Integrability in the AdS/CFT correspondence: classical and BFKL solutions

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Abstract

In this thesis, the algebraic curve classification of the AdS/CFT correspondence is investigated. A novel method of assigning algebraic curves to Wilson loop minimal surfaces is proposed, different from the usual construction from flat connection monodromy, which is trivial for such cases. The new definition is shown to be meaningful, by recovering the original solution with the existing reconstruction formulae. Two examples, namely the null cusp and quark–antiquark potential Wilson loops, are worked out. Additionally, two examples of solutions dual to correlation functions are compared. Even though described by the same quasi-momentum, they are shown to have different algebraic curves.

Further, the algebraic curve formalism is applied to the description of a scattering process with an exchange of the BFKL pomeron. A dual string solution with appropriate conserved charges is constructed, along with a semi-classical expansion of the Regge trajectory intercept. The position of cuts on the algebraic curve is determined from the reality conditions, and results in the expected values of the conserved charges.

W poniższej rozprawie badane są krzywe algebraiczne pojawiające się w korespondencji AdS/CFT. Zaproponowana zostaje nowa metoda przypisywania krzywych algebraicznych do powierzchni minimalnych dualnych do pętli Wilsona, dla których typowa konstrukcja oparta na monodromii płaskiej koneksji nie znajduje zastosowania. Poprzez odtworzenie oryginalnych rozwiązań przy użyciu istniejących wzorów rekonstrukcyjnych pokazane zostaje, że nowa konstrukcja krzywych nie jest trywialna. Szczegółowo zostają przedyskutowane przykłady pętli Wilsona w kształcie przecięcia dwóch linii światłopodobnych oraz potencjału kwark–antykwark. Ponadto porównane zostają dwa przykłady rozwiązań dualnych do funkcji korelacji, które, chociaż opisane przez ten sam kwazipęd, okazują się mieć różne krzywe algebraiczne.

W drugiej części formalizm krzywej algebraicznej zostaje zastosowany do opisu rozpraszania cząstek z wymianą pomeronu BFKL. Zaproponowane zostaje dualne rozwiązanie strunowe, a także rozwinięcie klasyczne wyrazu wolnego trajektorii Regge. Opisana zostaje stosowna krzywa algebraiczna, której cięcia zostają zidentyfikowane w oparciu o warunki rzeczywistości, prowadząc do oczekiwanych wartości zachowanych ładunków.

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Chapter 1

Introduction

The AdS/CFT correspondence is an astonishing conjecture stating that two, seemingly unrelated, theories of physics are actually identical. These theories, the string theory in $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, differ not only in the dimensionality of spaces on which they are defined, but even more in character: one contains gravity, and the other is a gauge theory without neither gravity, nor even a mass scale. And yet, there is a heuristic argument by which they should coincide, followed by thousands of meaningful results published over less than twenty years that support this claim.

This intrinsic beauty would be by itself a reason to study this correspondence, but AdS/CFT has very interesting applications as well. Although neither of the two theories is exactly a theory of our world, $\mathcal{N} = 4$ SYM resembles quantum chromodynamics in many aspects. Moreover, it is just one (albeit the best investigated) member of the family of gauge–gravity dualities; what we learn about one, may be fruitful for understanding another, in which the realistic theories may feature more explicitly.

What makes AdS/CFT special (but not unique) is the phenomenon of integrability. Both theories of the correspondence have been discovered to be exactly solvable. Not only this means that more quantities can be computed in either, but also suggests even stronger that the correspondence is true. It may even turn out to be the technique to prove it.

In this thesis, we begin by introducing all the necessary physical context and mathematical formalism. The extent is more or less tailored to the scope of the subsequent results, and by no means exhaustive. Whatever this thesis is lacking in this aspect, can be found in one of the reviews [AGM+99, Nast07, IntR10] or excellent dissertations [Vice08, Viei08, Łuko10].

The central object that will be of our interest throughout this thesis is the algebraic curve, a complex variety that appears in the study of many integrable systems. Besides its theoretical importance, it has a practical aspect in the sense that usually it is defined by a function called quasi-momentum that encodes the conserved charges of the system. Most importantly, charges like energy or spin can be obtained from quasi-momentum asymptotics at special values, without in fact solving the non-linear equations of motion of the theory.

As the algebraic curve for strings is traditionally defined in terms of a parallel transport of a matrix along the worldsheet, there are configurations, such as Wilson loop minimal surfaces, for which this construction trivially fails. Studying these objects, we propose a new construction, which is entirely local, ie. does not require the knowledge of the full worldsheet. We also validate the result by reconstructing the original configurations from principles of analytic properties of functions defined on algebraic curves.

Further, we identify a pair of solutions dual to correlation functions. These do have non-contractible contours, and the usual procedure assigns the same algebraic curve to both. Surprisingly, we note that our construction predicts different algebraic curves, which again allows for a reconstruction of the original solution. This calls for a discussion of the usually implied one-to-one relation between the quasi-momenta and algebraic curves.

We also apply the formalism to describe a particular process of high-energy scattering. We modify a known string solution, the GKP curve, so that its non-zero conserved charges correspond to the quantum numbers of the exchanged virtual particle, the BFKL pomeron. Knowing the algebraic curve for the original solution, we modify it accordingly. The position of cuts needs to be determined anew, but the reality conditions are enough to fix it. The result produces the expected values of the conserved charges and integral equations.

Technical remarks

This thesis is based on the following two journal articles:

- *Surprises in the AdS algebraic curve constructions: Wilson loops and correlation functions*
  Romuald A. Janik and Paweł Laskoś-Grabowski
  Nuclear Physics B **861** (2012) 361, `arXiv:1203.4246 [hep-th]`
- *Approaching the BFKL pomeron via integrable classical solutions*
  Romuald A. Janik and Paweł Laskoś-Grabowski
  Journal of High Energy Physics **1401** (2014) 074, `arXiv:1311.2302 [hep-th]`

They are referred to as [JLG12, JLG13], respectively.

The structure of this thesis was designed to separate the introductory and review parts and the new results, and this goal was achieved to a first approximation. Specifically, chapter 2 is an introduction to the AdS/CFT correspondence and is rather sketchy, whereas chapter 3 reviews some of the existing results and techniques of integrability in the context of AdS/CFT. In section 3.2 there is a new result, namely the derivation of the asymptotics (3.50) in the $AdS_3 \times S^1$ case, which for convenience of the reader is located in its right context. The introductory part is presented in a bottom-up order, namely we first introduce the theories involved in the AdS/CFT correspondence before outlining the correspondence itself, and first discuss the appearance of some aspects of integrability before hinting at the full wealth of this formalism in AdS/CFT.

Subsequently, chapter 4 consists of the analysis of algebraic curves for Wilson loops and correlation functions, and is based exclusively on [JLG12]. The algebraic curve of the BFKL pomeron appears in chapter 5, which is based on [JLG13]. It contains a derivation of the solution to the GKP folded string equations, which is nothing new, but is again helpful to understand the modifications that are applied to it later. Finally, chapter 6 is a with a brief summary and a review of possible open problems, followed by an appendix A on elliptic functions, which appear copiously throughout the text.

This thesis uses the 'mostly plus' signature for the Minkowski metric; the metric itself is denoted by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \dots)$. The same symbol is used as η_{AB} for the metric of the AdS_d embedding space $\mathbb{R}^{d+1}$, therefore having an additional component $\eta_{d,d} = -1$. The worldsheet coordinates are $\tau, \sigma \equiv \sigma^0, \sigma^1$, respectively time-like and space-like in Lorentzian case. The light-cone coordinates are denoted by $w, \bar{w}$ and defined as

$$w = \sigma + \tau \qquad \bar{w} = \sigma - \tau \qquad \text{(Lorentzian worldsheet)} \tag{1.1}$$

$$w = \sigma + i\tau \qquad \bar{w} = \sigma - i\tau \qquad \text{(Euclidean worldsheet)} \tag{1.2}$$

and the following shorthands are used for the derivatives

$$\partial \equiv \frac{\partial}{\partial w} \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{w}} \tag{1.3}$$

Some effort has been made to keep the notation as unambiguous as possible, but at times this would require violating consistency with the literature. And so, depending on context, g may denote either the rescaled coupling, or genus, or isometry group element, whereas J is either flat connection or spin of an operator. We hope that the appropriate meaning of any given symbol will be clear from context. Also we want to warn the reader that there are several quantities for which there is no universally accepted definitions, and for instance many papers use ν that is different by a factor of two with respect to this thesis. The same applies to the older convention on the rescaled coupling g (2.22).

All figures were made with either Mathematica or Ti*k*Z.

Chapter 2

AdS/CFT correspondence

This chapter introduces the two physical theories that appear in the most popular variant of the AdS/CFT correspondence, which is also the one that is studied in this thesis. Then, the correspondence itself is outlined from a general angle, as well as for the specific results that will prove useful later on. Some broader context is given, albeit only slightly. Any comprehensive discussion would be beyond the scope of this thesis due to the extremely fertile nature of the subject.

2.1 String theory in $AdS_5 \times S^5$

String theory is an idea which began emerging in the 1960s, that generalised the notion of elementary point-like particle. The basic objects now acquired finite dimensions, and in case of just one dimension were called strings. Theories of their higher-dimensional analogues, branes (generalised from two-dimensional membranes), did not achieve much success, but branes themselves very often appear as background objects in string theory, for instance with open strings attached to them. Over the years, string theory was suspected to be the theory of strong interactions (its original purpose), the theory of quantum gravity (which it is, despite some caveats and criticisms), and even the theory of everything (which is still the highest hope invested in it). The most secure statement is that it has proven to be a very useful and powerful, even if unwieldy and controversial, tool that has produced significant results in many branches of physics and even provided some insights in mathematics. A classical textbook on the topic is [Polc98], while an excellently accessible introduction is [Tong09].

Just as a trajectory of a particle in spacetime is a worldline parameterised by one variable, the evolution of a string is described by a two-dimensional worldsheet embedded in the spacetime and parameterised by two coordinates, called τ, σ. The Lagrangean formalism determines the motion of a string by the condition of minimising the area of the worldsheet. For a string moving in flat space, whose worldsheet is given in target space coordinates as $X^\mu(\sigma,\tau)$, the induced metric on the worldsheet is

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\,\eta_{\mu\nu} \tag{2.1}$$

where η is the Minkowski metric. The action would be then proportional to the area of the worldsheet

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det\gamma} = \frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-(\dot{X})^2(X')^2 + (\dot{X}\cdot X')^2} \tag{2.2}$$

where dot and prime respectively denote differentiation in τ, σ, and the prefactor $\frac{1}{2\pi\alpha'}$ can be interpreted as string tension. This action, called Nambu–Goto action, leads to highly non-linear equations of motion, so an important development was made by introducing the Polyakov action, in which $\gamma_{\alpha\beta}(\sigma,\tau)$ is promoted to a variable in its own right

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det\gamma}\gamma^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \tag{2.3}$$

Of course, the equations of motion as a whole would be just as troublesome as for the Nambu–Goto action, but the gauge symmetries of the Polyakov action allow for a very specific choice of gauge. Namely, by using the reparameterisation invariance, one can impose the conformal gauge condition, in which the metric γ is proportional (by a scalar factor) to the Minkowski metric. Further, Weyl invariance

$$\gamma_{\alpha\beta}(\sigma,\tau) \mapsto \Theta(\sigma,\tau)^2\,\gamma_{\alpha\beta}(\sigma,\tau) \tag{2.4}$$

can be used to impose the flat gauge, $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, in which the equations of motion are just wave equations

$$\partial_\alpha\partial^\alpha X^\mu = 0 \tag{2.5}$$

This simplicity is slightly obfuscated by what remains of the γ equations of motion after the gauge choice. These constraint equations, called the Virasoro constraints, amount to the vanishing of the stress-energy tensor and read in general

$$\partial_\alpha X_\mu\,\partial_\beta X^\mu - \tfrac{1}{2}\,\eta_{\alpha\beta}\,\eta^{\gamma\delta}\,\partial_\gamma X_\mu\,\partial_\delta X^\mu = 0 \tag{2.6}$$

In the conformal gauge their explicit form is

$$(\dot{X})^2 + (X')^2 = \dot{X}\cdot X' = 0 \tag{2.7}$$

respectively for the diagonal and off-diagonal components. Also, in the light-cone coordinates (1.1), in which the diagonal terms of the metric tensor vanish, and the off-diagonal terms are equal, the diagonal components of the constraints read

$$(\partial X)^2 = (\bar{\partial}X)^2 = 0 \tag{2.8}$$

while the off-diagonal are satisfied trivially.

A string theory course would proceed by considering excitations (vibrations) of a string, quantising them, and arriving at two peculiarities. Firstly, the ground state of a string is a tachyon, a particle of imaginary mass. Secondly, by a representation-theoretical argument, the first excited states should be massless, which is achieved only if the target space is 26-dimensional. In fact, from the point of view of 'reality' of string theory, the latter is less problematic than the former, as all the superfluous dimensions can belong to a compact manifold of an imperceivably small size.

So far, the excitations have only been bosonic, therefore describing forces without matter. Fermions are introduced by means of supersymmetry, and it turns out that for such theories the tachyon is no longer present, whereas the critical dimension is 10. Moreover, there are five different types of superstring theories, differing by field content, gauge symmetry group, and the existence of open strings. We will not go into any further detail here, as our calculations on the string theory side will be purely classical and the full supersymmetric Lagrangean will not even be needed. Still, from now on, we will formally be working with type IIB (closed) strings.

We will also work on a non-flat background, namely the product space $AdS_5 \times S^5$ of a five-dimensional anti-de Sitter space and a five-dimensional sphere. They are spaces of constant negative and positive curvature, respectively, which can be parameterised as embeddings in six-dimensional flat spaces with different metric signatures. Explicitly, for general number of dimensions d

$$\eta_{AB} Y^A Y^B = -Y_0^2 + Y_1^2 + \cdots + Y_{d-1}^2 - Y_d^2 = -1 \tag{2.9}$$
$$\delta_{AB} X^A X^B = X_1^2 + \cdots + X_{d+1}^2 = 1 \tag{2.10}$$

for the AdS_d, S^d, respectively.

There are several useful ways of parameterising the AdS space in terms of d independent coordinates, two of which can be expressed as follows

$$\begin{aligned} Y_0 &= \frac{x_0}{z} = \cosh\rho \sin t & \qquad Y_d &= \frac{1}{2z}(1 + z^2 + x^\mu x_\mu) = \cosh\rho\cos t \\ Y_i &= \frac{x_i}{z} = n_i \sinh\rho & \qquad Y_{d-1} &= \frac{1}{2z}(-1 + z^2 + x^\mu x_\mu) = n_{d-1}\sinh\rho \end{aligned} \tag{2.11}$$

The coordinates t, ρ are respectively the global time and the radial coordinate. The time is not periodic, so the space should be actually understood as a universal cover of AdS_d, depicted in fig. 2.1 for AdS_2. n_i, n_{d-1}, obeying $n_i \cdot n_i + n_{d-1}^2 = 1$, span a sphere of dimension $d-2$, which appears here as a factor of AdS_d. Any parameterisation of this sphere, together with t, ρ, form the global AdS coordinates, in which the induced metric reads

$$ds^2 = d\rho^2 - \cosh^2\rho \, dt^2 + \sinh^2\rho \, d\Omega_{d-2}^2 \tag{2.12}$$

In further discussion, a parameterisation of AdS_3 subspace will be used, and can be achieved by setting all but one Y_i, and therefore also n_i, to zero. Then, say,

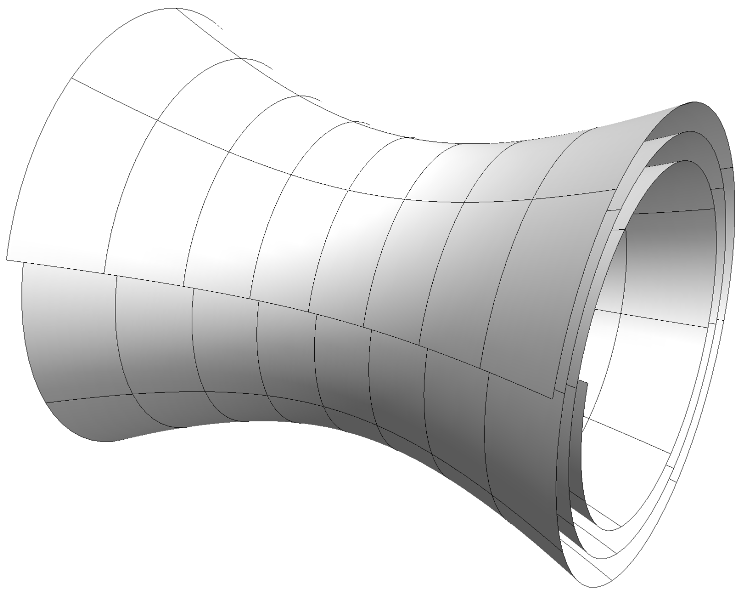

Figure 2.1: A rendering of an anti-de Sitter space, precisely AdS_2 as embedded in $\mathbb{R}^3$. The coordinate going around the 'reel' is the global time, and it is not periodic, therefore a universal cover of the space is drawn in the form of overlapping sheets, not unlike a roll of toilet paper.

$n_1^2 + n_{d-1}^2 = 1$ spans a S^1 factor which can be parameterised by an angular variable ψ so that

$$Y_1 = \sinh\rho\cos\psi \qquad\qquad Y_{d-1} = \sinh\rho\sin\psi \tag{2.13}$$

and the metric reads

$$ds^2 = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\psi^2 \tag{2.14}$$

The region of large ρ, where the (covering) space has the geometry of $\mathbb{R}_t \times S^{d-2}$, is the boundary of AdS_d of dimension $d-1$.

The other set of coordinates x_μ, z are the Poincaré coordinates in which the metric reads

$$ds^2 = \frac{1}{z^2}(dx^\mu dx_\mu + dz^2) \tag{2.15}$$

where the summation uses Minkowski signature. The coordinate $z > 0$ measures the distance from the boundary, and the metric can be understood as a Minkowski metric with an additional deformation depending solely on that distance. Restriction to the AdS_3 subspace amounts just to setting the superfluous x_μ to zero.

The Polyakov action on $AdS_5 \times S^5$ is straightforwardly

$$S = \frac{R^2}{4\pi\alpha'}\int d^2\sigma \left(-\partial_\alpha Y_A \partial^\alpha Y^A + \Lambda(Y_A Y^A + 1) - \partial_\alpha X_A \partial^\alpha X^A + \tilde{\Lambda}(X_A X^A - 1)\right) \tag{2.16}$$

where the Lagrange multipliers $\Lambda, \tilde{\Lambda}$ are introduced as usual to enforce the embedding conditions, and the radius R of both AdS and the sphere appears. The Virasoro constraints now read

$$\dot{Y}_A \dot{Y}^A + Y'_A Y'^A + \dot{X}_A \dot{X}^A + X'_A X'^A = \dot{Y}_A Y'^A + \dot{X}_A X'^A = 0 \tag{2.17}$$

Note that due to the presence of a curved background (or equivalently, the Lagrange multiplier terms in the action), the equations of motion are hard to solve even in the conformal gauge.

The action is necessarily invariant under the embedding space symmetries, $SO(2,4) \times SO(6)$ for the AdS and spherical parts, respectively. The conserved charges corresponding to these symmetries are [Tsey03, (2.13)]

$$\begin{aligned} S_{AB} &= \frac{R^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, (Y_A \dot{Y}_B - Y_B \dot{Y}_A) \\ J_{AB} &= \frac{R^2}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, (X_A \dot{X}_B - X_B \dot{X}_A) \end{aligned} \tag{2.18}$$

2.2 $\mathcal{N} = 4$ supersymmetric Yang–Mills theory

We will introduce the maximally supersymmetric four-dimensional Yang–Mills theory only in a very sketchy manner. The reason is that this thesis does not technically rely on any calculation in this theory, and it is only required as one of the sides of the AdS/CFT correspondence, and subsequently serves to motivate some of our work.

$\mathcal{N} = 4$ SYM, as it is commonly abbreviated, is a Yang–Mills theory with $SU(N_\mathrm{c})$ gauge group that was obtained in [BSS77] as a dimensional reduction of $\mathcal{N} = 1$ ten-dimensional SYM. Its particle content encompasses a gauge field, four fermionic fields, and six scalar fields, all interacting with a coupling constant g_YM. The scalars are quite often grouped in three pairs, whose complex combinations are called X, Y, Z.

This theory has a remarkable property of being conformally invariant, even at the quantum level. As a consequence, the beta function vanishes and there is no mass scale in the theory. The bosonic symmetry group of $\mathcal{N} = 4$ SYM is $SU(2,2) \times SU(4)$, where $SU(2,2)$ is the four-dimensional conformal group. Meanwhile $SU(4) \simeq SO(6)$ is an additional symmetry transforming scalar fields into one another, the so-called R-symmetry. Taking into account the supersymmetric generators, the full symmetry group is the projective special unitary group $PSU(2,2|4)$.

$\mathcal{N} = 4$ SYM, as a $SU(N_\mathrm{c})$ gauge theory, can be considered a close cousin of quantum chromodynamics, only a better-behaved one. It is much more symmetric, which makes it easier to handle, which, however, comes at a cost. It differs from QCD in many crucial aspects, most notably by the lack of mass scale or confinement, or the presence of supersymmetry, which is not present in QCD. Nevertheless, the relation between the two theories is one of the motivations to study $\mathcal{N} = 4$ SYM, in which the fields are characterised by anomalous dimension, ie. the number Δ that appears in the two-point function

$$\langle O(x)\bar{O}(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}} \tag{2.19}$$

and is the eigenvalue of the dilatation operator D, much like the energy is the eigenvalue of the Hamiltonian. A substantial part of what is meant by 'solving the $\mathcal{N} = 4$ SYM' is to determine the spectrum of anomalous dimension of fields. This is a non-trivial task, especially because the anomalous dimension receives quantum corrections. One of the striking and hardly coincidental resemblances is that the full anomalous dimension of twist-two operators in $\mathcal{N} = 4$ SYM matches the highest-transcendentality part of the QCD result. There is so far no theoretical understanding to this relation.

The observables of interest in $\mathcal{N} = 4$ SYM include not only the two-point functions, but also more complicated correlation functions, which contain the so-called structure constants of the theory. Another type of observable is the Wilson loop expectation value, ie. an ordered exponential of the gauge field along some closed contour C

$$W(C) = \frac{1}{N_\mathrm{c}} \operatorname{tr} P \exp \oint_C A_\mu dx^\mu \tag{2.20}$$

Also, as operators, albeit non-local, Wilson loops can enter correlation functions with other objects as well.

2.3 The duality

The keystone of the thesis is the AdS/CFT correspondence, also more generally called the gauge–gravity duality, which was initially introduced in [Mald97] and soon amplified by [GKP98, Witt98]. The very rough statement of the correspondence is that the two theories introduced in this chapter are actually two descriptions of the same reality.

Before going into details, note that this relation is indeed remarkable. The theories are very different at first glance: the string theory is a five-dimensional (sweeping the compactified subspace under the rug) theory of pure gravity, whereas $\mathcal{N} = 4$ SYM is a four-dimensional theory of gauge interactions with no mass scale. However, the symmetry groups of both theories do coincide, therefore giving a humble clue in favour of the correspondence.

AdS/CFT is usually motivated by painting a picture of a stack of N_c parallel $3+1$-dimensional massive branes at negligibly small distances from one another in a ten-dimensional space. Taking the low-energy limit, it can be argued that the open strings with endpoints on the branes decouple from the theory in the bulk, which becomes a free supergravity theory. The open strings transform as $U(N_\mathrm{c})$ fields due to the number of stacked branes on which their endpoints are, and the description of massless open string excitations actually matches the $\mathcal{N}=4$ SYM.

From the other point of view, the branes can be considered as sources for supergravity fields. Then it turns out that for the observer at an infinite distance from the branes, the closer the objects are to the branes, the lower their energy appears. Thus, in the low-energy limit, the region close to the branes decouples from the rest, which is again described by free supergravity. As it stands, we have two low-energy descriptions of the same situation as two pairs of decoupled theories, one of which is the same in both cases. Therefore it only makes sense to identify the other theories, that is, $\mathcal{N}=4$ SYM, and closed strings in a curved background. Its geometry is determined by the massive branes to be, unsurprisingly at this point, $AdS_5 \times S^5$.

It is crucial to correctly identify the parameters of the two corresponding theories. We get

$$\lambda \equiv g_{\mathrm{YM}}^2 N_\mathrm{c} = \frac{R^2}{(\alpha')^2} \qquad\qquad g_{\mathrm{string}} = g_{\mathrm{YM}}^2 = \frac{\lambda}{N_\mathrm{c}} \tag{2.21}$$

where λ is the newly introduced 't Hooft coupling and g_{string} is the string interaction coupling constant. Since $\mathcal{N}=4$ is exactly conformal even at the quantum level, λ remains an arbitrary dimensionless parameter. For convenience, very often a traditional rescaled coupling constant is used

$$g^2 = \frac{\lambda}{16\pi^2} \tag{2.22}$$

The parameter space of AdS/CFT is two-dimensional, usually parameterised by λ and $1/N_\mathrm{c}$.

It needs to be specified in what region of the parameter space the AdS/CFT correspondence holds. The modest, most widely accepted variant is confined to the planar limit, ie. $N_\mathrm{c} \to \infty$ with fixed λ. The name comes from the fact that for general number of colours N_c, a Feynman graph of $\mathcal{N}=4$ SYM is proportional to N_c to the power of minus the Euler characteristic (or genus) of the graph. Therefore, the leading contribution in the limit is given by the graphs with the lowest Euler characteristic, ie. planar. On the stringy side, by (2.21) the planar limit corresponds to non-interacting strings.

Note that in the planar limit, keeping the coupling (which in AdS/CFT context almost universally means 't Hooft coupling λ) small corresponds to the perturbative regime on the gauge theory side, or the dual theory of quantum free strings. Conversely, large coupling means either strongly coupled $\mathcal{N}=4$ SYM or classical free strings. This is perhaps the biggest appeal, aside from its philosophical beauty, of the AdS/CFT correspondence: by mapping strongly coupled regime of one theory to the classical regime of the other, it allows to treat the quantities that are hard to compute straightforwardly by translating the system by some AdS/CFT dictionary to a situation that will be easy to handle. For this reason, AdS/CFT is called a weak-strong duality.

This fact, while of utmost practical importance, also presents a dramatic drawback to any attempt to actually prove the AdS/CFT. The correspondence still has a status of conjecture, and even a more rigorous derivation would be a heuristic requiring a leap of faith. On one hand, there is already a huge body of results that act as clues in favour of the correspondence, and make this leap of faith rather non-demanding, but on the other it is not universally accepted if AdS/CFT will ever be proven (or even what it would exactly mean to prove it). Notably, one piece of evidence is the discovery of integrability of both theories of the correspondence.

Let us mention in passing that there also exist other variants of the correspondence, for instance strings on $AdS_4 \times CP^3$ corresponding to a three-dimensional CFT, and a class of dualities with different spaces of compactified dimension. The dualities, in which the string theory is four-dimensional, can be perceived as attempts in the direction of explaining 'our' four-dimensional gravity as a gauge theory, to mend the long-standing disparity between the Standard Model and general relativity. Also worth mentioning are some attempts at formulating what would be the AdS/QCD correspondence, or some even more down-to-earth, even if speculative applications to condensed matter theory. There also is a fertile field of research of the properties of the quark–gluon plasma, where a non-zero temperature of the boundary theory corresponds to the presence of a black brane in the bulk.

For the forthcoming discussion, we need to specify a few entries of the AdS/CFT dictionary. Namely, the boundary theory Wilson loop expectation values correspond to minimal surfaces in the bulk spanned by these loops, and the same goes for correlation functions of a large class of non-protected operators. Mathematically, expectation values of the Wilson loops or the correlators are essentially the exponentials of the areas of the respective surfaces. In case of the Wilson loops, the minimal surface can be viewed as the worldsheet of an open string, whose propagating endpoints trace out the loop. Two-point function minimal surface corresponds to a string state emitted at the position of one operator, propagating out into the bulk and back to the boundary to be absorbed at the position of the other operator. For local operators, the string would be point-like at the boundary; for Wilson loops it would be shaped like the loop, and obviously may change shape in the course of evolution. Viewed in the Poincaré patch, the minimal surfaces are drawn into the bulk by the scaling factor of the metric (2.15).

Another important part is the identification of the charges on both sides of the correspondence. The string charges defined as (2.18) translate to the boundary theory conformal group generators as follows [Tsey03, (2.20)]

$$S_{\mu\nu} = M_{\mu\nu} \qquad S_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu) \qquad S_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu) \qquad S_{54} = D \tag{2.23}$$

with $M_{\mu\nu}, P_\mu, K_\mu, D$ being respectively the Lorentz boosts and rotations, translations, special conformal transformations, and dilatation. The energy is $\frac{1}{2}(K_0 + P_0)$, and its relation to the dilatation is expressed as [DO02]

$$U \cdot \frac{1}{2}(P_0 + K_0) \cdot U^{-1} = -iD \tag{2.24}$$

which means that the respective eigenvalues coincide and it is sound to interpret the anomalous dimension as the conformal equivalent of $AdS_5 \times S^5$ energies with respect to the global AdS time.

2.4 The BFKL pomeron

For any gauge theory, aside from finding its spectrum, it is most important to determine the particle scattering amplitudes, how do they depend on the type of particles involved, as well as the interaction parameters. In practice, the analysis is performed in some specific parameter regime. One of those is the Regge limit of inelastic scattering, in which the particle energy (Mandelstam variable s) is very large in comparison with the cut-off scale of the theory, and the energy transfer (Mandelstam t) is fixed.

For such processes, the so-called Regge behaviour is expected and also confirmed by experimental data; namely, that the amplitude is proportional to $s^{\alpha(t)}$, where $\alpha(t)$ is called Regge trajectory. $\alpha(t)$ is the position of a pole in the partial-wave expansion of the amplitude, and is interpreted as an exchange of a virtual composite particle. This particle, a reggeized gluon, or reggeon, can be thought of as a number of gluons, which in turn exchange gluons between themselves, described by a ladder diagram.

The Regge trajectory is in fact approximated by a linear function, $\alpha(t) = j + \alpha' t$, where α' is the Regge slope and for string theory is related to the string tension. However, for scattering, the intercept j is more relevant and has been an object of investigation in various settings. One of them, in which t is large (but fixed and much smaller than s), ie. of order of the cut-off, is called the BFKL regime [Lipa76, KLF77, BL78]. The dynamics are dominated by a BFKL pomeron, which is a bound state of two reggeized gluons.

This discussion works for both QCD and $\mathcal{N} = 4$ SYM, and in fact the leading-order contributions to the pomeron from the BFKL equation coincide [KL02]. Even though the agreement does not hold to next orders, there is a lot of motivation to study the pomeron intercept in CFT.

To leverage the power of the AdS/CFT correspondence in this setting, a dictionary between the two theories is required. The direction of the collision defines the longitudinal plane in the spacetime, whereas the other two coordinates form the transverse plane. Somewhat in agreement with expectation, the BFKL equation is invariant under the $\mathfrak{sl}(2, \mathbb{C})$ symmetry of the transverse plane. Therefore, any quantity, such as the intercept, will be a function of the relevant principal continuous series representation of this algebra, parameterised by

$$h = \frac{1+n}{2} + i\nu \qquad \bar{h} = \frac{1-n}{2} + i\nu \tag{2.25}$$

with n integer and ν real. Now, the dictionary is provided by an identification between the generators of symmetries of the transverse plane and the isometries of a relevant subspace of AdS_5 [BST07]

$$J_0 = \frac{1}{2}(-iD + M_{12}) \qquad J_+ = \frac{1}{2}(P_1 - iP_2) \qquad J_- = \frac{1}{2}(K_1 + iK_2) \tag{2.26}$$

$$\bar{J}_0 = \frac{1}{2}(-iD - M_{12}) \qquad \bar{J}_+ = \frac{1}{2}(P_1 + iP_2) \qquad \bar{J}_- = \frac{1}{2}(K_1 - iK_2) \tag{2.27}$$

where the eigenvalues of $J_0, \bar{J}_0$ are respectively $h + m, \bar{h} + \bar{m}$, for (independent) integers $m, \bar{m}$. Therefore, the relation to the string charges will read

$$-iD = 1 + 2i\nu + m + \bar{m} \qquad S_{12} = n \tag{2.28}$$

In turn, as the BFKL Hamiltonian is proportional to the boost operator in the longitudinal plane, the intercept will be given by the relevant string charge as

$$j = -iS_{03} \tag{2.29}$$

Chapter 3

Aspects of integrability

The concept of integrability is notoriously difficult to delineate, and even the best established textbooks on the subject [BBT03] warn of its fragmented nature, promising understanding of 'profound unity' only later in the course of study. However, an integrable (in the Liouville sense) system itself is a well defined notion, and means a system with as many conserved charges as possible. This means n for systems with phase space of finite dimension $2n$, and infinity for the infinite-dimensional systems. Note that precise statements in the latter case are stil rather problematic.

We will not separately introduce the the integrability formalism before applying it to the relevant physical setup, as we are going to apply it only once. Instead, we are going to discuss it already in context of AdS/CFT, so that the logic behind it is more palpable.

3.1 Integrable strings in $AdS_3 \times S^1$

Consider a string worldsheet completely contained in a subspace of the full $AdS_5 \times S^5$, namely $AdS_3 \times S^1$. The Polyakov action is obtained by setting $Y_{2,3}$ to 0 and parameterising the S^1 by an angular variable ϕ. The result is (sans the Lagrange multiplier term)

$$S = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \left(-\partial_\alpha Y_A \partial^\alpha Y^A - \partial_\alpha \phi \partial^\alpha \phi \right) \tag{3.1}$$

To any point in AdS_3 a matrix field can be associated as follows

$$g = \begin{pmatrix} Y_0 + Y_1 & Y_5 - Y_4 \\ -Y_5 - Y_4 & Y_0 - Y_1 \end{pmatrix} \tag{3.2}$$

Calculating $\det g$ immediately yields the hyperboloid constraint (2.9), and thus $g \in SL(2, \mathbb{R})$. Then, the following current can be introduced

$$j_\alpha = g^{-1} \partial_\alpha g \tag{3.3}$$

It belongs to the corresponding Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by definition, but its tracelessness can be also seen by an explicit matrix calculation

$$\begin{aligned}\operatorname{tr} j_\alpha &= \operatorname{tr} g^{-1}\partial_\alpha(\exp\log g) = \operatorname{tr} g^{-1}\int_0^1 dv\, e^{v\log g}(\partial_\alpha \log g)\, e^{(1-v)\log g}\\ &= \operatorname{tr}\int_0^1 dv\,\partial_\alpha \log g = \partial_\alpha \operatorname{tr}\log g = \partial_\alpha \log\det g = \partial_\alpha 0 = 0\end{aligned} \tag{3.4}$$

where firstly the derivative of a matrix exponential [WILC67, (4.1)] was used, and the integral became trivial due to the cyclicity of trace. Using these quantities, the action can be rewritten as

$$S = \frac{\sqrt{\lambda}}{4\pi}\int_0^{2\pi} d\sigma \int d\tau\,\left(-\tfrac{1}{2}\operatorname{tr} j_\alpha j^\alpha - \partial_\alpha\phi\partial^\alpha\phi\right) \tag{3.5}$$

where the trace term just equals the corresponding term of (3.1), what can be easily seen using its intermediate form, $\frac{1}{2}\operatorname{tr}\partial_\alpha(g^{-1})\partial^\alpha g$. The hyperboloid constraint is satisfied automatically, and thus no Lagrange multiplier is introduced. This form of the action is called the principal chiral model, and is invariant under the following global symmetry of g

$$g \mapsto U_L g U_R \tag{3.6}$$

where the constant matrices $U_{L,R}$ immediately cancel in the trace term. Note that the current j_α is the Noether conserved current corresponding to the U_R part of the symmetry.

The Virasoro constraints for this action take the form

$$(-\tfrac{1}{2}\operatorname{tr} j_\alpha j_\beta - \partial_\alpha\phi\,\partial_\beta\phi) - \tfrac{1}{2}\eta_{\alpha\beta}\,\eta^{\gamma\delta}(-\tfrac{1}{2}\operatorname{tr} j_\gamma j_\delta - \partial_\gamma\phi\,\partial_\delta\phi) = 0 \tag{3.7}$$

In the light-cone coordinates, with

$$j = g^{-1}\partial g \qquad\qquad \bar{j} = g^{-1}\bar{\partial} g \tag{3.8}$$

the components read explicitly

$$\tfrac{1}{2}\operatorname{tr} j^2 = -(\partial\phi)^2 \qquad\qquad \tfrac{1}{2}\operatorname{tr}\bar{j}^2 = -(\bar{\partial}\phi)^2 \tag{3.9}$$

There always exists a gauge in which the right-hand sides are constant, which is in turn dictated by what solution is chosen for the ϕ equation of motion

$$\partial\bar{\partial}\phi = 0 \tag{3.10}$$

Note that for traceless 2×2 matrices such as $j, \bar{j}$, $\operatorname{tr} j^2 = -2\det j$, therefore the constraints can be rewritten as

$$\det j = (\partial\phi)^2 \qquad\qquad \det\bar{j} = (\bar{\partial}\phi)^2 \tag{3.11}$$

effectively determining eigenvalues of $j, \bar{j}$. There are two equations of motion for j

$$\partial\bar{j} + \bar{\partial} j = 0 \qquad\qquad \partial\bar{j} - \bar{\partial} j + [j,\bar{j}] = 0 \tag{3.12}$$

where the former can be checked, even if laboriously, to exactly reproduce the equations in the original variables

$$\partial_\alpha \partial^\alpha Y_A + (\partial_\beta Y_B \, \partial^\beta Y^B) Y_A = 0 \tag{3.13}$$

complete with the second term arising from the hyperboloid constraint Lagrange multiplier. The latter of (3.12) is trivially satisfied due to (3.8). However, it would be non-trivial if the action (3.5) would be introduced in terms of j, without a priori specifying their relation to g.

Now, let us define a connection

$$J = \frac{j}{1-x} \qquad\qquad \bar{J} = \frac{\bar{j}}{1+x} \tag{3.14}$$

that additionally depends on a complex spectral parameter x. The flatness condition

$$\forall x \quad \partial \bar{J} - \bar{\partial} J + [J, \bar{J}] = 0 \tag{3.15}$$

is satisfied if and only if both equations of motion (3.12) are satisfied simultaneously. Specifically, the left-hand side of the flatness condition is a linear combination of the left-hand sides of the equations of motion, while also yielding them under either the limit $x \to \infty$ in the leading term or the substitution $x = 0$, respectively.

With a flat connection, one can consider a parallel transport of the connection along some path C on the worldsheet

$$\Omega(C; x) = P \exp \int_C J \, dw + \bar{J} \, d\bar{w} \tag{3.16}$$

It has a crucial property of being independent of continuous deformation of path C, which is a consequence of flatness of J. Adapting from [AFSG97, §2], we will prove this, starting from a statement equivalent to (3.16)

$$\Omega' + (J w' + \bar{J} \bar{w}') \, \Omega = 0 \tag{3.17}$$

where in this paragraph prime denotes differentiation with respect to the variable υ smoothly parameterising C. Using the identities for $(\Omega^{-1})', (\delta\Omega)'$ that follow immediately, we can vary Ω to $\delta\Omega$ and write

$$\begin{aligned}(\Omega^{-1}\delta\Omega)' &= -\Omega^{-1} \, \delta(J w' + \bar{J} \bar{w}') \, \Omega \\ &= -\Omega^{-1} \left((\partial J \delta w + \bar{\partial} J \delta \bar{w}) w' + J \delta w' + (\partial \bar{J} \delta w + \bar{\partial} \bar{J} \delta \bar{w}) \bar{w}' + \bar{J} \delta \bar{w}' \right) \Omega\end{aligned} \tag{3.18}$$

We can integrate this quantity along the length of C. The $\delta w', \delta\bar{w}'$ terms can be integrated by parts, and the boundary term will vanish due to the usual assumption that $\delta w, \delta \bar{w}$ vanish at the limits of integration. The remaining terms can be rearranged to read

$$\Omega^{-1}\delta\Omega = \int d\upsilon \, \Omega^{-1} \left((w' \delta \bar{w} - \bar{w}' \delta w)(\partial \bar{J} - \bar{\partial} J + [J, \bar{J}]) \right) \Omega \tag{3.19}$$

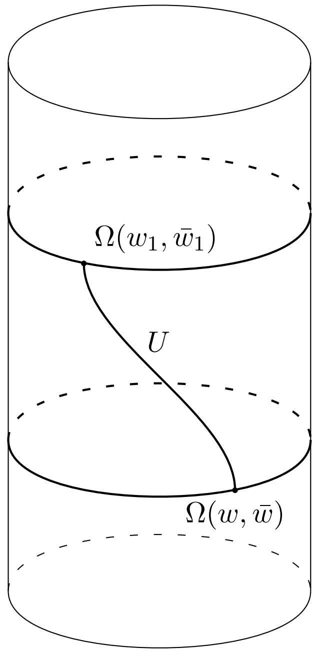

Figure 3.1: A monodromy Ω is a parallel transport of the connection $J, \bar{J}$, here around the string worldsheet. As changing the reference point $w, \bar{w}$ amounts to a similarity transformation with respect to U, the eigenvalues of Ω are conserved.

where one of the parentheses contain the flatness condition. Therefore, for flat connections $\delta\Omega = 0$.

A parallel transport along a closed path is called a monodromy (see fig. 3.1), and for these the path independence justifies the notation $\Omega(w, \bar{w}; x)$, as Ω depends only on a reference point at which the path starts and ends (and the homotopy class of the contour). Another consequence of path independence is that if the path is contractible to a point, the monodromy is trivial, $\Omega = I$. However, not all monodromies are trivial, as the worldsheet may exhibit topological features, such as punctures or holes, that the path may encircle and therefore be non-contractible. A typical situation in which this is observed is when the path goes around the worldsheet of a closed string, which has the topology of a cylinder.

Yet another crucial consequence of path independence is the following law of monodromy transformation between different reference points

$$\Omega(w_1, \bar{w}_1; x) = U\,\Omega(w_0, \bar{w}_0; x)\,U^{-1} \tag{3.20}$$

where U is some matrix, corresponding to the parallel transport of the connection between $w_0, \bar{w}_0$ and $w_1, \bar{w}_1$. This is a similarity transformation and implies that at any two reference points the eigenvalues of Ω are the same. In other words, Ω exhibits isospectral evolution, and the eigenvalues are constants of motion. They depend on the spectral parameter, so their values at any x are conserved, therefore yielding an infinite set of conserved quantities. This is how integrability appears for any system for which a flat connection can be constructed.

The monodromy matrix satisfies the following equations

$$\partial\Omega + [J, \Omega] = \bar{\partial}\Omega + [\bar{J}, \Omega] = 0 \tag{3.21}$$

which can be considered an infinitesimal version of (3.20). Alternatively, one may consider (3.17), taking into account that for a closed path both endpoints are differentiated simultaneously. Precisely, Ω can be represented as a product of two parallel transports $\Omega_1\Omega_2^{-1}$, both from the same point to $w_0, \bar{w}_0$, on opposite sides of the worldsheet. Then, from (3.17)

$$\partial\Omega = (\partial\Omega_1)\Omega_2^{-1} + \Omega_1(\partial\Omega_2^{-1}) = -J\Omega_1\Omega_2^{-1} + \Omega_1\Omega_2^{-1}J = -[J, \Omega] \tag{3.22}$$

and identically for $\bar{\partial}, \bar{J}$. (3.21) is called the Lax equation and is another common feature of integrable systems. In particular, for any Liouville integrable system, matrices satisfying it can always be constructed, although such construction relies on the knowledge of the conserved quantities and therefore does not uncover any new information on the system.

The equation (3.21) can be seen as an operator commutation relation

$$[\partial + J, \Omega] = [\bar{\partial} + \bar{J}, \Omega] = 0 \tag{3.23}$$

Together with the fact that $\partial + J, \bar{\partial} + \bar{J}$ also commute (their commutator is again the flatness condition) we obtain a set of mutually commuting operators. Therefore, there exists a basis of eigenvectors of Ω such that its elements are also eigenvectors of $\partial + J, \bar{\partial} + \bar{J}$; we say that the three operators are simultaneously diagonalisable. Any eigenvector of Ω is also an eigenvector of its logarithmic derivative

$$L(w, \bar{w}; x) = -\frac{\partial}{\partial x} \log \Omega(w, \bar{w}; x) \tag{3.24}$$

and thus the operator L also satisfies the Lax equations (3.21).

In the present case, $\Omega \in SL(2, \mathbb{R})$ as an ordered exponential of an element of the relevant Lie algebra. In particular, it is unimodular (has unit determinant), so its conserved eigenvalues can be written as $e^{\pm ip(x)}$, where p is called the quasi-momentum. Consequently, the eigenvalues of L are $\pm p'(x)$.

Let us also introduce the following auxiliary system of linear equations

$$\partial\Psi + J\Psi = 0 \qquad\qquad \bar{\partial}\Psi + \bar{J}\Psi = 0 \tag{3.25}$$

This system generically has two linearly independent two-component vector solutions, called wave functions (note that this is just a convenient name, as the object is still classical). They can be arranged as columns of a matrix $\hat{\Psi}$, which subsequently also satisfies (3.25). Note that the solution is determined up to a constant matrix U, as

$$\hat{\Psi} \mapsto \hat{\Psi}U \tag{3.26}$$

maps solutions to one another. This ambiguity is essentially the freedom of choice of integration constants in a differential equation.

The solution can be used to express the connection components as

$$J = -(\partial\hat{\Psi})\hat{\Psi}^{-1} \qquad\qquad \bar{J} = -(\bar{\partial}\hat{\Psi})\hat{\Psi}^{-1} \tag{3.27}$$

where the expressions are invariant under (3.26). Substituting these in the flatness condition satisfies it identically, therefore it can be interpreted as a consistency condition for the existence of solutions of the auxiliary system. The determinant of $\hat{\Psi}(w, \bar{w}; x)$ does not depend on $w, \bar{w}$, as can be seen from

$$\partial \log \det \hat{\Psi} = -\operatorname{tr} J = 0 \qquad\qquad \bar{\partial} \log \det \hat{\Psi} = -\operatorname{tr} \bar{J} = 0 \tag{3.28}$$

using (3.27) and transforming it as in (3.4). This is not surprising, as the definition of the parallel transport (3.16) resembles a formal solution to (3.25), so $\hat{\Psi}(w, \bar{w}; x)$ can be understood as a parallel transport matrix from some fixed point to $w, \bar{w}$. In particular, it can be used to assemble monodromies. Indeed, a quantity $\hat{\Psi}_1(w, \bar{w}; x)\hat{\Psi}_2(w, \bar{w}; x)^{-1}$ satisfies the Lax equation straightforwardly from (3.25). Conversely, a monodromy acts on wave functions as a transport operator.

At $x = 0$, the physical quantities can be recovered

$$j = -(\partial\hat{\Psi})\hat{\Psi}^{-1}\Big|_{x=0} \qquad\qquad \bar{j} = -(\bar{\partial}\hat{\Psi})\hat{\Psi}^{-1}\Big|_{x=0} \tag{3.29}$$

More importantly, taking (3.25) at this value, expressing $j, \bar{j}$ in terms of g, and left-multiplying sidewise by g, one gets

$$g(\partial\hat{\Psi}) + (\partial g)\hat{\Psi}\Big|_{x=0} = 0 \qquad\qquad g(\bar{\partial}\hat{\Psi}) + (\bar{\partial} g)\hat{\Psi}\Big|_{x=0} = 0 \tag{3.30}$$

so consequently $g\hat{\Psi}(w, \bar{w}; 0)$ is a constant matrix. This ambiguity also encompasses (3.26) and is fully consistent with the U_L symmetry of (3.6). Bearing this in mind, one usually writes

$$g = \sqrt{\det \hat{\Psi}}\, \hat{\Psi}^{-1}\Big|_{x=0} \tag{3.31}$$

where the additional factor is a constant that guarantees that g is unimodular, as $\hat{\Psi}(w, \bar{w}; 0)$ in general is not.

The reconstruction formulae (3.29), (3.31) seem rather tautological at this point, as to obtain g one needs a solution of (3.25), which in turn is given in terms of g. These will become more useful in the context of the following sections. Precisely, we will show how to obtain Ψ, and consequently g throught (3.31), in a purely algebraical fashion, ie. without resorting to solving any differential equation. This will be achieved with help of the algebraic curve.

Let us finally mention how the string energy and spin appear in the asymptotic behaviour of the quasi-momentum. The auxiliary linear system (3.25) can be expanded to the leading term in x at either zero or infinity, and then formally solved for the respective asymptotic expression of Ω. Its trace, which by definition equals $2\cos p(x)$, can be compared with the expressions for the charges (2.18) in terms of integrals of $j, \bar{j}$ currents. As the series expansions in x are required to match, the leading terms of p are discovered [KZ04, (3.28), (3.30)] to read

$$p(x) = -\frac{2\pi(E-S)}{\sqrt{\lambda}}x + O(x^2) \qquad (x \to 0) \tag{3.32}$$

$$p(x) = \frac{2\pi(E+S)}{\sqrt{\lambda}\,x} + O(x^{-2}) \qquad (x \to \infty) \tag{3.33}$$

3.2 The algebraic curve

Using any operator L satisfying the Lax equations, an algebraic curve can be defined as a set of pairs of complex numbers obeying the characteristic equation of L

$$\det(\tilde{y} - L) = 0 \tag{3.34}$$

The equation depends only on the eigenvalues of L, which in turn do not depend on the worldsheet coordinates. Note that this object is different from the spectral curve that is also a quantity of interest [SCHÄ10], which is defined by the characteristic equation of Ω (which also satisfies the Lax equation). The analytic properties of the spectral curve are much more complicated, in part due to the fact that Ω has essential singularities near $x = \pm 1$, where the connection diverges; L has only poles there.

The following analysis is valid for such L that (3.34) is rational in x. Moreover, to remove the multiple poles at $x = \pm 1$, a birational transformation in $\tilde{y}$ will be performed, so that the algebraic curve in the new variable y will have the following polynomial (hyperelliptic) form

$$y^n = \prod_i (x - a_i) \tag{3.35}$$

where n is the $\tilde{y}$-degree of (3.34), or, equivalently, the order of the matrix L. This can be done more explicitly for $n = 2$, where (3.34) reads

$$\tilde{y}^2 = -\det L = p'(x)^2 = r(x) \tag{3.36}$$

where the $\tilde{y}^1$ term is proportional to $\operatorname{tr} L$ and therefore vanishes. The right-hand side $r(x)$ is a rational function of x, for which a rational perfect square $Q(x)^2$ can be chosen so that $r(x)Q(x)^2$ is a polynomial without double zeroes. Therefore, multiplying (3.36) sidewise by $Q(x)^2$ and birationally transforming

$$\tilde{y} \mapsto y = Q(x)\tilde{y} \tag{3.37}$$

we obtain the desired polynomial form

$$y^2 = r(x)Q(x)^2 \tag{3.38}$$

which is a desingularisation of the original algebraic curve.

The algebraic curve can be understood as an n-sheeted cover of the complex plane. The sheets are unified at branch points where $y = 0$, therefore, at the roots of the x-polynomial in (3.35). If the polynomial is of odd degree, there is an additional branch point at infinity, and all branch points are pairwise connected by cuts. The cuts correspond to the choice of branch in a radical expression for y in terms of x.

The solutions that fall in the category described above, ie. admitting a desingularised, rational algebraic curve, are called finite-gap solutions. This traditional nomenclature stems from the study of the integrable Korteweg–de Vries equation, and in our analysis corresponds to the finite genus g of the algebraic curve. It is believed that any classical string solution can be obtained as a limit of a sequence of solutions of increasing but finite genus [DV06].

One of the crucial advantages of introducing the algebraic curve is that all eigenvalues and eigenvectors of L that depend on a complex parameter x can be treated as single functions defined on the curve. Except for the branch points, for any complex x there are n points of the curve lying above it, ie. n values of y satisfying (3.35), and each of them corresponds to one eigenvalue and one eigenvector of L. Such points will be denoted as $x^{(i)}$ for i-th sheet, but only when specifying the sheet is explicitly needed. In other cases, we will slightly abuse the notation, when by x we will mean any (fixed) of the $x^{(i)}$.

In fact, the eigenvector Ψ of L is almost uniquely determined by the properties of meromorphic functions on the algebraic curve. To see this, we start with determining the number of poles of Ψ, by considering a square of the determinant of the matrix $\left(\Psi(x^{(1)}); \ldots; \Psi(x^{(n)})\right)$. Such an expression has a double pole for each pole of some $\Psi(x^{(i)})$, and as a rational function of x it has as many poles as zeroes. In turn, it has zeroes precisely where its two columns coincide, or, equivalently, at branch points unifying two sheets of the curve. Consequently, Ψ has as many poles as half the number of branch points, that is, $g + n - 1$ by the Riemann–Hurwitz formula.

Note that in one case that is examined later, this is seemingly violated, when a degenerate algebraic curve with no finite-size cuts corresponds to a solution with two poles. However, this algebraic curve has two point-like points of degeneracy, which can be viewed as limits of two cuts, and in this sense solution should be related to a genus-1 curve, which is precisely the case.

For the Lax operator of $AdS_3 \times S^1$ string worldsheets the situation is as simple as possible, as the algebraic curve has just $n = 2$ sheets. The eigenvalues of L that the sheets correspond to are $\pm p'(x)$, and accordingly $x^{\pm}$ will denote a point of the algebraic curve lying above x on the respective sheet, assuming there is no branch point at x. Passing to the corresponding point on the other sheet is equivalent to changing the sign of the eigenvalue $\pm p'(x)$.

By choosing a particular normalisation of the eigenvector, $n - 1$ of the poles can be precisely determined. We define the normalised eigenvector Ψ_{n} to have 1 as its first component. If L is non-degenerate and with distinct eigenvalues in the $x \to \infty$ limit (meaning that there is no branch point there), its eigenvector will behave around $\infty^{(i)}$ (the point lying above infinity on the i-th sheet) as $e_i + O(x^{-1})$, where all components of e_i are zero except the i-th. The corresponding normalised eigenvector is obtained by dividing all components by the first, which results in the following behaviour

$$\begin{aligned}\Psi_{\mathrm{n}}(\infty^{(1)}) &= \big(1, O(x^{-1}), O(x^{-1}), \dots\big)\\ \Psi_{\mathrm{n}}(\infty^{(i)}) &= \big(1, \underbrace{O(1), \dots, O(1)}_{i-2}, O(x), \underbrace{O(1), \dots, O(1)}_{n-i}\big) \qquad 2 \le i \le n\end{aligned} \tag{3.39}$$

Therefore, $n-1$ poles appear at $\infty^{(i)}$ in the i-th component of the normalised eigenvector for $i > 1$. These do not actually carry any information on the system described by the algebraic curve, and are just artifacts of the chosen gauge, ie. normalisation.

The remaining g poles are called dynamical, as their position can vary with the worldsheet variables. Now, considering the non-trivial components ψ_i of the normalised eigenvector, we know that each of them has a single pole at $\infty^{(i)}$, at most g dynamical poles, and one zero at $\infty^{(1)}$. The Riemann–Roch theorem allows stating that the space of such functions is one-dimensional, unless the set of dynamical poles belongs to some very special case (see [Vice08, Def. 2.51/1.5.21] and the preceding discussion; we mention this caveat purely for completeness, as our investigation will not be affected by it). This means that all ψ_i are unique up to a multiplicative constant, and thus the whole Ψ_{n} is determined up to a left-multiplication by a constant diagonal matrix. This conclusion is not constructive itself, ie. it does not provide a recipe for construction of ψ_i, but allows for stating that any function constructed to meet the prerequisites is already of the most general form.

In the $AdS_3 \times S^1$ case, the normalised eigenvector will have just one non-trivial component ψ that will be required to vanish at ∞^+ and diverge at ∞^-, as well as at all of the g dynamical poles.

The relation between L and the operators $\partial + J, \bar{\partial} + \bar{J}$, with which it is simultaneously diagonalisable, allows to expect that the algebraic curve will be a suitable framework for constructing (as opposed to solving for) the wave function of (3.25). Indeed, Ψ will be necessarily proportional to Ψ_{n}, but because $\partial + J, \bar{\partial} + \bar{J}$ are differential operators, the proportionality factor f_{BA} in

$$\Psi(w, \bar{w}; x) = f_{\mathrm{BA}}(w, \bar{w}; x)\Psi_{\mathrm{n}}(w, \bar{w}; x) \tag{3.40}$$

will exhibit a non-trivial dependence on the worldsheet.

To learn about the properties of f_{BA}, one can examine the first component of the auxiliary system equation, which yields

$$\frac{\partial f_{\mathrm{BA}}}{f_{\mathrm{BA}}} = -(J\Psi_{\mathrm{n}})_1 \tag{3.41}$$

with the barred equation employing the exact same analysis. Singular behaviour is possible wherever either of the quantities on the right-hand side is divergent. At $x \to \infty$, Ψ_{n} has poles, but J vanishes and the product is regular. At a dynamical pole $\gamma(w, \bar{w})$ of Ψ_{n}, J is regular, but the residue of the right-hand side depends on w, so this case needs to be treated in more detail. By the fact that L and $\partial + J$ commute, Ψ_{n} and $(\partial + J)\Psi_{\mathrm{n}}$ are the eigenvectors of L with the same eigenvalue, and therefore are proportional. The scaling between them is easily determined from their first components to be $(J\Psi_{\mathrm{n}})_1$, so we can write

$$\partial \Psi_{\mathrm{n}} = \left(-J + (J\Psi_{\mathrm{n}})_1\right) \Psi_{\mathrm{n}} \tag{3.42}$$

and examine second-order poles of both sides at $\gamma(w, \bar{w})$. Considering the component(s) of Ψ_{n} that do have a pole at γ with residue $\mathfrak{p}$, the second-order poles read

$$\frac{\mathfrak{p}\,\partial\gamma}{(x-\gamma)^2} = \left(0 + \frac{\mathfrak{r}}{x-\gamma}\right) \frac{\mathfrak{p}}{x-\gamma} \tag{3.43}$$

so the residue $\mathfrak{r}$ of $(J\Psi_{\mathrm{n}})_1$ (in general, containing contributions of more components diverging at γ) equals $\partial\gamma$. Therefore, locally at $\gamma(w, \bar{w})$, (3.41) reads

$$\partial \log f_{\mathrm{BA}} = -\frac{\partial\gamma}{x-\gamma} + \text{regular} = \partial \log(x-\gamma) + \text{regular} \tag{3.44}$$

Consequently, f_{BA} vanishes at the dynamical poles.

Finally, (3.41) has a pole at $x = 1$ emerging from J (and the same holds for $x = -1$ in the barred equation), so an exponential essential singularity of f_{BA} is expected there. Its coefficient is derived in [DV06, §D] for strings propagating in $\mathbb{R}_t \times S^3$, and the result is proportional to the eigenvalue of $j, \bar{j}$. It is in turn dictated by the Virasoro constraints akin to (3.11) and never vanishes, as the solutions are expected to propagate in time. In our case, however, this eigenvalue may vanish for solutions completely contained in AdS_3, providing no information on the singular behaviour. It turns out that some statement about it can still be made from very general principles.

Consider a relation [BBT03, (3.15)] between the Lax operator and $J, \bar{J}$ that stems from a comparison of the poles in both terms of the Lax equation. In our case, where $J, \bar{J}$ have one pole each, this will read

$$J = [P_+(L, x)]^-_{x=1} \qquad\qquad \bar{J} = [P_-(L, x)]^-_{x=-1} \tag{3.45}$$

where $P_\pm(\tilde{y}, x)$ are some functions, polynomial in the first variable, with coefficients rational in the second, and $[\cdot]^-$ denotes the polar part at the given point. Unfortunately, in the general case, the form of the polynomials $P_\pm$ cannot be significantly restricted. We can only say that their coefficients will have non-negative powers of $(1 \mp x)$, so that powers of L, which already has the relevant pole, will be matched with the left-hand sides of (3.45), where the dependence is explicitly $1/(1 \mp x)$. Now, turning to the $n = 2$ case, the equations can be considered in the basis of diagonal L

$$J = [P_+(\mathrm{diag}(p', -p'), x)]^-_{x=1} \qquad \bar{J} = [P_-(\mathrm{diag}(p', -p'), x)]^-_{x=-1} \tag{3.46}$$

Now, $L^2 = p'(x)^2\, I$, where $p'(x)^2$ is rational for finite-gap solutions, so all terms can be expressed as L or I with a rational factor, and the polynomials $P_\pm$ effectively have degree 1. Taking into account that the projection $[\cdot]^-$ of rational functions preserves only the respective pole, we finally write

$$J = \left[c_+ L + \frac{c'_+ I}{1-x}\right]^-_{x=1} \qquad \bar{J} = \left[c_- L + \frac{c'_- I}{1+x}\right]^-_{x=-1} \tag{3.47}$$

where $c_\pm, c'_\pm$ are constant and the x-dependence of the coefficients matches the left-hand sides, remembering that L itself has poles at $x = \pm 1$. As all $J, \bar{J}, L$ are traceless, $c'_\pm$ need to vanish.

Applying (3.45) in (3.41), we obtain on the right-hand side

$$[P_+(L, x)]^- \Psi_{\mathrm{n}}(x^\pm) = [P_+(\pm\tilde{y}, x)]^- \Psi_{\mathrm{n}}(x^\pm) = \frac{\pm c_+ y}{1-x}\, \Psi_{\mathrm{n}}(x^\pm) \tag{3.48}$$

as $\Psi_{\mathrm{n}}(x^\pm)$ is an eigenvector of L with eigenvalue $\pm\tilde{y}$, whose polar part is proportional to the desingularised variable y. Locally, (3.41) reads thus

$$\frac{\partial f_{\mathrm{BA}}}{f_{\mathrm{BA}}} = -\frac{\pm c_+ y}{1-x} + \text{regular} \tag{3.49}$$

and the local behaviour is

$$f_{\mathrm{BA}} \propto \exp\left(-\frac{\pm c_+ y}{1-x}\, w - \frac{\pm c_- y}{1+x}\, \bar{w}\right) \tag{3.50}$$

where the second term follows from the analysis of the barred equation at $x = -1$. It is crucial to note that each of the terms is in fact a priori valid only in the neighbourhood of $x = \pm 1$.

We have established that the function $f_{\mathrm{BA}}(w, \bar{w}; x)$ vanishes at g dynamical poles and has essential singularities of a prescribed type at x lying over ± 1. Such functions are known as Baker–Akhiezer functions, and there are general formulae for their construction. Moreover, the Riemann–Roch theorem again allows for a conclusion that in the present case their space is one-dimensional.

Consequently, Ψ is now fully determined up to overall factors of the components, which may still depend on the worldsheet variables. However, the $x \to \infty$ form of (3.25) is

$$\partial\Psi = \bar{\partial}\Psi = 0 \tag{3.51}$$

so that all components of the wave function should be constant in this limit. This imposes a final condition in the reconstruction procedure, after which only purely constant factors in each component remain ambiguous.

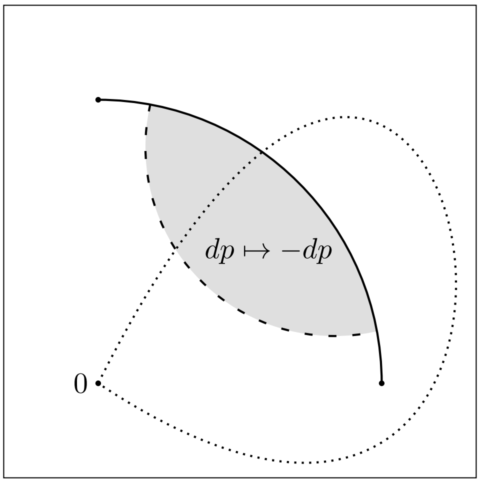

Figure 3.2: If the sign of dp is flipped over some region of the complex plane, partially bordered by the cut (solid), the cut of the redefined quasi-momentum appears along the rest of the border of the region (dashed). Integrating along the (dotted) contours from both sides, unimodularity is seen to be preserved.

The upshot of this analysis is that given an algebraic curve of genus g, g dynamical poles, and the polynomials $P_\pm$, a solution of (3.25) can be determined purely from analyticity conditions of meromorphic functions on algebraic curves. The values of the eigenvector at points $x^{(i)}$ correspond to the linearly independent solutions of the system.

In the $n = 2$ case, the reconstructed eigenvector can be meaningfully used with (3.31), by assembling a matrix $\hat{\Psi} = (\Psi(x^+); \Psi(x^-))$. The ambiguity of the overall constant factors in the components of Ψ amounts to left-multiplying $\hat{\Psi}$ by a constant diagonal matrix, which is encompassed by the U_R symmetry (3.6) of g. The only other undetermined constants in the reconstruction are $c_\pm$ of (3.50), and they can be absorbed by the diffeomorphism symmetry, ie. rescaling the variables $w, \bar{w}$. Also note that as the solution matrix of (3.25) was established to have a constant determinant, the same needs to be true about the reconstructed $\hat{\Psi}$.

Large parts of the discussion seem to be also applicable to systems in which the Lax operator is of order $n > 2$, for instance, for larger subsectors of $AdS_5 \times S_5$. In such cases, the auxiliary linear system would be a first-order equation for an n-component vector, therefore with n linearly independent solutions. Each of those correspond to one of n eigenvalues of L, and consequently to one of n sheets of the algebraic curve. The only point in the analysis that directly relied on $n = 2$ was the cap on the order of polynomials $P_\pm$ of (3.45). In general, some more intricate analysis would be needed to tell if (and at what degree) such cap appears for larger n. A natural guess would be $n - 1$, but even this would mean the presence of polynomials of unknown coefficients in (3.50). Still, the worldsheet-dependence of the exponents would remain linear.

Let us make a final note that while the algebraic curve equation (3.35) does fix the position of branch points, the exact location of cuts that connect them is not determined. Considering the behaviour of the quasi-momentum in the neighbourhood of x lying on the cut

$$p\left((x+\epsilon)^{-}\right) \approx p\left((x-\epsilon)^{+}\right) \tag{3.52}$$

we use the unimodularity of Ω

$$1 = \det \Omega = e^{ip(x+\epsilon)} e^{-ip(x+\epsilon)} \approx e^{ip(x+\epsilon)} e^{ip(x-\epsilon)} \tag{3.53}$$

to obtain the following condition

$$p(x+\epsilon) + p(x-\epsilon) = 2\pi n \qquad\qquad n \in \mathbb{Z} \tag{3.54}$$

that relates the values of p on the same sheet (or, simply, on the complex plane) on the opposite sides of the cut. However, this condition does not fix the position of the cut either. Namely, the positions of the cuts are determined by the branch choice of the square root in dp, which can be arbitrarily chosen for different regions of the complex plane. Let $d\tilde{p}$ differ from dp by sign only in some region partially bordered by a cut, and for x on that part of the cut we have

$$\begin{aligned} p(x+\epsilon) + p(x-\epsilon) &= \int_0^{x+\epsilon} dp + \int_0^{x-\epsilon} dp = \\ &= \int_0^{X+\epsilon} d\tilde{p} + \int_0^{X-\epsilon} d\tilde{p} = \tilde{p}(X+\epsilon) + \tilde{p}(X-\epsilon) \end{aligned} \tag{3.55}$$

where the contours of integration do not cross any discontinuities. X, where the contours leave the region of sign difference between $dp, d\tilde{p}$, is the new location of the cut, for which (3.54) still holds (see fig. 3.2).

3.3 The Bethe ansatz

The Bethe ansatz, in one of its numerous forms, is an all-important tool of integrability, this time in the quantum sense. As there is virtually no consensus about the definition of quantum integrability, it may very often appear that precisely the systems allowing a Bethe ansatz description are considered quantum integrable. Its original application [Beth31] was to exactly determine the eigenstates and eigenvalues of the Hamiltonian of a one-dimensional spin chain.

A notable difficulty of such systems is that the dimension of the Hilbert space grows exponentially with number of sites of the spin chain, so any kind of brute-force approach to diagonalising the Hamiltonian would be thwarted for all but the smallest examples, even with the computational power of today. In its original context, the idea of the Bethe ansatz was to consider a number of excitations (spin-down sites) of a vacuum (all sites spin-up) state of the spin chain, propose a particularly parameterised wave function of those excitations, and derive the equations that the parameters need to satisfy so that the wave function is an eigenstate of the Hamiltonian. The result are the so-called Bethe equations which, even if quite complicated, do not grow exponentially in complexity. The sets of numbers solving the equations are collectively called Bethe roots, are in general complex, and are related to the momenta of the spin chain excitations.

The Bethe ansatz emerges somewhat naturally in the $\mathcal{N} = 4$ SYM, where a typical object of study is the spectrum of an operator of the type $\operatorname{tr} Z^J$ with some 'impurities,' like covariant derivatives or other SYM fields, inserted in the trace. Even on the extremely superficial level, a trace of such form resembles a spin chain with excitations, and indeed the Bethe ansatz has been applied to such objects. More precisely, the one-loop correction to the dilatation operator was found to coincide with the spin chain Hamiltonian [MZ02].

The AdS/CFT correspondence, as it predicts the same features both for its gauge theory and string theory sides, would demand that the integrable structures of both sides agree. So far, the string theory is classically integrable by the algebraic curve description and the gauge theory has its Bethe ansatz. In a sequence of papers [KMMZ04, KZ04, BKS04] a semi-classical limit of the gauge theory Bethe ansatz was shown to match the algebraic curve for a sequence of subsectors of respective theories, finally concluding with a full theory analysis in [BKSZ05]. The idea is to consider a scaling limit, in which the spin chain length and excitation number are infinite, but their ratio fixed, under which the Bethe roots condense as finite-length cuts on the complex plane.

The Bethe roots density ρ (also called particle density, given the origin of Bethe roots as magnon momenta) is supported only on the cuts and defines a function p of the spectral variable that is continuous everywhere on the complex plane, except for the cuts. Its (imaginary) discontinuity on the cuts is proportional to the density, and the constant of proportionality can be determined from the fact that the density is normalised to the rescaled excitation number, or total spin [KMMZ04, (2.14)]

$$\int \rho(u)\, du = S \tag{3.56}$$

integrated over all cuts. Also, the values of p on the opposite sides of the cut satisfy

$$p(x+\epsilon) + p(x-\epsilon) = 2\pi n \qquad\qquad n \in \mathbb{Z} \tag{3.57}$$

where the right-hand side has its origin in taking the logarithm of the Bethe equation. By the comparison of the asymptotic behaviour to (3.32), (3.33), the function p is determined to coincide with the string quasi-momentum, with (3.57) corresponding to (3.54).

To obtain the quasi-momentum from the scaling limit of the Bethe ansatz, a few integral conditions are imposed. For p to be single-valued, the following needs

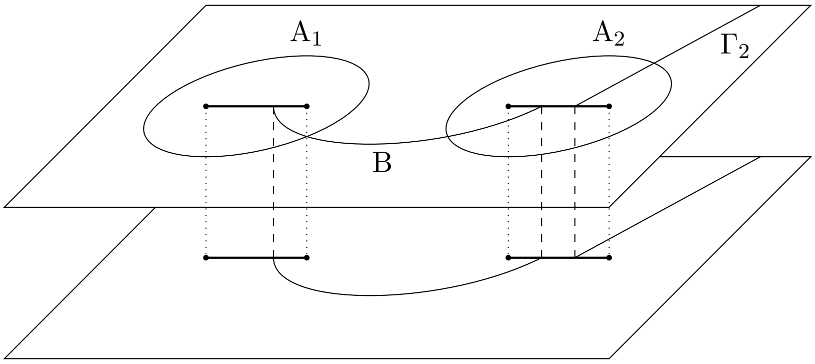

Figure 3.3: The integrals of the quasi-momentum differential need to attain specific values for several contours defined for the cuts of an algebraic curve, here of genus 1. The integrals along the A-cycles (going around the cuts) vanish, while the B-cycles (passing through pairs of cuts) and Γ-contours (linking cuts to infinities on either side) are even multiples of π.

to be satisfied at each cut

$$\int_{\mathrm{A}} dp = 0 \tag{3.58}$$

where the A-cycle is a contour going around the cut. Note that not all A-cycle integrals are independent, as one of them can be obtained as a sum of all others by a deformation of their integration contours. (3.57) corresponds to

$$\int_{\Gamma} dp = 2\pi n \tag{3.59}$$

where the Γ-contour is a contour from infinity to the cut and back. Alternatively, if p is considered on algebraic curve, the Γ-contour connects infinities on the opposite sheets of the curve via a given cut (see fig. 3.3). Obviously, given the origin of (3.58), (3.59), these will need to be satisfied also for dp obtained from the algebraic curve.

The classical integrability of string theory was expected to remain at the quantum level. Nevertheless, the Bethe ansatz in this case was considerably harder to develop, and indeed most of the approaches, starting with [AFS04], have been described as heuristic. However, in fact, this is how ansätze in general are introduced, and because the proposed equations resemble the Bethe equations, the use of the name 'Bethe ansatz for strings' is fully justified. Moreover, an underlying long-range spin chain description has been reverse-engineered later [Beis04].

For strings in $AdS_3 \times S^1$, or the $\mathfrak{sl}(2)$ sector, the Bethe ansatz equations read [BS05, (2.48)]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{\substack{j=1\\ j\neq k}}^{M} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^{-1} \frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_k^- x_j^+}} \sigma^2(x_k, x_j) \tag{3.60}$$

with the dressing factor [BS05, (2.46)]

$$\sigma(x_k, x_j) = \frac{1 - \frac{g^2}{x_k^- x_j^+}}{1 - \frac{g^2}{x_k^+ x_j^-}} \left(\frac{1 - \frac{g^2}{x_k^- x_j^+}}{1 - \frac{g^2}{x_k^+ x_j^+}} \frac{1 - \frac{g^2}{x_k^+ x_j^-}}{1 - \frac{g^2}{x_k^- x_j^-}} \right)^{i(u_k - u_j)} \tag{3.61}$$

and the usual shorthands of $x_j = x(u_j), x_j^{\pm} = x(u_j \pm \frac{i}{2})$, where the relation between the Bethe roots (particle rapidities) and spectral parameter is given by the Zhukovsky transformation [BS05, (2.28)]

$$x(u) = \frac{u}{2} + \frac{1}{2}\sqrt{u^2 - 4g^2} \qquad\qquad u(x) = x + \frac{g^2}{x} \tag{3.62}$$

We will now obtain the semi-classical limit of (3.60) in the non-standard situation in which $J = 0$ and the left-hand side is trivial. The right-hand side can be expressed in terms of Bethe roots only, which then are rescaled as $u \mapsto gu$. To express the $g \to \infty$ limit, the leading terms of a series expansion are taken, and the result is most conveniently obtained by taking a logarithm of both sides, and thus

$$2i\pi n = \frac{i}{g} \sum_{\substack{j=1 \\ j \neq k}}^{M} \frac{-2x_k x_j (x_k x_j - 1)}{(x_k^2 - 1)(x_j^2 - 1)(x_k - x_j)} \tag{3.63}$$

where on the left-hand side the logarithm ambiguity appears, and the Bethe roots have been expressed back in terms of the (rescaled) spectral parameter. The continuum limit replaces the sum by an integral or, more precisely, the principal value integral to account for excluding the $j = k$ divergent contribution, and weights the integrand with the particle density

$$\pi n = \frac{1}{g} \fint \frac{-xx'(xx' - 1)}{(x^2 - 1)(x'^2 - 1)(x - x')} \rho(u')\, du' \tag{3.64}$$

where $du' = \left(1 - (x')^{-2}\right) dx'$ follows from (3.62) after rescaling and the integration is understood over all cuts. A slightly simpler form can be obtained for a case where all cuts are pairwise symmetric under $x \mapsto -x$. Performing this substitution in one cut of each such pair, therefore mapping it on its counterpart, we obtain with odd ρ

$$\pi n = -\frac{2}{g} \fint \frac{x\rho(u')\, dx'}{x^2 - (x')^2} \tag{3.65}$$

integrating over the 'remaining' cuts.

Finally, we can fix the relation of ρ to the discontinuity of the quasi-momentum. Consider the following integral

$$\int_C \operatorname{disc} p \, du = \oint_C p(x) \left(1 - \frac{1}{x^2}\right) dx \tag{3.66}$$

taken along, and then around, all cuts. The contours can be deformed so that they encircle infinity and 0, where poles of respectively $p(x)$ and $-p(x)x^{-2}$ will be picked up. Using (3.32), (3.33), we write

$$\int_C \operatorname{disc} p\, du = \left(\oint_0 + \oint_\infty\right) p(x) \left(1 - \frac{1}{x^2}\right) dx =$$
$$= 2\pi i \left(\frac{2\pi(E-S)}{\sqrt{\lambda}} - \frac{2\pi(E+S)}{\sqrt{\lambda}}\right) = -\frac{8\pi^2 i S}{\sqrt{\lambda}} \tag{3.67}$$

By comparison with (3.56) we conclude

$$\rho = \frac{i\sqrt{\lambda}}{8\pi^2} \operatorname{disc} p \tag{3.68}$$

We can also obtain an integral expression of energy, for which we first express the magnon momentum [BS05, (2.30)]

$$P_k = -i \log \frac{x_k^+}{x_k^-} \tag{3.69}$$

as a leading term in g in the scaling limit

$$P(x) = \frac{1}{g} \frac{x}{(x^2-1)} \tag{3.70}$$

Inserting this into the dispersion relation, we obtain in the leading order

$$\varepsilon(x) = \sqrt{1 + 16 g^2 \sin^2 \frac{P}{2}} = \frac{x^2+1}{x^2-1} \tag{3.71}$$

Therefore we can confirm that

$$\int_C \rho\, \varepsilon\, du = \int_C \rho(x) \left(1 + \frac{1}{x^2}\right) dx = -\frac{1}{2}\left(-(E-S) - (E+S)\right) = E \tag{3.72}$$

by essentially flipping a sign with respect to (3.67). In fact, the reality conditions imposed by the physical sense of these two integrals are the main criterion for determining the exact location of the cuts.

3.4 Further results in AdS/CFT

The preceding sections offer just a glimpse into the numerous results and techniques that integrability offers in the context of AdS/CFT, recently summarised in a sizeable review [IntR10]. In fact, the discovery of integrability itself on both sides of the correspondence is a major piece of evidence in favour of the conjecture. At the level of computation, integrability can be seen as a tool to enable moving further away from the weak and strong coupling ends of the planar limit. Ultimately, it may allow for an interpolation between these two regions and, perhaps, proving the correspondence.

Let us sketch out a subset of what else integrability has to offer in AdS/CFT. First, the algebraic/spectral curves can be perturbatively quantised, by introducing very short cuts between the sheets. The charges that could be computed from the quasi-momentum asymptotics will receive their one-loop corrections.

From the spin chain perspective, the Bethe ansatz results are valid up to the order at which there appears wrapping, that is, interaction over the whole length of the spin chain. Lüscher corrections allow for including the leading wrapping effects in the computation. Next, the so-called Y-system [GKV09, GKKV09] was developed, that encodes the all-order energy of a state in the form of infinite number of equations. These were in turn reduced to a useful form of finite system of non-linear integral equations [GKLV11].

As for the BFKL pomeron, it should be noted that integrability appears also in the QCD case [Korc95]. In AdS/CFT, on the weak coupling side, the investigation mostly utilises the relation between the intercept and the anomalous dimensions of the so-called twist-two local operators [KLR+07, BJŁ08, ŁRV09]. In these calculations, the wrapping corrections played a crucial role.

At strong coupling, a series of results for increasingly better approximations of the intercept also partially relied on the relation to the anomalous dimension [CGP12, KL13]. Recently, using the newly proposed quantum spectral curve [GKLV13], a number of hitherto missing coefficients of the expansion have been computed [GLSV14, (6.8), §6.4]

$$
\begin{aligned}
j(\nu) = 2 - \frac{2+2\nu^2}{\sqrt{\lambda}} \Bigg(1 + \frac{\frac{1}{2}}{\sqrt{\lambda}} + \frac{-\frac{1}{8}+\frac{3}{2}\nu^2}{\lambda} \\
+ \frac{-1-3\zeta_3+(\frac{21}{8}-3\zeta_3)\nu^2}{\lambda^{3/2}} + \frac{-\frac{361}{128}-9\zeta_3+(\frac{51}{16}-9\zeta_3)\nu^2+\frac{21}{4}\nu^4}{\lambda^2} \\
+ \frac{-\frac{447}{64}-\frac{39}{2}\zeta_3-(\frac{13}{64}+45\zeta_3+\frac{15}{2}\zeta_5)\nu^2+(\frac{137}{8}-\frac{51}{2}\zeta_3-\frac{15}{2}\zeta_5)\nu^4}{\lambda^{5/2}} + \cdots \Bigg)
\end{aligned}
\tag{3.73}
$$

Chapter 4

Extending the algebraic curve classification

4.1 Algebraic curve for Wilson loops

As mentioned above, Wilson loop observables of the boundary theory are translated by the AdS/CFT correspondence to the minimal surfaces in the bulk spanned by these loops. As such, these surfaces do not have any topological features, so all closed loops on these surfaces are contractible to a point. Consequently, monodromies defined for such surfaces are equal to the unit matrix, by the argument of path independence. This suggests that an attempt to repeat the algebraic curve analysis as outlined in the previous chapter, inspired by its success for closed strings, is bound to fail miserably, as the quasi-momentum is identically zero and does not define any algeraic curve.

The idea now is to propose a different way of assigning an algebraic curve to a given solution, which would be applicable regardless of the presence of non-contractible worldsheet loops. Due to this assumption, the procedure will be purely local, in opposition to the traditional construction that required knowledge of the whole worldsheet to construct non-trivial monodromies. Using the machinery described in the previous chapter, we will show how the original solutions can be reconstructed from these newly defined, very simple algebraic curves.

However, the procedure has also introduced the auxiliary linear system (3.25), which is still well-defined for the Wilson loop minimal surfaces. If it is solvable, the wave function, or strictly speaking, a solution matrix $\hat{\Psi}$, can be used to construct a Lax operator. Namely, for any matrix $A(x)$ that does not depend on $w, \bar{w}$

$$L = \hat{\Psi}(w, \bar{w}; x)\, A(x)\, \hat{\Psi}(w, \bar{w}; x)^{-1} \tag{4.1}$$

satisfies the Lax equations (3.21), what can be straightforwardly seen using (3.27).

The algebraic curve was defined (3.34) as the characteristic polynomial of a Lax operator, and L defined by (4.1) can be used for this purpose. Notably, the present case does not require desingularisation (3.37), so $\tilde{y} = y, Q(x) = 1$. However, by

$$\det(y - \hat{\Psi} A \hat{\Psi}^{-1}) = \det(\hat{\Psi}(y - A)\hat{\Psi}^{-1}) = \det(y - A) \tag{4.2}$$

the characteristic polynomial of L is the same as for A, which is heretofore arbitrary (note that this equation is invariant under (3.26)). As we expect the algebraic curve to essentially encode crucial information on the solution for which it is defined, A needs to be governed by some clever criterion if the whole construction is supposed to make any sense.

Such criterion is inspired by the fact that Lax operators in general are assumed to be rational in x. Therefore, we will always choose the simplest possible A such that L is non-trivial and polynomial (one of the cases discussed actually relies on the distinction between polynomial and rational). More precisely, the usual ansatz will be

$$A(x) = f(x) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

for some function f. This further reinforces the connection with the original framework, as the eigenvalues of L, identical by construction to those of A, will differ only in sign. Note that the algebraic curve equation will then read

$$y^2 = f(x)^2 \tag{4.4}$$

An ultimate argument which would demonstrate that such definition of the Lax operator is meaningful would be the success of the reconstruction procedure outlined in section 3.2. There is one detail which needs to be reformulated in the present case, due to different analytic behaviour of L, namely the form of the essential singularities of the Baker–Akhiezer factor (3.50). Specifically, (3.47) reads as follows

$$J = \left[\frac{c_+ L + c'_+ I}{1-x}\right]^-_{x=1} \qquad \bar{J} = \left[\frac{c_- L + c'_- I}{1+x}\right]^-_{x=-1} \tag{4.5}$$

as the present L is polynomial in x, and $c'_\pm$ vanish just as previously. However, as $\tilde{y} = y$, the end result (3.50) will remain the same.

In the following sections, we consider two cases of Wilson loops: the null cusp and the quark–antiquark potential. For each of them we will firstly write out the algebraic curve by performing the following steps:

- write down the auxiliary linear system (3.25) by specifying g (see below) and using (3.8), (3.14);
- solve the system and verify that the solution matrix determinant is constant in $w, \bar{w}$, as well as (3.31);

- by (4.1), construct the Lax operator L and the algebraic curve.

Subsequently, starting from the algebraic curve, we will reconstruct the wave function as $\Psi = f_{\mathrm{BA}}\Psi_{\mathrm{n}}$, demanding that:

- f_{BA} vanishes at the dynamical poles of Ψ_{n} and locally behaves as (3.50),
- Ψ_{n} behaves as (3.39) under $x \to \infty$,
- components of Ψ become independent of $w, \bar{w}$ under $x \to \infty$.

The result will be compared with the original solution.

For pedagogical reasons, the cusp is presented in an expository manner, perhaps even with superfluous detail. For the quark–antiquark potential we notice and discuss some caveats. Also, an analogous procedure will be later applied to solutions corresponding to a particular correlation function, but the motivation and the apparent peculiarities will be entirely different than for Wilson loops. Note that throughout the whole chapter the underlying solutions have an Euclidean worldsheet metric, and thus the light-cone coordinates are expressed as (1.2).

Before we proceed, we use the parameterisations (2.11), (2.13) to obtain some useful forms of g from (3.2). In the Poincaré coordinates the result is

$$g_{\text{Poincaré}} = \frac{1}{z}\begin{pmatrix} x_0 + x_1 & 1 \\ x_0^2 - x_1^2 - z^2 & x_0 - x_1 \end{pmatrix} \tag{4.6}$$

$$g_{\text{Euclidean}} = \frac{1}{z}\begin{pmatrix} ix_0 + x_1 & 1 \\ -x_0^2 - x_1^2 - z^2 & ix_0 - x_1 \end{pmatrix} \tag{4.7}$$

where the latter form follows by Wick-rotating the time-like coordinate x_0. In the global coordinates, one can obtain

$$g_{\text{global}} = \begin{pmatrix} e^{it}\cosh\rho & e^{-i\psi}\sinh\rho \\ e^{i\psi}\sinh\rho & e^{-it}\cosh\rho \end{pmatrix} \tag{4.8}$$

by direct substitution and then using (3.6) with $U_L = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} i & -1 \\ 1 & -i \end{smallmatrix}\right), U_R = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -i \\ -i & 1 \end{smallmatrix}\right)$. It should be noted that the appearance of complex numbers in matrices that were defined as real is not troublesome, as they still belong to some representation of the same Lie group, determined by $U_{L,R}$.

4.2 The null cusp contour

The null cusp, or two intersecting light-like lines, spans a surface whose parameterisation is explicitly given in [RT07, (2.3–5)] as

$$\begin{aligned} x_0 &= e^{-\alpha\sigma-\beta\tau}\cosh(\beta\sigma - \alpha\tau) & \qquad z &= \sqrt{2}\, e^{-\alpha\sigma-\beta\tau} \\ x_1 &= e^{-\alpha\sigma-\beta\tau}\sinh(\beta\sigma - \alpha\tau) & & \end{aligned} \tag{4.9}$$

where $\alpha^2+\beta^2=0$. The form of z shows that the surface approaches the boundary when the combination $\alpha\sigma+\beta\tau$ tends to infinity; the intersecting lines forming the cusp are seen when simultaneously $\beta\sigma-\alpha\tau\to\pm\infty$. The domain of $w,\bar{w}$ is the full complex plane, so all loops on this surface are contractible just as expected. The surface is embedded in the Poincaré patch of Minkowskian AdS_3, so taking the appropriate form (4.6) of the element g we arrive at

$$g=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\alpha\tau+\beta\sigma} & e^{\alpha\sigma+\beta\tau} \\ -e^{-\alpha\sigma-\beta\tau} & e^{\alpha\tau-\beta\sigma}\end{pmatrix} \tag{4.10}$$

and the following connection components

$$J=\frac{(1+i)(\alpha-i\beta)}{4(1-x)}\begin{pmatrix} i & \mathfrak{e} \\ \mathfrak{e}^{-1} & -i\end{pmatrix} \qquad \bar{J}=\frac{(1+i)(\beta-i\alpha)}{4(1+x)}\begin{pmatrix} -i & \mathfrak{e} \\ \mathfrak{e}^{-1} & i\end{pmatrix} \tag{4.11}$$

with $\mathfrak{e}=e^{\sigma(\alpha-\beta)+\tau(\alpha+\beta)}$, and the flatness condition can be easily verified.

The equations of the auxiliary linear system, just as any first-order linear differential equations for two-component vectors

$$0=\begin{pmatrix} f_1 \\ f_2\end{pmatrix}'+\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} f_1 \\ f_2\end{pmatrix} \tag{4.12}$$

can be transformed to second-order equations for the components. Expressing one component completely in terms of the other

$$f_2=-\tfrac{f_1'+af_1}{b} \tag{4.13}$$

allows to straightforwardly arrive at

$$f_1''+(a+d-\tfrac{b'}{b})f_1'+(a'-\tfrac{ab'}{b}+ad-bc)f_1=0 \tag{4.14}$$

As the currents $J,\bar{J}$ are traceless by definition, the $a+d$ subexpression will vanish for all auxiliary linear systems.

For the cusp, also the subexpression '$ad-bc$' will vanish, which is not a coincidence, given that the right-hand sides of the Virasoro constraints (3.11) vanish for this solution. The equations for the upper component look as follows

$$0=\partial^2 f_1-\frac{(1-i)(\alpha-i\beta)}{2}\partial f_1-\frac{i(\alpha-i\beta)^2}{4(1-x)}f_1 \tag{4.15}$$

$$0=\bar{\partial}^2 f_1+\frac{(1-i)(\beta-i\alpha)}{2}\bar{\partial} f_1-\frac{i(\beta-i\alpha)^2}{4(1+x)}f_1 \tag{4.16}$$

and as the coefficients are constant, they are elementary to solve, even by hand. The result is a linear combination of the exponentials

$$\begin{aligned} f_1 \propto \exp\frac{\alpha-i\beta}{4}\left((1-i)\pm(1+i)\frac{\sqrt{1-x^2}}{1-x}\right)w \\ \times\exp\frac{i\alpha-\beta}{4}\left((1-i)\pm(1+i)\frac{\sqrt{1-x^2}}{1+x}\right)\bar{w} \end{aligned} \tag{4.17}$$

for all four possible choices of signs, with four different coefficients. However, two of them will vanish, due to the fact that there are two expressions for the lower component of the type (4.13) and they need to coincide. Taking this into account, one can finally write down the independent solutions

$$\Psi_\pm = e^{\pm\frac{1+i}{4}\left(-w(\alpha-i\beta)\frac{\sqrt{1-x^2}}{1-x}+\bar{w}(i\alpha-\beta)\frac{\sqrt{1-x^2}}{1+x}\right)} \times \begin{pmatrix} e^{\frac{1-i}{4}(w(\alpha-i\beta)+\bar{w}(i\alpha-\beta))} \\ e^{-\frac{1-i}{4}(w(\alpha-i\beta)+\bar{w}(i\alpha-\beta))}(-ix\pm\sqrt{1-x^2}) \end{pmatrix} \tag{4.18}$$

The structure of the solutions is indeed suggestive of an underlying algebraic curve, namely by the presence of the square-root terms and the fact that the two solutions differ precisely by a consistent sign change in those terms, ie. by choosing a different branch for the root. Moreover, the exponential prefactor is reminiscent of the Baker–Akhiezer function structure.

The solutions can be now arranged in a matrix as $\hat{\Psi} = \frac{1}{\sqrt{2}}(\Psi_+; -\Psi_-)$. We see that $\det\hat{\Psi} = \sqrt{1-x^2}$ is indeed constant, and (3.31) is satisfied, as $\hat{\Psi}^{-1}|_{x=0} = g$ without any symmetry transformation. Now, as mentioned before, the Lax operator can be constructed according to (4.1), where the simplest choice of A is, perhaps unsurprisingly, $A = \sqrt{1-x^2}\,\mathrm{diag}(1,-1)$ which leads to a Lax matrix L which indeed is polynomial in x,

$$L = \begin{pmatrix} ix & e^{-\frac{i-1}{2}(w(\alpha-i\beta)+\bar{w}(i\alpha+\beta))} \\ e^{\frac{i-1}{2}(w(\alpha-i\beta)+\bar{w}(i\alpha+\beta))} & -ix \end{pmatrix} \tag{4.19}$$

and the algebraic curve equation reads

$$y^2 = 1 - x^2 \tag{4.20}$$

which is a genus=0 case. Once again, the seemingly arbitrary factor $\sqrt{1-x^2}$ in A was in fact dictated by the requirement of L being polynomial. Also, by construction, $\Psi_\pm$ are the eigenvectors of L with respective eigenvalues $\pm\sqrt{1-x^2}$. Finally note that under $x \to \infty$, the leading term of L is non-degenerate, specifically diagonal with distinct eigenvalues.

Reconstruction

We now turn to the task of reconstructing the solution just from the algebraic curve $y^2 = 1 - x^2$ according to the procedures outlined earlier. We use the following uniformisation of the curve

$$x = \frac{1-z^2}{1+z^2} \qquad\qquad y = \frac{2z}{1+z^2} \tag{4.21}$$

under which passing to the other sheet is achieved by the transformation $z \mapsto -z$. The branch points $x = 1, -1$ correspond to $z = 0, \infty$, respectively, while $x = \infty^\pm$, the two points lying above infinity, correspond to $z = \pm i$. The genus of the curve is 0, therefore the normalized eigenvector will have no dynamical poles. As the eigenvalues of L remain defined and distinct in the $x \to \infty$ limit, the normalized eigenvector will have distinct asymptotic behaviours at $\infty^{\pm}$, and without loss of generality we choose them to be

$$\Psi_{\mathrm{n}}(w,\bar{w};\infty^{+}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Psi_{\mathrm{n}}(w,\bar{w};\infty^{-}) = \begin{pmatrix} 1 \\ \infty \end{pmatrix} \tag{4.22}$$

Thus, we posit that the lower component of Ψ_{n} is proportional to $\frac{z-i}{z+i}$.

As for the Baker–Akhiezer functions, we first note that under our uniformisation the asymptotic form (3.50) simplifies to

$$f_{\mathrm{BA}}(w,\bar{w};x) \propto \exp(-\tfrac{c_{+}}{z} w - c_{-} z \bar{w}) \tag{4.23}$$

It does not exhibit any singular behaviour except for the desired one at the branch points. As we already know that there are no dynamical poles, this is a full, correct expression of the spectral parameter dependence of the prefactor. All that now remains to be determined is the worldsheet-independence of the full eigenvector at infinity, which amounts to fixing the unknown functions $q_{1,2}$ in

$$\Psi = f_{\mathrm{BA}} \Psi_{\mathrm{n}} = e^{-\frac{c_{+}}{z} w - c_{-} z \bar{w}} q_1(w,\bar{w}) \begin{pmatrix} 1 \\ q_2(w,\bar{w}) \frac{z-i}{z+i} \end{pmatrix} \tag{4.24}$$

At ∞^{+}, ie. $z = i$, the upper component yields

$$e^{i c_{+} w - i c_{-} \bar{w}} q_1(w,\bar{w}) = 1 \tag{4.25}$$

with a similar result from the lower component at ∞^{-}, ie. $z = -i$

$$e^{-i c_{+} w + i c_{-} \bar{w}} q_1(w,\bar{w}) q_2(w,\bar{w}) = 1 \tag{4.26}$$

Solving these two equations gives a final result

$$\Psi = e^{-\frac{c_{+}}{z} w - c_{-} z \bar{w}} \begin{pmatrix} e^{-i c_{+} w + i c_{-} \bar{w}} \\ e^{i c_{+} w - i c_{-} \bar{w}} \frac{z-i}{z+i} \end{pmatrix} \tag{4.27}$$

To see if this reconstructed expression coincides with (4.18), we first need to replace the z variable by x, precisely

$$z = \frac{y}{1+x} = \frac{\sqrt{1-x^2}}{1+x} \qquad \frac{1}{z} = \frac{y}{1-x} = \frac{\sqrt{1-x^2}}{1-x} \tag{4.28}$$

$$\frac{z-i}{z+i} = \frac{(z-i)^2}{z^2+1} = -x - iy = -x - i\sqrt{1-x^2} \tag{4.29}$$

where we implicitly choose a $+\sqrt{1-x^2}$ branch of the solution, therefore expecting to recover Ψ_{+}. Comparing the form of x-dependence of the exponents, we identify constants $c_{\pm}$ as

$$c_{+} = \tfrac{1+i}{4}(\alpha - i\beta) \qquad c_{-} = -\tfrac{1+i}{4}(i\alpha - \beta) \tag{4.30}$$

It is now manifest that the remaining exponents also agree and the result exactly reproduces Ψ_+, up to a factor of i in the lower component. This is the residual ambiguity that was expected.

What needs to be stressed at this point, as it is a salient feature of the idea, is that the reconstruction procedure is indeed almost automatic. In this fairly generic example, essentially the only information needed to carry out the procedure was the equation of the algebraic curve. It had two distinct points lying above $x = \infty$, which corresponds to L being non-singular (diagonalisable) in this limit, so the asymptotic behaviour we were able to impose on Ψ was constraining enough not to leave any ambiguities.

4.3 The quark–antiquark potential contour

The contour considered here was introduced in [RY98, MALD98] as a limit of a rectangular Wilson loop, under which it was deformed to consist of infinite, parallel straight lines at a spacelike separation l. A conformally flat parameterisation of the corresponding worldsheet was found in [CHR09, (2.21)]

$$z = z_0 \operatorname{cn}\sigma \qquad x_0 = \frac{z_0}{\sqrt{2}}\tau \qquad x_1 = \frac{z_0}{\sqrt{2}}F(\sigma) \tag{4.31}$$

$z_0 = \Gamma(\frac{1}{4})^2(2\pi)^{-\frac{3}{2}}l$ [CHR09, (2.9)] is the maximum bulk extent of the surface, reached at $\sigma = 0$, whereas the derivative of F is known due to [CHR09, (2.8)]

$$\frac{\partial x_1}{\partial\sigma} = \frac{\partial x_1}{\partial z}\frac{\partial z}{\partial\sigma} = -\frac{\operatorname{cn}^2\sigma}{\operatorname{sn}\sigma\operatorname{dn}\sigma}(-\frac{z_0}{\sqrt{2}})\operatorname{sn}\sigma\operatorname{dn}\sigma = \frac{z_0}{\sqrt{2}}\operatorname{cn}^2\sigma \tag{4.32}$$

and by (A.23) for $k = \frac{1}{\sqrt{2}}$

$$F(\sigma) = 2E(\operatorname{am}\sigma|\tfrac{1}{2}) - \sigma \tag{4.33}$$

The special functions (elliptic integrals, Jacobi elliptic functions) are discussed in the appendix A.2, and the suppressed parameter of the Jacobi functions is $\frac{1}{2}$ throughout this section. The domain is now only a strip corresponding to $z \propto \operatorname{cn}\sigma \geq 0$, or $|\sigma| \leq K(\frac{1}{2})$. The surface approaches the boundary at the ends of this interval, and its profile is plotted in fig. 4.1.

Taking in this case the Euclidean signature form of g (4.7), the procedure is completely analogous to the case of the cusp, albeit noticeably harder due to the special functions involved. The expressions for $J, \bar{J}$ exhibit no simple dependence on $w, \bar{w}$, so we can consider the following linear combinations of the equations

$$\partial_\sigma\Psi + J_\sigma\Psi = (\partial\Psi + J\Psi) + (\bar{\partial}\Psi + \bar{J}\Psi) = 0 \tag{4.34}$$

$$\partial_\tau\Psi + J_\tau\Psi = i(\partial\Psi + J\Psi) - i(\bar{\partial}\Psi + \bar{J}\Psi) = 0 \tag{4.35}$$

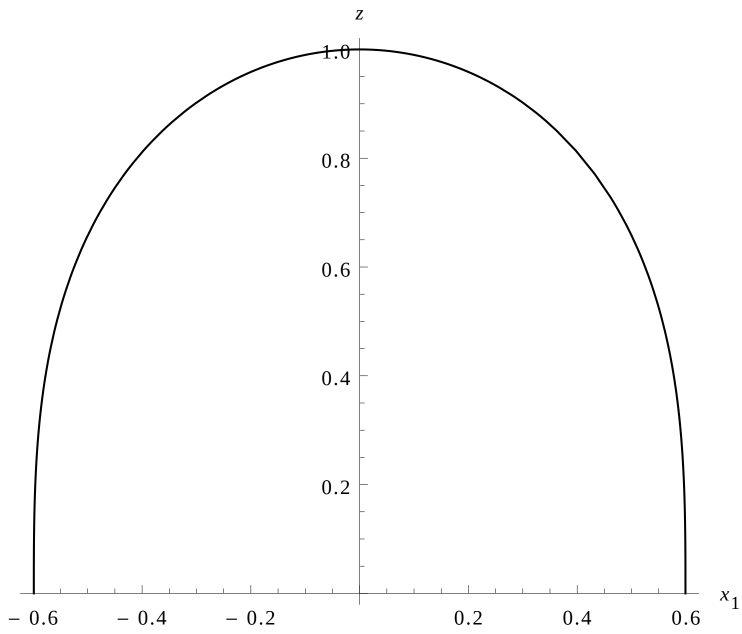

Figure 4.1: A slice of the quark–antiquark potential minimal surface of constant x_0, for $z_0 = 1$. The full surface is obtained by translating the profile perpendicular to the page.

determined by (1.2). The determinants of $J_{\sigma,\tau}$ no longer vanish, but still, with

$$J_\sigma = \frac{1}{2(1-x^2)\,\mathrm{cn}^2\,\sigma}\begin{pmatrix} \mathfrak{p}\,\mathrm{cn}^2\,\sigma - \mathfrak{q} - x\mathfrak{p} & (\mathrm{cn}^2\,\sigma - x)\sqrt{2}/z_0 \\ z_0(x(\mathfrak{p}^2 + 2\,\mathrm{cn}^2\,\sigma) - \mathfrak{r})/\sqrt{2} & -\mathfrak{p}\,\mathrm{cn}^2\,\sigma + \mathfrak{q} + x\mathfrak{p} \end{pmatrix} \tag{4.36}$$

$$J_\tau = \frac{i}{2(1-x^2)\,\mathrm{cn}^2\,\sigma}\begin{pmatrix} x(\mathfrak{p}\,\mathrm{cn}^2\,\sigma - \mathfrak{q}) - \mathfrak{p} & (x\,\mathrm{cn}^2\,\sigma - 1)\sqrt{2}/z_0 \\ z_0(\mathfrak{p}^2 + 2\,\mathrm{cn}^2\,\sigma - x\mathfrak{r})/\sqrt{2} & -x(\mathfrak{p}\,\mathrm{cn}^2\,\sigma - \mathfrak{q}) + \mathfrak{p} \end{pmatrix} \tag{4.37}$$

where $\mathfrak{p} = i\tau + F(\sigma), \mathfrak{q} = 2\,\mathrm{cn}\,\sigma\,\mathrm{dn}\,\sigma\,\mathrm{sn}\,\sigma, \mathfrak{r} = \mathfrak{p}^2\,\mathrm{cn}^2\,\sigma - 2\mathfrak{p}\mathfrak{q} - 2\,\mathrm{cn}^4\,\sigma$, the equation in τ constructed along the lines of (4.14) is remarkably simple

$$\partial_\tau^2 f_1 = \frac{x}{2(1-x^2)} f_1 \tag{4.38}$$

The one in σ is more complicated and has no obvious solution

$$0 = \partial_\sigma^2 f_1 - \frac{2x\,\mathrm{dn}\,\sigma\,\mathrm{sn}\,\sigma}{\mathrm{cn}\,\sigma(\mathrm{cn}^2\,\sigma - x)}\partial_\sigma f_1 + \left(\frac{x}{2(1-x^2)} + \frac{1}{\mathrm{cn}^2\,\sigma - x}\right) f_1 \tag{4.39}$$

However, we can find an explicit solution in τ

$$f_{1,\pm} = C_\pm(\sigma)\exp \pm \frac{\sqrt{x(1-x^2)}}{\sqrt{2}(1-x^2)}\tau \tag{4.40}$$

then use it with (4.13) in (4.36) to obtain two first-order equations for each of $C_\pm$. They do coincide and take the following form

$$\frac{\partial_\sigma C_\pm(\sigma)}{C_\pm(\sigma)} = -\frac{\mathrm{dn}\,\sigma\,\mathrm{sn}\,\sigma}{\mathrm{cn}\,\sigma(x\,\mathrm{cn}^2\,\sigma - 1)} \mp \left(\frac{i}{\sqrt{2}\sqrt{x(1-x^2)}} + \frac{i\sqrt{x(1-x^2)}}{\sqrt{2}x(x\,\mathrm{cn}^2\,\sigma - 1)}\right) \tag{4.41}$$

that can even be integrated by hand. Namely substituting $v = \operatorname{cn}\sigma$ in the first term gives a rational integrand $1/(v(xv^2-1))$, while the integral of $1/(x\operatorname{cn}^2\sigma - 1)$ after substituting $v = \operatorname{am}\sigma$ becomes explicitly equal by definition to the elliptic integral $\frac{1}{x-1}\Pi(\frac{x}{x-1};\operatorname{am}\sigma|\frac{1}{2})$. Using this result, as well as (4.13) with (4.40), the complete solution may be at last assembled as

$$\Psi_\pm = \frac{\sqrt{x\operatorname{cn}^2\sigma-1}}{\operatorname{cn}\sigma} e^{\pm\frac{x\tau - i\sigma + i(x+1)\Pi(\frac{x}{x-1};\operatorname{am}\sigma|\frac{1}{2})}{\sqrt{2}\sqrt{x(1-x^2)}}} \times \begin{pmatrix} 1 \\ z_0\frac{\sqrt{2}x\operatorname{cn}\sigma\operatorname{dn}\sigma\operatorname{sn}\sigma \pm i\sqrt{x(1-x^2)}\operatorname{cn}^2\sigma}{x\operatorname{cn}^2\sigma-1} - z_0\frac{i\tau+F(\sigma)}{\sqrt{2}} \end{pmatrix} \tag{4.42}$$

Again we see the telltale signs of the algebraic curve structure, namely an exponential prefactor, and the difference between the two solutions being a branch choice of some square root (here of $x(1-x^2)$).

The $\sqrt{x\operatorname{cn}^2\sigma-1}$ factor looks puzzling, however, as on a first glance it may give rise to a σ-dependent branch point that would go beyond any familiar algebraic curve description. It turns out, though, that the exponential part also has a non-trivial asymptotics at $x = \operatorname{cn}^{-2}\sigma$. Let us examine the series expansions of the terms of (4.41) in the vicinity of this point. The first term (which gives rise to the explicit root in (4.42)) is already expanded, while the third has a regular part and a simple pole with a coefficient of

$$\begin{aligned} \frac{i\sqrt{x(1-x^2)}}{\sqrt{2}x} &= \frac{i\operatorname{cn}^2\sigma}{\sqrt{2}}\sqrt{\frac{\operatorname{cn}^4\sigma-1}{\operatorname{cn}^6\sigma}} \\ &= \frac{i\operatorname{cn}^2\sigma}{\sqrt{2}}\sqrt{-2\frac{\operatorname{dn}^2\sigma\operatorname{sn}^2\sigma}{\operatorname{cn}^6\sigma}} = -|\operatorname{sn}\sigma|\frac{\operatorname{dn}\sigma}{\operatorname{cn}\sigma} \end{aligned} \tag{4.43}$$

where we have assumed that the principal branch of a square root gives positive multiplicities of i for negative arguments, and used the fact that $\operatorname{cn}\sigma, \operatorname{dn}\sigma$ are positive (and even) for our range of σ. $\operatorname{sn}\sigma$ is odd, though, and there are two mutually opposite values of σ that correspond to $x = \operatorname{cn}^{-2}\sigma$. Therefore, for any choice of signs in (4.41), there is exactly one choice of one of those two values, for which the polar part of the third term will cancel the first term, while for the other value it will double its contribution. Then, integrating the expansion term by term (in which the regular part converges, as for the full integrand the result was well defined), the former situation means that the problematic contribution vanishes. The latter, namely $\operatorname{sgn}\sigma = \mp 1$ for $\Psi_\pm$, will double the coefficient of the relevant term in $\log C_\pm$, and $C_\pm$ will exhibit a linear, instead of radical, behaviour at $x = \operatorname{cn}^{-2}\sigma$.

A complement of this situation appears in the lower component of (4.42), where the first term has a pole at $x\operatorname{cn}^2\sigma - 1$. However, by the same argument and calculation, there is one sign of σ that cancels the pole, and it is the same that cancels the root in the prefactor. The opposite σ preserves the pole, and the corresponding linear vanishing term in the prefactor, which is in fact another sign indicating an algebraic curve structure, more precisely of non-zero genus.

However, in this case the $x \to 0$ limit is harder to handle, as both $\Psi_\pm$ tend to the same values and $\det(\Psi_+;\Psi_-) \propto \sqrt{x(1-x^2)}$ vanishes. Another solution matrix can be chosen, though, namely

$$\left(\frac{\Psi_+ - \Psi_-}{2\sqrt{x(1-x^2)}}; \frac{i}{z_0}\Psi_\pm\right) \tag{4.44}$$

which has unit determinant and its inverse is equal to g under $x \to 0$ limit, again confirming (3.31). This can be seen by employing the expansion

$$\Pi(\tfrac{x}{x-1}; \operatorname{am}\sigma|\tfrac{1}{2}) = \sigma + (F(\sigma) - \sigma)x + O(x^2) \tag{4.45}$$

where both terms are needed to capture the x^1 order of the numerator of the exponent in (4.42).

To construct the Lax operator by (4.1), we do not have to choose (4.44) as $\hat{\Psi}$, as we want the result to be as simple as possible. Instead, taking $\hat{\Psi} = (\Psi_+;\Psi_-)$ and $A(x) = \sqrt{x(1-x^2)}\operatorname{diag}(1,-1)$ we again obtain a result explicitly polynomial in x, namely

$$L = -\frac{i}{\sqrt{2}\operatorname{cn}^2\sigma}\begin{pmatrix} x(\mathfrak{p}\operatorname{cn}^2\sigma - \mathfrak{q}) - \mathfrak{p} & (x\operatorname{cn}^2\sigma - 1)\sqrt{2}/z_0 \\ z_0(\mathfrak{p}^2 + 2x^2\operatorname{cn}^2\sigma - x\mathfrak{r})/\sqrt{2} & -(x(\mathfrak{p}\operatorname{cn}^2\sigma - \mathfrak{q}) - \mathfrak{p}) \end{pmatrix} \tag{4.46}$$

using the same shorthands as in (4.36). Note that the link with (4.44) can be still made, as it is related to $\hat{\Psi}$ by right-multiplication by a matrix constant in σ, τ, so for a different (non-diagonal) A we would still obtain the same, (comparatively) simple L. The algebraic curve that arises is

$$y^2 = x(1-x^2) \tag{4.47}$$

of genus 1, again directly related to the $\sqrt{x(1-x^2)}$ factor in A that was chosen to render L polynomial. Note that the algebraic curve (4.47) is very simple, what is especially striking when compared with the original solution (4.31), (4.33) given in terms of special functions.

Again, $\Psi_\pm$ are by construction the eigenvectors of L with respective eigenvalues $\pm\sqrt{x(1-x^2)}$, but a novel feature appears in $x \to \infty$. In this limit, L becomes degenerate with leading-order term $-iz_0x^2\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. This is related to the fact that $x = \infty$ is a branch point of the algebraic curve (4.47), which will have a significant impact on our subsequent analysis, in which the asymptotics at that point play a crucial role.

Reconstruction

The algebraic curve (4.47) can be uniformised employing theta functions with quasi-periods $2K = 2K(\frac{1}{2}), 2iK' = 2iK'(\frac{1}{2}) = 2iK$, namely

$$x(z) \propto \frac{\theta(z-K)\theta(z+K)}{\theta(z-iK)\theta(z+iK)} \tag{4.48}$$

$$y(z) \propto \frac{\theta(z)\theta(z-K)\theta(z+K+iK)}{\theta(z-iK)\theta(z+iK)^2} \tag{4.49}$$

with precise definitions given in the appendix A.3 for the nome $e^{-\pi}$. x, y are both of the form (A.33) and therefore doubly periodic. x has a double zero at K and a double pole at iK, so the points above $z = K, iK$ are the zero and infinity of the algebraic curve, respectively. The normalisation of x is chosen so that $x(0) = 1$, and consequently $x(K + iK) = -1$, as

$$\frac{x(K+iK)}{x(0)} = -e^{\pi}\frac{\theta(iK)^4}{\theta(K)^4} = -e^{\pi}\frac{\theta_3(\frac{\pi}{2})^4}{\theta_3(\frac{i\pi}{2})^4} = -e^{\pi}\frac{\theta_4(0)^4}{\theta_2(0)^4 e^{\pi}} = -1 \tag{4.50}$$

by (A.29), (A.26), and (A.27). y has simple zeroes at $0, K, K + iK$ and a triple pole at iK, thus the structure of poles and zeroes of both sides of the algebraic curve equation is identical. Therefore, a normalisation of y can be chosen so that this equation is exactly satisfied, for instance by taking the equation at $z = K$, where $y^2 \approx x$ and equating the leading coefficients on both sides. Then, the normalisation constant c_y is determined by

$$c_y^2\frac{2(z-K)^2\theta'(0)^2\theta(K)^2\theta(iK)^2}{\theta(K-iK)^2\theta(K+iK)^4} = \frac{2(z-K)^2\theta'(0)^2}{\theta(K-iK)\theta(K+iK)}\frac{\theta(iK)\theta(-iK)}{\theta(K)\theta(-K)} \tag{4.51}$$

where the latter factor on the right-hand side is the normalisation constant of x. By (A.29), $c_y = -ie^{\pi}\theta(K+iK)^2/\theta(K)^2$. Also by (A.29), x, y are respectively even and odd in z, so passing to the other sheet is achieved by $z \to -z$. The points in the fundamental domain for which this is an involution are $z = 0, K, iK, K + iK$, for which $x = 1, 0, \infty, -1$, respectively, confirming that these are the branch points of the curve.

As the present curve is of genus 1, one dynamical pole is expected to appear in the normalised eigenvector. Its position will turn out to appear as a zero of the Baker–Akhiezer function, so we start by examining its asymptotics. Because this case study is already complicated as it is, we will now make an 'educated guess' for $c_\pm$ of (3.50), or rather determine it from (4.5) and the known expression (4.46) for L. Starting from $\frac{L}{1\mp x}$, all x-dependence in the numerators of the components can be removed by omitting the terms regular at $x = \pm 1$, ie. proportional to $x \mp 1$, and the remaining polar parts are equal to $-2\sqrt{2}iJ, 2\sqrt{2}i\bar{J}$ for the respective choice of sign. From (4.5) then

$$c_+ = -c_- = \frac{i}{2\sqrt{2}} \tag{4.52}$$

and the prescription for the Baker–Akhiezer factor gives

$$\exp -\frac{i}{2\sqrt{2}}\left(\frac{y}{1-x}w+\frac{-y}{1+x}\bar{w}\right)=\exp\frac{1}{\sqrt{2}}\left(\frac{x}{y}\tau-i\frac{x^2}{y}\sigma\right) \tag{4.53}$$

which a priori is valid only in the neighbourhoods of $x=\pm 1$, ie. $z=0,K+iK$. These are the only points where f_{BA} may (and should) be singular, and from the definition of x,y one sees that the coefficient of τ has poles at these exact points, because of cancellations of theta functions in x/y. However, the coefficient of σ has an additional pole at $z=iK$, which proves that (4.53) is not a valid global expression for the Baker–Akhiezer function. This can be resolved by replacing $x^2/(\sqrt{2}y)$ with

$$G(z)=-\tfrac{1}{2}(\delta(z)+\delta(z-K-iK)) \tag{4.54}$$

where δ is the logarithmic derivative of θ (see appendix). This trick preserves the local structure, ie. residues at $z=0,K+iK$, at the cost of forfeiting periodicity in the imaginary direction. To remedy this, an additional factor can be introduced, so that

$$f_{\mathrm{BA}}=M(\sigma,\tau)\frac{\theta(z-\gamma(\sigma,\tau))}{\theta(z-\gamma_0)}\exp\left(\frac{x\tau}{\sqrt{2}y}+iG(z)\sigma\right) \tag{4.55}$$

Now, after shifting z by the imaginary quasi-period $2iK$, we apply (A.29), (A.32) that give rise to additional exponential factors. These should cancel, as the whole function should be periodic, and this is satisfied if

$$\gamma(\sigma,\tau)=\gamma_0-i\sigma \tag{4.56}$$

After setting $\gamma_0=0$ for simplicity, all that remains unknown in f_{BA} is an overall factor M independent of x.

So far we have secured the proper singularity structure of f_{BA}, as well as its vanishing at $z=\gamma$, which thus is the dynamical pole of the normalised eigenvector. Now, we demand that its lower component is a well-defined function with poles at $x=\infty$ and $z=\gamma$, and therefore its form can be restricted to

$$\psi=r_0(\sigma,\tau)+r_1(\sigma,\tau)(\delta(z-iK)-\delta(z+i\sigma)) \tag{4.57}$$

where the residues add up to zero to guarantee periodicity by (A.34).

There remains one criterion provided by the procedure that has not been applied yet, namely the worldsheet independence of the full eigenvector Ψ in the $x\to\infty$ limit. As opposed to the case of the cusp, where the algebraic curve had two distinct points lying above infinity, it has a branch point instead and there is only one asymptotic condition to be imposed. To partially make amends for this shortage of information, we take it at the second order, namely expanding

$$f_{\mathrm{BA}}=M(\sigma,\tau)\left(f_0(\sigma,\tau)+(z-iK)f_1(\sigma,\tau)+\cdots\right) \tag{4.58}$$

$$\psi=\frac{\psi_{-1}(\sigma,\tau)}{z-iK}+\psi_0(\sigma,\tau)+\cdots \tag{4.59}$$

with $\psi_{-1,0}, f_{0,1}$ being the relevant Laurent coefficients of $\psi, f_{\mathrm{BA}}/M$ at $z = iK$, we demand that all the coefficients explicitly spelled out in

$$\Psi = f_{\mathrm{BA}} \begin{pmatrix} 1 \\ \psi \end{pmatrix} = M \begin{pmatrix} 0 & + & f_0 & + & \cdots \\ \frac{f_0 \psi_{-1}}{z-iK} & + & f_1 \psi_{-1} + f_0 \psi_0 & + & \cdots \end{pmatrix} \tag{4.60}$$

be constant. Heretofore unknown functions are now set to

$$M(\sigma,\tau) = \frac{C_1}{f_0(\sigma,\tau)} \qquad r_1(\sigma,\tau) \equiv \psi_{-1}(\sigma,\tau) = \frac{C_2}{C_1} \tag{4.61}$$

$$\psi_0(\sigma,\tau) = \frac{C_3 - M(\sigma,\tau)\psi_{-1}(\sigma,\tau) f_1(\sigma,\tau)}{M(\sigma,\tau) f_0(\sigma,\tau)} = \frac{C_3}{C_1} - \frac{f_1(\sigma,\tau)}{f_0(\sigma,\tau)} \frac{C_2}{C_1} \tag{4.62}$$

$$r_0(\sigma,\tau) = \frac{C_3}{C_1} - \frac{f_1(\sigma,\tau)}{f_0(\sigma,\tau)} \frac{C_2}{C_1} - \frac{C_2}{C_1} \left(\frac{\theta''(0)}{2\theta'(0)} - \delta(iK + i\sigma) \right) \tag{4.63}$$

where r_0 has been obtained from ψ_0. This concludes the reconstruction.

The result as presented is definitely less satisfactory than in the case of the cusp, as the relation to the originally obtained vectors $\Psi_\pm$ is completely obscured. However, some agreement should be expected, firstly due to the fact that the Jacobi elliptic functions can be expressed in terms of theta functions, and one could try to find some analytic relation to the expression (4.42). Secondly, for any z, the solution ceases to be well-defined at $\sigma = \pm K$, as a factor of $\theta(iK + i\sigma)$ is present in f_0, a denominator of M. Thus the domain of the solution is $|\sigma| < K$, just as for the original solution. It should be noted, however, that this is a direct consequence of the judicious choice of γ_0, and other values of this parameter would cause at least shifting of the domain strip, if not complexifying the whole solution.

In lieu of showing an actual analytic equality between the two results, consider the position of the dynamical pole given by $z = -i\sigma$. Then $x(-i\sigma)$, including the normalisation, can be expressed as

$$x(-i\sigma) = \left(\frac{\theta_3(\frac{\pi}{2})\theta_3(\frac{i\pi\sigma}{2K} + \frac{i\pi}{2})}{\theta_3(\frac{i\pi}{2})\theta_3(\frac{i\pi\sigma}{2K} + \frac{\pi}{2})} \exp \frac{\pi\sigma}{2K} \right)^2 \tag{4.64}$$

where the supressed nome of θ_3 is $q = e^{-\pi}$ and the properties of quasi-periodicity (A.25) and evenness have been applied. Next, the half-period shifts (A.26) can be used

$$x(-i\sigma) = \left(\frac{\theta_4(0)\theta_2(\frac{i\pi\sigma}{2K})}{\theta_2(0)\theta_4(\frac{i\pi\sigma}{2K})} \right)^2 = \mathrm{cn}^2\, i\sigma = \frac{1}{\mathrm{cn}^2\, \sigma} \tag{4.65}$$

where subsequently (A.12) and (A.19) have been used (keeping in mind that $k = \frac{1}{\sqrt{2}}$). We thus see that the position of the dynamical pole exactly reproduces the zero of the prefactor in (4.42).

Finally, a convincing cross-check can be made with numerics. Namely, for

$$C_1 = \sqrt{1-x} \qquad C_2 = i\sqrt{2} C_1 \qquad C_3 = 0 \tag{4.66}$$

the expressions agree very well for any z, σ, τ. The freedom of choice of $C_{1,2}$ corresponds to the already familiar ambiguity of the reconstruction procedure. However, the lack of criteria to fix C_3 is a new feature of this particular algebraic curve and stems from the branch point at infinity, although the fact that the proper value is so specific vaguely suggests that we might have overlooked some less obvious constraint.

4.4 Other elliptic reconstructions

In this section, we consider elliptic algebraic curves in more general context, assuming only that there is no branch point at infinity (and neither at zero), as opposed to the quark–antiquark case. Even without fully specifying the uniformisation of the curve, fairly refined expressions for the corresponding string worldsheet appear. As a consequence, the curve

$$y^2 = (x^2 - 1)(x^2 - a^2) \tag{4.67}$$

will be shown to reproduce a folded string solution.

Assume that the uniformisation is done on a lattice in $\mathbb{C}$ with half-periods ω, ω', and x, y are respectively even and odd. Then the branch points will appear where $z \mapsto -z$ is an involution, that is, at $z = 0, \omega, \omega', \omega + \omega'$. The points corresponding to $x = \pm 1$, denoted as $z = I^{\pm}$, will necessarily be among these four. The pairs of points above $x = 0, \infty$, denoted respectively as $z = 0^{\pm}, \infty^{\pm}$, will be related as $0^+ = -0^-, \infty^+ = -\infty^-$.

The eigenvector on this algebraic curve will have the following form

$$\Psi = q_1 e^{\delta(z-I^+)w + \delta(z-I^-)\bar{w}} \frac{\theta(z-\gamma)}{\theta(z-\gamma_0)} \begin{pmatrix} 1 \\ q_2 \frac{\theta(z-\infty^+)}{\theta(z-\infty^-)} \frac{\theta(z+\infty^+-\infty^--\gamma)}{\theta(z-\gamma)} \end{pmatrix} \tag{4.68}$$

where $q_{1,2}, \gamma$ are functions of $w, \bar{w}$. The Baker–Akhiezer prefactor has essential singularities of the type (3.50) at $I^{\pm}$, and vanishes at the dynamical pole γ. The lower component of the normalised eigenvector vanishes at one of the infinities (here at ∞^+), and has a pole at the other (∞^-), as well as the dynamical one. The remaining theta function is chosen so that the product is periodic.

Periodicity needs to be restored in the prefactor, though. Again, shifting the argument by the complex quasi-period $2\omega'$, applying (A.29), (A.32), and demanding that the arising exponentials cancel, we fix

$$\gamma = w + \bar{w} + \gamma_0 \tag{4.69}$$

where γ_0 remains an unspecified reference point.

Demanding that the eigenvector is constant at $\infty^{\pm}$, we are able to fix $q_{1,2}$.

Inserting the result back into Ψ, we obtain

$$\Psi = \begin{pmatrix} \frac{e^{\delta(z-I^+)w+\delta(z-I^-)\bar{w}}}{e^{\delta(\infty^+-I^+)w+\delta(\infty^+-I^-)\bar{w}}} \frac{\theta(z-\gamma)}{\theta(z-\gamma_0)} \frac{\theta(\infty^+-\gamma_0)}{\theta(\infty^+-\gamma)} \\ q\frac{e^{\delta(z-I^+)w+\delta(z-I^-)\bar{w}}}{e^{\delta(\infty^--I^+)w+\delta(\infty^--I^-)\bar{w}}} \frac{\theta(z-\infty^+)}{\theta(z-\infty^-)} \frac{\theta(z+\infty^+-\infty^--\gamma)}{\theta(z-\gamma_0)} \frac{\theta(\infty^--\gamma_0)}{\theta(\infty^+-\gamma)} \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ q\Psi_2 \end{pmatrix} \tag{4.70}$$

where q is now constant. Note that the $w, \bar{w}$ dependence is present only in the exponentials and as γ.

A matrix of eigenvectors can be arranged as $\hat{\Psi} = (q^+\Psi(z); q^-\Psi(-z))$ for any constants $q^\pm$. Note that this is not a most general choice, as any linear combination of $\Psi(\pm z)$ is a solution of the auxiliary linear system at $x(z)$. By (3.31)

$$g = \frac{1}{\sqrt{\det \hat{\Psi}(0)}} \begin{pmatrix} q^-q\Psi_2(0^-) & -q^-\Psi_1(0^-) \\ -q^+q\Psi_2(0^+) & q^+\Psi_1(0^+) \end{pmatrix} \tag{4.71}$$

By comparison with (4.8), ratios or products of the components of this matrix will contain expressions for the global AdS_3 coordinates of the corresponding solution. Explicitly we arrive at

$$e^{2it} = \frac{q^-q}{q^+} \frac{\Psi_2(0^-)}{\Psi_1(0^+)} \qquad e^{-2i\psi} = \frac{q^-}{q^+q} \frac{\Psi_1(0^-)}{\Psi_2(0^+)} \tag{4.72}$$

$$\cosh^2 \rho = \frac{q^+q^-q}{\det \hat{\Psi}(0)} \Psi_1(0^+)\Psi_2(0^-) \tag{4.73}$$

The last expression is particularly worth attention. Even without examining the exact form of $\det \hat{\Psi}(z)$, it is guaranteed not to depend on $w, \bar{w}$, and its value at 0 is a constant. Now, both terms in the parentheses of

$$\det \hat{\Psi}(0) = q^+q^-q \left(\Psi_1(0^+)\Psi_2(0^-) - \Psi_1(0^-)\Psi_2(0^+)\right) \tag{4.74}$$

have exactly the same exponential part. It can be thus cancelled with the exponential part of the numerator in (4.73), and worldsheet dependence of $\cosh^2 \rho$ will appear only through γ, that is, as $w+\bar{w}$. Therefore, this reconstruction scheme may produce only configurations in which the radial coordinate is independent of τ. This does not indicate any shortcomings of the whole formalism, but rather of some details of the current derivation, in which for instance $\hat{\Psi}$ was not taken in the most general form.

We will apply these steps to the algebraic curve (4.67) with the following unformisation

$$x(z) \propto \frac{\theta(z - 0^+)\theta(z - 0^-)}{\theta(z - \infty^+)\theta(z - \infty^-)} \tag{4.75}$$

$$y(z) \propto \frac{\theta(z - I^+)\theta(z - I^-)\theta(z - A^+)\theta(z + A^-)}{\theta(z - \infty^+)^2\theta(z - \infty^-)^2} \tag{4.76}$$

where $0^\pm = \mp\frac{\omega}{2}, \infty^\pm = \pm(\omega' - \frac{\omega}{2})$, the branch points are $I^- = 0, I^+ = \omega, A^+ = \omega', A^- = \omega + \omega'$, and both functions are periodic by (A.33). By (A.29), $x(I^-) =$

$-x(I^+)$, so a normalisation for x can be chosen so that indeed $x(I^\pm) = \pm 1$. Moreover, similarly $x(A^-) = -x(A^+)$, and a proper (although highly non-trivial) choice of half-periods will lead to $x(A^\pm) = \pm a$. Then, the poles and zeroes on both sides of (4.67) coincide, so we conclude that y can be normalised accordingly. Finally, by (A.29) x, y are respectively even and odd as required.

Considering the expression for t (4.72), the $w, \bar{w}$-dependence remains only in the exponential factor and the following ratio

$$\frac{\theta(0^- + \infty^+ - \infty^- - \gamma)}{\theta(0^+ - \gamma)} = -e^{-\frac{i\pi}{\omega}(-\frac{\omega}{2}-\gamma)} \tag{4.77}$$

by (A.29), as here $0^- + \infty^+ - \infty^- = 0^+ + 2\omega'$. Collecting all terms in the exponents and applying (A.32), the coefficients of $w, \bar{w}$ in the exponents turn out to be mutually opposite, and the whole expression can be written as

$$e^{2it} \propto \frac{q^- q}{q^+} e^{c_t(w-\bar{w})} \tag{4.78}$$

and the overall proportionality can be set to 1 by a judicious choice of $q, q^\pm$. Similiarly, for ψ (4.72), the only non-constant theta factors are

$$\frac{\theta(0^- - \gamma)}{\theta(0^+ + \infty^+ - \infty^- - \gamma)} = -e^{\frac{i\pi}{\omega}(\frac{\omega}{2}-\gamma)} \tag{4.79}$$

as $0^+ + \infty^+ - \infty^- = 0^- - 2\omega + 2\omega'$. Collecting all the exponents, analogous behaviour appears and thus

$$e^{-2i\psi} \propto \frac{q^-}{q^+ q} e^{c_\psi(w-\bar{w})} \tag{4.80}$$

where there is still enough freedom in $q, q^\pm$ to set the overall proportionality to 1. We have then established that $t = c_t \tau, \psi = c_\psi \tau$.

Finally, as mentioned before, ρ depends only on σ. Together with the obtained form of t, ψ, this suffices to state that the solution corresponding to the algebraic curve (4.67) is the folded string (see also section 5.1).

It needs to be stressed that this is not the only solution that can be obtained within the described reconstruction procedure for the elliptic curves, even if it is slightly limited. Another example is the generalised quark–antiquark potential that was reconstructed in [JLG12, §B.3] out of an algebraic curve defined as

$$y^2 = (x^2 - 1)(x - a)(x + \tfrac{1}{a}) \tag{4.81}$$

4.5 Ambiguities for correlation functions

Consider two correlation functions: $\langle \operatorname{tr} \bar{Z}^J \operatorname{tr} Z^J \rangle$ of two local operators and $\langle W_\circ \operatorname{tr} Z^J \rangle$ of a local operator with a circular Wilson loop. Their dual minimal surfaces stretch between two insertion points in the former case, described by a propagation of a point-like (BMN) string, and between a circular contour and an insertion point in the latter. The quasi-momentum of the BMN string is [GV07, (31)]

$$p(x) = \frac{2\pi \mathcal{J} x}{x^2 - 1} \tag{4.82}$$

where $\mathcal{J} = J/\sqrt{\lambda}$. For the correlation function with $W_\circ$, the contour defining the monodromy can be shifted as close as possible to the $\operatorname{tr} Z^J$ insertion point, and by the argument of path independence the quasi-momentum will be precisely the same. In fact, this argument holds for correlation functions of $\operatorname{tr} Z^J$ with arbitrary, possibly very complex operators. This raises a question of how is it possible to encompass within the algebraic curve approach more than one distinct solutions with the exact same monodromy and, consequently, quasi-momentum.

Explicit parameterisations of the minimal surfaces take the simplest form, if an insertion point is sent to spatial infinity beforehand, which can always be done by virtue of conformal symmetries of the boundary theory. This corresponds to the dual string propagating indefinitely outwards into the bulk, either from a point at the origin in the BMN case, or from a circle centered at the origin in the Wilson loop case, if another symmetry is used to appropriately translate the remaining operator. The BMN solution then reads simply

$$z = e^{\mathcal{J}\tau} \qquad x_{0,1} = 0 \qquad \phi = i\mathcal{J}\tau \tag{4.83}$$

with $\tau \in \mathbb{R}$, while the correlator with the Wilson loop corresponds to a surface [Zare02, (4.7), (4.12)]

$$\begin{aligned} x_0 &= \frac{\tilde{\mathcal{J}} e^{\mathcal{J}\tau}}{\cosh(\tilde{\mathcal{J}}\tau + \operatorname{arsinh}\mathcal{J})} \cos\sigma & z &= \left(\tilde{\mathcal{J}} \tanh(\tilde{\mathcal{J}}\tau + \operatorname{arsinh}\mathcal{J}) - \mathcal{J}\right) e^{\mathcal{J}\tau} \\ x_1 &= \frac{\tilde{\mathcal{J}} e^{\mathcal{J}\tau}}{\cosh(\tilde{\mathcal{J}}\tau + \operatorname{arsinh}\mathcal{J})} \sin\sigma & \phi &= i\mathcal{J}\tau \end{aligned} \tag{4.84}$$

where $\tilde{\mathcal{J}} = \sqrt{1+\mathcal{J}^2}$, on a domain of $\tau \geq 0$ (approaching the boundary Wilson loop at 0) and periodic $|\sigma| \leq \pi$. The surface is plotted in fig. 4.2. In both cases both the target space is Euclidean, so g is given by (4.7).

The BMN solution yields an almost trivial auxiliary linear system

$$J = -\frac{i\mathcal{J}}{2(1-x)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \bar{J} = \frac{i\mathcal{J}}{2(1+x)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.85}$$

with immediate solutions

$$\Psi_+ = e^{\frac{i\mathcal{J}}{2}\left(-\frac{w}{1-x} + \frac{\bar{w}}{1+x}\right)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Psi_- = e^{-\frac{i\mathcal{J}}{2}\left(-\frac{w}{1-x} + \frac{\bar{w}}{1+x}\right)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.86}$$

$\hat{\Psi} = (\Psi_+; -\Psi_-)$ has unit determinant, reproduces g directly from (3.31), and for trivial $A = \operatorname{diag}(1,-1)$ gives a polynomial Lax matrix $L = \operatorname{diag}(-1,1)$. The corresponding algebraic curve is then

$$y^2 = 1 \tag{4.87}$$

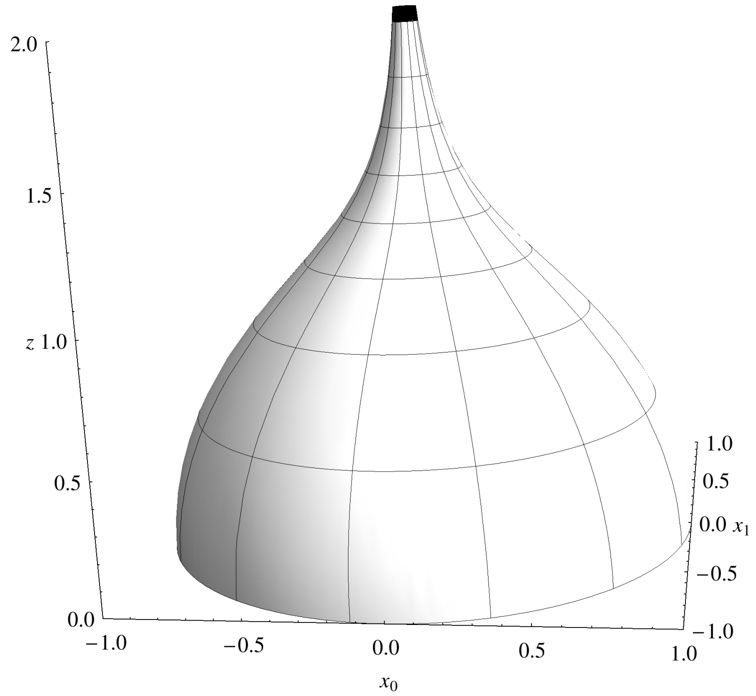

Figure 4.2: The minimal surface (4.84) of the correlation function of a circular Wilson loop and a local operator at spatial infinity. For large values of $\mathcal{J}$, the surface resembles a cylinder steadily contracting as propagating into the bulk. Here $\mathcal{J} = \frac{1}{5}$.

with no poles nor cuts, therefore consisting of two disconnected copies of a complex plane. By extension, one might say that its genus is -1.

For the surface (4.84), the situation is somewhat similar to the quark–antiquark potential Wilson loop, where the equations in $w, \bar{w}$ were harder to treat than their linear combinations (4.34). Here we obtain

$$J_\sigma = -\frac{1}{(1-x^2)\mathfrak{s}^2}\begin{pmatrix} i(\mathcal{J}x\mathfrak{s}^2 + 1 + \mathcal{J}^2) & \tilde{\mathcal{J}}\mathfrak{e}^{-1}\left(\mathfrak{c} + (\mathcal{J}-x)\mathfrak{s}\right) \\ \tilde{\mathcal{J}}\mathfrak{e}\left(\mathfrak{c} - (\mathcal{J}-x)\mathfrak{s}\right) & -i(\mathcal{J}x\mathfrak{s}^2 + 1 + \mathcal{J}^2) \end{pmatrix} \tag{4.88}$$

$$J_\tau = -\frac{1}{(1-x^2)\mathfrak{s}^2}\begin{pmatrix} -(\mathcal{J}\mathfrak{s}^2 + x(1+\mathcal{J}^2)) & i\tilde{\mathcal{J}}\mathfrak{e}^{-1}\left(x\mathfrak{c} + (\mathcal{J}x-1)\mathfrak{s}\right) \\ i\tilde{\mathcal{J}}\mathfrak{e}\left(x\mathfrak{c} - (\mathcal{J}x-1)\mathfrak{s}\right) & \mathcal{J}\mathfrak{s}^2 + x(1+\mathcal{J}^2) \end{pmatrix} \tag{4.89}$$

where $\mathfrak{e} = e^{i\sigma + \mathcal{J}\tau}$, $\mathfrak{s} = \sinh\tilde{\mathcal{J}}\tau$, $\mathfrak{c} = \tilde{\mathcal{J}}\cosh\tilde{\mathcal{J}}\tau$. The equation in σ constructed by (4.14) reads

$$0 = \partial_\sigma^2 f_1 + i\partial_\sigma f_1 + \frac{\mathcal{J}^2x^2 + \mathcal{J}x(1-x^2)}{(1-x^2)^2} f_1 \tag{4.90}$$

Its characteristic equation has a notable feature, namely its discriminant is a perfect square

$$\Delta = -\left(\frac{1 + 2\mathcal{J}x - x^2}{1-x^2}\right)^2 \tag{4.91}$$

so the two solutions obtained do not differ by choosing a branch of some square root

$$f_{1,+} = C_+(\tau) \exp \frac{i(x^2 - \mathcal{J}x - 1)\sigma}{1 - x^2} \qquad f_{1,-} = C_-(\tau) \exp \frac{i\mathcal{J}x\sigma}{1 - x^2} \tag{4.92}$$

Again, we use these with (4.13) in (4.89) and obtain

$$\frac{\partial_\tau C_+(\tau)}{C_+(\tau)} = \frac{\mathcal{J}x^2}{1 - x^2} - \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau \tag{4.93}$$

$$\frac{\partial_\tau C_-(\tau)}{C_-(\tau)} = -\frac{\mathcal{J}}{1 - x^2} - \frac{1 + \mathcal{J}^2}{\sinh^2 \tilde{\mathcal{J}}\tau (\mathcal{J} - x + \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau)} \tag{4.94}$$

where in both cases the second terms are logarithmic derivatives and the integration is elementary. Finally the solutions can be assembled as

$$\Psi_+ = \frac{1}{\tilde{\mathcal{J}}} e^{\mathcal{J}\frac{\tau - ix\sigma}{1 - x^2}} \begin{pmatrix} \tilde{\mathcal{J}} e^{-i\sigma - \mathcal{J}\tau} / \sinh \tilde{\mathcal{J}}\tau \\ i(\mathcal{J} - x - \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau) \end{pmatrix} \tag{4.95}$$

$$\Psi_- = e^{-\mathcal{J}\frac{\tau - ix\sigma}{1 - x^2}} \begin{pmatrix} \mathcal{J} - x + \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau \\ -i\tilde{\mathcal{J}} e^{i\sigma + \mathcal{J}\tau} / \sinh \tilde{\mathcal{J}}\tau \end{pmatrix} \tag{4.96}$$

These solutions do not exhibit the typical structural traits witnessed for other cases, as the differences between them are not as tangible as just choosing a branch of a square root. On the other hand, though, Ψ_- has a dynamical pole (explicit in a factorisation in which the upper component is 1), which would hint at a non-zero genus of the algebraic curve, whereas the trivial quasi-momentum suggests otherwise.

A good choice of the matrix of solutions is $\hat{\Psi} = (i\tilde{\mathcal{J}}\Psi_+; -\Psi_-)$, whose determinant equals $1 + 2\mathcal{J}x - x^2$, and subsequently $\hat{\Psi}^{-1}|_{x=0} = g$. Setting $A = (1 + 2\mathcal{J}x - x^2)\,\mathrm{diag}(1, -1)$, a polynomial Lax matrix arises

$$L = \frac{1}{\mathfrak{s}^2} \begin{pmatrix} -2(1 + \mathcal{J}^2) - (1 + 2\mathcal{J}x - x^2)\mathfrak{s}^2 & 2ie^{-i\sigma - \mathcal{J}\tau} \tilde{\mathcal{J}}(\mathfrak{c} + (\mathcal{J} - x)\mathfrak{s}) \\ 2ie^{i\sigma + \mathcal{J}\tau} \tilde{\mathcal{J}}(\mathfrak{c} - (\mathcal{J} - x)\mathfrak{s}) & 2(1 + \mathcal{J}^2) + (1 + 2\mathcal{J}x - x^2)\mathfrak{s}^2 \end{pmatrix} \tag{4.97}$$

using the same shorthands as in (4.89), and, by construction, $\Psi_\pm$ are again eigenvectors of L with respective eigenvalues $\pm(1 + 2\mathcal{J}x - x^2)$. Note that the scalar factor in A is itself a polynomial, so if one was to apply the traditional, relaxed criterion of rational L, the simplest choice of A would be just $\mathrm{diag}(1, -1)$, and the algebraic curve equation would be identical to the pathological BMN case considered earlier in this section. However, with the above choice, the equation is

$$y^2 = (1 + 2\mathcal{J}x - x^2)^2 \tag{4.98}$$

and the curve is still degenerate. Namely, the right-hand side of the equation has only double zeroes (at $x = \mathcal{J} \pm \tilde{\mathcal{J}}$) and no other features, which means that the curve has no actual branch cuts, only two points of unification. This degeneracy is yet another sign that this case is far from generic, and relates to the fact that (4.95)–(4.96) do not have the typical structure.

Reconstruction

For a degenerate algebraic curve like the one at hand (4.98), the whole formalism employed before breaks down. Nondegenerate curves have branch cuts, over which the analytical structure of functions can be meaningfully continued from one sheet of the curve to the other. In the present case, there are only two isolated points at which the sheets, and consequently the functions defined on them, are identified.

On the other hand, such curves can be understood as limits of particular nondegenerate curves with cut length converging to zero. From this point of view, the reconstruction procedure could be performed before taking the limit of the result. For the case at hand, the curve would need to have two cuts and thus be elliptic (genus-1). Both for simplicity and diversity of approach, we instead choose to perform some sort of 'hand-crafted' reconstruction on the degenerate curve, in which we will use insights drawn from the actual elliptic case. The cost of this approach is that the procedure may not be unique.

The outline of the procedure is then as follows: we will introduce two distinct vector functions, each on one of the sheets of the curve. Firstly, without calling it a 'Baker–Akhiezer function,' we will determine an exponential prefactor that will have the familiar local behaviour around $x = \pm 1$ (3.50). Secondly, just as with the elliptic case, we will introduce two poles, one at infinity and the other dynamical, but we will need to distribute them between the two vectors. Subsequently, the requirement that in $x \to \infty$ the functions are worldsheet-independent will be applied. Lastly, the functions will be required to coincide at the identification points $x = \mathcal{J} \pm \tilde{\mathcal{J}}$, as a remnant of continuity between the sheets.

Examining the polar part of $\frac{L}{1\mp x}$ (4.5), by dropping all regular terms one arrives at $-4iJ, -4i\bar{J}$ for the respective choice of sign. Therefore in (3.50) $c_\pm = \frac{i}{4}$, and

$$\Psi \propto \exp -\frac{i}{4}\left(\frac{yw}{1-x} + \frac{y\bar{w}}{1+x}\right) \tag{4.99}$$

Again, this cannot hold globally, as $y = \pm(1+2\mathcal{J}x-x^2)$ depending on the sheet, and using the full expression for y would cause an essential singularity at infinity. Instead, we use local values of y, that is $y(1) = \pm 2\mathcal{J}$ in the first term and $y(-1) = \mp 2\mathcal{J}$ in the second, thus

$$\Psi \propto \exp \mp\frac{i\mathcal{J}}{2}\left(\frac{w}{1-x} - \frac{\bar{w}}{1+x}\right) = \exp \pm\mathcal{J}\frac{\tau - ix\sigma}{1-x^2} \tag{4.100}$$

Distributing the poles is the most speculative step of this procedure. We will not put them in the same vector, as this would render the other trivial, ie. constant save for the scalar exponential factor already specified. We will then put the pole at $x = \infty$ in Ψ_+ and the dynamical one in Ψ_-, mainly because we know that we want to reproduce (4.95)–(4.96) (the reversed choice leads to results that do not solve the original auxiliary linear system). We then write

$$\Psi_+ = e^{\mathcal{J}\frac{\tau - ix\sigma}{1-x^2}} q^+(\sigma,\tau) \begin{pmatrix} 1 \\ q_1^+(\sigma,\tau)(x - q_2^+(\sigma,\tau)) \end{pmatrix} \tag{4.101}$$

$$\Psi_- = e^{-\mathcal{J}\frac{\tau - ix\sigma}{1-x^2}} q^-(\sigma,\tau)(x - q_1^-(\sigma,\tau)) \begin{pmatrix} 1 \\ \frac{q_2^-(\sigma,\tau)}{x - q_1^-(\sigma,\tau)} \end{pmatrix} \tag{4.102}$$

At $x \to \infty$, where the exponential factors vanish, the vectors should be worldsheet-independent, and we infer that the coefficients of x^1-terms are constant: $q^- = q^+ q_1^+ = 1$, leaving

$$\Psi_+ = e^{\mathcal{J}\frac{\tau - ix\sigma}{1-x^2}} \begin{pmatrix} q^+(\sigma,\tau) \\ x - q_2^+(\sigma,\tau) \end{pmatrix} \qquad \Psi_- = e^{-\mathcal{J}\frac{\tau - ix\sigma}{1-x^2}} \begin{pmatrix} x - q_1^-(\sigma,\tau) \\ q_2^-(\sigma,\tau) \end{pmatrix} \tag{4.103}$$

The final step is to require that the two vectors be equal at the identification points $x = \mathcal{J} \pm \tilde{\mathcal{J}}$

$$\Psi_+(\sigma,\tau;\mathcal{J} \pm \tilde{\mathcal{J}}) = \Psi_-(\sigma,\tau;\mathcal{J} \pm \tilde{\mathcal{J}}) \tag{4.104}$$

These two vector equations yield a system of four linear equations in the four unknown functions $q^+, q_2^+, q_{1,2}^-$ that solves to

$$q^+ = -\tilde{\mathcal{J}} e^{-i\sigma - \mathcal{J}\tau} / \sinh \tilde{\mathcal{J}}\tau \qquad q_2^+ = \mathcal{J} - \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau \tag{4.105}$$

$$q_2^- = \quad \tilde{\mathcal{J}} e^{i\sigma + \mathcal{J}\tau} / \sinh \tilde{\mathcal{J}}\tau \qquad q_1^- = \mathcal{J} + \tilde{\mathcal{J}} \coth \tilde{\mathcal{J}}\tau \tag{4.106}$$

The assembled result differs from (4.95)–(4.96) only by an overall constant factor ($-\tilde{\mathcal{J}}$ for Ψ_+, -1 for Ψ_-), and also by a factor of i in the lower components, consistently for both $\Psi_\pm$. The latter is an already familiar ambiguity of the reconstruction procedure that is also exhibited by its variant for degenerate curves, whereas the former corresponds to the global AdS_3 rotation.

Chapter 5

Application to the BFKL pomeron

In this chapter we will discuss how the strong coupling BFKL pomeron can be described in the algebraic curve framework. The motivation is to prepare the ground for a possible description of the pomeron in terms of Bethe ansatz or related (Y-system, FiNLIE) machinery, and to this end an understanding of particle content of excited states is needed. Even though there are fundamental conceptual problems, like the existence of continuous quantum numbers j, ν which do not easily reconcile with the particle interpretation, its semi-classical limit can be obtained as the algebraic curve.

We identify a suitable string dual, such that its non-zero conserved charges match (2.28). For simplicity, we consider a highest-weight state, so that $m = \bar{m} = 0$, also with vanishing conformal spin $n = 0$. We achieve this by modifying a well-known solution, namely the folded string. As a by-product (or consistency check), we derive an expansion of $j(\nu)$ that matches the semi-classical limit of (3.73), but can be continued to arbitrary order.

Subsequently, the folded string algebraic curve is modified accordingly, and the integral constraints and asymptotics of the quasi-momentum are discussed. We identify the cuts on the curve due to reality conditions and argue that the relevant limit of Bethe ansatz equations holds.

5.1 The dual string configuration

A good starting point of the construction is the well-known solution with only two non-vanishing conserved charges, the GKP folded string [GKP02]. It appears in the global AdS_3 coordinates

$$\begin{aligned} Y_0 &= \cosh\rho \sin t & Y_1 &= \sinh\rho\cos\psi & Y_3 &= Y_4 = 0 \\ Y_5 &= \cosh\rho\cos t & Y_2 &= \sinh\rho\sin\psi & & \end{aligned} \tag{5.1}$$

by imposing the following simple dependence of the time and angular coordinate

$$t = \kappa\tau \qquad\qquad \psi = \chi\tau \tag{5.2}$$

and demanding that the radial coordinate do not depend on τ and be periodic, $\rho \equiv \rho(\sigma) = \rho(\sigma + 2\pi)$. Here, we will derive its useful parameterisation given in [Tsey10, §4.1]. Of the two Virasoro constraints one is trivial, while the other reads

$$(\rho')^2 = \kappa \cosh^2\rho - \chi^2 \sinh^2\rho \tag{5.3}$$

The variables ρ, σ are already separate, and an integration by substituting $\upsilon = i\rho$ yields an explicit elliptic integral, so we obtain a relation

$$\kappa\sigma = iF(i\rho|1 - \tfrac{\chi^2}{\kappa^2}) \tag{5.4}$$

which can be inverted by the Jacobi amplitude function. Now, as we are chiefly interested in computing the charges (2.18), we will transform the following

$$\begin{aligned} \sinh\rho &= -i\underbrace{\sin\mathrm{am}}_{\mathrm{sn}}(-i\kappa\sigma|1 - \tfrac{\chi^2}{\kappa^2}) \\ &= -k\,\mathrm{sd}(\chi\sigma|k^2) = \frac{k}{\sqrt{1-k^2}}\,\mathrm{cn}(\chi\sigma + K(k^2)|k^2) \end{aligned} \tag{5.5}$$

where we have used (A.19), (A.20), (A.18), and $k = \frac{\kappa}{\chi}$. The function is odd in σ, as manifestly seen from its penultimate form. Subsequently, we impose the requirement of periodic ρ, and as the period of cn is $4K(k^2)$, we finally fix the relation between the two parameters

$$\chi = \frac{2K(k^2)}{\pi} \qquad\qquad \kappa = \frac{2kK(k^2)}{\pi} \tag{5.6}$$

Finally, let us comment that the equations of motion do not restrict the form of the solution any further. They turn out to read

$$0 = \kappa^2\cosh\rho + (\rho')^2\cosh\rho + \rho''\sinh\rho \tag{5.7}$$

$$0 = \chi^2\sinh\rho + (\rho')^2\sinh\rho + \rho''\cosh\rho \tag{5.8}$$

It can be separated into two independent equations, one for $(\rho')^2$, which reproduces (5.3), and one for ρ'', which reads

$$\rho'' = (\kappa^2 - \chi^2)\sinh\rho\cosh\rho \tag{5.9}$$

and is a consequence of (5.3) by differentiating its sides with respect to σ.

In calculation of the charges (2.18), the dependence on τ factorises out. In most of the cases, the integrand is then proportional to $\sinh\rho\cosh\rho$, which is odd, and the integral will vanish, as the limits of integration can be shifted to $-\pi, \pi$ due to periodicity. The remaining non-zero charges are then

$$S_{12} = \sqrt{\lambda}\,\frac{K(k^2)}{\pi^2}\int_0^{2\pi}\sinh^2\rho\,d\sigma \qquad S_{50} = \sqrt{\lambda}\,\frac{kK(k^2)}{\pi^2}\int_0^{2\pi}\cosh^2\rho\,d\sigma \tag{5.10}$$

By (5.5), (A.23)

$$\int_0^{2\pi}\sinh^2\rho\,d\sigma = \frac{k^2}{1-k^2}\frac{\pi}{2K(k^2)}\left(-\frac{1-k^2}{k^2}\cdot 4K(k^2) + \frac{1}{k^2}\cdot 4E(k^2)\right) \tag{5.11}$$

where we have used quasi-periodicity (A.6), (A.21) of $E(\operatorname{am} v|k^2)$. Finally,

$$S_{12} = \sqrt{\lambda}\,\frac{2}{\pi}\left(\frac{E(k^2)}{1-k^2} - K(k^2)\right) \qquad S_{50} = \sqrt{\lambda}\,\frac{2}{\pi}\frac{kE(k^2)}{1-k^2} \tag{5.12}$$

To conclude this introduction, let us comment on the physical meaning of the parameter k. Its range is the real interval $(0,1)$, which interpolates between the limits of very short string, when $k \to 0$, and infinitely long string, when $k \to 1$. Both spin and energy tend to respectively 0 and infinity in these limits.

Our aim now is to construct a solution whose only non-vanishing charges will be S_{54}, S_{03}. This can be achieved by a simple relabelling of the embedding coordinates. What needs to be taken into account, though, is that the coordinates still need to obey the embedding constraint (2.9). This is achieved by complexifying some of the coordinates, and the result is

$$Y_4 = i\cosh\rho\sin t \qquad Y_0 = i\sinh\rho\sin\psi \qquad Y_1 = Y_2 = 0 \tag{5.13}$$

$$Y_5 = \cosh\rho\cos t \qquad Y_3 = \sinh\rho\cos\psi \tag{5.14}$$

Most notably, the Virasoro constraints take the same form as before, leading to the same solution and analysis of ρ, and thus the same relation of κ, χ inside t, ψ to k. We dub this solution the BFKL folded string.

By the same argument, there again are only two non-zero charges, this time

$$S_{54} = -i\sqrt{\lambda}\,\frac{kK(k^2)}{\pi^2}\int_0^{2\pi}\cosh^2\rho\,d\sigma = -i\sqrt{\lambda}\,\frac{2}{\pi}\frac{kE(k^2)}{1-k^2} \tag{5.15}$$

$$S_{03} = i\sqrt{\lambda}\,\frac{K(k^2)}{\pi^2}\int_0^{2\pi}\sinh^2\rho\,d\sigma = i\sqrt{\lambda}\,\frac{2}{\pi}\left(\frac{E(k^2)}{1-k^2} - K(k^2)\right) \tag{5.16}$$

Using (2.28), (2.29), these are related to the relevant parameters as

$$S_{54} = D = 2\nu \qquad S_{03} = M_{03} = ij \tag{5.17}$$

where the integer terms in D have been dropped, as they will become negligible in the strong coupling limit, where ν and the right-hand side of (5.15) scale with $\sqrt{\lambda}$.

Now, a series expansion of (5.15) can be inverted to yield k in terms of $\nu/\sqrt{\lambda}$, and after plugging into (5.16) we get

$$\frac{j}{\sqrt{\lambda}} = -2\left(\frac{\nu}{\sqrt{\lambda}}\right)^2 - 3\left(\frac{\nu}{\sqrt{\lambda}}\right)^4 - \frac{21}{2}\left(\frac{\nu}{\sqrt{\lambda}}\right)^6 - \frac{391}{8}\left(\frac{\nu}{\sqrt{\lambda}}\right)^8 + \cdots \tag{5.18}$$

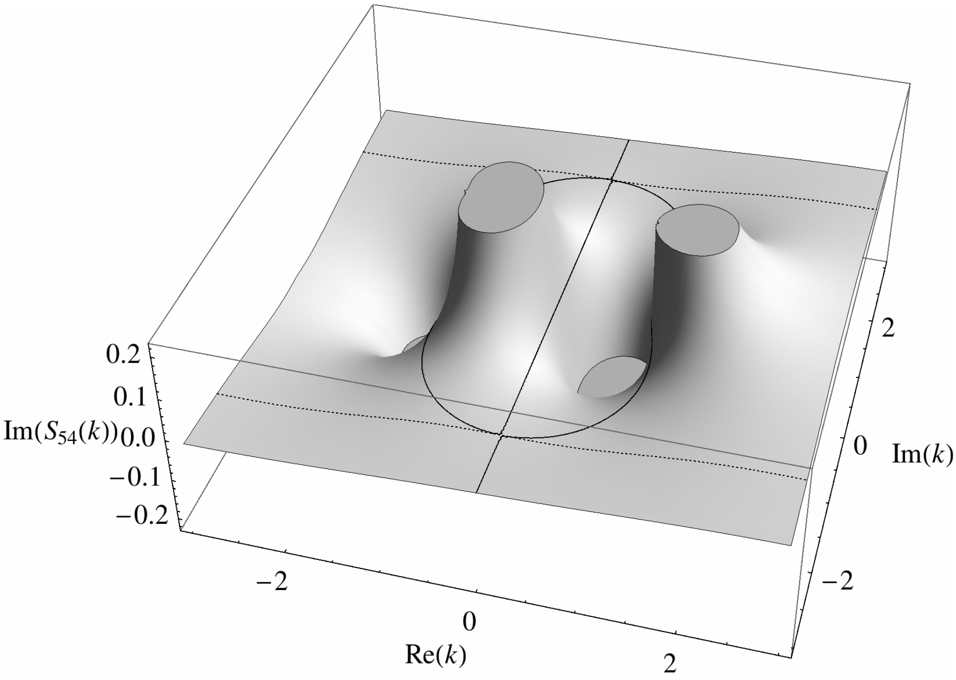

Figure 5.1: Plot of $\operatorname{Im} S_{54}$ (5.15). Mesh lines are at $z = 0$, therefore marking the values of argument k for which S_{54} is real. Dotted mesh lines are at $\operatorname{Im} k = \pm 2.18$.

This expansion could be continued to an *a priori* arbitrary order, but we have notably achieved full agreement with (3.73), which, after dividing sidewise by $\sqrt{\lambda}$ and keeping only ν^{2n}/λ^n terms on the right-hand side, gives first three terms of (5.18). This confirms that we have identified a classical string solution that explicitly realises the classical limit of the strong coupling analog of the BFKL pomeron intercept.

Finally, we can make note on how the similarity relation (2.24) is apparent here, realised by $SO(4,2)$ rotations by $\frac{\pi}{2}$ in the planes 13, 02, and 04 of the embedding space. By using explicit rotation generators that are zero in all components except for

$$(M_{\mu 0})_{\mu 0} = (M_{\mu 0})_{0\mu} = i \qquad\qquad (M_{\mu\nu})_{\mu\nu} = -(M_{\mu\nu})_{\nu\mu} = 1 \tag{5.19}$$

the similarity matrix is

$$U = e^{\frac{\pi}{2}M_{42}} e^{\frac{\pi}{2}M_{20}} e^{\frac{\pi}{2}M_{31}} = \exp\frac{\pi}{2}\left(\frac{4}{3\sqrt{3}}(M_{42} + M_{20} + M_{40}) + M_{31}\right) \tag{5.20}$$

and it indeed maps a vector $(Y_0, Y_1, Y_2, 0, 0, Y_5)$ to $(iY_2, 0, 0, Y_1, iY_0, Y_5)$, therefore transforming the GKP folded string into its BFKL counterpart.

What does not follow through from the original case is the restriction of k to the real axis. The proper locus of k on the complex plane is determined by the reality conditions that the $\mathfrak{sl}(2,\mathbb{C})$ representation theory dictates for the charges. The constraints

$$S_{54} = 2\nu \in \mathbb{R} \qquad\qquad S_{03} = ij \in i\mathbb{R} \tag{5.21}$$

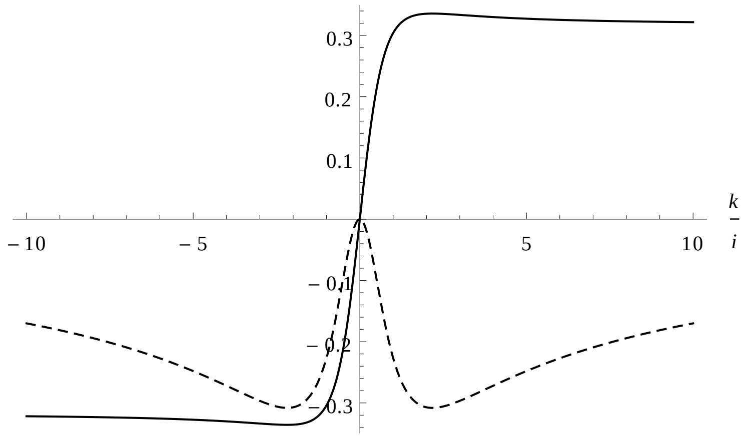

Figure 5.2: Values of $\nu/\sqrt{\lambda}$ (solid), $j/\sqrt{\lambda}$ (dashed) for imaginary k.

are satisfied on the imaginary axis, $k \in i\mathbb{R}$, what becomes evident from the expression for the charges once noticing that in such case $E(k^2), K(k^2) \in \mathbb{R}$. Further investigation, albeit largely numerical, indicates that this may only be a partial solution. As seen in fig. 5.1, there are curved paths in the complex k plane branching out of the imaginary axis and leading to $k = 1$, along which S_{54} is still real. The value of k at which the paths appear can be numerically found as a root of $\operatorname{Im} S_{54}$ just off the imaginary axis, and equals approximately

$$k_* \approx \pm 2.18i \tag{5.22}$$

This corresponds to the supplementary mesh lines of fig. 5.1. Similar behaviour is apparent in $\operatorname{Re} S_{03}$, although the essentially complex paths that appear at the same k_* have a slightly different shape.

The behaviour of ν, j for imaginary k, cf. fig. 5.2, can be described as follows. For positive $\operatorname{Im} k$, ν rises from 0 at $k = 0$ to some maximum ν_*, and then falls off to a finite limit of $\sqrt{\lambda}/\pi$ at infinity. j falls off from 0 to some minimum j_* and rises to 0 at infinity. As for negative $\operatorname{Im} k$, ν is odd and j is even. Now, to determine the values of argument that correspond to the extremal values ν_*, j_*, we can calculate the k-derivatives of the respective functions and using (A.7) obtain

$$\frac{dj}{dk} = \frac{ik}{2}\frac{d\nu}{dk} = \sqrt{\lambda}\,\frac{2}{\pi}\frac{k}{(1-k^2)^2}(2E(k^2) - (1-k^2)K(k^2)) \tag{5.23}$$

The value of k at which this expression vanishes is not known analytically, but numerically it turns out to be k_*.

From the point of view of representation theory ν is not expected to be bounded, and it would be ideal to investigate the behaviour along the curved paths of essentially complex k. Graphing the functions suggests that ν can reach arbitrarily large values there, positive in the upper half-plane, negative in the lower. However, j acquires an imaginary part. Another obstacle, of purely technical nature, is that the

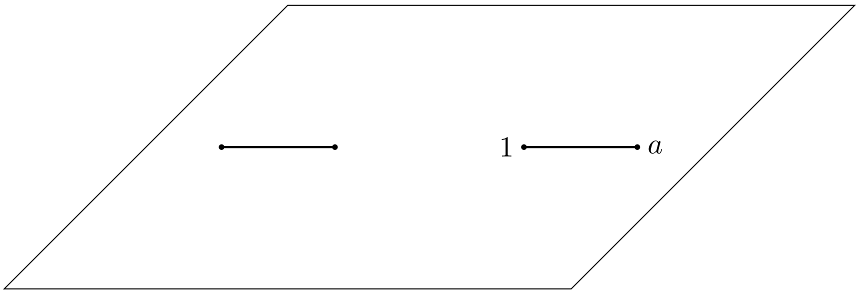

Figure 5.3: Position of cuts of the algebraic curve for the GKP folded string. a is real and larger than 1.

location of these paths is not analytically known, not even the exact value of the initial point k_*.

In the following discussion, k of the BFKL string will be then purely imaginary, and implicitly also $0 \leq \operatorname{Im} k \leq \operatorname{Im} k_*$.

5.2 Algebraic curve analysis

We have just learned that a string solution with charges relevant to BFKL is intimately related to the GKP string. Anticipating an equally close relation between their algebraic curves, we start by continuing over from section 4.4, where the curve for the GKP string has been identified as (4.67). The cuts between the sheets extend between 1 and real a, as well as between $-1, -a$, as seen in fig. 5.3. Accordingly, the quasi-momentum differential reads

$$dp = \frac{Ax^2 + B}{a(1-x^2)\sqrt{1-x^2}\sqrt{1-\frac{x^2}{a^2}}}dx \tag{5.24}$$

where the structure of the square root has been altered in anticipation of the following derivation. The possible sign change and ambiguity is absorbed in the unknown constants A, B. Their values are constrained by (3.58), (3.59). In the present case of genus 1, the two cuts map to one another when reflected over the origin, and the differential is even. This implies that the values of the two A-cycles will be mutually opposite, as also evidenced by the possibility to continuously deform one to the other, only with a different directionality. The same will hold for both Γ-contours, and their sum is equal to the only B-cycle, which will subsequently vanish identically. Therefore the only actually non-trivial equations are the proper values of the A-cycle (ie. vanishing) and the Γ-contour. These are enough to establish A, B.

To this end, we decompose dp as follows

$$\frac{dp}{dx} = \frac{A+B}{a}\left(\frac{x^2}{(1-x^2)\sqrt{1-x^2}\sqrt{1-\frac{x^2}{a^2}}} + \frac{a^2}{a^2-1}\frac{\sqrt{1-\frac{x^2}{a^2}}}{\sqrt{1-x^2}} - \frac{a^2}{a^2-1}\frac{\sqrt{1-\frac{x^2}{a^2}}}{\sqrt{1-x^2}}\right) + \frac{B}{a}\frac{1}{\sqrt{1-x^2}\sqrt{1-\frac{x^2}{a^2}}} \tag{5.25}$$

and integrate. Here, the elliptic integrals appear straight from their definitions, while the remainder gives a quite compact expression

$$p(x) = \frac{a(A+B)}{a^2-1}\left(\frac{x\sqrt{1-\frac{x^2}{a^2}}}{\sqrt{1-x^2}} - E(\arcsin x|a^{-2})\right) + \frac{B}{a}F(\arcsin x|a^{-2}) \tag{5.26}$$

The A-cycle can be deformed to follow the imaginary axis and close off by a semicircle at infinity. As the integrand dp vanishes for the latter part, the value of the A-cycle will be equal to $2p(i\infty)$ (due to p being odd). Instead of directly taking the limit of (5.26), we use an equivalent expression obtained by symbolic definite integral of dp, that is

$$\oint_{\mathrm{A}} dp = \frac{2i}{a^2-1}\big((A+B)E(1-a^2) - (a^2A+B)K(1-a^2)\big) = 0 \tag{5.27}$$

We require it to vanish, which allows us to express one of the constants in terms of the other.

The Γ-contour is in turn equal to the difference of p evaluated at the opposite sheets of the curve at one point of the cut. This is equal to double the value at that point, as the integrand changes sign when passing between the two sheets, and in our case, it is comparatively simple to evaluate at the branch point $x = a$, where the first term of (5.26) simply vanishes. Immediately using (A.10), we get

$$\int_\Gamma dp = 2p(a) = -2\left(\frac{A+B}{a^2-1}E(a^2) + AK(a^2)\right) = -2A\frac{KK' - EK' - KE'}{K' - E'} \tag{5.28}$$

where the last step follows from substituting B obtained from (5.27), and the shorthands K, K', E, E' denote respectively $K(a^2), K(1-a^2), E(a^2), E(1-a^2)$. In the numerator, one side of the Legendre relation (A.8) can be spotted, so the condition on Γ-contours reads

$$\frac{A\pi}{K(1-a^2) - E(1-a^2)} = 2\pi n \tag{5.29}$$

The constants in the quasi-momentum are thus

$$A = 2n(K(1-a^2) - E(1-a^2)) \qquad B = 2n(E(1-a^2) - a^2K(1-a^2)) \tag{5.30}$$

These values play an important role, as they appear in the identities that relate the asymptotic behaviour of p to the charges. Here we obtain

$$p(x) = xp'(0) + \cdots = \frac{B}{a}x + O(x^2) \qquad (x \to 0) \tag{5.31}$$

$$p(x) = \frac{1}{x}\lim_{v\to 0}\left(p(\tfrac{1}{v})\right)' + \cdots = -\frac{A}{x} + O(x^{-2}) \qquad (x \to \infty) \tag{5.32}$$

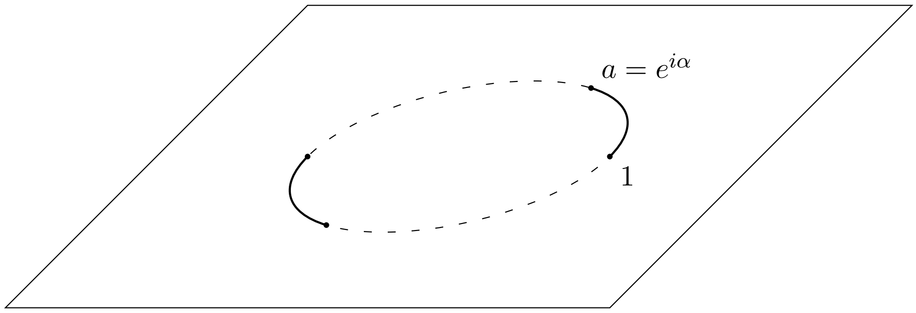

Figure 5.4: Position of cuts for the BFKL folded string. a lies on the unit circle.

In both expressions there is an ambiguity of overall sign, which is resolved by the requirement that p is continuous outside of the cuts. Above it has been already taken into account, by examining the behaviour of $\arg dp$ along some line between 0 and infinity, for instance just next to the real axis, therefore parallel to the cuts. By comparison with (3.32), (3.33), we obtain

$$E = \frac{n\sqrt{\lambda}}{2\pi}\frac{a-1}{a}(E' + aK') = \frac{n\sqrt{\lambda}}{2\pi}\frac{a-1}{a}(aE(1-a^{-2}) + K(1-a^{-2})) \tag{5.33}$$

$$S = \frac{n\sqrt{\lambda}}{2\pi}\frac{a+1}{a}(E' - aK') = \frac{n\sqrt{\lambda}}{2\pi}\frac{a+1}{a}(aE(1-a^{-2}) - K(1-a^{-2})) \tag{5.34}$$

using the same shorthands as (5.28), with the second step following from (A.9). These expressions manifestly reproduce a special case of the results of [GSSV11, (2.4), (2.6)] in which one of the branch points is set to 1.

Now we are able to connect the algebraic curve description to the string description, by comparing the charges in the last forms of (5.33), (5.34) to (5.12). They in fact do coincide by (A.11) for $\tilde{k} = \frac{1}{a}$ (and $n = 1$), therefore

$$a = \frac{1+k}{1-k} \tag{5.35}$$

For $k \in (0, 1)$, a is indeed a real number larger than 1.

Now, the whole discussion, and most importantly the association of the algebraic curve (4.67) to the folded solution will be also valid for the BFKL string. The physical values of a will be necessarily different, but the relation (5.35) still holds. In the small $\nu < \nu_*$ region, where unambiguously k is purely imaginary, $|a| = 1$. We express them in terms of a new, real parameter α

$$k = i \tan\frac{\alpha}{2} \qquad\qquad a = e^{i\alpha} \tag{5.36}$$

The position of cuts will have to be determined anew, according to the reality conditions of the particle density, which this time are imposed by (5.21). Instead of searching for the most general solution, we will venture the most natural guess of the cuts lying along the unit circle, and verify it. The situation is depicted in fig. 5.4.

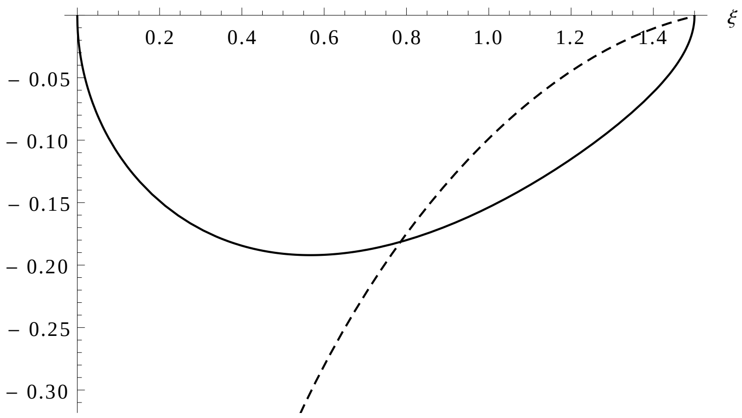

Figure 5.5: The integrands of (5.40), (5.41) (multiplied by i), i.e. $\rho(e^{i\xi})\frac{du}{d\xi}/\sqrt{\lambda}$ (solid), $i\varepsilon\rho(e^{i\xi})\frac{du}{d\xi}/\sqrt{\lambda}$ (dashed), for $\alpha = \frac{3}{2}$.

The analysis is done entirely numerically. A crucial point lies just at the beginning, namely one needs to guarantee that dp is continuous outside of the cuts, and the default choices of square root branches made by the software do not coincide with them in general. Therefore the expression (5.24) needs to be multiplied by a step function that will change its sign over certain regions, and in our investigation these were determined to be

$$x : |x| < 1, 0 < \arg(x^2 - 1) < \arg(a^2 - 1) \tag{5.37}$$

The required values of A-cycle and Γ-contour have been confirmed, and the discontinuity of p turned out to be purely imaginary. Parameterising the cut as $x = e^{i\xi}$, the infinitesimal element $du = (1 - \frac{1}{x^2})dx = (1 - \frac{1}{x^2})ix\, d\xi$ turns out to be real, while the dispersion relation (3.71) is purely imaginary on the unit circle.

To complete the particle description, the charges need to be expressed as integrals over particle density. We reverse-engineer the asymptotics of quasi-momentum by replacing in (3.32), (3.33) the quantities that correspond by comparing (5.12) to (5.16), (5.15). We obtain

$$p(x) = \frac{2\pi(j - 2i\nu)}{\sqrt{\lambda}} x + O(x^2) \qquad (x \to 0) \tag{5.38}$$

$$p(x) = \frac{2\pi(j + 2i\nu)}{\sqrt{\lambda}\, x} + O(x^{-2}) \qquad (x \to \infty) \tag{5.39}$$

and for the density integrals

$$\frac{i\sqrt{\lambda}}{8\pi^2} \int \operatorname{disc} p\, du = -\frac{\sqrt{\lambda}}{2\pi^2} \int_0^\alpha \operatorname{Im} p(e^{i\xi})(1 - e^{-2i\xi}) i e^{i\xi} d\xi = j \tag{5.40}$$

$$\frac{i\sqrt{\lambda}}{8\pi^2} \int \varepsilon \operatorname{disc} p\, du = -\frac{\sqrt{\lambda}}{2\pi^2} \int_0^\alpha \operatorname{Im} p(e^{i\xi})(1 + e^{-2i\xi}) i e^{i\xi} d\xi = 2i\nu \tag{5.41}$$

The conditions (5.21) are satisfied due to the integrands being purely real and imaginary in the respective cases, while the integrals indeed evaluate to their proper values obtained from (5.16), (5.15). The integrands are plotted in fig. 5.5. The continuum Bethe equation (3.65) is also discovered to hold for $n = -1$.

There also appears a subtlety related to the interpretation of (3.56), which in the case at hand means that the density is normalised to a negative quantity, ie. negative definite. It is currently unknown if this should be resolved by some kind of continuum Baxter equation interpretation, or just by a redefinition of the quantum numbers of elementary particles. The latter would be justified by the fact that the solution does not in fact belong to the standard $\mathfrak{sl}(2)$ sector, but instead to its reshuffled and complexified variant.

Chapter 6

Conclusions and outlook

The most concise summary of the results presented in this thesis is as follows: firstly, we have proposed and tested a method with which the algebraic curve classification of AdS/CFT can be broadened from spinning strings to Wilson loops minimal surfaces, and possibly also two-point correlation function duals. Secondly, an algebraic curve has been proposed for the processes involving the BFKL pomeron, that serves as a semi-classical limit of the relevant Bethe ansatz equations.

More precisely, for the Wilson loops, where the conventional construction is impossible as all monodromies are trivial, we have defined a Lax operator from a solution of an auxiliary linear problem, fixing any ambiguity by some general constraints on the analytical structure of the operator. Subsequently, we have considered two examples, for which we have proven that the algebraic curve arising from such Lax operator properly describes the original solution, ie. it can be reconstructed, up to the usual free coefficients, purely from the analytic properties of functions defined on algebraic curves.

For the correlation functions, we have highlighted a puzzling feature, namely the lack of one-to-one correspondence between the quasi-momenta and the algebraic curves. We have considered two solutions of the same quasi-momentum, to which our procedure assigned, and again meaningfully, different algebraic curves. This is in contrast to the monodromy-based approach in which the two solutions would be not distinguished.

Finally, we have found a stringy dual to the BFKL pomeron by simply demanding that the relevant conserved charges were non-zero. The algebraic curve equation for this solution is already known, and we have determined the position of branch cuts from the reality conditions.

There is a number of possible directions of further research and open questions left by the results presented. For completeness, we note that one of them would be an improvement on the analytic structure of the quark–antiquark reconstruction, to show that the result agrees with (4.42) not only numerically. Also, the unresolved ambiguity of the coefficient C_3 of the reconstruction requires a deeper insight, as such feature is bound to reappear in the reconstruction from any curve with a branch point at infinity. Possible ideas include introducing some birational transformation of the curve, or perhaps a modification of the original [BBT03] analysis, which from the beginning relies on an assumption that the algebraic curve has n distinct points above infinity.

An obvious continuation is to consider different solutions in context of the new algebraic curve definition, and some work in this direction has already been done [Ryan12, Cagn13]. However, it would be most interesting to see a solution to which corresponds a non-degenerate curve of genus 2 (or larger). Also, for the stringy solutions of [Ryan12], a comparison with the traditionally-derived curves would be in order.

In general, the methodology could be also applied to other sectors of the full string theory (or even the full string theory itself). Especially the larger ones would be interesting, but several obstacles can appear. One is visible already at this point, namely the larger rank of the Lax operator will correspond with the much more general form of the $x = \pm 1$ asymptotics of the Baker–Akhiezer prefactor than (3.50).

A major development for Wilson loops would be to somehow extend the notion of monodromy to Wilson loop surfaces or open string worldsheets. This could be done by introducing some new object that would enter the ordered exponential whenever the defining contour would touch the boundary of the worldsheet. Reflection matrices like this were already considered in [MV06], although our very limited attempts at providing this new definition along these lines failed.

Let us note at this point an interesting paper [Deke13] in which the author proposes yet another scheme of assigning the algebraic curves to a broad class class of solutions, which turns out to encompass all cases considered here and in [Ryan12]. As this scheme does not require introducing any functions as in (4.3), it can be considered to be less arbitrary than ours (although we do not agree that ours is arbitrary). There are also some issues with the new scheme, like an appearance of two related but different algebraic curves for one of the solutions considered, and the discussion is not complete yet.

As for the configurations dual to the correlation functions, we could boldly extrapolate from the evidence we have got and state that a given quasi-momentum can correspond to a number of algebraic curves that differ from one another by the presence of degeneration points. This, of course, requires a much deeper insight, and more relevant results for comparison could be gathered by considering two-point functions of the operator $\operatorname{tr} Z^J$ with more complicated objects.

In general, analyticity arguments on degenerate curves would require a detailed treatment. Using for instance the argument by which a curve with degeneration points is a limit of a curve with very short cuts, some more or less general properties of eigenvectors defined on such curves could be derived. This could lead to developing a method less haphazard than the reconstruction of section 4.5.

Another, even more puzzling question arises in context of correlation functions beyond two points. By picking contours of different homotopy classes (ie. not equivalent by continuous deformation), which is possible for higher-point functions (at least three classes for 3-point, at least seven, including the ‘pants decomposition’ contours, for 4-point, etc.), there will naively appear a number of different quasi-momenta and corresponding algebraic curves, possibly of different genera. How the algebraic curve description would work for such surfaces is well beyond even the scope of speculation at this point.

In the study of the BFKL pomeron algebraic curve, one possible continuation would be to examine the behaviour at larger values of imaginary parameter k, that is, beyond k_*. Our results indicate that to obtain arbitrary large values of the intercept, k is required to belong to an essentially complex contour, whose shape is not known analytically. It would be beneficial to investigate this region and the behaviour of the solution in it, as well as to decide if the region of large, purely imaginary parameter is physical and why.

Another proposal would be to look for a stringy dual with non-vanishing conformal spin n and discuss the relevant algebraic curve. Finally, our results are barely an invitation to apply in this case the powerful integrability machinery that relies on the particle description, along the lines of the Y-system.

Appendix A

Elliptic functions

This appendix essentially sums up the useful formulae of [HMF10].

There is more than one convention of notation for the elliptic integrals and theta functions. We consequently use exactly one of them. For practical reasons, it always is the one used by the respective functions in Mathematica. Consequently, the complete elliptic integrals also use the Mathematica convention instead of the traditional one, namely

$$K_{\text{Mathematica}}(k^2) \equiv K_{\text{textbook}}(k) \tag{A.1}$$

and the same for E (we do not use the complete integral of the third kind). The traditional notation is also used in [HMF10], so the formulae quoted here are recast to match the rest of the text.

Second arguments of the Jacobi and theta functions (that is, the modulus or parameter for Jacobi functions, and nome or lattice parameter for theta functions) is often suppressed, unless there are explicit transformations in it. Its present value should be clear from context. Most notably, the whole subsection 4.3 uses $k = \frac{1}{\sqrt{2}}$ for Jacobi functions and $\tau = i$ (or $q = e^{-\pi}$) for theta functions. On the other hand, the modulus or parameter of incomplete elliptic integrals is never suppressed, to avoid confusion with respective complete integrals.

A.1 Elliptic integrals

Elliptic integrals were introduced to deal with the fact that the arc length of an ellipse cannot be expressed by elementary functions. The incomplete elliptic integrals, respectively of the first, second, and third kind, are defined as follows [HMF10,

§19.2(ii)]

$$F(\phi|k^2) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{A.2}$$

$$E(\phi|k^2) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{A.3}$$

$$\Pi(n;\phi|k^2) = \int_0^\phi \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{A.4}$$

Obviously, all three vanish at $\phi = 0$, while at $\phi = \frac{\pi}{2}$ they give the complete elliptic integrals, respectively

$$K(k^2) = F(\tfrac{\pi}{2}|k^2) \qquad E(k^2) = E(\tfrac{\pi}{2}|k^2) \qquad \Pi(n|k^2) = \Pi(n;\tfrac{\pi}{2}|k^2) \tag{A.5}$$

They are quasi-periodic in the first argument, for instance [HMF10, (19.2.10)]

$$E(n\pi \pm \phi|k^2) = 2nE(k^2) \pm E(\phi|k^2) \tag{A.6}$$

Derivatives of the complete integrals read [HMF10, §19.4(i)]

$$\frac{dK(k^2)}{dk} = \frac{E(k^2)-(1-k^2)K(k^2)}{k(1-k^2)} \qquad \frac{dE(k^2)}{dk} = \frac{E(k^2)-K(k^2)}{k} \tag{A.7}$$

There are several transformations relating the different kinds of elliptic integrals. For complete integrals, we mention the Legendre's relation [HMF10, (19.7.1)]

$$E(k^2)K(1-k^2) + K(k^2)E(1-k^2) - K(k^2)K(1-k^2) = \tfrac{\pi}{2} \tag{A.8}$$

and [HMF10, (19.7.2)]

$$K(1-k^{-2}) = kK(1-k^2) \qquad\qquad E(1-k^{-2}) = \tfrac{1}{k}E(1-k^2) \tag{A.9}$$

We also use the reciprocal modulus transformation of incomplete integrals [HMF10, (19.7.4)] that with $\sin\psi = \frac{1}{k}\sin\phi \leq 1$ reads

$$F(\phi|k^{-2}) = kF(\psi|k^2) \qquad E(\phi|k^{-2}) = \tfrac{1}{k}(E(\psi|k^2) - (1-k^2)F(\psi|k^2)) \tag{A.10}$$

Note that we use it for a case where $\sin\phi = a = k$, thus $\psi = \frac{\pi}{2}$ and the right-hand side integrals simplify to their complete counterparts.

Reversing in [HMF10, §18.8(ii)] the role of k, k', renaming k to $\tilde{k}$ and then k_1 to k, we obtain the following form of the descending Landen transformation: for $k = \frac{1-\tilde{k}}{1+\tilde{k}}$,

$$K(1-\tilde{k}^2) = (1+k)K(k^2) \qquad E(1-\tilde{k}^2) = \frac{2}{1+k}E(k^2) + (k-1)K(k^2) \tag{A.11}$$

A.2 Jacobi elliptic functions

This is a set of twelve special functions that can be defined by their relation to the theta-functions (see below), for instance [HMF10, (22.2.5)]

$$\operatorname{cn}\sigma = \frac{\theta_4(0,q)}{\theta_2(0,q)}\frac{\theta_2(\upsilon,q)}{\theta_4(\upsilon,q)} \tag{A.12}$$

where $\upsilon = \frac{\pi\sigma}{2K(k^2)}$ and $q = \exp(-\pi K(1-k^2)/K(k^2))$. We do not rely much on these definitions, and instead several relations are crucial for our calculations, even if they are not mentioned explicitly. For instance [HMF10, (22.6.1)]

$$\operatorname{sn}^2\sigma + \operatorname{cn}^2\sigma = k^2\operatorname{sn}^2\sigma + \operatorname{dn}^2\sigma = 1 \tag{A.13}$$

and [HMF10, §22.3(i)]

$$\operatorname{cn}'\sigma = -\operatorname{sn}\sigma\operatorname{dn}\sigma \qquad \operatorname{sn}'\sigma = \operatorname{cn}\sigma\operatorname{dn}\sigma \qquad \operatorname{dn}'\sigma = -k^2\operatorname{sn}\sigma\operatorname{cn}\sigma \tag{A.14}$$

The remaining (rarely used) functions are defined by the following mnemonic [HMF10, (22.2.10)]

$$\mathrm{pq} = \frac{\mathrm{pr}}{\mathrm{qr}} = \frac{1}{\mathrm{qp}} \tag{A.15}$$

where p, q, r are any of the letters c, s, d, n, and any function named with repeated letters is constantly equal to 1. Finally, the Jacobi amplitude function is defined as

$$\operatorname{am}\sigma = \arcsin\operatorname{sn}\sigma \tag{A.16}$$

and turns out to be the inverse function of the incomplete elliptic integral of the first kind [HMF10, (22.16.10–11)]

$$\phi = \operatorname{am}\sigma \qquad \Leftrightarrow \qquad \sigma = F(\phi|k^2) \tag{A.17}$$

The functions transform to one another under half-period shifts [HMF10, §22.4.(i)]

$$\operatorname{cn}(\sigma + K(k^2)) = -\sqrt{1-k^2}\operatorname{sd}\sigma \tag{A.18}$$

imaginary rotation in the argument [HMF10, §22.6(iv)]

$$\operatorname{cn}(i\sigma|k^2) = \frac{1}{\operatorname{cn}(\sigma|1-k^2)} \qquad \operatorname{sc}(i\sigma|k^2) = i\operatorname{sn}(\sigma|1-k^2) \tag{A.19}$$

and inverse modulus transformation [HMF10, §22.17(i)]

$$\operatorname{sc}(\sigma|k^{-2}) = k\operatorname{sd}(\tfrac{\sigma}{k}|k^2) \tag{A.20}$$

The amplitude is quasi-periodic [HMF10, (22.16.2)]

$$\operatorname{am}(\sigma + 2K(k^2)) = \pi + \operatorname{am}\sigma \tag{A.21}$$

From [HMF10, (22.16.3)] we see that

$$\operatorname{am} 0 = 0 \qquad\qquad \operatorname{am}' \sigma = \operatorname{dn} \sigma \tag{A.22}$$

and we can integrate

$$\begin{aligned}\int \operatorname{cn}^2 \sigma \, d\sigma &= -\frac{1-k^2}{k^2}\sigma + \frac{1}{k^2}\int \operatorname{dn}^2 \sigma \, d\sigma = -\frac{1-k^2}{k^2}\sigma + \frac{1}{k^2}\int \sqrt{1-k^2\sin^2 \upsilon}\, d\upsilon \\ &= -\frac{1-k^2}{k^2}\sigma + \frac{1}{k^2}E(\operatorname{am}\sigma|k^2)\end{aligned} \tag{A.23}$$

by substitution $\upsilon = \operatorname{am}\sigma$.

A.3 Theta functions

This is a set of four functions defined by their Fourier expansions. We will mostly use one of them [HMF10, (20.2.3)]

$$\theta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz \tag{A.24}$$

where $q = e^{i\pi\tau}$ is called the nome, and τ ($\operatorname{Im}\tau > 0$) the quasi-period. The points $(0, \pi, \pi\tau, \pi + \pi\tau)$ define the lattice or the fundamental parallelogram on the domain. The theta functions are quasi-periodic on the lattice [HMF10, (20.2.8)], ie.

$$\theta_3(z + (m + n\tau)\pi, q) = q^{-n^2} e^{-2inz}\theta_3(z,q) \tag{A.25}$$

for $m, n \in \mathbb{Z}$. Moreover, θ_3 is odd.

The theta functions transform to one another under half-period shifts [HMF10, §20.2(iii)]

$$\theta_3(z + \tfrac{\pi}{2}) = \theta_4(z) \qquad\qquad \theta_3(z + \tfrac{\pi\tau}{2}) = \theta_2(z)e^{-iz-i\pi\tau/4} \tag{A.26}$$

There also holds a very curious equality at specific values of both arguments, namely for $\tau = i = \tau', z = 0$ [HMF10, (20.7.31)]

$$\theta_2(0, e^{-\pi}) = \theta_4(0, e^{-\pi}) \tag{A.27}$$

In the main text we predominantly use a theta function with rescaled and shifted argument, so that the fundamental parallelogram is explicitly spanned by half-periods ω, ω', so $\tau = \omega'/\omega$, and $\theta(0) = 0$. Specifically

$$\theta(z) = \theta_3\left(\frac{\pi z}{2\omega} - \frac{1+\tau}{2}\pi, e^{i\pi\tau}\right) \tag{A.28}$$

and its second argument is always suppressed. It behaves as follows under argument reflection and quasi-period shifts

$$\theta(z + 2\omega) = \theta(z) \qquad\qquad \theta(-z) = \theta(z + 2\omega') = -e^{-i\pi z/\omega}\theta(z) \tag{A.29}$$

We define also a logarithmic derivative

$$\delta(z) = \frac{\theta'(z)}{\theta(z)} = \frac{\pi}{2\omega\theta(z)}\,\theta_3'\left(\frac{\pi z}{2\omega} - \frac{1+\tau}{2}\pi, e^{i\pi\tau}\right) \tag{A.30}$$

so that by construction its residue at $z = 0$ is 1. To obtain its quasi-periodicity properties we differentiate (A.29) with respect to z

$$-\theta'(-z) = \theta'(z + 2\omega') = \tfrac{i\pi}{\omega} e^{-i\pi z/\omega}\theta(z) - e^{-i\pi z/\omega}\theta'(z) \tag{A.31}$$

and divide the result side by side by (A.29) again to obtain

$$\delta(z + 2\omega) = \delta(z) \qquad\qquad -\delta(-z) = \delta(z + 2\omega') = \delta(z) - \tfrac{i\pi}{\omega} \tag{A.32}$$

By (A.25), the expression

$$\prod_{i=1}^{n} \frac{\theta_3(z - A_i)}{\theta_3(z - B_i)}, \text{ with } \sum_{i=1}^{n}(A_i - B_i) = m\pi, m, n \in \mathbb{Z} \tag{A.33}$$

is periodic in both directions. By (A.29), the same holds for its analog defined in terms of θ instead of θ_3 (with $\sum(A_i - B_i) = 2m\omega, m \in \mathbb{Z}$), which also has the advantage that it has manifest zeroes at $z = A_i$ and poles at $z = B_i$ (A_i do not need to be pairwise distinct, and B_i similarly). By (A.32),

$$\sum_{i=1}^{n} R_i\delta(z - B_i), \text{ with } \sum_{i=1}^{n} R_i = 0, n \in \mathbb{Z} \tag{A.34}$$

is also periodic in both directions. These two expressions are extremely useful for defining doubly-periodic functions, the former with manifest zeroes and poles, and the latter with manifest poles with specified residues R_i.

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