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A thesis submitted for the degree of Doctor of Philosophy in Physics

General Mass Scheme for Jet Production in QCD

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Revised July 2012

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Preface

Quantum chromodynamics (QCD) is – as we believe – the correct theory of the strong interactions, with quarks and gluons being its fundamental degrees of freedom. Although there are many puzzles remaining unsolved, it is very successful in describing various aspects of the modern high energy data. Theoretical predictions are based on two major issues. The most important one is the asymptotic freedom, which asserts that the value of the strong coupling constant is relatively low at high energy scales. It enables us to use perturbation theory in calculations concerning scattering amplitudes. However, there are no free quarks and gluons in the nature – they are all bounded in colourless hadrons, thus the perturbative calculations are not the whole story, as the hadrons are clearly of non-perturbative nature. Therefore, the second basic issue are factorization theorems, which allow for a separation of a process to a non-perturbative bound state physics and calculable in QCD hard scattering amplitudes. Technically the former is described in terms of various distribution functions and distribution amplitudes, which so far are most reliably taken from experiments.

Phenomenologically the most important non-perturbative input comprises parton distribution functions (PDFs). Historically, they appeared in a description of inclusive lepton-hadron deep inelastic scattering (DIS) as an element of the parton model. Nowadays they are used also in the other high energy experiments like proton-proton collisions at LHC for instance. In order to obtain PDFs one has to fit the theoretically calculated cross section with suitable analytical parametrizations of PDFs to the real data. Most often the electron-proton HERA data are used in this procedure. There are several groups making an effort in extracting PDFs, e.g. CTEQ group [5] or MRST group [41] to mention only the most known.

One may ask the question: what is the difference between various sets of PDFs? There are at least a few odds. The first are different functional parametrizations and different statistical methods used in fitting the data. However the main difference is connected to a scheme in which the actual cross section is calculated. This issue is inseparably related to heavy quarks and the problem of scales in QCD. Namely, there is a difficulty with finite-order perturbative calculations if there are a few external parameters (scales) that are very different. Those scales can be fixed e.g. by an external energy or by the masses of the quarks. Actually it is the case in reality, as we have six quark flavours with three of them being marginally heavier than the others. Moreover there is also substantial mass splittings between the heavy quarks. In some situations, this difficulty can be solved by means of the renormalization group methods, however there does not exist a uniform perturbative expansion suitable for all the scales. Therefore one has to choose a specific *scheme*. Intuitively it corresponds to a situation, where our measurement resolution is too small to distinguish some tiny details and to large to see the whole thing. We can however always change the instrument to get different insights. The same is true for the schemes in perturbative QCD.

For many years, a completely massless scheme have been the most standard in treating DIS scattering. Actually, the scheme was massless in the sense of neglecting the mass parameters in calculations, but changing the number of flavours in the same time. We shall see the details later, however even intuitively we see that such an approach is very limited in accuracy. It can be satisfactory only in certain, relatively narrow ranges of kinematic space. Therefore also the other schemes were used, treating the heavy quarks in a more ordered way. In particular, the schemes for charm, and bottom quarks were used. Those calculations are very accurate in a suitable kinematic range, but – again – those ranges are limited. Further development must have led to a composite scheme, that is to change the number of flavours on one hand (like in the massless scheme mentioned in the beginning of the paragraph) and to keep the masses finite on the other. The most common name for the approach of this type is a "general-mass scheme" or "variable flavour number scheme". We shall see such a solution in details below in this work. So far the general-mass schemes were used in inclusive processes, both in extracting the PDFs and predicting experimental outcome.

There is however another very important class of high energy processes, namely the production of jets. Since one measures also the spatial distribution of the outgoing particles it can give much more information about underlying parton dynamics. Loosely speaking, a jet – a collimated bunch of hadrons – is a remnant of a parton ejected from the center of collision. Thus by analysing the momentum and energy of the jet, we get almost direct access to the parton level subprocess. It allows for more precise measurements of some quantities, for instance strong coupling constant (e.g. the analysis performed by ZEUS collaboration using dijet production in DIS [1] and by H1 collaboration using inclusive-jet, dijets and trijet analysis [32]). The jets production processes are also used to obtain the parton distribution functions (together with the inclusive data). Theoretical calculations needed to this procedure are again scheme dependent. In case of jets, there is however much less theoretical development concerning heavy quarks. There are several Monte-Carlo (MC) programs using massless quarks, e.g. NLOJET++ [42], DISENT [9], both for hadron-hadron and lepton-hadron collisions (the last for neutral current without Z^0-exchange). For heavy quarks, there are some calculations for inclusive-jet and two jets production at NLO [26] in a scheme with fixed number of flavours. It should be remarked, that we mean here strict QCD calculations, not a model-based ones. For jets, the former are much more involved and require special treatments of singularities that appear at NLO (and higher) orders.

In this work, we propose a solution intended to fill the gap in existing heavy flavour treatments. It is a general-mass scheme for jets production processes, based on some solutions available on the market. We concentrate herein on DIS processes with neutral current interactions. Further extensions are possible, as we shortly discuss in Chapter 5. The developments we are presenting are essentially theoretical. However, in order to support the validity of our calculations we give some sample numerical results using a dedicated MC program. It is a part of a larger project that is currently under development.

The material is organised as follows. First, in Chapter 1 we recall the basic formalism we shall use throughout, including factorization theorems and jets treatment. Chapter 2 is devoted to existing approaches to heavy quarks in inclusive DIS processes and its problems. Notably, it introduces the general-mass solution, which we later apply to jets. Those two chapters possess mainly introductory character. Next, in Chapter 3, we re-analyze so called dipole subtraction method for jets, assuming the most general situation of massive partons, including possible initial state heavy quarks. Finally we gather all the pieces and construct the general-mass scheme for jets in Chapter 4. We introduce tons of symbols throughout this work. Some of them may look messy, however this accounts for the precise theoretical formulation of the material. In order to facilitate the reading we put some of them in a Nomenclature. The technical details that are not essential in the main text are listed in the appendices.

Acknowledgements

In the first place, I would like to thank Prof. Wojciech Słomiński for his patience, substantial help and many hours of joint discussions. The words of gratitude belong also to Prof. Michał Praszałowicz, the head of Particle Theory Department I had a great pleasure to be a member as PHD student.

I thank my wife Iwona for constant support and faith in me, especially when everything was going wrong. The same is true about her family: Brunon and Basia, Marek and Ela together with their children.

I own special thanks to Mirek Trociuk, my high school teacher and Ola Łubnicka, who created the scientific atmosphere that brought me to this point.

Finally, I am grateful to my parents Maria and Jerzy, my brothers Damian and Mateusz, and my little sister Kasia, for everything.

Chapter 1

Hadrons, partons and jets in QCD

1.1 Factorization theorems

Although QCD has incredible amount of successes, the theory is still not solved. For instance, there is a colour confinement hypothesis, stating that all observable particles are colour singlets. This conjecture has very strong experimental evidence; so far free quark or gluon has not been found. However, such a property has not been derived yet from QCD, although there are several theoretical clues, both perturbative and non-perturbative. Moreover, there does not exist a complete description of composed objects like hadrons in terms of the fundamental QCD degrees of freedom (i.e. quarks and gluons). For example, it is known that many features of a proton can be explained by assuming that it is build of three quarks u, u, d. Their masses (i.e. the parameters in QCD lagrangian; mass is poorly defined quantity for an unobservable particle) are about a few MeV. On the other hand, the proton mass is well defined and can be measured – it turns out to be around 1 GeV, clearly not about three times the masses of constituents. This is an evidence of very important non-perturbative phenomenon, namely spontaneous chiral symmetry breaking. It generates so called constituent quark mass, which should be about one third of the proton mass. Such a value cannot be described by perturbation theory, the tool which at present is best understood and under control. There are much more problems in describing hadrons within perturbative QCD.

All these features draw hadrons as a very complicated, non-perturbative objects. Nevertheless, there are possibilities to get certain insight into the structure of hadrons using perturbation theory. As it was already mentioned, QCD has a property of being asymptotically free, i.e. at very short distances the QCD coupling is very weak, giving some chances to use perturbation theory. This is a key observation leading to modern high-energy experiments; collisions of particles with higher energies can probe smaller space-time volumes. However, since we probe only a small part of a colourless hadron, we can hope to have perturbative description only partially – the rest must be somehow parametrized, or obtained by other methods. This is in fact a very loose description of famous factorization theorems, which we now shall recall in some details. In our introduction we shall try to give mostly necessary results, but we recall also some completely elementary facts concerning factorization.

We are mainly concentrated on lepton-hadron deep inelastic processes throughout. Moreover, in this section we limit ourselves to inclusive processes only. There are essentially two possible approaches to factorization, which percolate at some stages. Both have its own cons and pros.

First one, relies on the operator product expansion (OPE) [54] and is historically the first approach to factorization [7], of course except Feynman's parton formalism considered before QCD had been born. Although OPE allows for very systematic treatment of all terms that can appear, its applicability is rather limited to the inclusive processes only.

Second approach is based on general power counting theorems [40, 39] and methods developed in [21]. Let us recall the basic ideas, as they shall be important later, when we discuss more complicated topics. For a review see e.g. [17, 15].

Consider a generic unpolarized boson-hadron cut amplitude, as shown in Fig. 1.1A. We denote proton momentum as P and boson as q, with $q^2 = -Q^2$. Moreover, we assume that the boson virtuality Q^2 is much larger than all the quark masses (including possible heavy quarks) and that the Bjorken variable $x_B = Q^2/P \cdot q$ is fixed. The situation where Q^2 is of the same order as the mass of a given heavy quark will be discussed in the next chapter. It turns out that all the leading contributions to the cut amplitude can be characterized by the cut amplitudes that have the form showed in Fig. 1.1B. The upper blob has all the internal momenta off-shell by order Q^2 and thus is called a hard part. Note, that although some of the internal lines are cut and hence on-shell, they effectively can be treated as off-shell lines by virtue of the optical theorem. The lower part in Fig. 1.1B, the soft part, incorporates hadronic states and two partonic lines joining it with the hard part. Those lines are either quark or gluon lines with virtuality much lower than Q^2 and momenta collinear to the hadronic momentum. It should be mentioned that the internal blob of the soft part, can still have UV singularities, see below.

The contributions that have structure described above are called twist-2, as they correspond in OPE language to a series of matrix elements of local operators with twist[1] equal to 2. Contributions which have more than two lines joining hard and soft parts have higher twist. Recall that such higher twist contributions are suppressed by m^2/Q^2, where m^2 is the mass of the heaviest quark taken into account. There are several complications (see e.g. [15]), however the general picture is as just described.

Note, that the two lines joining both parts cannot correspond to a heavy quark with mass of the order of Q^2, due to the assertion that they have low virtuality comparing to Q^2. This fact shall be important later on.

Now, we come to more precise definitions of the soft part and its connection to the rest of the process. As is commonly known, the soft part can be parametrized in terms of parton distribution functions (PDF) (we shall interchangeably call it parton density) inside a hadron. In order to proceed we introduce light-cone coordinates; any four-vector v can be decomposed as

$$v^\mu = v^+ \tilde{n}^\mu + v^- n^\mu + v_T^\mu, \tag{1.1}$$

where

$$v^+ = v \cdot n, \quad v^- = v \cdot \tilde{n} \tag{1.2}$$

with two light-like vectors n,$\tilde{n}$ defined as

$$n = \frac{1}{\sqrt{2}} (1, 0, 0, -1), \quad , \tilde{n} = \frac{1}{\sqrt{2}} (1, 0, 0, 1). \tag{1.3}$$

[1] In OPE formalism twist is the difference between spin and canonical dimension of the operator.

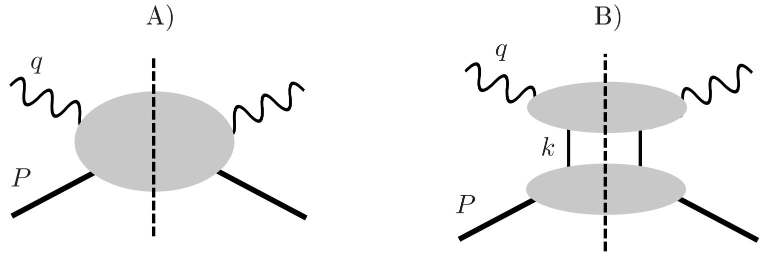


Figure 1.1: A) Cut Feynman amplitude for unpolarized boson-hadron process. B) Leading regions of the cut amplitude for large virtuality of the boson. The lines connecting upper and lower parts have low virtuality and can be light quarks or a gluon.

Let us now assume that the momentum k joining the hard and soft parts is parametrized using the light-cone variables and that suitable frame is chosen, such that P^+ is large $\sim \sqrt{Q^2}$. Then, since k have small virtuality comparing to Q^2 its k^- and k_T components can be neglected in the hard part. Then, the connection of the two parts can be realised as an integral over the k^+ component. The rest of the momentum integration (i.e transverse and "minus" components) are embodied in the definition of PDF, where they cannot be neglected. Its external lines (connecting it with the hard part) effectively lie on the light-cone.

All these remarks lead to the following definitions of the parton distributions. For the quark density we have

$$f_q^{(B)}(x) = \frac{1}{4\pi} \int dy^- e^{-ixP^+y^-} \left\langle P \left| \overline{\psi}_q(y^-n)\, \gamma^+ \left[y^-n, 0\right] \psi_q(0) \right| P \right\rangle \tag{1.4}$$

and for gluon

$$f_g^{(B)}(x) = \frac{1}{2\pi x P^+} \int dy^- e^{-ixP^+y^-} \left\langle P \left| F_A^{+\mu}(y^-n) \left[y^-n, 0\right]_{AB} F_{B\,\mu}^{+}(0) \right| P \right\rangle. \tag{1.5}$$

Let us now explain the above notation. First, there are quark field operators ψ_q and the gluon field strength operator $F_C^{\mu\nu} = \partial^\mu A_C^\nu - \partial^\nu A_C^\mu + g\, f_{CDE}\, A_D^\mu A_E^\nu$. All these fields are unrenormalized, thus the PDFs defined in such a way are the bare ones as indicated by the superscript[2]. Parameter x corresponds to a fraction of "plus" component of hadron momentum that is transferred to the hard part, that is we assume $k^+ = xP^+$ and is fixed. Next, y is a space-time point we integrate over, with however fixed $y^+ = 0$; the integration over y^- in disentangled while the one over y_T is performed (or hidden). Finally, there is a gauge link in order to make the definitions gauge invariant. It reads in the present case

$$\left[y^-n, 0\right] = \mathbb{P} \exp\left\{ ig \int_0^{y^-} dz\, A_C^+(z)\, t^C \right\}, \tag{1.6}$$

where the path joining both points is chosen to be a straight line. In particular, when we use light-cone gauge defined as $A \cdot n = 0$ the gauge link is a unity operator (it is

[2] Since we follow here mainly [15] and other papers of this author, we use the term "bare" in the sense of "unrenormalized". It has nothing to do with the IR unsafe PDFs, which actually are not needed in the formalism.

useful in some general considerations). The last remark concerning (1.4), (1.5) is that only connected diagrams should be taken into account.

As already mentioned, the parton distribution functions defined above contain UV divergences. Required renormalization concerns not only the elementary fields, but also the bilocal quark or gluon operators itself. As is well known, the renormalization introduces additional dependence on apriori unspecified mass scale μ_r.

It can be proved that the relation between the bare densities and the renormalized ones has the form [16, 15]

$$f_a^{(R)}\left(x,\mu_r^2\right)=\sum_b\int\frac{dz}{z}\,K_{ab}\left(\frac{z}{x},\alpha_s\left(\mu_r^2\right);\varepsilon\right)f_b^{(B)}\left(z\right), \tag{1.7}$$

where the renormalization kernel K_{ab} is a perturbatively calculable quantity. Note that we have introduced dimensional UV regulator ε defined as

$$D=4-2\varepsilon, \tag{1.8}$$

where D is the spacetime dimension. The summation in (1.7) goes over all possible kinds of the lines joining the soft and hard parts (exact sets shall be defined in the next chapter). The kernel K_{ab} can be calculated by considering the same objects as f_a but with the hadronic states replaced by the partonic ones. Thus we define the quantity $\mathcal{F}_{ab}$, which we refer to as a density of parton b inside a parton a. The definition is exactly the same as for f_b with the hadronic state replaced by the on-shell state a. The quantities $\mathcal{F}_{ab}$ can be calculated perturbatively in QCD with the help of special Feynman rules [16, 17] – we shall use them for massive quarks in Chapter 4.2. Actually, we have to again distinguish between the bare $\mathcal{F}_{ab}^{(B)}$ and the renormalized one $\mathcal{F}_{ab}^{(R)}$, however the relation between the two remains the same as (1.7). This allows to obtain K_{ab} once specific renormalization scheme is chosen (see also below).

Since the bare densities $f_a^{(B)}$ are defined by means of the bare fields only, they are completely independent on the renormalization scale. Therefore it is relatively straightforward to derive an evolution equation for the densities. It reads

$$\frac{d}{d\log\mu_r}\,f_a^{(R)}\left(x,\mu_r^2\right)=\sum_b\int\frac{dz}{z}\,P_{ab}\left(\frac{z}{x},\alpha_s\right)f_b^{(R)}\left(z,\mu_r^2\right), \tag{1.9}$$

where the evolution kernel P_{ab} is related to the renormalization kernel by the formula

$$P_{ab}\left(\frac{z}{x},\alpha_s\right)=2\alpha_s\,\frac{\partial K_{ab,\,1}\left(\frac{z}{x},\alpha_s\right)}{\partial\alpha_s}, \tag{1.10}$$

with $K_{ab,\,n}$ defined by the Laurent expansion

$$K_{ab}\left(z,\alpha_s;\varepsilon\right)=\delta\left(z-1\right)\delta_{ab}+\sum_{n=1}^{\infty}\left(\frac{1}{\varepsilon}\right)^n K_{ab,\,n}\left(z,\alpha_s\right). \tag{1.11}$$

For example, in the $\overline{\mathrm{MS}}$ scheme with N_f flavours we obtain

$$P_{ab}\left(z,\alpha_s\right)=\delta\left(z-1\right)\delta_{ab}+\frac{\alpha_s}{2\pi}\,P_{ab}^{(1)}\left(z\right)+\mathcal{O}\left(\alpha_s^2\right), \tag{1.12}$$

where $P_{ab}^{(1)}$ are famous lowest order splitting functions. They read

$$P_{qq}^{(1)}\left(z\right)=C_F\left(\frac{1+z^2}{1-z}\right)_+, \tag{1.13}$$

$$P_{gg}^{(1)}(z) = 2C_A \left[\left(\frac{1}{1-z} \right)_+ + \frac{1-z}{z} - 1 + z(1-z) \right] + \delta(1-z) \left(\frac{11}{6} C_A - \frac{2}{3} N_f T_R \right), \quad (1.14)$$

$$P_{gq}^{(1)}(z) = T_R \left[1 - 2z(1-z) \right], \quad (1.15)$$

$$P_{qg}^{(1)}(z) = C_F \, \frac{1 + (1-z)^2}{z}. \quad (1.16)$$

The "plus" distribution is defined in a standard way as

$$h_+(z) = h(z) - \delta(1-z) \int_0^1 dy \, h(y). \quad (1.17)$$

Note, that the support is $[0,1]$ – we pay attention to this detail, since we shall often use distributions with different supports (see also Appendix B.3). Since the splitting functions $P_{ab}^{(1)}$ are often used in this thesis we drop the superscript in what follows

$$P_{ab}^{(1)}(z) \equiv P_{ab}(z). \quad (1.18)$$

We turn also attention to our convention of ordering the subscripts. The notation ab corresponds to a splitting process $a \to b$, where parton b takes the fraction z of the original momentum. The physical interpretation of the functions P_{ab} is then such, that it gives a probability density for such a splitting.

Let us now come back to the factorization. Once the renormalization of the PDFs and of the hard part is done, we can finally write the factorization formula. In what follows we drop the renormalization indication in the hadronic PDFs

$$f_a^{(R)}\left(z, \mu_r^2\right) \equiv f_a\left(z, \mu_r^2\right). \quad (1.19)$$

The factorization theorem takes the following form

$$d\sigma\left(P, q; x_B, Q^2\right) = \sum_a \int_{x_B}^1 \frac{dz}{z} \, f_a\left(z, \mu_f^2, \mu_r^2\right) \, d\hat{\sigma}_a\left(zP, q; Q^2, \mu_f^2, \mu_r^2\right) + \mathcal{O}\left(\frac{m^2}{Q^2}\right). \quad (1.20)$$

Here $d\sigma$ corresponds to a differential DIS inclusive cross section, while $d\hat{\sigma}_a$ is a partonic cross section which is infra-red (IR) finite. Besides UV singularities, there are also divergences which originate in zero mass of the gluons and there are two sorts of them: the soft singularities and the collinear ones. They remain even after renormalization, however the soft and mixed soft-collinear divergences are cancelled between different contributions (we shall take up this issue in the next section). What remains are the collinear ones. The factorization procedure asserts, that they can be included in PDFs as it is essentially a nonperturbative object and we shall never calculate it using perturbation theory. Such a procedure is at the expense of introducing additional factorization scale μ_f. Apriori it is arbitrary scale and one often sets it equal to the renormalization scale. Moreover, there is certain freedom in choosing actually subtracted terms. Such a prescription defines the factorization scheme. Once it is specified, we can unambiguously derive $d\hat{\sigma}_a$ as follows. We use the factorization formula (1.20) at the partonic level (compare to derivation of the kernel K_{ab}). Thus, we have

$$d\sigma_a^{(R)}\left(p,q;x,Q^2,\mu_r^2\right)=\sum_b\int_x^1\frac{dz}{z}\left[S\left(\mu_r^2,\mu_f^2\right)\mathcal{F}_{ab}^{(R)}\left(z,\mu_r^2\right)+S_{ab}\left(z,\mu_r^2,\mu_f^2\right)\right]$$
$$d\hat{\sigma}_b\left(zp,q;Q^2,\mu_f^2,\mu_r^2\right)+\mathcal{O}\left(\frac{m^2}{Q^2}\right),\quad (1.21)$$

where the functions $\mathcal{F}_{ab}^{(R)}$ are the renormalized densities of parton inside a parton discussed before (we indicated also that unsubtracted cross section is renormalized). The quantities S and S_{ab} define our factorization scheme, see below. The above equation can be solved order by order, calculating $d\sigma_a^{(R)}$ and $\mathcal{F}_{ab}^{(R)}$ to a desired order.

As an illustration, let us consider completely massless case. Choosing $\overline{\mathrm{MS}}$ scheme to define PDFs we get at the lowest nontrivial order

$$\mathcal{F}_{ab}^{\overline{\mathrm{mMS}}}\left(x,\mu_r^2\right)=\delta\left(1-x\right)\delta_{ab}+\frac{\alpha_s\left(\mu_r^2\right)}{2\pi}\left(-\frac{1}{\varepsilon}\right)P_{ab}\left(x\right).\quad (1.22)$$

The superscript $\overline{\mathrm{mMS}}$ explicitly indicates that we use $\overline{\mathrm{MS}}$ renormalization scheme *and* completely massless calculation. Since $\overline{\mathrm{MS}}$ can be also used in a massive case, we feel a necessity to distinguish both situations as we shall encounter them in one place later on. We see that there is a collinear pole $1/\varepsilon$ in the result, which cancels the similar pole in $d\sigma_a$. Next, if we choose the factorization scheme to be $\overline{\mathrm{MS}}$, we have

$$S_{ab}\left(z,\mu_r^2,\mu_f^2\right)=0,\quad (1.23)$$

$$S\left(\mu_r^2,\mu_f^2\right)=\frac{1}{\Gamma\left(1-\varepsilon\right)}\left(\frac{4\pi\mu_r^2}{\mu_f^2}\right)^{\varepsilon}.\quad (1.24)$$

Let us conclude this section by giving some summarizing remarks. First is that hadronic PDFs are essentially nonperturbative, and have to be obtained from experiment, lattice calculations or low energy effective models. Most reliable are those obtained by global fits to data (e.g. [38]). Moreover, PDFs are scheme dependent, and as such are unphysical. Therefore one have to be careful when mixing PDFs obtained by one method with calculations in some other scheme, as the reminder ($\mathcal{O}\left(\ldots\right)$ terms) in factorization theorem can become large.

1.2 Jets in QCD

In the previous section we have considered the factorization theorem essentially for inclusive DIS scattering. One of the elements of the actual proof of the factorization property is the cancellation of the soft singularities. In this section, we take a closer look at this problem. In particular, we describe a method allowing for this cancellation in case when the process is not fully inclusive but consist in jets. This shall be a very general presentation of the topic and it will evolve throughout the whole dissertation. We follow [9] in this introduction.

Before we start, let us introduce some notation. The n-particle invariant phase space (PS) shall be denoted as

$$d\Phi_n\left(p,q;p_1,\ldots,p_n\right)\equiv d\Phi_n\left(p,q;\{p_i\}_{i=1}^n\right),\quad (1.25)$$

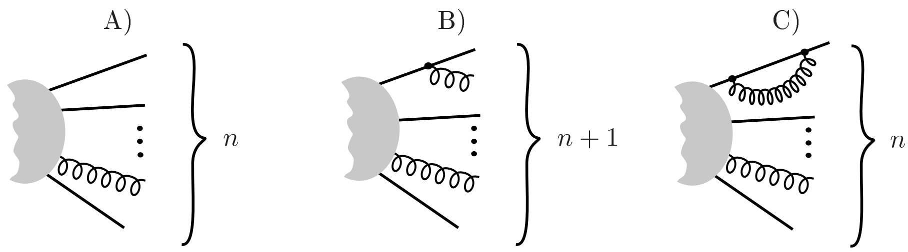


Figure 1.2: Illustrative presentation of the amplitudes for n-jet production. A) LO amplitude, B) real emission corrections, C) virtual corrections.

where p and q are incoming momenta. On the right hand side we have used mathematicians' notation for sets, as it will often allow to make formulae shorter. The phase space can be expressed as

$$d\Phi_n\left(p,q;\{p_i\}_{i=1}^n\right) = (2\pi)^D\,\delta^{(D)}\left(p+q-\sum_{i=1}^n p_i\right)\prod_{i=1}^n d\Gamma_i \tag{1.26}$$

in terms of the invariant measures for a particle i

$$d\Gamma_i \equiv d\Gamma\left(p_i\right) = \frac{d^D p_i\,\delta_+\left(p_i^2-m_i^2\right)}{(2\pi)^{D-1}}. \tag{1.27}$$

All the definitions are written in D space-time dimensions.

A tree-level amplitude with incoming momenta p, q and n outgoing states shall be denoted as $\mathcal{M}_n\left(p,q;\{p_i\}_{i=1}^n\right)$. At this stage all possible colour or spin indices are suppressed. If relevant, we will adorn the amplitude by various symbols and/or indices, for example we will put a hat if we consider external fermions to underline that we work with a matrix. Very often we will refer to a part of the amplitude, for example when two external legs are replaced by one. Then the reminder is referred to as *reduced amplitude*.

Let us now switch to the actual matter of this section. We start with the very schematic description of NLO calculation for n-jets. Suppose for simplicity that there are no initial state hadrons, e.g. electron-positron annihilation. A detailed formulae for DIS shall be given in Section 4.3.

To NLO accuracy, the total cross section can be written as

$$\sigma_n = \sigma_n^{\mathrm{LO}} + \sigma_n^{\mathrm{NLO}}. \tag{1.28}$$

The leading order contribution reads (Fig. 1.2A)

$$\sigma_n^{\mathrm{LO}} = \int d\Phi_n\,\left|\mathcal{M}_n\right|^2 F_n, \tag{1.29}$$

where $\mathcal{M}_n$ and $d\Phi_n$ are explained above (we suppress all momenta dependence), while F_n is certain (generalized) function that gives us an observable we are interested in (i.e it may include step-functions for kinematic cuts, delta functions for differential cross section, jet algorithms etc.). We shall refer to F_n as a *jet function*. We describe its properties in detail later.

The next-to-leading order term has in turn the following form

$$\sigma_n^{\mathrm{NLO}} = \sigma_n^R + \sigma_n^V, \tag{1.30}$$

where σ^R represents the real corrections, i.e the ones connected to the emissions of additional on-shell particles in the final state (Fig. 1.2B). Next, σ^V corresponds to loop corrections to $\mathcal{M}_n$ (Fig. 1.2C). The last can be written as

$$\sigma_n^V = \int d\Phi_n \, \mathcal{M}_n^{(\mathrm{loop})\,2} \, F_n. \tag{1.31}$$

The notation is symbolic here, $\mathcal{M}_n^{(\mathrm{loop})\,2}$ is actually an interference between the tree level amplitude and the one containing loop corrections. For the real corrections we write

$$\sigma_n^R = \int d\Phi_{n+1} \, |\mathcal{M}_{n+1}|^2 \, F_{n+1}. \tag{1.32}$$

As already stated in the previous section, higher order calculations in QCD lead to divergences. First, there are UV singularities, which are removed by renormalization and we do not consider them here any more. Second, there are mentioned IR singularities coming from vanishing propagators due to almost zero energy of massless particles or collinear emissions. We shall define them precisely in Section 3.2. Both kinds of singularities appear in σ^R and σ^V and are regularized e.g. dimensionally. However the physical cross section, which does not distinguish between the soft or collinear emissions, has to be finite. Therefore IR singularities have to cancel between both terms in cross sections (except possible pure collinear singularities connected with initial state emissions which are removed by factorization). It is precisely stated by means of the KLN theorem (Kinoshita-Lee-Nauenberg) and its extensions, see e.g. [48, 49] and references therein to the original papers. In what follows we assume that the jet cross section under consideration is infra-red safe, that is it fulfils all the assumptions of the KLN theorem.

This however requires to impose some restrictions on the jet functions. Namely, if one of the final state gluons in $(n+1)$-particle phase space is soft (its four-momentum vanishes) we must have $F_{n+1} = F_n$. Similarly, if two of the final state partons become collinear, their F_{n+1} function must also coincide with F_n. On the other hand, if we enter a singular region in n-particle phase space F_n must vanish. Those rules can be extended to initial state partons and massive partons as well.

Now, since we know that IR singularities cancel, there remains the problem of technical nature, which however is of great importance. Namely, both corrections σ^R and σ^V are integrated over different phase spaces with different jet functions. Analytical calculations are here extremely difficult and impractical, thus one often uses Monte Carlo methods. The problem is now to cancel the singularity that appears during numerical integration in σ^R with analytical singularities in σ^V, e.g. $1/\varepsilon$ poles.

Historically the first method was so called phase space slicing method. It can be illustrated by simple mathematical example (e.g. [36]). Suppose we have the following finite expression

$$I = \lim_{\kappa \to 0} \left\{ \int_0^1 dx \, \frac{h(x)}{x^{1-\kappa}} - \frac{1}{\kappa} h(0) \right\}, \tag{1.33}$$

where the dependence on x in h is very complicated but such that the integral exists. The first term in curly bracket corresponds to a real contribution regularized dimensionally, while the second term is the corresponding soft pole in "virtual correction". Both singularities cancel as actually the real value of the integral is

$$I = \int_0^1 dx \, \frac{h(x) - h(0)}{x}. \tag{1.34}$$

Suppose however, that we want to cancel them numerically. To this end, we divide the integration domain $\int_0^1 \ldots = \int_0^\delta \ldots + \int_\delta^1 \ldots$, with $\delta \ll 1$. Since $h(x)$ is regular enough, we can approximate $h(x) \approx h(0)$ for $x \in [0, \delta]$. Then, after simple steps we get

$$I \approx h(0) \log \delta + \int_\delta^1 dx \, \frac{h(x)}{x}. \tag{1.35}$$

Note, that the singularities cancelled and the integral can be now performed numerically with removed regularization, i.e. we set $\kappa = 0$. There is however a disadvantage as the result is approximate.

Another method, advocated in this work, is the subtraction method [36]. One constructs an auxiliary cross section

$$\sigma^{\text{sub}} = \int d\Phi_{n+1} \left|\mathcal{M}_{n+1}^{\mathsf{sub}}\right|^2 F_n \tag{1.36}$$

which mimics all the singularities of σ^R, i.e $\sigma^{\text{sub}} = \sigma^R$ in the singular regions of PS (note, there is F_n for n partons). Beside those points of phase space it can be anything that have the properties of a cross section. On the other hand, it must be chosen in such a way, that the analytical integration over one-particle subspace is possible. That is, if we write PS schematically as

$$d\Phi_{n+1} = d\Phi_n \otimes d\phi, \tag{1.37}$$

we must be able to perform $\int d\phi \left|\mathcal{M}_{n+1}^{\mathsf{sub}}\right|^2$ analytically. It leads then in dimensional regularization to poles of the form $1/\varepsilon$ which cancel those in virtual corrections due to the KLN theorem. The procedure of calculating NLO contribution using this method can be summarized as follows

$$\begin{aligned}\sigma^{\mathsf{NLO}} &= \left(\sigma^R - \sigma^{\text{sub}}\right) + \left(\sigma^V + \sigma^{\text{sub}}\right) \\ &= \int d\Phi_{n+1} \left(\left|\mathcal{M}_{n+1}\right|^2 F_{n+1} - \left|\mathcal{M}_{n+1}^{\mathsf{sub}}\right|^2 F_n \right) \\ &\quad + \int d\Phi_n \left\{ \mathcal{M}_n^{(\mathsf{loop})\,2} + \int d\phi \left|\mathcal{M}_{n+1}^{\mathsf{sub}}\right|^2 \right\} F_n. \end{aligned} \tag{1.38}$$

In the second line, due to IR properties of the jet functions, we can perform the integration in four dimensions and it is finite. In the third line a cancellation of the poles takes place and after that we can set $D = 4$.

This method has an obvious advantage, namely it is exact. Second, all the integrals over one-particle subspace have to be made only once and they are universal. This can be also generalized to higher orders, we however need much more subtraction terms.

A particular choice for σ^{sub} is realized in [9, 25] for massless partons, and in [10] for massive quarks in the final state (with some restrictions discussed in 3.1). This specific choice is called dipole subtraction term. Actually, a solid part of this work is devoted to generalizing this approach to completely massive case, such that one can practically apply massive factorization procedure described in the Chapter 2.

The dipole method has, however, also some drawbacks. First, it is relatively complicated, as we shall see. Moreover, it is unlikely to be generalized easily to higher orders. The reason is that it operates on the amplitudes squared and the number of subtraction terms increases rapidly. There is some hope connected with so called antenna method which constructs subtraction terms at the amplitude level, see e.g. [33]. Second problem, which actually concerns the subtraction procedure in general, is that of numerical nature. Namely, depending on implementation, there may be some problems when performing the integration in the second line of (1.38). Thus effectively, one may be forced to use a support in a form of a slicing-like method.

1.3 Quark masses in QCD

In the previous sections we did not pay special attention to the quark masses. Here we recall some basic facts connected with their inclusion in perturbative calculations. The following material is essential to the whole work. In some parts we rely on [13].

Today we know six flavours of quarks with the following masses[3] [43]:

$$m_u = 1.7\text{-}3.1\,\mathrm{MeV}, \quad m_d = 4.1\text{-}5.7\,\mathrm{MeV}, \quad m_s \approx 100\,\mathrm{MeV}, \tag{1.39}$$

$$m_c \approx 1.29\,\mathrm{GeV}, \quad m_b \approx 4.19\,\mathrm{GeV}, \quad m_t \approx 172.9\,\mathrm{GeV}. \tag{1.40}$$

Recall now, that the basic requirement to be in a perturbative regime, is that the typical energy scale, say Q, satisfies $Q \gg \Lambda_{\mathrm{QCD}}$. Since $\Lambda_{\mathrm{QCD}} \approx 200\,\mathrm{MeV}$ we can safely neglect the masses of u, d, s quarks in perturbative calculations. If the scale is high enough, we can also make such an approximation with the other quarks.

On the other hand, apriori we do not known if there exist heavier quarks. Similar situation used to be before the discovery of the top quark. Thus, the question was about the relevance of field theoretic calculation, where some of the quarks are possibly missed. The solution to this problem is formulated by means of so called decoupling theorem [4]. It states that for a Feynman amplitude with a typical momentum scale Q we can drop all the diagrams with quark mass $m \gg Q$, doing error $\mathcal{O}(Q/m)$. Let us now assume, that the remaining number of quark flavours is N_f, thus all the renormalized parameters (masses, couplings etc.) in such an effective theory are calculated using this number. In general, the renormalized parameters in the effective theory with $N_f + 1$ flavours are different.

The problem however arises, when the masses are not extremely different, as actually happens for charm and bottom quarks. For instance, when the scale is close to m_c, we can make a mistake of the order $m_c/m_b \approx 30\%$ (for an example see e.g. [13]). Fortunately, there is a better method than such an uncontrolled decoupling. It reduces to the last in the limit of very large masses. It is a special renormalization scheme existing in the literature as CWZ (Collins-Wilczek-Zee) renormalization scheme [19, 45, 14]. In order to define its basics let us introduce an *active number of quarks* N_a. It is a number of quarks lighter than the fixed external energy scale (note, that we do not have to set those masses to zero). The CWZ scheme consist in the subschemes characterized by N_a. In each subscheme the renormalization is done according to the following points:

a) the graphs with internal lines being active are renormalized using $\overline{\mathrm{MS}}$

b) the graphs with at least one internal heavy quark line (inactive) are renormalized by zero-momentum subtraction

c) masses of heavy quarks are usually defined as the pole masses

[3] As the free quark states are unobservable, these are just parameters obtained in $\overline{\mathrm{MS}}$ scheme at scale about 2 GeV.

This scheme possesses several important properties (see e.g. [13]). For us two of them are the most important. First is that it satisfies manifest decoupling. That is, if the external scale is much smaller than the masses of inactive quarks, the renormalized parameters of a subscheme with N_a active flavours are the same as in effective theory with $N_f = N_a$. Hence we can just drop all the diagrams with inactive quarks. The second important property is that the evolution of the renormalized parameters in each subscheme is exactly the same as in $\overline{\rm MS}$ with $N_f = N_a$, in particular the evolution kernels are massless.

This last property is of great importance in this thesis. As we have seen in Section 1.1, the operational definition of parton distribution functions includes a renormalization scheme. Since we are going to treat factorization with the heavy quarks it is convenient to define PDFs in CWZ scheme. Then, due to the second property, such PDFs undergo the standard DGLAP evolution equation in each subscheme. We shall discuss it in details in Section 2.4, while in Section 4.2 we calculate some of them in this scheme.

There is one more comment in order. The purpose of introducing such a scheme, is to be able to evolve a given parameter through all applicable scales without loosing accuracy. It is realized by switching the schemes at given *switching points*. Therefore, we have to state a matching conditions at those points[4] in order to have a starting parameters in evolution. Such conditions were obtained even up to three loops for the coupling (using effective theory formalism [12]) and up to two loops for PDFs [8].

In the end, let us introduce some more notation we shall use throughout. First, we often need to distinguish between heavy and light flavours. Thus we define $N_f = N_q + N_{\mathbf{Q}}$, where q is a generic light quark, while $\mathbf{Q}$ corresponds to heavy quarks. Sometimes we refer to light partons number, which is simply $N_l = N_q + 1$, as gluon is always light. If we want to refer to all the quark flavours, but including gluon, we use the symbol N_f'. For all the defined symbols, we introduce the sets, containing corresponding flavours and their anti-flavours. The sets shall be denoted by blackboard font, for instance $\mathbb{N}_f$, $\mathbb{N}_l$ etc.

[4] In general, one should distinguish between the switching point and a matching point. The first is the point in which the transition between the schemes takes place. The second is a point used to recalculate parameters from one scheme to another. In this thesis we set them equal.

Chapter 2

Inclusive DIS with heavy quarks

2.1 Introduction

As we have seen in Section 1.3, there are certainly some complications when there are heavy quarks with masses that are neither marginally large nor negligibly small. The problems are even more evident in the processes which require factorization. In Section 1.1 we recalled the factorization theorem assuming that the masses can be neglected. In case, when they cannot, such a treatment is obviously very inaccurate. In this chapter, we shall analyse this issue in more details in the context of inclusive DIS scattering.

First, in the next section we recall the simplest possible way of including heavy quarks, actually treating them as massless partons. This scheme, often called zero-mass variable flavour number scheme (ZM-VFNS) is most often used in phenomenological analysis of DIS processes. However, as we shall see, it is inaccurate in non-asymptotic regions of energy scale. That section is also devoted to introducing some notation which we use in this and the next chapters. Further, in Section 2.3 we briefly describe more accurate treatment, however aiming at completely different kinematic regime than the latter. This second solution is often referred to as fixed-flavour number scheme (FFNS), and takes all the effects of heavy quarks into account. The problem is however, that as the energy scale increases, such a prediction becomes less accurate, unless we go to higher orders of perturbation theory. Needless to say, such a massive high-order calculations are much more involved and time-consuming than the massless ones, not to mention generalizations to exclusive processes.

Therefore, it is desirable to have a scheme which is applicable at intermediate energy scales and contains both above schemes as a limiting cases. Such solutions were indeed developed [2, 50, 8], however with explicit treatment of inclusive processes only. What is worth emphasizing, the approach cited as [2] was proved to all orders of perturbation theory [15]. We shall briefly describe this approach, referred to as ACOT (Aivazis-Collins-Olness-Tung) scheme, in Section 2.4. It is based on CWZ renormalization scheme for parton densities and can be easily generalized to another IR safe cross sections.

For a short review of the mentioned treatments of heavy quark production in inclusive DIS see e.g. [51, 52].

We remark, that although this chapter is considered to be introductory, we discuss also a new improvement of existing methods at the end of Section 2.4.

2.2 Zero-mass variable flavour number scheme

Let us start by defining our object of interest in this chapter. We shall be concentrated here mainly on the structure functions parametrizing the cross section for inclusive DIS processes, notably $F_2\left(x_B, Q^2\right)$, and its dependence on the photon virtuality Q^2. Remember, that the structure functions are obtained by means of a suitable projection of hadronic tensor $W^{\mu\nu}$, defined as usual in terms of matrix element of electroweak currents sandwiched between hadron states

$$W^{\mu\nu}\left(q, P; x_B, Q^2\right) = \frac{1}{4\pi}\sum_{\text{spin}}\sum_{P_x}\int d\Phi_1\left(q, P; P_X\right)\left\langle P\left|j^{\dagger\,\mu}\left(0\right)\right|P_X\right\rangle\left\langle P_X\left|j^{\nu}\left(0\right)\right|P\right\rangle, \tag{2.1}$$

where the second sum goes over all final states P_X. The projection is made using suitable base tensors made of the vectors P, q and the metric tensor. Neglecting the hadron mass we get for F_2

$$F_2\left(x_B, Q^2\right) = \frac{2\,x_{\mathrm{B}}}{D-2}\left(-W_{\mu}^{\mu} + (D-1)\,\frac{2\,x_B}{P\cdot q}W_{\mu\nu}P^{\mu}P^{\nu}\right). \tag{2.2}$$

If we replace the hadronic state by a parton, such a tensor is called the partonic tensor. We shall denote it as $w^{\mu\nu}$. Both tensors are related by means of factorization theorem – we shall give some examples below. We do not give further details related to other structure functions and related issues as they are all standard (for the precise definitions incorporating quark and target masses see [3]). Such limited considerations are completely enough to elucidate the basic problems with heavy quark masses, as we shall see.

Before we proceed, let us recall, that we denote a generic heavy quark by symbol $\mathbf{Q}$. The light quarks are denoted as q, there should be no confusion since this is only used in this meaning as a subscript.

Let us start further considerations by noting, that the simplest possible approach to heavy quarks is when $Q^2 \to \infty$ with x_B fixed, such that *all* the existing heavy quark masses can be neglected. Then, the precise predictions are given by the factorization theorem (1.20), which is exact. All the quarks (including heavy quarks) are treated as massless partons having corresponding PDFs. Such situation is obviously not very plausible. In practice the energy scales do not tend to infinity, moreover many interesting phenomena exist at lower scales. Secondly, we have several heavy quarks with large mass splittings, as discussed in Section 1.3. On the other hand, when Q^2 is much smaller than the mass of a given heavy quark, it may be dropped from calculations due to decoupling theorem mentioned also in Section 1.3.

These two marginally different situations ($Q^2 \gg m_{\mathbf{Q}}^2$ and $m_{\mathbf{Q}}^2 \gg Q^2$) motivate the following simplest scheme of treating "heavy" quarks:

a) completely decouple given heavy quark $\mathbf{Q}$ when $m_{\mathbf{Q}}^2 > Q^2$, i.e. treat it as infinitely heavy

b) treat $\mathbf{Q}$ as a massless parton with associated PDF, when $Q^2 > m_{\mathbf{Q}}^2$

We have assumed here that the factorization and renormalization scales are equal to Q. If there are several heavy quarks, we have the composite scheme, with subschemes characterized by an active number of flavours N_a. Thus we have a set of parton distribution functions $f_a^{(N_a)}$ and couplings $\alpha_s^{(N_a)}$. We note, that this scheme is a special kind of CWZ scheme mentioned earlier, in which PDFs are defined. All the masses are however set to zero. Since CWZ satisfies manifest decoupling, we just drop the inactive quarks, and each subscheme is effectively a $\overline{\text{MS}}$ scheme with N_a flavours and corresponding DGLAP massless evolution of PDFs. As already mentioned, the schemes with different N_a are actually different renormalization schemes, they differ by finite terms and a relation between schemes with N_a and $N_a + 1$ flavours can be stated.

Here the switching point is usually chosen to be $\mu_{\text{th}} = m_{\mathbf{Q}}$, where $\mathbf{Q}$ is $(N_a + 1)$-th flavour. It is convenient, since then the heavy quark density $f_{\mathbf{Q}}^{(N_a+1)}$ is zero at the threshold[1]. It follows from two facts. First is just a precise form of the relation between PDFs in two subschemes [18]. Second is that below μ_{th} it is suppressed by power of $\Lambda_{\text{QCD}}/m_{\mathbf{Q}}$ due to decoupling theorem. Thus we have the continuity condition

$$f_{\mathbf{Q}}^{(N_a)}\left(z,\mu_{\text{th}}^2\right) = f_{\mathbf{Q}}^{(N_a+1)}\left(z,\mu_{\text{th}}^2\right) = 0 \tag{2.3}$$

Then, above the threshold it is evolved using DGLAP equations with $N_a + 1$ flavours starting from zero value.

As already mentioned in the introduction, such a scheme is called zero-mass variable flavour number scheme (ZM-VFNS). Corresponding factorization theorem takes the form

$$W_{\mu\nu}^{(N_a)}\left(q,P;x_B,Q^2\right) = \sum_{a\in\mathbb{N}_a} f_a^{(N_a)}\left(\mu_f^2\right) \otimes \hat{w}_{\mu\nu}^{(N_a)}\left(q,p_a;\frac{Q^2}{\mu_f^2}\right), \tag{2.4}$$

where we explicitly denoted the dependence on the factorization scale (equal here to the renormalization scale). We also introduced the convolution symbol, which simplifies the notation; it is defined here as

$$f \otimes w = \int_{x_B}^{1} \frac{d\xi}{\xi}\, f\left(\xi\right) w\left(\frac{x_B}{\xi}\right). \tag{2.5}$$

In (2.4) $p_a = \xi P$, nevertheless we leave p_a as this notation is more general. As we vary the scale, the active number of partons changes. Such a formula is actually valid up to corrections of order $\mathcal{O}\left(m_{N_a}^2/Q^2\right)$, where m_{N_a} would be the mass of the heaviest active quark, if we did not set it to zero. Therefore, in reality such an approach is unreliable for Q^2 around the masses of heavy quarks. Moreover, as we reach the region of validity of (2.4) for one heavy quark, say charm, we simultaneously can enter the region of inapplicability for the beauty quark. Thus, only at really asymptotic regimes this scheme is correct, as we remarked earlier.

To illustrate this approach, consider now a calculation of F_2 structure function in this scheme up to order α_s. Let us assume we work in the scheme with $N_a = 4$, that is besides gluon, u, d and s quarks, which are always massless, we have also charm c

$$\mathbb{N}_a = \left\{g,u,\overline{u},d,\overline{d},s,\overline{s},c,\overline{c}\right\}. \tag{2.6}$$

Then, to this order

$$\frac{1}{x_B} F_2\left(x_B,Q^2\right) = \sum_{a\in\mathbb{N}_a} f_a\left(\mu_f^2\right) \otimes \left[C_a^{(0)}\left(\frac{Q^2}{\mu_f^2}\right) + C_a^{(1)}\left(\frac{Q^2}{\mu_f^2}\right)\right]. \tag{2.7}$$

[1] It is however true only at leading and next to leading order, see [8].

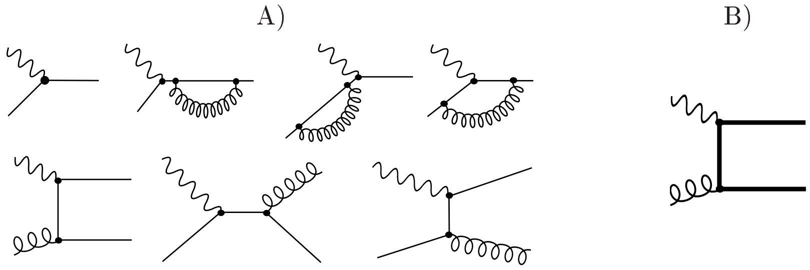


Figure 2.1: A) Feynman diagrams contributing to structure functions in ZM-VFNS up to order α_s^1; B) The same for FFNS. There is only boson-gluon fusion at order α_s^1. Thick line corresponds to a heavy quark.

We are actually interested in charm contribution to F_2, which can be picked up from the above equation. Setting $\mu_f^2 = Q^2$ (and the same for renormalization scale) we have

$$F_2^c\left(x_B, Q^2\right) = \left[f_c\left(Q^2\right) + f_{\bar{c}}\left(Q^2\right)\right] \otimes \left[C_c^{(0)} + C_c^{(1)}\right] + f_g\left(Q^2\right) \otimes C_g^{(1)}, \qquad (2.8)$$

where the result for coefficients C_i in massless $\overline{\text{MS}}$ scheme is well known (e.g. [28], for corresponding diagrams see Fig. 2.1A) and reads

$$C_c^{(0)}(z) = e_c^2\, z\delta(1-z), \qquad (2.9)$$

$$C_c^{(1)}(z) = e_c^2\,\frac{\alpha_s}{2\pi}\,C_F\left[\frac{1+z}{1-z}\left(\log\frac{1-z}{z} - \frac{3}{4}\right) + \frac{1}{4}(9+5z)\right]_+, \qquad (2.10)$$

$$C_g^{(1)}(z) = e_c^2\,\frac{\alpha_s}{2\pi}\left[2P_{gq}(z)\log\frac{1-z}{z} + 8z(1-z) - 1\right]. \qquad (2.11)$$

The splitting function P_{gq} and "plus" distribution were defined in Section 1.1.

The behaviour of this solution will be explicitly demonstrated in Section 2.4, where we present some plots comparing this NLO calculation to other schemes.

Although – as we have just seen – such a scheme is very simplified, it is still most commonly used in PDFs global fits to data (e.g. CTEQ fits [38] and earlier). Its great advantage is simplicity and practicality. It should be also mentioned that it was very successful in describing large amount of modern high energy data.

2.3 Fixed flavour number scheme

Let us now present another approach, which is applicable when Q^2 is about the heavy quark mass $m_{\mathbf{Q}}^2$. Actually, it is a generalization of the previous scheme, where $\mathbf{Q}$ is inactive, but has finite mass. Thus we have N_a massless partons undergoing massless evolution and one heavy flavour, which can be only produced dynamically. For example, at LO in DIS it is the boson-gluon fusion (BGF) process depicted in Fig. 2.1B.

The factorization theorem in this case takes the form

$$W_{\mu\nu}\left(q, P; x_B, Q^2, m_{\mathbf{Q}}^2\right) = \sum_{a\in\mathbb{N}_a} f_a\left(\mu_f^2\right) \otimes \hat{w}_{\mu\nu}\left(q, p_a; \frac{Q^2}{\mu_f^2}, \frac{m_{\mathbf{Q}}^2}{\mu_f^2}\right) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}^2}{m_{\mathbf{Q}}^2}\right). \qquad (2.12)$$

Here, the hard scale is given by the heavy quark mass and on the contrary to (2.4) N_a does not change. Therefore such a scheme is called fixed flavour number scheme (FFNS) and was pioneered in [28, 27, 37, 29]. We note, that here the convolution symbol is defined as in (2.5) but the integration limits depend on quark masses (we shall see the example below).

In order to discuss some of its properties, let us again consider the explicit result, namely the contribution to F_2 coming from the charm quark. As already mentioned, the situation where charm connects the hard and soft parts is suppressed by $\Lambda^2_{\mathrm{QCD}}/m_c^2$, thus the perturbative calculation for F_2^c starts at α_s^1 with BGF process (Fig. 2.1)

$$\frac{1}{x_B} F_2^c\left(x_B, Q^2\right) = f_g\left(Q^2\right) \otimes C_g^{(1)}\left(\frac{m_c^2}{Q^2}\right), \tag{2.13}$$

with coefficient given by (e.g. [47])

$$C_g^{(1)}\left(z,\rho\right) = e_c^2 \frac{\alpha_s}{2\pi} \left\{ \left[2P_{gq}\left(z\right) + 4\rho z^2\left(1-3z\right) - 8\rho^2 z^3\right] \log\frac{1+v}{1-v} \right. \\ \left. + \left(4z\left(1-z\right)\left(2-\rho\right)-1\right) z\, v \right\}, \tag{2.14}$$

where we abbreviated $\rho = m_c^2/Q^2$ and $v = \sqrt{1-4\rho z/\left(1-z\right)}$ is the velocity of the charm quark in the photon-gluon CM frame. Now the lower limit on the convolution is $z_{\min} = x_B\left(1+4\rho\right)$.

Let us discuss now this result. First, let us note that it contains the powers of m_c^2/Q^2, which are lacking in ZM-VFNS treatment (higher twists). Therefore indeed it is reliable calculation when Q^2 is of the order of m_c^2. Now, the question is what is the behaviour of this solution when the scale is much larger. In this case, we find that

$$C_g^{(1)}\left(z,\rho\right) = e_c^2 \frac{\alpha_s}{2\pi} P_{gq}\left(z\right) \log\rho + \mathcal{O}\left(\rho\right). \tag{2.15}$$

Thus we see, that we have a potentially large logarithm of the heavy quark mass and the hard scale ratio. Such logarithms appear in every next order of perturbation theory, typically

$$C_a^{(m)} = \sum_{k=0}^{m} c_{a,\,k}^{(m)} \log^k \rho, \tag{2.16}$$

what makes such an expansion unreliable. The solution is to resum all the powers of α_s in front of the given power of logarithm, i.e. to suitably rearrange the above series. Then, we actually arrive at the zero-mass scheme with charm being a massless parton. However, one has to bear in mind that it happens at a price of loosing control of the terms $\mathcal{O}\left(m_c^2/Q^2\right)$ (actually, if we do not track higher twist terms, which is not easy and so far has not been solved). In the next section we shall present some plots comparing this scheme to ZM-VFNS.

There is one more comment in order. One can ask when this fixed flavour approach fails, since logarithm is a very slowly increasing function. In [29] it was argued, that the cross sections calculated in this approach at NLO are stable even for relatively large scales, however one has to use a special sets of PDFs, namely so called dynamical PDFs (see e.g. [53]). Such an approach however does not solve the basics of the problem, therefore we shall not follow this path in this thesis.

2.4 ACOT scheme

As we already anticipated in the introduction, there exist solutions which contain ZM-VFNS and FFNS as a special cases. In the following section we describe one of them, the so called ACOT scheme [2, 15]. We believe it is the best solution that can be easily generalized to less inclusive processes, in particular jets. This is in contrast to other approaches like [50, 8]. As we have already remarked, it has been proved for inclusive DIS to all orders in [15].

Basic assumption of the scheme is that the PDFs are defined using CWZ renormalization scheme and that the masses relevant to actual energy scale are kept finite. This results in the higher twist errors of the order of $\Lambda^2_{\mathrm{QCD}}/Q^2$ over the whole kinematically allowed region of Q^2. We shall see how it works in practice below.

Consider again the hadronic tensor $W^{\mu\nu}$ and suppose for simplicity that there is only one heavy quark $\mathbf{Q}$. The factorization is realised actually by two different theorems [15]. The first one is essentially the same as (2.12), i.e. it is applicable when $Q^2 \lesssim m^2_{\mathbf{Q}}$. The second one, is when $Q^2 \gtrsim m^2_{\mathbf{Q}}$, that is both theorems have an overlap region. Let us analyse the second case. The theorem under consideration has the following form

$$W_{\mu\nu}\left(q, P; x_B, Q^2, m^2_{\mathbf{Q}}\right) = \sum_{a\in\mathbb{N}_a} f_a\left(\mu^2_f\right) \otimes \hat{w}_{\mu\nu}\left(q, p_a; \frac{Q^2}{\mu^2_f}, \frac{m^2_{\mathbf{Q}}}{\mu^2_f}\right) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right). \quad (2.17)$$

Superficially it is almost the same as (2.12), however there are differences. First is that in this scheme (i.e. above some switching point $\mu_{\mathrm{th}} \sim m_{\mathbf{Q}}$) the set of active quarks $\mathbb{N}_a$ does include the quark $\mathbf{Q}$. Second difference is subtle. It is connected with IR finite partonic tensor. To see this let us calculate it to the first order in α_s. Recall, that it is done with factorization (2.17), but on the partonic level (let us set all the scales equal to Q)

$$w^{\mu\nu}_a\left(q, p_a; Q^2, m^2_{\mathbf{Q}}\right) = \sum_{b\in\mathbb{N}_a} \mathcal{F}^{\mathrm{CWZ}}_{ab}\left(\frac{Q^2}{m^2_{\mathbf{Q}}}\right) \otimes \hat{w}^{\mu\nu}_b\left(q, p_b; \frac{m^2_{\mathbf{Q}}}{Q^2}\right). \quad (2.18)$$

We denoted that the parton densities inside a parton are renormalized using CWZ. To the first order it becomes (below we drop all the arguments, vector indices and CWZ superscript for transparency)

$$w^{(0)}_a + w^{(1)}_a = \sum_{b\in\mathbb{N}_a} \left(\mathcal{F}^{(0)}_{ab} + \mathcal{F}^{(1)}_{ab}\right) \otimes \left(\hat{w}^{(0)}_b + \hat{w}^{(1)}_b\right) + \mathcal{O}\left(\alpha^2_s\right). \quad (2.19)$$

Thus, the zeroth order partonic tensor is IR safe

$$w^{(0)}_a = \hat{w}^{(0)}_a. \quad (2.20)$$

Solving further the recurrence we get for a light quark

$$w^{(1)}_q = \mathcal{F}^{(1)}_{qq} \otimes w^{(0)}_q + \hat{w}^{(1)}_q, \quad (2.21)$$

for heavy quark

$$w^{(1)}_{\mathbf{Q}} = \mathcal{F}^{(1)}_{\mathbf{QQ}} \otimes w^{(0)}_{\mathbf{Q}} + \hat{w}^{(1)}_{\mathbf{Q}} \quad (2.22)$$

and for a gluon

$$w^{(1)}_g = \mathcal{F}^{(1)}_{gq} \otimes w^{(0)}_q + \mathcal{F}^{(1)}_{g\mathbf{Q}} \otimes w^{(0)}_{\mathbf{Q}} + \hat{w}^{(1)}_g. \quad (2.23)$$

We have used the fact that zeroth order densities are trivial

$$\mathcal{F}^{(0)}_{ab} = \mathbf{1}\,\delta_{ab}. \tag{2.24}$$

Equations (2.21)-(2.23) can be now solved for IR safe quantities occurring in (2.17).

Projecting the hadronic tensor suitably to get F_2 structure function we now have

$$\begin{aligned}\frac{1}{x_B}F_2 = \sum_{q\in\mathbb{N}_q} f_q \otimes \left(C^{(0)}_q + C^{(1)}_q - \mathcal{F}^{(1)}_{qq}\otimes C^{(0)}_q\right) \\ + f_{\mathbf{Q}} \otimes \left(C^{(0)}_{\mathbf{Q}} + C^{(1)}_{\mathbf{Q}} - \mathcal{F}^{(1)}_{\mathbf{QQ}}\otimes C^{(0)}_{\mathbf{Q}}\right) \\ + f_g \otimes \left(C^{(1)}_g - \sum_{q\in\mathbb{N}_q}\mathcal{F}^{(1)}_{gq}\otimes C^{(0)}_q - \mathcal{F}^{(1)}_{g\mathbf{Q}}\otimes C^{(0)}_{\mathbf{Q}}\right).\end{aligned} \tag{2.25}$$

The parton densities are renormalized in $\overline{\text{MS}}$ scheme here since we are above the switching point and $\mathbf{Q}$ is treated as active parton. Thus for massless partons we have precisely the result (1.22) while for the heavy quarks we get

$$\mathcal{F}^{\overline{\text{MS}}}_{g\mathbf{Q}}\left(x, \frac{Q^2}{m^2_{\mathbf{Q}}}\right) = \frac{\alpha_s}{2\pi}\log\left(\frac{Q^2}{m^2_{\mathbf{Q}}}\right)P_{gq}(x), \tag{2.26}$$

$$\mathcal{F}^{\overline{\text{MS}}}_{\mathbf{QQ}}\left(x, \frac{Q^2}{m^2_{\mathbf{Q}}}\right) = \frac{\alpha_s}{2\pi}C_F\left\{\frac{1+x^2}{1-x}\left[\log\left(\frac{Q^2}{m^2_{\mathbf{Q}}}\right) - 2\log(1-x) - 1\right]\right\}_+. \tag{2.27}$$

Those results were partially calculated in [45], but we had to re-derive them as explained in Section 4.2. The first one is however well known (e.g. [2]), while the second was obtained e.g. in [35] by different method (by comparing asymptotic expressions).

In order to better understand this result, let us pick up only heavy quark contribution to F_2, where $\mathbf{Q}$ is e.g. a charm quark. That is we consider

$$\frac{1}{x_B}F^{\mathbf{Q}}_2 = f_{\mathbf{Q}}\otimes\left(C^{(0)}_{\mathbf{Q}} + C^{(1)}_{\mathbf{Q}} - \mathcal{F}^{(1)}_{\mathbf{QQ}}\otimes C^{(0)}_{\mathbf{Q}}\right) + f_g\otimes\left(C^{(1)}_g - \mathcal{F}^{(1)}_{g\mathbf{Q}}\otimes C^{(0)}_{\mathbf{Q}}\right). \tag{2.28}$$

First note, that the coefficients of order α^1_s without a hat are actually finite. For instance $C^{(1)}_g$ is precisely the one given in (2.14). However they are not IR safe as discussed in the previous section. However, when $Q^2 \gg m^2_{\mathbf{Q}}$ they become IR safe by construction, thanks to the subtraction terms terms $\mathcal{F}^{(1)}_{\mathbf{QQ}}\otimes C^{(0)}_{\mathbf{Q}}$ and $\mathcal{F}^{(1)}_{g\mathbf{Q}}\otimes C^{(0)}_{\mathbf{Q}}$. Therefore in this limit such $F^{\mathbf{Q}}_2$ becomes equal to the one obtained in ZM-VFNS scheme. This is illustrated in Fig. 2.2.

Now let us consider what happens when $Q^2 \gtrsim m^2_{\mathbf{Q}}$, i.e. just above the matching point, which for convenience is chosen to be precisely at $\mu_{\text{th}} = m_{\mathbf{Q}}$ (see Section 2.2). Then, the ACOT scheme should reproduce the FFNS scheme with $\mathbf{Q}$ being inactive. Indeed it is the case here. First, let us note that the evolution equations for all PDFs are standard DGLAP equations (1.9) with massless splitting functions. This is a simple consequence of choosing CWZ scheme to define PDFs. Since it may be not obvious that in a massive calculation we may have massless evolution, we prove this fact in Section 4.2.2. Next, due the above choice of the switching point, the density $f_{\mathbf{Q}}$ is zero there. Therefore,

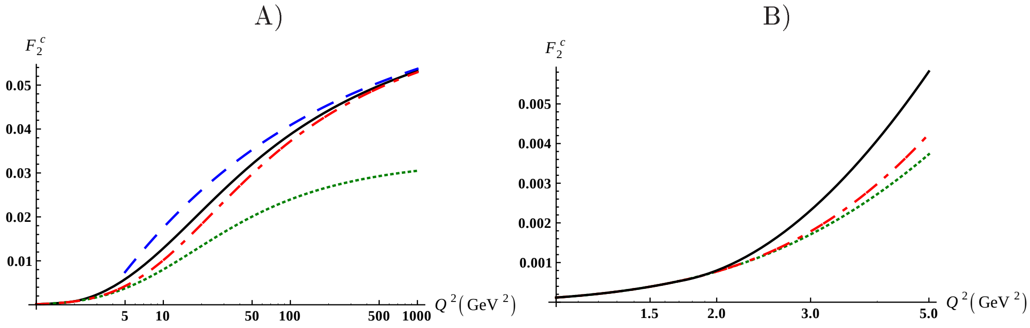


Figure 2.2: A) Charm contribution at order α_s^1 to F_2 structure function in different schemes (dashed: ZM-VFNS, solid: ACOT, dot-dashed: ACOT-χ, dotted: FFNS). Calculation is done for $x_B = 0.05$ and CTEQ LO PDF set. Factorization and renormalization scales are all set to Q^2. B) The same around the matching point $\mu_{\text{th}}^2 = m_c^2 = 1.69\,\text{GeV}^2$ (we do not show the dashed line here as it does not make sense in this region). The ACOT scheme curve is obtained without quark-scattering NLO contribution, as it was originally introduced.

solving evolution equation for $f_{\mathbf{Q}}\left(\mu^2\right)$ just above the matching point, i.e. for $\mu^2 \gtrsim m_{\mathbf{Q}}^2$ (which is straightforward) we get at LO

$$f_{\mathbf{Q}}\left(\mu^2\right) \approx \frac{\alpha_s}{2\pi} f_g\left(m_{\mathbf{Q}}^2\right) \otimes P_{g\mathbf{Q}} \log \frac{\mu^2}{m_{\mathbf{Q}}^2} = f_g\left(m_{\mathbf{Q}}^2\right) \otimes \mathcal{F}_{g\mathbf{Q}}^{(1)}\left(\frac{\mu^2}{m_{\mathbf{Q}}^2}\right). \tag{2.29}$$

Analysing (2.28) in the same regime we find that most of the terms cancel and what remains is just $f_g \otimes C_g^{(1)}$, i.e. FFNS. Moreover, we can go with this formula below the matching point. There it becomes exactly FFNS. This is due to CWZ renormalization scheme. It turns out, that $\mathcal{F}_{g\mathbf{Q}}^{(1)}$ vanishes below the matching point (see Section 4.2.2), the same is true for $f_{\mathbf{Q}}$. Hence, what remains is again just the boson-gluon fusion process. This is illustrated in Fig. 2.2, with dropped NLO quark-scattering contribution as originally done by the authors of [2]. This have been later fixed in [35] and confirmed that this contribution is usually negligible. We also confirm this fact by explicit calculation using our MC program in Section 4.5.

There are several subtle points, which we have skipped above. First, there is certain freedom concerning the factorization theorem with heavy quarks ([2], see also [15] for more details). This fact was used in Ref. [34] where the version of factorization theorem with massless initial state partons was considered. The scheme just mentioned for inclusive processes is called SACOT scheme (Simplified ACOT). It is usefull in the context of higher order calculations, as it may considerably simplify the situation. The freedom mentioned above can be used also differently, namely we set the initial state masses to zero only in the terms $C_{\mathbf{Q}}^{(0)}$ above, i.e. we leave the mass in $f_{\mathbf{Q}} \otimes C_{\mathbf{Q}}^{(1)}$. This approach is elucidated in Section 4.3.2, where we discuss this for jet production.

Next, there is a problem that $C_{\mathbf{Q}}^{(0)}$ does not have correct threshold behaviour for heavy quark production. It is obvious, since it is a different mechanism. There are some approaches in the literature, which try to incorporate some artificial scaling variables in order to fix it. For instance, so called χ-prescription assumes replacement

$$x \longleftrightarrow \chi = x\left(1 + \frac{m_{\mathbf{Q}}^2}{Q^2}\right) \tag{2.30}$$

in functions involving $C_{\mathbf{Q}}^{(0)}$, i.e. where $\mathbf{Q}$ enters the reduced matrix element. Then, the physical threshold for $\mathbf{Q}$ production is incorporated. We show an example calculation in Fig. 2.2, as ACOT-χ.

Although such prescriptions are allowed by the freedom we have discussed, we find them impractical as far as jet production is concerned. Moreover, there is a more natural approach. Notice, that when all the occurrences of $C_{\mathbf{Q}}^{(0)}$ in (2.28) are the same, the cancellation taking place around the matching point is the most effective. We mean here not only the form of $C_{\mathbf{Q}}^{(0)}$, but also the way it is convoluted. Then, it is easy to see that they can be the same only when the mass of the initial state is set to zero in $C_{\mathbf{Q}}^{(0)}$ (what is allowed due to the freedom we have discussed). Otherwise, even if $C_{\mathbf{Q}}^{(0)}$ are everywhere the same the convolutions are different, since the integrals have different limits. For example the convolutions $f_g \otimes \mathcal{F}_{g\mathbf{Q}}^{(1)}$ and $f_{\mathbf{Q}} \otimes C_{\mathbf{Q}}$ are not the same (in the operational sense) when $m_{\mathbf{Q}} \neq 0$. On the other hand, when $m_{\mathbf{Q}} = 0$ both lower bounds in the convolutions are just x_B whilst the upper ones equal to 1. We turn attention, that the only dependence on mass that remains in subtraction terms is hidden under logarithms in $\mathcal{F}_{g\mathbf{Q}}^{(1)}$ and $\mathcal{F}_{\mathbf{QQ}}^{(1)}$.

In summary, we interpret LO term $f_{\mathbf{Q}} \otimes C_{\mathbf{Q}}^{(0)}$ as an 'asymptotic' expression appearing after resummation of logarithms, thus it should be subtracted around the matching point leaving only BGF mechanism. Since it is the asymptotic expression the initial state mass is set to zero. Accordingly, we set $m_{\mathbf{Q}} = 0$ in $C_{\mathbf{Q}}^{(0)}$ appearing in subtraction terms. This allows for complete cancellation around the matching point as shown in Fig. 2.2. We stress that we do not set all initial state masses to zero. We shall come back to this issue in Section 4.3.2.

Chapter 3

Massive dipole subtraction method

3.1 Introduction

The dipole subtraction method is a specific realization of the subtraction method described in Section 1.2. The subtraction term is constructed as a sum of so-called dipoles, motivated by very general behaviour of matrix elements. It was first developed in an extensive paper by S. Catani and M. Seymour [9] in 1996. Their method is applicable for lepton-lepton, lepton-hadron and hadron-hadron processes, also with the possible identified partons in a final state. However, it was developed for massless partons only. The method of [9] was later extended in [23] to a general case of massive quarks (also in the initial state), however for the processes, where the photons radiate off the fermions. Moreover, they used a finite photon mass in order to regularize IR singularities. Therefore, their results are not sufficient for QCD processes, where not only gluons are emitted from initial quarks, but also the gluons can split into $q\overline{q}$ and gg final state pairs. This was the reason for a joint work of the authors of Refs. [9, 23] and independently [44]. In [10] they developed the dipole method for massive partons, however, they resign to take into account the masses of possible initial state heavy quarks, as they allow only the standard (massless) factorization theorem. We note, that they treat as massless not only the initial state splitting processes like $q \to qg$, but also they use massless quarks in $g \to q\overline{q}$ splitting. However, as we have seen in the preceding sections, taking into account the masses of possible initial state heavy quarks, especially taking into account massive $g \to q\overline{q}$ splitting, is essential if we want to get consistent and reliable predictions for large range of external scale.

In the present chapter we develop the fully massive dipole subtraction method, which allows for NLO calculations of neutral current DIS processes. We take into account all the masses of the quarks, including possible initial states. Our method is a generalization of the one mentioned above [10], therefore we try to keep similar notation. Most of the material presented in this chapter is new. It should however be mentioned, that in order to calculate a full jet cross section the material should be supplemented by the results of [10] which do not involve initial states and are not treated here.

The chapter is arranged as follows. Before we explain in details (Section 3.3) how the dipole subtraction method is constructed, we must learn how the matrix elements

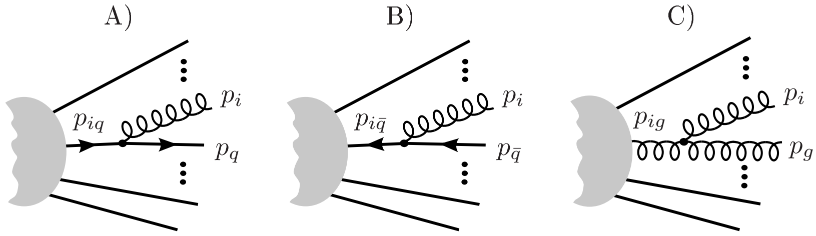


Figure 3.1: Emission of a soft gluon from A) quark, B) anti-quark and C) gluon. The lines without arrows correspond to any type of particle, while dots denote the rest of final states. Shaded blob corresponds to a reduced amplitude $\hat{\mathcal{M}}_n$.

behave in the soft and collinear limits, especially when the quarks are massive. We shall see that in the context of this work, it is desirable to consider so called quasi-collinear limit instead of the usual collinear one. This is done in Section 3.2. Next, we describe a special kinematics, that has to be introduced (Section 3.4) in order to factorize the $(n+1)$-particle phase space into n-particle and a subspace that after integration leads to singularities. Using these variables, we define the dipole splitting functions in Section 3.5. They are the core of the subtraction terms and mimic the true singular behaviour of the matrix elements. Next, in Section 3.6, we describe in details the phase space factorization procedure. Finally, we integrate all the dipoles over the factorized subspace in Section 3.7.

We note, that the fully massive dipole subtraction method presented below, is still not sufficient to make reasonable calculations. As we shall see, there are potential collinear singularities that have to be factorized into PDFs. This step shall be done in the next Chapter 4.

3.2 Singular behaviour of tree-level matrix elements

Let us start with the investigation of singularities, that appear in the tree-level matrix elements. They emerge from two different kinematic regions (which can however overlap). First one is so called soft region, connected with an emission of a gluon with zero energy. Second region, actually more complicated, is the collinear region (more precisely – quasi-collinear, see below). Essentially, we follow [10], however we give more details and present some results, which do not appear in the literature explicitly.

3.2.1 Soft limit

The content of this section is essentially well known, although most often the masses of quarks are neglected. In some parts we follow [9, 10, 22].

Let us consider a generic $(n+1)$-particle amplitude $\hat{\mathcal{M}}_{n+1}$. The hat reminds that it generally is an object with spinor and colour indices, which are suppressed. Alternatively we may think, that colour or spin indices can be pulled out by treating $\hat{\mathcal{M}}$ as a vector in colour and helicity space and projecting it onto suitable basis vectors. Let us assume for a moment that all the partons are final states.

Let us now suppose that i-th particle is a gluon that is emitted from an off-shell quark q with mass m (we assume also that all the other partons are on-shell). This situation is depicted in Fig. 3.1. The contribution to full amplitude from this kind of emission can be written as

$$\hat{\mathcal{M}}_{n+1}^{(q)}\left(p_1,\ldots,p_{n+1}\right)=\varepsilon_\mu^*\left(\nu\right)\left(-ig\right)\hat{t}^A\,\overline{u}\left(p_q\right)\gamma^\mu\,i\,\frac{\not{p}_{iq}+m}{p_{iq}^2-m^2}\,\widetilde{\hat{\mathcal{M}}}_n\left(\left\{p_k\right\}_{k\in\mathbb{A}'}\right), \tag{3.1}$$

where $\hat{t}^A$ are standard matrices of colour group generators, $\varepsilon^\mu\left(\nu\right)$ is a photon polarization vector with helicity ν and $\overline{u}\left(p_q\right)$ is an adjoint spinor for a quark with momentum p_q. Here we have defined

$$p_{iq}^\mu=p_i^\mu+p_q^\mu \tag{3.2}$$

and the set $\mathbb{A}$ is defined as

$$\mathbb{A}'=\left\{1,\ldots,n+1\right\}\setminus\left\{i,q\right\}\cup\left\{iq\right\}, \tag{3.3}$$

that is we have removed the partons i, q from the original matrix element and replace by single iq. Note, however that at this stage the leg p_{iq} in the amplitude on the RHS of (3.1) is off-shell. This fact is marked by a tilde adorning the amplitude. Also, in (3.1), we have suppressed a spin index in the external spinor.

The soft limit is reached when for any fixed four-vector r^μ we have [9]

$$p_i^\mu=\lambda r^\mu,\quad \lambda\to 0. \tag{3.4}$$

Then, using simple spinor algebra and Dirac equation we get from (3.1)[1]

$$\hat{\mathcal{M}}_{n+1}^{(q)}\left(p_1,\ldots,p_{n+1}\right)\underset{\lambda\to 0}{\longrightarrow}\frac{1}{\lambda}\,g\,\varepsilon_\mu^*\left(\nu\right)\hat{t}^A\,\frac{p_q^\mu}{r\cdot p_q}\,\hat{\mathcal{M}}_n\left(\left\{p_k\right\}_{k\in\mathbb{A}}\right), \tag{3.5}$$

where now

$$\mathbb{A}=\left\{1,\ldots,n+1\right\}\setminus\left\{i\right\}, \tag{3.6}$$

Note that now the momentum of a quark p_{iq} left after removing the gluon is on-shell, since

$$p_{iq}^\mu\underset{\lambda\to 0}{\longrightarrow}p_q^\mu. \tag{3.7}$$

In complete analogy, we obtain the contribution from the emission from an anti-quark $\overline{q}$

$$\hat{\mathcal{M}}_{n+1}^{(\overline{q})}\left(p_1,\ldots,p_{n+1}\right)\underset{\lambda\to 0}{\longrightarrow}\frac{1}{\lambda}\,g\,\varepsilon_\mu^*\left(\nu\right)\hat{\mathcal{M}}_n\left(\left\{p_k\right\}_{k\in\mathbb{A}}\right)\left(-\hat{t}^A\right)\frac{p_{\overline{q}}^\mu}{r\cdot p_{\overline{q}}}. \tag{3.8}$$

Finally, we have to consider the situation when the soft gluon is emitted from another off-shell gluon, as in Fig. 3.1C. The result can be simply obtained if one uses the fact that the gluon propagator becomes transverse for $\lambda\to 0$, resulting in

$$\hat{\mathcal{M}}_{n+1}^{(g)}\left(p_1,\ldots,p_{n+1}\right)\underset{\lambda\to 0}{\longrightarrow}\frac{1}{\lambda}\,g\,\varepsilon_\mu^*\left(\nu\right)\left(-if_{ABC}\right)\frac{p_g^\mu}{r\cdot p_g}\,\hat{\mathcal{M}}_n^C\left(\left\{p_k\right\}_{k\in\mathbb{A}}\right). \tag{3.9}$$

We see, that all the contributions $\hat{\mathcal{M}}_{(n+1)}^{(q)}$, $\hat{\mathcal{M}}_{(n+1)}^{(\overline{q})}$, $\hat{\mathcal{M}}_{(n+1)}^{(g)}$ have the same structure, except the colour factors. This reflects the fact that the soft gluon has a very long wavelength and thus is insensitive to the spin structure of the emitting particle.

In order to write the full amplitude with the soft gluon emission in a uniform fashion, let us introduce the *colour operator* $\hat{T}_j^A$ for a parton j, which generates pertinent colour

[1] The replacement of the gluon-quark (or photon-quark) vertex γ^μ by $2p_q^\mu$ is the eikonal approximation.

structure. Its action is most transparently defined introducing the orthonormal basis in the colour space; for n particles in the final state and m particles in the initial state we define

$$\left(\bigotimes_{a=1}^{m} |c_a\rangle\right) \otimes \left(\bigotimes_{j=1}^{n} |d_j\rangle\right) \equiv |c_1, \ldots, c_m; d_1, \ldots, d_n\rangle \equiv |\{c_a\}; \{d_j\}\rangle , \tag{3.10}$$

such that

$$\left\langle \hat{\mathcal{M}}_n (\{p_a\}; \{p_j\}) \middle| \{c_a\}; \{d_j\} \right\rangle = \frac{1}{\prod_a \sqrt{n_{c_a}}} \hat{\mathcal{M}}_n^{c_1,\ldots,c_m,d_1,\ldots,d_n} (\{p_a\}; \{p_j\}) , \tag{3.11}$$

where c_a is a colour charge of an initial state parton a whereas d_j is the same for final state parton j. On the LHS we treat the amplitude as a vector in the colour space, normalized in such a way that colour averaged amplitude squared can be written as

$$\left|\overline{\mathcal{M}}_n (\{p_a\}; \{p_j\})\right|^2 = \left\langle \hat{\mathcal{M}}_n (\{p_a\}; \{p_j\}) \middle| \hat{\mathcal{M}}_n (\{p_a\}; \{p_j\}) \right\rangle . \tag{3.12}$$

In the above formulae the range of the index numbering the elements of sets $\{.\}$ is dropped for transparency; we shall often do this, if it does not lead to a confusion. Once we have chosen the colour basis, we can define the colour operators as follows

$$\hat{T}_j^A |\{c_a\}; d_1, \ldots, d_j, \ldots, d_n\rangle = \begin{cases} t^A_{d_j b} |\{c_a\}; d_1, \ldots, b, \ldots, d_n\rangle , & j = q \\ -t^A_{b d_j} |\{c_a\}; d_1, \ldots, b, \ldots, d_n\rangle , & j = \overline{q} \\ -i f_{A d_j B} |\{c_a\}; d_1, \ldots, B, \ldots, d_n\rangle , & j = g \end{cases} \tag{3.13}$$

if the operator acts on a final state, and

$$\hat{T}_a^A |c_1, \ldots, c_a, \ldots, c_m; \{d_j\}\rangle = \begin{cases} -t^A_{b c_a} |c_1, \ldots, b, \ldots, c_m; \{d_j\}\rangle , & a = q \\ t^A_{c_a b} |c_1, \ldots, b, \ldots, c_m; \{d_j\}\rangle , & a = \overline{q} \\ i f_{A c_a B} |c_1, \ldots, B, \ldots, c_m; \{d_j\}\rangle , & a = g \end{cases} \tag{3.14}$$

for operation on initial state. The action of the final state colour operators is evident from our derivation above, eqs. (3.5)-(3.9), while the action of the initial state operators can be easily obtained using crossing symmetry. Let us note, that

$$\hat{T}_k^2 \equiv \sum_A \hat{T}^A \hat{T}^A = \begin{cases} C_F, & k = q, \overline{q} \\ C_A, & k = g. \end{cases} \tag{3.15}$$

Due to colour conservation we have also the following property

$$\left(\sum_a \hat{T}_a^A + \sum_j \hat{T}_j^A\right) \left|\hat{\mathcal{M}}_n (\{p_a\}; \{p_j\})\right\rangle = 0. \tag{3.16}$$

Using the above notation, we can write the complete amplitude with the soft gluon emission (now we take into account the possible initial states) as

$$\hat{\mathcal{M}}^A_{n+1} (\{p_a\}; \{p_j\}) \underset{\lambda \to 0}{\longrightarrow} \frac{1}{\lambda} g\, \varepsilon^*_\mu (\nu)\ \hat{J}^{\mu\, A} (r)\, \hat{\mathcal{M}}_n \left(\{p_a\}; \{p_k\}_{k \in \mathbb{A}}\right) , \tag{3.17}$$

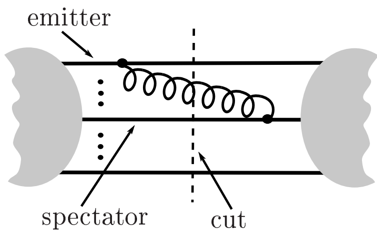


Figure 3.2: Definition of the emitter and spectator. The last one is a parton that recoils emitted particle on the other side of the cut. Both emitter and spectator can be final or initial states.

where so called insertion (or eikonal) current J^μ is defined as

$$\hat{J}^{\mu\,A}(r) = \sum_a \hat{T}_a^A \frac{p_a^\mu}{r \cdot p_a} + \sum_{j \neq i} \hat{T}_j^A \frac{p_j^\mu}{r \cdot p_j} = \sum_{I \neq i} \hat{T}_I^A \frac{p_I^\mu}{r \cdot p_I}, \tag{3.18}$$

where in the last step we introduced the index I that runs over both initial and final states.

Let us now square the amplitude and sum/average over colours and spins. First, the eikonal current squared and summed over A can be written as

$$\sum_A \hat{J}^{\mu\,A}(r)\,\hat{J}_\mu^A(r) = \sum_{I \neq i} \frac{1}{r \cdot p_I} \sum_{K \neq I} \hat{T}_I \cdot \hat{T}_K \left(\frac{2p_I \cdot p_K}{r \cdot (p_I + p_K)} - \frac{m_I^2}{r \cdot p_I} \right). \tag{3.19}$$

This form was obtained by partial fractioning the expressions of the type $1/(r \cdot p_I)(r \cdot p_K)$ and using colour conservation (3.16). The amplitude squared thus takes the following form (in D dimensions)

$$\left|\overline{\mathcal{M}}_{n+1}(\{p_a\};\{p_j\})\right|^2 \underset{\lambda \to 0}{\longrightarrow} -\frac{1}{\lambda^2} 8\pi\mu_{\mathrm{r}}^{2\varepsilon}\alpha_s \sum_{I \neq i} \frac{1}{r \cdot p_I} \sum_{K \neq I} \left(\frac{p_I \cdot p_K}{r \cdot (p_I + p_K)} - \frac{m_I^2}{2\,r \cdot p_I} \right) \left\langle \hat{\mathcal{M}}_n(\{p_a\};\{p_j\}) \right| \hat{T}_I \cdot \hat{T}_K \left| \hat{\mathcal{M}}_n(\{p_a\};\{p_j\}) \right\rangle. \tag{3.20}$$

Above formula is the key for constructing dipole subtraction terms, although as we shall see in Section 3.3 there are several points to overcome.

For further convenience, let us introduce the following notation for the colour-correlated amplitudes

$$\left\langle \hat{\mathcal{M}}_n(\{p_a\};\{p_j\}) \right| \hat{T}_I \cdot \hat{T}_K \left| \hat{\mathcal{M}}_n(\{p_a\};\{p_j\}) \right\rangle \equiv \left|\mathcal{M}_n(\{p_a\};\{p_j\})\right|_{I,K}^2. \tag{3.21}$$

It will allow for more compact formulae later on.

In the end of this section, let us introduce a nomenclature following [9] that we shall use throughout. A particle which emits a gluon (or in general any other parton) we call an *emitter*. Further, as far as one considers the amplitude squared, an emitted particle is recoiled on the other side of the cut by a parton that we call a *spectator* (Fig. 3.2). There is a symmetry between all emitter-spectator cases, as is evident e.g. from (3.20). In general, we can distinguish the following cases

- final state emitter - final state spectator (FE-FS)

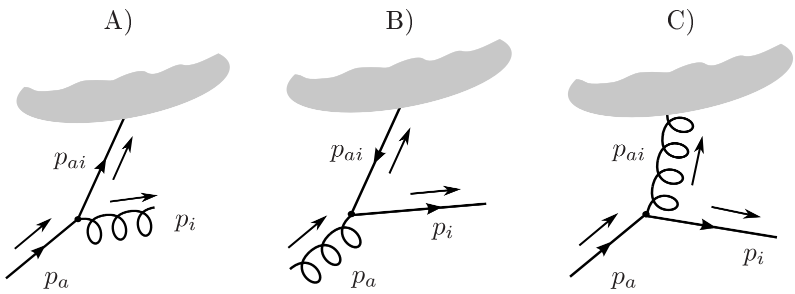


Figure 3.3: Initial state splitting processes with massive quark $\mathbf{Q}$: A) $\mathbf{Q} \to \mathbf{Q}g$ splitting, B) $g \to \overline{\mathbf{Q}}\mathbf{Q}$, C) $\mathbf{Q} \to g\mathbf{Q}$.

- final state emitter - initial state spectator (FE-IS)
- initial state emitter - final state spectator (IE-FS)
- initial state emitter - initial state spectator (IE-IS).

In the present work we shall consider the first three classes, since we are so far concentrated on lepton-hadron processes only.

3.2.2 Quasi-collinear limit

Let us now consider another region where a tree-level amplitude can be divergent. The propagators, as those in Fig. 3.1, can become infinite when the momenta of the massless partons are collinear. If the quarks are massive there is no true collinear singularity, however the singularity arises again when the mass can be neglected comparing to external energy scales. Therefore in view of the present work it is convenient to consider so called *quasi-collinear limit* [11, 10] and *quasi-collinear singularity*. In the following subsections we recall this idea in details. We pay particular attention to the case of an emission from the initial state with heavy quark masses taken into account. This is the elementary case, which – as far as we are concerned – has not been given explicitly in the literature.

3.2.2.1 Initial state emitter case

Let us start with the case where an initial state parton a with momentum p_a emits another parton i with momentum p_i. Three possible in QCD cases, that involve quarks are shown in Fig. 3.3. Such an emission, where there are massive partons is not considered in [10, 11] while in [23] only photon radiation off fermions is worked out. Therefore, we shall give brief, but detailed discussion.

Let us first introduce the shortcut notation (Fig. 3.3)

$$p^{\mu}_{ai} = p^{\mu}_{a} - p^{\mu}_{i}. \tag{3.22}$$

The momentum p_{ai} is off-shell, while for p_i and p_a we assume the on-shell conditions

$$p_i^2 = m_i^2, \quad p_a^2 = m_a^2. \tag{3.23}$$

In order to define the quasi collinear limit, let us introduce the Sudakov parametrization of momenta. We decompose p_i into a component parallel to p_a and a transverse one. To this end we choose auxiliary light-like four-vector n,

$$n^2 = 0. \tag{3.24}$$

Then the Sudakov decomposition reads

$$p_i^\mu = (1-x)\, p_a^\mu + k_T^\mu - \frac{k_T^2 + (1-x)^2\, m_a^2 - m_i^2}{1-x} \frac{n^\mu}{2\, n \cdot p_a}. \tag{3.25}$$

The transverse component is perpendicular to p_a and n,

$$k_T \cdot p_a = k_T \cdot n = 0. \tag{3.26}$$

Accordingly, due to (3.22), we have

$$p_{ai}^\mu = x p_a^\mu - k_T^\mu + \frac{k_T^2 + (1-x)^2\, m_a^2 - m_i^2}{1-x} \frac{n^\mu}{2\, n \cdot p_a}. \tag{3.27}$$

The variable x has an obvious physical interpretation. It is the fraction of the original momentum p_a that enters the reduced matrix element (the shaded blobs in Fig. 3.3).

Now, consider the denominator of one of the propagators from Fig. 3.3. It reads

$$p_{ai}^2 - m_{\underline{ai}}^2 = \frac{k_T^2 + x\,(1-x)\, m_a^2 - x\, m_i^2 - (1-x)\, m_{\underline{ai}}^2}{1-x}, \tag{3.28}$$

where $m_{\underline{ai}}$ is the on-shell mass of the parton corresponding to momentum p_{ai}. Usual collinear limit is defined by $|k_T| \to 0$. Then however, the inverse propagator (3.28) is in general nonzero. In order to make it zero, we use a uniform rescaling

$$|k_T| \to \lambda\, |k_T|\,, \quad m_q \to \lambda m_q, \quad \lambda \to 0, \tag{3.29}$$

for $q = a, i, \underline{ai}$, such that the propagator (3.28) indeed becomes zero. The limits (3.29) define advocated quasi-collinear behaviour.

Let us now switch to the more specific cases. Let us start with the splitting process showed in Fig. 3.3A, namely

$$\mathbf{Q}\,(p_a) \to \mathbf{Q}\,(p_{ai})\, g\,(p_i)\,. \tag{3.30}$$

In this case we have

$$m_a = m_{\underline{ai}} = m_{\mathbf{Q}} \equiv m, \quad m_i = 0. \tag{3.31}$$

The amplitude can be written as

$$\hat{\mathcal{M}}_{n+1}\,(p_a; p_i, \ldots) = \widetilde{\hat{\mathcal{M}}}_n^\dagger\,(p_{ai}; \ldots)\; i\, \frac{\not{p}_{ai} + m}{p_{ai}^2 - m^2} \left(-ig\gamma^\mu \hat{t}^A\right) u^s\,(p_a)\, \varepsilon_\mu^*\,(\nu)\,. \tag{3.32}$$

The notation used above is similar to the one used in Section 3.2.1. We adorned by tilde the reduced matrix element on the RHS in order to underline that it has amputated leg corresponding to the off-shell momentum p_{ai}. Spinor superscript s refers to a spin state. Squaring the amplitude and summing/averaging over colour and spin we get

$$\left|\overline{\hat{\mathcal{M}}_{n+1}\,(p_a; p_i, \ldots)}\right|^2 = 2\pi\alpha_s \mu_r^{2\varepsilon}\, \frac{C_F}{N_c}\, d_{\mu\nu}\,(p_i; n)\; \widetilde{\hat{\mathcal{M}}}_n^\dagger\,(p_{ai}; \ldots)\, \frac{\hat{\Gamma}^{\mu\nu}}{\left(p_{ai}^2 - m^2\right)^2}\, \widetilde{\hat{\mathcal{M}}}_n\,(p_{ai}; \ldots)\,, \tag{3.33}$$

where

$$\hat{\Gamma}^{\mu\nu} = (\not{p}_{ai} + m)\,\gamma^{\mu}\,(\not{p}_a + m)\,\gamma^{\nu}\,(\not{p}_{ai} + m)\,, \tag{3.34}$$

and the polarization tensor for the gluon with momentum p_i reads

$$d^{\mu\nu}(p_i; n) = -g^{\mu\nu} + \frac{p_i^{\mu} n^{\nu} + p_i^{\nu} n^{\mu}}{p_i \cdot n}. \tag{3.35}$$

Next, we apply the Sudakov decomposition and the limit (3.29). It leads to

$$d_{\mu\nu}\hat{\Gamma}^{\mu\nu} = -\lambda^2 \left[(D-2)(1-x)\left(p_{ai}^2 - m^2\right) + \frac{4xk_T^2}{(1-x)^2} \right] \not{p}_a + \mathcal{O}\left(\lambda^3\right). \tag{3.36}$$

Thus, in the quasi-collinear limit we finally obtain

$$\left|\overline{\hat{\mathcal{M}}}_{n+1}(p_a; p_i, \ldots)\right|^2 \to -\frac{1}{\lambda^2}\, 8\pi\alpha_s \mu_r^{2\varepsilon}\, \frac{1}{x}\, \frac{1}{p_{ai}^2 - m^2} \\ \overline{\hat{\mathcal{M}}}_n^{\dagger}(p_{\underline{ai}}; \ldots)\, \hat{P}_{\mathbf{QQ}}(x)\, \overline{\hat{\mathcal{M}}}_n(p_{\underline{ai}}; \ldots)\,, \tag{3.37}$$

where the splitting matrix reads (the unit matrix in helicity space is suppressed)

$$\hat{P}_{\mathbf{QQ}}(x) = C_F \left(\frac{1+x^2}{1-x} - \varepsilon(1-x) + \frac{2x\,m^2}{p_{ai}^2 - m^2} \right). \tag{3.38}$$

The momentum $p_{\underline{ai}}$ refers to limiting, on-shell version of p_{ai}, i.e. $p_{\underline{ai}} = xp_a$ and $p_{\underline{ai}}^2 = 0$ in the limit (3.29).

There are several comments concerning (3.37). First, there is a factor $1/x$, which comes from the conversion

$$\hat{\mathcal{M}}_n^{\dagger}(p_{ai}; \ldots)\,\not{p}_a \hat{\mathcal{M}}_n(p_{ai}; \ldots) \to \frac{1}{x}\left|\hat{\mathcal{M}}_n\left(p_{\underline{ai}}; \ldots\right)\right|^2. \tag{3.39}$$

Second, we adorned the amplitudes in (3.37) by a bar, since we included factors from colour and spin averages. Finally, since splitting matrices act in helicity space, in general there are spin correlations. In this case however the splitting matrix is diagonal.

Let us now move to the next splitting case (Fig. 3.4B)

$$g(p_a) \to \overline{\mathbf{Q}}(p_{ai})\,\mathbf{Q}(p_i) \tag{3.40}$$

where we have

$$m_a = 0, \quad m_{\underline{ai}} = m_i = m_{\mathbf{Q}} \equiv m. \tag{3.41}$$

Analogous calculation to the one above leads again to (3.37) with however different splitting matrix

$$\hat{P}_{g\mathbf{Q}}(x) = T_R \left[1 - \frac{2}{1-\varepsilon} \left(x(1-x) + \frac{x\,m^2}{p_{ai}^2 - m^2} \right) \right]. \tag{3.42}$$

It is again diagonal in helicity (identity matrix was dropped). We recall, that the convention for naming the splitting functions was given in Section 1.1.

Finally, let us turn to the process from Fig. 3.3C

$$\mathbf{Q}(p_a) \to g(p_{ai})\,\mathbf{Q}(p_i) \tag{3.43}$$

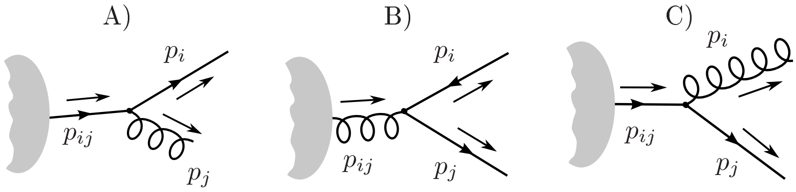


Figure 3.4: Final state splitting processes with massive quark $\mathbf{Q}$: A) $\mathbf{Q} \to \mathbf{Q}g$, B) $g \to \overline{\mathbf{Q}}\mathbf{Q}$, C) $\mathbf{Q} \to g\mathbf{Q}$.

with the masses

$$m_a = m_i = m_{\mathbf{Q}} \equiv m, \quad m_{\underline{ai}} = 0. \tag{3.44}$$

As one can expect, the behaviour of the matrix element is the same as before with different splitting matrix. Now it reads

$$\left(\hat{P}_{\mathbf{Q}g}(x)\right)^{\mu\nu} = C_F (1-\varepsilon)\left(-xg^{\mu\nu} - 4\frac{k_T^\mu k_T^\nu}{x\, p_{ai}^2}\right). \tag{3.45}$$

We see, that this time it is not diagonal in helicity, thus we cannot simply factorize the matrix element squared as before. Note, that during the derivation, we used the following relation between the colour factors

$$\frac{T_R}{N_c} = \frac{C_F}{N_A}. \tag{3.46}$$

This was used to transform the colour average from the one over quark colours to the one over gluon colour states. The factor $(1-\varepsilon)$ follows from the spin conversion.

There is one more possible splitting process, which involves only gluons

$$g(p_a) \to g(p_{ai})\, g(p_i). \tag{3.47}$$

Since it does not involve massive partons we can use the result from [9]. The splitting process is symmetric with respect to exchange of partons, thus we can overtake their final state formula and convert to our initial state kinematics. We get

$$\left(\hat{P}_{gg}(x)\right)^{\mu\nu} = 2C_A\left[-g^{\mu\nu}\left(\frac{x}{1-x} + \frac{1-x}{x}\right) - (D-2)\, x\frac{k_T^\mu k_T^\nu}{p_{ai}^2}\right]. \tag{3.48}$$

The splitting matrices introduced above reduce to the well known splitting functions after averaging over helicities, taking massless limit, and setting $D = 4$.

3.2.2.2 Final state emitter

Let us now analyze the case, where initially off-shell final state parton is split into partons i and j with the momenta p_i and p_j respectively, as showed in Fig. 3.4.

The Sudakov parametrization of final state momenta takes the following form

$$p_i^\mu = zp_{\underline{ij}}^\mu + k_T^\mu - \frac{k_T^2 + z^2 m_{\underline{ij}}^2 - m_i^2}{z}\frac{n^\mu}{2\,p_{\underline{ij}}\cdot n}, \tag{3.49}$$

$$p_j^\mu = (1-z)\, p_{\underline{ij}}^\mu - k_T^\mu - \frac{k_T^2 + (1-z)^2 m_{\underline{ij}}^2 - m_j^2}{1-z}\frac{n^\mu}{2\,p_{\underline{ij}}\cdot n}. \tag{3.50}$$

These equations are actually defining formulae for the four-vector $p_{\underline{ij}}$, i.e. in the quasi-collinear limit (3.29) (with rescaling masses suitably to this section of course) p_i and p_j are parallel to $p_{\underline{ij}}$. For further purposes we introduce

$$p_{ij}^{\mu} = p_i^{\mu} + p_j^{\mu}. \tag{3.51}$$

Note the difference between $p_{\underline{ij}}$ and p_{ij}.

The case of final state quasi-collinear emission is covered in [10]. Since derivation is completely analogous to that of the previous subsection, we limit ourselves to listing the formulae as we shall need them later.

The matrix element squared behaves as

$$\left|\overline{\hat{\mathcal{M}}}_{n+1}\left(p_a;p_i,p_j\ldots\right)\right|^2 \to -\frac{1}{\lambda^2}\,8\pi\alpha_s\mu_r^{2\varepsilon}\,\frac{1}{p_{ij}^2-m_{\underline{ij}}^2} \\ \overline{\hat{\mathcal{M}}}_n^{\dagger}\left(p_a;p_{\underline{ij}},\ldots\right)\,\hat{P}_{\underline{ij},i}\left(z\right)\,\overline{\hat{\mathcal{M}}}_n\left(p_a;p_{\underline{ij}},\ldots\right). \tag{3.52}$$

Note, that this time there is no analog of $1/x$ factor comparing to (3.37). The splitting matrices are as follows

$$\hat{P}_{\mathbf{QQ}}\left(z\right) = C_F\left(\frac{1+z^2}{1-z} - \varepsilon\left(1-z\right) + \frac{2\,m^2}{p_{ij}^2-m^2}\right), \tag{3.53}$$

$$\hat{P}_{\mathbf{Q}g}\left(z\right) = C_F\left(\frac{1+\left(1-z\right)^2}{z} - \varepsilon z + \frac{2\,m^2}{p_{ij}^2-m^2}\right), \tag{3.54}$$

$$\left(\hat{P}_{g\mathbf{Q}}\left(z\right)\right)^{\mu\nu} = T_R\left(-g^{\mu\nu} - 4\,\frac{k_T^{\mu}k_T^{\nu}}{p_{ij}^2}\right), \tag{3.55}$$

$$\left(\hat{P}_{gg}\left(z\right)\right)^{\mu\nu} = 2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right) - \left(D-2\right)\frac{k_T^{\mu}k_T^{\nu}}{p_{ij}^2}\right]. \tag{3.56}$$

The mass $m \equiv m_{\mathbf{Q}}$ above refers to pertinent heavy quark $\mathbf{Q}$ involved in splitting process.

The splitting matrices (3.53)-(3.56) are in general different than those from the previous subsection, as one could expect. Here for instance there is a symmetry $z \to (1-z)$ for $\hat{P}_{\mathbf{QQ}}$ and $\hat{P}_{\mathbf{Q}g}$ splittings as can be seen from Fig. 3.4A, C. It is not the case for initial state splitting. The universal objects, suitable for initial and final state, are four-dimensional, massless and averaged versions of the matrices presented in this section.

3.3 Construction of dipoles

As we have seen in Sections 3.2.1, 3.2.2, a tree level amplitude squared can be written in the both singular regions in the following schematic form

$$\left|\overline{\hat{\mathcal{M}}}_{n+1}\right|^2 \to 8\pi\alpha_s\mu_r^{2\varepsilon}\,\frac{1}{\mathcal{S}}\,\hat{V}\otimes\left|\overline{\hat{\mathcal{M}}}_n\right|^2, \tag{3.57}$$

where $\mathcal{S}$ represents adequate scalar propagator, $\hat{V}$ encodes the information about soft/collinear splitting process leading to singularities, and the convolution sign realizes spin and colour correlations.

The above structure can be used to construct subtraction term in the dipole method. Recall that such a term is just a fake cross section, that becomes equal to the real one in the singular regions (cf. Section 1.2). First problem to overcome, is that away from the soft limit there is no momentum conservation in reduced matrix element $\hat{\mathcal{M}}_n$ (recall, we removed the soft gluon, which was legitimate only in the strict soft limit), consequently (3.57) cannot be calculated as an usual cross section. Second, one could construct the subtraction term by just adding the limiting formulae for soft and collinear behaviour, in order to mimic both regions. The problem is however, that there is an overlap region, with a collinear and a soft particle at the same time. It is evident for example from (3.56), when $z \to 0$ we have double soft-collinear singularity. Thus adding both kinds of limiting formulae leads to double counting of soft singularities.

The solution given in [9] is the following. The subtraction term which mimics the $(n+1)$-particle matrix element squared, is given by (for simpler notation we assume only one initial state parton a)

$$\mathcal{D}\left(p_a;\{p_i\}_{i=1}^{n+1}\right) = \sum_{i=1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n+1}\Bigg\{\mathcal{D}^{\text{IE-FS}}_{i,j,a}\left(\tilde{p}_{\underline{ai}};\{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right) + \mathcal{D}^{\text{FE-IS}}_{i,j,a}\left(\tilde{p}_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right) + \sum_{\substack{k=1\\ k\neq i,j}}^{n+1}\mathcal{D}^{\text{FE-FS}}_{i,j,k}\left(p_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right)\Bigg\} \tag{3.58}$$

where

$$\mathcal{D}^{\text{IE-FS}}_{i,j,a}\left(\tilde{p}_{\underline{ai}};\{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right) = -\frac{1}{\mathcal{S}_{a,i}}\frac{1}{x} \left\langle\hat{\mathcal{M}}_n\left(\tilde{p}_{\underline{ai}};\{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right)\right|\frac{\hat{T}_j\cdot\hat{T}_{\underline{ai}}}{\hat{T}^2_{\underline{ai}}}\,\hat{V}^{\text{IE-FS}}_{a\to\underline{ai}\,i,\,j}\left|\hat{\mathcal{M}}_n\left(\tilde{p}_{\underline{ai}};\{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right)\right\rangle, \tag{3.59}$$

$$\mathcal{D}^{\text{FE-IS}}_{i,j,a}\left(\tilde{p}_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right) = -\frac{1}{\mathcal{S}_{i,j}}\frac{1}{x} \left\langle\hat{\mathcal{M}}_n\left(\tilde{p}_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right)\right|\frac{\hat{T}_a\cdot\hat{T}_{\underline{ij}}}{\hat{T}^2_{\underline{ij}}}\,\hat{V}^{\text{FE-IS}}_{\underline{ij}\to i\,j,\,a}\left|\hat{\mathcal{M}}_n\left(\tilde{p}_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right)\right\rangle, \tag{3.60}$$

$$\mathcal{D}^{\text{FE-FS}}_{i,j,k}\left(p_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right) = -\frac{1}{\mathcal{S}_{i,j}} \left\langle\hat{\mathcal{M}}_n\left(p_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right)\right|\frac{\hat{T}_k\cdot\hat{T}_{\underline{ij}}}{\hat{T}^2_{\underline{ij}}}\,\hat{V}^{\text{FE-FS}}_{\underline{ij}\to i\,j,\,k}\left|\hat{\mathcal{M}}_n\left(p_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right)\right\rangle. \tag{3.61}$$

Let us now carefully explain the notation. First, $\mathcal{S}_{i,j}$, $\mathcal{S}_{a,i}$ are pertinent inverse scalar propagators

$$\mathcal{S}_{i,j} = (p_i+p_j)^2 - m^2_{\underline{ij}}, \tag{3.62}$$

$$\mathcal{S}_{a,i} = (p_a-p_i)^2 - m^2_{\underline{ai}}. \tag{3.63}$$

Next, some of the momenta are adorned by a tilde. They are new momenta constructed in such a way that momentum conservation holds in $\hat{\mathcal{M}}_n$. In what follows, we shall refer to them as *dipole momenta*. In soft or quasi-collinear limit they behave as

$$\tilde{p}_{\underline{ij}} \to p_{\underline{ij}}, \quad \tilde{p}_{\underline{ai}} \to p_{\underline{ai}}, \dots \tag{3.64}$$

They are constructed in Section 3.4. The sets of indices are defined as

$$\mathbb{X}_{\text{IE-FS}} = \{1, \dots, n+1\} \setminus \{i, j\} \cup \{\tilde{j}\}, \tag{3.65}$$

$$\mathbb{X}_{\text{FE-IS}} = \{1, \dots, n+1\} \setminus \{i, j\} \cup \{\tilde{\underline{ij}}\}, \tag{3.66}$$

$$\mathbb{X}_{\text{FE-FS}} = \{1, \dots, n+1\} \setminus \{i, j, k\} \cup \left\{\tilde{\underline{ij}}, \tilde{k}\right\}, \tag{3.67}$$

where the tilde over the parton symbol means that corresponding momentum should be marked by tilde, e.g.

$$p^\mu_{\tilde{j}} \equiv \tilde{p}^\mu_j, \quad p^\mu_{\tilde{\underline{ij}}} \equiv \tilde{p}^\mu_{\underline{ij}}. \tag{3.68}$$

Finally, the objects $\hat{V}$ are *dipole splitting functions*. They are matrices acting in the helicity space, their form being close to the usual splitting matrices. They become the latter in the quasi-collinear limit on one hand and fulfil soft limit without double counting of soft singularities, on the other. We shall construct them in Section 3.5.

In the end of this section, let us check that provided the dipole splitting matrix tends to the true splitting matrix obtained in Section 3.2, i.e. if

$$\hat{V}_{A\to BC,\,D} \to \hat{P}_{AB}, \tag{3.69}$$

we indeed recover correct quasi-collinear behaviour. Clearly, since the dependence on the spectator parton is lost in $\hat{V}$, as shows the above formula, we can make use of the colour conservation (3.16) (reduced matrix elements do not depend on the spectator in this limit). Thus, the colour correlations vanish and the colour factors cancel yielding the required result.

3.4 Dipole kinematics

In the following section we construct an explicit realization of the dipole momenta that are on-shell and fulfil momentum conservation away from the soft limit. It should be pointed out that there is no unique solution - their precise form depends on the kinematic variables one is going to use. The latter have to be defined in such a way, that one can easily control soft and collinear limits.

We concentrate here on the FE-IS and IE-FS cases, since the situation when all the particles (emitter and spectator) are in the final state is fully covered in [23, 10].

3.4.1 Final State Emitter - Initial State Spectator

The situation we want to describe is the following. A final state particle p_i is emitted from another final state parton, which after emission has the momentum p_j. Afterwards it is absorbed by an initial state p_a (a spectator). Let us introduce the following notation we shall use throughout (Fig. 3.5A)

$$\mathcal{P}^\mu = p^\mu_i + p^\mu_j, \tag{3.70}$$

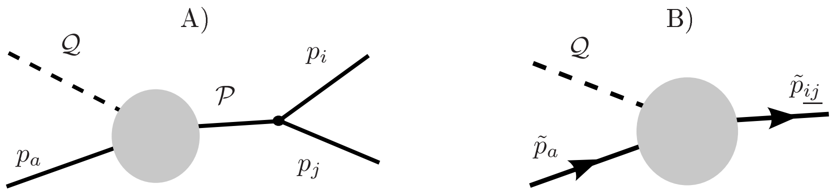


Figure 3.5: A) the momenta assignment in FE-IS case, dashed line represents a relative momentum transfer; B) reduced diagram for FE-IS case, the new tilded momenta are on-shell and fulfil momentum conservation.

$$\mathcal{Q}^\mu = \mathcal{P}^\mu - p_a^\mu. \tag{3.71}$$

That is, $\mathcal{P}$ is the total dipole momentum and $\mathcal{Q}$ is the relative (with respect to the dipole) momentum transfer. It must not be confused with q – the total momentum transfer to the hard process. We assume that $\mathcal{Q}$ is space like $\mathcal{Q}^2 < 0$ and that the particles are on-shell

$$p_a^2 = m_a^2, \quad p_{i,j}^2 = m_{i,j}^2. \tag{3.72}$$

As already mentioned in Section 3.3, in dipole method we have to introduce the new momenta $\tilde{p}_{\underline{ij}}$, $\tilde{p}_a$ (Fig. 3.5B), in such a way that in the soft limit (defined here by $p_i \to 0$) we have $\tilde{p}_{\underline{ij}} \to p_j$ and $\tilde{p}_a \to p_a$. In the quasi-collinear limit $\tilde{p}_{\underline{ij}}$ approaches the fixed collinear direction of p_i and p_j (see Section 3.2.2.2). Of course in QCD reality, the soft limit concerns only the gluons, however on general ground it is enough to have $m_{\underline{ij}} = m_j$ at this stage. We shall analyse the soft and quasi-collinear behaviour of the tilded momenta later, in Sections 3.4.1.3, 3.4.1.4.

One of the possible forms of the dipole momenta is

$$\tilde{p}_{\underline{ij}}^\mu = \tilde{w}\mathcal{P}^\mu - \tilde{u}p_a^\mu, \tag{3.73}$$

$$\tilde{p}_a^\mu = \tilde{p}_{\underline{ij}} - \mathcal{Q}^\mu = (\tilde{w} - 1)\,\mathcal{P}^\mu - (\tilde{u} - 1)\,p_a^\mu. \tag{3.74}$$

In the next subsection we shall fix arbitrary at this stage parameters $\tilde{u}$ and $\tilde{w}$ in such a way that $\tilde{p}_{\underline{ij}}$ fulfil boundary conditions mentioned above. Notice, that we should have $\tilde{u} \to 0$, $\tilde{w} \to 1$ in the soft limit. Note also that we have the explicit momentum conservation.

In order to control the quasi-collinear behaviour, let us also introduce the "angular" variable (definition and notation is due to [9])

$$\tilde{z} = \frac{p_i \cdot p_a}{\mathcal{P}_a} \tag{3.75}$$

where

$$\mathcal{P}_a \equiv \mathcal{P} \cdot p_a. \tag{3.76}$$

Note, that neither $\tilde{u}$ nor $\tilde{z}$ are the Sudakov variables used in Section 3.2.2.2, although they are obviously related. We shall state the relation between both kinds in Section 3.4.1.4.

In what follows we shall refer to $\tilde{u}$, $\tilde{w}$ and $\tilde{z}$ variables as *dipole variables.*

3.4.1.1 Standard dipole kinematics

A already mentioned, we require that the dipole momenta are on-shell. This gives us two conditions

$$\tilde{p}_{\underline{ij}}^2 = m_{\underline{ij}}^2, \tag{3.77}$$

$$\tilde{p}_a^2 = m_a^2, \tag{3.78}$$

which readily can be solved yielding solutions for $\tilde{u}$ and $\tilde{w}$ in terms of the invariants made up of vectors p_a, p_i, p_j. We however shall use more natural quantities in our approach, namely $\mathcal{P}^2$, $\mathcal{Q}^2$ and $\mathcal{P}_a$. In terms of these invariants the solution reads

$$\tilde{u} = \frac{\left(\mathcal{P}_a - \mathcal{P}^2\right)\mathcal{R}}{2v\mathcal{Q}^2\mathcal{P}_a} + \frac{\mathcal{Q}^2 + m_{\underline{ij}}^2 - m_a^2}{2\mathcal{Q}^2} \tag{3.79}$$

$$\tilde{w} = -\frac{\left(\mathcal{P}_a - m_a^2\right)\mathcal{R}}{2v\mathcal{Q}^2\mathcal{P}_a} + \frac{\mathcal{Q}^2 + m_{\underline{ij}}^2 - m_a^2}{2\mathcal{Q}^2}, \tag{3.80}$$

where

$$\mathcal{Q}^2 = \left(\mathcal{P} - p_a\right)^2 = \mathcal{P}^2 - 2\mathcal{P}_a + m_a^2, \tag{3.81}$$

$$\mathcal{R} = \sqrt{\left(\mathcal{Q}^2 - m_{\underline{ij}}^2 - m_a^2\right)^2 - 4m_a^2 m_{\underline{ij}}^2}, \tag{3.82}$$

and

$$v = \sqrt{1 - \frac{m_a^2\mathcal{P}^2}{\mathcal{P}_a^2}}. \tag{3.83}$$

We shall later interpret v as a velocity of p_a in the CM $(\mathcal{Q}, p_a)$ system.

Note, that if $m_a = 0$ we have $v = 1$ and

$$\tilde{u} = \frac{\mathcal{P}^2 - m_{\underline{ij}}^2}{2\mathcal{P}_a}, \tag{3.84}$$

$$\tilde{w} = 1, \tag{3.85}$$

which agrees with [10].

Summarizing, if we are given p_a, p_i and p_j (thus we know $\mathcal{P}^2$, $\mathcal{P}_a$), we can obtain $\tilde{u}$ and $\tilde{w}$ and thus reconstruct $\tilde{p}_{\underline{ij}}$ and $\tilde{p}_a$, which are needed in dipole subtraction term. We shall refer to this approach as a *standard dipole kinematics.*

One can consider also the equations (3.77), (3.78) as some kind of equations of state, and $\tilde{u}$, $\tilde{w}$, $\mathcal{P}^2$, $\mathcal{P}_a$ as "thermodynamical" parameters. We can consider for instance $\mathcal{P}_a = \mathcal{P}_a\left(\mathcal{P}^2, \tilde{u}\right)$ and $\tilde{w} = \tilde{w}\left(\mathcal{P}^2, \tilde{u}\right)$, etc. Since some of the relations of this type are very useful, we list a few of them in Appendix A.1 together with some relations between the derivatives.

3.4.1.2 Kinematics with additional invariant (gamma-kinematics)

It turns out that it is convenient to introduce an additional invariant, let us denote it $\tilde{\gamma}$. This will allow us to derive simpler formulae when we integrate the dipole splitting functions.

There are several possibilities to choose $\tilde{\gamma}$, for example it can be $\mathcal{Q}^2$ or even $\tilde{p}_{\underline{ij}} \cdot \tilde{p}_a$, however we shall use the following

$$\tilde{\gamma} = \tilde{p}_{\underline{ij}} \cdot p_a. \tag{3.86}$$

According to (3.73) we have

$$\tilde{\gamma} = \tilde{w}\mathcal{P}_a - \tilde{u}m_a^2. \tag{3.87}$$

This equation, together with the two on-shell conditions (3.77), (3.78) gives the solution for $\tilde{w}$, $\mathcal{P}^2$ and $\mathcal{P}_a$ in terms of $\tilde{u}$ and $\tilde{\gamma}$,

$$\tilde{w} = \frac{\chi + \delta}{2\tilde{\gamma} + m_{\underline{ij}}^2}, \tag{3.88}$$

$$\mathcal{P}_a = \frac{(\chi - \delta)\left(\tilde{\gamma} + \tilde{u}m_a^2\right)}{\tilde{u}\left(2\tilde{\gamma} + \tilde{u}m_a^2\right) + m_{\underline{ij}}^2}, \tag{3.89}$$

$$\mathcal{P}^2 = \frac{(\chi - \delta)^2}{\tilde{u}\left(2\tilde{\gamma} + \tilde{u}m_a^2\right) + m_{\underline{ij}}^2}, \tag{3.90}$$

where

$$\chi = \tilde{\gamma} + \tilde{u}\left(\tilde{\gamma} + m_a^2\right) + m_{\underline{ij}}^2, \tag{3.91}$$

$$\delta = \sqrt{\tilde{u}m_a^2\left(2\tilde{\gamma} + \tilde{u}m_a^2 - m_{\underline{ij}}^2\left(\tilde{u} - 2\right)\right) + \tilde{\gamma}^2\left(1 - \tilde{u}\right)^2}. \tag{3.92}$$

Let us now make a very general analysis of the bounds on $\tilde{u}$ and $\tilde{\gamma}$ variables. Suppose, we have the following bounds on the "standard" invariants

$$\mathcal{P}_-^2 \le \mathcal{P}^2 \le \mathcal{P}_+^2, \tag{3.93}$$

$$\mathcal{P}_{a\,-}\left(\mathcal{P}^2\right) \le \mathcal{P}_a \le \mathcal{P}_{a\,+}\left(\mathcal{P}^2\right). \tag{3.94}$$

We shall give explicit expressions for those bounds later in Section 3.6, however immediately we can write

$$\mathcal{P}_-^2 = \left(m_i + m_j\right)^2. \tag{3.95}$$

The rest of the limits depend on the specific "external" kinematical case, thus we do not give them here. Using (3.90), (3.89) we can convert (3.93), (3.94) to the following inequalities

$$\tilde{u}_-\left(\tilde{\gamma}\right) \le \tilde{u} \le \tilde{u}_+\left(\tilde{\gamma}\right), \tag{3.96}$$

$$\tilde{\gamma}_- \le \tilde{\gamma} \le \tilde{\gamma}_+. \tag{3.97}$$

This procedure is however more complicated than it looks. This is because in the most general case, in different regions of $[\tilde{\gamma}_-, \tilde{\gamma}_+]$ the bounds on $\tilde{u}$ are obtained from different conditions (3.93)-(3.94). This shall be discussed in details in Section 3.6.4. However, in the most interesting cases the lower bound on $\tilde{u}$ is always obtained by solving (3.90) for the lower limit on $\mathcal{P}^2$. When $\tilde{u} = \tilde{u}_-$ we shall encounter singularities, therefore let us give here the result

$$\tilde{u}_-\left(\tilde{\gamma}\right) = \frac{m_{\underline{ij}}^2\rho_-^2 + \left(\tilde{\gamma} + m_{\underline{ij}}^2\right)\left(4\tilde{\gamma}\mathcal{P}_-^2 - \upsilon_-\right) + \rho_-\vartheta_-\tilde{\gamma}\tilde{v}_{\underline{ij}}}{m_a^2\upsilon_- - 4\tilde{\gamma}\mathcal{P}_-^2\left(\tilde{\gamma} + 2m_a^2\right)}, \tag{3.98}$$

where

$$\tilde{v}_{\underline{ij}} = \sqrt{1 - \frac{m_a^2 m_{\underline{ij}}^2}{\tilde{\gamma}^2}}, \tag{3.99}$$

$$\rho_- = 2\tilde{\gamma} + m_{\underline{ij}}^2 + \mathcal{P}_-^2, \tag{3.100}$$

$$\upsilon_- = \rho_-^2 - 4m_a^2\mathcal{P}_-^2, \tag{3.101}$$

$$\vartheta_- = \sqrt{\left(\rho_- - 2\mathcal{P}_-\right)^2 - 4m_a^2\mathcal{P}_-^2}. \tag{3.102}$$

We turn attention to the variable $\tilde{v}_{\underline{ij}}$ - we shall often encounter quantities of this type (with different masse) later on. The upper bounds on $\tilde{u}$ and $\tilde{\gamma}$ will be discussed later.

The result of this section will be useful for the integration of dipole splitting functions in Section 3.7. In the following we shall refer to the above solution as the *gamma-kinematics*.

3.4.1.3 Soft and collinear limits of the dipole variables

In this section we shall investigate the behaviour of the dipole variables in singular regions. This is crucial for constructing the dipole splitting functions.

Recall that the soft limit is defined as

$$p_i^\mu = \lambda l^\mu, \quad \lambda \to 0, \tag{3.103}$$

where l is any fixed four-vector and p_i must be a final state gluon. This implies, that $m_{\underline{ij}} = m_j$ and of course $m_i = 0$. Then it can be shown using e.g. (3.79) that in the soft limit

$$\tilde{u} \to 0, \quad \mathcal{P}^2 \to m_{\underline{ij}}^2. \tag{3.104}$$

Moreover

$$\tilde{z} \to 0, \tag{3.105}$$

as immediately follows from its definition.

Next let us consider the quasi-collinear limit. The Sudakov parametrization of the final state momenta has the general form

$$p_i^\mu = z\tilde{p}_{\underline{ij}}^\mu + k_T^\mu + \alpha_1 \frac{n^\mu}{2\tilde{p}_{\underline{ij}}\cdot n}, \tag{3.106}$$

$$p_j^\mu = (1-z)\,\tilde{p}_{\underline{ij}}^\mu - k_T^\mu + \alpha_2 \frac{n^\mu}{2\tilde{p}_{\underline{ij}}\cdot n}, \tag{3.107}$$

where α_i are functions of k_T^2 and relevant masses - the details are not important here (see Section 3.2.2.2). This decomposition is slightly different than those in Section 3.2.2.2, as here we use $\tilde{p}_{\underline{ij}}$ instead of $p_{\underline{ij}}$. They are equivalent in the quasi-collinear limit defined by the rescaling (3.29). In this limit $\alpha_i = \mathcal{O}\left(\lambda^2\right)$ and we have

$$p_i^\mu = z\,\tilde{p}_{\underline{ij}}^\mu + \lambda k_T^\mu + \mathcal{O}\left(\lambda^2\right), \tag{3.108}$$

$$p_j^\mu = (1-z)\,\tilde{p}_{\underline{ij}}^\mu - \lambda k_T^\mu + \mathcal{O}\left(\lambda^2\right). \tag{3.109}$$

Hence

$$\mathcal{P}^2 = \mathcal{O}\left(\lambda^2\right) \;\to\; 0, \tag{3.110}$$

$$\mathcal{P}_a = \mathcal{O}\left(\lambda^0\right), \tag{3.111}$$

and from (3.79) we get

$$\tilde{u} = \mathcal{O}\left(\lambda^2\right) \ \rightarrow \ 0. \tag{3.112}$$

Similarly, since $\tilde{w} \to 1$, by contracting (3.108) with p_a we get

$$\tilde{z} = z + \mathcal{O}\left(\lambda\right). \tag{3.113}$$

Thus in the quasi-collinear limit $\tilde{z}$ becomes the Sudakov z.

3.4.1.4 Relation between dipole and the Sudakov variables

In order to better understand the meaning of the dipole variables $\tilde{u}$ and $\tilde{z}$, let us relate them to the Sudakov variables of Section 3.2.2.2. We shall see, that they are similar, but not equal; they reduce to the Sudakov variables in the massless limit as one could expect.

Let us start with the variable $\tilde{u}$. In general, the Sudakov parametrization of the momentum $\tilde{p}_a$ has the form

$$\tilde{p}_a^\mu = \overline{x} p_a^\mu + \overline{k}_T^\mu + \overline{\beta} \frac{\overline{n}^\mu}{2 p_a \cdot \overline{n}}, \tag{3.114}$$

where $\overline{\beta}$ is a factor depending on the pertinent mass configuration and is irrelevant in this discussion. Note, that $\overline{k}_T$, $\overline{n}$ and $\overline{\beta}$ are adorned by the bar sign in order to underline that they are in general different from those in (3.106). The parameter $\overline{x}$ is different than the one introduced in (3.27) (see also Section 3.4.2.4), although it has similar physical interpretation: it is a fraction of the original momentum p_a that enters the reduced matrix element, defined with initial $\tilde{p}_a$ momentum. Comparing this with (3.74), we see that

$$\overline{x} = 1 - \tilde{u} + (\tilde{w} - 1)\frac{\mathcal{P} \cdot \overline{n}}{p_a \cdot \overline{n}}. \tag{3.115}$$

As a special case we get

$$\overline{x} \doteq 1 - \tilde{u}, \qquad \text{for } m_a = 0, \tag{3.116}$$

as can be easily seen from (3.85). Thus, in the massless initial state case, $\tilde{u}$ is just the fraction of initial state momentum p_a. In order to get more transparent result it is convenient to use the CM $(p_a, \mathcal{Q})$ frame (see Section 3.6.1.1 for explicit formulae) with $\overline{n}^\mu = (1, 0, 0, 1)$. We obtain

$$\overline{x} = 1 - \tilde{u} + (\tilde{w} - 1)\frac{\mathcal{P}^2}{\mathcal{P}_a (1 - v)}, \tag{3.117}$$

where v is given in (3.83).

For further purposes let us investigate $\tilde{u} \to 0$ behaviour with fixed $\tilde{\gamma}$. Then, as shown in Eqs. (3.88)-(3.92) $\tilde{w}$, $\mathcal{P}^2$, $\mathcal{P}_a$ depend on $\tilde{u}$. Using the gamma-kinematics and expanding in $\tilde{u}$ we get very transparent result

$$1 - \overline{x} = \tilde{u} \underline{\tilde{v}_{ij}} + \mathcal{O}\left(\tilde{u}^2\right), \tag{3.118}$$

with $\underline{\tilde{v}_{ij}}$ defined in (3.99). Thus, as far as we consider the soft behaviour in gamma-kinematics, the Sudakov $1 - x$ is basically the same as $\tilde{u}$, however with different slope in the general massive case. In the quasi-collinear limit both variables are equal.

Let us now turn to the $\tilde{z}$ variable and its relation to z. Again, using (3.106), the definition of $\tilde{z}$ and the CM $(p_a, \mathcal{Q})$ frame, we get

$$z = \frac{2\tilde{z}\mathcal{P}^2 - (1-v)\left(m_i^2 - m_j^2 + \mathcal{P}^2\right)}{2v\left[\tilde{w}\mathcal{P}^2 - \tilde{u}\mathcal{P}_a(1-v)\right]}. \tag{3.119}$$

In the case of massless initial state, both variables are equal

$$z \doteq \tilde{z}, \qquad \text{for } m_a = 0, \tag{3.120}$$

as we could already guess from (3.113). Let us investigate the relation (3.119) close to the $\tilde{u} \to 0$ limit with $\tilde{\gamma}$ fixed. Using the gamma-kinematics and expanding in $\tilde{u}$ we get

$$z = \tilde{z}\,\frac{1}{\tilde{v}_{\underline{ij}}} - \frac{m_a^2\left(m_i^2 - m_j^2 + m_{\underline{ij}}^2\right)}{2\tilde{v}_{\underline{ij}}\left(1 + \tilde{v}_{\underline{ij}}\right)\tilde{\gamma}^2} + \mathcal{O}(\tilde{u}). \tag{3.121}$$

Notice, that due to the constant term (of the order $\mathcal{O}\left(\tilde{z}^0\right)$), the soft limit is indeed possible only within specific configuration of masses, as already stated in Section 3.4.1. For instance, the radiation of gluon from a final state massive quark corresponds to the configuration where $m_i = 0$, $m_{\underline{ij}} = m_j$. Then there is no constant term and p_i can become zero.

3.4.1.5 Dipole variable as a free parameter

In Section 3.6 we shall consider $\tilde{u}$ as a free parameter in order to properly factorize the phase space (more precisely it will be a convolution variable). Let us refer to it as u (without tilde) in this context. In order to make it possible, we have to drop the on-shell condition for $\tilde{p}_{\underline{ij}}$; we denote this off-shell vector as $\tilde{p}_{\underline{ij}}(u)$. During this procedure we have to keep some other invariants fixed.

Let us start with $\mathcal{P}^2$ and $\mathcal{P}_a$ kept constant. Then we have

$$\tilde{p}_{\underline{ij}}^{\mu}\left(u, \mathcal{P}^2, \mathcal{P}_a\right) = w\left(u, \mathcal{P}^2, \mathcal{P}_a\right)\mathcal{P}^{\mu} - u p_a^{\mu}, \tag{3.122}$$

$$\tilde{p}_a^{\mu}\left(u, \mathcal{P}^2, \mathcal{P}_a\right) = \left(w\left(u, \mathcal{P}^2, \mathcal{P}_a\right) - 1\right)\mathcal{P}^{\mu} - (u-1)\,p_a^{\mu}, \tag{3.123}$$

where

$$w\left(u, \mathcal{P}^2, \mathcal{P}_a\right) = \frac{\mathcal{P}^2 - \mathcal{P}_a(1-u-r)}{\mathcal{P}^2} \tag{3.124}$$

with

$$r = \sqrt{1 + u(u-2)v^2}. \tag{3.125}$$

We have obtained $w\left(u, \mathcal{P}^2, \mathcal{P}_a\right)$ solving the only one on-shell condition $\tilde{p}_a^2\left(u, \mathcal{P}^2, \mathcal{P}_a\right) = m_a^2$, which we assume to hold always.

On the other hand, we can keep variables $\mathcal{P}_a$ and $\tilde{\gamma}$ fixed. Then we get

$$w\left(u, \tilde{\gamma}, \mathcal{P}_a\right) = \frac{\tilde{\gamma} + u m_a^2}{\mathcal{P}_a}, \tag{3.126}$$

$$\mathcal{P}^2\left(u, \tilde{\gamma}, \mathcal{P}_a\right) = \left(\frac{\mathcal{P}_a}{\tilde{\gamma} + u m_a^2 - \mathcal{P}_a}\right)^2\left[2(1-u)\left(\mathcal{P}_a - \tilde{\gamma}\right) + u^2 m_a^2\right]. \tag{3.127}$$

We can use either method (also other are possible), depending which of the kinematic invariants we want to make the "external" ones. We shall return to this point with the explicit example later in Section 3.6.4. Let us now introduce the generic notation for these "external" invariants

$$\mathcal{X},\, \mathcal{Y} \in \left\{\mathcal{P}^2, \mathcal{P}_a, \tilde{\gamma}\right\}. \tag{3.128}$$

Since now u is a free parameter, we have to specify its bounds $u_{\min}$, $u_{\max}$. We have to choose the support in such a way that it includes the point $u = \tilde{u}$, that is for

$$u_{\min} \leq u \leq u_{\max} \tag{3.129}$$

we must have

$$\tilde{u} \in \left[u_{\min}, u_{\max}\right]. \tag{3.130}$$

It is natural to set

$$u_{\min} = \tilde{u}_-, \quad u_{\max} = \tilde{u}_+, \tag{3.131}$$

where $\tilde{u}_\pm$ are the bounds on $\tilde{u}$ discussed in Section 3.4.1.2.

3.4.2 Initial State Emitter - Final State Spectator

Now, in completely analogous way we treat the case, where the emission process occurs from the initial state parton. Due to the similarity of the procedure we give much less comments and concentrate rather on the differences to the previous case.

Consider the situation depicted in Fig. 3.6. We introduce new dipole momenta $\tilde{p}_{\underline{ai}}$ and $\tilde{p}_j$, where now the first one is a new initial state momentum that replaces p_a and p_i. The momentum $\tilde{p}_j$ replaces the spectator momentum p_j. The form of the new dipole momenta is completely the same as for the FE-IS case

$$\tilde{p}^\mu_j = \tilde{w}\mathcal{P}^\mu - \tilde{u}p^\mu_a, \tag{3.132}$$

$$\tilde{p}^\mu_{\underline{ai}} = \tilde{p}_j - \mathcal{Q}^\mu = \left(\tilde{w} - 1\right)\mathcal{P}^\mu - \left(\tilde{u} - 1\right)p^\mu_a, \tag{3.133}$$

but the form of $\tilde{u}$, $\tilde{w}$ is in general different due to different on-shell conditions (see below). Note, that again we have the explicit momentum conservation. In the next subsection we shall fix arbitrary at this stage parameters $\tilde{u}$ and $\tilde{w}$ in such a way that $\tilde{p}_{\underline{ai}}$ and $\tilde{p}_j$ fulfil

$$\tilde{p}^2_j = m^2_j, \tag{3.134}$$

$$\tilde{p}^2_{\underline{ai}} = m^2_{\underline{ai}}. \tag{3.135}$$

We shall also need $\tilde{z}$ variable, which is defined as in (3.75) without change. It is also useful to recollect the relative momentum already introduced in Section 3.2.2.1

$$p^\mu_{ai} = p^\mu_a - p^\mu_i. \tag{3.136}$$

Note the difference with $\tilde{p}_{\underline{ai}}$.

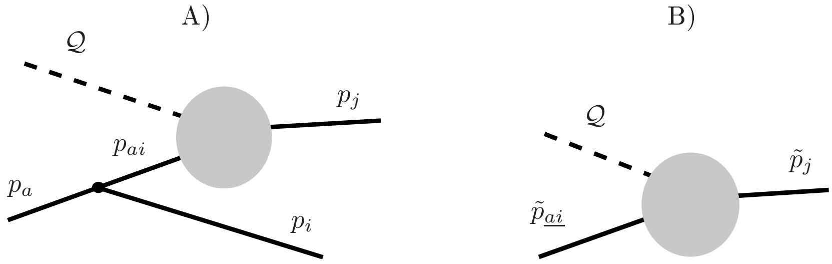


Figure 3.6: A) the momenta assignment in IE-FS case, dashed line represents a relative momentum transfer; B) reduced diagram for IE-FS case, the new tilded momenta are on-shell and fulfil momentum conservation.

3.4.2.1 Standard kinematics

Recall, that standard kinematics refers to a situation, where we express dipole variables $\tilde{u}$, $\tilde{w}$ through "standard" invariants $\mathcal{P}^2$, $\mathcal{P}_a$.

Solving the two on-shell conditions (3.134), (3.135) we obtain

$$\tilde{u} = \frac{\left(\mathcal{P}_a - \mathcal{P}^2\right)\mathcal{R}}{2v\mathcal{Q}^2\mathcal{P}_a} + \frac{\mathcal{Q}^2 + m_j^2 - m_{\underline{ai}}^2}{2\mathcal{Q}^2} \tag{3.137}$$

$$\tilde{w} = -\frac{\left(\mathcal{P}_a - m_a^2\right)\mathcal{R}}{2v\mathcal{Q}^2\mathcal{P}_a} + \frac{\mathcal{Q}^2 + m_j^2 - m_{\underline{ai}}^2}{2\mathcal{Q}^2}, \tag{3.138}$$

where now

$$\mathcal{R} = \sqrt{\left(\mathcal{Q}^2 - m_j^2 - m_{\underline{ai}}^2\right)^2 - 4m_{\underline{ai}}^2 m_j^2}, \tag{3.139}$$

while $\mathcal{Q}^2$ and v remain the same, i.e. they are given by (3.81), (3.83) respectively.

Note, that the above equations are almost identical to the corresponding equations from Section 3.4.1.1. Technically, one has to simply replace some occurrences (but not all!) of m_a by $m_{\underline{ai}}$ and $m_{\underline{ij}}$ by m_j.

3.4.2.2 Gamma-kinematics

Recall, that in gamma-kinematics we express dipole variable $\tilde{w}$ *and* invariants $\mathcal{P}^2$, $\mathcal{P}_a$ by $\tilde{u}$ and the additional invariant $\tilde{\gamma}$. It is a preparation for making $\tilde{u}$ a free parameter.

In the present case of the emission from initial state, we define this additional invariant as

$$\tilde{\gamma} = \tilde{p}_j \cdot p_a. \tag{3.140}$$

Thus, according to (3.132) we have

$$\tilde{\gamma} = \tilde{w}\mathcal{P}_a - \tilde{u}m_a^2. \tag{3.141}$$

This equation is superficially identical with the one for FE-IS case.

The expressions for $\tilde{w}$ and the "standard" invariants in terms of $\tilde{\gamma}$ and $\tilde{u}$ read

$$\tilde{w} = \frac{\chi + \delta}{2\tilde{\gamma} + m_a^2 - m_{\underline{ai}}^2 + m_j^2}, \tag{3.142}$$

$$\mathcal{P}_a = \frac{(\chi - \delta)\left(\tilde{\gamma} + \tilde{u}m_a^2\right)}{\tilde{u}\left(2\tilde{\gamma} + \tilde{u}m_a^2\right) + m_j^2}, \tag{3.143}$$

$$\mathcal{P}^2 = \frac{(\chi - \delta)^2}{\tilde{u}\left(2\tilde{\gamma} + \tilde{u}m_a^2\right) + m_j^2}, \tag{3.144}$$

where

$$\chi = \tilde{\gamma} + \tilde{u}\left(\tilde{\gamma} + m_a^2\right) + m_j^2, \tag{3.145}$$

$$\delta = \sqrt{\tilde{u}m_{\underline{ai}}^2\left(2\tilde{\gamma} + \tilde{u}m_a^2\right) + m_j^2\left(m_{\underline{ai}}^2 - m_a^2(1-\tilde{u})^2\right) + \tilde{\gamma}^2(1-\tilde{u})^2}. \tag{3.146}$$

Again they are very similar to the ones for FE-IS case.

The analysis of the bounds follows the same steps as in Section (3.4.1.2). Here we only collect the analogous formulae.

In further discussion only the lower bound on $\tilde{u}$ is relevant. It reads

$$\tilde{u}_-(\tilde{\gamma}) = \frac{m_j^2\rho_-^2 + \left(\tilde{\gamma} + m_j^2\right)\left(4\tilde{\gamma}\mathcal{P}_-^2 - \upsilon_-\right) + \rho_-\vartheta_-\tilde{\gamma}\tilde{v}_j}{m_a^2\upsilon_- - 4\tilde{\gamma}\mathcal{P}_-^2\left(\tilde{\gamma} + 2m_a^2\right)}, \tag{3.147}$$

where this time

$$\tilde{v}_j = \sqrt{1 - \frac{m_a^2 m_j^2}{\tilde{\gamma}^2}}, \tag{3.148}$$

$$\rho_- = 2\tilde{\gamma} + m_a^2 - m_{\underline{ai}}^2 + m_j^2 + \mathcal{P}_-^2, \tag{3.149}$$

$$\upsilon_- = \rho_-^2 - 4m_a^2\mathcal{P}_-^2, \tag{3.150}$$

$$\vartheta_- = \sqrt{\left(\rho_- - 2\mathcal{P}_-\right)^2 - 4m_{\underline{ai}}^2\mathcal{P}_-^2}. \tag{3.151}$$

The lower bound on $\mathcal{P}^2$ is the same as in the FE-IS case, $\mathcal{P}_-^2 = (m_i + m_j)^2$.

3.4.2.3 Dipole variable as a free parameter

In full analogy to the FE-IS case, later we shall treat the dipole variable $\tilde{u}$ as a free parameter u. Let us recall, that in order for this to work, we have to release one on-shell condition. More precisely the spectator $\tilde{p}_j$ must be off-shell. It drops into its physical mass when $u = \tilde{u}$.

All the relevant equations are analogous to the ones we have obtained for FE-IS case. Let us only list relevant solutions for single on-shell condition. For $w\left(u, \mathcal{P}^2, \mathcal{P}_a\right)$ it is the same as (3.124) with however

$$r = \sqrt{(u-1)^2 + \frac{m_{\underline{ai}}^2\mathcal{P}^2}{\mathcal{P}_a^2}}. \tag{3.152}$$

Further, $w\left(u, \tilde{\gamma}, \mathcal{P}_a\right)$ remains the same as (3.126) while

$$\mathcal{P}^2\left(u, \tilde{\gamma}, \mathcal{P}_a\right) = \left(\frac{\mathcal{P}_a^2}{\tilde{\gamma} + um_a^2 - \mathcal{P}_a}\right)^2\left[2(1-u)\left(\mathcal{P}_a - \tilde{\gamma}\right) + \left(u^2 - 1\right)m_a^2 + m_{\underline{ai}}^2\right]. \tag{3.153}$$

3.4.2.4 Soft and collinear limits

Let us now check the soft and quasi-collinear behaviour of the dipole variables in the case where the emitter comes from the initial state.

Recall, that the soft limit is reached when the momentum of the emitted gluon goes to zero. The precise definition was given e.g. in (3.103). This is of course possible only if $m_a = m_{\underline{ai}}$ and $m_i = 0$. Then, again $\tilde{u}$ and $\tilde{z}$ tend to zero

$$\tilde{u} \to 0, \quad \tilde{z} \to 0 \tag{3.154}$$

while this time

$$\mathcal{P}^2 \to m_j^2. \tag{3.155}$$

Thus, the soft behaviour in the IE-FS case is essentially the same as in FE-IS case.

Next let us consider the quasi-collinear limit. We have to use the Sudakov parametrization, but this time for the initial state momenta. Let us recall from Section 3.2.2.1 that

$$p_{ai}^\mu = x p_a^\mu - k_T^\mu + \alpha_1 \frac{n^\mu}{2p_a \cdot n}, \tag{3.156}$$

$$p_i^\mu = (1-x)\, p_a^\mu + k_T^\mu + \alpha_2 \frac{n^\mu}{2p_a \cdot n}, \tag{3.157}$$

where α_i is a function of k_T^2 and the relevant masses. Note, that the Sudakov x above is apriori completely different from $\overline{x}$ defined in Section 3.4.1.4 for the FE-IS case. The analogue of the last is however $\overline{x}$ defined as

$$\tilde{p}_{\underline{ai}}^\mu = \overline{x} p_a^\mu - \overline{k}_T + \overline{\alpha}\, \frac{\overline{n}^\mu}{2p_a \cdot \overline{n}} \tag{3.158}$$

and is interpreted as the fraction of incident momentum p_a entering the reduced matrix element. We shall come back to the above decomposition below, in Section 3.4.2.5.

The quasi-collinear limit is again defined as the rescaling (3.29) with respect to k_T defined in the decompositions (3.156), (3.157). We get in this limit

$$p_{ai}^\mu = x p_a^\mu - \lambda k_T^\mu + \mathcal{O}\left(\lambda^2\right) \tag{3.159}$$

$$p_i^\mu = (1-x)\, p_a^\mu + \lambda k_T^\mu + \mathcal{O}\left(\lambda^2\right). \tag{3.160}$$

Contracting the second equation with p_a we get

$$\tilde{z} = \mathcal{O}\left(\lambda^2\right) \;\to\; 0, \tag{3.161}$$

since

$$\mathcal{P}_a = \mathcal{O}\left(\lambda^0\right). \tag{3.162}$$

On the other hand, contracting (3.160) with p_j we obtain

$$\mathcal{P}^2 = 2\,(1-x)\,(1-\tilde{z})\,\mathcal{P}_a + \mathcal{O}\left(\lambda\right) \;\to\; 2\,(1-x)\,\mathcal{P}_a. \tag{3.163}$$

There are two remarks in order. First is that now $\tilde{z}$ tends to zero in quasi-collinear limit. Second, we note that quasi-collinear limit does not imply any particular limiting value of $\tilde{u}$. This is connected to the fact, that $\tilde{u}$ plays a role of the Sudakov variable. This connection shall be clarified in the next subsection.

3.4.2.5 Relation between dipole and the Sudakov variables

The analysis of the previous subsection leads to the conclusion, that in the IE-FS case, it is $\tilde{z}$ variable that controls the quasi-collinear limit. Thus we expect $\tilde{u}$ plays a role of the longitudinal Sudakov variable. To find the explicit relation we use (3.137), rescale the masses according to the definition of the quasi-collinear limit and use (3.162), (3.163). This leads to the expected result

$$\tilde{u} = 1 - x + \mathcal{O}(\lambda) . \tag{3.164}$$

Moreover, contracting (3.158) with $\mathcal{P}$, using (3.133), again (3.163) and rescaling all the masses we get

$$\overline{x} = x + \mathcal{O}(\lambda) . \tag{3.165}$$

Thus, in the quasi-collinear limit the two Sudakov longitudinal fractions are the same, as desired.

For completeness, let us also discuss the relation between the above variables in the soft limit, controlled by $\tilde{u} \to 0$ and $\tilde{z} \to 0$. For $\overline{x}$ we obtain analogous result to FE-IS case, namely we have in gamma-kinematics

$$1 - \overline{x} = \tilde{u}\tilde{v}_j + \mathcal{O}\left(\tilde{u}^2\right) , \tag{3.166}$$

with $\tilde{v}_j$ defined in (3.148). Next, for x variable, we obtain

$$1 - x = \frac{m_j^2}{\tilde{\gamma}\tilde{v}_j\left(1 - \tilde{v}_j\right)} \left(\frac{m_a^2 \tilde{v}_j^2}{\tilde{\gamma}\left(1 - \tilde{v}_j\right)} \tilde{u} - \tilde{z} \right) + \mathcal{O}\left(\tilde{u}\tilde{z}\right) . \tag{3.167}$$

Thus we see that $x, \overline{x} \to 1$ in the soft limit.

We note, that in order to obtain the above formulae the assumption $m_{\underline{ai}} = m_a$ has to be made. Moreover one has to drop the terms proportional to the mass of the gluon m_i.

3.5 Dipole splitting functions

Now we are ready to give the precise form of the dipole splitting matrices introduced in Section 3.3. Our functions are similar to those in [10], however they need modifications required by the massive initial state partons. In what follows, we treat FE-IS and IE-FS cases separately. The case FE-FS is completely covered in [10]. Moreover, the case of initial state $g \to gg$ splitting is also the same, thus it is not considered here.

3.5.1 Final state emitter - Initial state spectator

3.5.1.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

The assignment of the momenta is shown in Fig 3.7. Here we assume

$$m_i = 0, \quad m_j = m_{\underline{ij}} = m_{\mathbf{Q}} \equiv m \tag{3.168}$$

while we do not assume anything about m_a, since it can be either a quark (massive or massless) or a gluon.

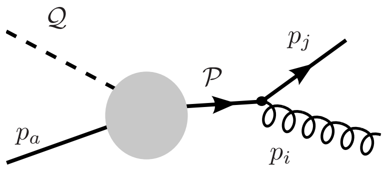


Figure 3.7: The final state splitting $\mathbf{Q} \to \mathbf{Q}g$ with the spectator p_a in the initial state. The spectator can be either massive quark or massless quark or gluon. The case with anti-quarks is formally identical.

We postulate the following dipole splitting matrix

$$\left(\hat{V}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}\right)^{ss'} = \delta^{ss'}\, 8\pi\mu_r^{2\varepsilon}\alpha_s C_F \left[\frac{2}{\tilde{u}\tilde{v}^2 + \tilde{z}} + (1-\varepsilon)\,\tilde{z} - 2 - \frac{m^2}{p_i \cdot p_j}\right], \tag{3.169}$$

where $\tilde{v}$ was defined in (3.99) and here takes the form

$$\tilde{v} \equiv \tilde{v}_{\mathbf{Q}} = \sqrt{1 - \frac{m_a^2 m^2}{\tilde{\gamma}^2}}. \tag{3.170}$$

Recall that μ_r is the mass scale that keeps the coupling constant dimensionless.

The origin of the dipole splitting function (3.169) can be understood as follows. First, recall the soft behaviour (3.20) when $p_i \to 0$. Here we are actually interested in one term from the sum in (3.20), namely in the expression

$$\frac{p_a \cdot p_j}{p_i\,(p_a + p_j)} - \frac{m^2}{2p_i \cdot p_j}. \tag{3.171}$$

We see, that the second term does match the last one in (3.169). Consider thus the first one

$$\frac{p_a \cdot p_j}{p_i\,(p_a + p_j)} = \frac{(1-\tilde{z})\,\mathcal{P}_a}{\tilde{z}\mathcal{P}_a + \frac{1}{2}\left(\mathcal{P}^2 - m^2\right)}. \tag{3.172}$$

Using the gamma-kinematics we have

$$\mathcal{P}^2 - m^2 = \frac{2\,\tilde{u}\,\tilde{v}^2\,\tilde{\gamma}^2\left(m^2\,(\tilde{u}-2) - \tilde{u}m_a^2 - 2\tilde{\gamma}\right)}{(\tilde{\gamma}+m^2)\,(m^2 + \tilde{u}\,(2\tilde{\gamma} + \tilde{u}m_a^2)) - \tilde{w}\,(2\tilde{\gamma}+m^2)\,(m^2 + \tilde{u}\,(\tilde{\gamma}+m_a^2) + \tilde{\gamma})}. \tag{3.173}$$

The leading behaviour in $\tilde{u}$ turn out to be

$$\mathcal{P}^2 - m^2 = 2\tilde{\gamma}\tilde{u}\tilde{v}^2 + \mathcal{O}\left(\tilde{u}^2\right). \tag{3.174}$$

Thus we see that $\tilde{u} \to 0$ limit indeed controls the soft behaviour. Moreover

$$\mathcal{P}_a = \tilde{\gamma} + \mathcal{O}\left(\tilde{u}\right). \tag{3.175}$$

Therefore (3.172) becomes

$$\frac{1}{\tilde{z} + \tilde{u}\tilde{v}^2} + \mathcal{O}\left(1\right) \tag{3.176}$$

in consistency with (3.169). On the other hand, when we reach quasi-collinear limit, we have also $\tilde{u} \to 0$ due to (3.112) and $\tilde{z} \to z$ according to (3.121). Thus the squared bracket in (3.169) equals exactly the splitting matrix for $\mathbf{Q} \to \mathbf{Q}g$ process (3.53).

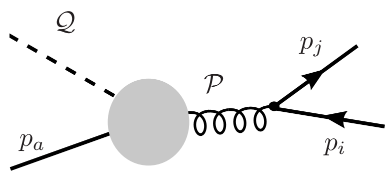


Figure 3.8: The final state splitting $g \to \mathbf{Q}\overline{\mathbf{Q}}$ with the spectator p_a in the initial state. The spectator can be either massive quark or massless quark or gluon.

There is one more comment in order. As shown in Section 3.4.1.4, our variable $\tilde{u}$ is equivalent to $1 - \tilde{x}_i$ appearing in Ref. [10] (more precisely, they are the same when $m_a = 0$). Now, if we translate the corresponding dipole splitting function of [10] to our variable $\tilde{u}$ we get (3.169), but without $\tilde{v}^2$ in the denominator of the first term. However, precisely this coefficient is necessary in order to maintain the soft limit when $m_a \neq 0$, as argued above.

3.5.1.2 $g \to \mathbf{Q}\overline{\mathbf{Q}}$ splitting

Let us now turn to the process in Fig. 3.8. The configuration of masses is the following

$$m_{\underline{ij}} = 0, \quad m_i = m_j = m_{\mathbf{Q}} \equiv m, \tag{3.177}$$

In the present case the splitting matrix is correlated with the reduced matrix element in the helicity space (see Section 3.2.2.2). Therefore this case is less trivial than the previous one. Below, we describe the procedure which leads to a suitable dipole splitting function, which can be later integrated over the one-particle subspace.

First, recall the real splitting matrix (3.55)

$$\left(\hat{P}_{g\mathbf{Q}}\right)^{\mu\nu} = T_R \left(-g^{\mu\nu} - 4\frac{k_T^\mu k_T^\nu}{\mathcal{P}^2}\right), \tag{3.178}$$

where k_T^μ is the Sudakov four-vector transverse to $\tilde{p}_{\underline{ij}}$. The strategy in choosing $\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}}}$ is to replace (3.178) by an expression leading to (3.178) in the quasi-collinear limit (here there is no soft singularity) and is suitable for analytical integration. Those requirements are satisfied by the following form

$$\left(\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right)^{\mu\nu} = 8\pi\mu_r^{2\varepsilon}\alpha_s T_R \left(-g^{\mu\nu} - 4\frac{\mathcal{C}^{\mu\nu}}{\mathcal{P}^2}\right), \tag{3.179}$$

where we shall refer to the tensor $\mathcal{C}^{\mu\nu}$ as a *correlation tensor*. By assumption, it is transverse to the collinear direction

$$\tilde{p}^\mu_{\underline{ij}}\mathcal{C}_{\mu\nu} = \tilde{p}^\nu_{\underline{ij}}\mathcal{C}_{\mu\nu} = 0. \tag{3.180}$$

In what follows, we shall find a suitable form of this tensor.

To this end, it is instructive to investigate the corresponding tensor used in [10]. Let us denote it by $\mathcal{C}^{\mu\nu}_*$. It has the form

$$\mathcal{C}^{\mu\nu}_* = \left[\tilde{z}p_i^\mu - (1-\tilde{z})\,p_j^\mu\right]\left[\tilde{z}p_i^\nu - (1-\tilde{z})\,p_j^\nu\right]. \tag{3.181}$$

It can be easily shown using on-shell condition for $\tilde{p}_{\underline{ij}}$ (see e.g. (A.1)) that

$$\tilde{p}_{\underline{ij}}^{\mu}\left(\tilde{z}p_{i\,\mu}-(1-\tilde{z})\,p_{j\,\mu}\right)=\frac{1}{2}\left(1-2\tilde{z}\right)\frac{m_a^2\tilde{u}^2}{\tilde{w}}. \tag{3.182}$$

Therefore, if we assume that initial state is massless, $\mathcal{C}_*^{\mu\nu}$ is the correct tensor. However it is no longer the case when $m_a \neq 0$. Nevertheless it gives us a hint what modification should be used. It can be easily checked, that the following tensor possesses the required property

$$\mathcal{C}^{\mu\nu}=\left[\tilde{z}p_i^{\mu}-(1-\tilde{z})\,p_j^{\mu}-\frac{\tilde{u}m_a^2}{2\tilde{w}\mathcal{P}_a}\left(p_i^{\mu}-p_j^{\mu}\right)\right]\left[\tilde{z}p_i^{\nu}-(1-\tilde{z})\,p_j^{\nu}-\frac{\tilde{u}m_a^2}{2\tilde{w}\mathcal{P}_a}\left(p_i^{\nu}-p_j^{\nu}\right)\right]. \tag{3.183}$$

It obviously reduces to $\mathcal{C}_*^{\mu\nu}$ in the case of massless initial state. Note, the tensor (3.183) is constructed as a dyadic formed from the vector

$$\mathcal{V}^{\mu}=\tilde{z}p_i^{\mu}-(1-\tilde{z})\,p_j^{\mu}-\frac{\tilde{u}m_a^2}{2\tilde{w}\mathcal{P}_a}\left(p_i^{\mu}-p_j^{\mu}\right) \tag{3.184}$$

which is orthogonal to $\tilde{p}_{\underline{ij}}$

$$\tilde{p}_{\underline{ij}}\cdot\mathcal{V}=0. \tag{3.185}$$

Now, what remains, is to check, that (3.179) with (3.183) indeed reproduce the correct quasi-collinear behaviour. Using the Sudakov decomposition for p_i, p_j and the results of Section 3.4.1.4 we find that

$$\mathcal{C}^{\mu\nu}=\left[\tilde{p}_{\underline{ij}}^{\mu}\left(2z-1\right)+k_T^{\mu}\right]\left[\tilde{p}_{\underline{ij}}^{\nu}\left(2z-1\right)+k_T^{\nu}\right]+\mathcal{O}\left(\left|k_T\right|^3\right). \tag{3.186}$$

Since terms with $\tilde{p}_{\underline{ij}}$ do not give contribution, $\mathcal{C}^{\mu\nu}$ is of the order $\mathcal{O}\left(\left|k_T\right|^2\right)$. Of the same order is the denominator in (3.179), thus we indeed obtain the splitting matrix (3.178).

3.5.1.3 $g \to gg$ splitting

Although this subprocess involves the massless partons only (Fig. 3.9)

$$m_i=m_j=m_{\underline{ij}}=0, \tag{3.187}$$

it is not legitimate to take the massless dipole splitting function as in [10] or [9]. The reason originates in non-zero mass of the spectator $m_a \neq 0$. This case is paradoxically the most complicated one and in fact accommodates both of the cases described in Sections 3.5.1.1, 3.5.1.2.

Let us first recall the true splitting matrix (3.56)

$$\left(\hat{P}_{gg}\right)^{\mu\nu}=2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right)+2\left(1-\varepsilon\right)\frac{k_T^{\mu}k_T^{\nu}}{\mathcal{P}^2}\right], \tag{3.188}$$

where z is the Sudakov parameter. Recall, that this gives the behaviour of matrix element squared in the quasi-collinear limit. However, there are also the soft singularities when $z = 0,\,1$, what corresponds to vanishing four-momentum of the gluon i or j. The construction of dipole splitting function for these singularities is analogous to the one

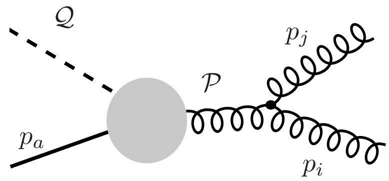


Figure 3.9: The final state splitting $g \to gg$ with the spectator p_a in the initial state. The spectator can be either massive quark or massless quark or gluon.

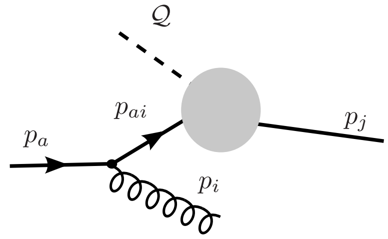


Figure 3.10: The initial state splitting $\mathbf{Q} \to \mathbf{Q}g$ with the spectator p_j in the final state. The spectator can be either massive quark or massless quark or gluon.

described in Section 3.5.1.1. Namely, in terms of our dipole variables we replace the denominators in 3.188 as follows

$$\frac{z}{1-z} \to \frac{\tilde{z}}{1-\tilde{z}+\tilde{u}} = \frac{1}{1-\tilde{z}+\tilde{u}} - 1 + \mathcal{O}(\tilde{u})\,, \tag{3.189}$$

$$\frac{1-z}{z} \to \frac{1-\tilde{z}}{\tilde{z}+\tilde{u}} = \frac{1}{\tilde{z}+\tilde{u}} - 1 + \mathcal{O}(\tilde{u})\,. \tag{3.190}$$

Notice, there is no $\tilde{v}_{\underline{ij}}^2$ factor multiplying $\tilde{u}$, since now simply $\tilde{v}_{\underline{ij}} = 1$.

Consider now the transverse part in (3.188). It is treated in exactly the same way as in Section 3.5.1.2 by means of the correlation tensor. Therefore, the dipole splitting function takes the following form

$$\left(\hat{V}^{\text{FE-IS}}_{g\to gg,\,a}\right)^{\mu\nu} = 16\pi\mu_r^{2\varepsilon}\alpha_s C_A \left[-g^{\mu\nu}\left(\frac{1}{1-\tilde{z}+\tilde{u}} + \frac{1}{\tilde{z}+\tilde{u}} - 2\right) + 2\,(1-\varepsilon)\,\frac{\mathcal{C}^{\mu\nu}}{\mathcal{P}^2}\right], \tag{3.191}$$

where $\mathcal{C}^{\mu\nu}$ is given in (3.183).

3.5.2 Initial State Emitter - Final State Spectator

3.5.2.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

Let us start by looking at the configuration of masses in the considered case (Fig. 3.10)

$$m_a = m_{ai} = m_{\mathbf{Q}} \equiv m, \quad m_i = 0. \tag{3.192}$$

The dipole splitting matrix can be constructed in similar manner as in FE-IS case in Section 3.5.1.1. Let us first analyse the soft behaviour, which our dipole splitting

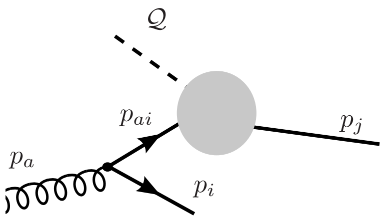


Figure 3.11: The initial state splitting $g \to \mathbf{Q}\overline{\mathbf{Q}}$ with the spectator p_j in the final state. The spectator can be either massive quark or massless quark or gluon.

function should possess. Now, we are interested in the following term from the sum in (3.20)

$$\frac{p_a \cdot p_j}{p_i\,(p_a + p_j)} - \frac{m^2}{2p_i \cdot p_a}. \tag{3.193}$$

The first term is exactly the same as in FE-IS case (3.171), while in the second term there is a different scalar propagator. In pertinent kinematics, it can be shown in close analogy to Section 3.5.1.1, that the first term becomes

$$\frac{p_a \cdot p_j}{p_i\,(p_a + p_j)} = \frac{1}{\tilde{z} + \tilde{u}\tilde{v}_j^2} + \mathcal{O}\,(1)\,. \tag{3.194}$$

Next, we have to look at the quasi-collinear behaviour, in particular at the splitting matrix (3.38)

$$\left(\hat{P}_{\mathbf{QQ}}\right)^{ss'} = \delta^{ss'}\,C_F \left(\frac{1+x^2}{1-x} - \varepsilon\,(1-x) + \frac{2xm^2}{p_{ai}^2 - m^2}\right). \tag{3.195}$$

Recalling that $x = 1 - \tilde{u} + \mathcal{O}\left(\lambda^2\right)$ and $\tilde{z} = \mathcal{O}\left(\lambda^2\right)$, where $\lambda \to 0$ controls the quasi-collinear limit, we see that the following dipole splitting matrix has the required behaviour

$$\left(\hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j}\right)^{ss'} = \delta^{ss'}\,8\pi\mu_r^{2\varepsilon}\alpha_s C_F \left[\frac{2}{\tilde{u}\tilde{v}_j^2 + \tilde{z}} + (1-\varepsilon)\,\tilde{u} - 2 - \frac{(1-\tilde{u})\,m^2}{p_i \cdot p_a}\right]. \tag{3.196}$$

3.5.2.2 $g \to \mathbf{Q}\overline{\mathbf{Q}}$ splitting

The corresponding mass configuration reads

$$m_a = 0, \quad m_{ai} = m_i = m_{\mathbf{Q}} \equiv m. \tag{3.197}$$

For convenience, let us recall the splitting matrix obtained in Section 3.2.2.1 in (3.42)

$$\left(\hat{P}_{g\mathbf{Q}}\right)_{ss'} = \delta_{ss'}\,T_R \left[1 - \frac{2}{1-\varepsilon}\left(x\,(1-x) + \frac{xm^2}{p_{ai}^2 - m^2}\right)\right]. \tag{3.198}$$

Since in this case there are no soft singularities, we can easily find the dipole splitting matrix. Here it is just

$$\left(\hat{V}^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j}\right)_{ss'} = 8\pi\mu_{\mathrm{r}}^{2\varepsilon}\alpha_s\,\delta_{ss'} T_R \left[1 - \frac{2}{1-\varepsilon}\left(\tilde{u}\,(1-\tilde{u}) + \frac{(1-\tilde{u})\,m^2}{p_{ai}^2 - m^2}\right)\right]. \tag{3.199}$$

It coincides with (3.198) (up to the factors) in the quasi-collinear limit (due to (3.164)).

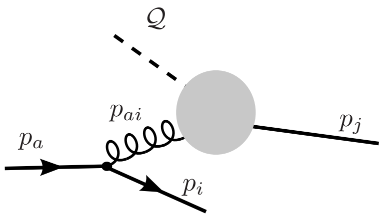


Figure 3.12: The initial state splitting $\mathbf{Q} \to g\mathbf{Q}$ with the spectator p_j in the final state. The spectator can be either massive quark or massless quark or gluon.

3.5.2.3 $\mathbf{Q} \to g\mathbf{Q}$ and $\overline{\mathbf{Q}} \to g\overline{\mathbf{Q}}$ splittings

In this case the configuration of masses is the following (Fig. 3.12)

$$m_a = m_i = m_{\mathbf{Q}} \equiv m, \quad m_{\underline{ai}} = 0. \tag{3.200}$$

Again we face the correlations of the dipole splitting matrix and reduced matrix element. Using the experience gained in the FE-IS case, we can deduce the following form

$$\left(\hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\, j}\right)^{\mu\nu} = 8\pi\mu_{\mathrm{r}}^{2\varepsilon}\alpha_s C_F\left(1-\varepsilon\right)\left[-g^{\mu\nu}\left(1-\tilde{u}\right) - \frac{4}{1-\tilde{u}}\frac{\mathcal{C}^{\mu\nu}}{p_{ai}^2}\right], \tag{3.201}$$

where the correlation tensor is again the dyadic

$$\mathcal{C}^{\mu\nu} = \mathcal{V}^{\mu}\mathcal{V}^{\nu}, \tag{3.202}$$

formed from the following vectors (they are different than in the FE-IS case)

$$\mathcal{V}^{\mu} = \left(1-\tilde{z}\right)p_i^{\mu} - \tilde{z}p_j^{\mu} - \frac{\left(\tilde{w}-1\right)\left[m^2 - m_j^2 + \mathcal{P}^2\left(1-2\tilde{z}\right)\right]}{2\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\mathcal{P}^{\mu}. \tag{3.203}$$

For later convenience let us note that

$$\tilde{p}_{\underline{ai}}\cdot\mathcal{P} = \mathcal{P}^2\left(\tilde{w}-1\right) - \mathcal{P}_a\left(\tilde{u}-1\right). \tag{3.204}$$

The vectors $\mathcal{V}$ are constructed in such a way that

$$\tilde{p}_{\underline{ai}}\cdot\mathcal{V} = 0 \tag{3.205}$$

as can be easily checked using on-shell conditions. When $m_a = 0$ it reduces to

$$\mathcal{V}_*^{\mu} = \left(1-\tilde{z}\right)p_i^{\mu} - \tilde{z}p_j^{\mu}, \tag{3.206}$$

which is the same as used in [10].

Let us now prove, that (3.201) indeed possesses correct quasi-collinear behaviour (soft limit does not exist here), i.e. that it reduces to

$$P_{\mathbf{QQ}}^{\mu\nu} = C_F\left(1-\varepsilon\right)\left(-xg^{\mu\nu} - \frac{4}{x}\frac{k_T^{\mu}k_T^{\nu}}{p_{ai}^2}\right). \tag{3.207}$$

First, recall from Section 3.4.2.4, that then $\tilde{z} = \mathcal{O}\left(\lambda^2\right)$ and $\tilde{u} \to 1-x$. Moreover it turns out that

$$\tilde{w} - 1 = \lambda^2 \frac{m^2\left(\mathcal{P}^2 - 2\mathcal{P}_a\right)}{4\mathcal{P}_a^2} + \mathcal{O}\left(\lambda^4\right). \tag{3.208}$$

Thus, the term proportional to $\mathcal{P}^\mu$ in (3.203) vanishes. The same is true for the term $\tilde{z}p_j$ since p_j is the fixed spectator momentum. Now, using the Sudakov decomposition (3.157) we see that

$$\mathcal{V}^\mu = (1-x)\, p_a^\mu + \lambda k_T^\mu + \mathcal{O}\left(\lambda^2\right). \tag{3.209}$$

However

$$(1-x)\, p_a^\mu = \tilde{p}_{\underline{ai}}^\mu + \mathcal{O}\left(\lambda^2\right). \tag{3.210}$$

The momentum $\tilde{p}_{\underline{ai}}$ corresponds to a massless gluon, thus only the term proportional to k_T survives due to the Ward identity (the dipole splitting matrix is contracted with the reduced matrix element). Finally, the propagator also behaves as $\mathcal{O}\left(\lambda^2\right)$

$$p_{ai}^2 = \left(p_a - p_i\right)^2 = \lambda^2\, 2\left(m^2 - \tilde{z}\mathcal{P}_a\right). \tag{3.211}$$

Putting all the pieces together we indeed recover (3.207). Another check can be made after the integration of dipole splitting matrix, as we shall see in Section 3.7.3.3.

3.6 Phase space factorization

As explained in Section 1.2, the dipole subtraction method requires separation of the $(n+1)$-particle PS into n-particle PS and a subspace, which after integration leads to IR singularities. In the following section we present complete treatment of the two cases: FE-IS and IE-FS in the most general situation of non-zero quark masses, including initial state. Our treatment is close to the one in [23, 10]. The third possible configuration with final state emitter and final state spectator, is developed in [10] and need not be modified, as it does not involve the initial state at all.

Before we start, let us make some general comments. First, since in the new factorized phase space the initial state is modified as described in Section 3.4 (with the help of a continuous parameter), actually we obtain a whole family of phase spaces. Therefore it is natural to expect that the form of factorization will be a convolution rather than a simple product. Second, when the initial state is massive, one parameter is not enough to fix the new (Lorentz transformed) frame. This leads to some complications which we shall also discuss below.

3.6.1 Preliminaries

In order to derive the phase space factorization formula, let us start with the $(n+1)$-particle PS and write it as follows

$$d\Phi_{n+1}\left(q, p_a; p_1, \ldots, p_{n+1}\right) = d\Phi_2\left(\mathcal{Q}, p_a; p_i, p_j\right) \prod_{k\in\mathbb{M}} d\Gamma_k, \tag{3.212}$$

where

$$\mathcal{Q}^\mu = q^\mu - \sum_{k\in\mathbb{M}} p_k^\mu \tag{3.213}$$

and the set of indices is defined as

$$\mathbb{M} = \{1, \dots, n+1\} \setminus \{i, j\} . \tag{3.214}$$

Our notation for phase space ingredients was introduced in Section 1.2.

So far, in (3.212) we have only disentangled two-particle PS for two chosen final state partons i and j. Further development is addressed to the next subsections.

3.6.1.1 Two-particle phase space

First, let us calculate two-particle phase space disentangled above. The result is well known and represents usual two-body phase space slightly adjusted to our notation.

Using center of mass frame of momenta $\mathcal{Q}$ and p_a (denote it as $\mathrm{CM}(\mathcal{Q}, p_a)$) and notation introduced in Section 3.4 we have

$$\begin{aligned} d\Phi_2 (\mathcal{Q}, p_a; p_i, p_j) \\ &= \frac{d^D p_i \, \delta_+ \left(p_i^2 - m_i^2\right)}{(2\pi)^{D-1}} \frac{d^D p_j \, \delta_+ \left(p_j^2 - m_j^2\right)}{(2\pi)^{D-1}} (2\pi)^D \delta^D (\mathcal{Q} + p_a - p_i - p_j) \\ &\doteq \delta \left(\mathcal{P}^2 - 2\sqrt{\mathcal{P}^2} E_i + m_i^2 - m_j^2\right) \frac{|\vec{p}_i|}{E_i} (|\vec{p}_i| \sin\theta)^{D-4} \frac{d\vec{p}_i^{\,2} d(\cos\theta) \, d\Omega_{D-2}}{4 (2\pi)^{D-2}} . \end{aligned} \tag{3.215}$$

When deriving this result we used (A.48) from the Appendix A.3. Utilizing delta function we get

$$d\Phi_2 (\mathcal{Q}, p_a; p_i, p_j) = \frac{d\Omega_{D-2}}{4 (2\pi)^{D-2}} \overline{p} \, (\hat{p} \sin\theta)^{D-4} \, d(\cos\theta) , \tag{3.216}$$

where

$$\overline{p} = \frac{\hat{p}}{\mathcal{P}^2} = \frac{1}{2} \sqrt{1 - 2\left(\overline{m}_i^2 + \overline{m}_j^2\right) + \left(\overline{m}_i^2 - \overline{m}_j^2\right)^2} \tag{3.217}$$

with

$$\overline{m}_q^2 = \frac{m_q^2}{\mathcal{P}^2}, \quad q = i, j \tag{3.218}$$

The variable $\hat{p}$ is the modulus of the outgoing particles three-momentum in $\mathrm{CM}(\mathcal{Q}, p_a)$ and $d\Omega_{D-2}$ is the solid angle element on hyperplane perpendicular to z axis (Fig. 3.13). In general we can parametrize it using

$$d\Omega_N = \prod_{k=1}^{N-1} \sin^{N-1-k} \theta_k \, d\theta_k . \tag{3.219}$$

We require at this stage

$$\mathcal{P}^2 \geq \mathcal{P}_-^2 = (m_i + m_j)^2 . \tag{3.220}$$

For completeness and future use we give also the energies and the momenta of the particles in $\mathrm{CM}(\mathcal{Q}, p_a)$

$$E_a = \frac{\mathcal{P}_a}{\sqrt{\mathcal{P}^2}}, \tag{3.221}$$

$$|\vec{p}_a| = \frac{\sqrt{\mathcal{P}_a^2 - m_a^2 \mathcal{P}^2}}{\sqrt{\mathcal{P}^2}}, \tag{3.222}$$

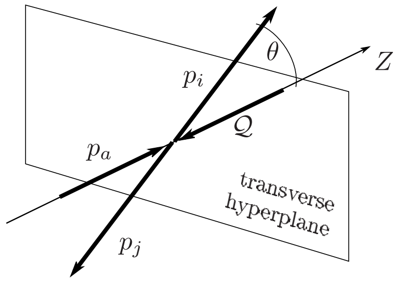


Figure 3.13: Center of mass system of momenta $\mathcal{Q}$ and p_a used to calculate two-particle phase space.

$$E_{\mathcal{Q}} = \sqrt{\mathcal{P}^2} - E_a = \frac{\mathcal{P}^2 - \mathcal{P}_a}{\sqrt{\mathcal{P}^2}}, \tag{3.223}$$

$$E_i = \frac{m_i^2 - m_j^2 + \mathcal{P}^2}{2\sqrt{\mathcal{P}^2}}, \tag{3.224}$$

$$E_j = \sqrt{\mathcal{P}^2} - E_i = \frac{m_j^2 - m_i^2 + \mathcal{P}^2}{2\sqrt{\mathcal{P}^2}}. \tag{3.225}$$

3.6.1.2 Two-particle phase space in dipole variables

Since the dipole splitting functions are expressed among others in special variables $\tilde{u}$ and $\tilde{z}$, we start by trading $\cos\theta$ for $\tilde{z}$ in (3.216). This is easily done using the definition of $\tilde{z}$ (3.75) in CM $(\mathcal{Q}, p_a)$. We get

$$d\Phi_2\left(\mathcal{Q}, p_a; p_i, p_j\right) \equiv d\Phi_2\left(\mathcal{P}^2, \mathcal{P}_a\right) = \frac{d\Omega_{D-2}}{4\left(2\pi\right)^{D-2}} v^{3-D} \left(\mathcal{P}^2\right)^{\frac{D}{2}-2} \left[\left(\tilde{z}_+ - \tilde{z}\right)\left(\tilde{z} - \tilde{z}_-\right)\right]^{\frac{D}{2}-2} d\tilde{z}, \tag{3.226}$$

where

$$\tilde{z}_- \leq \tilde{z} \leq \tilde{z}_+ \tag{3.227}$$

with

$$\tilde{z}_\pm = \frac{1}{2}\left(1 + \overline{m}_i^2 - \overline{m}_j^2 \pm 2v\overline{p}\right). \tag{3.228}$$

and $v = \left|\vec{p}_a\right| / E_a$ defined in (3.83).

Now, using the results of Section 3.4.1.2, we can express the invariants $\left\{\mathcal{P}^2, \mathcal{P}_a\right\}$ in terms of the dipole variable $\tilde{u}$ and one of the other invariants $\tilde{\gamma}$, $\mathcal{P}_a$ or $\mathcal{P}^2$. That is one can use the relations of the type

$$\mathcal{Y} = \mathcal{Y}\left(\tilde{u}, \mathcal{X}\right), \tag{3.229}$$

where $\mathcal{X}, \mathcal{Y} \in \left\{\mathcal{P}^2, \mathcal{P}_a, \tilde{\gamma}\right\}$ and $\mathcal{X} \neq \mathcal{Y}$ of course. Let thus write generically

$$d\Phi_2 \equiv d\Phi_2\left(\mathcal{X}, \tilde{u}\right). \tag{3.230}$$

At this stage it is not necessary to decide what $\mathcal{X}$ and $\mathcal{Y}$ are, neither to insert their precise forms, which in fact depend on the kinematical configuration (FE-IS or IE-FS case). Those distinct cases are treated in the next subsections.

3.6.2 Final state emitter - initial state spectator

Now we are ready to derive the factorized form of two-particle phase space. We start with FE-IS case, but IE-FS goes in the same fashion.

There are several possible approaches to phase space factorization. One of the simplest possibilities is to "go backward" from (3.226) using some elementary identities. This is the path we choose in this dissertation.

Let us consider the phase space (3.230) and insert the following delta function

$$d\Phi_2\left(\mathcal{X},\tilde{u}\right) = \int du\,\delta\left(u-\tilde{u}\right)\,d\Phi_2\left(\mathcal{X},u\right). \tag{3.231}$$

Next, let us convert $\delta\left(u-\tilde{u}\right)$ into the on-shell delta function for $\tilde{p}^{\mu}_{\underline{ij}}$ defined in (3.73). According to Section 3.4.1.5, we mean by this, that $\tilde{u}$ is replaced by u and $\tilde{w} \equiv w$ is obtained as a function of u and two "external" invariants $\mathcal{X}$, $\mathcal{Y} \in \left\{\mathcal{P}^2, \mathcal{P}_a, \tilde{\gamma}\right\}$, see Section 3.4.1.5. Thus, we use the following identity

$$\frac{\delta\left(u-\tilde{u}\right)}{\mathcal{J}\left(u,\mathcal{X}\right)} = \delta_+\left(\tilde{p}^2_{\underline{ij}}\left(u,\mathcal{X},\mathcal{Y}\right) - m^2_{\underline{ij}}\right), \tag{3.232}$$

where

$$\mathcal{J}\left(u,\mathcal{X}\right) = \left|\frac{\partial\tilde{p}^2_{\underline{ij}}\left(u,\mathcal{X},\mathcal{Y}\right)}{\partial u}\right| \equiv \left|\frac{\partial\tilde{p}^2_{\underline{ij}}}{\partial u}\right|_{\mathcal{X},\mathcal{Y}}. \tag{3.233}$$

Note, that we calculate the above derivative with the invariants $\mathcal{X}$, $\mathcal{Y}$ fixed (as explicitly denoted using "thermodynamical" notation) and next used $\mathcal{Y} = \mathcal{Y}\left(\tilde{u},\mathcal{X}\right) = \mathcal{Y}\left(u,\mathcal{X}\right)$ due to the delta function.

Finally, we insert D-dimensional delta function for momentum conservation with the appropriate integration

$$d\Phi_2\left(\mathcal{X},\tilde{u}\right) = \int du \int d^D\tilde{p}_{\underline{ij}}\;\delta_+\left(\tilde{p}^2_{\underline{ij}} - m^2_{\underline{ij}}\right)\,d\Phi_2\left(\mathcal{X},u\right)$$
$$\mathcal{J}\left(u,\mathcal{X}\right)\,\delta^D\left(\mathcal{Q} + \tilde{p}_a\left(u,\mathcal{X},\mathcal{Y}\right) - \tilde{p}_{\underline{ij}}\right). \tag{3.234}$$

Recall, that

$$\tilde{p}^{\mu}_a\left(u,\mathcal{X},\mathcal{Y}\right) = w\left(u,\mathcal{X},\mathcal{Y}\right)\mathcal{P}^{\mu}_a - u\,p^{\mu}_a, \tag{3.235}$$

where $w\left(u,\mathcal{X},\mathcal{Y}\right)$ was obtained using single on-shell condition $\tilde{p}^2_a = m^2_a$.

Equation (3.234) is just an identity, however it has already the required factorized form. To see this more clearly, let us introduce the following measure

$$d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j,\,a}\left(u,\mathcal{X}\right) = \frac{1}{4\left(2\pi\right)^{D-1}}\,v^{3-D}\,\left(\mathcal{P}^2\right)^{\frac{D}{2}-2}\mathcal{J}\,\left[\left(\tilde{z}_+ - \tilde{z}\right)\left(\tilde{z} - \tilde{z}_-\right)\right]^{\frac{D}{2}-2}\,d\Omega_{D-2}d\tilde{z}, \tag{3.236}$$

where we implicitly understand, that all the quantities above should be expressed through u and $\mathcal{X}$. Using this notation we can cast (3.234) into

$$d\Phi_2\left(\mathcal{Q},p_a;p_i,p_j\right) = \int du\,\left(2\pi\right)^D\delta^D\left(\mathcal{Q} + \tilde{p}_a\left(u,\mathcal{X},\mathcal{Y}\right) - \tilde{p}_{ij}\right)\,d\Gamma\left(\tilde{p}_{\underline{ij}}\right)\,d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j}\left(u,\mathcal{X}\right)$$
$$= \int du\,d\Phi_1\left(\mathcal{Q},\tilde{p}_a\left(u,\mathcal{X},\mathcal{Y}\right);\tilde{p}_{\underline{ij}}\right)\,d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j}\left(u,\mathcal{X}\right). \tag{3.237}$$

Finally, using (3.212) we can write the factorization formula for $(n+1)$-particle phase space

$$d\Phi_{n+1}\left(q, p_a; \{p_m\}_{i=1}^{n+1}\right) = \int du\, d\Phi_n\left(\mathcal{Q}, \tilde{p}_a\left(u, \mathcal{X}\right); \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right) d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j}\left(u, \mathcal{X}\right), \tag{3.238}$$

where the set $\mathbb{X}_{\text{FE-IS}}$ is defined as

$$\mathbb{X}_{\text{FE-IS}} = \{1, \ldots, n+1\} \setminus \{i, j\} \cup \{\tilde{\underline{ij}}\}. \tag{3.239}$$

Recall that the tilde above $\underline{ij}$ means that corresponding momentum should be adorned by a tilde, i.e. $p_{\tilde{\underline{ij}}} \equiv \tilde{p}_{\underline{ij}}$.

In what follows, we shall refer to the phase space $d\Phi_n\left(q, \tilde{p}_a\left(u, \mathcal{X}, \mathcal{Y}\right); \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right)$, as a *skewed* phase space.

3.6.3 Initial state emitter - final state spectator

The treatment of the IE-FS case is completely analogous to FE-IS. This time we have an initial state (an emitter) given by the vector $\tilde{p}_{\underline{ai}}$ (3.133) and a spectator given by $\tilde{p}_j$ (3.132). Therefore the phase space factorization formula takes the form

$$d\Phi_{n+1}\left(q, p_a; \{p_m\}_{i=1}^{n+1}\right) = \int du\, d\Phi_n\left(\mathcal{Q}, \tilde{p}_{\underline{ai}}\left(u, \mathcal{X}, \mathcal{Y}\right); \{p_m\}_{m\in\mathbb{X}_{\text{IE-FS}}}\right) d\phi^{\text{IE-FS}}_{a\to\underline{ai}\,i,\,j}\left(u, \mathcal{X}\right), \tag{3.240}$$

where

$$\mathbb{X}_{\text{IE-FS}} = \{1, \ldots, n+1\} \setminus \{i, j\} \cup \{\tilde{\underline{j}}\}. \tag{3.241}$$

The measure is defined again as in (3.236) with the jacobian

$$\mathcal{J}\left(u, \mathcal{X}\right) = \left|\frac{\partial \tilde{p}_j^2\left(u, \mathcal{X}, \mathcal{Y}\right)}{\partial u}\right| \equiv \left|\frac{\partial \tilde{p}_j^2}{\partial u}\right|_{\mathcal{X},\mathcal{Y}}. \tag{3.242}$$

All the relevant kinematics is worked out in Section 3.4.2.

3.6.4 Factorization of three-particle phase space

In this subsection we present explicit formulae for the factorization of three-particle phase space as an illustration of our approach. We shall use them later in constructing a Monte Carlo program for numerical calculations.

We demonstrate the construction for FE-IS case (for IE-FS it is analogous). The phase space factorization formula reads

$$\begin{aligned} d\Phi_3\left(q, p_a; p_i, p_j, k\right) &= \int du\, d\Phi_2\left(q, \tilde{p}_a\left(u, \mathcal{X}, \mathcal{Y}\right); k, \tilde{p}_{\underline{ij}}\right) d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j,\,a}\left(u, \mathcal{X}\right) \\ &= \int du\, \left(2\pi\right)^D \delta^D\left(q + \tilde{p}_a\left(u, \mathcal{X}, \mathcal{Y}\right) - \tilde{p}_{\underline{ij}} - k\right) d\Gamma\left(\tilde{p}_{\underline{ij}}\right) d\Gamma\left(k\right) d\phi^{\text{FE-IS}}_{\underline{ij}\to i\,j,\,a}\left(u, \mathcal{X}\right). \end{aligned} \tag{3.243}$$

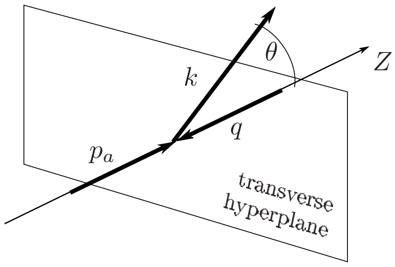


Figure 3.14: Center of mass system of momenta q and p_a used to calculate factorized three-particle phase space.

We assume, that the initial state momenta q and p_a are fixed. Note however, that nevertheless the CM energy of the $\tilde{p}_{\underline{ij}}$ and k system is not fixed for fixed u. To see this explicitly, let us choose for example $\mathcal{X} = \mathcal{P}^2$, $\mathcal{Y} = \mathcal{P}_a$, we have then for the CM energy

$$\tilde{s} = \left(q + \tilde{p}_a\left(u, \mathcal{P}^2, \mathcal{P}_a\right)\right)^2 = s\,(w-u) + (w-1)\left(\mathcal{P}^2 - m_k^2 - 2\mathcal{P}_a\right) + u\left(q^2 + m_a^2\right), \tag{3.244}$$

where $w = w\left(u, \mathcal{P}_a, \mathcal{P}^2\right)$. Above

$$s = (q + p_a)^2 \tag{3.245}$$

and we used the following relations

$$q \cdot \mathcal{P} = \frac{s - m_k^2 + \mathcal{P}^2 - 2\mathcal{P}_a}{2}, \tag{3.246}$$

$$q \cdot p_a = \frac{s - q^2 - m_a^2}{2}, \tag{3.247}$$

which originate in momentum conservation in the form

$$q^\mu + p_a^\mu = k^\mu + \mathcal{P}^\mu. \tag{3.248}$$

Further we have the on-shell condition $\tilde{p}_{\underline{ij}}^2 = m_{\underline{ij}}^2$ in $d\Phi_2\left(q, \tilde{p}_a\left(u, \mathcal{P}^2, \mathcal{P}_a\right); k, \tilde{p}_{\underline{ij}}\right)$, which allows to express for instance $\mathcal{P}_a = \mathcal{P}_a\left(u, \mathcal{P}^2\right)$. Thus, in order to have fixed the energy of the $\tilde{p}_{\underline{ij}}$ and k system, we need u *and* $\mathcal{P}^2$ to be fixed in (3.244). In general they could be u and any other of the invariants $\left\{\mathcal{P}^2, \mathcal{P}_a, \tilde{\gamma}\right\}$.

We shall now be concentrated on the skewed phase space. Let us derive explicit expressions for the choice $\mathcal{X} = \mathcal{P}^2$, $\mathcal{Y} = \mathcal{P}_a$. Due to the comment made above it is reasonable to choose CM$\,(q, p_a)$ as a reference frame (Fig. 3.14). In order to express $d\Phi_2\left(q, \tilde{p}_a; k, \tilde{p}_{ij}\right)$ as a function of the invariants $\mathcal{P}^2$ and $\mathcal{P}_a$, we represent the kinematic parameters $\left|\vec{k}\right|$ and $\cos\theta$ accordingly (we can use some of the results from subsection 3.6.1.1)

$$\left|\vec{k}\right| = \frac{\sqrt{s\left[s - 2\left(m_k^2 + \mathcal{P}^2\right)\right] + \left(m_k^2 - \mathcal{P}^2\right)^2}}{2\sqrt{s}} \equiv \hat{k}, \tag{3.249}$$

$$\cos\theta = \frac{\left(m_a^2 - q^2\right)\left(m_k^2 - \mathcal{P}^2\right) - s\left(s + \mathcal{P}^2 - m_k^2 + m_a^2 - 4\mathcal{P}_a - q^2\right)}{4s\,\hat{p}_a\hat{k}}, \tag{3.250}$$

where

$$\hat{p}_a = \frac{\sqrt{s\left[s - 2\left(m_a^2 + q^2\right)\right] + \left(m_a^2 - q^2\right)^2}}{2\sqrt{s}}. \tag{3.251}$$

Calculating the jacobian

$$\left| \frac{D\left(\left|\vec{p}_i^2\right|, \cos\theta\right)}{D\left(\mathcal{P}^2, \mathcal{P}_a\right)} \right| = \frac{s + m_k^2 - \mathcal{P}^2}{2s\,\hat{p}_a \hat{k}} \tag{3.252}$$

we get

$$d\Phi_2\left(q, \tilde{p}_a\left(u, \mathcal{P}^2, \mathcal{P}_a\right); k, \tilde{p}_{\underline{ij}}\right) = \\ \frac{d\Omega_{D-2}}{4\left(2\pi\right)^{D-2}}\,\delta\left(\tilde{p}_{\underline{ij}}^2\left(u, \mathcal{P}^2, \mathcal{P}_a\right) - m_{\underline{ij}}^2\right) \frac{\hat{p}_a^{3-D}}{\sqrt{s}} \left[\left(\mathcal{P}_a^+ - \mathcal{P}_a\right)\left(\mathcal{P}_a - \mathcal{P}_a^-\right)\right]^{\frac{D}{2}-2} d\mathcal{P}^2\, d\mathcal{P}_a, \tag{3.253}$$

where the bounds on $\mathcal{P}_a$ integration read

$$\mathcal{P}_a^{\pm} = \frac{\left(s + m_a^2 - q^2\right)\left(s - m_k^2 + \mathcal{P}^2\right)}{4s} \pm \hat{p}_a \hat{k}. \tag{3.254}$$

Moreover

$$\left(m_i + m_j\right)^2 \leq \mathcal{P}^2 \leq \left(\sqrt{s} - m_k\right)^2. \tag{3.255}$$

Finally, we can perform one of the remaining integrations due to the on-shell delta function

$$d\Phi_2\left(q, \tilde{p}_a\left(u, \mathcal{P}^2, \mathcal{P}_a\right); k, \tilde{p}_{\underline{ij}}\right) = \frac{d\Omega_{D-2}}{4\left(2\pi\right)^{D-2}} \left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial \mathcal{P}_a}\right|_{u,\mathcal{P}^2}^{-1} \frac{\hat{p}_a^{3-D}}{\sqrt{s}} \\ \left[\left(\mathcal{P}_a^+\left(\mathcal{P}^2\right) - \mathcal{P}_a\left(u, \mathcal{P}^2\right)\right)\left(\mathcal{P}_a\left(u, \mathcal{P}^2\right) - \mathcal{P}_a^-\left(\mathcal{P}^2\right)\right)\right]^{\frac{D}{2}-2} d\mathcal{P}^2 \tag{3.256}$$

where we suppressed all the functional dependence on fixed variables, such as s or q^2.

Recall, that the formula (3.256) is actually used in the convolution (3.243) and corresponds to $\mathcal{X} = \mathcal{P}^2$, $\mathcal{Y} = \mathcal{P}_a\left(u, \mathcal{P}^2\right)$. It means that all the invariants in $d\phi_{\underline{ij}\to i\,j}^{\text{FE-IS}}$ have to be expressed accordingly in terms of u and $\mathcal{P}^2$. The advantage of the above choice $\mathcal{X} = \mathcal{P}^2$, $\mathcal{Y} = \mathcal{P}_a$ is that one can relatively easily obtain the bounds on u and $\mathcal{P}^2$. In practice however, it is more convenient to keep u and $\tilde{\gamma}$ fixed when integrating over $d\phi_{\underline{ij}\to i\,j}^{\text{FE-IS}}$, i.e. to choose $\mathcal{X} = \tilde{\gamma}$. Thus, let us now derive needed formulae. It is actually straightforward, one needs only to insert appropriate jacobian. We obtain

$$d\Phi_2\left(q, \tilde{p}_a\left(u, \tilde{\gamma}, \mathcal{P}_a\right); k, \tilde{p}_{\underline{ij}}\right) = \frac{d\Omega_{D-2}}{4\left(2\pi\right)^{D-2}} \left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial \mathcal{P}_a}\right|_{u,\tilde{\gamma}}^{-1} \left(\frac{\partial \mathcal{P}^2}{\partial \tilde{\gamma}}\right)_{\mathcal{P}_a} \frac{\hat{p}_a^{3-D}}{\sqrt{s}} \\ \left[\left(\mathcal{P}_a^+\left(u, \tilde{\gamma}\right) - \mathcal{P}_a\left(u, \tilde{\gamma}\right)\right)\left(\mathcal{P}_a\left(u, \tilde{\gamma}\right) - \mathcal{P}_a^-\left(u, \tilde{\gamma}\right)\right)\right]^{\frac{D}{2}-2} d\tilde{\gamma}. \tag{3.257}$$

Using relations (A.7)-(A.9) from the Appendix A.1 and comparing the above formula with (3.256), we can prove that (3.257) is indeed correct.

The nontrivial issue in this choice of $\mathcal{X}$ and $\mathcal{Y}$ is however to find analytically the correct support for the variables u and $\tilde{\gamma}$ (see also Section 3.4.1.5). For an explicit example, see the next subsection. The problems exist because the function $\tilde{\gamma}\left(\mathcal{P}^2, \mathcal{P}_a\right)$ (see 3.87) can have a minimum in $\mathcal{P}_a$ *inside* the region $[\mathcal{P}_{a\,-}, \mathcal{P}_{a\,+}]$ defined by (3.254). If one does not need analytical expressions for the bounds and skewed phase space integration is performed via Monte Carlo, those problems are not essential. Unfortunately, in some cases we do need the analytical expressions for the bounds, namely when we find "plus" distributions in our calculations. Thus, it seems that there is a problem: on one hand we want to have "simple" formulae of integrated dipoles (achieved when $\mathcal{X} = \tilde{\gamma}$), on the other we want to have analytical expressions for the support of $\tilde{u}$ variable. Both requirements seem to be in a contradiction. However, there is a very simple solution. We can just mix both approaches, i.e. we can generate PS using choice $\mathcal{X} = \mathcal{P}^2$ while integrate dipoles using $\mathcal{X} = \tilde{\gamma}$ with however some care when treating the "plus" distributions. To summarize, it is a technical problem which can be solved in one or the other way and we do not discuss it further.

For the IE-FS case, one can obtain analogous formulae. Basically, one should replace $\tilde{p}_a$ by $\tilde{p}_{\underline{ai}}$ and $\tilde{p}_{ij}$ by $\tilde{p}_j$ in the above equations, according to Subsection 3.6.3.

3.6.4.1 Explicit examples

Since the presented phase space factorization procedure is rather non-trivial, it requires careful verification. In this paragraph we shall check our results against the usual expression for the three-particle phase space, which we derive in A.3.

First, notice that pure, fully integrated phase space $\int d\Phi_3\left(q, p_a; p_i, p_j, k\right)$ depends only on the masses m_i, m_j, m_k and the CM energy s. We have checked numerically using (3.256) and (3.257) directly inside (3.243), that the result depends neither on m_a nor $m_{\underline{ij}}$. Moreover, it precisely equals three-body phase space obtained by standard method described in the Appendix A.3. The sample results are presented in Fig. 3.15 for $\mathcal{X} = \mathcal{P}^2$, $\mathcal{Y} = \mathcal{P}_a$ and in Fig. 3.16 for $\mathcal{X} = \tilde{\gamma}$, $\mathcal{Y} = \mathcal{P}_a$. The solid angle was integrated out analytically. We show also the supports in these cases. Note in particular the complexity of the support in u and $\tilde{\gamma}$ variables, as we have discussed in the previous subsection.

3.7 Integration of the dipoles

As explained in Section 3.1, the cancellation of soft poles in virtual corrections requires integration of the dipoles over the measure $d\phi$. In the following, we shall compute all the necessary integrals and disentangle the soft poles.

Let us first make some general remarks. To be specific, let us concentrate on FE-IS case. Recall, that the unintegrated dipole splitting functions are - as a matter of fact - matrices in the helicity space. Let us also recall that they are sandwiched between the reduced matrix elements (see Section 3.3); the complete general dipole has the form

$$\mathcal{D}^{\text{FE-IS}} = -\frac{1}{\mathcal{S}}\overline{\hat{\mathcal{M}}}^{\dagger}\left(q, \tilde{p}_a; \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right) \hat{C}\hat{V}\,\overline{\hat{\mathcal{M}}}\left(q, \tilde{p}_a; \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right), \tag{3.258}$$

where $\mathcal{S}$ is a scalar propagator relevant to the splitting case, $\hat{V}$ is a pertinent dipole splitting matrix. Recall that $\mathbb{X}$ is the set enumerating the corresponding final state momenta entering the reduced matrix element. The matrix $\hat{C}$ is the adequate colour correlation matrix. Suppose we want to integrate the dipole over the subspace $d\phi$. We have

$$\int d\phi\, \mathcal{D}^{\text{FE-IS}} = -\overline{\hat{\mathcal{M}}}^{\dagger}\left(q, \tilde{p}_a; \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right) \hat{C}\hat{I}^{\text{FE-IS}}\,\overline{\hat{\mathcal{M}}}\left(q, \tilde{p}_a; \{p_m\}_{m\in\mathbb{X}_{\text{FE-IS}}}\right), \tag{3.259}$$

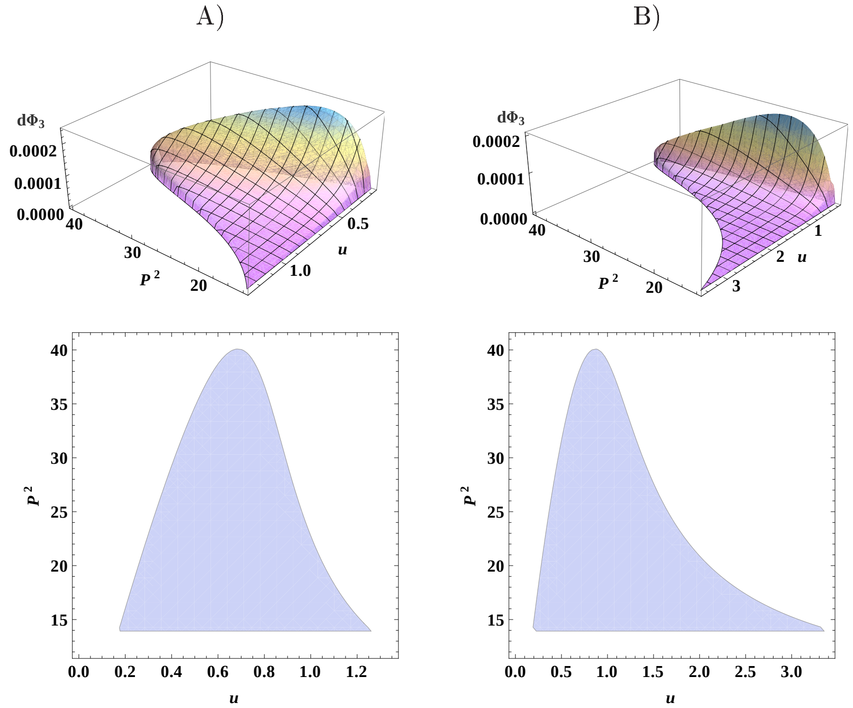


Figure 3.15: Sample results for the three-particle phase space in terms of u and $\mathcal{P}^2$ variables and FE-IS case. All the plots are made for $s = 60\,\mathrm{GeV}^2$, $m_k^2 = 2\,\mathrm{GeV}^2$, $m_i^2 = 3\,\mathrm{GeV}^2$, $m_j^2 = 4\,\mathrm{GeV}^2$ and fractional dimension $D = 4.3$. In column A) we show surface plot (top) and the support (bottom) for $m_a^2 = 5\,\mathrm{GeV}^2$, $m_{\underline{ij}}^2 = 1\,\mathrm{GeV}^2$; column B) - the same for $m_a^2 = 15\,\mathrm{GeV}^2$, $m_{\underline{ij}}^2 = 0$. Notice, that in general $u_{\max} > 1$ contrary to the massless case. The solid angle was integrated out.

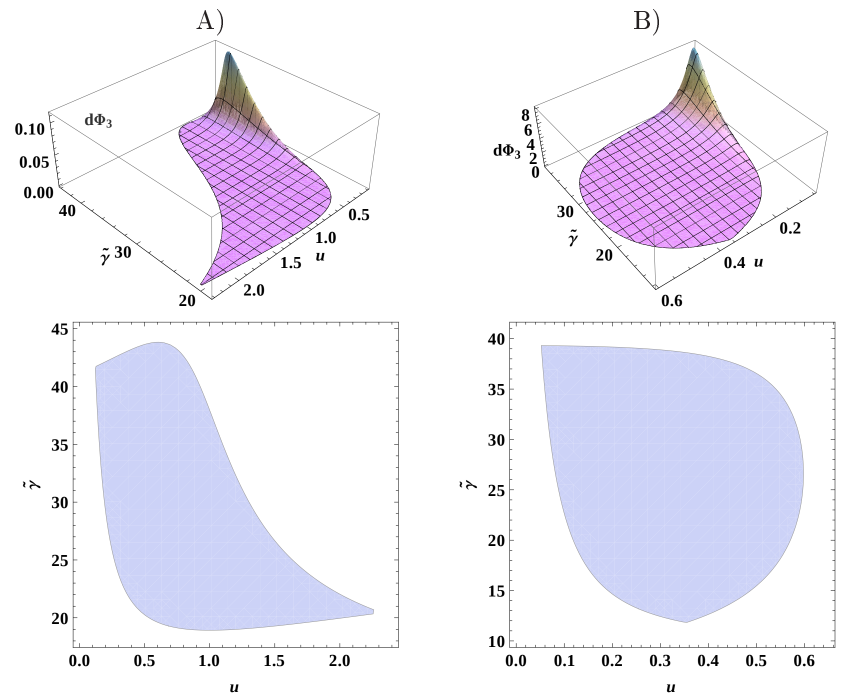


Figure 3.16: Same as Fig. 3.15 but in terms of u and $\tilde{\gamma}$ variables. In column A) $m_a^2 = 15\,\mathrm{GeV}^2$, $m_{\underline{ij}}^2 = 5\,\mathrm{GeV}^2$, in column B) $m_a^2 = 5\,\mathrm{GeV}^2$, $m_{\underline{ij}}^2 = 10\,\mathrm{GeV}^2$.

where

$$\hat{I}^{\text{FE-IS}} = \int d\phi \, \frac{1}{\mathcal{S}} \hat{V}. \tag{3.260}$$

The integral is in general correlated (in helicity space) with the matrix element. However, it turns out that it is superficial. As we shall see, after integration the correlations vanish and we are left with

$$\int d\phi \, \mathcal{D}^{\text{FE-IS}} = -I^{\text{FE-IS}} \, \hat{C} \left| \hat{\overline{\mathcal{M}}} \left(q, \tilde{p}_a; \{p_m\}_{m \in \mathbb{X}_{\text{FE-IS}}} \right) \right|^2, \tag{3.261}$$

with

$$I^{\text{FE-IS}} = \int d\phi \, \frac{1}{\mathcal{S}} \left\langle \hat{V} \right\rangle, \tag{3.262}$$

where $\langle . \rangle$ denotes helicity average in D dimensions. The reduced matrix element in (3.261) is averaged over initial state polarizations of parton $\tilde{p}_a$ which is the same as for p_a. It is not the case for IE-FS, see below.

For initial state emitter, the overall situation is very similar. The only difference at this stage is due to the fact, that the number of polarizations of the new initial state $\tilde{p}_{\underline{ai}}$ is different. That is we have

$$\int d\phi \, \mathcal{D}^{\text{IE-FS}} = -I^{\text{IE-FS}} \, \hat{C} \otimes \left| \hat{\overline{\mathcal{M}}} \left(q, \tilde{p}_{\underline{ai}}; \{p_m\}_{m \in \mathbb{X}_{\text{IE-FS}}} \right) \right|^2, \tag{3.263}$$

where now the reduced matrix element is averaged over polarizations of $\underline{ai}$. Therefore, we have to include a spin-transition factor into the dipole splitting functions or into the integral. In our case, we have included it in the splitting function. Note, this is different than in [10], where dipole splitting functions are defined without these factors. More precisely, the dipole integral is then

$$I_*^{\text{IE-FS}} = \frac{n_{\underline{ai}}}{n_a} \int d\phi \, \frac{1}{\mathcal{S}} \left\langle \hat{V} \right\rangle, \tag{3.264}$$

where $n_{\underline{ai}}$, n_a are number of spin states of partons $\underline{ai}$ and a respectively. In our case it is

$$I^{\text{IE-FS}} = \int d\phi \, \frac{1}{\mathcal{S}} \left\langle \hat{V} \right\rangle. \tag{3.265}$$

We note, that although those spin transition factors equal to 1 or $1-\varepsilon$, they are essential as the integrals are in general singular.

3.7.1 The notation

Let us introduce some helpful notation we shall use throughout this section. We define scaled, dimensionless masses for a quark q as follows

$$\eta_q^2 = \frac{m_q^2}{2\tilde{\gamma}}. \tag{3.266}$$

In this fashion, it is also useful to define other scaled quantities. For any quantity X with the dimension of the mass squared we define

$$\eta_X^2 = \frac{X}{2\tilde{\gamma}}. \tag{3.267}$$

For example, we shall use

$$\eta^2_{\mathcal{P}^2} = \frac{\mathcal{P}^2}{2\tilde{\gamma}}, \tag{3.268}$$

$$\eta^2_{\mathcal{P}_a} = \frac{\mathcal{P}_a}{2\tilde{\gamma}} \tag{3.269}$$

and so on. It is also useful to define $\tilde{u}$-scaled variables of this kind

$$\tilde{\eta}^2_X = \frac{\eta^2_X}{\tilde{u}} = \frac{X}{2\tilde{u}\tilde{\gamma}}. \tag{3.270}$$

The motivation in introducing such a notation is that we want to disentangle $\tilde{u}$ dependence as much as possible, since it can lead to the singularities. This point shall be clarified later in this section.

It is convenient to introduce a reduced dipole integral $\tilde{I}$, defined as follows

$$I = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu_r^2}{2\tilde{\gamma}}\right)^{\varepsilon} \tilde{I}, \tag{3.271}$$

where the integrated dipole I is defined in (3.262). That is we pull out the standard factors relevant to $\overline{\text{MS}}$ calculations.

3.7.2 Final state emitter - Initial state spectator

3.7.2.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

In that case the dipole splitting matrix (3.169) is diagonal in helicity. Therefore there are no correlations between the dipole and the reduced matrix element. Averaged dipole splitting function reads simply

$$\left\langle V^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}\right\rangle = 8\pi\mu_r^{2\varepsilon}\alpha_s C_F \left[\frac{2}{\tilde{u}\tilde{v}^2+\tilde{z}} + (1-\varepsilon)\,\tilde{z} - 2 - \frac{m^2}{p_i\cdot p_j}\right]. \tag{3.272}$$

Let us define the integral of the dipole function

$$I^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \int d\phi^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}\, \frac{1}{2p_i\cdot p_j} \left\langle \hat{V}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}\right\rangle. \tag{3.273}$$

Note we included the propagator. Let us express the dipole splitting function, propagator and the subspace $d\phi$ via the scaled variables introduced in Section 3.7.1. The propagator reads

$$2p_i\cdot p_j = \mathcal{P}^2 - m^2 = 2\tilde{\gamma}\left(\eta^2_{\mathcal{P}^2} - \eta^2\right) = 2u\tilde{\gamma}\,\tilde{\eta}^2_{\mathcal{P}^2-m^2}, \tag{3.274}$$

where we introduced rescaled inverse propagator

$$\tilde{\eta}^2_{\mathcal{P}^2-m^2} = \frac{\eta^2_{\mathcal{P}^2-m^2}}{u} = \frac{\mathcal{P}^2-m^2}{2\tilde{\gamma}u}. \tag{3.275}$$

Note, that we use u instead of $\tilde{u}$, due to phase space factorization procedure. We turn special attention that $\tilde{\eta}_{\mathcal{P}^2-m^2}$ is rescaled also by u comparing to similar quantities. This fact is marked by the notation with the tilde (see also 3.7.1). The reason to make such rescaling is to disentangle possible $u = 0$ singularity; in this limit the propagator in Fig. 3.7 vanishes. We shall analyse the support in u in more detail below.

Let us now rewrite the sub-space $d\phi$ (3.236) using scaled variables. We have (after integration over azimuthal angles which is trivial)

$$d\phi^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \frac{2\,(2\tilde{\gamma})^{1-\varepsilon}}{(4\pi)^{2-\varepsilon}\,\Gamma(1-\varepsilon)}\,\eta^2_{\mathcal{J}}\,\left(\eta^2_{\mathcal{P}^2}\right)^{-\varepsilon} v^{2\varepsilon-1}\,\left[(\tilde{z}_+ - \tilde{z})\,(\tilde{z} - \tilde{z}_-)\right]^{-\varepsilon} d\tilde{z}, \tag{3.276}$$

where the velocity reads

$$v = \frac{\tilde{v}}{1 + 2u\eta^2_a}, \tag{3.277}$$

with (compare (3.99))

$$\tilde{v} = \sqrt{1 - 4\eta^2\eta^2_a}. \tag{3.278}$$

The scaled jacobian (3.233)

$$\eta^2_{\mathcal{J}} = \frac{\mathcal{J}}{2\tilde{\gamma}} \tag{3.279}$$

is given in Appendix (A.2.1.1). In the considered case the bounds on $\tilde{z}$ are

$$\tilde{z}_\pm = \frac{u\,\tilde{\eta}^2_{\mathcal{P}^2-m^2}\left(1 + 2u\eta^2_a \pm \tilde{v}\right)}{2\eta^2_{\mathcal{P}^2}\left(1 + 2u\eta^2_a\right)}. \tag{3.280}$$

Now we are ready to integrate the dipole splitting function over $d\tilde{z}$. As anticipated already in Section 3.7.1, it is convenient to define the reduced integral $\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}$ as

$$I^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \frac{\alpha_s}{2\pi}\,\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{4\pi\mu_r^2}{2\tilde{\gamma}}\right)^{\varepsilon}\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a}, \tag{3.281}$$

i.e. we pull out the standard factors appearing in the $\overline{\text{MS}}$ NLO calculations. Using the integrals calculated in Appendix B.1 (integrals $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$) we obtain

$$\begin{aligned}\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \frac{2C_F\eta^2_{\mathcal{J}}\left(\eta^2_{\mathcal{P}^2}\right)^{\varepsilon}}{u^{1+2\varepsilon}\left(\tilde{\eta}^2_{\mathcal{P}^2-m^2}\right)^{1+2\varepsilon}}\Bigg\{&\frac{1}{v}\mathcal{F}\left(\mathcal{A}(u)\,;-\varepsilon\right)\\ &- B(1-\varepsilon)\left[\frac{\eta^2}{\eta^2_{\mathcal{P}^2}} + \frac{\tilde{\eta}^2_{\mathcal{P}^2-m^2}}{\eta^2_{\mathcal{P}^2}}\,u\left(1 + \frac{\tilde{\eta}^2_{\mathcal{P}^2-m^2}}{4\eta^2_{\mathcal{P}^2}}uv\,(\varepsilon-1)\right)\right]\Bigg\},\end{aligned} \tag{3.282}$$

where

$$\mathcal{A}(u) = \frac{2\tilde{v}\tilde{\eta}^2_{\mathcal{P}^2-m^2}}{2\eta^2_{\mathcal{P}^2}\tilde{v}^2\left(1 + 2u\eta^2_a\right) + \tilde{\eta}^2_{\mathcal{P}^2-m^2}\left(1 + 2u\eta^2_a - \tilde{v}\right)}. \tag{3.283}$$

The function $\mathcal{F}$ can be expressed as a hypergeometric function and is precisely defined in (B.21); we also refer to Appendix B.1 for some of its useful properties. We turn attention to our special notation of Euler's beta function with two equal arguments

$$B(1-\varepsilon, 1-\varepsilon) \equiv B(1-\varepsilon)\,. \tag{3.284}$$

Remember that w, $\eta^2_{\mathcal{P}^2}$, $\eta^2_{\mathcal{P}^2-m^2}$, $\eta^2_{\mathcal{P}_a}$ should all be considered as the functions of u. We do not write it explicitly for more transparency (the explicit expressions are collected in Appendix A.2.1.1).

Let us now analyse the support of the integral in u (recall, that it will be convoluted with the reduced matrix element). According to (3.98) we have for the lower bound in our mass configuration

$$u_- = 0 \tag{3.285}$$

and in general the upper bound u_+ is different than one (we do not give it explicitly here). We see that the integral (3.282) is singular at the point $u = u_-$ in $D = 4$. This is the soft singularity. In order to regularize this singularity we define

$$\kappa = -\varepsilon, \quad \kappa > 0. \tag{3.286}$$

and disentangle the divergence using the "plus" distribution $f_{[0,u_+]}$ defined as (see also Appendix B.3)

$$f(u) = f_{[0,u_+]}(u) + \delta(u) \int_0^{u_+} f(u)\, du. \tag{3.287}$$

Notice, that our distribution has the support $[0, u_+]$, rather than commonly used $[0, 1]$.

Let us now write (3.282) as

$$\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \frac{1}{u^{1-2\kappa}} K(u;\kappa), \tag{3.288}$$

where $K(u;\kappa)$ is now free from the singularity. Using (3.287) we have

$$\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g,\,a} = \left(\frac{1}{u}\right)_{[0,u_+]} K(u;0) + \delta(u) K(0;\kappa) \left(\frac{1}{2\kappa} + \log u_+\right) + \mathcal{O}(\kappa). \tag{3.289}$$

The logarithm of u_+ reflects our definition of the "plus" distribution with a non-standard support. The two distinct limits of the quantity K read

$$K(u;0) = \frac{2C_F\eta^2_{\mathcal{J}}}{\tilde{\eta}^2_{\mathcal{P}^2-m^2}} \left\{\frac{1}{v}\log(1+\mathcal{A}(u)) - \left[\frac{\eta^2}{\eta^2_{\mathcal{P}^2}} + \frac{u\tilde{\eta}^2_{\mathcal{P}^2-m^2}}{\eta^2_{\mathcal{P}^2}}\left(1 - \frac{uv\,\tilde{\eta}^2_{\mathcal{P}^2-m^2}}{4\eta^2_{\mathcal{P}^2}}\right)\right]\right\}, \tag{3.290}$$

$$K(0;\kappa) = -2C_F\tilde{v}^{4\kappa}\eta^{-2\kappa}\left(B(1+\kappa) - \frac{1}{\tilde{v}}\mathcal{F}(\mathcal{A};\kappa)\right), \tag{3.291}$$

where

$$\mathcal{A} \equiv \mathcal{A}(0) = \frac{2\tilde{v}}{1+2\eta^2-\tilde{v}}. \tag{3.292}$$

Let us write the full result as

$$\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g} = J(u) + \delta(u)\left(J_{\text{pole}} + J_{\text{finite}}\right). \tag{3.293}$$

The subsequent pieces read

$$J_{\text{pole}} = -C_F\frac{1}{\kappa}\left(1 - \frac{1}{\tilde{v}}\log(1+\mathcal{A})\right), \tag{3.294}$$

$$J_{\text{finite}} = C_F\left[2 + \frac{1}{\tilde{v}}\mathcal{F}_1(\mathcal{A}) - \left(1 - \frac{1}{\tilde{v}}\log(1+\mathcal{A})\right)\log\frac{u_+^2\tilde{v}^4}{\eta^2}\right], \tag{3.295}$$

$$J(u) = \left(\frac{1}{u}\right)_{[0,u_+]} K(u;0)\,, \tag{3.296}$$

where $\mathcal{F}_1$ is the coefficient in expansion

$$\mathcal{F}(\mathcal{A};\kappa) = \log(1+\mathcal{A}) + \kappa\,\mathcal{F}_1(\mathcal{A}) + \mathcal{O}(\kappa)\,, \tag{3.297}$$

calculated in Appendix B.2.

Let us now check, if in the limit of massless initial state parton we recover the result of [10]. When $m_a \to 0$, we have in particular

$$v = \tilde{v} = 1, \quad w = 1, \tag{3.298}$$

$$\eta^2_{\mathcal{P}^2} = u + \eta^2, \quad \tilde{\eta}^2_{\mathcal{P}^2 - m^2} = 1, \quad \eta^2_{\mathcal{J}} = 1, \tag{3.299}$$

$$\mathcal{A}(u) = \frac{1}{u+\eta^2}. \tag{3.300}$$

Now, the terms in decomposition (3.293) read (we have adorned them by a star to underline that they are evaluated for the massless spectator)

$$J^*_{\text{pole}} = -C_F\,\frac{1}{\kappa}\left(1 - \log\frac{1+\eta^2}{\eta^2}\right), \tag{3.301}$$

$$J^*_{\text{finite}} = C_F\left[2 + \mathcal{F}_1\left(\frac{1}{\eta^2}\right) + \left(1 - \log\frac{1+\eta^2}{\eta^2}\right)\log\eta^2\right], \tag{3.302}$$

$$J^*(u) = 2C_F\left(\frac{1}{u}\right)_+ \left\{\log\frac{1+u+\eta^2}{u+\eta^2} - 1 + \frac{u^2}{4\,(u+\eta^2)^2}\right\}. \tag{3.303}$$

Now, we use some effort to reshuffle the terms between the above equations using (B.74) in order to get the same "plus" distributions as in [10]. We get

$$\begin{aligned}
\tilde{I}^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g}\Big|_{m_a^2=0} &= 2C_F\left(\frac{1}{u}\right)_+ \log\left(1+u+\eta^2\right) \\
&+ C_F\left[\frac{1}{u}\left(\frac{u^2}{2\,(u+\eta^2)} - 2\log\left(u+\eta^2\right) - 2\right)\right]_+ \\
&+ C_F\,\delta(u)\left[\frac{1}{\kappa}\left(\log\left(1+\eta^2\right) - 1\right) + \left(\frac{1}{2} - \frac{1}{\kappa}\right)\log\eta^2 - \frac{1}{2}\log^2\eta^2 + \frac{3}{2} - \frac{2}{3}\pi^2\right. \\
&\left. + \frac{1}{2}\log\left(1+\eta^2\right)\left(1+\log\left(1+\eta^2\right)\right) - 2\log\eta^2\log\left(1+\eta^2\right) - 4\mathrm{Li}_2\left(-\eta^2\right) + \frac{\eta^2}{2\,(1+\eta^2)}\right].
\end{aligned} \tag{3.304}$$

We note that the u-dependent part is identical to the one in [10]. The same is true for the endpoint contributions, which are finite in massless limit. We obtain full agreement once we expand the $\eta^{2\kappa}$ term[2] which multiplies some poles and finite factors in [10]. Such a factor was introduced there, in order to keep possibility to obtain massless results at any time. After expanding $\eta^{2\kappa}$ it is no longer possible, since the limits $\kappa \to 0$ and $\eta^2 \to 0$ do not commute.

[2] adapted to our notation.

In the end, let us give two remarks. First, note an interesting trick. As far as we consider soft and soft/collinear poles in a massive calculation, i.e. the end-point terms containing $\log\eta^2$, the corresponding poles in massless calculation can be always recovered by means of the following correspondence rules

$$\log\eta^2 = \frac{1}{\kappa}\left(\eta^{2\kappa}-1\right) + \mathcal{O}\left(\kappa\right) \;\to\; -\frac{1}{\kappa}, \tag{3.305}$$

$$\frac{1}{\kappa}\log\eta^2 + \frac{1}{2}\log^2\eta^2 = \frac{1}{\kappa^2}\left(\eta^{2\kappa}-1\right) + \mathcal{O}\left(\kappa\right) \;\to\; -\frac{1}{\kappa^2}. \tag{3.306}$$

Thus, using this method we can check soft singularities in a massive calculation against corresponding massless results, which are either well known or are simple to obtain. This of course does not work for the finite terms.

Second observation is that we cannot set the mass of heavy quark $\mathbf{Q}$ literarly to zero in our integrated dipole function (3.293). This is because we have not kept track of $\eta^{2\kappa}$ factors and they are expanded, see more extensive discussion below in Section 3.7.3.1. We give separate formula in Appendix B.4, as the calculation requires some tools introduced below in Subsection 3.7.2.3.

3.7.2.2 $g \to \mathbf{Q}\overline{\mathbf{Q}}$ splitting

Let us first recall, that in this case we have the following configuration of masses

$$m_i = m_j = m_{\mathbf{Q}} \equiv m, \quad m_{\underline{ij}} = 0. \tag{3.307}$$

For convenience, we recall also that the dipole splitting matrix derived in Section 3.5.1.2 reads

$$\left(\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right)^{\mu\nu} = 8\pi\mu_r^{2\varepsilon}\alpha_s T_R\left(-g^{\mu\nu} - 4\frac{\mathcal{C}^{\mu\nu}}{\mathcal{P}^2}\right), \tag{3.308}$$

where

$$\mathcal{C}^{\mu\nu} = \left[\tilde{z}p_i^\mu - \left(1-\tilde{z}\right)p_j^\mu - \frac{\tilde{u}m_a^2}{2\tilde{w}\mathcal{P}_a}\left(p_i^\mu - p_j^\mu\right)\right]\left[\tilde{z}p_i^\nu - \left(1-\tilde{z}\right)p_j^\nu - \frac{\tilde{u}m_a^2}{2\tilde{w}\mathcal{P}_a}\left(p_i^\nu - p_j^\nu\right)\right]. \tag{3.309}$$

Let us define the integral of the dipole splitting matrix as

$$\mathcal{K}^{\mu\nu} = \int d\phi^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\,\frac{1}{\mathcal{P}^2}\left(\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right)^{\mu\nu}, \tag{3.310}$$

where the detailed expression for the measure $d\phi$ is derived in Section 3.6. Due to the Lorentz invariance, the integral can be decomposed as

$$\mathcal{K}^{\mu\nu} = -g^{\mu\nu}A_1 + \frac{\tilde{p}^\mu_{\underline{ij}}\tilde{p}^\nu_a + \tilde{p}^\nu_{\underline{ij}}\tilde{p}^\mu_a}{\tilde{p}_{\underline{ij}}\cdot\tilde{p}_a}A_2 + \tilde{p}^\mu_{\underline{ij}}\tilde{p}^\nu_{\underline{ij}}A_3 + \tilde{p}^\mu_a\tilde{p}^\nu_a A_4. \tag{3.311}$$

Since the momentum $\tilde{p}_{\underline{ij}}$ is the one of the massless gluon, the A_2, A_3 terms do not contribute to the full subtraction term. This is due to the gauge invariance and the following Ward identity

$$\tilde{p}^\mu_{\underline{ij}}\mathcal{M}_\mu = 0. \tag{3.312}$$

Here the reduced amplitude $\mathcal{M}_\mu$ corresponds to the shaded blob in Fig. 3.8. Let us now turn to the A_4 coefficient. Contracting (3.311) with $\tilde{p}^\mu_{\underline{ij}}\tilde{p}^\nu_{\underline{ij}}$ we get

$$A_4 = \int d\phi \, \frac{1}{\mathcal{P}^2 \left(\tilde{p}_{\underline{ij}} \cdot \tilde{p}_a\right)^2} \tilde{p}^\mu_{\underline{ij}}\tilde{p}^\nu_{\underline{ij}} \left(\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}}}\right)_{\mu\nu} = 0, \tag{3.313}$$

due to the transversality property of $\mathcal{C}^{\mu\nu}$ (3.180) and the on-shell condition

$$\tilde{p}^2_{\underline{ij}} = 0. \tag{3.314}$$

Thus, only A_1 contributes to the full subtraction term. It can be disentangled by taking the average over helicities of the gluon $\tilde{p}_{\underline{ij}}$ by means of the polarization tensor

$$d^{\mu\nu}\left(\tilde{p}_{\underline{ij}}; p_a\right) = -g^{\mu\nu} + \frac{\tilde{p}^\mu_{\underline{ij}} p^\nu_a + \tilde{p}^\nu_{\underline{ij}} p^\mu_a}{\tilde{p}_{\underline{ij}} \cdot p_a} - m_a^2 \frac{\tilde{p}^\mu_{\underline{ij}}\tilde{p}^\nu_{\underline{ij}}}{\left(\tilde{p}_{\underline{ij}} \cdot p_a\right)^2}. \tag{3.315}$$

Here p_a is used as an auxiliary vector. In practice - again due to the transversality of $\mathcal{C}^{\mu\nu}$ - the gauge terms in $d^{\mu\nu}$ can be omitted.

Summarizing, only the average of the dipole splitting function contributes. This average reads

$$\left\langle V^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right\rangle = \frac{1}{D-2} d_{\mu\nu}\left(\tilde{p}_{\underline{ij}}; p_a\right)\left(V^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right)^{\mu\nu}. \tag{3.316}$$

The only non-trivial (although straightforward) point is to evaluate the average of $\mathcal{C}^{\mu\nu}$. It turns out to be simple

$$d_{\mu\nu}\left(\tilde{p}_{\underline{ij}}; p_a\right)\mathcal{C}^{\mu\nu} = \mathcal{P}^2\left(\tilde{z}_+ - \tilde{z}\right)\left(\tilde{z} - \tilde{z}_-\right) \tag{3.317}$$

Hence

$$\left\langle V^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a}\right\rangle = 8\pi\mu_r^{2\varepsilon}\alpha_s T_R\left[1 - \frac{2}{1-\varepsilon}\left(\tilde{z}_+ - \tilde{z}\right)\left(\tilde{z} - \tilde{z}_-\right)\right], \tag{3.318}$$

where the bounds $\tilde{z}_\pm$ on $\tilde{z}$ variable are given in Section 3.6.1.2. Let us list their form for the present mass configuration:

$$\tilde{z}_\pm = \frac{1}{2}\left(1 \pm v\overline{p}\right), \tag{3.319}$$

where $\overline{p}$ was defined in (3.217) and reads here

$$\overline{p} = \sqrt{1 - \frac{4m^2}{\mathcal{P}^2}}. \tag{3.320}$$

Now, we shall perform the integral over the measure $d\phi$. Note, that in the present case

$$\tilde{v}_{\underline{ij}} = 1. \tag{3.321}$$

Further

$$v = \frac{\tilde{\gamma}}{\tilde{\gamma} + u m_a^2} \tag{3.322}$$

and thus the measure reads

$$d\phi^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}},\,a} = \frac{2\left(2\tilde{\gamma}\right)^{1-\varepsilon}}{\left(4\pi\right)^{2-\varepsilon}\Gamma\left(1-\varepsilon\right)}\left(u\tilde{\eta}^2_{\mathcal{P}^2}\right)^{-\varepsilon}\eta^2_{\mathcal{J}}\left(1+2u\eta_a^2\right)^{1-2\varepsilon}\left[\left(\tilde{z}_+-\tilde{z}\right)\left(\tilde{z}-\tilde{z}_-\right)\right]^{-\varepsilon}d\tilde{z}. \tag{3.323}$$

Let us recall, that $\eta^2_{\mathcal{J}}$ is the scaled jacobian $\mathcal{J}$, that has different form depending on the choice of the "external" kinematic variables. The form we use is given in Appendix A.2.1.2. We define the pertinent integral as

$$I^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}},\,a} = \int d\phi^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}},\,a}\,\frac{1}{\mathcal{P}^2}\left\langle\hat{V}^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}},\,a}\right\rangle \tag{3.324}$$

and introduce - similarly as before - the integral without the standard factors, $\tilde{I}^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}},\,a}$, see (3.271) for definition. In terms of the integrals defined and calculated in the Appendix B.1 we have (again converting $\varepsilon\to-\kappa$)

$$\begin{aligned}\tilde{I}^{\text{FE-IS}}_{g\to\mathbf{Q\overline{Q}}} &= \frac{T_R}{u^{1-\kappa}\tilde{\eta}^2_{\mathcal{P}^2}}\,\eta^2_{\mathcal{J}}\left(1+2u\eta_a^2\right)^{1+2\kappa}\left[\mathcal{I}_2-\frac{2}{1+\kappa}\mathcal{I}_7\right] \\ &= \frac{T_R\,B\left(1+\kappa\right)}{u^{1-\kappa}\tilde{\eta}^2_{\mathcal{P}^2}}\,\eta^2_{\mathcal{J}}\overline{p}^{1+2\kappa}\left[1+\frac{\overline{p}^2}{\left(3+2\kappa\right)\left(1+2u\eta_a^2\right)^2}\right].\end{aligned} \tag{3.325}$$

We see that there is a potential singularity due to the denominator $\frac{1}{u}$. However, it is not the case as long as we deal with massive quarks. To see this, check the lower bound on u. It reads

$$u_- = \frac{8\eta^2}{1+16\eta^2\left(\eta^2-\eta_a^2\right)+\left(1+4\eta^2\right)\sqrt{\left(1-4\eta^2\right)^2-16\eta^2\eta_a^2}} \tag{3.326}$$

and equals zero only for $m=0$ (not even for $m_a=0$). The lack of the "formal" singularity, however, does not make necessarily life easier. This is because for large momenta $u_-\to 0$ and thus it is desirable to control the potential singular behaviour. Possible solution to this problem, is the following (see [10]). Regularize this potential singularity by introducing the "plus" distribution with the support extending from non-zero value, namely (see also Appendix B.3)

$$f\left(u\right) = f_{[u_-,u_+]}\left(u\right)+\delta\left(u-u_-\right)\int_{u_-}^{u_+}f\left(u\right)du. \tag{3.327}$$

It has the property that

$$f_{[u_-,u_+]}\underset{m\to 0}{\longrightarrow}f_{[0,u_+]}, \tag{3.328}$$

In the context of (3.325), the above procedure is realised by the following replacement

$$\frac{1}{u} = \left(\frac{1}{u}\right)_{[u_-,u_+]}+\delta\left(u-u_-\right)\frac{1}{\kappa}\left(u_+^\kappa-u_-^\kappa\right). \tag{3.329}$$

At this stage we are still free to literally set $m=0$ – then we get the soft pole (since $u_-=0$). However in practice, if we want to deal with massive quarks, we have to expand the numerator in κ and we get

$$\frac{1}{u} = \left(\frac{1}{u}\right)_{[u_-,u_+]}+\delta\left(u-u_-\right)\log\frac{u_+}{u_-}. \tag{3.330}$$

With this substitution we obtain

$$\tilde{I}^{\text{FE-IS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,a} = \left(\frac{1}{u}\right)_{[u_-,u_+]} K\left(u\right) + \delta\left(u-u_-\right) K\left(u_-\right) \log\frac{u_+}{u_-}, \tag{3.331}$$

where

$$K\left(u\right) = T_R \frac{\eta^2_{\mathcal{J}}}{\tilde{\eta}^2_{\mathcal{P}^2}}\, \overline{p} \left[1 + \frac{\overline{p}^2}{3\left(1+2u\eta_a^2\right)^2}\right]. \tag{3.332}$$

Note, we have set $\kappa = 0$, as it was legitimate.

One can wonder about the usefulness of such a decomposition. Normally in such situation, we have $\delta\left(u\right)$ and a soft part can be added to virtual correction under the same integrand, since for $u=0$ the dipole momentum $\tilde{p}_a$ becomes the usual p_a (i.e. the skewed PS becomes a usual one). Here however, the skewed phase space is generated from $\tilde{p}_a\left(u_-\right) \neq p_a$. Therefore the considered end-point contribution cannot be added to virtual part under the same PS. One can extend the support of the distribution (3.327) in order to include $u=0$ point. But this is not necessary. Consider a situation, where all the integrations are made by MC. The above endpoint integral is binned into different bins than the one for virtual corrections, since their PS do not coincide. However, when $\eta^2 \to 0$ the bins start to overlap and the cancellation can occur.

3.7.2.3 $g \to gg$ splitting

Finally, let us consider pure gluonic splitting, with a possible massive spectator. For convenience, let us recall the dipole splitting function introduced in Section 3.5.1.3

$$\left(\hat{V}^{\text{FE-IS}}_{g\to gg,\,a}\right)^{\mu\nu} = 16\pi\mu_r^{2\varepsilon}\alpha_s C_A \left[-g^{\mu\nu}\left(\frac{1}{1-\tilde{z}+\tilde{u}} + \frac{1}{\tilde{z}+\tilde{u}} - 2\right) + 2\left(1-\varepsilon\right)\frac{\mathcal{C}^{\mu\nu}}{\mathcal{P}^2}\right], \tag{3.333}$$

where $\mathcal{C}^{\mu\nu}$ is given for instance in (3.309). The masses are $m_i = m_j = m_{\underline{ij}} = 0$ and $m_a \neq 0$ in general.

In complete analogy to Section 3.7.2.2 we can prove that the contribution to the integral of the dipole splitting matrix comes from its average

$$I^{\text{FE-IS}}_{g\to gg,\,a} = \int d\phi^{\text{FE-IS}}_{g\to gg,\,a}\,\frac{1}{\mathcal{P}^2}\left\langle \hat{V}^{\text{FE-IS}}_{g\to gg,\,a}\right\rangle, \tag{3.334}$$

where this time

$$d\phi^{\text{FE-IS}}_{g\to gg,\,a} = \frac{2\left(2\tilde{\gamma}\right)^{1-\varepsilon}}{\left(4\pi\right)^{2-\varepsilon}\Gamma\left(1-\varepsilon\right)} \left(u\tilde{\eta}^2_{\mathcal{P}^2}\right)^{-\varepsilon}\eta^2_{\mathcal{J}}\left(1+2u\eta_a^2\right)^{1-2\varepsilon}\left[\left(\tilde{z}_+-\tilde{z}\right)\left(\tilde{z}-\tilde{z}_-\right)\right]^{-\varepsilon} d\tilde{z}. \tag{3.335}$$

The average itself is easily calculated in the same manner as in Section 3.7.2.2 and reads

$$\left\langle V^{\text{FE-IS}}_{g\to gg,\,a}\right\rangle = 16\pi\mu_{\text{r}}^{2\varepsilon}\alpha_s C_A\left[\frac{1}{1-\tilde{z}+\tilde{u}} + \frac{1}{\tilde{z}+\tilde{u}} - 2 + \left(\tilde{z}_+-\tilde{z}\right)\left(\tilde{z}-\tilde{z}_-\right)\right]. \tag{3.336}$$

Here

$$\tilde{z}_\pm = \frac{1\pm v}{2}, \tag{3.337}$$

while the rest of the quantities, like $\eta^2_{\mathcal{J}}$, $\tilde{\eta}^2_{\mathcal{P}^2}$ etc., are given in Appendix A.2.1.2.

In complete analogy to previous cases we define the scaled integral $\tilde{I}^{\text{FE-IS}}_{g\to gg,\,a}$, see (3.271) for definition. Using integrals from Appendix B.1 we get

$$\tilde{I}^{\text{FE-IS}}_{g\to gg,\,a} = 2C_A \frac{\eta^2_{\mathcal{J}}}{\tilde{\eta}^{2(1-\kappa)}_{\mathcal{P}^2}} \frac{1}{u^{1-\kappa}} \left(1+2u\eta_a^2\right) \left[\mathcal{F}\left(\mathcal{A}_1\left(u\right);\kappa\right) - \mathcal{F}\left(\mathcal{A}_2\left(u\right),\kappa\right) + B\left(1+\kappa\right)v\left(\frac{1+\kappa}{2\left(3+2\kappa\right)}v^2-2\right)\right], \quad (3.338)$$

where

$$v = \frac{1}{1+2u\eta_a^2}, \quad (3.339)$$

$$\mathcal{A}_1\left(u\right) = \frac{1}{u}\frac{1}{1+\left(1+2u\right)\eta_a^2}, \quad (3.340)$$

$$\mathcal{A}_2\left(u\right) = \frac{-1}{1+u\left(1+\left(1+2u\right)\eta_a^2\right)}. \quad (3.341)$$

The difference of $\mathcal{F}$ functions in (3.338) can be further simplified, due to the relation

$$\mathcal{F}\left(\mathcal{A}_1;\kappa\right) = -\mathcal{F}\left(\mathcal{A}_2,\kappa\right). \quad (3.342)$$

following the Pfaff transformation[3] (see Appendix B.1) and the relation

$$\frac{\mathcal{A}_1}{1+\mathcal{A}_1} = -\mathcal{A}_2. \quad (3.343)$$

Now, we have to disentangle the soft singularity that appears due to the fact, that in the present case $u_- = 0$. This time however - apart from the singular factor $1/u$ - there is also a singularity hidden in function $\mathcal{F}$, due to the behaviour of $\mathcal{A}_1$ as a function of u; we shall see this more precisely in a moment. Therefore we expect a double soft/collinear pole.

Since the expansion in κ of the function $\mathcal{F}\left(\mathcal{A}_1;\kappa\right)$ is not analytic at $u=0$, we have to put it inside the "plus" distribution, i.e. we make the following replacement

$$\frac{1}{u^{1-\kappa}}\mathcal{F}\left(\mathcal{A}_1\left(u\right);\kappa\right) = \left(\frac{\mathcal{F}\left(\mathcal{A}_1\left(u\right);0\right)}{u}\right)_{[0,u_+]} + \delta\left(u\right)\int_0^{u_+} dy\, \frac{1}{y^{1-\kappa}}\mathcal{F}\left(\mathcal{A}_1\left(y\right);\kappa\right) + \mathcal{O}\left(\kappa\right). \quad (3.344)$$

The integration above is rather non-trivial when $m_a \neq 0$, because $\mathcal{A}_1$ is a nontrivial function of u.

In order to simplify the procedure, let us define the special variable, which plays an analogous role as u,

$$\mathfrak{r} = \frac{1}{\mathcal{A}_1} = u\,\left(1+\left(1+2u\right)\eta_a^2\right). \quad (3.345)$$

Thus u can be replaced by

$$u = \mathfrak{r}\,\mathcal{B}\left(\mathfrak{r}\right), \quad (3.346)$$

[3] This can be also deduced already from (3.336) by simply noting the symmetry of the first two integrals.

where

$$\mathcal{B}(\mathfrak{r}) = \frac{2}{1+\eta_a^2+\sqrt{1+\eta_a^2\left[\eta_a^2+2\left(1+4\mathfrak{r}\right)\right]}}. \tag{3.347}$$

Notice that

$$\mathcal{B}(\mathfrak{r}) = 1 \quad \text{for } m_a = 0 \tag{3.348}$$

and

$$\mathcal{B}(0) = \frac{1}{1+\eta_a^2}. \tag{3.349}$$

Let us now define the "plus" distribution using this new variable

$$\frac{1}{\mathfrak{r}^{1-\kappa}}\mathcal{F}\left(\frac{1}{\mathfrak{r}};\kappa\right) = \left[\frac{\mathcal{F}\left(\frac{1}{\mathfrak{r}};0\right)}{\mathfrak{r}}\right]_+ + \delta(\mathfrak{r})\int_0^1 dy\,\frac{1}{y^{1-\kappa}}\mathcal{F}\left(\frac{1}{y};\kappa\right) + \mathcal{O}(\kappa). \tag{3.350}$$

Note, we defined the usual "plus" distribution, i.e. the support is $[0,1]$. This accounts for simpler evaluation of the integral – it is calculated up to order $\mathcal{O}(\kappa)$ in Appendix B.1 as Integral M. In the end we get

$$\frac{1}{\mathfrak{r}^{1-\kappa}}\mathcal{F}\left(\frac{1}{\mathfrak{r}};\kappa\right) = \frac{1}{\mathfrak{r}}\log(1+\mathfrak{r}) + \left[\frac{1}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right]_+ + \delta(\mathfrak{r})\left(\frac{1}{2\kappa^2}-\frac{\pi^2}{4}\right) + \mathcal{O}(\kappa), \tag{3.351}$$

where we used the relation

$$\left[\frac{1}{\mathfrak{r}}\log(1+\mathfrak{r})\right]_+ = \frac{1}{\mathfrak{r}}\log(1+\mathfrak{r}) - \delta(\mathfrak{r})\frac{\pi^2}{12} \tag{3.352}$$

following Appendix B.3.

To be consistent, we also use the new variable $\mathfrak{r}$ for the rest of the terms in (3.338). We can easily transform the end-point delta function $\delta(\mathfrak{r})$ into $\delta(u)$. More attention has to be paid to practical integration of the "plus" distributions in the new variable $\mathfrak{r}$. A useful formula is given in Appendix B.3.

Let us now collect the singular and non-singular pieces and decompose the integral similarly as before

$$\tilde{I}^{\text{FE-IS}}_{g\to gg,\,a} = J(u) + \delta(u)\left[J_{\text{finite}} + J_{\text{pole}}\right], \tag{3.353}$$

For the subsequent parts we obtain

$$J_{\text{pole}} = 2C_A\left[\frac{1}{\kappa^2}-\frac{11}{6\kappa}-\frac{1}{\kappa}\log\left(1+\eta_a^2\right)\right], \tag{3.354}$$

$$J_{\text{finite}} = 2C_A\left\{\frac{67}{18}-\frac{\pi}{2}+\left[\frac{1}{2}\log\left(1+\eta_a^2\right)+\frac{11}{6}\right]\log\left(1+\eta_a^2\right)\right\}, \tag{3.355}$$

$$J(u) = 2C_A\frac{\eta_{\mathcal{J}}^2}{\mathcal{B}(\mathfrak{r})\,\tilde{\eta}_{\mathcal{P}^2}^2}\left(1+2\mathfrak{r}\mathcal{B}(\mathfrak{r})\,\eta_a^2\right) \left[\frac{2}{\mathfrak{r}}\log(1+\mathfrak{r}) + \left(\frac{2}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right)_+ + \frac{v}{6}\left(v^2-12\right)\left(\frac{1}{\mathfrak{r}}\right)_+\right], \tag{3.356}$$

where the functional dependence $\mathfrak{r}(u)$ was skipped.

It can be easily checked, that for $m_a = 0$ we arrive at the known result from [9]. We turn attention, that we have an additional pole with a logarithm containing the spectator mass comparing to [9]. This is however perfectly right, as for every FE-IS case, there is a corresponding IE-FS case. The last can have similar singularities to those in (3.354) since we allow for massive quark there.

3.7.3 Initial State Emitter - Final State Spectator

3.7.3.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

Now we switch to the IE-FS case. Let us start with $\mathbf{Q} \to \mathbf{Q}g$ splitting. Note, here we distinguish this case from $\mathbf{Q} \to g\mathbf{Q}$. In the former a gluon has momentum p_i and is radiated out, while in the last it enters a reduced matrix element. In FE-IS both cases were treated simultaneously.

Let us start with the precise definition of the dipole integral. We define

$$I^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = \int d\phi^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j}\, \frac{-1}{p_{ai}^2 - m^2} \left\langle \hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} \right\rangle, \tag{3.357}$$

where the dipole splitting matrix was defined in (3.196). The whole procedure is quite analogous to FE-IS case. The fact that we can use the averaged dipole splitting matrix is due to its diagonality. The propagator in (3.357) reads

$$p_{ai}^2 - m^2 = -2\tilde{z}\mathcal{P}_a. \tag{3.358}$$

The measure $d\phi$ reads here explicitly

$$d\phi^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = \frac{2\,(2\tilde{\gamma})^{1-\varepsilon}}{(4\pi)^{2-\varepsilon}\,\Gamma(1-\varepsilon)}\, \frac{\eta_{\mathcal{J}}^2\left(\eta_{\mathcal{P}^2}^2\right)^{-\varepsilon}}{\tilde{v}_j^{1-2\varepsilon}} \left(1+2u\eta^2\right)^{1-2\varepsilon} \left[(\tilde{z}_+ - \tilde{z})(\tilde{z}-\tilde{z}_-)\right]^{-\varepsilon} d\tilde{z}, \tag{3.359}$$

where

$$\tilde{z}_\pm = \frac{u\tilde{\eta}^2_{\mathcal{P}^2-m_j^2}\left(1+2u\eta^2 \pm \tilde{v}_j\right)}{2\eta^2_{\mathcal{P}^2}\left(1+2u\eta^2\right)}. \tag{3.360}$$

We delegate the explicit forms of $\eta_{\mathcal{J}}^2$, $\eta_{\mathcal{P}^2}^2$ etc. to the Appendix A.2.2.1.

Performing the integration over $d\tilde{z}$ (using Appendix B.1) we obtain for the reduced integral $\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j}$ defined as usual in (3.271)

$$\begin{aligned}
\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = -\frac{C_F\,\eta_{\mathcal{J}}^2\left(\tilde{\eta}^2_{\mathcal{P}^2-m_j^2}\right)^{2\kappa}\left(1+2u\eta^2\right)}{u^{1-2\kappa}\left(\eta^2_{\mathcal{P}^2}\right)^{\kappa}\tilde{v}_j^3\eta^2_{\mathcal{P}_a}} &\Bigg\{\mathcal{F}\left(\mathcal{A}_1(u);\kappa\right) \\
&+\frac{1}{2}\left[u\left(2-u(1+\kappa)\right)\tilde{v}_j^2 - 2\right]\mathcal{F}\left(\mathcal{A}_2(u);\kappa\right) \\
&+\frac{\eta^2\eta^2_{\mathcal{P}^2}\tilde{v}_j^2(1-u)\left(1+2u\eta^2\right)}{\tilde{\eta}^2_{\mathcal{P}^2-m_j^2}\eta^2_{\mathcal{P}_a}\left(1-\tilde{v}_j+2u\eta^2\right)}\mathcal{G}\left(\mathcal{A}_2(u);\kappa\right)\Bigg\},
\end{aligned} \tag{3.361}$$

where

$$\mathcal{A}_1(u) = \frac{2u\tilde{v}_j\tilde{\eta}^2_{\mathcal{P}^2-m_j^2}}{\eta^2_{\mathcal{P}^2}\left[2u\left(\eta^2+\tilde{v}_j^2\left(1+2u\eta^2\right)\right)+1-\tilde{v}_j\right]-\eta_j^2\left(1+2u\eta^2-\tilde{v}_j\right)}, \tag{3.362}$$

$$\mathcal{A}_2(u) = \frac{2\tilde{v}_j}{1+2u\eta^2-\tilde{v}_j}. \tag{3.363}$$

Recall, that the functions $\mathcal{F}$, $\mathcal{G}$ are defined in Appendix B.1.

Since we consider the emission of a gluon, we expect the soft singularity. Indeed there is $1/u$ term in (3.361), while the minimal value of u variable is $u_- = 0$. Therefore, we disentangle the singularity by means of the plus distribution (3.287). To this end define the regular (in the limit $u \to 0$ and $\eta^2 \neq 0$) quantity $K(u;\kappa)$ as

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = \frac{1}{u^{1-2\kappa}} K(u;\kappa) . \tag{3.364}$$

We obtain

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = \left(\frac{1}{u}\right)_{[0,u_+]} K(u;0) + \delta(u)\left(\frac{1}{2\kappa} + \log u_+\right) K(0;\kappa) , \tag{3.365}$$

where

$$\begin{aligned} K(u;0) = & -\frac{C_F \eta^2_{\mathcal{J}} \left(1+2u\eta^2\right)}{2\tilde{v}^3_j \eta^4_{\mathcal{P}_a} \tilde{\eta}^2_{\mathcal{P}^2-m^2_j}} \\ & \Big\{\eta^2_{\mathcal{P}_a} \tilde{\eta}^2_{\mathcal{P}^2-m^2_j} \left[2\log\left(1+\mathcal{A}_1(u)\right) - \log\left(1+\mathcal{A}_2(u)\right)\left(2+u\left(u-2\right)\tilde{v}^2_j\right)\right] \\ & + \frac{4\tilde{v}^3_j \eta^2 \eta^2_{\mathcal{P}^2}\left(1-u\right)\left(1+2u\eta^2\right)}{\left(1+2u\eta^2+\tilde{v}_j\right)\left(1+2u\eta^2-\tilde{v}_j\right)}\Big\}, \end{aligned} \tag{3.366}$$

$$\begin{aligned} K(0;\kappa) = -2C_F \left(\eta^2_j\right)^{-\kappa} \tilde{v}^{4\kappa-1}_j \Big[& \mathcal{F}\left(\mathcal{A}_1(0);\kappa\right) - \mathcal{F}\left(\mathcal{A}_2(0);\kappa\right) \\ & + \frac{1}{2}\left(1+\tilde{v}_j\right) \mathcal{G}\left(\mathcal{A}_2(0);\kappa\right)\Big]. \end{aligned} \tag{3.367}$$

When deriving (3.367) we used

$$\eta^2_{\mathcal{P}^2}\Big|_{u=0} = \eta^2_j, \quad \eta^2_{\mathcal{P}_a}\Big|_{u=0} = \frac{1}{2}, \tag{3.368}$$

$$\eta^2_{\mathcal{P}^2-m^2_j}\Big|_{u=0} = \tilde{v}^2_j, \quad \eta^2_{\mathcal{J}}\Big|_{u=0} = \tilde{v}^2_j. \tag{3.369}$$

Moreover

$$\mathcal{A}_1(0) = \frac{2\tilde{v}_j}{1+2\eta^2_j - \tilde{v}_j} \equiv \mathcal{A}_1, \tag{3.370}$$

$$\mathcal{A}_2(0) = \frac{2\tilde{v}_j}{1-\tilde{v}_j} \equiv \mathcal{A}_2. \tag{3.371}$$

Let us note interesting identity, which holds if $\eta^2_j = \eta^2_a$

$$\frac{1+\mathcal{A}_1}{1+\mathcal{A}_2} = \frac{1}{1+\mathcal{A}_1}. \tag{3.372}$$

We decompose the result into three parts

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j} = J(u) + \delta(u)\left(J_{\text{finite}} + J_{\text{pole}}\right). \tag{3.373}$$

For the subsequent pieces we get

$$J_{\text{pole}} = -C_F \frac{1}{\kappa}\left(1 + \frac{1}{\tilde{v}_j}\log\frac{1+\mathcal{A}_1}{1+\mathcal{A}_2}\right), \tag{3.374}$$

$$J_{\text{finite}} = -C_F \left(1 + \frac{1}{\tilde{v}_j} \log \frac{1+\mathcal{A}_1}{1+\mathcal{A}_2}\right) \log \frac{\tilde{v}_j^4 u_+^2}{\eta_j^2} - C_F \frac{1}{\tilde{v}_j} \left(\mathcal{F}_1 \left(\mathcal{A}_1\right) - \mathcal{F}_1 \left(\mathcal{A}_2\right) - \log \left(1+\mathcal{A}_2\right)\right), \quad (3.375)$$

$$J(u) = \left(\frac{1}{u}\right)_{[0,u_+]} K(u;0). \quad (3.376)$$

There are two comments in order here. First, there is no standard collinear pole of the form $\frac{1}{\kappa} P_{QQ}$ in our result. However, analysing Eq. (3.366) in the quasi-collinear limit, we find that there is a term of the form $\log \eta^2 P_{QQ}$ which plays analogous role. Therefore, our dipole function in the present form are not infrared safe. Later, in Chapter 4 we shall fix this using methods described in Section 2.4.

Next, our result (3.373)-(3.376) is not well suited for massless spectator case, i.e. for $m_j = 0$. This is seen e.g. by looking at $K(u;0)$ which should be finite for $u = 0$. However, if $m_j = 0$ we have

$$\log \left(1 + \mathcal{A}_2 (u)\right) = -\log \left(u\eta^2\right) + \mathcal{O}\left(\eta^2\right). \quad (3.377)$$

In order to assure the smooth limit, we could have kept track of factors $\eta_j^{2\kappa}$ and do not expand them, which however is not easy in the fully massive case. Another solution, is simply to have distinct formulae, with m_j set to zero literally from the very beginning. This is reasonable, since if we dealt with completely massless spectator (like gluon or quarks u, d, s) we should have put the factors $\eta_j^{2\kappa}$ to zero manually. On the other hand, when we deal with heavy quark spectator, we nevertheless have to expand those factors manually.

Therefore, in the following we derive the corresponding formulae with a spectator assumed to be massless in the beginning.

Let us thus start with (3.361) with $\eta_j^2 = 0$. Now it becomes

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g,\,j}\Big|_{\eta_j^2=0} = -\frac{C_F\, \eta_{\mathcal{J}}^2 \left(\tilde{\eta}_{\mathcal{P}2}^2\right)^{\kappa} \left(1+2u\eta^2\right)}{u^{1-\kappa}\eta_{\mathcal{P}_a}^2} \Bigg\{ \mathcal{F}\left(\mathcal{A}_1^*(u);\kappa\right) + \frac{1}{2}\left[u\left(2-u\left(1+\kappa\right)\right)-2\right] \mathcal{F}\left(\mathcal{A}_2^*(u);\kappa\right) + \frac{(1-u)\left(1+2u\eta^2\right)}{2\eta_{\mathcal{P}_a}^2} \mathcal{G}\left(\mathcal{A}_2^*(u);\kappa\right) \Bigg\}, \quad (3.378)$$

where

$$\mathcal{A}_1^*(u) = \mathcal{A}_1(u)\big|_{\eta_j^2=0} = \frac{1}{u}\frac{1}{1+(1+2u)\eta^2}, \quad (3.379)$$

$$\mathcal{A}_2^*(u) = \mathcal{A}_2(u)\big|_{\eta_j^2=0} = \frac{1}{u\eta^2}. \quad (3.380)$$

We see, that the functions $\mathcal{A}_1^*(u)$, $\mathcal{A}_2^*(u)$ do not behave well in the $u \to 0$ limit and we have to include $\mathcal{F}$, $\mathcal{G}$ functions in the "plus" distribution.

First split (3.378) into three distinct parts

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g}\Big|_{\eta_j^2=0} = K_1 + K_2 + K_3, \quad (3.381)$$

where K_1 , K_2, K_3 are the first, the second and the third term in (3.378) correspondingly.

Let us start with K_1. We follow similar steps of Section 3.7.2.3 for FE-IS case. We define the variable

$$\mathfrak{r} = \frac{1}{\mathcal{A}_1^*(u)} = u\left(1 + (1+2u)\,\eta^2\right). \tag{3.382}$$

Note it is the same as in Section (3.345) if we replace $\eta_a^2 \longleftrightarrow \eta^2$. Thus we can immediately use the other results from that section, in particular, we have

$$\frac{1}{u^{1-\kappa}}\,\mathcal{F}\left(\mathcal{A}_1^*(u)\,;\kappa\right) = \frac{\mathcal{B}^{\kappa-1}(\mathfrak{r})}{\mathfrak{r}^{1-\kappa}}\,\mathcal{F}\left(\frac{1}{\mathfrak{r}};\kappa\right), \tag{3.383}$$

where $\mathcal{B}(\mathfrak{r})$ is given in (3.347) with replacement $\eta_a^2 \longleftrightarrow \eta^2$ and

$$\frac{1}{\mathfrak{r}^{1-\kappa}}\mathcal{F}\left(\frac{1}{\mathfrak{r}};\kappa\right) = \frac{1}{\mathfrak{r}}\log(1+\mathfrak{r}) + \left[\frac{1}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right]_+ + \delta(\mathfrak{r})\left(\frac{1}{2\kappa^2} - \frac{\pi^2}{4}\right) + \mathcal{O}(\kappa). \tag{3.384}$$

Thus we get

$$\begin{aligned} K_1 = &-\frac{C_F\,\eta_{\mathcal{J}}^2\left(1+2\mathfrak{r}\mathcal{B}(\mathfrak{r})\,\eta^2\right)}{\eta_{\mathcal{P}_a}^2\mathcal{B}(\mathfrak{r})}\left[\frac{1}{\mathfrak{r}}\log(1+\mathfrak{r}) + \left[\frac{1}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right]_+\right] \\ &-\delta(\mathfrak{r})\,C_F\,\left(1+\eta^2\right)\left[\frac{1}{\kappa^2} - \frac{1}{\kappa}\log\left(1+\eta^2\right) - \frac{1}{2}\left(\pi^2 - \log^2\left(1+\eta^2\right)\right)\right]. \end{aligned} \tag{3.385}$$

Next consider K_2. Here the rescaling of the soft variable u is trivial, namely we define

$$\mathfrak{s} = \frac{1}{\mathcal{A}_2^*(u)} = u\eta^2. \tag{3.386}$$

We can thus obtain much of the results by replacing $\mathcal{B}(\mathfrak{r})$ above by $1/\eta^2$. We obtain

$$\begin{aligned} K_2 = &-\frac{C_F\,\eta_{\mathcal{J}}^2\,(1+2\mathfrak{s})\,\eta^2}{\eta_{\mathcal{P}_a}^2}\,\frac{1}{2}\left[\frac{\mathfrak{s}}{\eta^2}\left(2-\frac{\mathfrak{s}}{\eta^2}\right) - 2\right]\left[\frac{1}{\mathfrak{s}}\log(1+\mathfrak{s}) + \left[\frac{1}{\mathfrak{s}}\log\frac{1}{\mathfrak{s}}\right]_+\right] \\ &+\delta(\mathfrak{s})\,C_F\,\eta^2\left[\frac{1}{\kappa^2} - \frac{1}{\kappa}\log\eta^2 - \frac{1}{2}\left(\pi^2 - \log^2\eta^2\right)\right]. \end{aligned} \tag{3.387}$$

Finally, we calculate K_3. We include $\mathcal{G}$ in the "plus" distribution. Introducing the variable $\mathfrak{s}$ as above, we make the replacement

$$\frac{1}{\mathfrak{s}^{1-\kappa}}\mathcal{G}\left(\frac{1}{\mathfrak{s}};\kappa\right) = \left[\frac{1}{\mathfrak{s}^{1-\kappa}}\mathcal{G}\left(\frac{1}{\mathfrak{s}};\kappa\right)\right]_+ + \delta(\mathfrak{r})\,\mathcal{J}_2(\kappa), \tag{3.388}$$

where the integral $\mathcal{J}_2(\kappa)$ is calculated up to the order $\mathcal{O}(\kappa)$ in Appendix B.1 as Integral N. Explicitly we get

$$\begin{aligned} \frac{1}{\mathfrak{s}^{1-\kappa}}\mathcal{G}\left(\frac{1}{\mathfrak{s}};\kappa\right) &= \left(\frac{1}{\mathfrak{s}}\,\frac{1}{1+\mathfrak{s}}\right)_+ + \delta(\mathfrak{s})\left(\frac{1}{2\kappa} - \log 2\right) + \mathcal{O}(\kappa) \\ &= \left(\frac{1}{\mathfrak{s}}\right)_+\frac{1}{1+\mathfrak{s}} + \delta(\mathfrak{s})\,\frac{1}{2\kappa} + \mathcal{O}(\kappa), \end{aligned} \tag{3.389}$$

where we have used the relation (B.74). Gathering all the pieces we obtain

$$K_3 = -\frac{C_F\,\eta_{\mathcal{J}}^2\,(1+2\mathfrak{s})^2}{2\eta_{\mathcal{P}_a}^4}\,\frac{\eta^2-\mathfrak{s}}{1+\mathfrak{s}}\left(\frac{1}{\mathfrak{s}}\right)_+ - C_F\,\delta(\mathfrak{s})\,\eta^2\left(\frac{1}{\kappa}-\log\eta^2\right). \tag{3.390}$$

Let us now collect the full integral. We decompose it as in (3.373). We however adorn the quantities by a star to underline that they are evaluated for the massless spectator. We get

$$J^*_{\text{pole}} = -C_F\frac{1}{\kappa}\left(1+\log\eta^2-\log\left(1+\eta^2\right)\right), \tag{3.391}$$

$$J^*_{\text{finite}} = -\frac{1}{2}C_F\left[\log^2\left(1+\eta^2\right)-\log\eta^2\left(\log\eta^2+2\right)\right], \tag{3.392}$$

$$\begin{aligned} J^*(u) = -\frac{C_F\,\eta_{\mathcal{J}}^2}{\eta_{\mathcal{P}_a}^2}\Bigg\{&\frac{1+2\mathfrak{r}\mathcal{B}(\mathfrak{r})\,\eta^2}{\mathcal{B}(\mathfrak{r})}\left[\frac{1}{\mathfrak{r}}\log(1+\mathfrak{r})+\left[\frac{1}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right]_+\right] \\ &+\frac{1}{2}(1+2\mathfrak{s})\,\eta^2\left[\frac{\mathfrak{s}}{\eta^2}\left(2-\frac{\mathfrak{s}}{\eta^2}\right)-2\right]\left[\frac{1}{\mathfrak{s}}\log(1+\mathfrak{s})+\left[\frac{1}{\mathfrak{s}}\log\frac{1}{\mathfrak{s}}\right]_+\right] \\ &-\frac{\left(\mathfrak{s}-\eta^2\right)(1+2\mathfrak{s})^2}{2\eta_{\mathcal{P}_a}^2\,(1+\mathfrak{s})}\left(\frac{1}{\mathfrak{s}}\right)_+\Bigg\}. \end{aligned} \tag{3.393}$$

We skip functional dependence on u for $\mathfrak{s}$ and $\mathfrak{r}$ in the above equation.

Later, in Section 4.4 we will need the above formulae in the limit $\eta \to 0$. Then, the "plus" distributions in $\mathfrak{r}$ variable become straightforwardly the distributions in u due to (3.382). This is not the case for "plus" distributions in $\mathfrak{s}$ variable. The relation is the following: for the distribution originating in $\mathcal{F}$ function

$$\begin{aligned} &\eta^2\left[\frac{1}{\mathfrak{s}}\log\frac{1+\mathfrak{s}}{\mathfrak{s}}\right]_+ = \eta^2\frac{1}{\mathfrak{s}}\log(1+\mathfrak{s})+\eta^2\left[\frac{1}{\mathfrak{s}}\log\frac{1}{\mathfrak{s}}\right]_+-\delta(u)\frac{\pi^2}{12} \\ &= \left[\frac{1}{u}\log\frac{1+u\eta^2}{u\eta^2}\right]_{[0,u_+]}-\delta(u)\left[\text{Li}_2\left(-u_+\eta^2\right)+\frac{1}{2}\log^2\left(u_+\eta^2\right)+\frac{\pi^2}{12}\right]+\mathcal{O}(\kappa), \end{aligned} \tag{3.394}$$

and for the distribution originating in $\mathcal{G}$ function

$$\begin{aligned} \eta^2\left[\frac{1}{\mathfrak{s}}\frac{1}{1+\mathfrak{s}}\right]_+ &= \eta^2\left(\frac{1}{\mathfrak{s}}\right)_+\frac{1}{1+\mathfrak{s}}+\delta(u)\log 2 \\ &= \left[\frac{1}{u}\frac{1}{1+u\eta^2}\right]_{[0,u_+]}-\delta(u)\log\frac{1+u_+\eta^2}{2u_+\eta^2}+\mathcal{O}(\kappa). \end{aligned} \tag{3.395}$$

It is interesting to note, that in the limit $\eta^2 \to 0$ the above relations effectively leads to the opposite sign in front of $\log\eta^2\left(\log\eta^2+2\right)$ in Eq. (3.392).

3.7.3.2 $g \to \mathbf{Q}\overline{\mathbf{Q}}$ splitting

The integral of the dipole splitting function (3.199) is defined as

$$I^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j} = \int d\phi^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j}\,\frac{-1}{p_{ai}^2-m^2}\left\langle\hat{V}^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j}\right\rangle. \tag{3.396}$$

The averaged dipole splitting function reads (averaging is trivial, since the dipole matrix is diagonal, see (3.199))

$$\left\langle \hat{V}^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j} \right\rangle = 8\pi\mu_r^{2\varepsilon}\alpha_s\, T_R \left[1 - \frac{2\left(1-\tilde{u}\right)}{1-\varepsilon}\left(\tilde{u} + \frac{m^2}{p_{ai}^2 - m^2}\right)\right]. \tag{3.397}$$

Here, the one particle subspace reads

$$d\phi^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} = \frac{2\left(2\tilde{\gamma}\right)^{1-\varepsilon}}{\left(4\pi\right)^{2-\varepsilon}\Gamma\left(1-\varepsilon\right)}\left(\eta^2_{\mathcal{P}^2}\right)^{-\varepsilon}\eta^2_{\mathcal{J}}\left[\left(\tilde{z}_+ - \tilde{z}\right)\left(\tilde{z} - \tilde{z}_-\right)\right]^{-\varepsilon} d\tilde{z}. \tag{3.398}$$

The scaled jacobian $\eta^2_{\mathcal{J}}$ and other scaled quantities are explicitly listed in Appendix A.2.2.2. Note that

$$v = 1 \tag{3.399}$$

in (3.398) (since the mass of the initial state $m_a = 0$), and consequently the bounds on $\tilde{z}$ are

$$\tilde{z}_\pm = \frac{\eta^2 - \eta_j^2}{2\eta^2_{\mathcal{P}^2}} + \frac{1}{2}\left(1 \pm 2\overline{p}\right). \tag{3.400}$$

The propagator in (3.396) is the same as in (3.358). Evaluating the integrals over $d\tilde{z}$ we obtain for the reduced integral

$$\tilde{I}^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j} = \frac{T_R}{1+\kappa}\,\frac{\eta^2_{\mathcal{J}}\left(\eta^2_{\mathcal{P}^2}\right)^{\kappa}\left(2\overline{p}\right)^{2\kappa}}{2\left(\eta^2_{\mathcal{P}_a}\right)^2}\left\{\eta^2_{\mathcal{P}_a}\left(1 - 2u\left(1-u\right) + \kappa\right)\mathcal{F}\left(\mathcal{A}\left(u\right);\kappa\right)\right.$$
$$\left. + \frac{2\eta^2\eta^2_{\mathcal{P}^2}\left(1-u\right)}{\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\left(1-2\overline{p}\right)}\mathcal{G}\left(\mathcal{A}\left(u\right);\kappa\right)\right\}, \tag{3.401}$$

where

$$\mathcal{A}\left(u\right) = \frac{4\eta^2_{\mathcal{P}^2}\overline{p}}{\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\left(1-2\overline{p}\right)}. \tag{3.402}$$

Since the integral (3.401) is finite (for finite mass m), we can safely set $\kappa = 0$. Then we get

$$\tilde{I}^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}},\,j} = \frac{-T_R\,\eta^2_{\mathcal{J}}}{2\left(\eta^2_{\mathcal{P}_a}\right)^2}\left\{-\eta^2_{\mathcal{P}_a}\left(1-2u\left(1-u\right)\right)\log\left(1+\mathcal{A}\left(u\right)\right)\right.$$
$$\left. - \frac{8\eta^2\left(\eta^2_{\mathcal{P}^2}\right)^2\overline{p}\left(1-u\right)}{\left[\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\left(1-2\overline{p}\right)\right]\left[\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\left(1+2\overline{p}\right)\right]}\right\}. \tag{3.403}$$

In Section 4.4, we will check that the leading behaviour is of the form $\log\eta^2\, P_{gq}$, when the mass m becomes negligible comparing to other scales. Actually it is already apparent in the above formula. Such an analysis will be starting point towards factorization of the mass singularity.

As an another preparatory step, let us note, that our unintegrated dipole splitting function (3.199) is exactly the same as the one in [10], when we set $m = 0$. Some caution however must be paid, since our dipole splitting function already includes spin conversion factor (we defined quasi-collinear limit with reduced matrix element already averaged over spins). Since in Chapter 4 we shall need our result for $m = 0$ set in the beginning, according to what we stated above, we will not have to calculate it as it can be just taken from [10].

3.7.3.3 $\mathbf{Q} \to g\mathbf{Q}$ and $\overline{\mathbf{Q}} \to g\overline{\mathbf{Q}}$ splittings

In complete analogy to the splittings $g \to \mathbf{Q}\overline{\mathbf{Q}}$ and $g \to gg$ in the FE-IS case, it can be shown, that the contribution of the integrated dipole splitting matrix defined in Section 3.5.2.3 reduces to the average over the helicities in D dimensions

$$I^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} = \int d\phi^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j}\, \frac{-1}{p_{ai}^2} \left\langle \hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} \right\rangle . \tag{3.404}$$

Similarly as before we introduce $\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j}$, see (3.271) for definition.

Let us recall the dipole splitting matrix for convenience (note, that we inserted the spin conversion factor in front)

$$\left(\hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j}\right)^{\mu\nu} = 8\pi\mu_r^{2\varepsilon}\alpha_s C_F\,(1-\varepsilon)\left[-g^{\mu\nu}\,(1-\tilde{u}) - \frac{4}{1-\tilde{u}}\,\frac{\mathcal{C}^{\mu\nu}}{p_{ai}^2}\right], \tag{3.405}$$

with $\mathcal{C}^{\mu\nu} = \mathcal{V}^\mu\mathcal{V}^\nu$ and

$$\mathcal{V}^\mu = (1-\tilde{z})\,p_i^\mu - \tilde{z}p_j^\mu - \frac{(\tilde{w}-1)\left[m^2 - m_j^2 + \mathcal{P}^2\,(1-2\tilde{z})\right]}{2\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\,\mathcal{P}^\mu. \tag{3.406}$$

The average over helicities is performed using the following polarization tensor

$$d^{\mu\nu}\left(\tilde{p}_{\underline{ai}};p_a\right) = -g^{\mu\nu} + \frac{\tilde{p}^\mu_{\underline{ai}}p^\nu_a + \tilde{p}^\nu_{\underline{ai}}p^\mu_a}{\tilde{p}_{\underline{ai}}\cdot p_a} - m_a^2\,\frac{\tilde{p}^\mu_{\underline{ai}}\tilde{p}^\nu_{\underline{ai}}}{\left(\tilde{p}_{\underline{ai}}\cdot p_a\right)^2}, \tag{3.407}$$

corresponding to the gluon with momentum $\tilde{p}_{\underline{ai}}$. As an auxiliary vector we choose p_a, however in practice only the metric tensor gives contribution. The average reads

$$\langle\mathcal{C}^{\mu\nu}\rangle = \frac{1}{D-2}\left[\frac{(1-\tilde{u})\,\mathcal{P}_a}{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\right]^2 \mathcal{P}^2\,(\tilde{z}_+ - \tilde{z})\,(\tilde{z}-\tilde{z}_-)\,, \tag{3.408}$$

where the denominator reads

$$\tilde{p}_{\underline{ai}}\cdot\mathcal{P} = \mathcal{P}^2\,(\tilde{w}-1) - \mathcal{P}_a\,(\tilde{u}-1)\,. \tag{3.409}$$

Using this, we get for the averaged dipole splitting matrix

$$\left\langle \hat{V}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j}\right\rangle = 8\pi\mu_r^{2\varepsilon}\alpha_s C_F\,(1-\varepsilon)\left[1-\tilde{u} - \frac{2}{1-\varepsilon}\,\frac{1-\tilde{u}}{p_{ai}^2}\left(\frac{\mathcal{P}_a}{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\right)^2 \mathcal{P}^2\,(\tilde{z}_+ - \tilde{z})\,(\tilde{z}-\tilde{z}_-)\right]. \tag{3.410}$$

Again, it is instructive to check whether in quasi-collinear limit (3.410) gives averaged splitting matrix

$$\left\langle \hat{P}_{\mathbf{Q}g}\,(x;\kappa)\right\rangle = C_F\,(1-\varepsilon)\left[x + \frac{2}{1-\varepsilon}\left(\frac{1-x}{x} + \frac{xm^2}{p_{ai}^2}\right)\right], \tag{3.411}$$

where x is the Sudakov variable defined e.g. in (3.156). To this end, note that in the quasi-collinear limit (3.29) we have

$$\langle\mathcal{C}^{\mu\nu}\rangle = \frac{1}{D-2}\,\lambda^2\left[\tilde{z}\mathcal{P}^2 - m^2\left(1+\left(\frac{\mathcal{P}^2}{2\mathcal{P}_a}\right)^2\right)\right], \tag{3.412}$$

where we have taken into account the fact that $\tilde{z} = \mathcal{O}\left(\lambda^2\right)$, more precisely

$$\tilde{z} = \lambda^2 \frac{m^2\left(1+(1-x)^2\right) - k_T^2}{2\,(1-x)\,\mathcal{P}_a}. \tag{3.413}$$

Recalling also that

$$p_{ai}^2 = \lambda^2\, 2\left(m^2 - \tilde{z}\mathcal{P}_a\right) = \lambda^2 \frac{k_T^2 - x^2 m^2}{1-x} \tag{3.414}$$

and $\mathcal{P}^2 = 2\,(1-x)\,\mathcal{P}_a + \mathcal{O}\left(\lambda^2\right)$ we indeed recover (3.411).

Now, let us perform the integration over the subspace $d\phi$. For pertinent mass configuration it reads

$$d\phi^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} = \frac{2\,(2\tilde{\gamma})^{1-\varepsilon}}{(4\pi)^{2-\varepsilon}\,\Gamma\,(1-\varepsilon)} \left(\eta^2_{\mathcal{P}^2}\right)^{-\varepsilon} \eta^2_{\mathcal{J}} v^{2\varepsilon-1} \left[(\tilde{z}_+ - \tilde{z})\,(\tilde{z} - \tilde{z}_-)\right]^{-\varepsilon} d\tilde{z}. \tag{3.415}$$

The scaled quantities like $\eta^2_{\mathcal{J}}$ etc. are explicitly given in the Appendix A.2.2.3.

Using the results from Appendix B.1 for the integrals over $d\tilde{z}$, we obtain

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} = (1+\kappa)\, C_F \frac{(1-u)\,(2\overline{p})^{2\kappa}\left(\eta^2_{\mathcal{P}^2}\right)^{\kappa} \eta^2_{\mathcal{J}}}{2v\eta^2_{\mathcal{P}_a}} \left[\mathcal{F}\left(\mathcal{A}\left(u\right);\kappa\right) + \frac{2v\eta^2_{\mathcal{P}^2}\eta^2_{\mathcal{P}_a}\overline{p}}{(1+\kappa)\left(\eta^2_{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\right)^2}\mathcal{H}\left(\mathcal{A}\left(u\right);\kappa\right)\right], \tag{3.416}$$

where the function $\mathcal{H}\left(\mathcal{A};\kappa\right)$ reads (see Integral L. in Appendix B.1)

$$\mathcal{H}\left(\mathcal{A};\kappa\right) = \mathcal{A}^2\, B\left(\kappa+2\right) F\left(2,\kappa+2,2\left(\kappa+2\right);-\mathcal{A}\right). \tag{3.417}$$

Moreover

$$\eta^2_{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}} = \eta^2_{\mathcal{P}_a}\left(1-u\right) + \eta_{\mathcal{P}^2}\left(w-1\right), \tag{3.418}$$

$$\mathcal{A}\left(u\right) = \frac{4v\overline{p}\eta^2_{\mathcal{P}^2}\eta^2_{\mathcal{P}_a}}{\eta^2_{\mathcal{P}_a}\left(\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\left(1-2v\overline{p}\right)\right) - 2\eta^2\eta^2_{\mathcal{P}^2}}, \tag{3.419}$$

where $\overline{p}$ is defined in (3.217) and is not simplified at all for the considered case, thus we do not replace it by its explicit form.

Inspection of (3.416) leads to the conclusion, that there are no singularities that should be regularized. The only possible singular behaviour comes from collinear logarithms. Therefore we are allowed to set $\kappa = 0$. Then (3.416) reduces to

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j} = -C_F \frac{(1-u)\,\eta^2_{\mathcal{J}}}{2v\eta^2_{\mathcal{P}_a}} \left[\frac{2v\eta^2_{\mathcal{P}^2}\eta^2_{\mathcal{P}_a}\overline{p}}{\left(\eta^2_{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\right)^2} - \log\left(1+\mathcal{A}\left(u\right)\right)\left(1 + \frac{\eta^2_{\mathcal{P}_a}\left(\eta^2 - \eta_j^2 + \eta^2_{\mathcal{P}^2}\right) - 2\eta^2\eta^2_{\mathcal{P}^2}}{\left(\eta^2_{\tilde{p}_{\underline{ai}}\cdot\mathcal{P}}\right)^2}\right)\right]. \tag{3.420}$$

Let us note, that the integrated dipole function has smooth behaviour for vanishing spectator mass $m_j \to 0$. In Chapter 4 we shall investigate the behaviour of (3.420) in the limit $m \to 0$. We shall find, that its leading behaviour is of the required form $\log \eta^2 \, P_{Qg}$, which is not obvious at this stage.

In the end of this section, let us give appropriate expression, for a case when initial state mass is set to zero from the very beginning. We shall need it in Chapter 4. Note, that our unintegrated dipole splitting function is now different than the corresponding one in [10], thus we have to make a separate calculation. This calculation is straightforward and much simpler than in the massive case, thus we skip all the details. We obtain

$$\tilde{I}^{\text{IE-FS}}_{\mathbf{Q}\to g\mathbf{Q},\,j}\Big|_{\eta^2=0} = (1+\kappa)\,(u+\eta_j)^{\kappa}\left[\left(\frac{1}{\kappa} + 2\log\frac{u}{u+\eta_j^2}\right) P_{qg}\,(1-u) - C_F\,\frac{4u}{1-u}\right]$$
$$= \left(\frac{1}{\kappa} + \log\frac{u^2}{u+\eta_j^2} + 1\right) P_{qg}\,(1-u) - C_F\,\frac{4u}{1-u}. \qquad (3.421)$$

3.7.4 Complete expressions for integrated dipoles

Let us now come back to the full expressions for dipoles, i.e. (3.59)-(3.61). We shall now introduce some useful notation, that will be helpful in Chapter 4. In order to better understand the motivation for this section, the reader might look at Section 4.3.1 in advance.

Let us recall, that when calculating the jet cross section via dipole subtraction method, say n-jet cross section with one initial state, we add and subtract the dipole contributions, which live in the $(n+1)$-particle phase space. Let us write this contributions as follows

$$\mathfrak{D}_n\,(p_a) = \sum_{\Pi(n+1|a)} \frac{1}{s_{\Pi(n+1|a)}} \int d\Phi_{n+1}\left(p_a; \{p_j\}_{j=1}^{n+1}\right) \mathcal{D}_{F_n}\left(p_a; \{p_i\}_{i=1}^{n+1}\right). \qquad (3.422)$$

Here the first summation is over all the configurations $\Pi\,(n+1|a)$ of $n+1$ partons in the final state, with parton a in the initial state. $s_{\Pi(n+1|a)}$ is a symmetry factor for identical particles in the final state for configuration $\Pi\,(n+1|a)$. To be precise, in order to define a real subtraction term as in Section 1.2 it should be convoluted with parton densities and equipped in some normalization factors. Moreover in (3.422) we used

$$\mathcal{D}_{F_n}\left(p_a; \{p_i\}_{i=1}^{n+1}\right) = \sum_{i=1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n+1}\Bigg\{\mathcal{D}^{\text{IE-FS}}_{i,j,a}\left(\tilde{p}_{\underline{ai}}; \{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right) F_n\left(\tilde{p}_{\underline{ai}}; \{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right)$$
$$+ \mathcal{D}^{\text{FE-IS}}_{i,j,a}\left(\tilde{p}_a; \{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right) F_n\left(\tilde{p}_a; \{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right)$$
$$+ \sum_{\substack{k=1\\ k\neq i,j}}^{n+1} \mathcal{D}^{\text{FE-FS}}_{i,j,k}\left(p_a; \{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right) F_n\left(p_a; \{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right)\Bigg\}. \qquad (3.423)$$

There are three distinct n-particle jet functions F_n, since for each type of dipole we have to generate a different set of momenta (the sets $\mathbb{X}$ are defined in (3.65)-(3.67)). The integration in (3.422) is over the full phase space – all the cuts (step functions) and Dirac deltas (for differential cross section) are hidden in F_n. In what follows, we shall use the more compact notation, namely we assume, that F_n and any amplitudes that appear have the same arguments as the phase space $d\Phi$. Hence we shall not write the arguments explicitly.

On the other hand, we can factorize the phase space in (3.422) and integrate the dipoles in order to obtain analytical poles, which in turn should cancel with virtual corrections. Then we get[4]

$$\mathfrak{D}_n(p_a) = -\sum_{\Pi(n+1|a)} \frac{1}{s_{\Pi(n+1|a)}} \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \Bigg\{ \int du\, I^{\text{IE-FS}}_{a\to \underline{ai}\, i,\, j}(u)\; d\mathfrak{S}_{(n)\,\underline{ai},j}\left(\tilde{p}_a(u);\{p_l\}_{l\in\mathbb{X}_{\text{IE-FS}}}\right)$$
$$+\int du\, I^{\text{FE-IS}}_{\underline{ij}\to i\, j,\, a}(u)\; d\mathfrak{S}_{(n)\,a,\underline{ij}}\left(\tilde{p}_a(u);\{p_l\}_{l\in\mathbb{X}_{\text{FE-IS}}}\right) + \sum_{\substack{k=1 \\ k\neq i,j}}^{n+1} I^{\text{FE-FS}}_{\underline{ij}\to i\, j,\, k}\, d\mathfrak{S}_{(n)\,k,\underline{ij}}\left(p_a;\{p_l\}_{l\in\mathbb{X}_{\text{FE-FS}}}\right)\Bigg\}, \tag{3.424}$$

where "pseudo" cross sections $d\mathfrak{S}$ are defined as

$$d\mathfrak{S}_{(n)\,I,J}(p;\{p_l\}) = \int d\Phi_n(p;\{p_l\})\; F_n\, \left|\overline{\mathcal{M}}_n\right|^2_{I,J}, \tag{3.425}$$

Recall, that the matrix elements squared with subscripts are correlated in colour space; they were defined in (3.21). We note, that there are also FE-FS contributions in (3.424), which were not given in the present work (they have to be taken from [10]). The phase space factorization in that case is a product of a measure $d\phi$ and the skewed phase space, i.e. there is no convolution over u.

It is possible to convert the sum in (3.424) over pairs (i,j) into a simpler sum (note also, that parton i appears only in the integrated dipole splitting functions as a process index). This procedure is described in details in [9] (sections 7.2 and 8.1), thus we only apply this method to our formula. To this end it is convenient to split (3.424) into three distinct terms

$$\mathfrak{D}_n(p_a) = \mathfrak{D}_n^{\text{IE-FS}}(p_a) + \mathfrak{D}_n^{\text{FE-IS}}(p_a) + \mathfrak{D}_n^{\text{FE-FS}}(p_a), \tag{3.426}$$

which correspond to the three contributions in (3.424). Then we obtain for IE-FS

$$\mathfrak{D}_n^{\text{IE-FS}}(p_a) = -\sum_{b\in\mathbb{N}'_f} \sum_{\Pi(n|b)} \frac{1}{s_{\Pi(n|b)}} \sum_{j=1}^{n} \int du\, I^{\text{IE-FS}}_{a,b,j}(u)\; d\mathfrak{S}_{(n)\,b,j}\left(\tilde{p}_b(u);\{p_l\}_{l=1}^n\right), \tag{3.427}$$

where we have changed the name of parton $\underline{ai}$ to b in order to completely remove appearance of parton i which in fact is not needed any more. We recall, that N'_f is the total number of flavours (including gluon). Moreover we changed the notation

$$I^{\text{IE-FS}}_{a,b,j} \equiv I^{\text{IE-FS}}_{a\to b\, i,\, j} \tag{3.428}$$

for the same purpose. Now, turn to FE-IS case:

$$\mathfrak{D}_n^{\text{FE-IS}}(p_a) = -\sum_{\Pi(n|a)} \frac{1}{s_{\Pi(n|a)}} \sum_{j=1}^{n} \int du\, I^{\text{FE-IS}}_{j,\,a}(u)\; d\mathfrak{S}_{(n)\,j,a}\left(\tilde{p}_a(u);\{p_l\}_{l=1}^n\right), \tag{3.429}$$

[4] For IE-FS and FE-IS contributions there is also a factor $1/x$, which we used to have outside the dipole splitting functions. Here we hide it for transparency.

where this time we converted $\underline{ij}$ to j and introduced

$$I_{j,\,a}^{\text{FE-IS}} = \begin{cases} I_{q\to qg,\,a}^{\text{FE-IS}} & j = q, \overline{q} \\ \frac{1}{2} I_{g\to gg,\,a}^{\text{FE-IS}} + \sum_{l\in\mathbb{N}_f} I_{g\to q_l\overline{q}_l,\,a}^{\text{FE-IS}} & j = g. \end{cases} \tag{3.430}$$

Finally, for FE-FS we have

$$\mathfrak{D}_n^{\text{FE-FS}}(p_a) = -\sum_{\Pi(n|a)} \frac{1}{s_{\Pi(n|a)}} \sum_{j=1}^{n} \sum_{\substack{k=1\\k\neq j}}^{n} I_{j,\,a}^{\text{FE-FS}}\, d\mathfrak{S}_{(n)\,k,j}\left(p_a; \{p_l\}_{l=1}^n\right), \tag{3.431}$$

with

$$I_{j,\,k}^{\text{FE-FS}} = \begin{cases} I_{q\to qg,\,k}^{\text{FE-FS}} & j = q, \overline{q} \\ \frac{1}{2} I_{g\to gg,\,k}^{\text{FE-FS}} + \sum_{l\in\mathbb{N}_f} I_{g\to q_l\overline{q}_l,\,k}^{\text{FE-FS}} & j = g. \end{cases} \tag{3.432}$$

The symbol N_f (without prime) is a number of all quark flavours (without a gluon) either light or heavy.

Now, we could insert the results for integrated dipole splitting functions to get explicit form for the kernels I, as in [9, 10]. Instead, we shall only write the general form, as we will need it later in Chapter 4.

First let us note, that we can write

$$\mathfrak{D}_n^{\text{IE-FS}}(p_a) + \mathfrak{D}_n^{\text{FE-IS}}(p_a) = \\ -\sum_{b\in\mathbb{N}_f'} \sum_{\Pi(n|b)} \frac{1}{s_{\Pi(n|b)}} \sum_{j=1}^{n} \int du\; \mathfrak{J}_{a,b,j}(u)\; d\mathfrak{S}_{(n)\,b,j}\left(\tilde{p}_b(u); \{p_l\}_{l=1}^n\right), \tag{3.433}$$

where

$$\mathfrak{J}_{a,b,j}(u) = -\left[I_{a,b,j}^{\text{IE-FS}}(u) + \delta_{ab} I_{j,\,a}^{\text{FE-IS}}(u)\right]. \tag{3.434}$$

Next, as we have seen earlier in this chapter, we have to disentangle end-point contributions, thus the above can be written as

$$\mathfrak{J}_{a,b,j}(u) = \mathfrak{J}_{a,b,j}^{\text{coll}}(u) + \mathfrak{J}_{a,b,j}^{\text{remn}}(u) + \left(\mathfrak{J}_{a,b,j}^{\text{pole}} + \mathfrak{J}_{a,b,j}^{\text{finite}}\right)\delta(u), \tag{3.435}$$

where $\mathfrak{J}^{\text{coll}}$ is a part that leads to collinear singularities, $\mathfrak{J}^{\text{remn}}$, $\mathfrak{J}^{\text{finite}}$ are the finite parts that will remain after cancellation of all singularities and $\mathfrak{J}^{\text{pole}}$ is a part that contains soft poles only. For FE-FS case the structure is similar to the coefficient in front of $\delta(u)$ in (3.435). Therefore, we can write the integrated dipole (3.424) in the following form

$$\mathfrak{D}_n(p_a) = \Bigg\{ \sum_{b\in\mathbb{N}_f'} \sum_{\Pi(n|b)} \frac{1}{s_{\Pi(n|b)}} \sum_{j=1}^{n} \\ \int du\; \left(\mathfrak{J}_{a,b,j}^{\text{coll}}(u) + \mathfrak{J}_{a,b,j}^{\text{remn}}(u)\right) d\mathfrak{S}_{(n)\,b,j}\left(\tilde{p}_b(u); \{p_l\}_{l=1}^n\right) \\ + d\mathfrak{S}_{(n)\,a}^{\text{dip-soft}}\left(p_a; \{p_l\}_{l=1}^n\right) \Bigg\}, \tag{3.436}$$

where we put the following contribution into a new symbol

$$d\mathfrak{S}^{\text{dip-soft}}_{(n)\,a}\left(p_a;\{p_l\}_{l=1}^{n}\right) = \sum_{j=1}^{n}\left[\sum_{b\in\mathbb{N}'_f}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\left(\mathfrak{J}^{\text{pole}}_{a,b,j}+\mathfrak{J}^{\text{finite}}_{a,b,j}\right)d\mathfrak{S}_{(n)\,b,j}\left(p_b;\{p_l\}_{l=1}^{n}\right) + \sum_{\Pi(n|a)}\frac{1}{s_{\Pi(n,a)}}\sum_{\substack{k=1\\k\neq j}}^{n} I^{\text{FE-FS}}_{j,\,a}\, d\mathfrak{S}_{(n)\,k,j}\left(p_a;\{p_l\}_{l=1}^{n}\right)\right]. \quad (3.437)$$

Let us comment this general structure of integrated dipole function. The term $d\mathfrak{S}^{\text{dip-soft}}$ contains all the soft singularities, that should cancel with the virtual contribution. All the collinear (for massless partons) and quasi-collinear singularities are hidden in $\mathfrak{J}^{\text{coll}}$. We shall need this formula in the next chapter, when we construct infra-red safe cross section.

Chapter 4

General Mass Scheme for Jets

4.1 Introduction

In the previous chapter we constructed dipole subtraction method taking into account possible massive initial state splittings. We have seen, that dipoles connected with this kind of splitting can be free from "standard" collinear singularities appearing as the poles. Instead, we encounter terms containing logarithms of a quark mass, which spoils accuracy of the predictions when the mass can be negligible. In this chapter we construct a scheme, which factorizes out those quasi-collinear singularities. It follows from the ACOT scheme outlined for inclusive processes in Section 2.4.

Our further work is divided into several parts. First, in Section 4.2, we derive one-loop parton densities in CWZ renormalization scheme. They were actually derived in [45], however in different scheme, so called dimensional reduction. In consequence, some of the results are different from those used in standard calculations in D dimensions. There are also some other problems connected with calculations in [45], therefore we had to rederive them again. Perhaps those results exist somewhere else in the literature, as they are only one-loop calculations with massive quark, nevertheless it is very instructive to obtain them consistently as they are crucial ingredients of our scheme. Basing on those parton densities, we construct in Section 4.3 the subtraction terms which remove potential collinear singularities from the dipoles. Finally, we demonstrate that in the kinematic limit when the masses can be neglected, we recover correct massless dipole formulae. This is one of the tests of our formalism and is done in Section 4.4.

We turn attention, that there are at least two kinds of subtraction terms in the present work, which should not be confused. First, there is a dipole subtraction term, which removes all the singularities from the tree matrix elements. Second, there is a collinear (or quasi-collinear) subtraction term, which removes potential collinear singularities from the dipole subtraction term.

4.2 Parton densities in CWZ renormalization scheme

The densities of parton inside a parton were defined in Section 1.1. Let us recall, that they are needed in order to calculate IR finite partonic cross section. Since in our approach we have masses in the initial state splittings, we have to have partonic PDFs where masses are not neglected. Moreover, as explained in Section 1.3, it is convenient to choose special renormalization scheme, designed for massive calculation (CWZ).

First we sketch derivation of bare densities at one-loop accuracy. Next, using CWZ counter terms we obtain the renormalized ones. As already mentioned, all the pieces were actually derived in [45]. However, there are several problems with those calculations. First, they are carried in a scheme, called dimensional reduction, where one uses analytic continuation to D dimensions for the integrals, while the tensors are purely four dimensional objects. Clearly, it must lead to different result for the unobservable objects like parton densities. More about dimensional reduction can be found e.g. in [30]. Therefore, we have to trace and recalculate diagrams which leads to contraction of metric tensors. However, there is another problem connected with calculations in [45]. Namely we encounter some misprints or errors, thus also the other diagrams have to be checked.

4.2.1 Unrenormalized parton densities

To one loop accuracy we can write

$$\mathcal{F}_{ab}\left(x\right) = \delta_{ab}\delta\left(1-x\right) + \mathcal{F}_{ab}^{(1)}\left(x\right) \tag{4.1}$$

As already remarked in Section 1.1 those objects can be calculated order by order in perturbation theory by means of Feynman rules. Besides the set of standard QCD rules, we have a few additional objects. We shall not describe them here in details, they can be found for instance in [16, 45, 17]. Instead we shall describe the ingredients of such calculations using the specific example below.

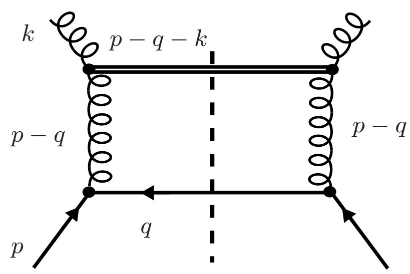


Figure 4.1: Cut Feynman diagram for the bare parton density $\mathcal{F}_{\mathbf{Q}g}$.

Let us thus start with the simplest parton density $\mathcal{F}_{\mathbf{Q}g}$. The one loop contribution has only one corresponding Feynman diagram showed in Fig. 4.1. Let us briefly describe its main elements. The double, so called eikonal line, arises from an expansion of the path exponential in (1.5). In Feynman gauge the gluons can be attached to this line also in the middle (see other PDFs). In what follows it is assumed that we use Feynman gauge as the light-cone gauge is less practical in actual calculations. The most top lines correspond to the operators and carry fixed "plus" component $k^{+} = xp^{+}$ of the initial state momentum p. The last is assumed to be on-shell. Since the eikonal line carries only "plus" component, once we cut it we must include an "on-shell" delta function. The

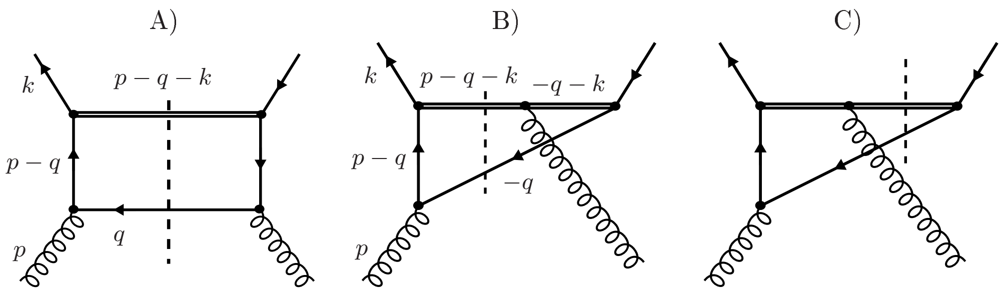


Figure 4.2: Cut Feynman's diagrams for the parton density $\mathcal{F}_{g\mathbf{Q}}$. Actually only A) and B) give contribution (see the main text).

whole expression corresponding to this diagram turns out to be

$$\mathcal{F}^{(1)}_{\mathbf{Q}g} = g^2 \mu_r^{2\varepsilon} C_F \int \frac{d^D q}{(2\pi)^{D-1}} \frac{1}{2} \mathrm{Tr}\left[\gamma^\mu \left(\not{q} + m\right) \gamma^\nu \left(\not{p} + m\right)\right] \delta\left(p^+ - q^+ - k^+\right)$$
$$\frac{\delta\left(q^2 - m^2\right)}{\left((p-q)^2 + i\epsilon\right)^2 \; k\cdot n} \left(k\cdot n\, g^\alpha_\mu - (p-q)^\alpha n_\mu\right)\left(k\cdot n\, g_{\alpha\nu} - (p-q)_\alpha n_\nu\right), \quad (4.2)$$

where we have already summed over spins and colours and where $m_{\mathbf{Q}} \equiv m$. The delta functions set

$$\delta\left(p^+ - q^+ - k^+\right) = \delta\left(p^+ - q^+ - xp^+\right) \quad \Rightarrow \quad q^+ = (1-x)\, p^+, \quad (4.3)$$

$$\delta\left(q^2 - m^2\right) = \delta\left(2q^+ q^- - q_T^2 - m^2\right) = \frac{1}{2p^+ \overline{x}} \delta\left(q^- - \frac{q_T^2 + m^2}{2p^+ \overline{x}}\right). \quad (4.4)$$

The integration measure with our choice of light-cone vectors reads

$$d^D q = d^{D-2} q_T \, dq^+ dq^- . \quad (4.5)$$

Performing the trace and integrating out the delta functions, we get

$$\mathcal{F}^{(1)}_{\mathbf{Q}g} = \frac{\alpha_s \mu_r^{2\varepsilon}\, 4C_F}{(2\pi)^{2(1-\varepsilon)}} \int d^{2(1-\varepsilon)} q_T \left\{ \frac{2m^2 x\,(1-x)}{\left(q_T^2 + x^2 m^2\right)^2} - \frac{2 - x\,(2 - x\,(1-\varepsilon))}{x\,\left(q_T^2 + x^2 m^2\right)} \right\}. \quad (4.6)$$

Using Integral D. from Appendix B.1 we finally obtain

$$\mathcal{F}^{(1)}_{\mathbf{Q}g} = \frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^\varepsilon C_F\, x^{-2\varepsilon-1} \left\{ 2\,(1-x) - \left[2 - x\,(2 - x\,(1-\varepsilon))\right] \frac{1}{\varepsilon} \right\}, \quad (4.7)$$

where we put some standard factors into one quantity

$$S(\varepsilon) = (4\pi)^\varepsilon\, \Gamma(1+\varepsilon). \quad (4.8)$$

Next, let us calculate the density $\mathcal{F}_{g\mathbf{Q}}$. At the one loop accuracy there are three distinct Feynman diagrams (Fig. 4.2). Thus we write

$$\mathcal{F}^{(1)}_{g\mathbf{Q}} = \mathcal{F}^{(A)}_{g\mathbf{Q}} + 2\mathcal{F}^{(B)}_{g\mathbf{Q}} + \mathcal{F}^{(C)}_{g\mathbf{Q}}, \quad (4.9)$$

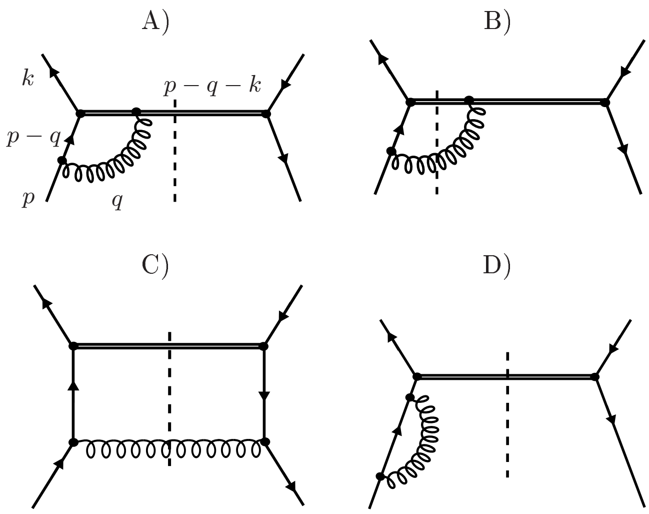


Figure 4.3: Cut Feynman's diagrams for the parton density $\mathcal{F}_{\mathbf{QQ}}$.

where the three terms correspond to the three diagrams (the factor of two in front of the second diagram is the symmetry factor). The calculation is very similar to the previous one, thus we skip the details. The result reads

$$\mathcal{F}_{g\mathbf{Q}}^{(A)} = \frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} T_R \left[\frac{1}{\varepsilon} + \frac{2x(1-x)}{1-\varepsilon}\right], \tag{4.10}$$

$$\mathcal{F}_{g\mathbf{Q}}^{(B)} = -\frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} T_R \frac{x(1-x)}{(1-\varepsilon)\varepsilon}, \tag{4.11}$$

$$\mathcal{F}_{g\mathbf{Q}}^{(C)} = 0.$$

The last diagram is simply zero, because of the kinematic argument. The cut eikonal line give $\delta(-q^+ - k^+)$ what results in $q^+ = -xp^+$, thus q^+ must lie in the past light-cone. On the other hand the anti-quark with momentum q is on-shell (see Fig. 4.2B) and thus we have $\delta_+ \left(q^2 - m^2\right)$. Therefore the common support shrinks to zero volume.

Let us now give the result for $\mathcal{F}_{\mathbf{QQ}}$ showed in Fig. 4.3. As before, we decompose

$$\mathcal{F}_{\mathbf{QQ}}^{(1)} = 2\left[\mathcal{F}_{\mathbf{QQ}}^{(A)} + \mathcal{F}_{\mathbf{QQ}}^{(B)}\right] + \mathcal{F}_{\mathbf{QQ}}^{(C)} + \mathcal{F}_{\mathbf{QQ}}^{(D)}. \tag{4.12}$$

The calculation of diagrams B and C go in a similar manner as before. They read

$$\mathcal{F}_{\mathbf{QQ}}^{(B)} = \frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \frac{1}{\varepsilon} x(1-x)^{-1-2\varepsilon}, \tag{4.13}$$

$$\mathcal{F}_{\mathbf{QQ}}^{(C)} = -\frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \, (1-x)^{-1-2\varepsilon} \left[2x - (1-x)^2 \left(\frac{1}{\varepsilon} - 1\right)\right]. \tag{4.14}$$

In the case of the diagram A we have full integration over the loop momenta q, i.e. the integrals over dq^+ and dq^- are not utilized by the delta functions. The integral over dq^- can however be easily carried out by the residue technique, while the integral over

dq^+ is left out, because it is actually divergent for $\varepsilon > 0$. It is the soft singularity, that should cancel between accompanying graph with real emissions (B and C). The details of this calculation are given correctly in [45]. Following the authors, it is convenient to introduce the dimensionless variable in order to parametrize this integral

$$\xi = 1 - \frac{q^+}{p^+}. \tag{4.15}$$

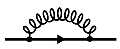
The result reads

$$\mathcal{F}^{(A)}_{\mathbf{QQ}} = -\frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \delta(1-x) \frac{1}{\varepsilon} \int_0^1 d\xi \, \xi \, (1-\xi)^{-1-2\varepsilon}. \tag{4.16}$$

Finally, we calculate the diagram D. The following expression appears as a part of the diagram

$$\frac{1}{2} \sum_s \overline{u}^s(p) \gamma^+ \frac{-i}{\not{p} - m} \Sigma(\not{p}, m) \, u^s(p), \tag{4.17}$$

where $\Sigma(\not{p}, m)$ is a quark self energy defined by



$$= -i\Sigma(\not{p}, m). \tag{4.18}$$

Since the full quark propagator can be written as (we neglect colour indices)

$$S(\not{p}, m) = \frac{i}{\not{p} - m - \Sigma(\not{p}, m)} = \frac{i Z_2^{\mathrm{OS}}}{\not{p} - m_{\mathrm{OS}}} \tag{4.19}$$

we have to one-loop accuracy

$$\Sigma(\not{p}, m_{\mathrm{OS}}) \approx \left(Z_2^{\mathrm{OS}} - 1\right)(\not{p} - m_{\mathrm{OS}}). \tag{4.20}$$

Above Z_2^{OS} is the quark field renormalization constant in the on-shell scheme, while m_{OS} is the pole mass. Inserting this to (4.17) we get simply

$$\mathcal{F}^{(D)}_{\mathbf{QQ}} = \left(Z_2^{\mathrm{OS}} - 1\right)\delta(1-x) = \left(\frac{\partial \Sigma(\not{p}, m_{\mathrm{OS}})}{\partial \not{p}}\right)_{p^2 = m^2_{\mathrm{OS}}} \delta(1-x). \tag{4.21}$$

The result for the quark self-energy or renormalization constant is well known (e.g. [20]). Thus we get

$$\mathcal{F}^{(D)}_{\mathbf{QQ}} = -\frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \delta(1-x) \left[\frac{1}{2\varepsilon} - 2\int_0^1 d\xi \, \xi \, (1-\xi)^{-1-2\varepsilon}\right]. \tag{4.22}$$

We replaced the renormalized mass for the bare one, as this is allowed to this order and simplifies the notation. Let us now rearrange the diagrams in a more logical manner

$$\mathcal{F}^{(A+B)}_{\mathbf{QQ}} = \frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \frac{1}{\varepsilon} \left[x(1-x)^{-1-2\varepsilon}\right]_+, \tag{4.23}$$

$$\mathcal{F}^{(C+D)}_{\mathbf{QQ}} = -\frac{\alpha_s}{2\pi} S(\varepsilon) \left(\frac{\mu_r^2}{m^2}\right)^{\varepsilon} C_F \left\{2\left[x(1-x)^{-1-2\varepsilon}\right]_+ + \left[(1-x)^{1-2\varepsilon}\right]_+ \left(1 - \frac{1}{\varepsilon}\right)\right\}, \tag{4.24}$$

where we have used the standard "plus" distribution.

Let us now finally turn into the density $\mathcal{F}_{gg}$. There are several diagrams, all of them do not involve masses except one (we do not depict them). It is the one with the gluon self energy graph with heavy quark loop. We must note, that it seems that calculation concerning those diagrams in [45] is incorrect. They conclude that only self energy corrections give contribution; it is not true as we know that one loop result has to have $P_{gg}(x)$, while the self energy corrections are proportional only to $\delta(1-x)$. Thus our strategy is the following. We do not calculate the massless loops as they lead to the standard, verified result for gluon-gluon splitting (1.14). We have to add to this result the heavy quark contribution to gluon self energy, which however is also well known. We shall do this in the next section at the level of renormalized PDFs already.

4.2.2 Renormalization of parton densities

Let us recall from Section 1.3, that in CWZ scheme we first decide whether given quark is active, or inactive. Next, for the diagrams with inactive quarks occurring in the loops we perform zero-momentum subtractions, while for the others the $\overline{\mathrm{MS}}$ renormalization. Therefore, some of the quantities calculated in the previous section have to be renormalized twice, in two different schemes.

Let us start, with $\overline{\mathrm{MS}}$ scheme, i.e. we preserve the masses of the quarks but use minimal subtraction (recall, that it is in contrast to $\overline{\mathrm{mMS}}$ prescription defined in Section 1.1, which refers to the scheme most often identified with $\overline{\mathrm{MS}}$, where the masses are set to zero). Renormalization is done simply by expanding our results in ε into Laurent series, with the exception of the factor $S(\varepsilon)$, and subtracting the pole parts. Let us list the results

$$\mathcal{F}_{g\mathbf{Q}}^{\overline{\mathrm{MS}}}(x) = \frac{\alpha_s}{2\pi} S(\varepsilon)\, T_R \log\left(\frac{\mu_r^2}{m^2}\right)(1-2x(1-x)), \tag{4.25}$$

$$\mathcal{F}_{\mathbf{Q}g}^{\overline{\mathrm{MS}}}(x) = \frac{\alpha_s}{2\pi} S(\varepsilon)\, C_F\, \frac{1+(1-x)^2}{x}\left[\log\left(\frac{\mu_r^2}{m^2}\right) - 2\log x - 1\right], \tag{4.26}$$

$$\mathcal{F}_{\mathbf{QQ}}^{\overline{\mathrm{MS}}}(x) = \frac{\alpha_s}{2\pi} S(\varepsilon)\, C_F \left\{\frac{1+x^2}{1-x}\left[\log\left(\frac{\mu_r^2}{m^2}\right) - 2\log(1-x) - 1\right]\right\}_+. \tag{4.27}$$

The massless $\overline{\mathrm{mMS}}$ result for $\mathcal{F}_{gg}$ was given in (1.22) with (1.14). We have to add to this the $\overline{\mathrm{MS}}$ renormalized gluon self-energy graph with heavy quark loop multiplied by tree level $\mathcal{F}_{gg}$ i.e. $\delta(1-x)$. The full result reads

$$\mathcal{F}_{gg}^{\overline{\mathrm{MS}}}(x) = \frac{\alpha_s}{2\pi} S(\varepsilon) \left\{ 2C_A \left[\left(\frac{1}{1-x}\right)_+ + \frac{1-x}{x} - 1 + x(1-x)\right] \right.$$
$$\left. + \delta(1-x)\left(\frac{11}{6}C_A - \frac{2}{3}N_f T_R - \frac{2}{3}T_R \log\frac{\mu_r^2}{m^2}\right)\right\}. \tag{4.28}$$

Now, we assume that the quarks appearing inside the loops are inactive. We do not consider here partonic densities where initial state is a heavy quark, because they are suppressed by $\mathcal{O}\left(\Lambda_{\mathrm{QCD}}^2/m^2\right)$ as discussed in Section 2.4 (they are convoluted with $f_{\mathbf{Q}}$ which is zero to leading twist). In order to renormalize the remaining diagrams by zero-momentum subtraction, we have to evaluate them first off shell and then set $p^2 = 0$. For

$\mathcal{F}_{g\mathbf{Q}}$ it is however trivial, since this diagram is already calculated with the gluon being on-shell. Thus we have

$$\mathcal{F}_{g\mathbf{Q}}^{\mathrm{mom}}\left(x\right)=0, \tag{4.29}$$

where the superscript denotes explicitly usage of zero-momentum subtraction for relevant diagrams. For $\mathcal{F}_{gg}$ the situation is also simple due to the same reason. We are left with

$$\mathcal{F}_{gg}^{\mathrm{mom}}\left(x\right)=\frac{\alpha_s}{2\pi}\,S\left(\varepsilon\right)\left\{2C_A\left[\left(\frac{1}{1-x}\right)_+ +\frac{1-x}{x}-1+x\left(1-x\right)\right]\right.$$
$$\left.+\,\delta\left(1-x\right)\left(\frac{11}{6}C_A-\frac{2}{3}N_f\,T_R\right)\right\}. \tag{4.30}$$

Note, that the usage of the superscript "mom" is conventional; it means that we perform zero-momentum subtractions only to the diagrams with heavy quark loop, but the others are renormalized by $\overline{\mathrm{MS}}$.

Summarizing, we denote above renormalized parton densities in common by $\mathcal{F}_{ab}^{\mathrm{CWZ}}$, where "mom" scheme should be used if a heavy quark is treated as inactive and $\overline{\mathrm{MS}}$ when it is active (leaving the mass finite).

Let us now check, that indeed the evolution equations for PDFs have massless kernels. This is actually completely straightforward. We have to look at the counterterms we have used to renormalize the above partonic densities (we consider only case when $\mathbf{Q}$ is active, otherwise it is power-suppressed). They are massless in $\overline{\mathrm{MS}}$ scheme. On the other hand, the evolution kernels are defined in (1.10). Therefore, already at this stage we see that these kernels are also massless. In order to derive their precise form, we have to go back and read off the coefficients in front of the poles. They turn out to be exactly the splitting functions P_{ab}.

4.3 Quasi-collinear subtraction terms for massive dipoles

Now, as we have the necessary ingredients, we can perform the mass factorization. However before we construct quasi-collinear subtraction terms for the dipoles, let us first define in details the IR safe cross section for DIS, assuming completely massless case. It will enable us to set up all the necessary notation, within the well known framework.

4.3.1 Massless treatment of factorization

Similar to Chapter 1.1, we consider a virtual neutral boson with momentum q interacting with a hadron having momentum P. We assume that the cross section was projected on suitable tensor structure, therefore we omit the vector indices.

Recall, that to NLO accuracy we have for n-jet cross section

$$\sigma_n=\sigma_n^{\mathrm{LO}}+\sigma_n^{\mathrm{NLO}}. \tag{4.31}$$

As we have the dipole formalism, let us now write the subsequent terms in more details comparing to Section 1.2. First, the leading order cross section is

$$\sigma_n^{\mathrm{LO}}\left(P,q;\left\{k_i^{(J)}\right\}_{i=1}^{n}\right)=\mathcal{N}\left(P,q\right)\sum_{a\in\mathbb{N}_f'}\sum_{\Pi(n|a)}\frac{1}{s_{\Pi(n|a)}}$$
$$\int dz\, f_a\left(z\right)\int d\Phi_n\left(p_a\left(z\right);\left\{p_j\right\}_{j=1}^{n}\right)\,F_n\,\left|\overline{\mathcal{M}}_n\right|^2, \quad (4.32)$$

where $k_i^{(J)}$ are the momenta of the reconstructed jets. At LO the jet momenta correspond exactly to the momenta of the final state partons p_j. The factor $\mathcal{N}\left(P,q\right)$ hides all the factors needed to obtain normalized cross section. The momentum p_a depends on the longitudinal momentum fraction z, as denoted explicitly above. For the rest of the notation we refer to Section 3.7.4. In particular we recall our shortcut convention: the function F_n, the amplitudes and dipoles have the same arguments as the phase space they belong.

The NLO contribution is a sum of virtual σ^V and real σ^R corrections. The real contribution is

$$\sigma_n^{R}\left(P,q;\left\{k_i^{(J)}\right\}_{i=1}^{n}\right)=\mathcal{N}\left(P,q\right)\sum_{a\in\mathbb{N}_f'}\sum_{\Pi(n+1|a)}\frac{1}{s_{\Pi(n+1|a)}}$$
$$\int dz\, f_a\left(z\right)\int d\Phi_{n+1}\left(p_a\left(z\right);\left\{p_j\right\}_{j=1}^{n+1}\right)\left\{F_{n+1}\,\left|\overline{\mathcal{M}}_{n+1}\right|^2-\mathcal{D}_{F_n}\right\}. \quad (4.33)$$

The sum over dipoles $\mathcal{D}_{F_n}$ was defined in (3.423).

The virtual contribution reads

$$\sigma_n^{V}\left(P,q;\left\{k_i^{(J)}\right\}_{i=1}^{n}\right)=\mathcal{N}\left(P,q\right)\sum_{a\in\mathbb{N}_f'}$$
$$\int dz\, f_a\left(z\right)\left\{\sum_{\Pi(n|a)}\frac{1}{s_{\Pi(n|a)}}\int d\Phi_n\left(p_a\left(z\right);\left\{p_k\right\}_{k=1}^{n}\right)\overline{\mathcal{M}}_n^{\mathrm{loop}\ 2}\right.$$
$$\left.+\,\mathfrak{D}_n\left(p_a\left(z\right)\right)-\mathfrak{C}_n^{\overline{\mathrm{mMS}}}\left(p_a\left(z\right)\right)\right\} \quad (4.34)$$

In order to explain appearance of the term $\mathfrak{C}_n^{\overline{\mathrm{mMS}}}$, let us recall, that massless integrated dipoles contain collinear poles of the form P_{ab}/κ, which do not cancel with $\overline{\mathcal{M}}_n^{\mathrm{loop}\ 2}$, as the soft poles do. By means of the factorization theorem, those collinear poles have to be removed. To this order of accuracy and in massless $\overline{\mathrm{MS}}$ scheme (denoted here by $\overline{\mathrm{mMS}}$) it is done by means of the following collinear subtraction term (see e.g. [9])

$$\mathfrak{C}_n^{\overline{\mathrm{mMS}}}\left(p_a\right)=-\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\int dx\left[\left(\frac{\mu_r^2}{\mu_f^2}\right)^{-\kappa}\mathcal{F}_{ab}^{\overline{\mathrm{mMS}}}\left(x\right)\right]$$
$$\int d\Phi_n\left(xp_b\left(z\right);\left\{p_k\right\}_{k=1}^{n}\right)\,F_n\,\left|\overline{\mathcal{M}}_n\right|^2 \quad (4.35)$$

where

$$\mathcal{F}_{ab}^{\overline{\mathrm{mMS}}}\left(x\right)=\frac{\alpha_s}{2\pi}\,S\left(\kappa\right)\,\frac{1}{\kappa}P_{ab}\left(x\right) \quad (4.36)$$

is the massless, renormalized parton density (see (1.22) in Section 1.1). We can further use the general form of $\mathfrak{D}_n$ obtained in Section 3.7.4, namely equation (3.436). In considered case this equation can be written as

$$\mathfrak{D}_n\left(p_a\right)=\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\sum_{j=1}^{n}\int dx\,\left[\mathfrak{J}_{a,b,j}^{\overline{\text{mMS}}}\left(1-x\right)+\mathfrak{J}_{a,b,j}^{\text{remn}}\left(1-x\right)\right]\,d\mathfrak{S}_{(n)\,b,j}\left(xp_b;\{p_l\}_{l=1}^{n}\right)+d\mathfrak{S}_{(n)\,a}^{\text{dip-soft}}\left(p_a;\{p_l\}_{l=1}^{n}\right)\quad(4.37)$$

where here we have used $\mathfrak{J}^{\text{coll}}$ in the form

$$\mathfrak{J}_{a,b,j}^{\text{coll}}\left(u\right)\equiv\mathfrak{J}_{a,b,j}^{\overline{\text{mMS}}}\left(u\right)=-\frac{\alpha_s}{2\pi}S\left(\kappa\right)\left(\frac{\mu_r^2}{2p_b\cdot p_j}\right)^{-\kappa}\frac{1}{\kappa}P_{ab}\left(u\right),\quad(4.38)$$

since in that form it appears in massless dipole formalism [9]. Note, we put the common factor $\Gamma\left(1-\kappa\right)\left(4\pi\right)^{-\kappa}$ we encountered in dipole integration into $S\left(\kappa\right)$ and replaced $\tilde{\gamma}$ by the scalar product $p_b\cdot p_j$[1]. Moreover, we replaced u by $1-x$ and $\tilde{p}_a\left(u\right)$ by xp_a, as we work in the massless limit. In order to write the collinear subtraction term $\mathfrak{C}_n^{\overline{\text{mMS}}}$ in similar form to (4.37) we can use the following trick

$$\mathfrak{C}_n^{\overline{\text{mMS}}}\left(p_a\right)=-\frac{\alpha_s}{2\pi}S\left(\kappa\right)\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\int dx\int d\Phi_n\left(xp_b;\{p_k\}_{k=1}^{n}\right)F_n\left\{\langle\overline{\mathcal{M}}_n|\left[\left(\frac{\mu_r^2}{\mu_f^2}\right)^{-\kappa}+\sum_{j=1}^{n}\frac{\hat{T}_j\cdot\hat{T}_b}{\hat{T}_b^2}\left(\frac{\mu_r^2}{2p_b\cdot p_j}\right)^{-\kappa}\right]\frac{1}{\kappa}P_{ab}\left(x\right)|\overline{\mathcal{M}}_n\rangle-\sum_{j=1}^{n}\langle\overline{\mathcal{M}}_n|\frac{\hat{T}_j\cdot\hat{T}_b}{\hat{T}_b^2}\left(\frac{\mu_r^2}{2p_b\cdot p_j}\right)^{-\kappa}\frac{1}{\kappa}P_{ab}\left(x\right)|\overline{\mathcal{M}}_n\rangle\right\},\quad(4.39)$$

that is we added and subtracted the term that is responsible for colour correlations in $d\mathfrak{S}_{(n)\,b,j}$. Due to the colour conservation the square bracket above reduces to

$$\left(\frac{\mu_r^2}{\mu_f^2}\right)^{-\kappa}+\sum_{j=1}^{n}\frac{\hat{T}_j\cdot\hat{T}_b}{\hat{T}_b^2}\left(\frac{\mu_r^2}{2p_b\cdot p_j}\right)^{-\kappa}=-\kappa\sum_{j=1}^{n}\frac{\hat{T}_j\cdot\hat{T}_b}{\hat{T}_b^2}\log\left(\frac{\mu_f^2}{2p_b\cdot p_j}\right)+\mathcal{O}\left(\kappa^2\right)\quad(4.40)$$

and in consequence, we obtain

$$\mathfrak{C}_n^{\overline{\text{mMS}}}\left(p_a\right)=\frac{\alpha_s}{2\pi}S\left(\kappa\right)\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\sum_{j=1}^{n}\int dx\,d\mathfrak{S}_{(n)\,b,j}\left(xp_b;\{p_l\}_{l=1}^{n}\right)\left[\log\left(\frac{\mu_f^2}{2p_b\cdot p_j}\right)-\left(\frac{\mu_r^2}{2p_b\cdot p_j}\right)^{-\kappa}\frac{1}{\kappa}\right]P_{ab}\left(x\right).\quad(4.41)$$

[1]Recall that $\tilde{\gamma}$ was originally defined as $p_a\cdot\tilde{p}_j$ for IE-FS case, however it was the definition relevant to $(n+1)$-particle matrix element; later, in Section 3.7.4, we changed tilded spectator to "normal" final state parton.

We see, that now the pole has exactly the same coefficient as in $\mathfrak{D}_n(p_a)$. Thus we get for the virtual contribution

$$\sigma_n^V\left(P,q;\left\{k_i^{(J)}\right\}_{i=1}^n\right) = \mathcal{N}(P,q)\sum_{a\in\mathbb{N}_f'}$$
$$\int dz\, f_a(z)\left\{\int d\mathfrak{S}_n^{\text{loop}}\left(p_a(z);\{p_l\}_{l=1}^n\right)+\int d\mathfrak{S}_{(n)\,a}^{\text{dip-soft}}\left(p_a(z);\{p_l\}_{l=1}^n\right)\right.$$
$$+\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\sum_{j=1}^n\int dx\,\left[\mathfrak{J}_{a,b,j}^{\text{remn}}(1-x,z)+\mathfrak{J}_{a,b,j}^{\text{fact}}(x,z)\right]$$
$$\left.\int d\mathfrak{S}_{(n)\,b,j}\left(xp_b(z);\{p_l\}_{l=1}^n\right)\right\}, \tag{4.42}$$

where

$$\mathfrak{J}_{a,b,j}^{\text{fact}}(x,z) = -\frac{\alpha_s}{2\pi}\,S(\kappa)\,\log\left(\frac{\mu_f^2}{2p_b(z)\cdot p_j}\right)P_{ab}(x) \tag{4.43}$$

is a remnant of cancellation of the collinear poles. Above in Eq. (4.42) we introduced

$$d\mathfrak{S}_n^{\text{loop}}\left(p_a;\{p_l\}_{l=1}^n\right) = \sum_{\Pi(n|a)}\frac{1}{s_{\Pi(n|a)}}\int d\Phi_n\left(p_a;\{p_k\}_{k=1}^n\right)\,\overline{\mathcal{M}}_n^{\text{loop }2}. \tag{4.44}$$

Note, that the sums over final state configurations and corresponding symmetry factors are contained in the definitions of $d\mathfrak{S}^{\text{loop}}$ and $d\mathfrak{S}^{\text{dip-soft}}$.

In the next subsection, we shall generalize above formulae to the massive case. We will see, that this is actually straightforward, once we have massive dipole kinematics and partonic PDFs with masses taken into account. There are however some subtleties similar to those discussed in Section 2.4.

4.3.2 Fully massive case

Now, basing on (4.35) we are going to construct analogous subtraction term with the massless functions $\mathcal{F}_{ab}^{\overline{\text{mMS}}}$ replaced by the massive ones $\mathcal{F}_{ab}^{\text{CWZ}}$. There is however a complication. Namely, the question is, whether to treat the tree level amplitude $F_n\left|\overline{\mathcal{M}}_n\right|^2$ in (4.35) as massive or not, and consequently what are the bounds on x convolution. This is not restricted by factorization itself as discussed in Section 2.4.

In the massive case, we cannot simply pass the simple fraction of p_a into $\left|\overline{\mathcal{M}}_n\right|^2$ as in (4.35), because partons a and b can have different types (a gluon or quark) and consequently different masses. Of course it is the "plus" component of p_a momentum, that should be actually passed to the reduced matrix element. However, equivalently this can be solved using the kinematics we developed for dipoles, i.e we can write

$$\tilde{\mathfrak{C}}_n^{\text{CWZ}}(p_a) = -\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\int du\,\mathcal{F}_{ab}^{\text{CWZ}}(1-u)$$
$$\int d\Phi_n\left(\tilde{p}_b(u);\{p_k\}_{k=1}^n\right)\,F_n\,\left|\overline{\mathcal{M}}_n\right|^2. \tag{4.45}$$

Now the reduced matrix element is calculated with an on-shell momentum $\tilde{p}_b$ fixed by the fraction $1-u$ and some additional invariants, as we learnt in Section 3.6.4. In the limit of vanishing mass $m_{\mathbf{Q}}$ it becomes

$$\tilde{\mathfrak{C}}_n^{\mathrm{CWZ}}\left(p_a\right) = -\sum_{b\in\mathbb{N}_f'}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}} \int dx\,\mathcal{F}_{ab}^{\mathrm{CWZ}}\left(x\right)\int d\Phi_n\left(xp_b;\{p_k\}_{k=1}^n\right)\,F_n\,\left|\overline{\mathcal{M}}_n\right|^2+\mathcal{O}\left(\eta^2\right) = \mathfrak{C}_n^{\mathrm{CWZ}}\left(p_a\right)+\mathcal{O}\left(\eta^2\right). \quad (4.46)$$

Note, the difference between $\tilde{\mathfrak{C}}_n^{\mathrm{CWZ}}\left(p_a\right)$ and $\mathfrak{C}_n^{\mathrm{CWZ}}\left(p_a\right)$. The first one is calculated with the full mass dependence in kinematics and the matrix element, while in the latter the only dependence on mass $m_{\mathbf{Q}}$ is in $\mathcal{F}_{ab}^{\mathrm{CWZ}}$. Nonetheless, both can be used as a collinear subtraction term, because the only singular dependence on mass is in $\mathcal{F}_{ab}^{\mathrm{CWZ}}$.

This ambiguity can be resolved analogously to the inclusive case discussed in Section 2.4. Namely, we expect, that around the switching point for a given heavy quark, the last can be mostly generated dynamically from lighter flavours. Thus, consistent formalism should lead to subtraction from σ_n^{LO} those contributions which originate in heavy quark-initiated processes. It is most effectively done, when we treat the *initial state quarks* as massless, both in σ_n^{LO} and in $\mathfrak{C}_n$. Only then this cancellation can be complete without introducing artificial scaling variables, as we have seen in Section 2.4. We underline, that it does not mean that we set the masses of heavy quarks to zero everywhere in σ_n^{LO} or $\mathfrak{C}_n$, thus still the diagrams like BGF are the dominant production channels. Moreover, we can have initial state massive quarks in other subprocesses, see below.

In order to clarify the above statements, let us discuss a specific example in more details. Consider the process

$$\gamma H \rightarrow 2\,J_{\mathbf{Q}}, \quad (4.47)$$

where H is a hadron and we denoted by $J_{\mathbf{Q}}$ a jet with possible heavy flavour $\mathbf{Q}$. Note, that actually the jets can be flavourless when there is $\mathbf{Q}\overline{\mathbf{Q}}$ pair, nevertheless they "feel" heavy quarks. In order to simplify the notation, let us assume that there is only one heavy quark and one light quark. According to factorization theorem, the cross section can be calculated as

$$\sigma_{\gamma H\rightarrow 2J_{\mathbf{Q}}} = \sum_{a\in\mathbb{N}_f'} f_a \otimes \hat{\sigma}_{a\gamma\rightarrow 2\,J_{\mathbf{Q}}}, \quad (4.48)$$

where the hat denotes IR safe cross section as usual. PDFs are defined in the composite CWZ scheme, thus for a given value of external scale, we have to specify the active number of flavours N_a. Hence, we have $f_a \equiv f_a^{(N_a)}$, where N_a increases as one crosses switching point. However, below the switching point for a given heavy quark $\mathbf{Q}$, $f_{\mathbf{Q}}=0$ up to the power corrections. Therefore, the summation can go over all the flavours and gluon $\mathbb{N}_f'$.

The relation defining IR safe cross section is

$$\sigma_{a\gamma\rightarrow 2\,J_{\mathbf{Q}}} = \sum_{b\in\mathbb{N}_f'} \mathcal{F}_{ab}^{\mathrm{CWZ}} \otimes \hat{\sigma}_{b\gamma\rightarrow 2\,J_{\mathbf{Q}}}. \quad (4.49)$$

More specifically, we have for a gluon initiated process at NLO (we drop "CWZ" indication in what follows and denote the order in α_s in superscript)

$$\begin{aligned}\sigma^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \sigma^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}} &= \sum_{b\in\mathbb{N}'_f} \left(\mathcal{F}^{(0)}_{gb} + \mathcal{F}^{(1)}_{gb}\right) \otimes \left(\hat{\sigma}^{(1)}_{b\gamma\to 2\,J_{\mathbf{Q}}} + \hat{\sigma}^{(2)}_{b\gamma\to 2\,J_{\mathbf{Q}}}\right) \\ &= \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \hat{\sigma}^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \sum_{b\in\mathbb{N}'_f} \mathcal{F}^{(1)}_{gb} \otimes \hat{\sigma}^{(1)}_{b\gamma\to 2\,J_{\mathbf{Q}}} \\ &= \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \hat{\sigma}^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \mathcal{F}^{(1)}_{gg} \otimes \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \mathcal{F}^{(1)}_{g\mathbf{Q}} \otimes \hat{\sigma}^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}} \end{aligned} \tag{4.50}$$

Here and below $\mathbf{Q} \equiv \left\{\mathbf{Q}, \overline{\mathbf{Q}}\right\}$ for more transparency. Solving this for the "hatted" quantity we get for a gluon initiated process

$$\hat{\sigma}^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}} = \sigma^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{gg} \otimes \sigma^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{g\mathbf{Q}} \otimes \sigma^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}}. \tag{4.51}$$

For a light quark initiated process we have

$$\begin{aligned}\sigma^{(1)}_{q\gamma\to 2\,J_{\mathbf{Q}}} + \sigma^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}} = \hat{\sigma}^{(1)}_{q\gamma\to 2\,J_{\mathbf{Q}}} &+ \hat{\sigma}^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}} \\ &+ \mathcal{F}^{(1)}_{qg} \otimes \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \mathcal{F}^{(1)}_{qq} \otimes \hat{\sigma}^{(1)}_{q\gamma\to 2\,J_{\mathbf{Q}}} \end{aligned} \tag{4.52}$$

and thus

$$\hat{\sigma}^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}} = \sigma^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{qg} \otimes \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{qq} \otimes \hat{\sigma}^{(1)}_{q\gamma\to 2\,J_{\mathbf{Q}}}. \tag{4.53}$$

Similar relation holds for a heavy quark in the initial state

$$\hat{\sigma}^{(2)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}} = \sigma^{(2)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{\mathbf{Q}g} \otimes \hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} - \mathcal{F}^{(1)}_{\mathbf{Q}\mathbf{Q}} \otimes \hat{\sigma}^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}}. \tag{4.54}$$

For the subprocesses contributing to different mechanisms of two jets production see Fig. 4.4.

Summarizing, the cross section on a hadronic target can be written in terms of bare partonic cross sections as follows

$$\begin{aligned}\sigma_{\gamma H\to 2\,J_{\mathbf{Q}}} = f_g \otimes &\left(\sigma^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \sigma^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}}\right) \\ &+ f_q \otimes \left(\sigma^{(1)}_{q\gamma\to 2\,J_{\mathbf{Q}}} + \sigma^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}}\right) + f_{\mathbf{Q}} \otimes \left(\sigma^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}} + \sigma^{(2)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}}\right) \\ &- \left[f_g \otimes \mathcal{F}^{(1)}_{gg} + f_q \otimes \mathcal{F}^{(1)}_{qg} + f_{\mathbf{Q}} \otimes \mathcal{F}^{(1)}_{\mathbf{Q}g}\right] \sigma^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} \\ &- \left[f_g \otimes \mathcal{F}^{(1)}_{g\mathbf{Q}} + f_{\mathbf{Q}} \otimes \mathcal{F}^{(1)}_{\mathbf{Q}\mathbf{Q}}\right] \sigma^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}}. \end{aligned} \tag{4.55}$$

The third and fourth lines contain collinear subtraction terms. Note, that the first two terms in the third line are pure poles and should be canceled by hand with similar poles appearing after integration of massless contributions to dipoles. Then we are left only with finite terms. When the external scale is very large, we have the IR safe cross section by construction,

$$\begin{aligned}\sigma_{\gamma H\to 2\,J_{\mathbf{Q}}} \underset{m_{\mathbf{Q}}\to 0}{\longrightarrow} f_g &\otimes \left(\hat{\sigma}^{(1)}_{g\gamma\to 2\,J_{\mathbf{Q}}} + \hat{\sigma}^{(2)}_{g\gamma\to 2\,J_{\mathbf{Q}}}\right) \\ &+ f_{\mathbf{Q}} \otimes \left(\hat{\sigma}^{(1)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}} + \hat{\sigma}^{(2)}_{\mathbf{Q}\gamma\to 2\,J_{\mathbf{Q}}}\right) + f_q \otimes \hat{\sigma}^{(2)}_{q\gamma\to 2\,J_{\mathbf{Q}}}. \end{aligned} \tag{4.56}$$

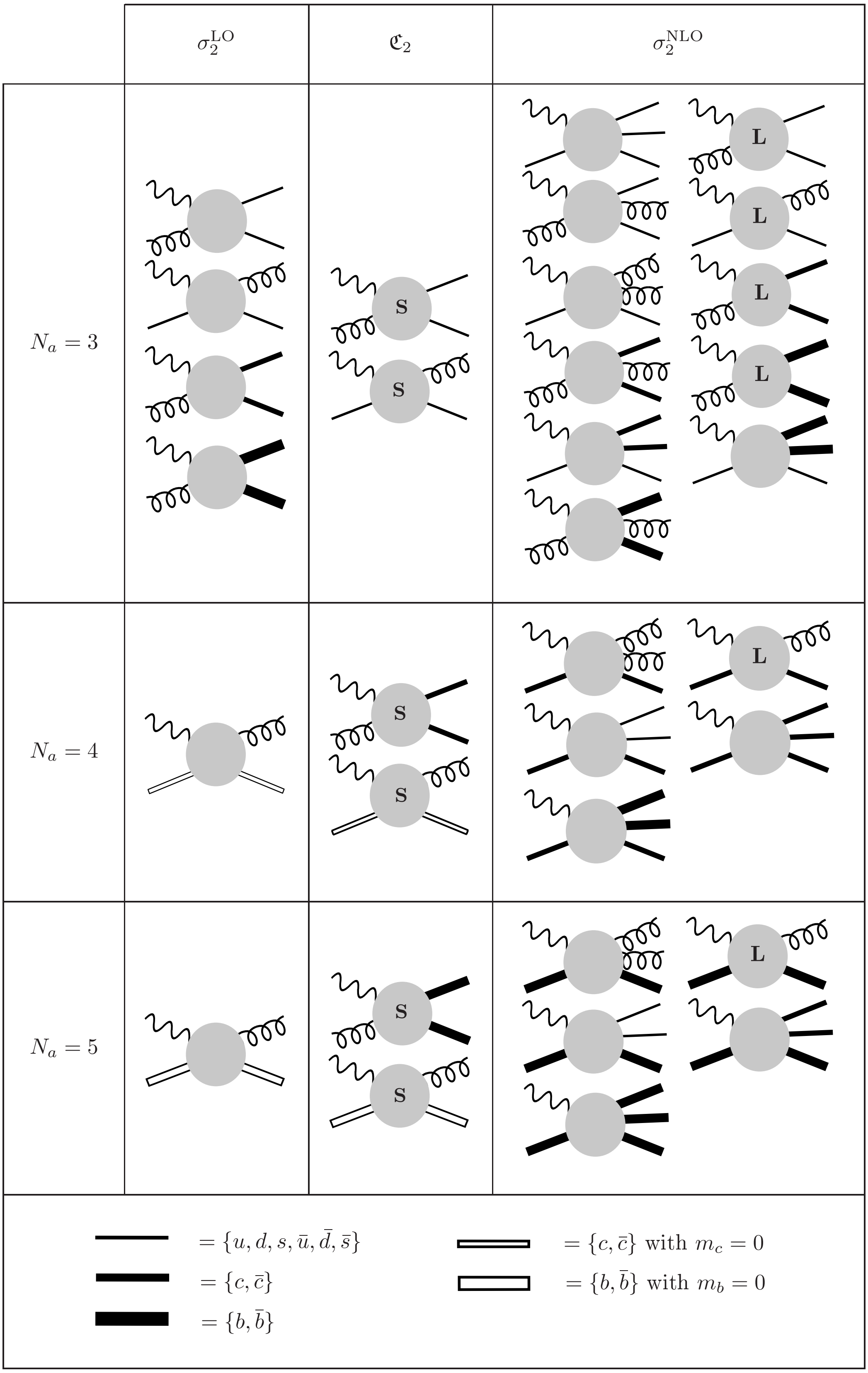


Figure 4.4: The diagram types occurring in a calculation of two-jets cross section with the heavy quark effects. The total number of flavours is $N_f = 6$. Shaded blob denotes tree level amplitude, the blob with **L** stands for loop corrections, while with **S** for a collinear subtraction term. Empty legs denote heavy quarks with masses set to zero, as corresponding diagrams are treated as asymptotic expressions.

Note the difference, now there are IR safe ("hatted") NLO partonic cross sections. On the other hand, slightly above the matching point for a quark $\mathbf{Q}$

$$f_{\mathbf{Q}} \otimes \sigma^{(1)}_{\mathbf{Q}\gamma\to 2\, J_{\mathbf{Q}}} - f_{\mathbf{Q}} \otimes \mathcal{F}^{(1)}_{\mathbf{Q}g} \otimes \sigma^{(1)}_{g\gamma\to 2\, J_{\mathbf{Q}}} - f_{\mathbf{Q}} \otimes \mathcal{F}^{(1)}_{\mathbf{QQ}} \otimes \sigma^{(1)}_{\mathbf{Q}\gamma\to 2\, J_{\mathbf{Q}}} \approx 0. \tag{4.57}$$

As explained above, those are the terms where it is reasonable to set the masses of initial state heavy quarks to zero in $\sigma^{(1)}_{\mathbf{Q}\gamma\to 2\, J_{\mathbf{Q}}}$.

There are two comments in order. First, we stress that setting initial state masses to zero in the above terms is not a limitation of the present method. We are free to keep a complete mass dependence. Second, both prescriptions define actually two different schemes, which should be used to obtain PDFs, as they are also scheme dependent. After this is done, one should actually compare calculations in both schemes. Nevertheless, a scheme just described seems to be more compatible with existing PDFs, which evolve according to massless DGLAP equations (also the bounds in the convolutions are massless).

The above example is also illustrated in a comics form in Fig. 4.4, where we take into account two heavy quarks c and b. We show what processes contribute if the scale crosses the switching points for $N_a = 3, 4, 5$.

4.4 Massless limit and consistency check

Let us now check, that introduced in the previous section subtraction terms indeed factorize the quasi-collinear singularities. This is done by applying the quasi-collinear subtraction terms and taking massless initial state limit in our IE-FS dipoles. As a result we should obtain exactly the massless version of the dipoles (more precisely with massless initial state, a spectator can be massive), with subtracted collinear poles according to $\overline{\mathrm{MS}}$ scheme. That is, we require

$$\lim_{m\to 0}\left[\mathfrak{D}^{\text{IE-FS}}_n(p_a) - \tilde{\mathfrak{C}}^{\text{CWZ}}_n(p_a)\right] = \mathfrak{D}^{\text{IE-FS}}_n(p_a)\Big|_{m=0} - \mathfrak{C}^{\overline{\text{mMS}}}_n(p_a)\,. \tag{4.58}$$

We note, that this equation is required to hold after the soft singularities were cancelled. Actually, it is enough to check (4.58) for integrated dipole splitting function only. To see this, let us recall, that

$$\mathfrak{D}^{\text{IE-FS}}_n(p_a) = -\sum_{b\in\mathbb{N}'_f}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\sum_{j=1}^{n}\int du\, I^{\text{IE-FS}}_{a,b,j}(u)\; d\mathfrak{S}_{(n)\,b,j}\left(\tilde{p}_b(u); \{p_l\}_{l=1}^n\right). \tag{4.59}$$

On the other hand, we can rewrite the collinear subtraction term in any scheme as

$$\mathfrak{C}_n(p_a) = -\sum_{b\in\mathbb{N}'_f}\sum_{\Pi(n|b)}\frac{1}{s_{\Pi(n|b)}}\sum_{j=1}^{n}\int du\, \mathcal{F}_{ab}(1-u)\; d\mathfrak{S}_{(n)\,b,j}\left(\tilde{p}_b(u)\right), \tag{4.60}$$

where we used the colour conservation in order to write $\mathfrak{C}_n$ in similar form to $\mathfrak{D}_n$. This is possible, since the kernel $\mathcal{F}_{ab}$ does not depend on j. Now, we can cast (4.58) into

$$\lim_{m\to 0}\left[I^{\text{IE-FS}}_{a,b,j}(u) - \mathcal{F}^{\text{CWZ}}_{ab}(1-u)\right] = I^{\text{IE-FS}}_{a,b,j}(u)\Big|_{m=0} - \mathcal{F}^{\overline{\text{mMS}}}_{ab}(1-u)\,. \tag{4.61}$$

Let us thus start with exploring the massless limit of our integrated dipole function corresponding to the initial state $g \to \mathbf{Q}\overline{\mathbf{Q}}$ splitting, i.e. we are interested in formula (3.403). First, we have to find the leading behaviour of the collinear logarithm

$\log\left(1+\mathcal{A}\left(u\right)\right)$, with

$$\mathcal{A}\left(u\right)=\frac{4\eta_{\mathcal{P}^2}^2\overline{p}}{\eta^2-\eta_j^2+\eta_{\mathcal{P}^2}^2\left(1-2\overline{p}\right)} \tag{4.62}$$

when the mass of the initial state goes to zero. To this end, we first obtain the following expansions

$$\eta_{\mathcal{P}^2}^2=\frac{\left(\eta_j^2+u\right)\left(2\eta^2+u-1\right)}{u-1}+\mathcal{O}\left(\eta^4\right), \tag{4.63}$$

$$\overline{p}=\frac{2\eta^2\eta_j^2+u\left(u-1\right)\left(u-\eta^2\right)}{2u\left(u-1\right)\left(u+\eta_j^2\right)}+\mathcal{O}\left(\eta^2\right), \tag{4.64}$$

with the help of

$$w=1+\frac{\eta^2}{1-u}+\mathcal{O}\left(\eta^4\right). \tag{4.65}$$

Consequently, we get

$$\log\left(1+\mathcal{A}\left(u\right)\right)=\log\frac{u^2}{u+\eta_j^2}-\log\eta^2+\mathcal{O}\left(\eta^2\right). \tag{4.66}$$

Taking the massless limit with the rest of the expression (3.403) we find the following answer

$$I_{g,\mathbf{Q},j}^{\text{IE-FS}}\left(u\right)=\frac{\alpha_s}{2\pi}\left[P_{gq}\left(u\right)\left(\log\frac{u^2}{u+\eta_j^2}-\log\eta^2\right)+2T_R\,u\left(1-u\right)\right]+\mathcal{O}\left(\eta^2\right). \tag{4.67}$$

Let us subtract now the collinear contribution

$$\lim_{m\to 0}\left[I_{g,\mathbf{Q},j}^{\text{IE-FS}}\left(u\right)-\mathcal{F}_{g\mathbf{Q}}^{\overline{\text{MS}}}\left(1-u\right)\right]$$
$$=\frac{\alpha_s}{2\pi}\left[P_{gq}\left(u\right)\left(\log\frac{u^2}{u+\eta_j^2}-\log\frac{\mu_f^2}{2p_j\cdot p_g}\right)+2T_R\,u\left(1-u\right)\right] \tag{4.68}$$

First, observe that now it is finite in $m\to 0$ limit. Here we use $\overline{\text{MS}}$ subscheme of CWZ, since we are in the region where the quark is treated as active parton. Now we have to check whether it equals to the RHS of (4.61). We have first (note the massless quark in the subscript)

$$I_{g,q,j}^{\text{IE-FS}}\left(u\right)=\frac{\alpha_s}{2\pi}S\left(\kappa\right)\left(\frac{\mu_r^2}{2p_j\cdot p_g}\right)^{-\kappa}\left[\left(\frac{1}{\kappa}+\log\frac{u^2}{u+\eta_j}\right)P_{gq}\left(u\right)+2T_R\,u\left(1-u\right)\right].$$

Note, this is the same result as in [10]. As already explained in Section 3.7.3.2, this is because when $m=0$ we have the same dipole splitting function. Next, we calculate

$$I_{g,q,j}^{\text{IE-FS}}\left(u\right)-\mathcal{F}_{gq}^{\overline{\text{mMS}}}\left(1-u\right)=\frac{\alpha_s}{2\pi}S\left(\kappa\right)\left(\frac{\mu_r^2}{2p_j\cdot p_g}\right)^{-\kappa}$$
$$\left[\left(\frac{1}{\kappa}-\frac{1}{\kappa}\left(\frac{2p_j\cdot p_g}{\mu_f^2}\right)^{-\kappa}+\log\frac{u^2}{u+\eta_j}\right)P_{gq}\left(u\right)+2T_R\,u\left(1-u\right)\right]$$
$$=\frac{\alpha_s}{2\pi}\left[\left(\log\frac{u^2}{u+\eta_j}-\log\frac{\mu_f^2}{2p_j\cdot p_g}\right)P_{gq}\left(u\right)+2T_R\,u\left(1-u\right)\right]. \tag{4.69}$$

Thus we see, that (4.68) and (4.69) are indeed identical.

Let us switch now to $\mathbf{Q} \to g\mathbf{Q}$ case. This time the logarithm behaves as

$$\log\left(1+\mathcal{A}\left(u\right)\right) = \log\frac{u^2}{\left(1-u\right)^2\left(u+\eta_j^2\right)} - \log\eta^2 + \mathcal{O}\left(\eta^2\right). \tag{4.70}$$

Analysing the rest of the expression (3.420) in the massless limit, we get

$$I^{\text{IE-FS}}_{\mathbf{Q},g,j}\left(u\right) = \frac{\alpha_s}{2\pi}\left[\left(\log\frac{u^2}{\left(1-u\right)^2\left(u+\eta_j^2\right)} - \log\eta^2\right)P_{qg}\left(1-u\right) - C_F\frac{4u}{1-u}\right] + \mathcal{O}\left(\eta^2\right). \tag{4.71}$$

Performing the subtraction we get

$$\lim_{m\to 0}\left[I^{\text{IE-FS}}_{\mathbf{Q},g,j}\left(u\right) - \mathcal{F}^{\overline{\text{MS}}}_{\mathbf{Q}g}\left(1-u\right)\right]$$
$$= \frac{\alpha_s}{2\pi}\left[\left(\log\frac{u^2}{\left(u+\eta_j^2\right)} - \log\frac{\mu_f^2}{2p_j\cdot p_q} + 1\right)P_{qg}\left(1-u\right) - C_F\frac{4u}{1-u}\right] \tag{4.72}$$

On the other hand, the massless result (3.421) reads

$$I^{\text{IE-FS}}_{q,g,j}\left(u\right) = \frac{\alpha_s}{2\pi}S\left(\kappa\right)\left(\frac{\mu_r^2}{2p_j\cdot p_q}\right)^{-\kappa}$$
$$\left[\left(\frac{1}{\kappa} + \log\frac{u^2}{u+\eta_j^2} + 1\right)P_{qg}\left(1-u\right) - C_F\frac{4u}{1-u}\right] \tag{4.73}$$

and as can be easily checked, again $I^{\text{IE-FS}}_{q,g,j}\left(u\right) - \mathcal{F}^{\text{m}\overline{\text{MS}}}_{qg}\left(1-u\right)$ equals precisely to (4.72).

Finally, let us turn to $\mathbf{Q} \to \mathbf{Q}g$ splitting case. This case is slightly harder. Let us first note that in this case the pertinent logarithms behave as

$$\log\left(1+\mathcal{A}_1\left(u\right)\right) = \log\frac{1+\eta_j^2}{\eta_j^2} + \mathcal{O}\left(\eta^2\right),$$

$$\log\left(1+\mathcal{A}_2\left(u\right)\right) = \log\frac{1}{u+\eta_j^2} - \log\eta^2 + \mathcal{O}\left(\eta^2\right).$$

Therefore, the whole expression (3.364) reduces to

$$I^{\text{IE-FS}}_{\mathbf{Q},\mathbf{Q},j}\left(u\right) = \frac{\alpha_s}{2\pi}\left(\frac{\mu_r^2}{2p_j\cdot p_q}\right)^{-\kappa}C_F\left\{\left(\frac{1}{u}\right)_+\left[-\left(1+\left(1-u\right)^2\right)\left(\log\eta^2 + \log\left(u+\eta_j^2\right)\right)\right.\right.$$
$$\left. - 2\log\frac{1+u+\eta_j^2}{u+\eta_j^2} - 2\left(1-u\right)\right]$$
$$- \delta\left(u\right)\left[\frac{1}{\kappa}\left(1+\log\eta^2+\log\left(1+\eta_j^2\right)\right) + \frac{1}{2}\log\eta^2\left(2+\log\eta^2\right)\right.$$
$$\left.\left. - \frac{1}{2}\log^2\left(1+\eta_j^2\right) - 2\text{Li}_2\left(\frac{1}{1+\eta_j^2}\right) + \frac{\pi^2}{3}\right]\right\} + \mathcal{O}\left(\eta^2\right) \tag{4.74}$$

Making the subtraction and rearranging the terms we get

$$\lim_{m\to 0}\left[I^{\text{IE-FS}}_{\mathbf{Q},\mathbf{Q},j}(u)-\mathcal{F}^{\overline{\text{MS}}}_{\mathbf{QQ}}(1-u)\right]=\frac{\alpha_s}{2\pi}S(\kappa)\left(\frac{\mu_r^2}{2p_j\cdot p_q}\right)^{-\kappa}C_F$$
$$\left\{-\frac{1}{C_F}P_{qq}(1-u)\log\left(\frac{\mu_f^2}{2p_j\cdot p_q}\right)+(2-u)\log\left(u+\eta_j^2\right)\right.$$
$$-2\left(\frac{1}{u}\right)_+\log\left(1+u+\eta_j^2\right)+4\left(\frac{\log u}{u}\right)_+ +2(u-2)\log u+u$$
$$-\delta(u)\left[\frac{1}{\kappa}\left(1+\log\eta^2+\log\left(1+\eta_j^2\right)\right)-\frac{3}{2}\log\eta^2+2\right.$$
$$\left.\left.+\frac{1}{2}\log\eta^2\left(2+\log\eta^2\right)-\frac{1}{2}\log^2\left(1+\eta_j^2\right)-2\text{Li}_2\left(\frac{1}{1+\eta_j^2}\right)+\frac{\pi^2}{3}\right]\right\}. \quad (4.75)$$

Now we can compare this with the calculation of [10], as in this case our dipole splitting function is precisely the same in the massless initial state limit. Massless $\overline{\text{MS}}$ subtraction procedure is analogous as before, thus we skip this step. We find, that indeed our result fulfils (4.61). However, as already mentioned, we have to drop the endpoint contribution first (the one proportional to delta function as it cancels with virtual corrections). Nevertheless some comparison can be made also with this part. First, we find that all the terms involving spectator do agree with [10]. Moreover, we can check, that the soft/collinear poles are also correct. This can be done by the help of the corresponding formulae (3.305), (3.306). The relation (4.58) is also true for our formula with the massless spectator assumed at the beginning.

The remaining initial state splitting process is $g\to gg$ with dipole function given in [10]. However, since there are only collinear poles it is trivial. The only observation is that now our collinear subtraction term has the end-point contribution with a logarithm of mass (4.28). This contribution cancels the singularity of the massive loop correction to gluon initial state leg.

4.5 Practical application and `MassJet` project

Let us now sketch a relatively simple example of NLO calculation using general-mass dipole formalism. It is very instructive and the results can be compared with existing calculations.

Consider heavy quark structure function discussed in Chapter 2. All relevant diagrams are shown in Fig. 2.1. We assume that there is a massive heavy quark $\mathbf{Q}$ in the initial state. That is, we consider also the quark scattering (QS) process $\gamma\mathbf{Q}\to\mathbf{Q}$ and its virtual and real corrections. Both have soft singularities and "live" on different phase spaces: the former on $d\Phi_2$ while the latter on $d\Phi_1$. The real correction is $\gamma\mathbf{Q}\to\mathbf{Q}g$, where the gluon can be emitted either from initial or final state. There is also boson-gluon fusion process $\gamma g\to\mathbf{Q}\overline{\mathbf{Q}}$ which does not have any singularities. However both BGF and QS have quasi-collinear emissions which have to be treated appropriately.

Since it is inclusive process, the jet function is just a unity. We need three dipoles, two for the initial state emission and one for the final state. We can write somewhat symbolically

$$F_{2\mathbf{Q}}^{\text{LO}}=f_{\mathbf{Q}}\otimes\int d\Phi_1\left|\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}}\right|^2, \quad (4.76)$$

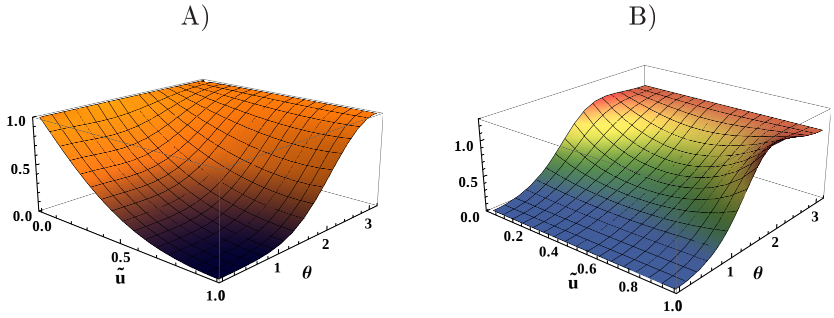


Figure 4.5: The ratio of a matrix elements for real emission and the corresponding dipole subtraction terms A) for quark scattering process, B) for boson-gluon fusion. Here θ is the angle in photon-parton CM frame. The edge $\tilde{u} = 0$ corresponds to the soft singularity, while $\theta = \pi$ to quasi-collinear. Note, that for QS the ratio is equal to 1 on both edges, while for BGF only on the one corresponding to quasi-collinear region. The plots are made for transverse projection of matrix elements, $Q^2 = 10^4$ GeV and the charm quark.

and for NLO

$$\begin{aligned} F_{2\mathbf{Q}}^{\text{NLO}} = f_{\mathbf{Q}} \otimes \int d\Phi_2 \left[|\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}g}|^2 - (V^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g} + V^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g}) \otimes |\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}}|^2 \right] \\ + f_g \otimes \int d\Phi_2 \left[\left|\mathcal{M}_{\gamma g\to\mathbf{Q}\overline{\mathbf{Q}}}\right|^2 - V^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}}} \otimes |\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}}|^2 \right] \\ + f_{\mathbf{Q}} \otimes \int d\Phi_1 \left[\mathcal{M}^{\text{loop}\,2}_{\gamma\mathbf{Q}\to\mathbf{Q}} + (I^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g} + I^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g} - I^{\text{coll}}_{\mathbf{Q}\mathbf{Q}}) \otimes |\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}}|^2 \right] \\ + f_g \otimes \int d\Phi_1 \left(I^{\text{IE-FS}}_{g\to\mathbf{Q}\overline{\mathbf{Q}}} - I^{\text{coll}}_{g\mathbf{Q}} \right) \otimes |\mathcal{M}_{\gamma\mathbf{Q}\to\mathbf{Q}}|^2 . \quad (4.77) \end{aligned}$$

The real matrix elements are easy to obtain, while the virtual corrections are given in [35] for a general massive case. We find that the pole part of $\mathcal{M}^{\text{loop}\,2}_{\gamma\mathbf{Q}\to\mathbf{Q}}$ cancels exactly with the pole part of $(I^{\text{IE-FS}}_{\mathbf{Q}\to\mathbf{Q}g} + I^{\text{FE-IS}}_{\mathbf{Q}\to\mathbf{Q}g})$. The singularities in the real corrections are also subtracted in a proper way as shown in Fig. 4.5.

Those analytical observations are confirmed by our C++ MC implementation of the general-mass scheme (Fig. 4.6) (see also below). We observe that in considered case:

a) phase space is factorized correctly, as we have checked it explicitly comparing suitable histograms

b) dipole splitting functions are chosen correctly, as the MC integration in the first line of (4.77) is finite and stable (we do not need an additional IR cutoff here)

c) integrated dipoles are correct; for BGF it can be checked explicitly since there are no soft poles, while for QS it can be checked by comparison with analytical calculations

In Fig. 4.6 we compare the present calculation with the one of Section 2.2. In the latter the NLO QS contributions were dropped as they are usually negligible (they are effectively

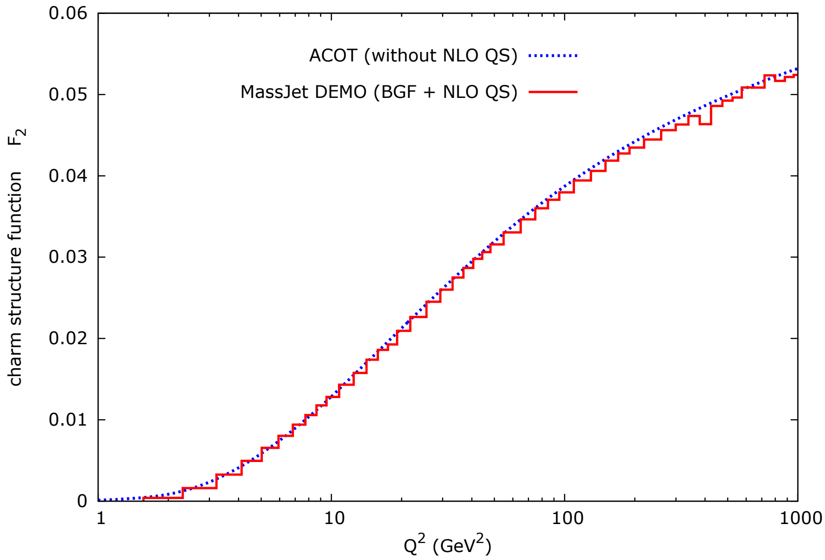


Figure 4.6: The comparison of charm structure function calculated semi-analytically in ACOT scheme (dotted) and using demo version of MC program `MassJet`. The latter includes NLO quark-scattering process with heavy quark in the initial state. The difference is negligible as effectively QS process is of higher order. The calculations are done for $x_B = 0.05$.

of higher order). Indeed we observe only little difference between our numerical results and the former one.

The working name of our C++ project is `MassJet`. It is based on FOAM MC algorithm [31], which is used to generate events and weights. The program is still under development, however we have already implemented most of the necessary features needed for NLO dijets calculations in DIS within our scheme. The major lack are fully massive virtual corrections, which, although calculated in the literature are not well suited to our purposes and still need some work.

Chapter 5

Summary and outlook

In the present work we have given a detailed description of the general method for calculating jets cross sections with heavy quark effects taken into account. In the first place, we have revised the dipole subtraction method and extended it to the case, where the initial state splitting processes may involve massive quarks. Such a situation have been treated only for QED-like processes before and is by no means suited to jets in QCD. In particular, this approach does not take into account gluon splitting into heavy quarks, what is of great importance due to largeness of gluon density. Inclusion of massive initial state partons is a minor change only superficially. In reality, we have to redefine most of the existing dipole splitting functions and the corresponding kinematics. Hence we had to perform all the integrals again.

Moreover, we supply the dipole formalism in the special subtraction terms. They are intended to factorize out the mass quasi-singularities and retain the massless DGLAP evolution for PDFs. This is done by the method based on firm theoretical background and could be in principle generalized to any desired order of perturbation theory. We have checked that our cross sections are free from collinear initial state singularities in the massless limit.

Presented results are actually enough to prepare a Monte-Carlo algorithm for calculating multi-jets cross section at NLO in DIS processes. They can be applied to both neutral and charged current reactions, since the information of the latter are buried inside a reduced matrix elements. Such generalizations are constrained mainly by the complexity of loop calculations in a fully massive case.

The method can be also extended to fragmentation processes and hadron-hadron reactions. This is actually not an extremely difficult task. In case of hadron-hadron scattering one needs to consider one additional case, namely initial state emitter with initial state spectator and corresponding kinematics. There is however another difficulty, namely so called DFT disease [24]. There is a confirmed violation of KLN theorem at two loops when there are two massive partons in the initial state. Such non-cancellation of IR singularities may invalidate factorization theorem at NNLO. Those and related subjects are left for the future studies.

We did not discuss the jet algorithms. There are several IR safe routines, including solutions for heavy quarks [46, 6]. We however realise, that this topic needs further verification as massive initial state quarks were not used in jet calculations before. We however do not expect any complications as any collinear safe observable can be extended to a quasi-collinear one [10].

In order to support our theoretical calculations, we have constructed a dedicated C++ modular MC program based on the FOAM algorithm. Although the project is under development, we have performed sample inclusive calculations confirming efficacy of our method. So far, we have implemented most (but not all) of the constituents needed for dijets calculations at NLO accuracy in our general-mass scheme with any number of heavy quarks. Still, some virtual corrections within a suitable renormalization scheme have to be taken into account.

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Appendix A

Kinematics

A.1 Thermodynamics of the invariants

We have the following "equations of state"

$$\tilde{u}^2 m_a^2 + \tilde{w}\left(\tilde{w}\mathcal{P}^2 - 2\mathcal{P}_a\tilde{u}\right) = \begin{cases} m_{\underline{ij}}^2 & \text{FE-IS} \\ m_j^2 & \text{IE-FS} \end{cases} \tag{A.1}$$

$$\left(1-\tilde{u}\right)^2 m_a^2 + \left(1-\tilde{w}\right)\left[\left(1-\tilde{w}\right)\mathcal{P}^2 - 2\mathcal{P}_a\left(1-\tilde{u}\right)\right] = \begin{cases} m_a^2 & \text{FE-IS} \\ m_{\underline{ai}}^2 & \text{IE-FS} \end{cases} \tag{A.2}$$

If we introduce the additional invariant $\tilde{\gamma}$ we have also

$$\tilde{\gamma} = \tilde{w}\mathcal{P}_a - \tilde{u}m_a^2. \tag{A.3}$$

We can obtain some useful relations between the partial derivatives. For example

$$\left(\frac{\partial \tilde{w}}{\partial \tilde{u}}\right)_{\tilde{\gamma}} \mathcal{P}_a - m_a^2 + \tilde{w}\left(\frac{\partial \mathcal{P}_a}{\partial \tilde{u}}\right)_{\tilde{\gamma}} = 0, \tag{A.4}$$

$$\left(\frac{\partial \tilde{w}}{\partial \tilde{u}}\right)_{\tilde{\gamma}} = \left(\frac{\partial \tilde{w}}{\partial \tilde{u}}\right)_{\mathcal{P}_a} + \left(\frac{\partial \tilde{w}}{\partial \mathcal{P}_a}\right)_{\tilde{u}}\left(\frac{\partial \mathcal{P}_a}{\partial \tilde{u}}\right)_{\tilde{\gamma}}, \tag{A.5}$$

$$\left(\frac{\partial \mathcal{P}^2}{\partial \tilde{u}}\right)_{\tilde{\gamma}} = \left(\frac{\partial \mathcal{P}^2}{\partial \tilde{u}}\right)_{\mathcal{P}_a}\left[1 - \left(\frac{\partial \tilde{u}}{\partial \mathcal{P}_a}\right)_{\mathcal{P}^2}\left(\frac{\partial \mathcal{P}_a}{\partial \tilde{u}}\right)_{\tilde{\gamma}}\right], \tag{A.6}$$

where all derivatives are evaluated with $\tilde{p}_{\underline{ij}}$ on-shell. Moreover, we note the following useful identities

$$\left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial \mathcal{P}_a}\right|_{u,\mathcal{P}^2}^{-1}\left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial u}\right|_{\mathcal{P}_a,\mathcal{P}^2} = \left|\frac{\partial \mathcal{P}_a}{\partial u}\right|_{\mathcal{P}^2,\tilde{p}_{\underline{ij}}^2=m_{\underline{ij}}^2}, \tag{A.7}$$

$$\left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial \mathcal{P}_a}\right|_{u,\tilde{\gamma}}^{-1}\left|\frac{\partial \tilde{p}_{\underline{ij}}^2}{\partial u}\right|_{\mathcal{P}_a,\tilde{\gamma}} = \left|\frac{\partial \mathcal{P}_a}{\partial u}\right|_{\tilde{\gamma},\tilde{p}_{\underline{ij}}^2=m_{\underline{ij}}^2}, \tag{A.8}$$

$$\left|\frac{\partial \mathcal{P}_a}{\partial u}\right|_{\mathcal{P}^2,\tilde{p}_{\underline{ij}}^2=m_{\underline{ij}}^2}\left(\frac{\partial \mathcal{P}^2}{\partial \tilde{\gamma}}\right)_u = \left|\frac{\partial \mathcal{P}_a}{\partial u}\right|_{\tilde{\gamma},\tilde{p}_{\underline{ij}}^2=m_{\underline{ij}}^2}\left(\frac{\partial \mathcal{P}^2}{\partial \tilde{\gamma}}\right)_{\mathcal{P}_a}. \tag{A.9}$$

For IE-FS case the formulae are analogous. All the identities are derived using well-known jacobian techniques. Note we use the thermodynamical notation for partial derivatives, i.e. subscripts denote fixed parameters.

If we have all three equations of state (A.1)-(A.3) satisfied, we can express $\mathcal{P}^2$ as follows

$$\mathcal{P}^2\left(\mathcal{P}_a,\tilde{\gamma}\right)=\frac{4\delta^2_{\mathcal{P}_a}\mathcal{P}_a\left(2\tilde{\gamma}+m^2_{\underline{ij}}\right)+m^2_a\left(2\delta^2_{\mathcal{P}_a}+m^2_{\underline{ij}}\right)^2}{m^2_a\left(m^2_{\underline{ij}}-\delta^2_{\mathcal{P}_a}\right)-2\tilde{\gamma}\delta_{\mathcal{P}_a}\left(\delta_{\mathcal{P}_a}+\tilde{v}_{\underline{ij}}\sqrt{\delta_{\mathcal{P}_a}+2m^2_a}\right)},\quad\text{for FE-IS case}\tag{A.10}$$

$$\begin{aligned}\mathcal{P}^2\left(\mathcal{P}_a,\tilde{\gamma}\right)=&-\Big\{m^4_a\left[m^2_a-2\left(m^2_{\underline{ai}}-m^2_j-2\delta^2_{\mathcal{P}_a}\right)\right]\\&+m^2_a\left[m^4_{\underline{ai}}-2m^2_{\underline{ai}}\left(m^2_j+2\delta^2_{\mathcal{P}_a}\right)+m^4_j+4m^2_j\delta^2_{\mathcal{P}_a}+4\left(\delta^4_{\mathcal{P}_a}-\tilde{\gamma}\mathcal{P}_a\right)\right]\\&+4\mathcal{P}_a\left[\tilde{\gamma}\left(m^2_{\underline{ai}}-2\delta^2_{\mathcal{P}_a}\right)-m^2_j\delta^2_{\mathcal{P}_a}\right]\Big\}\Big\{m^2_a\left(m^2_a-m^2_{\underline{ai}}-m^2_j+2\delta^2_{\mathcal{P}_a}\right)\\&+2\tilde{\gamma}\left[\delta^2_{\mathcal{P}_a}+\tilde{v}_j\sqrt{m^2_a\left(m^2_a-m^2_{\underline{ai}}+2\delta^2_{\mathcal{P}_a}\right)+\delta^4_{\mathcal{P}_a}}\right]\Big\}^{-1},\quad\text{for IE-FS case}\end{aligned}\tag{A.11}$$

where

$$\delta_{\mathcal{P}_a}=\sqrt{\tilde{\gamma}-\mathcal{P}_a}.\tag{A.12}$$

Let us also give $\tilde{w}$ and $\mathcal{P}_a$ in terms of $\tilde{u}$ and $\mathcal{P}^2$. In the FE-IS case we have

$$\mathcal{P}_a\left(\tilde{u},\mathcal{P}^2\right)=\frac{1}{2\tilde{u}\left(\tilde{u}-1\right)}\left\{\tilde{u}\left[\frac{1}{3}\sigma_3-\mathcal{P}^2-m^2_{\underline{ij}}+\tilde{u}m^2_a\right]-\frac{\sigma^2_3}{36\mathcal{P}^2}+m^2_{\underline{ij}}\right\},\tag{A.13}$$

$$\tilde{w}\left(\tilde{u},\mathcal{P}_a\right)=\frac{\sigma_3}{6\mathcal{P}^2},\tag{A.14}$$

where

$$\sigma_3=\frac{4\mathcal{P}^2}{\sigma_2}\left[\tilde{u}^2\left(3m^2_a+\mathcal{P}^2\right)+\left(1-\tilde{u}\right)\left(3m^2_{\underline{ij}}+\mathcal{P}^2\right)\right]+2\mathcal{P}^2\left(1+\tilde{u}\right)+\sigma_2,\tag{A.15}$$

$$\sigma_2=\left\{4\left[\left(\tilde{u}-2\right)\mathcal{P}^4\left[\mathcal{P}^2\left(1+\tilde{u}\right)\left(2\tilde{u}-1\right)-9\left(2\tilde{u}^2m^2_a+\left(\tilde{u}-1\right)m^2_{\underline{ij}}\right)\right]+\mathcal{P}^2\sigma_1\right]\right\}^{\frac{1}{3}},\tag{A.16}$$

$$\begin{aligned}\sigma_1=\mathcal{P}\Big\{&\mathcal{P}^2\left(\tilde{u}-2\right)^2\left(9\left(2m^2_a\tilde{u}^2+\left(\tilde{u}-1\right)m^2_{\underline{ij}}\right)-\mathcal{P}^2\left(2\tilde{u}^2+\tilde{u}-1\right)\right)^2\\&-4\left(3m^2_a\tilde{u}^2-3m^2_{\underline{ij}}\left(\tilde{u}-1\right)+\mathcal{P}^2\left(1+\tilde{u}\left(\tilde{u}-1\right)\right)\right)^3\Big\}^{\frac{1}{2}}.\end{aligned}\tag{A.17}$$

For IE-FS case we get

$$\mathcal{P}_a\left(\tilde{u},\mathcal{P}^2\right)=\frac{1}{2\tilde{u}\left(\tilde{u}-1\right)}\left\{\tilde{u}\left[\frac{1}{3}\sigma_3-\mathcal{P}^2-m^2_j+\left(\tilde{u}-1\right)m^2_a+m^2_{\underline{ai}}\right]-\frac{\sigma^2_3}{36\mathcal{P}^2}+m^2_j\right\},\tag{A.18}$$

where now

$$\sigma_3 = \frac{4\mathcal{P}^2}{\sigma_2}\left[\tilde{u}^2\left(3m_a^2+\mathcal{P}^2\right)-\tilde{u}\left(3\left(m_a^2-m_{\underline{ai}}^2+m_j^2\right)+\mathcal{P}^2\right)+3m_j^2+\mathcal{P}^2\right]+2\mathcal{P}^2\left(1+\tilde{u}\right)+\sigma_2, \tag{A.19}$$

$$\sigma_2 = \left\{4\left[\left(\tilde{u}-2\right)\mathcal{P}^4\left[\mathcal{P}^2\left(1+\tilde{u}\right)\left(2\tilde{u}-1\right)-9m_j^2\left(\tilde{u}-1\right)\right]\right.\right. \\ \left.\left.+9\mathcal{P}^4\tilde{u}\left[m_a^2\left(\left(3-2\tilde{u}\right)\tilde{u}-1\right)+m_{\underline{ai}}^2\left(\tilde{u}+1\right)\right]+\mathcal{P}^2\sigma_1\right]\right\}^{\frac{1}{3}} \tag{A.20}$$

$$\sigma_3 = \mathcal{P}\left\{\mathcal{P}^2\left[9\tilde{u}\left(m_a^2\left(\tilde{u}-1\right)\left(2\tilde{u}-1\right)-m_{\underline{ai}}^2\left(\tilde{u}+1\right)\right)+\left(\tilde{u}-2\right)\left(9m_j^2\left(\tilde{u}-1\right)-\mathcal{P}^2\left(\tilde{u}+1\right)\left(2\tilde{u}-1\right)\right)\right]^2\right. \\ \left.-4\left[3\tilde{u}\left(m_a^2\left(\tilde{u}-1\right)+m_{\underline{ai}}^2\right)-3m_j^2\left(\tilde{u}-1\right)+\mathcal{P}^2\left(\tilde{u}\left(\tilde{u}-1\right)+1\right)\right]^3\right\}^{\frac{1}{2}}$$

A.2 Explicit expressions for rescaled variables

We give here the explicit forms of the rescaled functions $\eta_X^2 = X/2\tilde{\gamma}$, jacobians and other variables used in the integrated dipoles. They are useful when analysing different limits of the integrals. In practise they are most convenient calculated directly from the invariants. Below we use $\eta^2 \equiv \eta_{\mathbf{Q}}^2$.

A.2.1 Final State Emitter - Initial State Spectator

A.2.1.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

$$w = \frac{1}{2\left(\eta^2+1\right)}\left[\sqrt{u\left(4\eta_a^2\left(2\eta^2+1\right)+u\left(1-4\eta^2\eta_a^2\right)+4\eta_a^4u-2\right)+1}\right. \\ \left.+2\left(\eta^2+\eta_a^2u\right)+u+1\right], \tag{A.21}$$

$$\eta_{\mathcal{P}^2}^2 = \frac{\left(\eta^2+1\right)\left(\eta^2+\eta_a^2u^2+u\right)}{-\eta^2-\eta_a^2u^2+u(2\eta_a^2w+w-1)+2\eta^2w+w}, \tag{A.22}$$

$$\tilde{\eta}_{\mathcal{P}^2-m^2}^2 = \frac{\tilde{v}^2\left(\eta^2(u-2)-\eta_a^2u-1\right)}{\left(2\eta^2+1\right)\left(\eta_a^2u^2-\eta^2(w-1)-w\right)-u\left(-2\eta^2+\left(\eta^2+1\right)\left(2\eta_a^2+1\right)w-1\right)}, \tag{A.23}$$

$$\eta_{\mathcal{P}_a}^2 = \frac{2\eta_a^2u+1}{2w}, \quad v = \frac{\tilde{v}}{2\eta_a^2u+1}, \tag{A.24}$$

$$\eta_{\mathcal{J}}^2 = \left|\frac{2\left(\eta_{\mathcal{P}_a}^4-\eta_a^2\eta_{\mathcal{P}^2}^2\right)\left(\sigma+(1-u)\left(\eta_{\mathcal{P}^2}^2-\eta_{\mathcal{P}_a}^2\right)\right)}{\eta_{\mathcal{P}^2}^2\sigma}\right|, \tag{A.25}$$

where

$$\sigma = \sqrt{(u-1)^2\eta_{\mathcal{P}_a}^4 - \eta_a^2(u-2)u\ \eta_{\mathcal{P}^2}^2}. \tag{A.26}$$

A.2.1.2 $g \to \mathbf{Q}\overline{\mathbf{Q}}$ and $g \to gg$ splittings

$$w = \frac{1}{2}\left(2u\eta_a^2 + \sqrt{u\left(4\eta_a^2 + 4u\eta_a^4 + u - 2\right) + 1} + u + 1\right), \tag{A.27}$$

$$\eta_{\mathcal{P}_a}^2 = \frac{\left(2u\eta_a^2 + 1\right)\left(2u\eta_a^2 + u - w + 1\right)}{2u\left(u\eta_a^2 + 1\right)}, \tag{A.28}$$

$$\tilde{\eta}_{\mathcal{P}^2}^2 = -\frac{u\eta_a^2 + 1}{u\left(\eta_a^2(u-2w) + 1\right) + (-u-1)w}, \tag{A.29}$$

$$v = \frac{1}{2\eta_a^2 u + 1} \tag{A.30}$$

and the jacobian $\eta_{\mathcal{J}}^2$ has the same form as in the previous subsection.

A.2.2 Initial State Emitter - Final State Spectator

A.2.2.1 $\mathbf{Q} \to \mathbf{Q}g$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}g$ splittings

$$w = \frac{2\left(\eta_j^2 + \eta^2 u\right) + \sqrt{u\left(4\eta^2 - 4\eta^2(u-2)\eta_j^2 + \left(4\eta^4 + 1\right)u - 2\right) + 1} + u + 1}{2\left(\eta_j^2 + 1\right)}, \tag{A.31}$$

$$\eta_{\mathcal{P}^2}^2 = \frac{\left(\eta_j^2 + 1\right)\left(\eta_j^2 + \eta^2 u^2 + u\right)}{(2w-1)\eta_j^2 + u\left(\eta^2(-u) + 2\eta^2 w + w - 1\right) + w}, \tag{A.32}$$

$$\eta_{\mathcal{P}_a}^2 = \frac{\left(2\eta^2 u + 1\right)\left(2\left(\eta_j^2 + \eta^2 u\right) - w\left(\eta_j^2 + 1\right) + u + 1\right)}{2\left(\eta_j^2 + \eta^2 u^2 + u\right)}, \tag{A.33}$$

$$\tilde{\eta}_{\mathcal{P}^2 - m_j^2}^2 = \tilde{v}_j^2\left(2\eta_j^2 + u\left(\eta^2 - \eta_j^2\right) + 1\right)\left[\left(2\eta_j^2 + 1\right)\left((w-1)\eta_j^2 - \eta^2 u^2 + w\right)\right.$$
$$\left. + u\left(-2\eta_j^2 + \left(2\eta^2 + 1\right)w\left(\eta_j^2 + 1\right) - 1\right)\right]^{-1}, \tag{A.34}$$

$$\eta_{\mathcal{J}}^2 = \left|\frac{2\left(\eta_{\mathcal{P}_a}^4 - \eta^2\eta_{\mathcal{P}^2}^2\right)\left(\sigma + (1-u)\left(\eta_{\mathcal{P}^2}^2 - \eta_{\mathcal{P}_a}^2\right)\right)}{\eta_{\mathcal{P}^2}^2\sigma}\right|, \tag{A.35}$$

where

$$\sigma = \sqrt{(u-1)^2\eta_{\mathcal{P}_a}^4 - \eta^2(u-2)u\ \eta_{\mathcal{P}^2}^2}. \tag{A.36}$$

A.2.2.2 $g \to \mathbf{Q\overline{Q}}$ splitting

$$w = \frac{2\eta_j^2 + \sqrt{4\eta^2\eta_j^2 + u\left(4\eta^2 + u - 2\right) + 1} + u + 1}{2\left(\eta_j^2 + 1\right) - 2\eta^2} \tag{A.37}$$

$$\eta_{\mathcal{P}^2}^2 = \frac{\left(-\eta^2 + \eta_j^2 + 1\right)\left(\eta_j^2 + u\right)}{(2w-1)\eta_j^2 + u(w-1) + w} \tag{A.38}$$

$$\eta_{\mathcal{P}_a}^2 = \frac{2\eta_j^2 + w\left(\eta^2 - \eta_j^2 - 1\right) + u + 1}{2\left(\eta_j^2 + u\right)} \tag{A.39}$$

$$\eta_{\mathcal{J}}^2 = \left|\frac{2\eta_{\mathcal{P}_a}^4\left[\sqrt{(u-1)^2\eta_{\mathcal{P}_a}^4 + \eta^2\eta_{\mathcal{P}^2}^2} + (1-u)\left(\eta_{\mathcal{P}^2}^2 - \eta_{\mathcal{P}_a}^2\right)\right]}{\eta_{\mathcal{P}^2}^2\sqrt{(u-1)^2\eta_{\mathcal{P}_a}^4 + \eta^2\eta_{\mathcal{P}^2}^2}}\right| \tag{A.40}$$

A.2.2.3 $\mathbf{Q} \to g\mathbf{Q}$ and $\overline{\mathbf{Q}} \to g\overline{\mathbf{Q}}$ splittings

$$\begin{aligned}\eta_{\mathcal{P}^2}^2 = 2\left(\eta^2 + \eta_j^2 + 1\right)^2\left(\eta_j^2 + \eta^2u^2 + u\right) \\ \Big\{u^2\left(-\left(\left(2\eta^2+1\right)\tilde{v}_j - 2\eta^2\left(\eta^2 - \eta_j^2 + 1\right) - 1\right)\right) \\ + 2u\left(\eta^2\left(\tilde{v}_j + 4\eta_j^2 + 1\right) + \eta_j^2\left(1 - \tilde{v}_j\right)\right) + 2\eta_j^2\left(\tilde{v}_j - \eta^2 + \eta_j^2 + 1\right) + \tilde{v}_j + 1\Big\}\end{aligned} \tag{A.41}$$

$$\eta_{\mathcal{P}_a}^2 = \frac{\left(2\eta^2u + 1\right)\left(u\left(\tilde{v}_j + 2\eta^2 + 1\right) - \tilde{v}_j + 2\eta_j^2 + 1\right)}{4\left(\eta_j^2 + \eta^2u^2 + u\right)} \tag{A.42}$$

$$w = \frac{u\left(-\tilde{v}_j + 2\eta^2 + 1\right) + \tilde{v}_j + 2\eta_j^2 + 1}{2\left(\eta^2 + \eta_j^2 + 1\right)} \tag{A.43}$$

$$\eta_{\mathcal{J}}^2 = \frac{2v\eta_{\mathcal{P}_a}^2\left((1-v)\eta_{\mathcal{P}_a}^2 - \eta_{\mathcal{P}^2}^2\right)}{\eta_{\mathcal{P}^2}^2} \tag{A.44}$$

A.3 Three-particle phase space in D dimensions

Let us consider

$$\int d\Phi_3\left(K; k_1, k_2, k_3\right) = \int (2\pi)^D\,\delta^D\left(K - k_1 - k_2 - k_3\right) \frac{d^Dk_1\,\delta_+\left(k_1^2 - m_1^2\right)}{(2\pi)^{D-1}}\,\frac{d^Dk_2\,\delta_+\left(k_2^2 - m_2^2\right)}{(2\pi)^{D-1}}\,\frac{d^Dk_3\,\delta_+\left(k_3^2 - m_3^2\right)}{(2\pi)^{D-1}}. \tag{A.45}$$

We introduce the notation

$$K^2 = M^2, \tag{A.46}$$

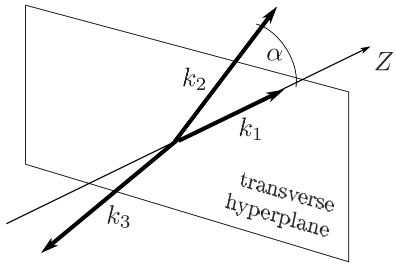

Figure A.1: Center of mass system of the momenta k_1, k_2 and k_3. We orient the frame in such a way that k_1 points towards $+Z$ axis. In D dimensions Ω_{D-2} is a solid angle on the transverse hyperplane and the orientation of the Z axis is given by $\Omega_{D-1}^{(+Z)}$.

$$m_{ab}^2 = (k_a + k_b)^2 . \tag{A.47}$$

Recall that

$$d\Gamma(k) = \frac{d^D k}{(2\pi)^{D-1}} \delta_+ \left(k^2 - m^2\right) = \frac{d^{D-1}k}{(2\pi)^{D-1} 2E_k} = \frac{\left|\vec{k}\right|^{D-2} d\left|\vec{k}\right| d\Omega_{D-1}}{(2\pi)^{D-1} 2E_k}$$
$$= \frac{1}{4(2\pi)^{D-1}} \frac{\left|\vec{k}\right|}{E_k} \left(\left|\vec{k}\right| \sin\theta\right)^{D-4} d\vec{k}^2 d(\cos\theta)\, d\Omega_{D-2} \tag{A.48}$$

where Ω_{D-1} is a solid angle on the $D-1$ dimensional sphere, while Ω_{D-2} is a solid angle on transverse hyperplane to Z axis.

Let us now choose CM(K) system and orient it in such a way that k_1 points towards $+Z$ axis (Fig. A.3). Integrating (A.45) and using (A.48) we get

$$\int d\Phi_3(K; k_1, k_2, k_3) = \int (2\pi)\, \delta\left((K - k_1 - k_2)^2 - m_3^2\right) \left(\frac{1}{4(2\pi)^{D-1}}\right)^2$$
$$\frac{\left|\vec{k}_1\right|^{D-3}}{E_1} \frac{\left|\vec{k}_2\right|}{E_2} \left(\left|\vec{k}_2\right| \sin\alpha\right)^{D-4} d\vec{k}_1^2 d\vec{k}_2^2 d(\cos\alpha)\, d\Omega_{D-2} d\Omega_{D-1}^{(+Z)}, \tag{A.49}$$

where $d\Omega_{D-1}^{(+Z)}$ is an solid angle representing orientation of $+Z$ axis in $D-1$ dimensional space. The delta function gives

$$0 = M^2 + m_1^2 + m_2^2 - m_3^2 - 2M(E_1 + E_2) + 2E_1E_2 - 2\left|\vec{k}_1\right|\left|\vec{k}_2\right| \cos\alpha \tag{A.50}$$

and thus

$$\int d\Phi_3(K; k_1, k_2, k_3) = \frac{2^{4-D}}{8(2\pi)^{2D-1}}$$
$$\left[4\vec{k}_1^2\vec{k}_2^2 - \left(M^2 + m_1^2 + m_2^2 - m_3^2 - 2M(E_1 + E_2) + 2E_1E_2\right)^2\right]^{\frac{D}{2}-2} dE_1 dE_2 d\Omega_{D-2} d\Omega_{D-1}^{(+Z)}, \tag{A.51}$$

or in terms of the invariants m_{ab}^2

$$\int d\Phi_3\left(K;k_1,k_2,k_3\right)=\frac{2^{4-D}}{32\left(2\pi\right)^{2D-1}}\frac{1}{M^{D-2}}\hat{a}^{\frac{D}{2}-2}\,dm_{23}dm_{13}d\Omega_{D-2}d\Omega_{D-1}^{(+Z)}, \tag{A.52}$$

where

$$\begin{aligned}\hat{a}=M^2&\left(m_3^2\left(m_1^2+m_2^2+m_{13}^2+m_{23}^2\right)+\left(m_1^2-m_{13}^2\right)\left(m_2^2-m_{23}^2\right)-m_3^2\left(m_3^2+M^2\right)\right)\\&-m_1^4m_2^2+m_1^2\left(m_{13}^2\left(m_{23}^2-m_3^2+m_2^2\right)+m_2^2\left(m_{23}^2+m_3^2-m_2^2\right)\right)\\&+m_{23}^2\left(m_3^2\left(m_{13}^2-m_2^2\right)-m_{13}^2\left(m_{13}^2+m_{23}^2-m_2^2\right)\right).\end{aligned} \tag{A.53}$$

In the derivation we used

$$E_1=\frac{M^2+m_1^2-m_{23}^2}{2M}, \tag{A.54}$$

$$E_2=\frac{M^2+m_2^2-m_{13}^2}{2M}, \tag{A.55}$$

$$E_3=\frac{M^2+m_3^2-m_{12}^2}{2M}, \tag{A.56}$$

which was obtained using relations of the type $\left(K-k_1\right)^2=\left(k_2+k_3\right)^2=m_{23}^2$ etc. in $\mathrm{CM}\left(K\right)$.

What remains is to find the support. Let us start with m_{23}^2. Clearly

$$\left(m_2+m_3\right)^2\leq m_{23}^2\leq\left(M-m_1\right)^2. \tag{A.57}$$

The upper bound comes from the fact that $m_{23}^2=M^2+m_1^2-2ME_1$ is maximal for $E_1=m_1$. Let us now switch to $\mathrm{CM}\left(k_2,k_3\right)$ for a moment (we shall mark by star quantities in this frame). We have

$$E_1^*=\frac{M^2-m_{23}^2-m_1^2}{2m_{23}}. \tag{A.58}$$

$$E_2^*=\frac{m_{23}^2-m_3^2+m_2^2}{2m_{23}} \tag{A.59}$$

$$E_3^*=\frac{m_{23}^2-m_2^2+m_3^2}{2m_{23}} \tag{A.60}$$

However

$$m_{13}^2=m_1^2+m_3^2+2E_1^*E_3^*-2\left|\vec{k}_1^*\right|\left|\vec{k}_3^*\right|\cos\beta^*, \tag{A.61}$$

where β^* is an angle between $\vec{k}_1$ and $\vec{k}_3$ in $\mathrm{CM}\left(k_2,k_3\right)$. Therefore the bounds on m_{13}^2 read

$$m_1^2+m_3^2+2E_1^*E_3^*-2\left|\vec{k}_1^*\right|\left|\vec{k}_3^*\right|\leq m_{13}^2\leq m_1^2+m_3^2+2E_1^*E_3^*+2\left|\vec{k}_1^*\right|\left|\vec{k}_3^*\right|. \tag{A.62}$$

Appendix B

Mathematical supplement

B.1 The integrals

In this chapter we list and solve some of the appearing integrals. We start with the standard special functions and next we switch to the integrals specific to this work. The space-time dimension is defined as $D = 4 - 2\epsilon = 4 + 2\kappa$, $\varepsilon, \kappa > 0$.

Integral A. (Euler's Gamma) For $\mathrm{Re}\,(z) > 0$ it is defined as

$$\Gamma\,(z) = \int_0^\infty dt\, t^{z-1} e^{-t} dt. \tag{B.1}$$

It has the following property

$$\Gamma\,(1+z) = z\Gamma\,(z)\,. \tag{B.2}$$

We also use the following incomplete Gamma function

$$\Gamma_\lambda\,(z) = \int_\lambda^\infty dt\, t^{z-1} e^{-t} dt. \tag{B.3}$$

We shall need also the following expression

$$\Gamma_\lambda\,(z) = \lambda^z e^{-\lambda} \int_0^\infty dw\; (1+w)^{z-1}\, e^{-\lambda w}. \tag{B.4}$$

Integral B. (Euler's Beta) The integral definition of Beta function reads

$$B\,(x,y) = \int_0^1 dt\; (1-t)^{y-1}\, t^{x-1} \tag{B.5}$$

for $\mathrm{Re}\,(x) > 0$ and $\mathrm{Re}\,(y) > 0$. It is related to Gamma function by means of the formula

$$B\,(x,y) = \frac{\Gamma\,(x)\,\Gamma\,(y)}{\Gamma\,(x+y)}. \tag{B.6}$$

If both arguments are equal, we use the following notation

$$B\,(x,x) \equiv B\,(x)\,. \tag{B.7}$$

Integral C. (Dilogarithm) This function is defined via the integral

$$\mathrm{Li}_2\left(z\right) = -\int_0^z dt\, \frac{\log\left(1-t\right)}{t}. \tag{B.8}$$

Its special value is

$$\mathrm{Li}_2\left(1\right) = \frac{1}{6}\pi^2. \tag{B.9}$$

Let us note the following useful properties (for suitable x)

$$\mathrm{Li}_2\left(x\right) + \mathrm{Li}_2\left(1-x\right) = \frac{\pi^2}{6} - \log x\, \log\left(1-x\right), \tag{B.10}$$

$$\mathrm{Li}_2\left(x\right) + \mathrm{Li}_2\left(\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2\left(-x\right). \tag{B.11}$$

Integral D. (Transverse integrals)

When calculating the parton densities we encounter the integrals of the following form

$$\int \frac{d^{D-2}q_T}{\left(q_T^2 + A\right)^N} = \pi^{\frac{D}{2}-1} A^{\frac{D}{2}-1-N} \frac{\Gamma\left(N+1-\frac{D}{2}\right)}{\Gamma\left(N\right)}. \tag{B.12}$$

Integral E. (Hypergeometric function) We define the hypergeometric function by means of the following series (for z lying in the unit circle)

$$F\left(a,b,c;z\right) = 1 + \frac{ab}{c}\, z + \frac{1}{2}\, \frac{a\left(a+1\right)b\left(b+1\right)}{c\left(c+1\right)}\, z^2 + \ldots. \tag{B.13}$$

More suitable integral definition involves Beta function

$$B\left(b, c-b\right) F\left(a,b,c;z\right) = \int_0^1 dt\, \left(1-t\right)^{c-b-1} t^{b-1} \left(1-zt\right)^{-a} \tag{B.14}$$

for such a, b, c that the integral exists.

Let us note some of the useful properties:

a) Gauss's theorem

$$F\left(a,b,c;1\right) = \frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)} \tag{B.15}$$

b) Pfaff transformation

$$F\left(a,b,c;z\right) = \frac{1}{\left(1-z\right)^a}\, F\left(a, c-b, c; -\frac{z}{1-z}\right) \tag{B.16}$$

c) contiguous relation

$$F\left(a+1,b,c;z\right) = \left(1 - \frac{b}{a}\right) F\left(a,b,c;z\right) + \frac{b}{a} F\left(a,b+1,c;z\right) \tag{B.17}$$

Integral F. Let us define the following integral

$$\mathcal{I}_1(y;\varepsilon) = \int_{z_-}^{z_+} dz\, \frac{(z_+ - z)^{-\varepsilon} (z - z_-)^{-\varepsilon}}{y+z}, \tag{B.18}$$

where $0 \le z_- < z_+ \le 1$ and y is such that the integral exists. The result is

$$\mathcal{I}_1(y;\varepsilon) = A(y)\, (z_+ - z_-)^{-2\varepsilon}\, B(1-\varepsilon, 1-\varepsilon)\, F(1, 1-\varepsilon, 2(1-\varepsilon); -A(y)), \tag{B.19}$$

where

$$A(y) = \frac{z_+ - z_-}{y + z_-}. \tag{B.20}$$

Let us introduce the following notation (recall $\kappa = -\varepsilon$)

$$\mathcal{F}(A;\kappa) = A\, B(1+\kappa)\, F(1, 1+\kappa, 2(1+\kappa); -A). \tag{B.21}$$

Its integral form reads

$$\mathcal{F}(A;\kappa) = A \int_0^1 dt\, \frac{[t(1-t)]^{\kappa}}{1+At}. \tag{B.22}$$

Note we have the following reflection relations

$$\mathcal{F}(A;\kappa) = -\mathcal{F}\left(-\overline{A};\kappa\right), \tag{B.23}$$

where

$$\overline{A} = \frac{A}{1+A}. \tag{B.24}$$

Proof. The integral (B.18) can be easily transformed into the form of (B.14) via the substitution

$$t = \frac{z - z_-}{z_+ - z_-}. \tag{B.25}$$

We get

$$\mathcal{I}_1(y;\varepsilon) = A(y)\, (z_+ - z_-)^{-2\varepsilon} \int_0^1 dt\, (1-t)^{-\varepsilon}\, t^{-\varepsilon}\, (1 + A(y)\, t)^{-1}, \tag{B.26}$$

and (B.19) easily follows from Integral E..

Integral G.

$$\mathcal{I}_2(\varepsilon) = \int_{z_-}^{z_+} dz\, (z_+ - z)^{-\varepsilon} (z - z_-)^{-\varepsilon} = (z_+ - z_-)^{1-2\varepsilon}\, B(1-\varepsilon). \tag{B.27}$$

Integral H.

$$\mathcal{I}_3(\varepsilon) = \int_{z_-}^{z_+} dz\, z\, (z_+ - z)^{-\varepsilon} (z - z_-)^{-\varepsilon} = \frac{1}{2}(z_+ + z_-)\, \mathcal{I}_2(\varepsilon). \tag{B.28}$$

Integral I. Consider now

$$\mathcal{I}_4(y;\varepsilon) = \int_{z_-}^{z_+} dz\, \frac{(z_+ - z)^{-\varepsilon}(z - z_-)^{-\varepsilon}}{z\,(y+z)}$$
$$= \frac{z_-^{-1} A_1(y)}{A_1(y) - A_2}\,(z_+ - z_-)^{-2\varepsilon}\,\{\mathcal{F}(A_1(y);-\varepsilon) - \mathcal{F}(A_2;-\varepsilon)\}\,, \quad \text{(B.29)}$$

where

$$A_1(y) = A(y)\,, \quad A_2 = \frac{z_+ - z_-}{z_-} \quad \text{(B.30)}$$

with $A(y)$ defined in (B.20).

Proof. We use the substitution (B.25) and decompose the integrand into the proper fractions. Finally use Integral F.

Integral J.

$$\mathcal{I}_5(\varepsilon) = \int_{z_-}^{z_+} dz\, \frac{(z_+ - z)^{-\varepsilon}(z - z_-)^{-\varepsilon}}{z} = (z_+ - z_-)^{-2\varepsilon}\,\mathcal{F}(A_2;-\varepsilon)\,. \quad \text{(B.31)}$$

Integral K.

$$\mathcal{I}_6(y;\varepsilon) = \int_{z_-}^{z_+} dz\, \frac{(z_+ - z)^{-\varepsilon}(z - z_-)^{-\varepsilon}}{z^2} = (z_+ - z_-)^{-2\varepsilon}\, z_-^{-1}\,\mathcal{G}(A_2;-\varepsilon)\,, \quad \text{(B.32)}$$

where we have defined the function

$$\mathcal{G}(A;\kappa) = A\,B(1+\kappa)\,F(2, 1+\kappa, 2(1+\kappa); -A)\,. \quad \text{(B.33)}$$

Let us note the following reflection formula

$$\mathcal{G}(A;\kappa) = -\left(1 - \overline{A}\right)\,\mathcal{G}\left(-\overline{A};\kappa\right)\,, \quad \text{(B.34)}$$

where the "bar" operation is defined in (B.24). We note however, that one has to verify if this operation is permitted for κ being close to zero as it is not always the case in practice.

Integral L.

$$\mathcal{I}_7(y;\varepsilon) = \int_{z_-}^{z_+} dz\, \frac{(z_+ - z)^{1-\varepsilon}(z - z_-)^{1-\varepsilon}}{(y+z)^2}$$
$$= (z_+ - z_-)^{1-2\varepsilon}\,\mathcal{H}(A(y);-\varepsilon)\,, \quad \text{(B.35)}$$

where $A(y)$ is as in (B.20) and we have introduced

$$\mathcal{H}(A;\kappa) = A^2\,B(2+\kappa)\,F(2, 2+\kappa, 2(2+\kappa); -A)\,. \quad \text{(B.36)}$$

Integral M. The following integral is needed to order $\mathcal{O}(\kappa)$

$$\mathcal{J}_1(\kappa) = \int_0^1 dy \, \frac{1}{y^{1-\kappa}} \mathcal{F}\left(\frac{1}{y};\kappa\right) = \frac{1}{2\kappa^2} - \frac{\pi^2}{6} + \mathcal{O}(\kappa) \,. \tag{B.37}$$

Proof. Using (B.22) we have

$$\mathcal{J}_1(\kappa) = \int_0^1 dy \int_0^1 dt \, \frac{y^{\kappa-1}\left[t(1-t)\right]^{\kappa}}{y+t}. \tag{B.38}$$

We can separate both the integrals using the following trick

$$\frac{1}{y+t} = \int_0^\infty d\lambda \, e^{-\lambda(y+t)}. \tag{B.39}$$

We get

$$\mathcal{J}_1(\kappa) = \int_0^\infty d\lambda \int_0^1 dt \, \left[t(1-t)\right]^{\kappa} e^{-\lambda t} \int_0^1 dy \, y^{\kappa-1} e^{-\lambda y}. \tag{B.40}$$

The last integral reads

$$\int_0^1 dy \, y^{\kappa-1} e^{-\lambda y} = \lambda^{-\kappa} \left(\Gamma(\kappa) - \Gamma_\lambda(\kappa)\right). \tag{B.41}$$

Now we have

$$\mathcal{J}_1(\kappa) = \mathcal{J}_{1a}(\kappa) - \mathcal{J}_{1b}(\kappa)\,, \tag{B.42}$$

where

$$\mathcal{J}_{1a}(\kappa) = \Gamma(\kappa)\Gamma(1-\kappa)\, B(2\kappa, 1+\kappa)\,, \tag{B.43}$$

$$\mathcal{J}_{1b}(\kappa) = \int_0^1 dt \, \left[t(1-t)\right]^{\kappa} \int_0^\infty d\lambda \, \lambda^{-\kappa} e^{-\lambda t} \Gamma_\lambda(\kappa)\,. \tag{B.44}$$

The first integral was evaluated using

$$\int_0^\infty d\lambda \, \lambda^{-\kappa} e^{-\lambda t} = \Gamma(1-\kappa)\, t^{\kappa-1} \tag{B.45}$$

and the definition of Beta function (B.5). Let us find the expansion in κ of $\mathcal{J}_{1b}$. First, note that using the alternative definition (B.4) of $\Gamma_\lambda(\kappa)$ it can be written as follows

$$\mathcal{J}_{1b}(\kappa) = \int_0^1 dt \, \left[t(1-t)\right]^{\kappa} \int_0^\infty dw \, \frac{(1+w)^{\kappa-1}}{1+w+t}, \tag{B.46}$$

where we performed the trivial integration over $d\lambda$. The remaining expression is finite for $\kappa = 0$, thus

$$\mathcal{J}_{1b}(\kappa) = \frac{\pi^2}{12} + \mathcal{O}(\kappa)\,. \tag{B.47}$$

On the other hand, according to Appendix B.2 we have

$$\mathcal{J}_{1a}(\kappa) = \frac{1}{2\kappa^2} - \frac{\pi^2}{12} + \mathcal{O}(\kappa)\,. \tag{B.48}$$

Thus we get (B.37).

Integral N. The following integral is needed to order $\mathcal{O}(\kappa)$

$$\mathcal{J}_2(\kappa) = \int_0^1 dy\, \frac{1}{y^{1-\kappa}}\, \mathcal{G}\left(\frac{1}{y};\kappa\right) = \frac{1}{2\kappa} - \log 2 + \mathcal{O}(\kappa)\,. \tag{B.49}$$

Proof. Let us first express the function $\mathcal{G}$ using (B.17) as follows

$$\mathcal{G}\left(\frac{1}{y};\kappa\right) = -\kappa\,\mathcal{F}\left(\frac{1}{y};\kappa\right) + \kappa\, B(\kappa+2,\kappa)\,\frac{1}{y} F\left(1,\kappa+2,2(1+\kappa);-\frac{1}{y}\right). \tag{B.50}$$

However

$$B(\kappa+2,\kappa)\,\frac{1}{y} F\left(1,\kappa+2,2(1+\kappa);-\frac{1}{y}\right) = \int_0^1 dt\, \frac{t^{\kappa+1}(1-t)^{\kappa-1}}{y+t}. \tag{B.51}$$

Therefore, we have to evaluate the following integral

$$\mathcal{J}_2^\star(\kappa) = \int_0^1 dy \int_0^1 dt\, \frac{y^{\kappa-1}t^{\kappa+1}(1-t)^{\kappa-1}}{(y+t)}. \tag{B.52}$$

Repeating several the same steps as in Integral M. we get

$$\mathcal{J}_2^\star(\kappa) = \mathcal{J}_{2a}^\star(\kappa) - \mathcal{J}_{2b}^\star(\kappa)\,, \tag{B.53}$$

where

$$\mathcal{J}_{2a}^\star(\kappa) = \Gamma(\kappa)\,\Gamma(1-\kappa)\,B(\kappa,1+2\kappa)\,, \tag{B.54}$$

$$\mathcal{J}_{2b}^\star(\kappa) = \int_0^1 dt\, t^{\kappa+1}(1-t)^{\kappa-1} \int_0^\infty dw\, \frac{(1+w)^{\kappa-1}}{1+w+t}. \tag{B.55}$$

In order to find the Laurent expansion in κ of $\mathcal{J}_{2b}^\star(\kappa)$, let us use the following trick

$$\mathcal{J}_{2b}^\star(\kappa) = \int_0^1 dt\, t^{\kappa+1}(1-t)^{\kappa} \int_0^\infty dw\, \frac{(1+w)^{\kappa-1}}{1+w+t} + \int_0^1 dt\, t^{\kappa+1}(1-t)^{\kappa-1} \int_0^\infty dw\, \frac{(1+w)^{\kappa-1}}{2+w}. \tag{B.56}$$

The first integral is finite, thus is of order $\mathcal{O}(1)$ and can be dropped; recall that $\mathcal{J}_2^\star$ is multiplied by κ, see (B.50). Therefore, up to required order we get (using definition of Beta function and performing elementary integral)

$$\mathcal{J}_{2b}^\star(\kappa) = \frac{\log 2}{\kappa} + \mathcal{O}(1)\,. \tag{B.57}$$

Using the result for Integral M. and gathering all the pieces we finally obtain (B.49).

B.2 Expansions in ε

In this Appendix we list some of the necessary expansions. The spacetime dimension is defined as $D = 4 - 2\varepsilon = 4 + 2\kappa$, $\varepsilon, \kappa > 0$.

Expansion A. (Euler's Gamma) The expansion has the following form

$$\Gamma\left(\varepsilon\right)=\frac{1}{\varepsilon}-\gamma+\frac{1}{12}\left(6\gamma^{2}+\pi^{2}\right)\varepsilon+\mathcal{O}\left(\varepsilon^{2}\right), \tag{B.58}$$

where γ is Euler's constant

$$\gamma\approx 0.577216. \tag{B.59}$$

It is useful to list also

$$\Gamma\left(1+\varepsilon\right)=1-\gamma\varepsilon+\mathcal{O}\left(\varepsilon^{2}\right). \tag{B.60}$$

Expansion B. (Euler's Beta) The following expansions are useful

$$B\left(1-\varepsilon\right)=1+2\varepsilon+\left(4-\frac{\pi^{2}}{6}\right)\varepsilon^{2}+\mathcal{O}\left(\varepsilon^{3}\right), \tag{B.61}$$

$$B\left(a\,\varepsilon,1-\varepsilon\right)=\frac{1}{a}\frac{1}{\varepsilon}+\frac{\pi^{2}}{6}\varepsilon+\mathcal{O}\left(\varepsilon^{2}\right). \tag{B.62}$$

Expansion C.

$$\mathcal{F}\left(A;\kappa\right)=\log\left(1+A\right)+\kappa\,\mathcal{F}_{1}\left(A\right)+\mathcal{O}\left(\kappa^{2}\right), \tag{B.63}$$

where

$$\mathcal{F}_{1}\left(A\right)=\log\left(1+A\right)\log\left(\frac{1+A}{A}\right)-\frac{1}{6}\pi^{2}+\mathrm{Li}_{2}\left(\frac{1}{1+A}\right)+\mathrm{Li}_{2}\left(-A\right). \tag{B.64}$$

Proof. We start with the integral definition of $\mathcal{F}\left(A;\kappa\right)$ (B.22) and expand integrand to desired order. The less trivial integrals to be calculated are

$$\int_{0}^{1}dt\,\frac{\log\left(1-t\right)}{1+At}=\frac{1}{A}\mathcal{J}_{a}-\frac{\log A\,\log\left(1+A\right)}{A}, \tag{B.65}$$

$$\mathcal{J}_{a}=\int_{\frac{1}{1+A}}^{1}dz\,\frac{\log\left[\left(1+A\right)\left(1-z\right)\right]}{z}=\log^{2}\left(1+A\right)-\frac{1}{6}\pi^{2}+\mathrm{Li}_{2}\left(\frac{1}{1+A}\right), \tag{B.66}$$

where we used (B.8), (B.9).

$$\int_{0}^{1}dt\,\frac{\log t}{1+At}=\frac{1}{A}\mathcal{J}_{b}-\frac{\log A\,\log\left(1+A\right)}{A}, \tag{B.67}$$

$$\mathcal{J}_{b}=\lim_{\eta\to 0}\int_{1+\eta}^{1+A}du\,\frac{\log\left(u-1\right)}{u}=\log A\,\log\left(1+A\right)+\mathrm{Li}_{2}\left(-A\right). \tag{B.68}$$

Expansion D.

$$\mathcal{G}\left(A;\kappa\right)=\frac{1}{1+A}-\kappa\,\frac{2+A}{1+A}\log\left(1+A\right)+\mathcal{O}\left(\kappa\right). \tag{B.69}$$

Expansion E. Let us also note the following expansion

$$F(1,\kappa,2\kappa+1;-A)=1-\kappa\log(1+A)+\mathcal{O}(\kappa)\,. \tag{B.70}$$

Proof. We use the definition (B.13)

$$\begin{aligned}F(1,\kappa,2\kappa+1;z)&=1+z\frac{\kappa}{2\kappa+1}+\frac{1}{2!}z^2\frac{2!\kappa(\kappa+1)}{(2\kappa+1)(2\kappa+2)}+\frac{1}{3!}z^3\frac{3!\kappa(\kappa+1)(\kappa+2)}{(2\kappa+1)(2\kappa+2)(2\kappa+3)}+\ldots\\&=1+\kappa\left[z+\frac{1}{2!}z^2\frac{2!}{2!}+\frac{1}{3!}z^3\frac{3!2!}{3!}+\ldots+\frac{1}{n!}z^n\,(n-1)!+\ldots\mathcal{O}(\kappa)\right]\\&=1+\kappa\sum_{n=1}^{\infty}\frac{z^n}{n}=1-\kappa\log(1-z)+\mathcal{O}\left(\kappa^2\right).\end{aligned} \tag{B.71}$$

B.3 The "plus" distribution

We define the generalized "plus" distribution $f_{[a,b]}$ as follows

$$f_{[a,b]}(u)=f(u)-\delta(u-a)\int_a^b dy\,f(y)\,. \tag{B.72}$$

The action on sufficiently smooth test function $\varphi(u)$ with the support $[u_{\min},u_{\max}]$ is thus

$$\begin{aligned}\int_{u_{\min}}^{u_{\max}}du\,f_{[a,b]}(u)\,\varphi(u)&=\int_{u_{\min}}^{u_{\max}}du\,f(u)\left[\varphi(u)-\varphi(a)\right]\\&+\varphi(a)\left[\int_{u_{\min}}^{a}dy\,f(y)+\int_b^{u_{\max}}dy\,f(y)\right],\end{aligned} \tag{B.73}$$

if $a\in[u_{\min},u_{\max}]$.

Note the following useful property

$$f_{[a,b]}(u)\,h(u)=\left[f(u)\,h(u)\right]_{[a,b]}+\delta(u-a)\int_a^b dy\,f(y)\left[h(y)-h(a)\right], \tag{B.74}$$

which is the simple consequence of the definition.

Now, suppose that u is a function of some other variable, i.e. there is the following transformation

$$u=u(z)\,, \tag{B.75}$$

$$u_{\min}=u(z_{\min})\,, \tag{B.76}$$

$$u_{\max}=u(z_{\max})\,. \tag{B.77}$$

Define also

$$z_a=u^{-1}(a)\,, \tag{B.78}$$

$$z_b=u^{-1}(b)\,. \tag{B.79}$$

Then, the action of the distribution in u (B.72) on a test function of z can be calculated as follows

$$\begin{aligned}\int_{z_{\min}}^{z_{\max}}dz\,f_{[a,b]}(u(z))\,\varphi(z)&=\int_{z_{\min}}^{z_{\max}}dz\,f(u(z))\left[\varphi(z)-\varphi(z_a)\frac{u'(z)}{u'(z_a)}\right]\\&+\varphi(z_a)\left[\int_{u_{\min}}^{a}dy\,f(y)+\int_b^{u_{\max}}dy\,f(y)\right].\end{aligned} \tag{B.80}$$

B.4 Dipole integral for massless FE-IS $\mathbf{Q} \to \mathbf{Q}g$ splitting with massive spectator

The starting point is the formula for integrated dipole (3.282) with $\eta^2 = 0$. It reads

$$\tilde{I}^{\text{FE-IS}}_{q\to qg,\, a} = \frac{2C_F \eta^2_{\mathcal{J}}}{u^{1-\kappa}\left(\tilde{\eta}^2_{\mathcal{P}^2}\right)^{1-\kappa}} \left\{ \frac{1}{v}\mathcal{F}\left(\mathcal{A}\left(u\right);\kappa\right) - B\left(1+\kappa\right)\left(1 - \frac{1}{4}v\left(1+\kappa\right)\right) \right\}, \tag{B.81}$$

where we replaced heavy quark indicator $\mathbf{Q}$ for q in the subscript of the integral. In considered limit

$$\mathcal{A}\left(u\right) = \frac{1}{u}\frac{1}{1+\left(1+2u\right)\eta^2_a}, \tag{B.82}$$

$$v = \frac{1}{1+2u\eta^2_a} \tag{B.83}$$

and $\tilde{\eta}^2_{\mathcal{P}^2}$, $\tilde{\eta}^2_{\mathcal{J}}$ are easily recovered from Appendix A.2.1.1. Follwoing Section 3.7.2.3 we introduce

$$u = \mathfrak{r}\,\mathcal{B}\left(\mathfrak{r}\right), \tag{B.84}$$

with $\mathcal{B}\left(\mathfrak{r}\right)$ given in Eq. (3.347). Rearranging the terms we get

$$\begin{aligned}\tilde{I}^{\text{FE-IS}}_{q\to qg,\, a} =& \frac{2C_F}{v\mathcal{B}\left(\mathfrak{r}\right)}\frac{\eta^2_{\mathcal{J}}}{\tilde{\eta}^2_{\mathcal{P}^2}}\left[\frac{1}{\mathfrak{r}}\log\left(1+\mathfrak{r}\right) + \left(\frac{1}{\mathfrak{r}}\log\frac{1}{\mathfrak{r}}\right)_+\right] - C_F\frac{\eta^2_{\mathcal{J}}}{\tilde{\eta}^2_{\mathcal{P}^2}}\left(2-\frac{v}{2}\right)\left(\frac{1}{u}\right)_{[0,u_+]} \\ &+ \delta\left(u\right)C_F\left[\frac{1}{\kappa^2} - \frac{1}{\kappa}\log\left(1+\eta^2_a\right) - \frac{1}{2}\left(\pi^2 - \log^2\left(1+\eta^2_a\right)\right) - \frac{3}{2\kappa} - \frac{3}{2}\log u_+ + \frac{7}{2}\right].\end{aligned} \tag{B.85}$$

It can be now confronted with [10] for massless spectator, $\eta^2_a = 0$, finding agreement.

Nomenclature

$\underline{ai}$	the parton that results in splitting process $a \to \underline{ai}\, i$, page 53
CM (p, q)	center of mass frame for momenta p and q, page 65
$\mathfrak{C}_n$	the (quasi-)collinear subtraction term for n-parton configuration, see equation (4.34), page 104
$\mathcal{D}_{F_n}$	the sum over all the dipoles with corresponding jet functions, see equation (3.423), page 92
$\tilde{p}_k^{\mu}$, $\tilde{u}$, $\tilde{w}$, $\tilde{z}$	dipole momentum and variables, page 47
η_X^2	the rescaled mass or any other quantity X; if X is a parton indicator, $X = i, j, a, \ldots$, then $\eta_X^2 = m_X^2/2\tilde{\gamma}$; if X is any other kinematic quantity $X = \mathcal{P}^2, \mathcal{P}_a, \ldots$ then $\eta_X = X/2\tilde{\gamma}$, see equation (3.267), page 74
$\tilde{\eta}_X^2$	$\tilde{\eta}_X^2 = \eta_X^2/u$, see equation (3.270), page 74
$f_a\,(z, \mu_r)$	the renormalized distribution function for a parton a inside a hadron, page 17
$\mathcal{F}_{ab}^{(R)}$	the parton inside a parton density renormalized in a scheme R, page 18
F_n	the jet function, page 19
$\tilde{\gamma}$	special invariant $\tilde{p}_{\underline{ij}} \cdot p_a$ for FE-IS or $\tilde{p}_{\underline{j}} \cdot p_a$ for IE-FS, page 48
$\overline{\text{mMS}}$	'massless $\overline{\text{MS}}$' – indication for a massless calculation in $\overline{\text{MS}}$ renormalization scheme, see equation (1.22), page 18
N_a	active number of flavours used to define CWZ subscheme, page 22
N_f'	the number of all quark flavours plus gluon, page 23
N_f	the total number of quark flavours (light and heavy), page 22
N_l	the number of light partons (i.e. light quarks and a gluon), page 23
N_q	the number of light quarks, page 23
$N_{\mathbf{Q}}$	the number of heavy quarks, page 23
$\mathbb{N}_x$	the set of partons corresponding to the index $x = f, q, \mathbf{Q}, l$ etc.
$\mathcal{P}$	total dipole momentum $p_i + p_j$, page 47

$\mathcal{P}_a$	the invariant $(p_i + p_j) \cdot p_a$, page 47
P_{ab}	ordinary splitting functions, page 17
p_{ai}	$p_a - p_i$, page 53
$\Pi\,(n\|a)$	configuration of n partons in final state, when the initial state is a, page 92
$\underline{ij}$	the parton that splits to i and j, page 47
$\mathcal{Q}$	dipole momentum transfer $\mathcal{P} - p_a$, page 47
$d\mathfrak{S}_{(n)\,I,J}$	the pseudo cross section, see equation (3.425), page 93
v	velocity of p_a in CM$(\mathcal{Q}, p_a)$ frame, page 48
$\tilde{v}_q$	$\sqrt{1 - m_a^2 m_q^2/\tilde{\gamma}^2}$, page 50
$\mathbb{X}_{\mathcal{T}}$	the set of indices for enumerating the final state momenta in factorized phase space, where $\mathcal{T}$ is either FE-IS, IE-FS or FE-FS, see equation (3.67), page 46
$\mathcal{X}$, $\mathcal{Y}$	the external invariants, see equation (3.128), page 53