucial to obtain the transport coefficient coming from triangle anomalies (2.43) and (2.44). The field equations corresponding to the above action are

$$\begin{aligned} G_{AB} - 6g_{AB} + 2\left[F_{AC}F^{C}{}_{B} + \frac{1}{4}g_{AB}F_{CD}F^{CD}\right] &= 0,\\ \nabla_B F^{AB} + \kappa\epsilon^{ABCDE}F_{BC}F_{DE} &= 0, \end{aligned} \tag{4.15}$$

where g_{AB} is the five-dimensional metric, G_{AB} is the five dimensional Einstein's tensor. These equations admit an AdS-Reissner-Nordström black-brane solution

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 V(r,m,q)\; u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu,\\ A &= \frac{\sqrt{3}q}{2r^2}u_\mu dx^\mu, \end{aligned} \tag{4.16}$$

where

$$u_\mu dx^\mu = -dv, \qquad \text{and} \qquad V(r,m,q) \equiv 1 - \frac{m}{r^4} + \frac{q^2}{r^6} \equiv 1 - M + Q^2, \tag{4.17}$$

with $\eta_{\mu\nu} = \mathrm{diag}(-+++)$ being the Minkowski-metric.

We shall assume the black-branes and the corresponding fluids to be non-extremal - this corresponds to the regime $0 < Q^2 < 2$ or $0 < M < 3$. Although there are indications that the hydrodynamic limit should be valid for extremal black-holes, the methods presented here are not sufficient to construct the solution there. Presumably there is a phase transition, which makes the construction more subtle.

Using the flat black-brane solutions with slowly varying spatial components of velocity, we intend to incorporate the finite but small magnetic field into the general framework of derivative expansion constructed above,

$$\begin{aligned} g_{AB} &= g^{(0)}_{AB} + g^{(1)}_{AB} + \dots,\\ A_M &= A^{(0)}_M + A^{(1)}_M + \dots, \end{aligned} \tag{4.18}$$

where $g^{(1)}_{AB}$ and $A^{(1)}_M$ contain the first spatial boundary derivative of the velocity

$$\begin{aligned} g^{(0)}_{AB}dx^A dx^B &= -2u_\mu(x)dx^\mu dr - r^2 V(r,m(x),q(x))\; u_\mu(x)u_\nu(x)dx^\mu dx^\nu + r^2 P_{\mu\nu}(x)dx^\mu dx^\nu,\\ A^{(0)}_M dx^M &= \frac{\sqrt{3}q(x)}{2r^2}u_\mu(x)dx^\mu. \end{aligned} \tag{4.19}$$

We supplement the gauge condition (4.10) by

$$A_r = 0. \tag{4.20}$$

4.2.1 First order calculation

In this section we construct the first order solutions of the metric and the gauge field in the derivative expansion. We follow the procedure constructed in [33, 34]. However, we include the external magnetic field which will give rise to a new transport coefficient [35].

4.2.2 Scalars Of SO(3)

The scalar components of first order metric and gauge field perturbations ($g^{(1)}$ and $A^{(1)}$ respectively) are parameterized by the functions $h_1(r)$, $k_1(r)$ and $w_1(r)$ as follows,[2]

$$\begin{aligned} \sum_i g^{(1)}_{ii}(r) &= 3r^2 h_1(r), \\ g^{(1)}_{vv}(r) &= \frac{k_1(r)}{r^2}, \\ g^{(1)}_{vr}(r) &= -\frac{3}{2}h_1(r), \\ A^{(1)}_v(r) &= \frac{\sqrt{3}w_1(r)}{2r^2}. \end{aligned} \tag{4.21}$$

Note that $g^{(1)}_{ii}(r)$ and $g^{(1)}_{vr}(r)$ are related to each other by the gauge choice $\mathrm{Tr}[(g^{(0)})^{-1}g^{(1)}] = 0$. We begin by finding the constraint equations that constrain various derivatives of velocities, temperature and charge that appear in the first order scalar sector. The constraint equations are obtained by contracting the Einstein's and Maxwell's equations with the vector dual to the one form dr. If we denote the Einstein's and Maxwell's equations by $E_{AB} = 0$ and $M_A = 0$, then there are three constraint relations. Two of them come from Einstein's equations. They are given by

$$g^{rr}E_{vr} + g^{rv}E_{vv} = 0, \tag{4.22}$$

$$g^{rr}E_{rr} + g^{rv}E_{vr} = 0 \ . \tag{4.23}$$

and the third constraint relation comes from Maxwell's equations and is given by

$$g^{rr}M_r + g^{rv}M_v = 0 \ . \tag{4.24}$$

[2]here i runs over the boundary spatial coordinates, v is the boundary time coordinate and r is the radial coordinate in the bulk

Equation (4.22) reduces to

$$\partial_v m^{(0)} = -\frac{4}{3} m_0 \partial_i \beta_i^{(0)} \ . \tag{4.25}$$

which is same as the conservation of energy on the boundary at the first order in the derivative expansion, i.e., the above equation is identical to the constraint

$$\partial_\mu T^{\mu\nu}_{(0)} = 0 \ , \tag{4.26}$$

on the allowed boundary data. The second constraint equation (4.23) in the scalar sector implies a relation between $h_1(r)$ and $k_1(r)$.

$$2\partial_i\beta_i^{(0)} r^5 + 12 r^6 h_1(r) + 4q_0 w_1(r) - m_0 r^3 h_1'(r) + 3r^7 h_1'(r) - r^3 k_1'(r) - 2q_0 r w_1'(r) = 0. \tag{4.27}$$

The constraint relation coming from Maxwell's equation gives

$$\partial_v \ q^{(0)} = -q_0 \ \partial_i \beta_i^{(0)} \ . \tag{4.28}$$

This equation can be interpreted as the conservation of boundary current density at the first order in the derivative expansion,

$$\partial_\mu J^\mu_{(0)} = 0. \tag{4.29}$$

We now proceed to find the scalar part of the metric dual to a fluid configuration which obeys the above constraints. Among the Einstein's equations four are $SO(3)$ scalars. Further the r and v-components of the Maxwell's equations constitute two other equations in this sector. Two specific linear combinations of the rr and vv components of the Einstein's equations constitute the two constraint equations in (4.25). Further, a linear combination of the r and v-components of the Maxwell's equations appear as a constraint equation in (4.28). Now among the six equations in the scalar sector we can use any three to solve for the unknown functions $h_1(r)$, $k_1(r)$, and $w_1(r)$. Moreover, we must make sure that the solution satisfies the rest. The simplest two equations among these dynamical equations are

$$5h_1'(r) + r h_1''(r) = 0, \tag{4.30}$$

which comes from the rr-component of the Einstein's equation and

$$6q_0 h_1'(r) + w_1'(r) - r w_1''(r) = 0, \tag{4.31}$$

which comes from the r-components of the Maxwell's equation. Solving (4.30) we get

$$h_1(r) = \frac{C^1_{h_1}}{r^4} + C^2_{h_1}, \tag{4.32}$$

where $C^1_{h_1}$ and $C^2_{h_1}$ are constants to be determined. We can set $C^2_{h_1}$ to zero as it will lead to a non-normalizable mode of the metric. We then substitute the solution for $h_1(r)$ from

(4.32) into (4.31) and solve the resultant equation for $w_1(r)$. The solution that we obtain is given by

$$w_1(r) = C^1_{w_1} r^2 + C^2_{w_1} - q_0 \frac{C^1_{h_1}}{r^4}. \tag{4.33}$$

Here again $C^1_{w_1}$, $C^2_{w_2}$ are constants to be determined. $C^1_{w_1}$ corresponds to a non-normalizable mode of the gauge field and, therefore, can be set to zero. Finally, plugging in these solutions for $h_1(r)$ and $w_1(r)$ into one of the constraint equations in (4.25) and then solving the subsequent equation we obtain

$$k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} + C_{k_1} - \frac{2q_0}{r^2} C^2_{w_1} + \left(\frac{2{q_0}^2}{r^6} - \frac{m_0}{r^4} \right) C^1_{h_1}. \tag{4.34}$$

The constants C_{k_1} and $C^2_{w_1}$ may be absorbed into redefinitions of mass (m_0) and charge (q_0) respectively and, hence, may be set to zero. Further, we can gauge away the constant $C^1_{h_1}$ by the following redefinition of the r coordinate,

$$r \to r \left(1 + \frac{C}{r^4}\right), \tag{4.35}$$

with C being a suitably chosen constant.

We conclude that all the arbitrary constants in this sector can be set to zero and therefore our solutions may be summarized as,

$$h_1(r) = 0, \quad w_1(r) = 0, \quad k_1(r) = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} \ . \tag{4.36}$$

In terms of the first order metric and gauge field this result reduces to

$$\begin{aligned} \sum_i g^{(1)}_{ii}(r) &= 0, \\ g^{(1)}_{vv}(r) &= \frac{2}{3} r \partial_i \beta_i^{(0)}, \\ g^{(1)}_{vr}(r) &= 0, \\ A^{(1)}_v(r) &= 0. \end{aligned} \tag{4.37}$$

Now, we proceed to solving the equations in the vector sector.

4.2.3 Vectors Of $SO(3)$

The vector components of metric and gauge field $g^{(1)}$ and $A^{(1)}$ are parameterized by the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ as follows,

$$\begin{aligned} g^{(1)}_{vi}(r) &= \left(\frac{m_0}{r^2} - \frac{q_0}{r^4}\right) j_i^{(1)}(r), \\ A^{(1)}_i(r) &= -\left(\frac{\sqrt{3} q_0}{2r^2}\right) j_i^{(1)}(r) + g_i^{(1)}(r). \end{aligned} \tag{4.38}$$

We intend to solve for the functions $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$.

The dynamical equation obtained from the vi-component of the Einstein's equations is given by

$$\left(q_0^2 - 3m_0 r^2\right) \frac{dj_i^{(1)}(r)}{dr} + 4\sqrt{3} q_0 r^2 \frac{dg_i(1)(r)}{dr} + \left(m_0 r^2 - q_0^2\right) r \frac{d^2 j_i^{(1)}(r)}{dr^2} = 0, \tag{4.39}$$

and the dynamical equation from the ith-component of the Maxwell's equation reads

$$\begin{aligned} & r\left[2\left(r^6 - m_0 r^2 + q_0^2\right) \frac{d^2 g_i^{(1)}}{dr^2} r^2 + \left(6r^7 + 2m_0 r^3 - 6q_0^2 r\right) \frac{dg_i^{(1)}(r)}{dr}\right] \\ & - \sqrt{3} q_0 r \left(r^6 - m_0 r^2 + q_0^2\right) \frac{d^2 j_i^{(1)}(r)}{dr^2} + \sqrt{3} q_0 \left(r^6 - 3m_0 r^2 + 5q_0^2\right) \frac{dj_i^{(1)}(r)}{dr} \\ & \quad = -48 q_0^2 \kappa r \omega_i. \end{aligned} \tag{4.40}$$

For any function $j_i^{(1)}(r)$, using (4.39), $g_i^{(1)}(r)$ may be expressed as

$$g_i^{(1)}(r) = (C_g)_i + \frac{1}{4\sqrt{3} q_0} \left(4m_0 j_i^{(1)}(r) - \frac{\left(m_0 r^2 - q_0^2\right) \frac{dj_i^{(1)}(r)}{dr}}{r}\right). \tag{4.41}$$

Here $(C_g)_i$ is an arbitrary constant. It corresponds to non-normalizable mode of the gauge field and, hence, may be set to zero. Using this expression for $g_i^{(1)}(r)$, we obtain the following differential equation for $j_i^{(1)}(r)$,

$$\begin{aligned} & r^2 \left(q_0^2 - m_0 r^2\right) \left(r^6 - m_0 r^2 + q_0^2\right) \frac{d^3 j_i^{(1)}(r)}{dr^3} \\ & + r\left(-11 q_0^4 - \left(5r^6 - 14 m_0 r^2\right) q_0^2 - m_0 r^4 \left(r^4 + 3m_0\right)\right) \frac{d^2 j_i^{(1)}(r)}{dr^2} \\ & + \left(35 q_0^4 + 5r^2 \left(r^4 - 6m_0\right) q_0^2 + 3m_0 r^4 \left(3r^4 + m_0\right)\right) \frac{dj_i^{(1)}(r)}{dr} \\ & = -\frac{288}{\sqrt{3}} r \omega_i q_0^3 \kappa \end{aligned} \tag{4.42}$$

The general structure of this equation can be encapsulated in the form

$$p_2(r) \frac{d^2 j_i{}'^{(2)}(r)}{dr^2} + p_1(r) \frac{dj_i{}'^{(1)}(r)}{dr} + p_0(r) j_i{}'(r) = s(r). \tag{4.43}$$

We reduced the order by introducing $j'(r) = \frac{dj(r)}{dr}$. One solution of the homogenous part of Eq. (4.42) is

$$j_1{}'(r) = \frac{4m_0 r^7 - 6r^5 \left(m_0 R^2 - R^6\right)}{\left(-R^6 + m_0 R^2 - m_0 r^2\right)^2}. \tag{4.44}$$

The general solution can be constructed using the variation of constants method

$$j_i{}'(r) = (C_j^1)_i j_{1i}{}'(r) - (C_j^2)_i j_{2i}{}'(r) - j_{1i}{}'(r)\int_r^\infty \frac{s(r) j_{2i}{}'(r)}{p_2(r)W(r)}\,dr + j_{2i}{}'(r)\int_r^\infty \frac{s(r) j_{1i}{}'(r)}{p_2(r)W(r)}\,dr, \tag{4.45}$$

where

$$j_{2i}{}'(r) = \int_r^\infty \frac{e^{-\int_R^x \frac{p_1(w)}{p_2(w)}dw}}{j'(x)^2}\,dx, \tag{4.46}$$

and $W(r)$ denotes the wronskian. Constants $(C_j^1)_i$ and $(C_j^2)_i$ are set by the requirement that the solution is finite on the horizon R and on the boundary. The solution to Eq. (4.42) is given by

$$\begin{aligned} j_i^{(1)}(r) = (C_j^1)_i &+ \frac{(C_j^2)_i r^2}{\frac{m_0}{r^2} - \frac{q_0^2}{r^4}} + \frac{r\,\partial_v \beta_i^{(0)}}{\frac{m_0}{r^2} - \frac{q_0^2}{r^4}} \\ &+ \frac{2\sqrt{3}\omega_i\, q_0^3 \kappa}{m_0\left(\frac{m_0}{r^2} - \frac{q_0^2}{r^4}\right) r^4} + \frac{6r^2 q_0(\partial_i q^{(0)} + 3q_0 \partial_v \beta_i^{(0)})}{R_0^7\left(\frac{m_0}{r^2} - \frac{q_0^2}{r^4}\right)} F_1\left(\frac{r}{R_0}, \frac{m_0}{R_0^4}\right). \end{aligned} \tag{4.47}$$

$(C_j^2)_i$ corresponds to a non-normalizable mode of the metric and is set to zero. $(C_j^1)_i$ can be absorbed into a redefinition of the velocities. The function $F_1(\frac{r}{R_0}, \frac{m_0}{R_0^4})$ is given by

$$F_1(\rho, M) \equiv \frac{1}{3}\left(1 - \frac{M}{\rho^4} + \frac{Q^2}{\rho^6}\right)\int_\rho^\infty dp \frac{1}{\left(1 - \frac{M}{p^4} + \frac{Q^2}{p^6}\right)^2}\left(\frac{1}{p^8} - \frac{3}{4p^7}\left(1 + \frac{1}{M}\right)\right), \tag{4.48}$$

where $Q^2 = M - 1$.

Substituting this result for $j_i^{(1)}(r)$ into Eq. (4.41) we obtain the following expression for $g_i^{(1)}(r)$,

$$\begin{aligned} g_i^{(1)}(r) =& \frac{1}{4\sqrt{3}q_0 R^8\left(R_0^6 + m_0\left(r^2 - R_0^2\right)\right)}\left[6q_0^2\left(br^3 + 2\sqrt{3}\omega q_0 \kappa\right) R_0^8\right. \\ &+ q_0(\partial_i q^{(0)} + 3q_0\partial_v \beta_i^{(0)})\left(\left(R^6 - m_0 R^2 + m_0 r^2\right) F_1^{(1,0)}(\rho, M) r^5\right. \\ &\left.\left. + 6F_1(\rho, M) R_0^3\left(m_0 - R_0^4\right) r^4\right)\right], \end{aligned} \tag{4.49}$$

where we use the notation $f^{(i,j)}(\alpha,\beta)$ to denote the partial derivative $\partial^{i+j} f/\partial\alpha^i\partial\beta^j$ of the function f.

Plugging $j_i^{(1)}(r)$ and $g_i^{(1)}(r)$ back into (4.38) we conclude that the first order metric and gauge field in the vector sector are given by

$$\begin{aligned} g_{vi}^{(1)}(r) &= r\partial_v \beta_i^{(0)} + \frac{2\sqrt{3}\omega_i q_0^3 \kappa}{m_0 r^4} + \frac{6r^2}{R_0^7} q_0(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)}) F_1\left(\frac{r}{R_0}, \frac{m_0}{R_0^4}\right), \\ A_i^{(1)}(r) &= \frac{-\sqrt{3}m_0 q_0(\partial_i q^{(0)} + 3q_0\partial_v\beta_i^{(0)}) F_1^{(1,0)}(\frac{r}{R_0}, \frac{m_o}{R_0}) r^7 + 36\omega_i q_0^3\kappa}{12 m_0 q_0 r^2}. \end{aligned} \tag{4.50}$$

4.2.4 Tensors Of $SO(3)$

The tensor components of the first order metric is parameterized by the function $\alpha_{ij}^{(1)}(r)$ such that

$$g_{ij}^{(1)} = r^2 \alpha_{ij}^{(1)}. \tag{4.51}$$

The gauge field does not have any tensor components, therefore, in this sector there is only one unknown function to be determined. There are no constraint equations in this sector and the only dynamical equation is obtained from the ij-component of the Einstein's equations. This equation is given by

$$r\left(r^6 - m_0 r^2 + q_0^2\right)\frac{d^2\alpha_{ij}(r)}{dr^2} - \left(-5r^6 + m_0 r^2 + q_0^2\right)\frac{d\alpha_{ij}(r)}{dr} = -6\sigma_{ij}^{(0)} r^4, \tag{4.52}$$

where

$$\sigma_{ij}^{(0)} = \frac{1}{2}\left(\partial_i \beta_j^{(0)} + \partial_j \beta_i^{(0)}\right) - \frac{1}{3}\partial_k \beta_k^{(0)} \delta_{ij}. \tag{4.53}$$

The solution to Eq. (4.52) is again obtained by demanding regularity at the future event horizon and appropriate normalizability at infinity. We have

$$\alpha_{ij}^{(1)} = \frac{2}{R_0}\sigma_{ij} F_2\left(\frac{r}{R_0}, \frac{m_0}{R_0^4}\right), \tag{4.54}$$

where the function $F_2(\rho, M)$ reads

$$F_2(\rho, M) \equiv \int_\rho^\infty \frac{p\left(p^2+p+1\right)}{(p+1)\left(p^4+p^2-M+1\right)} dp, \tag{4.55}$$

with $M \equiv m/R^4$ as before. The tensor part of the first order metric is determined to be

$$g_{ij}^{(1)} = \frac{2r^2}{R_0}\sigma_{ij} F_2\left(\frac{r}{R_0}, \frac{m_0}{R_0^4}\right). \tag{4.56}$$

4.2.5 The global metric/gauge field at first order

In this subsection, we gather the results of the previous sections to write down the entire metric and the gauge field accurate up to the first order in the derivative expansion. We get the metric as

$$\begin{aligned}
ds^2 &= g_{AB} dx^A dx^B \\
&= -2u_\mu dx^\mu dr - r^2\, V(r,m,q)\, u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu\, r\left[u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{3} u_\nu\right] dx^\nu + \frac{2r^2}{R} F_2(\rho, M)\sigma_{\mu\nu} dx^\mu dx^\nu \\
&\quad - 2u_\mu dx^\mu \left[\frac{2\sqrt{3}\kappa q^3}{m r^4}\omega_\nu + \frac{6qr^2}{R^7} P_\nu^\lambda \mathcal{D}_\lambda q F_1(\rho, M)\right] dx^\nu + \ldots, \\
A &= \left[\frac{\sqrt{3}q}{2r^2} u_\mu + \frac{6\kappa q^2}{m r^2}\omega_\mu + \frac{\sqrt{3}r^5}{2R^8} P_\mu^\lambda \mathcal{D}_\lambda q F_1^{(1,0)}(\rho, M)\right] dx^\mu + \ldots,
\end{aligned} \tag{4.57}$$

where we have defined

$$P_{\mu}^{\lambda}\mathcal{D}_{\lambda}q \equiv P_{\mu}^{\lambda}\nabla_{\lambda}q + 3(u^{\lambda}\nabla_{\lambda}u_{\mu})q. \tag{4.58}$$

We have come to the main goal of the above computation. We want to extract the energy-momentum tensor and the current of the boundary theory. For that we need to calculate the extrinsic curvature $K_{\mu\nu}$ to the surface at fixed r and appropriately regularize. Using the prescription (3.46) we get the following,

$$T_{\mu\nu} = P(\eta_{\mu\nu} + 4u_{\mu}u_{\nu}) - 2\eta_0\tau_{\mu\nu} + \dots, \tag{4.59}$$

where the fluid pressure P and the viscosity η_0 are given by the expressions

$$P \equiv \frac{R^4}{16\pi G_5}, \tag{4.60}$$

$$\eta_0 \equiv \frac{R^3}{16\pi G_5} = \frac{s}{4\pi}. \tag{4.61}$$

This is a very important result. As explained earlier the shear flow is dominating in the plasma. We were able to calculate the shear transport coefficient and it turns out to be very small. This confirms the indication that QGP is strongly coupled, at least for $\mathcal{N} = 4$ SYM. In a similar manner we can obtain the current sourced by the gauge field,

$$J_{\mu} = \lim_{r\to\infty} \frac{r^2 A_{\mu}}{8\pi G_5} = n\ u_{\mu} - \mathfrak{D}\ P_{\mu}^{\nu}\mathcal{D}_{\nu}n + \xi\ \omega_{\mu} + \dots, \tag{4.62}$$

where the charge density n, the diffusion constant, and the anomalous transport coefficient ξ are given by

$$n \equiv \frac{\sqrt{3}q}{16\pi G_5}, \tag{4.63}$$

$$\mathfrak{D} \equiv \frac{1+M}{4MR}, \tag{4.64}$$

$$\xi \equiv \frac{3\kappa q^2}{4\pi G_5 m}. \tag{4.65}$$

We can further complicate the above calculation by introducing a constant, external magnetic field. This allows us to get in touch with results obtained in Section 2.2, where we constrained two anomalous transport coefficients. The second transport coefficient in the gravity language takes the following form

$$\xi_B \equiv \frac{\sqrt{3}\,(3R^4 + m_0)\,q\kappa}{8\pi G_5 m_0 R^2} \tag{4.66}$$

Despite hydrodynamic reasoning is very rigid, it is always nice to arrive at physical results using different logic. Now, we can explicitly check the values (2.43) and (2.44) in the particular model. This is again a non-trivial result since the hydrodynamic derivation is model independent up to anomaly coefficient which is easily calculated from the perturbative quantum field theory. In order to compare this results with field theory predictions we have to rewrite them as functions of chemical potential and temperature. The chemical potential μ of the boundary theory is given by the difference between the value of the temporal component of the gauge field A_t at the horizon and its value at the boundary [33]. The temperature T of the boundary theory can be calculated using Hawking temperature,

$$\mu = A_t(R) - A_t(\infty) = -\frac{\sqrt{3}q}{2R^2}, \qquad T = \frac{R}{2\pi}\left[2 - \left(\frac{R_-}{R}\right)^2 - \left(\frac{R_-}{R}\right)^4\right]. \tag{4.67}$$

We have defined R to be the larger of the two positive roots (outer horizon) of $V(r, m, q)$ defined in Eq. (4.17) and R_- the smaller of its two positive roots (inner horizon). Manipulating $V(r, m, q)$ and Eq. (4.67) allows us to express the position of the horizon and the black hole mass in terms of hydrodynamic variables

$$R = \frac{\pi T}{2}\left(1 + \sqrt{1 + \frac{8}{3}\frac{\mu^2}{\pi^2 T^2}}\right), \qquad m = \frac{\pi^4 T^4}{2^4}\left(\sqrt{1 + \frac{8}{3}\frac{\mu^2}{\pi^2 T^2}} + 1\right)^3\left(3\sqrt{1 + \frac{8}{3}\frac{\mu^2}{\pi^2 T^2}} - 1\right). \tag{4.68}$$

Plugging this back to the gravitational expressions for ξ and ξ_B we recover the results (2.43) and (2.44).

The construction we presented is flexible enough to be continued to the next order. This leads to more complicated expressions for the metric and the gauge field plus a bunch of new transport coefficients. We present the expressions for the current and stress tensor in the Appendix A. For the technical details we refer the reader to [33, 34]. The above construction can be extended for a system with multiple, conserved $U(1)$ currents [89]. The dual description confirms the field theory prediction.

4.3 Gravity dual of boost-invariant fluids

The solutions constructed in the previous section are valid for general fluids. However, they are given as complicated integral expressions. It turns out that it is possible to have an explicit and still a non-trivial solution for boost-invariant flow described in Section 2.3. W will be working in the coordinates (2.45). The only non-vanishing components of the energy-momentum tensor are the diagonal ones, which depends only on τ alone. At this point we have three degrees of freedom, $T_{\tau\tau}$, T_{yy}, and $T_{x_\perp x_\perp}$. They can be further reduced to just one single function by imposing tracelessness

$$T^\mu{}_\mu = -T_{\tau\tau} + \frac{1}{\tau^2}T_{yy} + 2T_{xx} = 0, \tag{4.69}$$

and energy-momentum conservation,

$$\nabla_\mu T^{\mu\nu} = \tau \frac{d}{d\tau} T_{\tau\tau} + T_{\tau\tau} + \frac{1}{\tau^2} T_{yy} = 0. \tag{4.70}$$

The energy-momentum tensor can be expressed in the following unique form,

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon(\tau) & 0 & 0 & 0 \\ 0 & -\tau^3 \frac{d}{d\tau}\varepsilon(\tau) - \tau^2 \varepsilon(\tau) & 0 & 0 \\ 0 & 0 & \varepsilon(\tau) + \frac{1}{2}\tau \frac{d}{d\tau}\varepsilon(\tau) & 0 \\ 0 & 0 & 0 & \varepsilon(\tau) + \frac{1}{2}\tau \frac{d}{d\tau}\varepsilon(\tau) \end{pmatrix}. \tag{4.71}$$

$\varepsilon(\tau)$ can be interpreted as the energy density of the plasma. There is a further constraint on $\varepsilon(\tau)$ coming from the positive energy condition which states that for a time-like vector t^μ, the energy in any reference frame should be positive,

$$T_{\mu\nu} t^\mu t^\nu \geq 0. \tag{4.72}$$

This condition gives a set of inequalities,

$$\varepsilon(\tau) \geq 0, \qquad \varepsilon'(\tau) \leq 0, \qquad \tau\varepsilon'(\tau) \geq -4\varepsilon(\tau). \tag{4.73}$$

From now on we will concentrate on large τ asymptotics of $\varepsilon(\tau)$. Motivated by the Bjorken solution (2.52) we will assume the form of energy density to be

$$\varepsilon(\tau) \sim \frac{1}{\tau^s}. \tag{4.74}$$

Kinematics is not enough to determine the precise value of s. From (4.73) we can only say that it lies within the range $0 \leq s \leq 4$. To determine s we will concentrate on $\mathcal{N} = 4$ SYM although so far the discussion has been general and independent of particular features or coupling regime of an underlying conformal gauge theory. For $\mathcal{N} = 4$ SYM we have the AdS/CFT correspondence which will help us to single out s.

4.3.1 Gauge/gravity dual of non-viscous Bjorken flow

We want to write down an ansatz for the geometry that is dual to the boost-invariant plasma in the $\mathcal{N} = 4$ SYM. Before we do that we have to choose the coordinate system. For the general fluid configuration it was useful to use Eddington-Finkelstein (EF) ingoing coordinates, because they allow to impose regularity on the solutions. However, Fefferman-Graham (FG) coordinates are particularly useful in the holographic renormalization procedure, which we will perform later for a probe $D7$-brane. Thus, we will construct a gravity dual of Bjorken flow in both coordinate systems choosing a more convenient one for a particular purpose. Let us start with the FG coordinate system. The metric should respect all symmetries of plasma configuration, which leads to the following ansatz,

$$ds^2 = \frac{1}{z^2}\left(-e^{a(z,\tau)} d\tau^2 + e^{b(z,\tau)} \tau^2 dy^2 + e^{c(z,\tau)} dx_\perp^2\right) + \frac{dz^2}{z^2}. \tag{4.75}$$

We want to solve Einstein's equations for the three unknown functions with the boundary condition

$$a(z,\tau) = -z^4\varepsilon(\tau) + z^6 a_6(\tau) + \dots. \tag{4.76}$$

This condition comes from Eq. (3.45). In general we would expect a complicated system of non-linear partial differential equations. However, in this particular case we have a scaling variable which allows us to solve ordinary differential equation. To see that we plug the energy density in the form (4.74) to the Taylor series (4.76). The dominant contribution to $a_n(\tau)$ is

$$z^n a_n(\tau) \propto \left(\frac{z}{\tau^{\frac{s}{4}}},\right)^n, \tag{4.77}$$

which leads to the following scaling variable

$$v \equiv \frac{z}{\tau^{\frac{s}{4}}}. \tag{4.78}$$

With the scaling variable we proceed in a complete analogy with the derivative expansion. Here the role of an expansion parameter is played by the inverse powers of τ

$$\begin{aligned} a(z,\tau) &= a_0(v) + \frac{1}{\tau^\delta} a_1(v) + \dots, \\ b(z,\tau) &= b_0(v) + \frac{1}{\tau^\delta} b_1(v) + \dots, \\ c(z,\tau) &= c_0(v) + \frac{1}{\tau^\delta} c_1(v) + \dots, \end{aligned} \tag{4.79}$$

where δ is some positive but unspecified power. We want to find $a(z,\tau)$, $b(z,\tau)$, and $c(z,\tau)$ order by order. We insert the metric (4.75) into the Einstein's equations and take the limit $\tau \to \infty$ keeping v fixed. This means that we drop off from equations all terms with subleading τ contribution. We obtain the following set of coupled nonlinear equations,

$$\begin{aligned} v(\partial_v c)^2 = &- v\left[2(\partial_v a)(\partial_v c) + (\partial_v a)(\partial_v b) + 2(\partial_v b)(\partial_v c)\right] \\ &+ 6(\partial_v a) + 6(\partial_v b) + 12(\partial_v c), \end{aligned} \tag{4.80}$$

$$\begin{aligned} 3v(\partial_v c)^2 = \quad & 6(\partial_v b) + 12(\partial_v c) - v(\partial_v b)^2 - 2v(\partial_v b) \\ &- 4v(\partial_v c) - 2v(\partial_v b)(\partial_v c), \end{aligned} \tag{4.81}$$

$$\begin{aligned} 2vs(\partial_v c)^2 = &- 2vs(\partial_v^2 b) - 2s(\partial_v b) - 8(\partial_v a) + vs(\partial_v a)(\partial_v b) + 8(\partial_v b) \\ &- vs(\partial_v b)^2 - 4vs(\partial_v^2 c) - 4s(\partial_v c) + 2vs(\partial_v a)(\partial_v c). \end{aligned} \tag{4.82}$$

The above equations can be solved exactly. The solution reads

$$a(v) = A(v) - 2m(v), \tag{4.83}$$

$$b(v) = A(v) + (2s-2)m(v), \tag{4.84}$$

$$c(v) = A(v) + (2-s)m(v), \tag{4.85}$$

where

$$A(v) = \frac{1}{2}\left(\log(1+\sqrt{\frac{3s^2-8s+8}{24}}\,v^4) + \log(1-\sqrt{\frac{3s^2-8s+8}{24}}\,v^4)\right), \tag{4.86}$$

$$m(v) = \frac{1}{4}\sqrt{\frac{24}{3s^2-8s+8}}\left(\log(1+\sqrt{\frac{3s^2-8s+8}{24}}\,v^4) - \log(1-\sqrt{\frac{3s^2-8s+8}{24}}\,v^4)\right). \tag{4.87}$$

We see that the above geometry is potentially singular. This may be related to a coordinate singularity. However, it may as well be that there is a special point in the parameter space for which the geometry (4.75) is regular. To see that explicitly we have to calculate a curvature invariant,

$$\mathcal{R}^2 = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}. \tag{4.88}$$

For generic values of s this invariant is singular apart form the point

$$s = \frac{4}{3}. \tag{4.89}$$

The late time expansion of $\mathcal{N}=4$ SYM is described as a perfect fluid. Moreover, it is in agreement with AdS/CFT correspondence, which gives exact form of a dual geometry

$$ds^2 = \frac{1}{z^2}\left[-\frac{\left(1-\frac{\varepsilon_0}{3}\frac{z^4}{\tau^{4/3}}\right)^2}{1+\frac{\varepsilon_0}{3}\frac{z^4}{\tau^{4/3}}}d\tau^2 + \left(1+\frac{\varepsilon_0}{3}\frac{z^4}{\tau^{4/3}}\right)(\tau^2 dy^2 + dx_\perp^2)\right] + \frac{dz^2}{z^2}. \tag{4.90}$$

To get some intuition about geometry (4.90) let us take the standard Schwarzschild metric with $L=1$ (3.50) and perform a coordinate transformation

$$r = \frac{\sqrt{1+z^4/z_0^4}}{z}, \tag{4.91}$$

which leads to the following expression,

$$ds^2 = -\frac{(1-z^4/z_0^4)^2}{(1+z^4/z_0^4)z^2}\,dt^2 + (1+z^4/z_0^4)\frac{dx_i^2}{z^2} + \frac{dz^2}{z^2}. \tag{4.92}$$

We see that the geometry (4.90) is analogous to AdS-Schwarzchild geometry with the position of the horizon moving into the bulk,

$$z_0 = \sqrt[4]{\frac{3}{\varepsilon_0}}\cdot\tau^{\frac{1}{3}}. \tag{4.93}$$

We could interpret this result as a cooling of the dual plasma during an expansion process with temperature inversely proportional to the expansion time,

$$T = \frac{2^{\frac{1}{2}}\varepsilon_0^{\frac{1}{4}}}{3^{\frac{1}{4}}\pi}\tau^{-\frac{1}{3}}. \tag{4.94}$$

As a caveat we note that the naïve identifications of event horizons, within AdS/CFT correspondence, in the time dependent systems are not always true. In [90] it was shown that in the systems near equilibrium the event and apparent horizons are close to each other. Hence, we can use them as a measure of entropy in the boundary theory. However, in the far-from-equilibrium evolution we should use the quasi-local apparent horizon as opposed to the event horizon to measure the entropy. There are some issues associated with this prescription, since an apparent horizon is foliation dependent, and in some case discontinuous, but these issues are not yet fully understood.

4.3.2 Gauge/gravity dual of viscous boost-invariant plasma

The boost-invariant plasma admits shear expansion (2.53). We showed that in the fluid/gravity duals one can determine the value of shear viscosity from the energy-momentum tensor. Now we wan to examine that effect in the boost-invariant flows. The natural way to do that is to study subasymptotic behavior of of the metric coefficients (4.79). In order to setup a systematic expansion procedure one expands the left hand side of the appropriately rescaled Einstein's equations,

$$E_{AB} \equiv (\tau^{\frac{2}{3}} E_{\tau\tau}, \tau^{\frac{4}{3}} E_{\tau z}, \tau^{\frac{2}{3}} E_{zz}, \tau^{-\frac{4}{3}} E_{yy}, \tau^{\frac{2}{3}} E_{xx}), \tag{4.95}$$

in powers of $\tau^{-\frac{2}{3}}$. This was first found in [26] and the result is

$$\begin{aligned} a_1(v) &= 2\eta_0 \frac{(9+v^4)v^4}{9-v^8} \\ b_1(v) &= -2\eta_0 \frac{v^4}{3+v^4} + 2\eta_0 \log \frac{3-v^4}{3+v^4} \\ c_1(v) &= -2\eta_0 \frac{v^4}{3+v^4} - \eta_0 \log \frac{3-v^4}{3+v^4}. \end{aligned} \tag{4.96}$$

Unfortunately, at this order we cannot say much about the parameters η_0 and δ since the curvature is regular and takes the following form

$$\mathcal{R}^2 = R_0(v) + \frac{1}{\tau^{\delta}} R_1(v) + \dots \tag{4.97}$$

We expect that we have to go one order higher to extract some more information. Indeed, the subsubleading curvature invariant yields

$$\mathcal{R}^2 = R_0(v) + \frac{1}{\tau^{\delta}} R_1(v) + \frac{1}{\tau^{2\delta}} R_2(v) + \frac{1}{\tau^{\frac{4}{3}}} \tilde{R}_2(v) + \dots \tag{4.98}$$

The last two terms are always singular. Thus, the only way to have a finite structure is to tune them to cancel each other. This can be done requiring

$$\delta = \frac{2}{3}, \tag{4.99}$$

and setting the viscosity coefficient to the universal value $\frac{s}{4\pi}$. This result shows a generic feature for the boost-invariant duals. We have to check $n+1$-th order of the expansion to determine the n-th order transport coefficients. However, if we go to third order, apart from singularities that we can cancel by tuning appropriate transport coefficients, there are always logarithmic singularities coming from the peculiar choice of coordinate system. To get a consistent description of boost-invariant flow at third order one has to redo the above analysis in the Eddington-Finkelstein coordinates.

4.4 Regularity issues in boost-invariant plasma

FG coordinates create unremovable singularities in the curvature invariant $\mathcal{R}^2$. However, in Section 4.1 we argued that in the derivative expansion the structure of equations in EF coordinates give regular solutions to all orders. Because boost-invariant solutions are just subclass of general hydrodynamic solutions we expect that the boost-invariant setup in EF coordinates will be free of singularities [43, 91]. To support this intuition let us study the relation between these two coordinates systems in the case of perfect fluid. We saw that there are two equivalent dual descriptions (4.4) and (4.90). We relate these descriptions by the following coordinate transformation

$$\begin{aligned} \tilde{\tau} &= \tau\left\{1-\frac{1}{\tau^{2/3}}\left[\frac{3^{1/4}\pi}{4\sqrt{2}}+\frac{3^{1/4}}{2\sqrt{2}}\arctan\left(\frac{3^{1/4}}{\sqrt{2}}r\tau^{1/3}\right)+\frac{3^{1/4}}{4\sqrt{2}}\log\frac{r\tau^{1/3}-\frac{\sqrt{2}}{3^{1/4}}}{r\tau^{1/3}+\frac{\sqrt{2}}{3^{1/4}}}\right]\right\}, \\ r &= \frac{1}{z}\sqrt{1+\frac{z^4}{3\tau^{4/3}}}, \end{aligned} \tag{4.100}$$

where we set initial energy density ε_0 to one. We see that the transformation is singular when $z=3^{\frac{1}{4}}\tau^{\frac{1}{3}}$. This is precisely the locus where the logarithmic singularities were found. This is because the 'boosted black brane' metric is regular and invertible up to the black brane singularity, which is not the case with (4.90). Taking into account all possible symmetries we assume the following ansatz, which is the generalization of (4.4),

$$ds^2=-r^2a(\tilde{\tau},r)d\tilde{\tau}^2+2d\tilde{\tau}dr+r^2\tilde{\tau}^2e^{b(\tilde{\tau},r)}dy^2+r^2e^{c(\tilde{\tau},r)}dx_\perp^2 \tag{4.101}$$

Analogous to the FG case we introduce the scaling variable $r\tilde{\tau}^{1/3}$ and construct the metric coefficients for the large values of $\tilde{\tau}$. This scaling limit should be the same at the boundary but, because of the non-trivial r dependence in the relation between τ and $\tilde{\tau}$, it should differ in the bulk. Using the scaling variable we expand the metric coefficients as

$$\begin{aligned} a(\tilde{\tau},r) &= A(v)\exp\left(\sum_{k>0}a_k(v)\tilde{\tau}^{-2k/3}\right), \\ e^{b(\tilde{\tau},r)} &= B(v)\exp\left(\sum_{k>0}b_k(v)\tilde{\tau}^{-2k/3}\right), \\ e^{c(\tilde{\tau},r)} &= C(v)\exp\left(\sum_{k>0}c_k(v)\tilde{\tau}^{-2k/3}\right). \end{aligned} \tag{4.102}$$

To recover the black brane solution we choose the zeroth order coefficients to be

$$A(v) = 1 - \frac{4}{3v^4}, \qquad B(v) = C(v) = 1. \tag{4.103}$$

The equations of motion at a given order k are a system of ordinary second order differential equations for the 3 functions $a_k(v)$, $b_k(v)$ and $c_k(v)$. Each solution involves two integration constants. Two of the equations of motion are constraints. At each order $k > 0$ one of the constraints fixes one of the integration constants appearing at that order, and the other one fixes an integration constant left undetermined at order $k-1$. The 4 remaining integration constants can be fixed order by order by imposing metric regularity. It turns out that the potential singularity is located only at $v = \sqrt{2}/3^{1/4}$. Thus, the functions $b_k(v), c_k(v)$ must remain finite as $v \to \sqrt{2}/3^{1/4}$. In case of $a_k(v)$ the requirement should be that the product with $A(v)$ must be finite. This requirement can be satisfied by requiring that $a_k(v)$ itself be regular. Furthermore, asymptotic AdS behavior of the metric requires that these functions vanish as $v \to \infty$ (in the late proper time regime). The first order expressions in EF coordinates read

$$\begin{aligned}
a_1 &= -\frac{2}{3}\frac{2\times 3^{-1/2} + 2^{1/2}\times 3^{-1/4}v + v^2}{\left(2^{1/2}\times 3^{-1/4} + v\right)\left(2\times 3^{-1/2} + v^2\right)},\\
b_1 &= \frac{\pi}{\sqrt{2}3^{3/4}} - \frac{\sqrt{2}}{3^{3/4}}\arctan\left(\frac{3^{1/4}}{\sqrt{2}}v\right) - \frac{2\sqrt{2}}{3^{3/4}}\log v\\
&\quad + \frac{\sqrt{2}}{3^{3/4}}\log\left(\frac{\sqrt{2}}{3^{1/4}} + v\right) + \frac{1}{\sqrt{2}3^{3/4}}\log\left(\frac{2}{3^{1/2}} + v^2\right),\\
c_1 &= -b_1/2.
\end{aligned} \tag{4.104}$$

The present approach to the boost-invariant flow starts from a manifestly regular metric in the leading order (no logarithmic and power-like singularities at $v = \sqrt{2}/3^{1/4}$) and produces regular solutions up to the third order. Since the metric components, inverse metric components, and their derivatives are regular, all curvature invariants are non-singular. Indeed, from (4.101) it follows that the non-vanishing components of the inverse are $G^{rr} = r^2 e^{a(\tilde{\tau},r)}$, $G^{r\tilde{\tau}} = 1$, $G^{yy} = r^{-2}\tilde{\tau}^{-2}e^{-b(\tilde{\tau},r)}$, $G^{\perp\perp} = r^{-2}e^{-c(\tilde{\tau},r)}$. There is no terms $e^{-a(\tilde{\tau},r)}$, thus, there is no singularities, and we end up with perfectly regular dual of the boost-invariant flow.

As with all problems within classical gravity coordinate chart has crucial importance. It strongly depends on what physical problem we want to analyze. The disadvantage of the FG coordinate system is that the regularity analysis is not straightforward. However, for some procedures, like holographic renormalization, regularity at the horizon is not relevant. Moreover, as pointed out in [92] the explicit construction of the solutions for general fluids simplifies dramatically in FG coordinates and we recover manifest Lorentz invariance. Therefore, the choice of coordinate system strongly depends on phenomena we want to study. In the next chapter we will introduce fundamental matter in the boost-invariant setup and will find the FG coordinate system convenient.

Chapter 5

Flavors in plasmas

In the quest for constructing a gravity dual to some model that can approximate plasma physics we managed to reconstruct hydrodynamics purely from gravity. We can push the correspondence forward to introduce matter in the fundamental representation of the gauge group. In QCD, gluons transform in adjoint representation, quarks transform in the fundamental representation of the gauge group, whereas in $\mathcal{N} = 4$ super Yang-Mills all fields are in the adjoint. Evidently there is lack of proper flavor fields in our model so far. In order to obtain flavor fields holographically we will generalize the original AdS/CFT correspondence following the work of Karch and Katz [93] (see also [94–96]).

5.1 Brane degrees of freedom

To get some intuition how one can include fields in the fundamental representation let us first focus on the adojoint representation. We previously showed that constructing the gauge/gravity correspondence we start with a system with large number of $D3$-branes. We can attach open strings to these branes in such a way that one end lies on one brane and the second string end lies on the other. We have N_c^2 degrees of freedom which upon quantization give us fields transforming in the adjoint representation of $SU(N_c)$. The idea to obtain a fundamental representation is to modify the open strings degrees of freedom. This can be achieved by adding some new D-branes to the stack. Depending of what kind of branes we add we get different field theory interpretations.

We recall from the discussion in Section 3.4 that in type IIB string theory we have branes labeled by odd numbers. The first choice then is to add say M_c $D3$-branes separated by some distance. This procedure will produce massive vector field in the world-volume that transforms in the fundamental representation. However, this is not the kind of fields we need, since we want to obtain fermions not vectors. Let us give up on adding more $D3$-branes and start to add higher dimensional branes. To do that we have to choose the directions parallel to the stack of $D3$s. The choice determines how supersymmetric the system will be. We want to preserve supersymmetry to ensure stability. It turns out that introducing new D-branes breaks at least half of the supersymmetries. Analysis of intersecting D-branes shows that half of the supersymmetries is preserved if one D-brane is extended in four or eight directions, in which the other one is not [73]. One way of choosing directions filled by a D-brane is given in Table 5.1

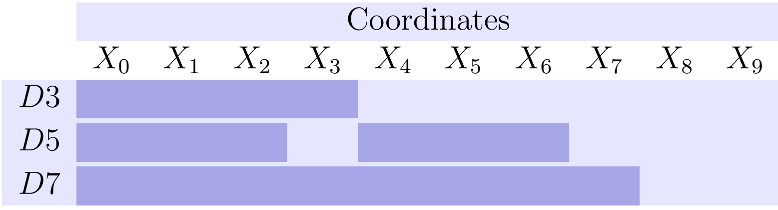

Table 5.1: Possible intersections of various D-brane systems. Blue color denotes the directions in which D-branes are extended. X_μ is a coordinate array in $\mathbf{R}^{9,1}$

We can consistently introduce fermionic fields by adding $D5$- or $D7$-branes. Note, however, that the physics is different in both cases. This comes from the dimensionality. If we attach one end of open string to a $D3$-brane and the other to $D5$-brane, then the resulting fermion moves in a (2+1) dimensional intersection. This can be interpreted as fermionic field restricted to some defect. This situation is closer to condensed matter systems than to QCD. Thus, in the rest of this Thesis we will focus on the $D7$-branes, which share a (3+1) dimensional space with $D3$-branes and the resulting fermionic matter is not restricted to any defect. Moreover, we can produce massive quarks separating $D3$ and $D7$-branes in orthogonal directions.

5.2 Gauge/gravity correspondence for the $D3/D7$-brane system

The field theory on a world-volume of a $D3/D7$-brane system can be reproduced by studying low-energy description of string excitations. As in the case of a single stack of $D3$-branes we have open strings with endpoints on the stack. We will denote them as (3-3). However, we have two new possibilities, (3-7), stretching between $D3$ and $D7$-branes, and (7-7), stretching between $D7$-branes. We are interested in large N_c limit with four-dimensional 't Hooft coupling λ and the number of $D7$-branes N_f being fixed. The eight-dimensional 't Hooft coupling

$$\lambda_8 = (2\pi)^4 \lambda \alpha'^2 \frac{N_f}{N_c} \tag{5.1}$$

which controls the (7-7) strings interaction vanishes in the low-energy limit $\alpha' \to 0$. (7-7) strings decouple and do not interact with (3-3) and (3-7) strings. Quantizing (3-3) strings we obtain $\mathcal{N} = 4$ SYM as previously, whereas (3-7) strings produce $\mathcal{N} = 2$ supersymmetric $U(N_c)$ gauge theory. The matter multiplet of that theory is called hypermultiplet. It is comprised of two complex scalars and a Dirac spinor. We will refer to that spinor as a ‘quark’. As a leftover of the above procedure we get scalar fields in the fundamental representation, which do not have counterparts in QCD.

Having found a field theory with fermionic matter we would like to repeat arguments invoked in original gauge/gravity duality. The first step is to find supergravity description of $D3/D7$ system. We expect that $D3$-branes will again act as a source for the metric and a self-dual five-form. The $D7$-branes will modify the metric and produce a non-trivial axion-dilaton profile. Unfortunately, the solution for a general $D3/D7$ system does not exist. The picture is especially complicated if the number of $D7$-branes is large, $N_f \approx N_c$. Therefore, N_f is usually taken to be of order 1. This is known as the probe limit, in which the effect from $D7$-branes is negligible. More precisely, in the limit $N_c \to \infty$, $D3$-branes source the metric and the five-form with $g_s N_c$, whereas the effect from $D7$-branes is controlled by the coupling $g_s N_f$. If N_f is small and $g_s \to 0$ we may ignore the backreaction, as well as the axion-dilaton profile. Thus, we are left again with supergravity with $AdS_5 \times S^5$ metric field plus a probe $D7$-brane described by a DBI action (3.18), extended along $AdS_5 \times S^3$.

The ingredients of the original Maldacena conjecture are modified. Therefore, the global symmetries in the system are different. The $D3/D7$ intersection breaks eight supercharges of original sixteen in $D3$-branes stack. Moreover, if we introduce non-zero mass for fermions, there is no supersymmetry transformation left. In addition to SUSY, we have global R-symmetry transformations, which are modified as well. It can be easily seen by looking at the geometry. From Table 5.1 we see that introducing a $D7$-brane breaks the rotational $SO(6)_R$ invariance in the $(X_4, X_5, X_6, X_7, X_8, X_9)$ directions to $SO(4)_R \times SO(2)_R$. The $SO(4)_R$ rotates in (X_4, X_5, X_6, X_7), while $SO(2)_R$ rotates in (X_8, X_9) plane. The $SO(2)_R$ may be identified with $U(1)_R$ on the field theory side. This symmetry is explicitly broken by a quark mass and acts as a chiral symmetry. For $m = 0$ the $D3/D7$ system is a point in the (X_8, X_9) plane. Introducing a non-zero mass we separate $D7$-brane from the stack of $D3$s. We have two points singled out in the (X_8, X_9) plane, labeled by the position of $D3$ and $D7$-branes. Two points on a plane are not $SO(2)_R$ invariant. Hence, we expect the conformal invariance to hold only for massless quarks. However, there is a caveat here. Moreover, if N_f is of order N_c, the scale is generated dynamically and the conformal invariance is lost.

5.2.1 $D7$-brane embeddings

The generalized form of duality provides tools to extract new field theory quantities, such as mass of the hypermultiplet or the meson spectra. For that we need the explicit form of the probe branes embedding. Evaluating the DBI action (3.18) with the appropriate tension (3.20) we get

$$S_{Dp} = N_f \frac{\lambda N_c}{2^5 \pi^6} \int d^8\xi \sqrt{-\det(P[G_{ab}]_{D7} + 2\pi\alpha' F_{ab})}. \tag{5.2}$$

From now on, we restrict ourselves to just one probe brane thus setting $N_f = 1$.

Throughout this Thesis we will be using various coordinate systems. We will start with standard coordinates for anti-de Sitter space (3.23) to get some intuition about $D7$-brane embedding. Next, we will rewrite the action in the Fefferman-Graham coordinates to perform holographic renormalization. Finally, we will switch to the boost-invariant geometry in the FG type coordinates to determine the embedding and renormalize holographically. The conventions are summarized in Table 5.2

$D3$-brane		
$D7$-brane		
Manifolds		
Coordinates		
$\mathbf{R}^{9,1}$		
X^M		
$\mathbf{R}^{3,1}$	S^5	
x^μ	$r = 1/z$, angular variables on S^5	
$\mathbf{R}^{3,1}$	S^3	S^2
x^μ	ρ, angular variables on S^3	w, angle

Table 5.2: $D3$- and $D7$-brane embeddings in the $AdS_5 \times S_5$, with conventions for world-volume coordinates

Placing the $D7$-brane parallel to the first eight coordinates its position and shape is described by X^8 and X^9. We can see that by rewriting the line element (3.23) in a way that shows the symmetries of the $D7$-brane,

$$
\begin{aligned}
ds_{10}^2 &= \frac{r^2}{L^2} dx^\mu dx_\mu + \frac{L^2}{r^2} d\rho^2 + L^2 \left[\rho^2 d\Omega_3^2 + (dX^8)^2 + (dX^9)^2\right] \\
&= \frac{r^2}{L^2} dx^\mu dx_\mu + \frac{L^2}{r^2} d\rho^2 + L^2(\rho^2 d\Omega_3^2 + dw^2 + w^2 d\Omega_1^2),
\end{aligned} \tag{5.3}
$$

where $r^2 = \sum (X^i)^2 = \rho^2 + (X^8)^2 + (X^9)^2$. The $D7$-brane is extended along the boundary directions and wraps three-sphere inside the original five-sphere. In the remaining two directions we have introduced polar coordinates. The motion in the w direction is equivalent with varying the mass of the multiplet in the boundary theory. Therefore, we are interested in finding a world-volume scalar which controls the brane dynamics in the radial direction. In principle this scalar depends on arbitrary world-volume coordinates. However, we require that it preserves certain symmetries. We want to preserve Lorentz symmetry, therefore, we forbid the scalar to depend on boundary coordinates. Moreover, we also assume $SO(4)$ symmetry of the three-sphere, in order to make the analysis feasible. As a result the ansatz for the scalar field depends only on the radial coordinate,

$$
w = \Phi(\rho). \tag{5.4}
$$

With the above ansatz we can calculate the induced metric on the $D7$-brane according to (3.17),

$$ds_{10}^2 = \frac{r^2}{L^2} dx^\mu dx_\mu + \frac{L^2}{r^2}\left[d\rho^2[1+\Phi'(\rho)] + \rho^2 d\Omega_3^2\right], \tag{5.5}$$

where prime denotes a derivative with respect to ρ. Calculating the determinant of that metric we can easily extract the DBI action density for a $D7$-brane,

$$\frac{S_{D7}}{V} = -2\pi^2 T_{D7} \int d\rho \rho^3 \sqrt{1+\Phi'(\rho)^2}. \tag{5.6}$$

The ground state configuration corresponds to the solution of the Euler-Lagrange equation,

$$\frac{d}{d\rho}\left[\frac{\rho^3}{\sqrt{1+\Phi'(\rho)}}\Phi'(\rho)\right] = 0. \tag{5.7}$$

We see that there is one analytic solution to that equation, namely $\Phi(\rho) = \text{const}$. We cannot solve Eq. (5.7) directly to obtain the full solution. However, we can extract asymptotic solution for $\rho \to \infty$. Close to the boundary the embedding behaves as

$$\Phi \xrightarrow[\rho\to\infty]{} m + \frac{c}{\rho^2} + ..., \tag{5.8}$$

where m and c are related to the bare quark mass m_q and chiral condensate $\langle\mathcal{O}\rangle$

$$m_q = \frac{m}{2\pi\alpha'}, \tag{5.9}$$

$$\langle\mathcal{O}\rangle = -\frac{N_c}{(2\pi\alpha')^3\lambda}\, c. \tag{5.10}$$

Let us briefly comment on this identification. In the asymptotic geometry, $\Phi(\rho)$ is precisely the separation of the $D3$s and the probe $D7$. The mass of the flavor fields is the string tension times the separation value, which gives precisely Eq. (5.9). To identify parameter c with the corresponding field theory quantity we recall the relation (3.41). m should act as a source to an operator with conformal dimension 3. Usually in a field theory m is a source to quark bilinear. Therefore, the operator $\mathcal{O}$ is a supersymmetric version of the quark bilinear, and it takes the schematic form,

$$\langle\mathcal{O}\rangle = \bar\psi\psi + q^\dagger \Phi q + m_q q^\dagger q. \tag{5.11}$$

We may find a bit puzzling that for the operator with $\Delta = 3$ the mass of corresponding scalar mode is negative according to Eq. (3.42). Non-zero mass breaks supersymmetry and the stability is no longer guaranteed. Fortunately, as shown by Breitenlohner and Freedman, scalar fields are stable if the mass is above certain negative value known as BF bound [97, 98]. In our case

$$m_{BF}^2 = -4 \geq -3 = \Delta(\Delta-4), \tag{5.12}$$

therefore, the scalar field is in a region that ensures stability. We note that in this theory there is no spontaneous chiral symmetry breaking, since the condensate (5.11) vanishes as m_q goes to zero.

5.2.2 $D7$-brane embedding in AdS/Schwarzschild geometry

A natural generalization of the previous construction is to place a probe $D7$-brane in the AdS/Schwarzschild geometry (3.50) [99]. This allows to study flavor fields in finite-temperature field theory. Moreover, it helps to get intuition how to embed a $D7$-brane in the boost-invariant geometry.

First of all, we note that in the geometries with a black hole in the bulk there are two possibilities for the topology of the $D7$-brane embedding. These are the so-called 'Minkowski embedding', when the brane doesn't touch the black hole, and 'black hole embedding', when the brane actually falls into the black hole. Moreover, we impose a condition which we use to identify physical solutions, the $D7$-brane embedding should have an interpretation as a renormalization group flow. This means that for each slice of the $D7$-brane geometry at fixed value of ρ^2, there is only one copy of the geometry $\mathbf{R}^4 \times S^3$ (see Fig. 5.1).

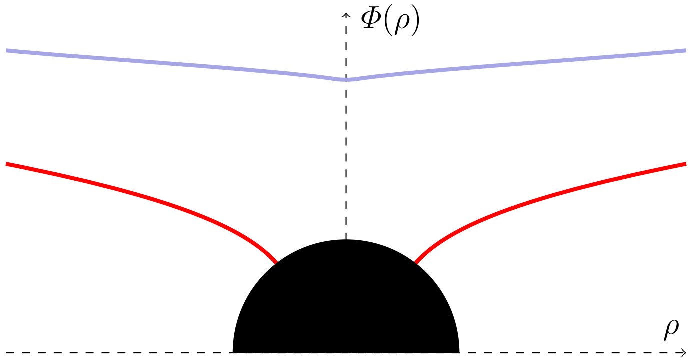

Figure 5.1: Schematic solutions of the $D7$-brane equations of motion. Blue line corresponds to Minkowski embedding and red line corresponds to black hole embedding.

For large quark mass, the solutions are very far from the black hole and become approximately constant embeddings as in the supersymmetric scenario [93]. This suggests the following perturbative analysis. For small ϵ we seek regular solutions

$$\Phi(\rho) = m + \epsilon f(\rho) \tag{5.13}$$

of the DBI action in AdS/Schwarzschild geometry

$$S_{D7/AdS/BH} = -2\pi^2 T_{D7} \int d\rho \rho^3 \left(1 - \frac{\epsilon (L^2 \pi T)^8}{16\left(\rho^2 + \Phi(\rho)^2\right)^4}\right) \sqrt{1 + \Phi'(\rho)} \tag{5.14}$$

The AdS black hole geometry asymptotically approaches $AdS_5 \times S^5$ geometry, thus the large ρ asymptotic solution coincides with the one found previously (5.8). The parameters m and c are taken as the boundary conditions for the EOMs. These equations are solved by means of shooting technique. We plot the condensate as a function of m in Figure 5.2. Solving perturbatively for the function $f(\rho)$ we obtain the following expression

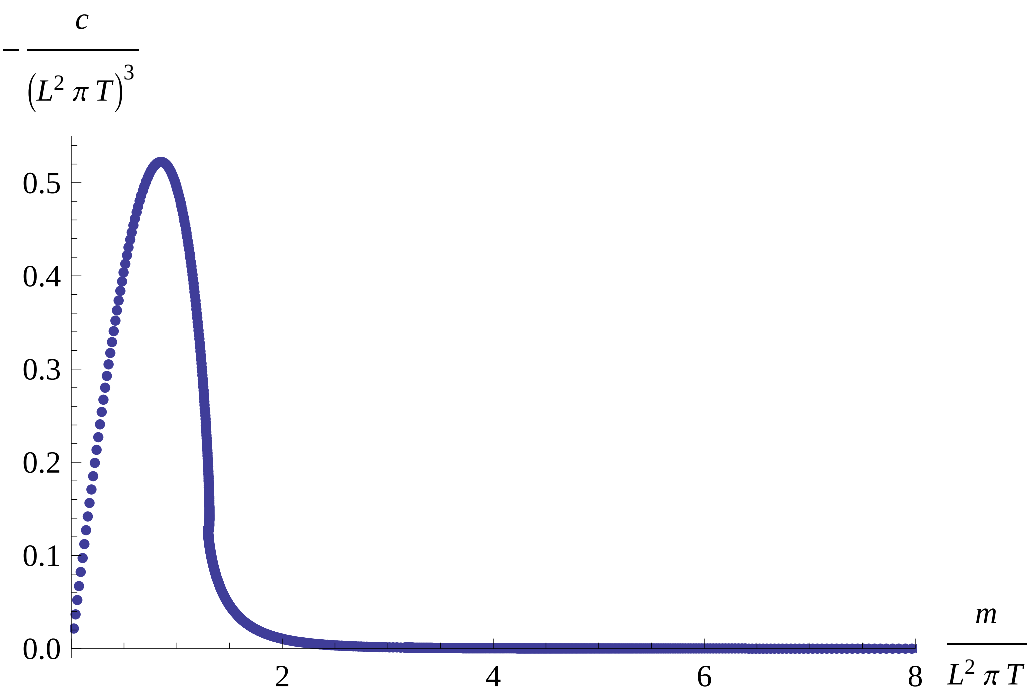

Figure 5.2: Chiral condensate as function of the quark mass for the static AdS/Schwarzschild background.

$$f(\rho) = -\frac{(L^2\pi T)^8 \left(3m^4 + 3\rho^2 m^2 + \rho^4\right)}{96m^5 \left(m^2 + \rho^2\right)^3}. \tag{5.15}$$

We can extract the value of the condensate,

$$c = -\frac{1}{96}\frac{(L^2\pi T)^8}{m^5}. \tag{5.16}$$

We have written (5.16) in a way that emphasizes the known fact that the temperature can be effectively removed from the equations by a suitable redefinition of the quark mass and condensate. In the present context, such a rescaling is not desirable. With (4.94) we conclude

$$c = -\frac{\varepsilon_0^2 L^{16}}{54m^5}\tau^{-\frac{8}{3}}. \tag{5.17}$$

5.2.3 $D7$-brane embedding in the boost-invariant geometry

Restricted to the scalar Φ, the action reads for our geometry (4.75) with the coefficients given by Eqs. (4.96)

$$S_{D7} = \frac{1}{2}\frac{N_c N_f}{(2\pi\ell_s^2)^4\lambda} V_x \int d\tau\, d\rho\, \tau\, \rho^3\, \mathbf{A} \sqrt{1 + \Phi'^2 - \mathbf{B}\frac{\dot{\Phi}^2}{(\rho^2+\Phi^2)^2}}, \tag{5.18}$$

$$\mathbf{A} \equiv \left(1 - \frac{v^8}{9}\right) \exp\left[-2\eta_0\, \varepsilon_0^{-\frac{1}{4}} \frac{v^8}{9-v^8}\tau^{-\frac{2}{3}}\right],$$
$$\mathbf{B} \equiv \frac{1+\frac{v^4}{3}}{(1-\frac{v^4}{3})^2} \exp\left[2\eta_0\, \varepsilon_0^{-\frac{1}{4}} v^4 \frac{9+v^4}{9-v^8}\tau^{-\frac{2}{3}}\right], \tag{5.19}$$

$$v \equiv \frac{\varepsilon_0^{\frac{1}{4}} L^2}{\tau^{\frac{1}{3}}\sqrt{\rho^2 + \Phi(\rho,\tau)^2}}, \tag{5.20}$$

where $V_x = \int dy\, d^2x_\perp$ is the infinite volume of spatial part of the boundary.

Since the viscous fluid geometry behaves similar to AdS/Schwarzschild with time-dependent temperature, we do not expect spontaneous symmetry breaking either, though going to small quark masses leaves the domain of validity of the geometry and it is thus hard to make a definite statement. We will therefore only consider 'Minkowski-type' embeddings that avoid the horizon at the center of the geometry.

For a given quark mass, in general regularity is only possible for a discrete set of values for the chiral condensate. In the regime under consideration, since no phase transition occurs, we expect $c = c(m)$ to be a one-valued function.

The equation of motion arising from (5.18) is a non-linear partial differential equation. We will solve it perturbatively by a late-time expansion,

$$\Phi(\rho,\tau) = m + \sum_{i=1}^{\infty} f_i(\rho)\tau^{-\frac{i}{3}}. \tag{5.21}$$

We use a fraction of 1/3 in the exponent because all exponents showing up in the background geometry (4.75) are integer multiples of one third. The ansatz reduces the equations of motion to the following (infinite) system of ordinary differential equations,

$$\rho^{-3}\partial_\rho(\rho^3 f_i'(\rho)) = \mathcal{I}_i(\rho) \tag{5.22}$$
$$\mathcal{I}_i = \frac{8m\varepsilon_0^2}{9(m^2+\rho^2)^5}\cdot \begin{cases} 1 & \text{if } i = 8 \\ -4\eta_0\varepsilon_0^{-1/4} & \text{if } i = 11 \\ 0 & \text{else; provided } i < 14 \end{cases}$$

The boundary behavior of solutions to (5.22) is

$$f_i(\rho) \xrightarrow[\rho\to\infty]{} m_i + \frac{c_i}{\rho^2}, \tag{5.23}$$

which becomes an exact solution when the inhomogeneous term vanishes, $\mathcal{I}_i = 0$. The first term, m_i, contributes $m_i \tau^{-i/3}$ to the bare quark mass. Since we do not accept a time-dependence of the bare parameters on physical grounds, we require $m_i = 0$. Thus in conjunction with regularity the value of c_i is completely fixed. In particular $\mathcal{I}_i = 0$ implies $c_i = 0$ or $f_i \equiv 0$.

To the considered order the solution is (see also Fig. 5.3),

$$\Phi(\rho,\tau) = m + c\,\frac{\rho^4 + 3\rho^2 m^2 + 3m^4}{(m^2+\rho^2)^3}, \tag{5.24}$$

with

$$c = -\frac{\varepsilon_0^2 L^{16}}{54 m^5}\tau^{-\frac{8}{3}}\left(1 - 4\eta_0 \varepsilon_0^{-\frac{1}{4}}\tau^{-\frac{2}{3}} + \dots\right). \tag{5.25}$$

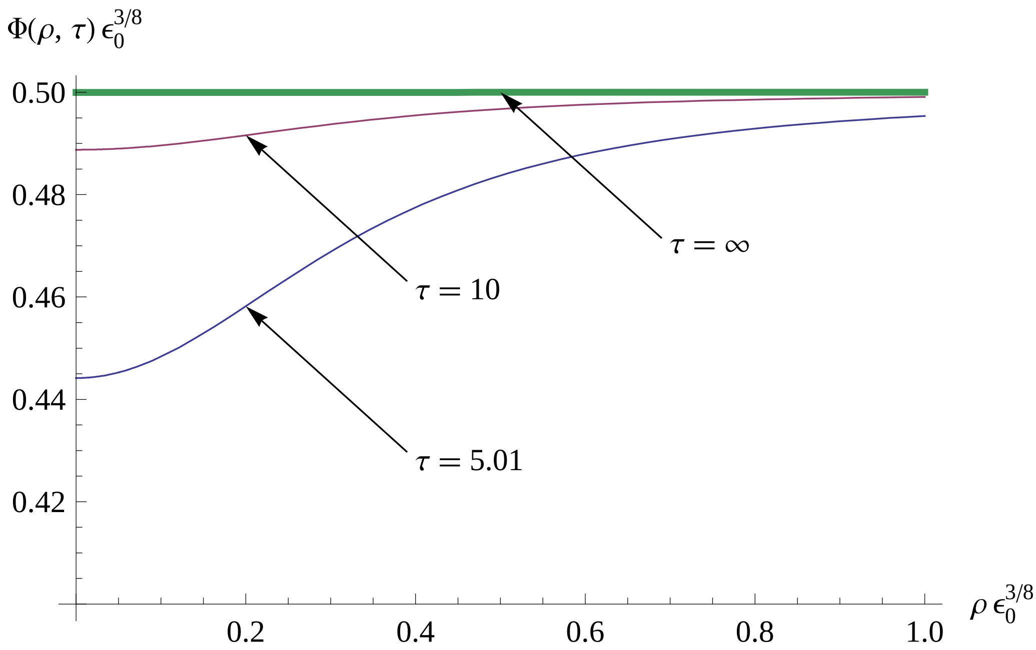

Figure 5.3: Embeddings with $m = \frac{1}{2}\varepsilon^{-3/8}$ at different times. For late times (bold), the supersymmetric embedding is approximated.

5.3 Holographic renormalization of a $D7$-brane

Thermodynamic properties of the $D3/D7$ system require the knowledge of the on-shell action [100]. We expect that we need to perform holographic renormalization procedure to get a finite action [101]. We outlined the general principles of this procedure in Section 3.5.3. In this section we will show how it works for a probe $D7$-brane.

To be consistent with Section 3.5.3 we perform two changes. We pass to Fefferman-Graham coordinate system and we use a different parametrization of the world-volume scalar. We rewrite the $AdS_5 \times S^5$ metric as

$$ds_{10}^2 = \frac{L^2}{z^2}(dx^\mu dx_\mu) + \frac{L^2}{z^2}(dz^2) + L^2(d\Psi + \sin^2\Psi d\Omega_1^2 + \cos^2\Psi d\Omega_3^2), \tag{5.26}$$

where we parametrized S^5 using

$$\begin{aligned} \rho &= r\cos\Psi, \\ w &= r\sin\Psi. \end{aligned} \tag{5.27}$$

Ψ is our new world-volume scalar, which depends on holographic coordinate z. We may interpret Ψ as an angle on S^5 in the region where $D7$-brane is wrapped. Ψ runs from zero, where $D7$ wraps the maximum volume S^3 to $\pi/2$ where S^3 collapses. We can translate the solution Φ to the new coordinates,

$$\Psi = \arcsin\left(\frac{z\Phi}{L^2}\right). \tag{5.28}$$

The asymptotic form for the solution Ψ reads

$$\Psi(z) = mz + \mathcal{O}(z^3). \tag{5.29}$$

In FG coordinates we identify the mass with the coefficient of the term linear in z.

The first step we have to take is to renormalize volume of AdS space,

$$V_{\mathrm{AdS}} = \int d^5x\sqrt{G_5}. \tag{5.30}$$

To keep the discussion general we don't assume anything about the boundary metric. We introduce a cut-off by extending the integration only to $z = \epsilon$. Next, we add counterterms, which cancel terms divergent after taking $\epsilon \to 0$. These counterterms must be built from data on the $\epsilon = z$ slice to preserve covariance. This means that we use the induced metric $\gamma_{\mu\nu}$ on the slice and the Ricci scalar constructed from γ. For the renormalization of four-dimensional anti-de Sitter spaces this leads to the following counterterms,

$$\begin{aligned} L_1 &= -\frac{1}{4}\sqrt{\gamma}, \\ L_2 &= -\frac{1}{48}\sqrt{\gamma}R_\gamma, \\ L_3 &= -\ln(\epsilon)\sqrt{\gamma}\frac{1}{32}(R_{ij}R^{ij} - \frac{1}{3}R_\gamma). \end{aligned} \tag{5.31}$$

The renormalized volume reads

$$V_{\mathrm{ren}} = \lim_{\epsilon\to 0}\left[V_{\mathrm{AdS}} + \int d^4x(L_1 + L_2 + L_3)\right]. \tag{5.32}$$

We note that if the boundary metric is flat, Ricci scalar on the slice ϵ vanishes and L_1 is sufficient to subtract volume divergence.

Apart from volume renormalization, we have a scalar $D7$-brane profile, which leads to divergences as well. The holographic renormalization proceeds as before leading to additional two counterterms,

$$\begin{aligned} L_4 &= \frac{1}{2}\sqrt{\gamma}\Psi^2 \\ , L_5 &= -\frac{1}{2}\ln(\epsilon)\sqrt{\gamma}(\Psi\Box_\gamma + \frac{1}{6}R_\gamma)\Psi, \end{aligned} \tag{5.33}$$

where $\Box_\gamma$ is d'Alambert operator on a slice. Finally, in the case of $D7$-branes one has to add finite counterterm

$$L_f = \alpha\gamma\Psi^4, \tag{5.34}$$

which doesn't cancel divergences but changes the on-shell action by a finite amount. We will fix α by requiring that the on-shell action vanishes. The renormalized action is

$$S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{reg}} + \sum L_i. \tag{5.35}$$

We are now in a position to do the computation. We start we the $AdS_5 \times S^5$ and generalize to the black hole and the boost-invariant geometries. The regularized action for a probe $D7$-brane in FG coordinates is

$$S_{\text{reg}} = \int_\epsilon^\infty \frac{1}{z^5}\cos^3\Psi(z)\sqrt{1 + z^2\Psi'(z)}. \tag{5.36}$$

This action is plainly divergent as $\epsilon \to 0$. We have to add proper counterterms, which in this case will be L_1, L_4, and L_f. We have to evaluate these counterterms on a hypersurface

$$ds_\epsilon^2 = \gamma_{\mu\nu}dx^\mu dx^\nu = \frac{1}{\epsilon^2}(-dt^2 + dx_i^2) = \frac{1}{\epsilon^2}(-d\tau^2 + \tau dy^2 + dx_\perp^2). \tag{5.37}$$

This leads to the following renormalized action,

$$S_{\text{ren}} = -2\pi^2 V_x T_{D7}\frac{m^4}{12}(12\alpha + 5), \tag{5.38}$$

where V is a four-dimensional volume. We noted that supersymmetry requires that the action evaluated on the solution vanishes. This amounts of fixing the remaining coefficient in the finite counterterm,

$$\alpha = -\frac{5}{12}. \tag{5.39}$$

The same procedure works for black-hole geometry

$$S_{\text{ren}} = -2\pi^2 V T_{D7}\frac{L^{16}\pi^8}{192m^4}T^7, \tag{5.40}$$

and for a probe $D7$-brane in the boost-invariant geometry,

$$S_{\text{ren}} = 2\pi^2 V_x T_{D7} \int d\tau\, \tau \left[-\frac{\varepsilon_0^2 L^{16}}{108 m^4 \tau^{8/3}} + \frac{\varepsilon_0^{7/4} \eta_0 L^{16}}{27 m^4 \tau^{10/3}} \right]. \tag{5.41}$$

We can compare the two results, since we argued that in the leading order we can interpret the boost-invariant geometry as a black-hole geometry. In the static case, the free energy density can be related to the regularized $D7$ action by means of a Wick rotation. In the conventions of [100] the time direction is periodically identified with $\beta = 1/T$. While it is clear that for a time-dependent background the substitution $1/T = \int dt$ is somewhat ill-defined, however, it is still interesting whether we can reproduce the asymptotic result (5.40). For comparison we perform this ill-defined step, namely replacing $\int d\tau\, \tau \mapsto 1/T$ and using (4.94). We find that there is an agreement.

5.4 Meson spectra

We pointed out in the introduction that one of the most important problems in strongly coupled gauge theories is the spectrum of hadrons. In this section we partly resolve this problem in the context of holography, namely we will calculate meson spectra. In the $D3/D7$ framework, meson spectra are determined from regular, normalizable solutions to the equations obtained from linearizing the full equations of motion of the $D7$-brane about the embedding solution that describes the position and shape of the brane [102]. This can be understood heuristically by noting that meson, which is two bounded quarks, corresponds to an open string attached to the brane with quarks at the end. For the finite temperature field theory dual to AdS/Schwarzschild geometry the meson spectrum was found using numerical techniques [99]. However, this analysis excludes corrections coming from viscosity. We will show how to improve this result.

In the following section we distinguish between four dimensional meson modes, which carry a '4d' label and eight dimensional fluctuations, which always start with a δ followed by a (Greek or Latin) capital letter, e.g. $\delta\Phi$ or δA^y. Our ansätze are products of spherical harmonics $\mathcal{Y}$ on the internal manifold, wave forms parallel to the boundary and radial parts, which describe the dependence on the holographic coordinates and are denoted by δ followed by a small letter.

5.4.1 Boost-invariance for the AdS geometry and scalar mesons

It is insightful to investigate how the boost-invariance changes wave forms of scalar and vector mesons in the conventional setting, where there is no time-dependence of the meson mass.

In four dimensional Minkowski space the massive Klein–Gordon equation assumes the form

$$\Box \Phi_{4d} = \left[-\frac{1}{\tau}\partial_\tau \tau \partial_\tau + \tau^{-2}\partial_y^2 + \partial_x^2 \right] \Phi_{4d} = M^2 \Phi_{4d}. \tag{5.42}$$

This implies that

$$\Phi_{4d} = (c_1\,\mathrm{J}_0(\omega\,\tau) + c_2\,\mathrm{Y}_0(\omega\,\tau))e^{\pm ik_\perp x_\perp}, \qquad k_\perp^2 = k_2^2 + k_3^2, \tag{5.43}$$

with J_0 and Y_0 being Bessel functions of first kind. Since a linear combination of the Bessel functions will appear frequently in our expressions we introduce the shorthand notation

$$\mathcal{F}_p[\omega] \equiv c_1\,\mathrm{J}_p(\textstyle\int\omega\,d\tau) + c_2\,\mathrm{Y}_p(\textstyle\int\omega\,d\tau). \tag{5.44}$$

We will not explicitly denote the time-dependence of $\mathcal{F}_0$ arising from the integral over τ. At this stage, the integral has been chosen for later convenience and gives $\omega\,\tau$ for constant frequencies. The eigenfrequencies $\omega \equiv \sqrt{M^2 + k_\perp^2}$ are to be determined in our holographic setup. We will thus assume $k_\perp = 0$ from the start to obtain the mass spectrum.

The 4d meson field is given as the boundary value of (linear, normalizable) fluctuations $\delta\Phi$, $\delta\Psi$ about the embedding solution $\Phi(\rho) \equiv m$,

$$X^8 = 0 + \delta\Psi(\rho,\tau), \qquad X^9 = \Phi(\rho) + \delta\Phi(\rho,\tau). \tag{5.45}$$

For the presentation of our ansatz, we will concentrate on the scalar mode $\delta\Phi$; the pseudoscalar mode $\delta\Psi$ will be treated analogously. With the following holographic ansatz,

$$\delta\Phi(\rho,\tau) = \delta\phi(\rho)\mathcal{F}_0[\omega]\mathcal{Y}^\ell(S^3), \tag{5.46}$$

the boundary value corresponding to quantum number ℓ is defined by

$$\Phi_{4d}^{(\ell)} = \lim_{\rho\to\infty} \rho^2 \frac{\delta\Phi(\rho,\tau)}{\mathcal{Y}^\ell(S^3)}. \tag{5.47}$$

Equation (5.46) is a natural modification of the ansatz given in [102] to separate the $D7$ equation of motion in anti-de Sitter space,

$$\left[-\frac{L^2}{(\rho^2+m^2)^2}\frac{1}{\tau}\partial_\tau\tau\partial_\tau + \frac{1}{\rho^3}\partial_\rho\rho^3\partial_\rho + \frac{1}{\rho^2}\Delta_{S^3}\right]\delta\Phi(\rho,\tau) = 0. \tag{5.48}$$

The radial equation obtained after separation reads

$$\left[\frac{1}{\rho^3}\partial_\rho\rho^3\partial_\rho + \frac{L^2\omega^2}{\rho^2+m^2} - \frac{\ell(\ell+2)}{\rho^2}\right]\delta\phi(\rho) = 0, \tag{5.49}$$

which is the well-known result of [102]. For simplicity, we will only consider the lowest Kaluza–Klein mode on the internal S^3, such that $\ell = 0$, $\mathcal{Y}^0 \equiv 1$.

The requirements of regularity in the interior ($\rho \to 0$) and vanishing at the boundary ($\rho \to \infty$), fix the modes completely. One obtains a discrete set of modes the lightest of which is given by

$$\mathcal{F}_0\left[\omega_0 = \frac{\sqrt{8}m}{L^2}\right] \cdot \frac{1}{m^2+\rho^2}. \tag{5.50}$$

5.4.2 Vector mesons

Our approach is capable to include vector mesons. We note that, in addition to a scalar field on the world-volume, one can include a gauge field. Matching degrees of freedom one can postulate that small fluctuations of the gauge field lead to the vector meson spectrum [103]. For a four-dimensional massive vector meson we have

$$\nabla_a F^{ab} = M^2 A^b_{4d}. \tag{5.51}$$

We assume that the solutions are still plane waves in the x_2, x_3-plane, i.e.,

$$A^a_{4d} = \xi^a(\tau) \exp ik_\perp x_\perp. \tag{5.52}$$

This yields the following component equations

$$-\tau\partial_\tau\left(\frac{1}{\tau}\partial_\tau A^{4d}_y\right) = (M^2 + k^2_\perp)A^{4d}_y, \tag{5.53}$$

$$-\partial^2_\tau A^{4d}_2 - \frac{1}{\tau}\partial_\tau A^{4d}_2 + ik_2\Big(\partial_\tau + \frac{1}{\tau}\Big)A^{4d}_\tau \tag{5.54}$$

$$+k_2k_3A^{4d}_3 = (M^2 + k^2_3)A^{4d}_2 \qquad \text{and } (2 \leftrightarrow 3),$$

$$A^{4d}_\tau = -\frac{i\partial_\tau(k_2A^{4d}_2 + k_3A^{4d}_3)}{\omega^2}. \tag{5.55}$$

Equation (5.53) can be treated separately. Its solution is

$$A^{4d}_y = \tau\mathcal{F}_1[\omega]e^{ik_\perp x_\perp}, \qquad A^{4d}_{\tau,x^2,x^3} = 0. \tag{5.56}$$

The others may be solved without loss of generality by turning the coordinate system such that $k_3 = 0$. Then it follows immediately that $A^{4d}_{2,3} = \xi_{2,3}\mathcal{F}_0[\omega_{2,3}] \exp ik_\perp x_\perp$. With this modified ansatz and plugging (5.55) into (5.54) we obtain

$$\left[-M^2 - k^2_3 + \omega^2_2\left(1 - \frac{k^2_2}{\omega^2}\right)\right]A^{4d}_2 + k_2k_3\left(1 - \frac{\omega^2_3}{\omega^2}\right)A^{4d}_3 = 0$$

$$\text{and } (2 \leftrightarrow 3), \tag{5.57}$$

which can only be satisfied for $\omega_2 = \omega_3 \equiv \omega_{23}$. Moreover, since it is a homogeneous system, $\omega_{23}(M, k_2, k_3)$ can be determined from degeneracy of the coefficient matrix. We shall not reproduce the final expression, but just note that $\omega_{23} = M$ when $k_2 = k_3 = 0$, which could also have been obtained directly from (5.54). We thus end up with the two solutions,

$$A^{4d}_{2,3} = \xi_{2,3}\mathcal{F}_0[\omega_{23}] \exp ik_\perp x_\perp. \tag{5.58}$$

Therefore, we adapt the holographic ansätze for meson modes found in [102] as follows

$$\begin{array}{ll}
\text{Type} & \\
\text{I} & \delta A_\alpha = \delta a_I^\pm(\rho)\mathcal{F}_0[\omega]e^{ik_\perp x_\perp}\mathcal{Y}_\alpha^{\ell,\pm}(S^3), \quad \alpha = 5,6,7; \\
\text{II}_y & \delta A_y = \delta a_{IIy}(\rho)\tau\mathcal{F}_1[\omega]e^{ik_\perp x_\perp}\mathcal{Y}^\ell(S^3); \\
\text{II}_{2,3} & \delta A_2 = \delta a_{II2}(\rho)\mathcal{F}_0[\omega]e^{ik_\perp x_\perp}\mathcal{Y}^\ell(S^3), \quad A_3 = 0; \\
& \qquad\qquad \text{and } (2 \leftrightarrow 3) \\
\text{III} & \delta A_\rho = \delta a_{III}(\rho)\mathcal{F}_0[\omega]e^{ik_\perp x_\perp}\mathcal{Y}^\ell(S^3), \\
& \delta A_\alpha = \delta \tilde{a}_{III}(\rho)\mathcal{F}_0[\omega]e^{ik_\perp x_\perp}\mathcal{Y}_\alpha^{\ell,\pm}(S^3);
\end{array} \tag{5.59}$$

with the respective other components set to zero. We will only consider modes of type II, which are the only modes dual to vector mesons and therefore most interesting.

In the ansätze, the dependence in the $0,1,2,3$ directions has been modified as compared to what can be found in [102]. The reason these changes are straight-forward is the following, the calculation of [102] only uses two important properties of the ansätze regarding derivatives in those directions,

$$\Delta_{4d}\delta A_I = M^2\delta A_I, \qquad I \in [0,\ldots,7] \tag{5.60}$$

$$g_{4d}^{ab}\partial_a\delta A_b = 0, \tag{5.61}$$

where $g_{4d} = \mathrm{diag}(-1,\tau^2,1,1)$ in our case, whereas in [102] it was a Minkowski metric. For our ansätze, the gauge condition (5.61) is either trivially obeyed or follows from (5.55). Moreover, it can be used to turn (5.60) into (5.51).

5.4.3 Viscous fluid geometry

Before coming to the actual holographic computation, we would like to discuss the general framework of late-time perturbative expansions that we use.

Since the viscous fluid geometry and our $D7$ embeddings are time-dependent, we do not expect, and do not see, a separation into a purely τ dependent and ρ dependent factor. This makes the problem very difficult to tackle analytically. We are helped by the property that at late proper times the geometry becomes pure AdS_5 with the corresponding $D7$-brane embedding. In this limit the simplest solution looks like (5.50), where $\mathcal{F}_0[\omega]$ is defined in Eq. (5.44). For smaller proper-times it is natural to treat the frequency appearing in (5.50) as depending on τ. However, as the equations do not allow for a separation of variables we have τ dependence also in the remaining part,

$$\tilde{\mathcal{F}}\Big[\omega(\tau)\Big]f(\rho,\tau) \tag{5.62}$$

where we allow for a general $\tilde{\mathcal{F}}$ which should reduce to $\mathcal{F}_0[\omega]$ for constant ω. We have, moreover, the expansions

$$\omega(\tau) = \omega + \frac{1}{\tau^{\frac{1}{3}}}\omega^{(1)} + \dots \tag{5.63}$$

$$f(\rho,\tau) = f^{(0)}(\rho) + \frac{1}{\tau^{\frac{1}{3}}} f^{(1)}(\rho) + \dots \tag{5.64}$$

Note that the above form is not unique. Redefining the coefficients of the expansions in an appropriate way, we may redefine the split (5.62). So in order to uniquely specify such an ansatz we have to supplement the usual regularity condition at $\rho = 0$ and Dirichlet boundary condition at $\rho = \infty$ by another condition which makes the split (5.62) unique. In this Thesis we will impose a condition on the profile of the mode $\delta\phi$ induced on the boundary,

$$\Phi_{4d}(\tau) \equiv \lim_{\rho\to\infty} \rho^2 \delta\phi(\rho,\tau) \tag{5.65}$$

Namely we will set

$$\Phi_{4d}(\tau) = \sqrt{\frac{\int \omega_{4d}(\tau)\, d\tau}{\omega_{4d}\, \tau}} \mathcal{F}_0[\omega_{4d}(\tau)], \tag{5.66}$$

which provides a definition of our frequency $\omega_{4d}(\tau)$. For constant $\omega_{4d}(\tau)$ this reduces of course to the pure AdS_5 result (5.50).

Our motivation for the above form (5.66) is that it arises as a WKB approximation to a Klein–Gordon equation with time dependent mass spectra,

$$\Box\Phi_{4d} = \left[-\frac{1}{\tau}\partial_\tau \tau \partial_\tau + \tau^{-2}\partial_y^2 + \partial_x^2\right]\Phi_{4d} = M_{4d}^2(\tau)\Phi_{4d} \tag{5.67}$$

We may separate variables by assuming a plane wave in the 2, 3 plane and obtain $\omega_{4d}^2(\tau) = M_{4d}^2(\tau) + k_\perp^2$. (Though we will assume $k_\perp = 0$, henceforth.) The remaining equation

$$-\frac{1}{\tau}\partial_\tau \tau \partial_\tau \Phi_{4d}(\tau) = \omega_{4d}^2(\tau)\Phi_{4d}(\tau), \tag{5.68}$$

can only be solved approximately, e.g., by the WKB approximation, which gives two linearly independent solutions,

$$\Phi_{4d}(\tau) \approx \sqrt{\frac{\int \omega_{4d}(\tau)\, d\tau}{\omega_{4d}\, \tau}} \mathcal{F}_0[\omega_{4d}(\tau)]. \tag{5.69}$$

The square root prefactor ensures that Abel's theorem is fulfilled, such that the wronskian for our ansatz is

$$W = \frac{\text{const}}{t}, \tag{5.70}$$

as it should be for the exact solution.

We have now all ingredients in place to actually calculate the meson spectrum for the time-dependent viscous fluid geometry. We expand the $D7$ action (5.18)[1], given by

$$\mathcal{L}_{DBI} = e^{\frac{\mathcal{A}}{2}+\frac{\mathcal{B}}{2}+\mathcal{C}}\rho^3\tau\Big[1+(\partial_\rho X^9)^2+(\partial_\rho X^8)^2+e^{-\mathcal{C}}(\partial_\rho A_2)^2 + \frac{e^{-\mathcal{B}}}{\tau^2}(\partial_\rho A_y)^2 - \frac{e^{-\mathcal{A}}L^4}{r^4}(\mathrm{I}) + (\text{quartic})\Big]^{1/2}$$

$$\begin{aligned}(\mathrm{I}) &= (\partial_\tau X^8)^2+(\partial_\tau X^9)^2+(\partial_\rho X^8)^2(\partial_\tau X^9)^2 \\ &\quad + (\partial_\tau X^8)^2(\partial_\rho X^9)^2 \\ &\quad - 2(\partial_\tau X^8)(\partial_\rho X^9)(\partial_\rho X^8)(\partial_\tau X^9) \\ &\quad + e^{-\mathcal{C}}(\mathrm{II}) + \frac{e^{-\mathcal{B}}}{\tau^2}(\mathrm{III}) \end{aligned} \tag{5.71}$$

$$\begin{aligned}(\mathrm{II}) &= (\partial_\tau A_2)^2+(\partial_\rho X^9)^2(\partial_\tau A_2)^2+(\partial_\tau X^9)^2(\partial_\rho A_2)^2 \\ &\quad - 2(\partial_\tau X^9)(\partial_\rho X^9)(\partial_\rho A_2)(\partial_\tau A_2)\end{aligned} \tag{5.72}$$

$$\begin{aligned}(\mathrm{III}) &= +(\partial_\tau A_y)^2+(\partial_\rho X^9)^2(\partial_\tau A_y)^2+(\partial_\tau X^9)^2(\partial_\rho A_y)^2 \\ &\quad - 2(\partial_\tau X^9)(\partial_\rho X^9)(\partial_\rho A_y)(\partial_\tau A_y)\end{aligned} \tag{5.73}$$

to quadratic order in fluctuations

$$\begin{aligned} X^9 &= \varPhi + \delta\varPhi, & X^8 &= 0+\delta\varPsi, \\ A_2 &= 0+\delta A_2, & A_y &= 0+\delta A_y. \end{aligned} \tag{5.74}$$

The resulting equation of motion is evaluated by performing a perturbative expansion in $\tau^{-1/3}$,

$$\begin{aligned}\delta\varPhi &= c_1\,\mathrm{J}_0\Big(\int\omega^{(\phi)}(\tau)\,d\tau\Big)\sum_{j=0}^{\infty}\delta\phi_j(\rho)\,\tau^{-j/3} \\ &\quad + c_2\,\mathrm{Y}_0\Big(\int\omega^{(\phi)}(\tau)\,d\tau\Big)\sum_{j=0}^{\infty}\delta\tilde{\phi}_j(\rho)\,\tau^{-j/3},\end{aligned} \tag{5.75}$$

$$\omega^{(\phi)} = \sum_{i=0}^{\infty}\omega_i\,\tau^{-\frac{i}{3}}, \tag{5.76}$$

and analogously for the other fluctuations[2]. We use the known asymptotic expansion of the Bessel functions and obtain schematically the following equation,

$$\begin{aligned}&[\text{polynomial in }\tau^{-1/3}]\cos\Big(\int\omega^{(\phi)}\,d\tau\Big) \\ &\quad + [\text{polynomial in }\tau^{-1/3}]\sin\Big(\int\omega^{(\phi)}\,d\tau\Big) = 0.\end{aligned} \tag{5.77}$$

[1]We do not write out those quartic terms that can only produce terms quartic in fluctuations.
[2]The equation of motion of A_y requires a slightly modified ansatz given in the Appendix B.

At any given order, the requirement that the coefficients of the polynomials vanish, provides a differential equation for $\delta\phi_i$ and $\delta\tilde{\phi}_i$ depending on ω_i (and lower order solutions). We have to go to order 6 before the viscosity η_0 enters the equations. To this order, the equations for $\delta\phi_i$ and $\delta\tilde{\phi}_i$ can be separated by choosing suitable linear combinations and yield $\delta\phi_i \equiv \delta\tilde{\phi}_i$, which is what is required for the WKB ansatz (5.69) to be applicable. We impose the boundary conditions

$$\begin{aligned} \delta\phi \xrightarrow[\rho\to\infty]{} 0, \qquad \delta\phi \xrightarrow[\rho\to 0]{} \text{finite}, \\ \rho^2\delta\phi \xrightarrow[\rho\to\infty]{} \sqrt{\frac{\int\omega^{(\phi)}\,d\tau}{\omega^{(\phi)}\tau}}\mathcal{F}_0[\omega^{(\phi)}]. \end{aligned} \tag{5.78}$$

The first two of these conditions pick regular normalizable solutions, the last ensures that meson solutions on the boundary (5.65) satisfy the constraint (5.70). Consequently, the conditions fix two integration constants and the frequency ω_i at each order in the perturbative expansion. The only remaining free constants are the overall factors c_1 and c_2 of our ansatz (5.75).

Each of the coefficient functions has to satisfy a differential equation that is best expressed with the substitutions

$$\begin{aligned} \delta\phi_i(\rho) &= (1-\mathbf{y})^{-n-1}\delta\phi_i(\mathbf{y}), \\ \mathbf{y} &= -\rho^2/m^2, \\ \omega_0^2 &= m^2((2n+3)^2-1). \end{aligned} \tag{5.79}$$

The lowest order equation then reads

$$\begin{gathered} \left[\mathbf{y}(1-\mathbf{y})\partial_{\mathbf{y}}^2 + (\mathbf{c}-(\mathbf{a}+\mathbf{b}-1)\mathbf{y})\partial_{\mathbf{y}} - \mathbf{a}\,\mathbf{b}\right]\phi_0(\mathbf{y}) = 0, \\ \mathbf{a} = -n-1, \qquad \mathbf{b} = -n, \qquad \mathbf{c} = 2. \end{gathered} \tag{5.80}$$

This is exactly the hypergeometric equation already encountered in [102]. The boundary conditions fix n to be a non-negative integer, thus yielding a discrete meson spectrum $M = \omega_0(n)$ and the solutions in terms of (degenerate) hypergeometric functions ${}_2\mathrm{F}_1$ are

$$h_1 = {}_2\mathrm{F}_1(-n-1, -n, 2; \mathbf{y}), \tag{5.81}$$

$$h_2 = (1-\mathbf{y})^{3+2n}\,{}_2\mathrm{F}_1(n+2, n+3, 2n+4; 1-\mathbf{y}). \tag{5.82}$$

Only h_1 is regular, such that $\delta\phi_0 \equiv h_1$.

Higher orders in perturbation theory produce inhomogeneous terms in the analogues of (5.80). Since it is a linear ordinary equation, the solution can still be obtained in closed form by standard methods. However, the resulting integrals are hard to solve in general. Since both solutions $h_{1,2}$ are rational functions of $\mathbf{y}$ and $\ln\mathbf{y}$, it is easy to do so for definite n. For the lowest five mesons $n = 0, \ldots, 4$ we give the solutions in the Appendix B.

The mass[3] of the lowest scalar meson mode is

$$\omega^{(\phi)} = \frac{4\pi}{\sqrt{\lambda}} \cdot \left[m_q - \frac{3\lambda^2\varepsilon_0}{80\pi^4\tau^{4/3}m_q^3} \cdot \left(1 - \frac{2\eta_0}{\tau^{\frac{2}{3}}\varepsilon_0^{\frac{1}{4}}}\right)\right]. \tag{5.83}$$

[3] defined by equation (5.66)

To the considered order, pseudoscalar modes $\delta\psi$ have exactly the same equations of motion and the spectrum is degenerate. Moreover, we note that the spectrum agrees with the adiabatic approximation even including the viscosity corrections. The reason for this might be that the bulk metric coefficients that enter the calculation for the scalar mesons can be expressed completely in terms of the energy density, whereas the components for the $y, 2, 3$ directions cannot.

The vector mesons deviate slightly from the scalar modes. For comparison we plot the mass ratio of scalar and vector modes in Figure 5.4.

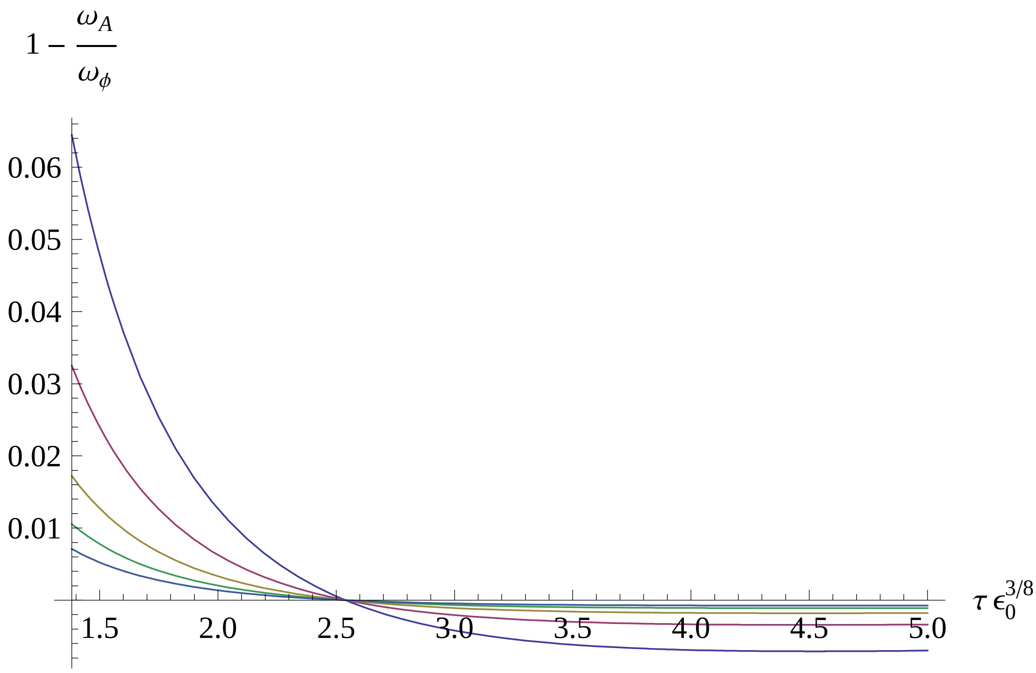

Figure 5.4: The plot shows the relative difference of the masses of (type II_y) vector and scalar mesons for $m = \varepsilon_0^{3/8}$.

The mass of the lowest vector mesons is given by

$$\omega^{(A^y)} = \frac{4\pi}{\sqrt{\lambda}} \cdot \left[m_q - \frac{7\lambda^2\varepsilon_0}{240\pi^4\tau^{4/3}m_q^3} \cdot \left(1 - \frac{6\eta_0}{7\tau^{\frac{2}{3}}\varepsilon_0^{\frac{1}{4}}}\right)\right], \tag{5.84}$$

$$\omega^{(A^{2,3})} = \frac{4\pi}{\sqrt{\lambda}} \cdot \left[m_q - \frac{7\lambda^2\varepsilon_0}{240\pi^4\tau^{4/3}m_q^3} \cdot \left(1 - \frac{18\eta_0}{7\tau^{\frac{2}{3}}\varepsilon_0^{\frac{1}{4}}}\right)\right]. \tag{5.85}$$

This agrees with the adiabatic approximation excluding the viscosity term. Since the metric components that enter the holographic computation, g_{yy} and g_{22}, agree only up to the viscosity terms, this deviation does not come as a surprise.

We plot the five lowest meson modes in Figure 5.5. The leading order term gives the exact supersymmetric spectrum that is approached for $\tau \to \infty$.

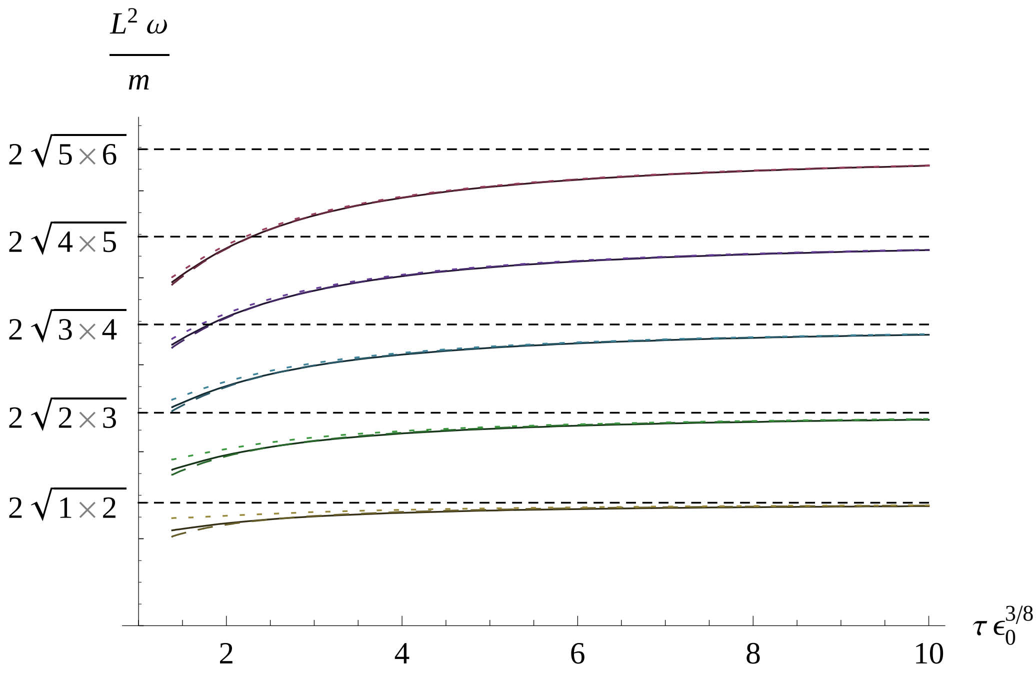

Figure 5.5: Late-time spectra for the viscous fluid geometry. The supersymmetric spectrum is shown as dashed horizontal lines. Scalar mesons are shown as continuous lines, vector modes are dashed.

5.4.4 Comparison to the adiabatic approximation

In this subsection, we will review some properties of low-temperature meson spectra for the static AdS black hole. Plugging in the time-dependence of the temperature into the static meson spectra, yields an estimate for the time-dependent spectrum, which we will refer to as adiabatic approximation. Using the similarity with the static black hole metric, we may read off the temperature from the position of the horizon

$$T(\tau) = \left(\frac{4\varepsilon_0}{3}\right)^{\frac{1}{4}} \frac{1}{\pi\tau^{\frac{1}{3}}} \left[1 - \frac{\eta_0}{2\varepsilon_0^{1/4}\tau^{2/3}}\right] \tag{5.86}$$

We will assume that the temperature dependence given in terms of Poincaré time t can be obtained from (5.86) by substituting τ for t

$$T_{AdS/BH}(t) = \left(\frac{4\varepsilon_0}{3}\right)^{\frac{1}{4}} \frac{1}{\pi\, t^{\frac{1}{3}}} \left[1 - \frac{\eta_0}{2\varepsilon_0^{1/4} t^{2/3}}\right]. \tag{5.87}$$

Note that we do not expect the resulting adiabatic meson spectra to accurately give the viscosity corrections. The reason for this is that even though the horizon position can be expressed completely in terms of the energy density, such that the Stefan–Boltzmann law holds, the bulk metric and energy-momentum tensor nevertheless contain additional

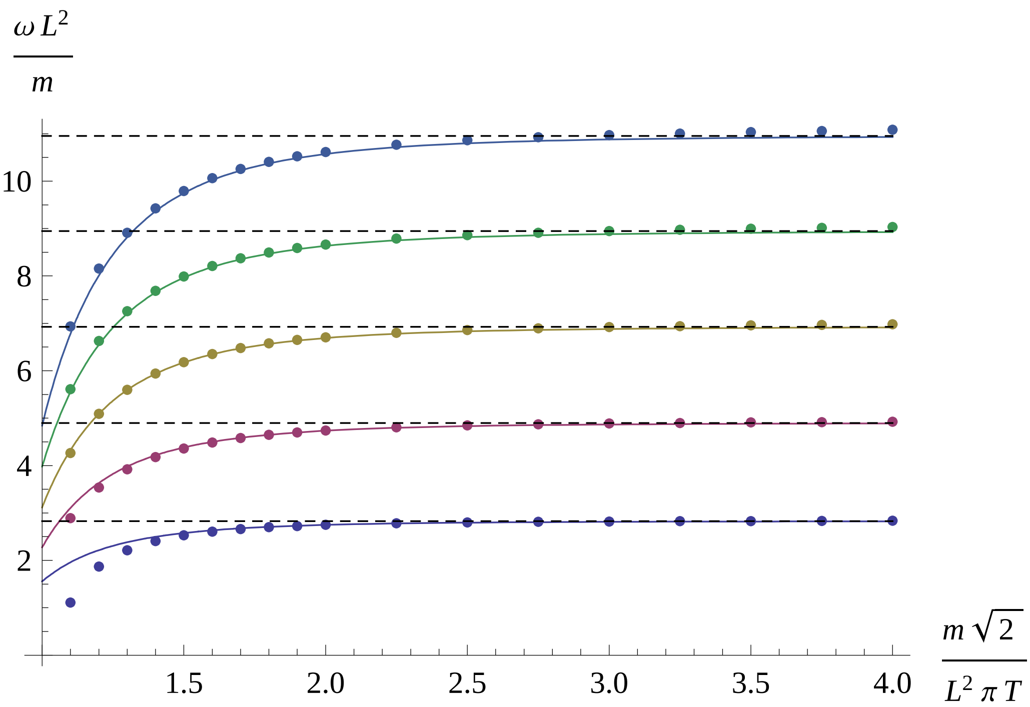

Figure 5.6: The plot shows the (pseudo)scalar meson spectrum. For small temperatures, the supersymmetric spectrum (dashed) is approached. The solid lines are asymptotic $T \to 0$ solutions, which are in good agreement with the numerical solutions (dots) for small temperatures.

viscosity terms that cannot be captured by the AdS/Schwarzschild geometry even when the geometry near the horizon and near the boundary is matched.

In Figure 5.6 we plot the numerical solution (dots) of the static case. We calculate the asymptotic solution in a low temperature expansion (solid curves), which agrees with the numerical calculation. Plugging the time dependence of the temperature (5.87) into this analytical approximation, we obtain the time-dependent meson spectrum in adiabatic approximation. The mass of the lowest scalar and vector modes are given by

$$\begin{aligned} \omega^{\text{ad.}}_{\phi,\psi} &= \frac{4\pi}{\sqrt{\lambda}}\left[m_q - \frac{9\lambda^2}{320}\frac{T^4}{m_q^3}\right] \\ &= \frac{4\pi}{\sqrt{\lambda}}\left[m_q - \frac{3\lambda^2}{80\pi^4}\frac{\varepsilon_0}{m_q^3}t^{-\frac{4}{3}}\left(1-\frac{2\eta_0}{t^{\frac{2}{3}}\varepsilon_0^{\frac{1}{4}}}\right)\right], \end{aligned} \tag{5.88}$$

$$\begin{aligned} \omega^{\text{ad.}}_{A^\mu} &= \frac{4\pi}{\sqrt{\lambda}}\left[m_q - \frac{7\lambda^2}{320}\frac{T^4}{m_q^3}\right] \\ &= \frac{4\pi}{\sqrt{\lambda}}\left[m_q - \frac{7\lambda^2}{240\pi^4}\frac{\varepsilon_0}{m_q^3}t^{-\frac{4}{3}}\left(1-\frac{2\eta_0}{t^{\frac{2}{3}}\varepsilon_0^{\frac{1}{4}}}\right)\right]. \end{aligned} \tag{5.89}$$

The mass of all of these modes decreases for increasing temperature. Moreover, the adiabatic scalar modes completely agree with our result (5.83), whereas the vector modes only agree in the leading contribution, the viscosity terms differ. We consider the agreement of the scalars accidental in the sense that it is a consequence of a certain property of the expanding plasma geometry, all metric coefficients entering the scalar equation of motion can be expressed purely in terms of the temperature, while additional viscosity terms only show up in those metric coefficients that end up in the equations for the vector modes.

An important assumption in our calculation has been that the Klein–Gordon equation is obeyed by scalar particles. This assumption can actually be proved employing the holographic equation of motion resulting from the linearization procedure. There is a parity symmetry $\rho \mapsto -\rho$ in the equation of motion. Since only 'Minkowski-type' solutions are considered, this will lead to even solutions, which have the following expansion,

$$\delta\Phi(\rho \to \infty, \tau) \sim \Phi_{4d}(\tau)\frac{1}{\rho^2} - \frac{L^4\Phi_{4d}(\tau)}{8}M^2(\tau)\frac{1}{\rho^4} + \dots, \tag{5.90}$$

because only normalizable solutions are allowed, such that the constant leading term vanishes. The subleading coefficient $M^2(\tau)$ has been multiplied by an additional factor $-L^4\Phi_{4d}/8$ for later convenience. Plugging above expansion into the eight-dimensional equation of motion, yields at leading order in $1/\rho$,

$$-\frac{1}{\tau}\partial_\tau(\tau\partial_\tau)\Phi_{4d} = M^2(\tau)\Phi_{4d}. \tag{5.91}$$

This establishes that at least for a background geometry dual to a hydrodynamic expansion up to and including viscosity, the scalar meson equation is a Klein–Gordon equation with time-dependent mass.

We will now assess the error of the WKB approximation by plugging our meson solutions into the four dimensional Klein–Gordon equation. The error should be smaller than τ^{-2} to be subleading to the viscosity contribution. We first determine the four dimensional meson solution by

$$\Phi_{4d}^{(\ell)} = \lim_{\rho\to\infty} \rho^2\delta\phi(\rho,\tau)\mathcal{F}_0[\omega^{(\phi)}], \tag{5.92}$$

where on the right hand side we plug in the mass (5.83) of the lowest holographic scalar meson solution, i.e., we set $\omega_{4d} \approx \omega^{(\phi)}$.

With (5.92) the error estimate $\Delta\omega$ can be obtained from the Klein–Gordon equation

$$\frac{1}{\tau}\partial_\tau\tau\partial_\tau\Phi_{4d}(\tau) = \left(\omega^{(\phi)}(\tau) + \Delta\omega(\tau)\right)^2\Phi_{4d}(\tau), \tag{5.93}$$

by linearizing in $\Delta\omega$. This yields

$$\Delta\omega(\tau) = \frac{\sqrt{2}L^{10}\varepsilon_0}{5m^4}\frac{1}{\tau^{10/3}} + \dots, \tag{5.94}$$

which is sufficiently small, that is subleading to the viscosity terms arising from the geometry. (Also note that the frequencies $\omega^{(\phi)}(\tau)$ and the meson mass obtained from the holographic expansion (5.90) agree up to and including order τ^{-2}.) However, when encoding hydrodynamic effects in the geometry that are of sufficiently high order, we would be forced to consider a better approximation for our ansatz, e.g., by using higher order WKB. Moreover, beyond a certain order the WKB ansatz is expected not to work anymore because the coefficients $\delta\phi_i$ and $\delta\tilde{\phi}$ are not expected to coincide to arbitrary order.

The main goal of this chapter was to study fundamental fields in the holographic dual of an expanding viscous fluid. We determined the $D7$ embedding and calculated the consequences of dynamical temperature for the chiral condensate to three orders. The leading order gives the supersymmetric solution, the subleading corresponds to the adiabatic approximation and the subsubleading order includes viscosity corrections going beyond the adiabatic approximation. Moreover, we calculated the meson spectrum and found that it agrees in the subleading order, though only the scalar mesons agree in the subsubleading order with the adiabatic approximation. The agreement crucially depends on the choice of ansatz defining the frequencies. We demonstrated that for our WKB ansatz solves the Klein–Gordon up to an error smaller than that inescapably introduced in the late-time expansion of the geometry. It would be interesting to find dual, expanding plasma description in more realistic systems and extend these calculations. Moreover, in the high enough temperature we expect mesons to be unstable and melt in the plasma. Another interesting extension of our analysis would be to study that transition following the logic of [104].

Chapter 6

Conclusions

The AdS/CFT correspondence provides tools to relate weakly coupled string theories with strongly coupled field theories. In this Thesis we focused on a particular regime of the correspondence, described by the equations of hydrodynamics. Let us briefly summarize the original results.

In Chapter 2 we provided field theory explanation for new terms in hydrodynamic expansion, related to triangle anomalies. These terms were first noted in the context of AdS/CFT correspondence. However, the lack of field theoretic understanding of such terms created a discrepancy between hydrodynamics and its dual description [105]. Motivated by this discrepancy, we generalized the notion of hydrodynamics and showed how the additional terms arise from underpinning quantum field theories. We used methods based on the entropy current. Assuming that the second law of thermodynamics is satisfied we derived constraints on the anomalous transport coefficients, which determined them completely.

In Chapter 4 we were able to derive transport coefficients, up to the second order in the derivative expansion for the fluid with one conserved current. A very small value of the shear viscosity over entropy density (4.61) seemed to confirm the assumption that QGP plasma in $\mathcal{N} = 4$ SYM is strongly coupled. Moreover, gravity construction indicated that there are parity breaking terms in the relativistic fluids, which originate from Chern-Simons terms in the supergravity action. These were matched against terms coming from anomalies in the dual theory, and we found perfect agreement. This solved the discrepancy noted in [105]. Finally, we focused on a subclass of conformal fluids which exhibits boost-invariance. We constructed an explicit gravity dual for the boost-invariant fluids and performed a detailed regularity analysis.

In Chapter 5 we showed how to introduce matter in a fundamental representation in the boost-invariant geometry. We followed known procedure of [93], which requires adding a new D-brane to the system. Studying small fluctuations around that brane we predicted a meson spectrum in the boost-invariant setup. A novel feature in this approach is the effect of viscosity included in the meson masses.

The AdS/CFT correspondence has been very useful in understanding properties of strongly coupled relativistic field theories, especially in the hydrodynamic regime. It is natural to ask what are possible extensions of its applicability. The most promising new applications seem to be in condensed matter physics. Systems to which holographic techniques might be applied, are those at quantum criticality. We may define a critical system by a Wilson-Fisher fixed point [106]. At the fixed point itself, the theory is scale invariant and possess the so-called Schrödinger symmetry, which can be mapped to the appropriate isometry of the dual metric [107]. Recently, there has recently been significant progress in constructing holographic duals for non-relativistic field theories (see [108, 109]). We can use them to study non-relativistic scale invariant fluids like the dilute Fermi gas at unitarity. There is a consistent procedure, which can be used to derive that metric from relativistic counterpart [110, 111]. Therefore, knowing a relativistic solution to the supergravity equations, which corresponds to some interesting physical system, we can transform it to non-relativistic solution. The field theory dual is in this case potentially more accessible experimentally.

Another interesting recent progress is the construction of holographic models of superfluidity [112] and superconductivity [113, 114]. Superfluid phase transitions are associated with spontaneous symmetry breaking while superconducting phase transitions with the Higgs mechanism. A detailed analysis of transport phenomena in both cases is still lacking. There is also no rigorous construction of gravity dual solutions, most solutions are purely numerical, apart from a very simplified situation [115].

From a practical point of view it would be remarkable, either to find a dual description of QCD, or to be able to engineer some field theory for which a dual theory exists. At the moment both cases seem to be very far from accomplishment. Hopefully we will overcome all obstacles to find a real link between string theory and experiment.

Appendix A

Second order fluid/gravity duality

Second order terms

List of two derivative terms		
1 of $SO(3)$	**3** of $SO(3)$	**5** of $SO(3)$
$\text{S1} = \partial_v^2 m$	$\text{V1}_i = \partial_i \partial_v m$	$\text{T1}_{ij} = \partial_i \partial_j m - \frac{1}{3}\ \text{s3}\ \delta_{ij}$
$\text{S2} = \partial_v \partial_i \beta_i$	$\text{V2}_i = \partial_v^2 \beta_i$	$\text{T2}_{ij} = \partial_{(i} l_{j)}$
$\text{S3} = \partial^2 m$	$\text{V3}_i = \partial_v l_i$	$\text{T3}_{ij} = \partial_v \sigma_{ij}$
$\text{ST1} = \partial_v \beta_i\, \partial_v \beta_i$	$\text{V4}_i = \frac{9}{5} \partial_j \sigma_{ji} - \partial^2 \beta_i$	$\text{TT1}_{ij} = \partial_v \beta_i\, \partial_v \beta_j - \frac{1}{3}\ \text{ST1}\ \delta_{ij}$
$\text{ST2} = l_i\, \partial_v \beta_i$	$\text{V5}_i = \partial^2 \beta_i$	$\text{TT2}_{ij} = l_{(i}\, \partial_v \beta_{j)} - \frac{1}{3}\ \text{ST2}\ \delta_{ij}$
$\text{ST3} = (\partial_i \beta_i)^2$	$\text{VT1}_i = \frac{1}{3} (\partial_v \beta_i)(\partial_j \beta^j)$	$\text{TT3}_{ij} = 2\, \epsilon_{kl(i}\, \partial_v \beta^k\, \partial_{j)} \beta^l + \frac{2}{3}\ \text{ST2}\ \delta_{ij}$
$\text{ST4} = l_i\, l^i$	$\text{VT2}_i = -\epsilon_{ijk}\, l^j\, \partial_v \beta^k$	$\text{TT4}_{ij} = \partial_k \beta^k\, \sigma_{ij}$
$\text{ST5} = \sigma_{ij}\, \sigma^{ij}$	$\text{VT3}_i = \sigma_{ij}\, \partial_v \beta^j$	$\text{TT5}_{ij} = l_i\, l_j - \frac{1}{3}\ \text{ST4}\ \delta_{ij}$
$\text{QS1} = \partial_v^2 q$	$\text{VT4}_i = l_i\, \partial_j \beta^j$	$\text{TT6}_{ij} = \sigma_{ik}\, \sigma^k_j - \frac{1}{3}\ \text{ST5}\ \delta_{ij}$
$\text{QS2} = \partial_i \partial_i q$	$\text{VT5}_i = \sigma_{ij}\, l^j$	$\text{TT7}_{ij} = 2\, \epsilon_{mn(i}\, l^m\, \sigma^n_{j)}$
$\text{QS3} = l_i \partial_i q$	$\text{QV1}_i = \partial_i \partial_v q$	$\text{QT1}_{ij} = \partial_i \partial_j q - \frac{1}{3}\ \text{QS2}\ \delta_{ij}$
$\text{QS4} = (\partial_i q)^2$	$\text{QV2}_i = \partial_i q \partial_k \beta^k$	$\text{QT2}_{ij} = \partial_{(i} q l_{j)} - \frac{1}{3}\ \text{QS3}\ \delta_{ij}$
$\text{QS5} = (\partial_i q)(\partial_v \beta_i)$	$\text{QV3}_i = \epsilon_{ijk} \partial_j l_k$	$\text{QT3}_{ij} = \partial_{(i} q \partial_{j)} q - \frac{1}{3}\ \text{QS4}\ \delta_{ij}$
	$\text{QV4}_i = \sigma_{ij} \partial_j q$	$\text{QT4}_{ij} = \partial_{(i} q \partial_v \beta_{j)} - \frac{1}{3}\ \text{QS5}\ \delta_{ij}$
	$\text{QV5}_i = \epsilon_{ijk} \partial_v \beta_j \partial_k q$	$\text{QT5}_{ij} = \epsilon_{(ikm} \partial_k q\ \sigma_{mj)}$

Table A.1: An exhaustive list of two derivative terms in made up from the mass, charge and velocity fields.

In order to present the results economically, we have dropped the superscript on the velocities β_i, the charge q, and the mass m, leaving it implicit that these expressions are only valid at second order in the derivative expansion. Moreover, we have defined the following quantities,

$$\begin{aligned} l_i &= \epsilon_{ijk}\partial_j\beta_k, \\ \sigma_{ij} &= \frac{1}{2}(\partial_i\beta_j + \partial_j\beta_k) - \frac{1}{3}\delta_{ij}\partial_k\beta_k \ . \end{aligned} \tag{A.1}$$

Weyl-covariant rearrangement of second order terms

We can arrange the terms in Table A.1 in the Weyl-covariant combinations, using the prescription of [116] (see also [117]).
There are six scalar/pseudo-scalar Weyl-covariant combinations given by

$$\begin{aligned} W_s^1 &\equiv \sigma_{\mu\nu}\sigma^{\mu\nu} = \mathrm{ST5}, \\ W_s^2 &\equiv \omega_{\mu\nu}\omega^{\mu\nu} = \frac{1}{2}\mathrm{ST4}, \\ W_s^3 &\equiv \mathcal{R} = 14\ \mathrm{ST1} + \frac{2}{3}\mathrm{ST3} - \mathrm{ST4} + 2\mathrm{ST5} - \frac{\mathrm{S3}}{m}, \\ W_s^4 &\equiv n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n = \frac{1}{q}\left[\mathrm{QS2} - \frac{3q}{4m}\mathrm{S3} + 18q\mathrm{ST1} + 5\mathrm{QS5}\right], \\ W_s^5 &\equiv n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\ \mathcal{D}_\nu n = \frac{1}{q^2}\left[\mathrm{QS4} + 6q\mathrm{QS5} + 9q^2\mathrm{ST1}\right], \\ W_s^6 &\equiv l^\mu\mathcal{D}_\mu q = \mathrm{QS3} + 3q\mathrm{ST2}. \end{aligned} \tag{A.2}$$

and five vector/pseudo-vector Weyl covariant combinations given by

$$\begin{aligned} (W_v)^1_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda\sigma_\nu{}^\lambda = \frac{5\mathrm{V4}}{9} + \frac{5\mathrm{V5}}{9} + \frac{5\mathrm{VT1}}{3} - \frac{5\mathrm{VT2}}{12} - \frac{11\mathrm{VT3}}{6}, \\ (W_v)^2_\mu &\equiv P^\nu_\mu\mathcal{D}_\lambda\omega_\nu{}^\lambda = \frac{5\mathrm{V4}}{3} - \frac{\mathrm{V5}}{3} - \mathrm{VT1} - \frac{\mathrm{VT2}}{4} + \frac{\mathrm{VT3}}{2}, \\ (W_v)^3_\mu &\equiv l^\lambda\sigma_{\mu\lambda} = \mathrm{VT5}, \\ (W_v)^4_\mu &\equiv n^{-1}\sigma_\mu{}^\lambda\mathcal{D}_\lambda n = \frac{1}{q}\left[\mathrm{QV4} + 3q\mathrm{VT3}\right], \\ (W_v)^5_\mu &\equiv n^{-1}\omega_\mu{}^\lambda\mathcal{D}_\lambda n = \frac{1}{2q}\left[\mathrm{QV3} + 3q\mathrm{VT2}\right]. \end{aligned} \tag{A.3}$$

In the tensor sector, there are nine Weyl-covariant combinations

$$
\begin{aligned}
WT^{(1)}_{\mu\nu} &= u^{\lambda}\mathcal{D}_{\lambda}\sigma_{\mu\nu} = \mathrm{TT1} + \frac{1}{3}\mathrm{TT4} + \mathrm{T3},\\
WT^{(2)}_{\mu\nu} &= -2\left(\omega^{\mu}{}_{\lambda}\sigma^{\lambda\nu} + \omega^{\nu}{}_{\lambda}\sigma^{\lambda\mu}\right) = \mathrm{TT7},\\
WT^{(3)}_{\mu\nu} &= \sigma^{\mu}{}_{\lambda}\sigma_{\lambda\nu} - \frac{1}{3}P^{\mu\nu}\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \mathrm{TT6},\\
WT^{(4)}_{\mu\nu} &= 4\left(\omega^{\mu}{}_{\lambda}\omega_{\lambda\nu} + \frac{1}{3}P^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta}\right) = \mathrm{TT5},\\
WT^{(5)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_{\alpha}\mathcal{D}_{\beta}n\\
&= \frac{1}{q}\left[\mathrm{QT1} + 8\mathrm{QT4} + 15q\mathrm{TT1} + q\mathrm{TT4} + 3q\mathrm{T3} + 3q\mathrm{TT6} + \frac{3q}{4}\mathrm{TT5}\right],\\
WT^{(6)}_{\mu\nu} &= n^{-2}\Pi^{\alpha\beta}_{\mu\nu}\mathcal{D}_{\alpha}n\mathcal{D}_{\beta}n = \frac{1}{q^2}\left[\mathrm{QT3} + 6q\mathrm{QT4} + 9q^2\mathrm{TT1}\right],\\
WT^{(7)}_{\mu\nu} &= \mathcal{D}_{\mu}l_{\nu} + \mathcal{D}_{\nu}l_{\mu} = 4\mathrm{TT2} + 2\mathrm{T2} - \mathrm{TT3},\\
WT^{(8)}_{\mu\nu} &= n^{-1}\Pi^{\alpha\beta}_{\mu\nu}l_{\alpha}\mathcal{D}_{\beta}n = \frac{1}{q}\left[\mathrm{QT2} + 3\ q\ \mathrm{TT2}\right],\\
WT^{(9)}_{\mu\nu} &= n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_{\alpha}\mathcal{D}_{\beta}n = \frac{1}{q}\left[\mathrm{QT5} - \frac{3}{2}\ q\ \mathrm{TT2} + \frac{3}{2}\ q\ \mathrm{TT3}\right].
\end{aligned}
\tag{A.4}
$$

The charge current at second order in derivative expansion

Using the prescription

$$
J^{(2)}_{\mu} = \lim_{r\to\infty}\frac{r^2 A^{(2)}_{\mu}}{8\pi G_5}\ , \tag{A.5}
$$

we find the following expression for the second order contribution to the current

$$
J^{(2)}_{i} = \left(\frac{1}{8\pi G_5}\right)\sum_{l=1}^{5}\mathcal{C}_l (W_v)^l_i, \tag{A.6}
$$

where the coefficients of the Weyl invariant terms $(W_v)^l_i$ are given by

$$
\begin{aligned}
\mathcal{C}_1 &= \frac{3\sqrt{3}R\sqrt{M-1}}{8M},\\
\mathcal{C}_2 &= \frac{\sqrt{3}R(M-1)^{3/2}}{4M^2},\\
\mathcal{C}_3 &= -\frac{3R\kappa(M-1)}{2M^2},\\
\mathcal{C}_4 &= \frac{1}{4}\sqrt{3}R\sqrt{M-1}\log(2) + \mathcal{O}(M-1),\\
\mathcal{C}_5 &= -\frac{\sqrt{3}R\sqrt{M-1}\,(M^2 - 48(M-1)\kappa^2 + 3)}{16M^2}.
\end{aligned}
\tag{A.7}
$$

Boundary Stress Tensor at second order

The expression for the second order contribution to the energy-momentum tensor reads

$$T_{\mu\nu} = \left(\frac{1}{16\pi G_5}\right)\sum_{l=1}^{9}\mathcal{N}_l\, WT^{(l)}_{\mu\nu}, \tag{A.8}$$

with $\mathcal{N}_l$ being the transport coefficients at second order in derivative expansion. These transport coefficients are given by

$$\begin{aligned}
\mathcal{N}_1 &= R^2\left(\frac{M}{\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{3+\sqrt{4M-3}}\right)+2\right),\\
\mathcal{N}_2 &= -\frac{MR^2}{2\sqrt{4M-3}}\log\left(\frac{3-\sqrt{4M-3}}{\sqrt{4M-3}+3}\right),\\
\mathcal{N}_3 &= 2R^2,\\
\mathcal{N}_4 &= \frac{R^2}{M}(M-1)\left(12(M-1)\kappa^2-M\right),\\
\mathcal{N}_5 &= -\frac{(M-1)R^2}{2M},\\
\mathcal{N}_6 &= \frac{1}{2}(M-1)R^2\Big(\log(8)-1\Big)+\mathcal{O}\left((M-1)^2\right),\\
\mathcal{N}_7 &= \frac{\sqrt{3}(M-1)^{3/2}R^2\kappa}{M},\\
\mathcal{N}_8 &= 0,\\
\mathcal{N}_9 &= 0.
\end{aligned} \tag{A.9}$$

Appendix B

Meson solutions

(Pseudo-)scalar mesons

$$\delta\Phi_0 = \mathcal{F}_0[\omega_0]\left[\frac{\left(8m^4+9\rho^2m^2+3\rho^4\right)\varepsilon_0L^8}{10m^4(m^2+\rho^2)^3\tau^{4/3}} - \frac{\left(13m^4+12\rho^2m^2+3\rho^4\right)\varepsilon_0^{3/4}\eta_0L^8}{10m^4(m^2+\rho^2)^3\tau^2} + \frac{1}{m^2+\rho^2}\right],$$

$$\omega_0 = \frac{2\sqrt{2}m}{L^2} + \left(-\frac{3L^6\tau^{-4/3}\varepsilon_0}{5\sqrt{2}m^3}\right)\times\left(1-\frac{2\eta_0}{\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$\delta\Phi_1 = \mathcal{F}_0[\omega_1]\left[\frac{\left(-12m^6+15\rho^2m^4+20\rho^4m^2+5\rho^6\right)\varepsilon_0L^8}{14m^4(m^2+\rho^2)^4\tau^{4/3}} + \frac{\left(19m^6-35\rho^2m^4-35\rho^4m^2-5\rho^6\right)\varepsilon_0^{3/4}\eta_0L^8}{14m^4(m^2+\rho^2)^4\tau^2} + \frac{\rho^2-m^2}{(m^2+\rho^2)^2}\right],$$

$$\omega_1 = \frac{2\sqrt{6}m}{L^2} + \left(-\frac{5\sqrt{3}L^6\tau^{-4/3}\varepsilon_0}{7\sqrt{2}m^3}\right)\times\left(1-\frac{2\eta_0}{\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$\delta\Phi_2 = \mathcal{F}_0[\omega_2]\left[\frac{\left(26m^8-140\rho^2m^6-22\rho^4m^4+55\rho^6m^2+11\rho^8\right)\varepsilon_0L^8}{30m^4(m^2+\rho^2)^5\tau^{4/3}}\right.$$
$$\left. - \frac{\left(41m^8-269\rho^2m^6+121\rho^6m^2+11\rho^8\right)\varepsilon_0^{3/4}\eta_0L^8}{30m^4(m^2+\rho^2)^5\tau^2} + \frac{m^4-3\rho^2m^2+\rho^4}{(m^2+\rho^2)^3}\right],$$

$$\omega_2 = \frac{4\sqrt{3}m}{L^2} + \left(-\frac{11L^6\tau^{-4/3}\varepsilon_0}{5\sqrt{3}m^3}\right)\times\left(1-\frac{2\eta_0}{\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$\delta\Phi_3 = \mathcal{F}_0[\omega_3]\left[\frac{\left(-134m^{10}+1506\rho^2m^8-1650\rho^4m^6-1045\rho^6m^4+342\rho^8m^2+57\rho^{10}\right)\varepsilon_0L^8}{154m^4(m^2+\rho^2)^6\tau^{4/3}}\right.$$
$$+ \frac{\left(211m^{10}-2784\rho^2m^8+3585\rho^4m^6+1805\rho^6m^4-912\rho^8m^2-57\rho^{10}\right)\varepsilon_0^{3/4}\eta_0L^8}{154m^4(m^2+\rho^2)^6\tau^2}$$
$$\left. + \frac{-m^6+6\rho^2m^4-6\rho^4m^2+\rho^6}{(m^2+\rho^2)^4}\right],$$

$$\omega_3 = \frac{4\sqrt{5}m}{L^2} + \left(-\frac{57\sqrt{5}L^6\tau^{-4/3}\varepsilon_0}{77m^3}\right)\times\left(1-\frac{2\eta_0}{\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$\delta\Phi_4 = \mathcal{F}_0[\omega_4]\Bigg[\frac{\left(68m^{12}-1279\rho^2m^{10}+3543\rho^4m^8-630\rho^6m^6-1566\rho^8m^4+203\rho^{10}m^2+29\rho^{12}\right)\varepsilon_0L^8}{78m^4(m^2+\rho^2)^7\tau^{4/3}} - \frac{\left(107m^{12}-2326\rho^2m^{10}+7057\rho^4m^8-1840\rho^6m^6-3161\rho^8m^4+638\rho^{10}m^2+29\rho^{12}\right)\varepsilon_0^{3/4}\eta_0L^8}{78m^4(m^2+\rho^2)^7\tau^2} + \frac{m^8-10\rho^2m^6+20\rho^4m^4-10\rho^6m^2+\rho^8}{(m^2+\rho^2)^5}\Bigg],$$

$$\omega_4 = \frac{2\sqrt{30}m}{L^2} + \left(-\frac{29\sqrt{\frac{5}{6}}L^6\tau^{-4/3}\varepsilon_0}{13m^3}\right) \times \left(1 - \frac{2\eta_0}{\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right).$$

Vector meson (Type $\mathrm{II}_{2,3}$)

$$(\delta a_{II2})_0 = \mathcal{F}_0[\omega_0]\left[\frac{\left(22m^4+21\rho^2m^2+7\rho^4\right)\varepsilon_0L^8}{30m^4(m^2+\rho^2)^3\tau^{4/3}} - \frac{\left(13m^4+12\rho^2m^2+3\rho^4\right)\varepsilon_0^{3/4}\eta_0L^8}{10m^4(m^2+\rho^2)^3\tau^2} + \frac{1}{m^2+\rho^2}\right],$$

$$\omega_0 = \frac{2\sqrt{2}m}{L^2} + \left(-\frac{7L^6\tau^{-4/3}\varepsilon_0}{15\sqrt{2}m^3}\right) \times \left(1 - \frac{18\eta_0}{7\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{II2})_1 = \mathcal{F}_0[\omega_1]\Bigg[\frac{\left(-118m^6+137\rho^2m^4+164\rho^4m^2+41\rho^6\right)\varepsilon_0L^8}{126m^4(m^2+\rho^2)^4\tau^{4/3}} + \frac{\left(19m^6-35\rho^2m^4-35\rho^4m^2-5\rho^6\right)\varepsilon_0^{3/4}\eta_0L^8}{14m^4(m^2+\rho^2)^4\tau^2} + \frac{\rho^2-m^2}{(m^2+\rho^2)^2}\Bigg],$$

$$\omega_1 = \frac{2\sqrt{6}m}{L^2} + \left(-\frac{41L^6\tau^{-4/3}\varepsilon_0}{21\sqrt{6}m^3}\right) \times \left(1 - \frac{90\eta_0}{41\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{II2})_2 = \mathcal{F}_0[\omega_2]\Bigg[\frac{\left(178m^8-880\rho^2m^6-106\rho^4m^4+315\rho^6m^2+63\rho^8\right)\varepsilon_0L^8}{180m^4(m^2+\rho^2)^5\tau^{4/3}} - \frac{\left(41m^8-269\rho^2m^6+121\rho^6m^2+11\rho^8\right)\varepsilon_0^{3/4}\eta_0L^8}{30m^4(m^2+\rho^2)^5\tau^2} + \frac{m^4-3\rho^2m^2+\rho^4}{(m^2+\rho^2)^3}\Bigg],$$

$$\omega_2 = \frac{4\sqrt{3}m}{L^2} + \left(-\frac{7\sqrt{3}L^6\tau^{-4/3}\varepsilon_0}{10m^3}\right) \times \left(1 - \frac{44\eta_0}{21\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{II2})_3 = \mathcal{F}_0[\omega_3]\Bigg[\frac{\left(-4666m^{10}+48234\rho^2m^8-52030\rho^4m^6-29975\rho^6m^4+9978\rho^8m^2+1663\rho^{10}\right)\varepsilon_0L^8}{4620m^4(m^2+\rho^2)^6\tau^{4/3}} + \frac{\left(211m^{10}-2784\rho^2m^8+3585\rho^4m^6+1805\rho^6m^4-912\rho^8m^2-57\rho^{10}\right)\varepsilon_0^{3/4}\eta_0L^8}{154m^4(m^2+\rho^2)^6\tau^2} + \frac{-m^6+6\rho^2m^4-6\rho^4m^2+\rho^6}{(m^2+\rho^2)^4}\Bigg],$$

$$\omega_3 = \frac{4\sqrt{5}m}{L^2} + \left(-\frac{1663L^6\tau^{-4/3}\varepsilon_0}{462\sqrt{5}m^3}\right) \times \left(1 - \frac{3420\eta_0}{1663\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{II2})_4 = \mathcal{F}_0[\omega_4]\Bigg[\frac{\left(1194m^{12}-20697\rho^2m^{10}+55709\rho^4m^8-10890\rho^6m^6-22928\rho^8m^4+2989\rho^{10}m^2+427\rho^{12}\right)\varepsilon_0L^8}{1170m^4(m^2+\rho^2)^7\tau^{4/3}}$$
$$-\frac{\left(107m^{12}-2326\rho^2m^{10}+7057\rho^4m^8-1840\rho^6m^6-3161\rho^8m^4+638\rho^{10}m^2+29\rho^{12}\right)\varepsilon_0^{3/4}\eta_0L^8}{78m^4(m^2+\rho^2)^7\tau^2}$$
$$+\frac{m^8-10\rho^2m^6+20\rho^4m^4-10\rho^6m^2+\rho^8}{(m^2+\rho^2)^5}\Bigg],$$
$$\omega_4 = \frac{2\sqrt{30}m}{L^2} + \left(-\frac{427L^6\tau^{-4/3}\varepsilon_0}{39\sqrt{30}m^3}\right)\times\left(1-\frac{870\eta_0}{427\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right).$$

Vector meson (Type II$_y$)

Vector mesons of type II$_y$ obey a different equation of motion (5.53), such that the WKB approximation yields a different result

$$A_y^{4d}(\tau) \approx \sqrt{\frac{\int \omega_{4d}(\tau)\, d\tau}{\omega_{4d}\,\tau}}\,\tau\,\mathcal{F}_1. \tag{B.1}$$

The corresponding holographic ansatz

$$\delta A_y = \delta a_{IIy}\,\tau\,\mathrm{J}_1(\textstyle\int \omega\, d\tau) + \delta\tilde{a}_{IIy}\,\tau\,\mathrm{Y}_1(\textstyle\int\omega\, d\tau) \tag{B.2}$$

gives rise to the following solutions for the five lowest mesons,

$$(\delta a_{IIy})_0 = \tau\mathcal{F}_1[\omega_0]\left[\frac{\left(22m^4+21\rho^2m^2+7\rho^4\right)\varepsilon_0L^8}{30m^4(m^2+\rho^2)^3\tau^{4/3}} - \frac{\left(11m^4+4\rho^2m^2+\rho^4\right)\varepsilon_0^{3/4}\eta_0L^8}{10m^4(m^2+\rho^2)^3\tau^2} + \frac{1}{m^2+\rho^2}\right],$$
$$\omega_0 = \frac{2\sqrt{2}m}{L^2} + \left(-\frac{7L^6\tau^{-4/3}\varepsilon_0}{15\sqrt{2}m^3}\right)\times\left(1-\frac{6\eta_0}{7\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{IIy})_1 = \tau\mathcal{F}_1[\omega_1]\Bigg[\frac{\left(-118m^6+137\rho^2m^4+164\rho^4m^2+41\rho^6\right)\varepsilon_0L^8}{126m^4(m^2+\rho^2)^4\tau^{4/3}}$$
$$+\frac{\left(81m^6-105\rho^2m^4-77\rho^4m^2-11\rho^6\right)\varepsilon_0^{3/4}\eta_0L^8}{42m^4(m^2+\rho^2)^4\tau^2}+\frac{\rho^2-m^2}{(m^2+\rho^2)^2}\Bigg],$$
$$\omega_1 = \frac{2\sqrt{6}m}{L^2} + \left(-\frac{41L^6\tau^{-4/3}\varepsilon_0}{21\sqrt{6}m^3}\right)\times\left(1-\frac{66\eta_0}{41\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$(\delta a_{IIy})_2 = \tau\mathcal{F}_1[\omega_2]\Bigg[\frac{\left(178m^8-880\rho^2m^6-106\rho^4m^4+315\rho^6m^2+63\rho^8\right)\varepsilon_0L^8}{180m^4(m^2+\rho^2)^5\tau^{4/3}}$$
$$-\frac{\left(129m^8-621\rho^2m^6+40\rho^4m^4+209\rho^6m^2+19\rho^8\right)\varepsilon_0^{3/4}\eta_0L^8}{60m^4(m^2+\rho^2)^5\tau^2}+\frac{m^4-3\rho^2m^2+\rho^4}{(m^2+\rho^2)^3}\Bigg],$$
$$\omega_2 = \frac{4\sqrt{3}m}{L^2} + \left(-\frac{7\sqrt{3}L^6\tau^{-4/3}\varepsilon_0}{10m^3}\right)\times\left(1-\frac{38\eta_0}{21\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right),$$

$$
\begin{aligned}
(\delta a_{IIy})_3 &= \tau\mathcal{F}_1[\omega_3]\Bigg[\frac{\left(-4666m^{10}+48234\rho^2m^8-52030\rho^4m^6-29975\rho^6m^4+9978\rho^8m^2+1663\rho^{10}\right)\varepsilon_0L^8}{4620m^4(m^2+\rho^2)^6\tau^{4/3}} \\
&\qquad + \frac{\left(3449m^{10}-34136\rho^2m^8+40675\rho^4m^6+15535\rho^6m^4-8368\rho^8m^2-523\rho^{10}\right)\varepsilon_0^{3/4}\eta_0L^8}{1540m^4(m^2+\rho^2)^6\tau^2} \\
&\qquad + \frac{-m^6+6\rho^2m^4-6\rho^4m^2+\rho^6}{(m^2+\rho^2)^4}\Bigg], \\
\omega_3 &= \frac{4\sqrt{5}m}{L^2} + \left(-\frac{1663L^6\tau^{-4/3}\varepsilon_0}{462\sqrt{5}m^3}\right)\times\left(1-\frac{3138\eta_0}{1663\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right), \\
(\delta a_{IIy})_4 &= \tau\mathcal{F}_1[\omega_4]\Bigg[\frac{\left(1194m^{12}-20697\rho^2m^{10}+55709\rho^4m^8-10890\rho^6m^6-22928\rho^8m^4+2989\rho^{10}m^2+427\rho^{12}\right)\varepsilon_0L^8}{1170m^4(m^2+\rho^2)^7\tau^{4/3}} \\
&\qquad - \frac{\left(891m^{12}-14718\rho^2m^{10}+40421\rho^4m^8-11920\rho^6m^6-14673\rho^8m^4+3014\rho^{10}m^2+137\rho^{12}\right)\varepsilon_0^{3/4}\eta_0L^8}{390m^4(m^2+\rho^2)^7\tau^2} \\
&\qquad + \frac{m^8-10\rho^2m^6+20\rho^4m^4-10\rho^6m^2+\rho^8}{(m^2+\rho^2)^5}\Bigg], \\
\omega_4 &= \frac{2\sqrt{30}m}{L^2} + \left(-\frac{427L^6\tau^{-4/3}\varepsilon_0}{39\sqrt{30}m^3}\right)\times\left(1-\frac{822\eta_0}{427\tau^{2/3}\sqrt[4]{\varepsilon_0}}\right).
\end{aligned}
$$

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