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Linear and nonlinear perturbations of Einstein equations with matter

PhD Thesis

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Oświadczenie

Ja, niżej podpisany Mieszko Rutkowski (nr indeksu: 1092232), doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. ,,Linear and nonlinear perturbations of Einstein equations with matter'' jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem dr hab. Andrzeja Rostworowskiego. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

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Streszczenie

W poniższej pracy rozważam fizyczne układy, w których pole grawitacyjne oddziałuje z materią różnego typu. Takie układy są w Ogólnej Teorii Względności opisywane równaniami Einsteina z materią. W pracy skupiam się na metodach perturbacyjnych, które są jednym ze sposobów radzenia sobie z dużym stopniem skomplikowania tych równań. W pierwszej części pracy prezentuję wyprowadzenie równań „master" dla liniowych zaburzeń układów grawitacyjnych oddziałujących z polem Maxwella i polem skalarnym w dowolnej liczbie wymiarów. Ten wynik jest owocem współpracy z A. Jansenem oraz A. Rostworowskim. W drugiej części pracy prezentuję wyprowadzenie równania „master" dla nieliniowych zaburzeń czasoprzestrzeni Reissnera–Nordströma. W trzeciej części pracy, używając nieliniowych metod perturbacyjnych, pokazuję, że pomimo obiecujących przesłanek, regularne rotujące grawastary nie są dobrymi kandydatami na źródło materialne dla rozwiązania Kerra. Ostatnia praca jest owocem współpracy z A. Rostworowskim.

Abstract

In this thesis I consider gravitational systems interacting with various types of matter. Such systems are described by the non–vacuum Einstein equations. I focus on perturbative methods, which are one of the ways to handle the complexity of these equations. Firstly, I present the derivation of the master equations for the linear perturbations of the Einstein–Maxwell–scalar systems in arbitrary dimensions. This result is an effect of the joint work with A. Jansen and A. Rostworowski. Secondly, I present the derivation of the master equations for the nonlinear perturbations of the Reissner–Nordström spacetime. Finally, using nonlinear perturbation methods I show that the regular rotating gravastars are not good candidates for the source of the Kerr metric, although they seemed to be a promising candidates for this role. The last results is an effect of the joint work with A. Rostworowski.

1 Introduction

1.1 Preface

This thesis is based on three publications [1, 2, 3] that I am an author or a co–author of. It consists of two parts: the goal of the first part (Sections 1–5) is to describe what articles [1, 2, 3] contain, to specify my contribution, describe the methods that were used and discuss the results. Thus, the first part of the thesis does not contain any new results, it is rather a guide on the aforementioned articles. The second part of the thesis contains articles [1, 2, 3]. Articles are not discusses in the chronological order ([2] appeared earlier that [1]), because [2] and [3] are closely related to each other and it is logical to present them one after another.

The first part is organised as follows: in Section 1 I provide a short introduction which summarises useful concepts from the theory of gravitational perturbations, in Sections 2, 3, 4 I discuss papers [1, 2, 3] and in Section 5 I summarize the thesis.

1.2 A note on the notation

Three papers [1, 2, 3] are closely related to each other both by the topic (perturbations of Einstein equations with matter) and by the mathematical apparatus. However, the perturbation theory of Einstein equations is a huge field, and these papers fall into three slightly distinct categories. The first paper [1] provides a formalism for dealing with linear perturbations in Einstein–Maxwell–scalar systems and it is mostly useful for the AdS/CFT community. The second paper [2] provides a formalism for nonlinear perturbations of Reissner–Nördstrom black holes and fits into the field of theoretical work on nonlinear perturbations in GR. It's conventions and notations are the same as in a paper by Rostworowski [4], since it is a generalisation of his scheme. The third paper [3] is about slowly rotating gravastars and fits into the field of slowly rotating stars in General Relativity, which is mostly based on Hartle formalism [5]. However, since it exploits works [4] and [6], it borrows notation and conventions from them. Unfortunately, people from different communities often use different nomenclature, conventions and notation. What's more, I was also introducing some changes between the papers, which was usually caused by finding the new notation or convention advantageous (eg. in [2] I was using angle θ as one of the variables on the sphere, but later I learned that it's simpler and more efficient to use $u = \cos(\theta)$ instead). These factors led to the differences in the notation and conventions between the papers. Throughout the text I will emphasise every discrepancy in notation and I hope that this inconvenience will not become confusing for the reader.

1.3 Perturbation Einstein Equations

General Relativity (GR), introduced more than a hundred years ago by Albert Einstein [7], is a classical theory o gravity. It can describe both vacuum systems, such as black holes or empty space outside the astrophysical objects and systems with matter such as stars, clouds of dust, accretion discs, the universe as a whole and more. Firstly even

Einstein himself did not believe in a possibility of finding nontrivial analytical solutions to his equations, but luckily he was wrong. Now we know a number of exact solutions to Einstein equations [8], out of which the most important for astrophysics and cosmology are the Schwarzschild [9] and Kerr [10] black holes and the cosmological solutions [11, 12, 13, 14, 15, 16]. However, due to an immense complexity of Einstein equations, all the exact solutions admit some symmetries to simplify the equations. Seeking for the exact solutions with little constraints on the symmetry of a system is usually doomed to failure. There are two main ways of dealing with this problem: one is to use computers to solve the equations for us. Due to a huge progress in computer efficiency and in numerical methods, simulations of colliding compact objects [17, 18, 19, 20], relativistic discs [21, 22], large scale inhomogeneous universe [23] and more are now possible. Numerical simulations, however, are not a cure for all the problems in GR. The second alternative to searching for the exact solutions are the perturbative methods. These methods base on the Taylor expansion of the metric tensor into series in some parameter. One of the very important outcomes of the perturbative methods is the Post–Newtonian (PN) formalism, which bases on the expansion of the metric into series in $\frac{1}{c}$ [24, 25]. This formalism is an important link in the process of obtaining gravitational wave signal templates - within it's range of applicability it is faster and easier to use than the numerical evolution of unperturbed Einstein equations. PN formalism, however, is applicable to the weak–field problems (i.e. metrics not too far from the Minkowski space). The methods I describe in this manuscript are useful to the different situations, namely to systems which are approximately (in zeroth perturbation order) a nontrivial solution to Einstein equations. As an illustrative example, we can take a slowly rotating, but massive Kerr black hole - it can be approximated by a Schwarzschild solution and some perturbation associated with the rotation, but it cannot be regarded as a perturbation of the Minkowski spacetime (at least close to a black hole). These methods find application in many branches of GR - stability analysis, rotating stars, cosmological structure formation, accretion discs, self force, AdS/CFT correspondence and more. Now I will introduce basic notions of the perturbation theory in GR.

Let's consider a metric g on a d-dimensional manifold $\mathcal{M}$. Einstein Equations are a set of differential equations for the components of g and functions of the matter components:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}\,, \tag{1}$$

where $R_{\mu\nu}$ and $R = R^{\mu}{}_{\mu}$ denote Ricci tensor and Ricci scalar of the metric g, respectively, Λ denotes the cosmological constant, $T_{\mu\nu}$ denotes the energy–momentum tensor and G and c denote the Newton's gravitational constant and the speed of light. We can organise Ricci tensor and Ricci scalar into the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. The interpretation of the Λ term in Einstein equations is ambiguous - it can be either regarded as a geometrical feature of the universe, or as an existence of the constant–density perfect fluid with an equation of state $p + \rho = 0$, where p and ρ stand for pressure and density, respectively. In the section about the rotating gravastars 4, we consider such a system. From now on we use geometric units, namely we set $G = c = 1$.

The main difficulty in solving (1) is the high nonlinearity of equations. Approximate methods are a way to get rid of these nonlinearities, but for a prize of the infinitely many equations to solve. To see this, let's assume that we already know a solution to (1) and let's denote it with $\bar{g}$ and $\bar{\Theta}$, where $\bar{\Theta}$ are some matter fields, which we do not specify for now. We will refer to $\bar{g}$ and $\bar{\Theta}$ as to the background or zeroth order solution and all the quantities with bar, such as Ricci tensor $\bar{R}_{\mu\nu}$ or Ricci scalar $\bar{R}$ will refer to the background solution. Now let's expand g and Θ, which are a new, unknown solution to Einstein equations into Taylor series around the background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{\infty} {}^{(i)}h_{\mu\nu}\epsilon^i \,, \tag{2}$$

$$\Theta = \bar{\Theta} + \sum_{i=1}^{\infty} {}^{(i)}\theta\epsilon^i \,, \tag{3}$$

where ${}^{(i)}h_{\mu\nu}$ and ${}^{(i)}\theta$ are metric and matter perturbations of order i and ϵ is a perturbation parameter - a quantity according to which the expansion is performed. Please note that there is no usual $i!$ term in the denominators of (2),(3). Unfortunately, different authors use different conventions and, what's even more unfortunate, my papers are also not consistent in this convention: [2] does not include and [3] does include $i!$ in the denominator. This is due to the fact, that the former is partially based on paper [4] and for the easy comparison of the formulas we follow it's conventions, whereas the latter uses extensively the general gauge transformations formalism from [6], whose conventions we follow. Fortunately, [1] contains only linear analysis and does not suffer from this confusion.

Metric g and matter fields Θ are now expressed by an infinite sum of it's perturbations ${}^{(i)}h$, ${}^{(i)}\theta$. We plug these expansions into Einstein equations and we obtain a system which looks in the following way:

$$G_{\mu\nu}(\bar{g} + {}^{(1)}h\epsilon + ...) + \Lambda(g_{\mu\nu} + {}^{(1)}h_{\mu\nu}\epsilon + ...) = 8\pi T_{\mu\nu}(\bar{g} + {}^{(1)}h\epsilon + ..., \bar{\Theta} + {}^{(1)}\theta\epsilon + ...) \,. \tag{4}$$

We can collect terms in (4) according to the powers of ϵ, and rewrite Einstein equations as an infinite system of differential equations for ${}^{(i)}h_{\mu\nu}$ and ${}^{(i)}\theta$ that from now on we call perturbation Einstein equations. One can notice that in every perturbation order (by order we mean a power of ϵ), Einstein equations follow the same universal pattern:

$$\delta G({}^{(i)}h)_{\mu\nu} + \Lambda {}^{(i)}h_{\mu\nu} - 8\pi {}^{(i)}t_{\mu\nu} = {}^{(i)}S_{\mu\nu} \,. \tag{5}$$

Perturbation of Einstein tensor δG is defined as:

$$\delta G(h)_{\mu\nu} = \Delta_L(h)_{\mu\nu} - \frac{1}{2}\left(\bar{g}^{\alpha\beta}\Delta_L(h)_{\alpha\beta} + h^{\alpha\beta}\bar{R}_{\alpha\beta}\right) - \frac{1}{2}\bar{R}h_{\mu\nu} \,, \tag{6}$$

where Δ_L denotes the so–called Lichnerowicz operator: $\Delta_L(h)_{\mu\nu} = \frac{1}{2}(-\bar{\nabla}^\alpha\bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h^\alpha{}_\alpha - 2\bar{R}_{\mu\alpha\nu\beta}h^{\alpha\beta} + \bar{\nabla}_\mu\bar{\nabla}^\alpha h_{\nu\alpha} + \bar{\nabla}_\nu\bar{\nabla}^\alpha h_{\mu\alpha})$ [4, 26, 27, 28]. By ${}^{(i)}t_{\mu\nu}$ we denote a perturbation of the energy–momentum tensor. It's explicit form depends on the matter

we consider. Please note that in [2] the energy–momentum tensor is traceless, therefore Ricci scalar is zero and equations simplify. The source terms ${}^{(i)}S_{\mu\nu}$ consist of all lower order quantities that give rise to ϵ^i and their construction is a purely algebraical task. Let's assume that we already know a solution up to order i:

$$\tilde{g} = \bar{g} + \sum_{j=0}^{i} {}^{(j)}h_{\mu\nu} \,, \tag{7}$$

$$\tilde{\Theta} = \bar{\Theta} + \sum_{j=0}^{i} {}^{(j)}\theta \,. \tag{8}$$

Source for Einstein equations of order $i+1$ is given by

$${}^{(i)}S_{\mu\nu} = [i+1] \left(-G_{\mu\nu}(\tilde{g}) + 8\pi T_{\mu\nu}(\tilde{g}, \tilde{\Theta}) \right) \,, \tag{9}$$

where $[k]\,(f)$ denotes the kth order expansion of f in ϵ. In the first order, there are no source terms and the system is homogeneous. If Einstein equations are supported by other equations (such as Maxwell equations), one has to construct sources for these additional equations as well.

What we have achieved so far, is changing the system of nonlinear equations (1) into the system of infinitely many, but linear equations (5). This linearity simplifies calculations and is a great advantage of perturbation scheme. Of course, equations have to be solved order by order, because for the construction of the sources at a given order, we need to know all the lower–order solutions.

1.4 Polar expansion

From now on we assume that the background metric is static and has a maximally symmetric patch of the dimension $n \equiv d-2$. It is well known that in 4 dimensions, perturbation Einstein equations for perturbations of static and spherically symmetric systems split into two sectors: axial and polar (in contrary to the cosmological perturbations, where the maximally symmetric patch is 3-dimensional and the third sector appears). Unfortunately the literature is not consistent on the names of the sectors (see Table 1 in [1]). In [2] and [3] we stick to *axial* and *polar* names, but in [1] we use *vector* and *scalar* instead. In the dimensions higher than 4 an additional sector appears and its usual name is *tensor*.

Axial perturbations of the Schwarzschild black hole were firstly treated in a classic paper by Regge and Wheeler [29] in which authors derived a master wave equation for the axial perturbations. The solution to master wave equation is sufficient to recover the solution to perturbation Einstein equations in a given sector and at a given order. Polar perturbations, due to the higher complexity, were waiting for their turn for more than 10 years and finally Zerilli [30] provided an analogue of the Regge–Wheeler master equation for the polar perturbations.

In papers [1, 2, 3] the background is assumed to be, respectively, a static black hole with a maximally symmetric horizon, Reissner–Nordström spacetime and the de Sitter

spacetime. Let me remind how to split metric and other tensors (e.g. field–strength tensor) into certain sectors in 4 dimensions. For that we will consider symmetric and antisymmetric tensors separately. We base this part on papers [31, 32, 33] and on the Appendix A of [1].

When the background metric is static and spherically symmetric, we expect Einstein equations to decouple into (t, r) and (θ, φ) parts. What's more, the perturbations split into three parts that transform differently under rotations. For a tensor T we can make the following division (see Eq. (11) in [32]):

$$T = \left(\begin{array}{c|c|c} \mathrm{S} & \mathrm{S} & \mathrm{V} \\ \hline \mathrm{S} & \mathrm{S} & \mathrm{V} \\ \hline \mathrm{V} & \mathrm{V} & \mathrm{T} \end{array} \right) , \tag{10}$$

where S transforms as a scalar, V transforms as a vector and T transforms as a tensor. We can now ask what scalars, vectors and tensors build those blocks? They should all be built of spherical harmonics Y_{lm} (where $l = 0, 1, ...$ and $m = 0, 1, ..., l$), since they are the eigenfunctions of a laplacian on a sphere. Therefore the scalar $S = Y_{lm}$ builds the scalar part of the tensor. The accessible vectors we can build out of Y_{lm} are:

$$V^1_a = D_a Y_{lm} \,, \tag{11}$$

$$V^2_a = \epsilon_{ab} D^b Y_{lm} \,, \tag{12}$$

where D_a denotes a covariant derivative compatible with a metric on a two sphere γ_{ab} and ϵ_{ab} is the Levi-Civita tensor. The accessible symmetric tensors are:

$$T^1_{ab} = D_a D_b Y_{lm} \,, \tag{13}$$

$$T^2_{ab} = Y_{lm} \gamma_{ab} \,, \tag{14}$$

$$T^3_{ab} = \frac{1}{2} \left(\epsilon^c_a D_b D_c Y_{lm} + \epsilon^c_b D_a D_c Y_{lm} \right) = \frac{1}{2} \left(D_a V^2_b + D_b V^2_a \right) \,. \tag{15}$$

The accessible antisymmetric tensors are:

$$T^4_{ab} = Y_{lm} \epsilon_{ab} \,, \tag{16}$$

$$T^5_{ab} = \frac{1}{2} \left(\epsilon^c_a D_b D_c Y_{lm} - \epsilon^c_b D_a D_c Y_{lm} \right) = \frac{1}{2} \left(D_a V^2_b - D_b V^2_a \right) \,. \tag{17}$$

Other antisymmetric tensors we could construct are either zero or proportional to T^4. Now the perturbation tensors are built out of the quantities listed above multiplied by unknown functions of t and r and put into the full metric in proper places according to (10). The key observation for the decoupling is that the objects we constructed behave differently under reflections: Y_{lm} gains multiplication factor $(-1)^l$ under the space inversion $(\theta, \varphi) \to (\pi - \theta, \pi - \varphi)$. Acting with a derivative on Y_{lm} or multiplying it by the metric γ_{ab} does not change this behaviour, but multiplying Y_{lm} by ϵ_{ab} changes the multiplication factor to $(-1)^{l+1}$. The sector transforming with $(-1)^\ell$ is called the polar sector and the sector transforming with $(-1)^{\ell+1}$ is called the axial sector. Since

Einstein equations do not contain multiplication by ϵ_{ab}, they preserve the parity of the perturbations and also split into two sectors. This allows to consider equations for both sectors separately. This can be also regarded as the helicity decomposition - perturbations in the polar sector have helicity $h = 0$ and in axial sector helicity $h = 1$. To sum up, for the symmetric tensor we have 7 polar (3 scalars S, 2 vectors V^1 and 1 tensor T^1 and 1 tensor T^2) and 3 axial components (2 vectors V^2 and 1 tensor T^3) and for the antisymmetric tensor we have 3 polar (1 scalar S and 2 vectors V^1) and 3 axial components (2 vectors V^2 and 1 tensor T^4). The explicit form of the polar expansion of the perturbations is provided in Eq. (15)-(24) of [2]. This expansion can be generalised to higher dimensions, where for $d > 4$ there appears another sector (see e.g. Appendix A in [1], where we follow Mukohyama [33]).

2 Master equations and stability of Einstein-Maxwell-scalar black holes

The classic papers [27, 28] by Kodama and Ishibashi provided the formalism to treat linear perturbations of Schwarzschild and Reissner–Nordström spacetimes in full generality (arbitrary dimension and topology of the horizon). In their approach, by clever transformations and combinations of linearized Einstein equations, they obtained the master scalar equations governing the system. This approach, however, becomes troublesome as the equations become more complex. For example, the general master equations for the Einstein–Maxwell–scalar systems were not known. Our article [1] fills this gap by utilising the "ansatz" approach used for the nonlinear perturbations by [4]. As a result, we obtain master scalar equations for the Einstein–Maxwell–scalar systems in $n+2$ dimensions, where n is the dimension of the maximally symmetric patch of the background spacetime (e.g. 2-dimensional sphere in the Schwarzschild case).

The paper [1] has been written by three authors: A. Jansen, A. Rostworowski and myself and we have different contributions to the different parts of the paper:

- The main result (master equations for the general system) is an effect of collaboration and teamwork between authors. However, the final version of the equations that includes a generalisation to the arbitrary dimension and topology was obtained by A. Jansen and he is the main contributor to this outcome.
- Treatment of the special cases and coordinate transformations (Appendices B and D): this part was done by myself apart from the planar case, which was done by A. Jansen.
- Applications of the derived formulas to stability and calculating quasinormal modes (Section 4, Appendix E): this part was done by A. Jansen.
- Comparison between our results and the results by Kovtun and Starinets (Appendix E): this part was done by A. Rostworowski.

Below I describe only the parts that I contributed to.

2.1 Linear perturbations of Einstein–Maxwell–scalar equations

Let's consider the Einstein–Maxwell–scalar system defined by the action:

$$S = \int d^{n+2}x\sqrt{-g}\left(R - 2\Lambda - \eta(\partial\phi)^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi)\right), \tag{18}$$

where by R is the Ricci scalar, Λ is the cosmological constant, η is an arbitrary constant, ϕ is the scalar field, $Z(\phi)$ is the arbitrary function of ϕ describing the coupling between the scalar and electromagnetic field, $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is the field–strength tensor of the electromagnetic field, and $V(\phi)$ is an arbitrary potential of the scalar field. In the equations, instead of F, we use the electromagnetic potential A, where $F = dA$.

As a background metric, we take a static $n+2$ dimensional metric with the n-dimensional maximally symmetric spatial patch. We work with Fefferman-Graham (FG) coordinates, which reduce to Schwarzschild coordinates in 4 dimensions: $(t, r, x_1, ..., x_n)$. Solution to the background Einstein–Maxwell–scalar equations with the aforementioned symmetry assumptions can be most generally written as:

$$ds^2 = -f(r)dt^2 + \frac{\zeta(r)^2}{f(r)}dr^2 + S(r)^2 dX^2_{(n,K)}\,, \tag{19}$$

$$A = a(r)dt\,, \tag{20}$$

$$\phi = \phi(r)\,, \tag{21}$$

where:

$$dX^2_{(n,K)} = \begin{cases} dx_1^2 + ... + dx_n^2 & \text{for } K = 0 \text{ (planar case)}\,, \\ d\Omega_{(n)} & \text{for } K = 1 \text{ (spherical case)}\,, \\ dH_{(n)} & \text{for } K = -1 \text{ (hyperbolic case)}\,. \end{cases} \tag{22}$$

2.2 Main results

We consider linear perturbations of the metric $\delta g_{\mu\nu}$, the electromagnetic potential δA_μ and the scalar field $\delta\phi$. We introduce auxiliary names for some coordinates for a clearer notation: $x_1 = x$, $x_2 = y$, $x_n = z$.

We choose the following form of the perturbations:

$$\begin{aligned} \delta g_{\mu\nu} &= \begin{pmatrix} h_{tt} & 1/2h_{tr} & h_{tx} & 0 & ... & 0 & h_{tz} \\ 1/2h_{tr} & h_{rr} & h_{rx} & 0 & ... & 0 & h_{rz} \\ h_{tx} & h_{rx} & h_{xx} & 0 & ... & 0 & h_{xz} \\ 0 & 0 & 0 & h_{yy} & 0 & 0 & h_{yz} \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ h_{tz} & h_{rz} & h_{xz} & h_{yz} & 0 & 0 & h_{zz} \end{pmatrix}, \\ \delta A_\mu &= \begin{pmatrix} a_t\,, & a_r\,, & a_x\,, & 0\,, & ...\,, & 0\,, & a_z \end{pmatrix}, \\ \delta\phi &= \varphi\,, \end{aligned} \tag{23}$$

where

$$\begin{aligned} h_{xx} &= \frac{1}{n}\left(h_+ - (n-1)k^2 h_-\right)\,, \\ h_{yy} = \cdots = h_{zz} &= \frac{1}{n}\left(h_+ + k^2 h_-\right)\,. \end{aligned} \tag{24}$$

All variables depend on t, r and x only. It turns out that such a choice is sufficient to solve the most general case. In particular: for the perturbations of the Schwarzschild black hole, the angular part of the solution can be decomposed into spherical harmonics Y_{lm}. However, equations for the $t-r$ part of the perturbations are the same for every m, therefore it is sufficient to assume the axial symmetry ($m = 0$) for the derivation

of the master equations. What's more, the perturbations $h_{t\alpha}$, $h_{r\alpha}$, $h_{x\alpha}$ and a_α where $\alpha = x_2, ..., x_{n-1}$ follow the same equations as h_{tz}, h_{rz}, h_{xz} and a_z. Also $h_{\alpha\beta}$, $\alpha, \beta = x_2, ..., x_{n-1}$ and $\alpha \neq \beta$ follow the same equation as h_{yz}. Therefore, we put $h_{t\alpha}$, $h_{r\alpha}$, $h_{x\alpha}$, a_α and $h_{\alpha\beta}$ to zero.

After expanding perturbations into eigenstates of the laplacian of the maximally symmetric patch, the dependence of the perturbations on x can be decoupled and all the variables depend on t and r only. Then perturbations (23) split into three sectors, which are not mixed by the homogeneous part of perturbation Einstein equations. Perturbations h_{tt}, h_{tr}, h_{rr}, h_{rx}, a_t, a_r, φ belong to the scalar sector (helicity 0), perturbations h_{tz}, h_{rz}, h_{xz} and a_z belong to the vector sector (helicity 1) and h_{yz} belongs to the tensor sector (helicity 2). For the details of the decomposition see Table 1 in [1].

The very important step of the whole calculation is dealing with the gauge transformations. We have two types of gauge transformations: coordinate gauge transformations generated by a gauge vector $\xi_\mu(t, r)$ and gauge transformation of the electromagnetic field $\lambda(t, r)$. The perturbations transform with gauge transformations as:

$$\begin{aligned} \delta g_{\mu\nu} &\to \delta g_{\mu\nu} - \bar{\nabla}_\mu \xi_\nu - \bar{\nabla}_\nu \xi_\mu \,, \\ \delta A_\mu &\to \delta A_\mu + \bar{\nabla}_\mu \lambda - \xi^\nu \bar{\nabla}_\nu A_\mu - A_\nu \bar{\nabla}_\mu \xi^\nu \,, \\ \delta\phi &\to \delta\phi - \xi^\nu \bar{\nabla}_\nu \phi \,. \end{aligned} \tag{25}$$

Now we have two options: we can either use the gauge freedom to choose a specific gauge or we can build gauge invariants out of perturbations (23). We choose the latter and we introduce gauge invariant quantities $\mathfrak{h}_{\mu\nu}$, $\mathfrak{a}_\mu$ and $\boldsymbol{\varphi}$ (see Eq. 3.4 in [1]). It turns out that we can construct 11 independent gauge invariant variables: $\mathfrak{h}_{tt}$, $\mathfrak{h}_{tr}$, $\mathfrak{h}_{rr}$, $\mathfrak{h}_{rx}$, $\mathfrak{a}_t$, $\mathfrak{a}_r$, $\boldsymbol{\varphi}$ in the scalar sector, $\mathfrak{h}_{tz}$, $\mathfrak{h}_{rz}$, $\mathfrak{a}_z$ in the vector sector and $\mathfrak{h}_{yz}$ in the tensor sector. These are so–called Detweiler gauge invariants [34]. Their construction, on example of the scalar sector is the following: we take variables h_{tt}, h_{tr}, h_{rr}, h_{rx}, a_t, a_r, φ and add to them linear combinations of the rest of variables and their derivatives: h_-, h_+, h_{tx}. The coefficients of the linear combinations are chosen in such a way that the gauge dependence of the whole expression vanishes. We do the same for the vector and tensor sector obtaining:

Tensor sector:

$$\mathfrak{h}_{yz} \equiv h_{yz} \,, \tag{26}$$

Vector sector:

$$\mathfrak{h}_{tz} \equiv h_{tz} - \partial_t h_{xz} \,, \tag{27}$$

$$\mathfrak{h}_{rz} \equiv h_{rz} - \partial_r h_{xz} + 2\frac{S'}{S} h_{xz} \,, \tag{28}$$

$$\mathfrak{a}_z \equiv a_z \,, \tag{29}$$

Scalar sector:

$$\mathfrak{h}_{tt} \equiv h_{tt} - 2\partial_t h_{tx} + \partial_t^2 h_- + \frac{f'}{2nSS'}(h_+ + k^2 h_-)\,, \tag{30}$$

$$\mathfrak{h}_{tr} \equiv h_{tr} - 2\partial_r h_{tx} + \partial_t\partial_r h_- + 2\frac{f'}{f}h_{tx} - \frac{f'}{f}\partial_t h_- - \frac{\zeta^2}{nfSS'}\partial_t(h_+ + k^2 h_-)\,, \tag{31}$$

$$\mathfrak{h}_{rr} \equiv h_{rr} - \frac{\zeta^2}{nfS'S}\partial_r(h_+ + k^2 h_-) + \left(\frac{\zeta^2}{2nf^2S^2S'}(Sf' + 2fS') - \eta\frac{\zeta^2}{n^2fS'^2}\phi'^2\right)(h_+ + k^2 h_-)\,, \tag{32}$$

$$\mathfrak{h}_{rx} \equiv h_{rx} - \frac{1}{2}\partial_r h_- + \frac{S'}{S}h_- - \frac{\zeta^2}{2nfSS'}(h_+ + k^2 h_-)\,, \tag{33}$$

$$\mathfrak{a}_t \equiv a_t - \partial_t a_x - \frac{a'}{2nSS'}(h_+ + k^2 h_-)\,, \tag{34}$$

$$\mathfrak{a}_r \equiv a_r - \partial_r a_x + \frac{a'}{2f}\partial_t h_- - \frac{a'}{f}h_{tx}\,, \tag{35}$$

$$\boldsymbol{\varphi} \equiv \varphi - \frac{\phi'}{2nSS'}(h_+ + k^2 h_-)\,. \tag{36}$$

The next three assumptions are the clue of our approach:

1. We assume that in each sector there exist master scalar variables. In the scalar sector, we have all kinds of fields: gravitational, electromagnetic and scalar, therefore we assume the existence of 3 master scalar functions: $\Phi_2^{(0)}$, $\Phi_1^{(0)}$ and $\Phi_0^{(0)}$. The upper index h in $\Phi_s^{(h)}$ stands for the helicity and the lower index s stands for the spin ($s = 2$ - gravity, $s = 1$ - Maxwell field, $s = 0$ - scalar field). In the vector sector, we don't have a scalar field, therefore we assume the existence of 2 master scalars: $\Phi_2^{(1)}$ and $\Phi_1^{(1)}$. In the tensor sector we have only gravitational field, therefore we assume to have only one master scalar $\Phi_2^{(2)}$.

2. We assume that these master scalars fulfil systems of wave equations coupled within each sector by a symmetric potential matrix $W_{s,s'}^{(h)}$ which depend on r only:

$$\Box\Phi_s^{(h)} - W_{s,s'}^{(h)}(r)\Phi_{s'}^{(h)} = 0\,, \tag{37}$$

   where we sum over s' index. For the scalar sector potential matrix has size 3x3, for the vector sector it has size 2x2 and for the tensor sector it is just one function.

3. We assume that the gauge invariant variables $\mathfrak{h}_{\mu\nu}$, $\mathfrak{a}_\mu$ and $\boldsymbol{\varphi}$ are the linear combinations of master scalar functions from their sector and their derivatives. The coefficients of these combination are functions of r only.

After plugging such an ansatz into linearised Einstein equations we are left with a system of equations for the coefficients of the linear combinations from the assumption 3. To be able to solve such a system, we use the assumption 2 to substitute second time derivatives of the master scalar functions, what makes potentials appear in the equations. It turns out that solving such a system of equations is almost purely algebraical task with only a few very simple ordinary differential equations to solve. As a result

we obtain the expressions for gauge invariants and potentials $W^{(h)}_{s\,s'}$. Expressions for the gauge invariants are too lengthy to write them down here and they can be found in [1] (equations (3.5), (3.7) and (C.1)-(C.5)). Altough the equations for the potentials are also quite obscure, we provide them below for an easier discussion. Potentials in the scalar sector read:

$$
\begin{aligned}
W^{(0)}_{0,0}(r) &= \frac{k^2}{S^2} + \frac{\phi'}{\mathcal{D}^2\zeta^2}\Bigg(\frac{\zeta^2 k^2}{nS'}\mathcal{A} + \mathcal{F}\mathcal{D}S\mathcal{V} + 2\eta f\phi'\left(\mathcal{F}\mathcal{P} + 4\zeta^4 k^2\left(k^2-(n-1)K\right)\right) + \\
&\quad 2\eta\mathcal{F}\mathcal{D}\phi'\left(Sf' + (n-2)fS'\right)\Bigg) - \frac{1}{4\eta\zeta^2}\left(\mathcal{V}' - 2a'^2 Z'^2/Z\right), \\
W^{(0)}_{1,1}(r) &= \frac{k^2}{S^2} + \frac{Za'^2}{\mathcal{D}^2\zeta^2}\Bigg(n^2 S'^2\mathcal{F}\left(Sf' - 2(n-1)fS'\right) + 2fn^2S^2Za'^2S'^2 + \\
&\quad 4f\zeta^2\left(nS'^2\left((2n-3)k^2 - n(n-1)K\right) + k^2\eta S^2\phi'^2\right) + 4\zeta^4k^4\Bigg) + \\
&\quad \frac{1}{Z}\left(\frac{Z'}{8\eta\zeta^2}\mathcal{V} + \frac{f(n-1)S'\phi'}{\zeta^2 S}Z' - \frac{fZ''\phi'^2}{2\zeta^2}\right) + \frac{2nfSS'Z'\phi'a'^2}{\mathcal{D}\zeta^2} - \\
&\quad \frac{(n-1)\left(nf'S' - f\eta S\phi'^2\right)}{\zeta^2 nS} + \frac{3fZ'^2\phi'^2}{4\zeta^2Z^2}, \\
W^{(0)}_{2,2}(r) &= \frac{k^2}{S^2} + \frac{n-1}{nS^2\mathcal{D}^2}\Bigg(4n^2\left(k^2-nK\right)fS^2a'^2ZS'^2 - 8n\zeta^4k^4K + 8\eta\zeta^2fS^2\phi'^2k^2\left(k^2-nK\right) + \\
&\quad 2n^2S'^2\mathcal{F}\left(Sf'\left(2k^2-nK\right) + 2fS'\left((n-2)k^2 - n(n-1)K\right)\right) + \\
&\quad 8\zeta^2\left(nS'\left(fS'\left(k^4 - n(n-2)k^2K + n^2(n-1)K^2\right) - k^4Sf'\right)\right)\Bigg), \\
W^{(0)}_{0,1}(r) &= -\frac{k\sqrt{Z}a'}{\sqrt{2}\mathcal{D}^2\zeta\sqrt{\eta}}\Bigg(\mathcal{A} + \mathcal{D}S\mathcal{V} + \frac{\mathcal{D}^2Z'}{SZ} + 2\eta\mathcal{D}Sf\phi'^2Z'/Z + \\
&\quad 4\eta f\phi'\left(\mathcal{P} - n(n-1)S'^2\mathcal{F} - 2\zeta^2S'\left((1-2n)k^2 + n(n-1)K\right)\right)\Bigg), \\
W^{(0)}_{0,2}(r) &= k\sqrt{k^2-nK}\frac{\sqrt{n-1}}{\sqrt{n}\sqrt{\eta}S\mathcal{D}^2}\Bigg(\mathcal{A} + \mathcal{D}S\mathcal{V} + 4\eta f\phi'\left(\mathcal{P} + 2\zeta^2nS'\left(k^2-(n-1)K\right)\right)\Bigg), \\
W^{(0)}_{1,2}(r) &= -\sqrt{k^2-nK}\frac{\sqrt{2}\sqrt{n-1}\sqrt{Z}a'}{\zeta\sqrt{n}S\mathcal{D}^2}\Bigg(2fn^2S^2Za'^2S'^2 + n^2Sf'S'^2\mathcal{F} + \\
&\quad 4f\zeta^2\left(nS'^2\left(k^2(n-2) - K(n-1)n\right) + k^2\eta S^2\phi'^2\right) + \mathcal{D}fnSS'\phi'\frac{Z'}{Z} + 4\zeta^4k^4\Bigg),
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
\mathcal{V}(r) &= -2\zeta^2 V' + a'^2 Z' \,. \\
\mathcal{F}(r) &= 2fS' - Sf' \,, \\
\mathcal{D}(r) &= 2\zeta^2 k^2 - nS'\mathcal{F} \,, \\
\mathcal{P}(r) &= \left(\eta S^2 \phi'^2 - nS'^2\right)\mathcal{F} \,, \\
\mathcal{A}(r) &= 4n\eta f S' S^2 Z a'^2 \phi' \,.
\end{aligned}
\tag{39}
$$

For the clear presentation, primes denote derivatives with respect to r when they act on functions of r, and they denote functional derivatives with respect to ϕ when they act on functionals of ϕ: V, Z or $\mathcal{V}$.

Potentials in the vector sector read:

$$
\begin{aligned}
W^{(1)}_{1,1}(r) &= \frac{k^2}{S^2} - \frac{f'S'}{\zeta^2 S} + (n-2)\left(\frac{K}{S^2} - \frac{fS'^2}{\zeta^2 S^2}\right) + \frac{Za'^2}{\zeta^2} + \frac{f\eta\phi'^2}{n\zeta^2} - \frac{1}{8\eta\zeta^2}\frac{Z'}{Z}\mathcal{V} \\
&\quad - \frac{Z'^2}{Z^2}\frac{f\phi'^2}{4\zeta^2} - \frac{Z'}{Z}\frac{fS'\phi'}{\zeta^2 S} + \frac{f\phi'^2 Z''}{2\zeta^2 Z} \,, \\
W^{(1)}_{1,2}(r) &= -\sqrt{k^2 - nK}\frac{\sqrt{Z}a'}{\zeta S} \,, \\
W^{(1)}_{2,2}(r) &= \frac{k^2}{S^2} - n\left(\frac{f'S'}{\zeta^2 S} - \frac{fS'^2}{\zeta^2 S^2} + \frac{K}{S^2}\right) + \eta\frac{f\phi'^2}{\zeta^2} \,,
\end{aligned}
\tag{40}
$$

and in the tensor sector:

$$
W^{(2)}(r) = \frac{k^2}{S^2} \,. \tag{41}
$$

As we put the background matter fields to zero, interaction potentials vanish and all the equations decouple. What's more, when $\sqrt{k^2 - nK} = 0$, wave equations also decouple, but expressions for the gauge invariants blow up. Due to this behaviour, these cases require special treatment.

2.3 Special cases

Let's consider special cases $\sqrt{k^2 - nK} = 0$. In our work [1] I was responsible for the spherical case $K = 1$ ($k^2 = n$), what translates into $\ell = 0$ or $\ell = 1$ case, where ℓ corresponds to the index of the scalar function $Y_{\ell m}$. The planar case was studied by A. Jansen and I refer to the paper [1], section B.3 to read about this case.

2.3.1 $\ell = 0$

For $\ell = 0$ vector and tensor sectors do not appear and we consider the scalar sector only. In this case there are no h_{tx}, h_{rx}, h_- and a_x variables and there is no ξ_x gauge component. Because of that, the gauge invariants used earlier are not gauge invariant anymore. Instead, we use ξ_t, ξ_r and λ to set h_{tr}, h_+ and a_t to zero and we are left with h_{tt}, h_{tr}, a_r, φ only. It turns out that we can fulfil Einstein equations by introducing

just one master scalar variable $\Phi_0^{(0)}$. However, in this case the wave equation for $\Phi_0^{(0)}$ is inhomogeneous:

$$\Box\Phi_0^{(0)} - W_{0,0}^{(0)}\Phi_0^{(0)} = \frac{c_0(a'^2 Z' - 2\zeta^2 V')}{4f\zeta\eta S^{n-1}S'} + \frac{c_0\left(Sf' + f(n-1)S'\right)\phi'}{f\zeta S^n S'} - \frac{c_0\eta\phi'^3}{\zeta n S^{n-2}S'^2}\,, \tag{42}$$

where c_0 is an arbitrary constant corresponding to a static perturbation of the background spacetime, such as the shift in mass for the Reissner–Nordström black hole. The other variables are given by:

$$\begin{aligned}
\varphi &= \Phi_0^{(0)}\,,\\
h_{rr} &= \frac{c_0\zeta^3}{f^2 S^{n-1}S'} + \frac{2\zeta^2\eta S\phi'}{fnS'}\Phi_0^{(0)}\,,\\
f\partial_r\left(\frac{h_{tt}}{f}\right) &= \left(\frac{S\left(2\zeta^2 V' - Z'a'^2 - 4\eta f'\phi'\right)}{2nS'} + \frac{2f\eta^2 S^2\phi'^3}{n^2 S'^2} - \frac{2(n-1)\eta f\phi'}{n}\right)\Phi_0^{(0)} +\\
&\quad -\frac{2f\eta S\phi'}{nS'}\partial_r\Phi_0^{(0)} + \frac{c_0\zeta\left(f\eta S^2\phi'^2 - nSf'S' - fn(n-1)S'^2\right)}{fnS^n S'^2},\\
\partial_t a_r &= \frac{a'h_{tt}}{2f} + \frac{1}{n}a'\left(\frac{nZ'}{Z} - \frac{\eta S\phi'}{S'}\right)\Phi_0^{(0)} - \frac{c_0\zeta a'}{2fS^{n-1}S'}\,.
\end{aligned} \tag{43}$$

To sum up, the only dynamical degree of freedom for $\ell = 0$ is the scalar field.

2.3.2 $\ell = 1$

For $\ell = 1$ the tensor sector does not appear. In the vector sector we don't have h_{xz} and the gauge invariants $\mathfrak{h}_{tz}$ and $\mathfrak{h}_{rz}$ are not gauge invariant anymore. We choose a gauge vector component ξ_z to set $h_{rz} = 0$. The only two variables left are h_{tz} and a_z. It turns out, that Einstein equations for them can be fulfilled by introducing one master scalar variable $\Phi_1^{(1)}$ fulfilling an inhomogeneous wave equation:

$$\Box\Phi_1^{(1)} - W_{1,1}^{(1)}\Phi_1^{(1)} = c_1\frac{\sqrt{Z}a'}{\zeta S^{n+1}}\,, \tag{44}$$

where c_1 is an arbitrary constant corresponding to the angular momentum of the Kerr–Newman spacetime.

In the scalar sector we treat $\ell = 1$ case in a slightly different way than in the $\ell = 0$ case. Now we don't have h_- variable and gauge invariants used earlier, again, are not

gauge invariant anymore and in this case they transform as:

$$
\begin{aligned}
\mathfrak{h}_{tt} &\to \mathfrak{h}_{tt} + \frac{f'}{SS'}\xi_x + 2\partial_t^2\xi_x \,, \\
\mathfrak{h}_{tr} &\to \mathfrak{h}_{tr} + 2\partial_t\partial_r\xi_x - \frac{2\left(Sf'S' + \zeta^2\right)}{fSS'}\partial_t\xi_x \,, \\
\mathfrak{h}_{rr} &\to \mathfrak{h}_{rr} + \frac{\zeta^2\left(Sf'S' + 2fS'^2 - \frac{2f\eta S^2\phi'^2}{n}\right)}{f^2S^2S'^2}\xi_x - \frac{2\zeta^2}{fSS'}\partial_r\xi_x \,, \\
\mathfrak{h}_{rx} &\to \mathfrak{h}_{rx} + \frac{\left(2S'^2 - \frac{\zeta^2}{f}\right)}{SS'}\xi_x - \partial_r\xi_x \,, \\
\mathfrak{a}_t &\to \mathfrak{a}_t - \frac{a'}{SS'}\xi_x \,, \\
\mathfrak{a}_r &\to \mathfrak{a}_r + \frac{a'}{f}\partial_t\xi_x \,, \\
\boldsymbol{\varphi} &\to \boldsymbol{\varphi} - \frac{\phi'}{SS'}\xi_x \,.
\end{aligned}
\tag{45}
$$

What's more, we do not have Einstein equations corresponding to the "-" index, but following Kodama and Ishibashi [28] we can keep the "-" equation as a gauge condition together with $\mathfrak{h}_+ = 0$ and $\mathfrak{h}_{tx} = 0$. It means that we can use the solution which is valid for $\ell > 1$ directly to the case $\ell = 1$ and still have Einstein equations fulfilled. However, some parts of this solution turn out to be a pure gauge for $\ell = 1$. By a proper choice of the gauge component $\xi_x = -S^2\Phi_2^{(0)}$ we can completely rule out the dependence of the metric and matter perturbations on the gravitational master scalar. Therefore there are two dynamical degrees of freedom for $\ell = 1$ in the scalar sector - electromagnetic and scalar.

2.4 Transformations

2.4.1 Regge–Wheeler gauge invariants

So far, we have used the Detweiler gauge–invariants. The other popular choice are the Regge–Wheeler gague invariants, which differ in the scalar sector. Below we provide how to translate one to onother.

We remind how the Detweiler gauge invariants are built: we choose certain perturbations (for the scalar sector these are h_{tt}, h_{tr}, h_{rr}, h_{rx}, a_t, a_r, φ) and add to them linear combinations of the rest of variables and their derivatives (in the scalar sector these are h_-, h_+, h_{tx}), so that the whole expressions do not depend on gauge. The Regge–Wheeler gauge invariants are constructed similarly, but instead of h_{rx}, we build a gauge invariant on h_+ and we use h_{rx} in linear combinations to eliminate the gauge dependence. It's important to understand the difference between Detweiler and Regge–Wheeler gauge invariants and Detweiler and the Regge–Wheeler gauge. When we speak about a specific gauge, we mean using the gauge freedom to put h_-, h_+, h_{tx} to zero in the Detweiler gauge and h_-, h_{rx}, h_{tx} to zero in the Regge–Wheeler gauge.

We can notice that in the Detweiler gauge variables h_{tt}, h_{tr}, h_{rr}, h_{rx}, a_t, a_r, φ correspond exactly to the Detweiler gauge invariants $\mathfrak{h}_{tt}$, $\mathfrak{h}_{tr}$, $\mathfrak{h}_{rr}$, $\mathfrak{h}_{rx}$, $\mathfrak{a}_t$, $\mathfrak{a}_r$, $\boldsymbol{\varphi}$ (analogously for the Regge–Wheeler case). Therefore, to compare the Regge–Wheeler and Detweiler gauge invariants it is sufficient to find the transformation between the Detweiler gauge and Regge–Wheeler gauge. It turns out that such a transformation corresponds to a gauge transformation given by a gauge vector $\zeta_\mu = (0, h^D_{rx}, 0, .., 0)$. Finally, we obtain the relations between the Detweiler and RW gauge invariants:

$$\begin{aligned}
\mathfrak{h}^{RW}_{tt} =& \mathfrak{h}^D_{tt} + \frac{ff'}{\zeta^2}\mathfrak{h}^D_{rx}\,,\\
\mathfrak{h}^{RW}_{tr} =& \mathfrak{h}^D_{tr} - 2\partial_t\mathfrak{h}^D_{rx}\,,\\
\mathfrak{h}^{RW}_{rr} =& \mathfrak{h}^D_{rr} + \left(\frac{2\zeta'}{\zeta} - \frac{f'}{f}\right)\mathfrak{h}^D_{rx} - 2\partial_r\mathfrak{h}^D_{rx}\,,\\
\mathfrak{h}^{RW}_{+} =& \frac{-2nfSS'}{\zeta^2}\mathfrak{h}^D_{rx}\,,\\
\mathfrak{a}^{RW}_{t} =& \mathfrak{a}^D_{t} - \frac{fa'}{\zeta^2}\mathfrak{h}^D_{rx}\,,\\
\mathfrak{a}^{RW}_{r} =& \mathfrak{a}^D_{r}\,,\\
\boldsymbol{\varphi}^{RW} =& \boldsymbol{\varphi}^D - \frac{f\phi'}{\zeta^2}\mathfrak{h}^D_{rx}\,.
\end{aligned} \tag{46}$$

We could obtain the same result by constructing the Regge–Wheeler gauge invariants from scratch and then searching for the relation to the Detweiler gauge invariants.

2.4.2 Eddingtion–Finkelstein coordinates

The results were derived in FG coordinates, but the wave equations are covariant and one can use them in any coordinates. Eddington–Finkelstein coordinates are especially useful, since they are regular on the horizon and they allow for easier computation of the quasinormal modes (there are only first order time derivatives in wave equations). Below we provide how the gauge invariants transform with the change of the background coordinates (gauge invariants are invariant to the transformations generated by the linear gauge vector ξ and the gauge function λ). The background metric in EF coordinates reads:

$$ds^2 = -f(r)dt^2 + 2\zeta(r)dtdr + S(r)^2 dX^2_{(n,K)}\,, \tag{47}$$

and it can be obtained by a transformation $g^{EF}_{\mu\nu} = L^\alpha_\mu L^\beta_\nu g^{FG}_{\alpha\beta}$, where L is given by:

$$(L^\mu_\nu) = \begin{pmatrix} 1 & -\frac{\zeta}{f} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{1}_n \end{pmatrix}\,. \tag{48}$$

In the same way we obtain expressions for the transformation of gauge invariants:

$$\mathfrak{h}^{EF}_{\mu\nu} = L^\alpha_\mu L^\beta_\nu \mathfrak{h}^{FG}_{\alpha\beta}\,, \tag{49}$$

$$\mathfrak{a}^{EF}_{\mu} = L^\alpha_\mu \mathfrak{a}^{FG}_{\alpha}\,, \tag{50}$$

Additionally, we have to change derivatives: $\partial_t \to \partial_t$, $\partial_r \to \partial_r + \frac{\zeta}{f}\partial_t$. As an example, let's take a vector sector. The transformation by L matrix reads:

$$\mathfrak{h}_{tz}^{EF} = \mathfrak{h}_{tz}^{FG}\,, \tag{51}$$

$$\mathfrak{h}_{rz}^{EF} = \mathfrak{h}_{rz}^{FG} - \frac{\zeta}{f}\mathfrak{h}_{tz}^{FG}\,, \tag{52}$$

$$\mathfrak{a}_{z}^{EF} = \mathfrak{a}_{z}^{FG}\,. \tag{53}$$

By plugging in explicit form of the gauge invariants (given by Eq. (3.7) in [1]) and applying transformation of derivatives, we finally obtain:

$$\mathfrak{h}_{tz}^{EF} \equiv \frac{nfSS'\Phi_2^{(1)}}{\zeta} + \frac{fS^2\partial_r\Phi_2^{(1)}}{\zeta} + S^2\partial_t\Phi_2^{(1)}\,, \tag{54}$$

$$\mathfrak{h}_{rz}^{EF} \equiv -S^2\partial_r\Phi_2^{(1)} - nSS'\Phi_2^{(1)} \tag{55}$$

$$\mathfrak{a}_{z}^{EF} \equiv \sqrt{k^2 - nK}\frac{S}{\sqrt{Z}}\Phi_1^{(1)}\,. \tag{56}$$

In the same manner one could obtain EF version of scalar gauge invariants (Eq. (C.1)-(C.5) in [1])

2.5 Summary

In the paper [1] we generalised the Kodama–Ishibashi results to the Einstein–Maxwell–scalar case. We also demonstrated the powerfulness of the "ansatz" approach to the perturbations of Einstein equations. This approach allows to tackle complicated cases, such as a scalar sector in the Einstein–Maxwel–scalar system, which was not fully resolved before our paper. The part of our paper which I did not discuss contains an applications of our equations - in Section 4 of [1] the linear stability of the vector sector is proven for the general Einstein–scalar theories and for the Gibbons–Maeda–Garfinkle–Horowitz–Strominger black hole. The possible extension of our model would be to find sources for the nonlinear wave equations for the higher–order perturbations of Einstein–Maxwell–scalar system following the scheme by [4].

3 Nonlinear perturbations of Reissner–Nordström black holes

3.1 Introduction

The aim of paper [2] is to provide a scheme for treating the nonlinear perturbations of the Reissner–Nördstrom spacetime. It is an extension of the nonlinear perturbation scheme of the Λ–vacuum spacetimes [4], or, from a different perspective, it is a generalisation of the Zerilli's work on linear perturbations of the Reissner–Nördstrom spacetime [35] to the nonlinear orders. In this case, the matter field denoted in the introduction as Θ is an electromagnetic field described by a field–strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ (or $F = dA$, where A is a vector electromagnetic potential). This is one of the places where the convention has been changed from paper to paper - in [1] we were using the vector potential, while in [2] I was using the field–strength tensor only. Einstein–Maxwell equations in this configuration take the form:

$$R_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{57}$$

$$\nabla^\mu F_{\mu\nu} = 0\,, \tag{58}$$

$$\nabla_{[\lambda} F_{\mu\nu]} = 0\,, \tag{59}$$

where square brackets denote antisymmetrization and $T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$. In (58) we used the fact that $T_{\mu\nu}$ is traceless, therefore $R_{\mu\nu}$ is traceless as well and the Ricci scalar is zero. Eq. (59) is the equation of motion for the electromagnetic field and (60) comes from the fact that F is an exterior derivative of the vector potenital. As a background $(\bar{g}, \bar{F})$ we take the Reissner–Nördstrom solution:

$$\bar{g} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2\,, \tag{60}$$

$$\bar{F} = \frac{Q}{r^2} dt \wedge dr\,, \tag{61}$$

where $d\Omega^2$ denotes a metric on a 2-sphere. From now on, we use $A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, to be consistent with a notation from [2]. I also apologise for the misprint in [2], where in the denominators of $\bar{F}_{tr}$ and F_{rt} a wrong power of r appeared. I use standard static coordinates (t, r, θ, φ), but the results of this paper can be easily transformed in the same manner as described earlier for the paper [1].

Now I follow a standard procedure. I expand metric and field–strength tensor into series in parameter ϵ:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i>0} {}^{(i)}h_{\mu\nu}\epsilon^i\,, \tag{62}$$

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + \sum_{i>0} {}^{(i)}f_{\mu\nu}\epsilon^i\,, \tag{63}$$

and plug them into Einstein equations (58)-(60). Perturbative form of these equations

can be organised as follows (equations (8)-(14) in [2]):

$$\Delta_L \left({}^{(i)}h\right)_{\mu\nu} - 8\pi {}^{(i)}t_{\mu\nu} = {}^{(i)}S^G_{\mu\nu}\,, \tag{64}$$

$$\bar{\nabla}^{\mu}{}^{(i)}f_{\mu\nu} - {}^{(i)}\Theta_\nu = {}^{(i)}S^M_\nu\,, \tag{65}$$

$${}^{(i)}f_{[\mu\nu,\lambda]} = 0\,, \tag{66}$$

where

$$\Delta_L \left({}^{(i)}h\right)_{\mu\nu} = \frac{1}{2}(-\bar{\nabla}^\alpha\bar{\nabla}_\alpha{}^{(i)}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu{}^{(i)}h^\alpha{}_\alpha - 2\bar{R}_{\mu\alpha\nu\beta}{}^{(i)}h^{\alpha\beta} + \bar{\nabla}_\mu\bar{\nabla}^{\alpha(i)}h_{\nu\alpha} + \\ + \bar{\nabla}_\nu\bar{\nabla}^{\alpha(i)}h_{\mu\alpha})\,, \tag{67}$$

$${}^{(i)}t_{\mu\nu} = 2{}^{(i)}f_{\alpha(\mu}\bar{F}^\alpha{}_{\nu)} - \frac{1}{2}{}^{(i)}f_{\alpha\beta}\bar{F}^{\alpha\beta}\bar{g}_{\mu\nu} + \left(\frac{1}{2}\bar{F}_{\alpha\sigma}\bar{F}_\beta{}^\sigma\bar{g}_{\mu\nu} - \bar{F}_{\mu\alpha}\bar{F}_{\nu\beta}\right){}^{(i)}h^{\alpha\beta} + \\ - \frac{1}{4}\bar{F}^{2(i)}h_{\mu\nu} - {}^{(i)}h_{\alpha(\mu}\bar{T}^\alpha{}_{\nu)}\,, \tag{68}$$

$${}^{(i)}\Theta_\nu = \bar{g}^{\alpha\beta}(\bar{F}_{\sigma\nu}{}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} + \bar{F}_{\beta\sigma}{}^{(i)}\delta\Gamma^\sigma_{\alpha\nu})\,, \tag{69}$$

$${}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\bar{g}^{\sigma\delta}\left(-\bar{\nabla}_\delta{}^{(i)}h_{\alpha\beta} + \bar{\nabla}_\alpha{}^{(i)}h_{\beta\delta} + \bar{\nabla}_\beta{}^{(i)}h_{\delta\alpha}\right)\,. \tag{70}$$

Since there are additional nonlinear equations for the matter content (59), there appears an additional source ${}^{(i)}S^M_\nu$. It's construction is analogous to the construction of the gravitational source ${}^{(i)}S^G_{\mu\nu}$ given by (9). Since Einstein equations obey Bianchi identities and tensor $F_{\mu\nu}$ obeys Jacobi identity, sources for the Einstein equations are not independent and they follow five identities (equations (30)–(34) in [2]).

For the further simplification, we polar–expand all the perturbations (since it's 3+1 dimensions, there are only two sectors: polar (called scalar in [1]) and axial (called vector in [1])). However, so far we have been dealing with symmetric tensors only and now we have the antisymmetric tensor $F_{\mu\nu}$. Polar expansion of the antisymmetric tensors is straightforward and it was discussed in the introduction. The explicit form of this expansions was provided e.g. by [30]. For the antisymmetric tensor $F_{\mu\nu}$, its polar components are F_{tr}, $F_{t\theta}$ and $F_{r\theta}$ and they expand as:

$$F_{tr}(t,r,\theta) = \sum_{0\le\ell} F_{\ell tr}(t,r)P_\ell(\cos\theta)\,, \tag{71}$$

$$F_{a\theta}(t,r,\theta) = \sum_{1\le\ell} F_{\ell a\theta}(t,r)\partial_\theta P_\ell(\cos\theta)\,, \quad a = t,r\,. \tag{72}$$

Axial components of $F_{\mu\nu}$ are $F_{t\varphi}$, $F_{\theta\varphi}$ and $F_{t\varphi}$ and they expand as:

$$F_{a\phi}(t,r,\theta) = \sum_{1\le\ell} F_{\ell a\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta)\,, \quad a = t,r\,, \tag{73}$$

$$F_{\theta\phi}(t,r,\theta) = \sum_{0\le\ell} F_{\ell\theta\phi}(t,r)\sin\theta P_\ell(\cos\theta)\,. \tag{74}$$

Finally, polar expansion of Einstein equations (65)-(67) yields:

$$
\begin{aligned}
{}^{(i)}E_{\ell\mu\nu} &= \Delta_L\big({}^{(i)}h\big)_{\ell\mu\nu} - 8\pi^{(i)}t_{\ell\mu\nu} = {}^{(i)}S^{G}_{\ell\ \mu\nu}\,, && (75)\\
{}^{(i)}J_{\ell\nu} &= \bar{\nabla}^{\mu(i)}f_{\ell\mu\nu} - {}^{(i)}\Theta_{\ell\nu} = {}^{(i)}S^{M}_{\ell\ \nu}\,, && (76)\\
{}^{(i)}f_{\ell(\mu\nu,\alpha)} &= 0. && (77)
\end{aligned}
$$

3.2 Gauge choice and perturbations

Metric and matter perturbations transform under the gauge transformations. We have two ways of dealing with this fact: either to choose a certain gauge, or to build gauge–invariants out of the variables. In this paper, we stick to the first method and we use the gauge freedom to set $h_{\ell t\theta} = h_{\ell r\theta} = h_{\ell -} = 0$ in the polar sector and $h_{\ell\theta\phi} = 0$ in the axial sector. It corresponds to the Regge–Wheeler gauge, which we described earlier with the discussion of [1].

Let us also write down how ${}^{(i)}h$ and ${}^{(i)}f$ transform under the gauge transformation generated by a gauge vector ${}^{(i)}\zeta$. Please note that to be consistent with [4] we use ζ instead of ξ to denote a gauge vector.

$$
\begin{aligned}
{}^{(i)}h_{\ell tt} &\to {}^{(i)}h_{\ell tt} + 2\partial_t{}^{(i)}\zeta_{\ell t} - AA'^{(i)}\zeta_{\ell r}\,, && (78)\\
{}^{(i)}h_{\ell tr} &\to {}^{(i)}h_{\ell tr} + \partial_r{}^{(i)}\zeta_{\ell t} + \partial_t{}^{(i)}\zeta_{\ell r} - \frac{A'}{A}{}^{(i)}\zeta_{\ell t}\,, && (79)\\
{}^{(i)}h_{\ell t\theta} &\to {}^{(i)}h_{\ell t\theta} + \partial_t{}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell t}\,, && (80)\\
{}^{(i)}h_{\ell rr} &\to {}^{(i)}h_{\ell rr} + 2\partial_r{}^{(i)}\zeta_{\ell\theta} + \frac{A'}{A}{}^{(i)}\zeta_{\ell r}\,, && (81)\\
{}^{(i)}h_{\ell r\theta} &\to {}^{(i)}h_{\ell r\theta} + \partial_r{}^{(i)}\zeta_{\ell\theta} - \frac{2}{r}{}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell r}\,, && (82)\\
{}^{(i)}h_{\ell +} &\to {}^{(i)}h_{\ell +} + 2A\frac{{}^{(i)}\zeta_{\ell r}}{r} - \ell(\ell+1)\frac{{}^{(i)}\zeta_{\ell\theta}}{r^2}\,, && (83)\\
{}^{(i)}h_{\ell -} &\to {}^{(i)}h_{\ell -} + {}^{(i)}\zeta_{\ell\theta}\,, && (84)\\
{}^{(i)}f_{\ell t\theta} &\to {}^{(i)}f_{\ell t\theta} + \frac{AQ}{r^2}{}^{(i)}\zeta_{\ell r}\,, && (85)\\
{}^{(i)}f_{\ell r\theta} &\to {}^{(i)}f_{\ell r\theta} + \frac{Q}{Ar^2}{}^{(i)}\zeta_{\ell t}\,, && (86)\\
{}^{(i)}f_{\ell tr} &\to {}^{(i)}f_{\ell tr} + Q\partial_r\left(\frac{A}{r^2}{}^{(i)}\zeta_{\ell r}\right) - \frac{Q}{r^2A}\partial_t{}^{(i)}\zeta_{\ell t}\,, && (87)
\end{aligned}
$$

and in axial sector:

$$
\begin{aligned}
&{}^{(i)}h_{\ell t\phi} \to {}^{(i)}h_{\ell t\phi} + \partial_t {}^{(i)}\zeta_{\ell\phi}\,, && (88)\\
&{}^{(i)}h_{\ell r\phi} \to {}^{(i)}h_{\ell r\phi} + \partial_r {}^{(i)}\zeta_{\ell\phi} - 2\frac{{}^{(i)}\zeta_{\ell\phi}}{r}\,, && (89)\\
&{}^{(i)}h_{\ell\theta\phi} \to {}^{(i)}h_{\ell\theta\phi} + {}^{(i)}\zeta_{\ell\phi}\,. && (90)\\
&{}^{(i)}f_{\ell t\phi} \to {}^{(i)}f_{\ell t\phi}\,, && (91)\\
&{}^{(i)}f_{\ell r\phi} \to {}^{(i)}f_{\ell r\phi}\,, && (92)\\
&{}^{(i)}f_{\ell\theta\phi} \to {}^{(i)}f_{\ell\theta\phi}\,. && (93)
\end{aligned}
$$

3.3 Master scalar equation for $\ell \geq 2$

Now we follow an "ansatz" approach proposed by Rostworowski [4] to deal with the nonlinear perturbations of Schwarzschild and AdS spacetimes. The idea is to use the experience gained from existing work on linear perturbation equations, in particular from Regge and Wheeler [29], Zerilli [30, 35], Kodama and Ishibashi [27, 28], and, similarly to [1], assume that:

1. In each sector there exist master scalar variables corresponding to the fields that appear in a given sector. In the polar sector we have two master scalars ${}^{(i)}\Phi_\ell^{\mathcal{P}}$ and ${}^{(i)}\Psi_\ell^{\mathcal{P}}$ corresponding to the gravitational and electromagnetic field, respectively. In the axial sector we also have two master scalars ${}^{(i)}\Phi_\ell^{\mathcal{A}}$ and ${}^{(i)}\Psi_\ell^{\mathcal{A}}$ corresponding to the gravitational and electromagnetic field, respectively.

2. We assume that these master scalars fulfil systems of inhomogeneous master wave equations coupled in each sector by a symmetric potential matrix V which depend only on r:

$$
\begin{cases}
r(-\bar{\Box} + V_{G\,\ell}^{\mathcal{P}/\mathcal{A}})\frac{{}^{(i)}\Phi_\ell^{\mathcal{P}/\mathcal{A}}}{r} + V_{MG\,\ell}^{\mathcal{P}/\mathcal{A}}{}^{(i)}\Psi_\ell^{\mathcal{P}/\mathcal{A}} = {}^{(i)}\tilde{S}_{G\,\ell}^{\mathcal{P}/\mathcal{A}}\,,\\
r(-\bar{\Box} + V_{M\,\ell}^{\mathcal{P}/\mathcal{A}})\frac{{}^{(i)}\Psi_\ell^{\mathcal{P}/\mathcal{A}}}{r} + V_{MG\,\ell}^{\mathcal{P}/\mathcal{A}}{}^{(i)}\Phi_\ell^{\mathcal{P}/\mathcal{A}} = {}^{(i)}\tilde{S}_{M\,\ell}^{\mathcal{P}/\mathcal{A}}\,,
\end{cases}
\tag{94}
$$

   where the $\bar{\Box}$ symbol denotes d'Alembert operator with respect to the background and $\mathcal{P}/\mathcal{A}$ correspond to polar and axial sector.

3. We assume that the scalar sources for the wave equations ${}^{(i)}\tilde{S}_{G\,\ell}^{\mathcal{P}/\mathcal{A}}$ are given by the linear combinations of the sources for the Maxwell–Einstein equations ${}^{(i)}S_{\mu\nu}^{G}$ and ${}^{(i)}S_{\nu}^{M}$ and their derivatives.

4. We assume that the perturbations ${}^{(i)}h_{\ell\mu\nu}$ and ${}^{(i)}f_{\ell\mu\nu}$ consist of two parts. The first part are the linear combinations of master scalar functions from the corresponding sector and their derivatives and the coefficients of these combinations depend on r only. This part is a general solution of the homogeneous part of Einstein–Maxwell equations. The second part are the terms corresponding to the particular solution of the nonlinear Einstein–Maxwell equations.

Such an approach of taking the existence of master equations as an ansatz is much more efficient than starting with the full form of perturbation Einstein equations and tediously manipulating them to obtain their simpler form. However, such an ansatz approach has also a drawback - it does not provide a proof that one finds all the solutions to Einstein equations.

The linear problem of perturbations of Reissner–Nördstrom spacetime has been solved by Zerilli [35]. As a result, Zerilli provided four wave equations: two for the scalar and two for the axial sector, each sector including electromagnetic and gravitational master scalar equation (Zerilli's paper contained minor misprints corrected by [36, 37]). We assumed that the potential matrix is symmetric (so the coupling potential is the same in both equations) - and it turns out to work. However, Zerilli in his equations has non–symmetric potential matrix. This discrepancy arises from the fact that any linear combination of master scalars also fulfils a wave equation with some potentials, but if we want the matrix to be symmetric, then the choice is unique.

To make assumption 2 more precise, in the polar sector it is sufficient to assume that the homogeneous part of ${}^{(i)}h_{\mu\nu}$ is expressed by ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and it's derivatives up to the second order and by ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ without any derivatives and that the homogeneous part of ${}^{(i)}f_{\mu\nu}$ is expressed by ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$, ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ and their first derivatives. In the axial sector it is sufficient to assume that the homogeneous part of ${}^{(i)}h_{\ell\,\mu\nu}$ is expressed by ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and it's first derivatives and that the homogeneous part of ${}^{(i)}f_{\mu\nu}$ is expressed by ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ only. To write it explicitly, polar variables are assumed to have the form:

$$
\begin{aligned}
{}^{(i)}h_{\ell\,tr} &= \alpha_{rr}\partial_r^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \alpha_{tr}\partial_t\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \alpha_{tt}\partial_t^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \alpha_t\partial_t{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \alpha_r\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \\
&\quad + \alpha_0{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\alpha_{\ell}\ , && (95)\\
{}^{(i)}h_{\ell\,rr} &= \beta_{rr}\partial_r^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \beta_{tr}\partial_t\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \beta_{tt}\partial_t^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \beta_t\partial_t{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \beta_r\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \\
&\quad + \beta_0{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\beta_{\ell}\ , && (96)\\
{}^{(i)}h_{\ell\,+} &= \gamma_{rr}\partial_r^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \gamma_{tr}\partial_t\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \gamma_{tt}\partial_t^{2}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \gamma_t\partial_t{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \gamma_r\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \\
&\quad + \gamma_0{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\gamma_{\ell}\ , && (97)\\
{}^{(i)}f_{\ell\,t\theta} &= \lambda_t\partial_t{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \lambda_r\partial_r{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \lambda_0{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\lambda_{\ell}\,, && (98)\\
{}^{(i)}f_{\ell\,r\theta} &= \kappa_t\partial_t{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \kappa_r\partial_r{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \kappa_0{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\kappa_{\ell}\,, && (99)
\end{aligned}
$$

and axial variables are assumed to have the form:

$$
\begin{aligned}
{}^{(i)}h_{t\varphi} &= \sigma_t\partial_t{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + \sigma_r\partial_r{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + \sigma_0{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + {}^{(i)}\sigma_{\ell}\,, && (100)\\
{}^{(i)}h_{t\varphi} &= \chi_t\partial_t{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + \chi_r\partial_r{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + \chi_0{}^{(i)}\Phi^{\mathcal{A}}_{\ell} + {}^{(i)}\chi_{\ell}\,, && (101)\\
{}^{(i)}f_{\theta\varphi} &= \delta_0{}^{(i)}\Psi^{\mathcal{A}}_{\ell} + {}^{(i)}\delta_{\ell}\,, && (102)
\end{aligned}
$$

where coefficients near master scalars, which we call homogeneous coefficients, are functions of r. Functions ${}^{(i)}\alpha_{\ell}$, ${}^{(i)}\beta_{\ell}$, ${}^{(i)}\gamma_{\ell}$, ${}^{(i)}\lambda_{\ell}$, ${}^{(i)}\kappa_{\ell}$, ${}^{(i)}\sigma_{\ell}$, ${}^{(i)}\chi_{\ell}$ and ${}^{(i)}\delta_{\ell}$, which we call inhomogeneous coefficients, are particular solutions to nonlinear perturbation Einstein equations (they are built out of the tensor and vector sources).

We didn't expand ${}^{(i)}h_{\ell\,tt}$, ${}^{(i)}f_{\ell\,tr}$, ${}^{(i)}f_{\ell\,t\varphi}$ and ${}^{(i)}f_{\ell\,r\varphi}$, because Einstein–Maxwell equa-

tions provide simple algebraic relations for them:

$$
{}^{(i)}h_{\ell\, tt} = 4A^{(i)}S_{\ell\,-} + A^{2(i)}h_{\ell\, rr}\,, \tag{103}
$$

$$
{}^{(i)}f_{\ell\, tr} = \partial_r{}^{(i)}f_{\ell\, t\theta} - \partial_t{}^{(i)}f_{\ell\, r\theta}\,, \tag{104}
$$

$$
{}^{(i)}f_{\ell\, t\varphi} = -\frac{\partial_t{}^{(i)}f_{\ell\,\theta\varphi}}{\ell(\ell+1)}\,, \tag{105}
$$

$$
{}^{(i)}f_{\ell\, r\varphi} = -\frac{\partial_r{}^{(i)}f_{\ell\,\theta\varphi}}{\ell(\ell+1)}\,. \tag{106}
$$

The next step is to plug ansatz of the form (96) - (103) to Einstein equations. Firstly, we consider a homogeneous part of the equations only. In the homogeneous part, there are homogeneous coefficients multiplied by master scalars and their derivatives. If we use the wave equations (e.g. substitute the second order time derivatives) we end up with a system of mostly algebraic equations with a simple ordinary differential equations for the homogeneous coefficients, similarly to [1]. These results for the polar sector are the following:

$$
\begin{aligned}
V^{\mathcal{P}}_{G\,\ell} = \tau^2\hat{V}^{\mathcal{P}}_{G\,\ell} = &\frac{\tau^2\left(-r^2A'^2 - 2A\left(-2A+\ell(\ell+1)+2\right) + \ell^2(\ell+1)^2\right)}{r^2\left(rA' - 2A + \ell(\ell+1)\right)^2} + \\
&+ \frac{8Q^2\tau^2A}{r^4\left(rA' - 2A + \ell(\ell+1)\right)^2}\,,
\end{aligned} \tag{107}
$$

$$
\begin{aligned}
V^{\mathcal{P}}_{M\,\ell} = &\frac{4Q^2\left(2A\left(2r^3A' + \tau^2r^2 + 4Q^2\right) - r^4A'^2 - 4r^2A^2 + \left(\ell(\ell+1)\right)^2r^2\right)}{r^6\left(rA' - 2A + \ell(\ell+1)\right)^2} + \\
&+ \frac{-rA' + \ell(\ell+1)}{r^2}\,,
\end{aligned} \tag{108}
$$

$$
V^{\mathcal{P}}_{MG\,\ell} = \tau\hat{V}^{\mathcal{P}}_{MG\,\ell} = \frac{2\tau Q\left(2A\left(r^3A' + 4Q^2 - 2r^2\right) - r^4A'^2 + \left(\ell(\ell+1)\right)^2r^2\right)}{r^5\left(rA' - 2A + \ell(\ell+1)\right)^2}\,, \tag{109}
$$

$$
\begin{aligned}
{}^{(i)}h_{\ell\, tr} = &- r\partial_{tr}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \left(\frac{rA'}{2A} - \frac{\tau^2}{rA' - 2A + \ell(\ell+1)}\right)\partial_t{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \\
&- \frac{2\tau Q\partial_t}{r\left(rA' - 2A + \ell(\ell+1)\right)}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\alpha_{\ell}\,,
\end{aligned} \tag{110}
$$

$$
\begin{aligned}
{}^{(i)}h_{\ell\, rr} = &- r\partial_{rr}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \left(-\frac{\tau^2}{rA' - 2A + \ell(\ell+1)} - \frac{rA'}{2A}\right)\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{r}{2A}V^{\mathcal{P}}_{MG\,\ell}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \\
&+ \frac{r}{2A}\left(\frac{A'}{r} + V^{\mathcal{P}}_{G\,\ell}\right){}^{(i)}\Phi^{\mathcal{P}}_{\ell} - \frac{2\tau Q}{r\left(rA' - 2A + \ell(\ell+1)\right)}\partial_r{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\beta_{\ell}\,,
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
{}^{(i)}h_{\ell\,+} = &- A\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{\left(\frac{2A(rA'-2A+2)}{rA'-2A+\ell(\ell+1)} - \ell(\ell+1)\right)}{2r}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \\
&- \frac{2\tau QA}{r^2\left(rA' - 2A + \ell(\ell+1)\right)}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\gamma_{\ell}\,,
\end{aligned} \tag{112}
$$

$$
{}^{(i)}f_{\ell\, t\theta} = \frac{A\tau}{4}\partial_r {}^{(i)}\Psi_\ell^{\mathcal{P}} - \frac{QA}{2r}\partial_r {}^{(i)}\Phi_\ell^{\mathcal{P}} + \frac{QA}{2r^2}{}^{(i)}\Phi_\ell^{\mathcal{P}} + {}^{(i)}\lambda_\ell\,, \tag{113}
$$

$$
{}^{(i)}f_{\ell\, r\theta} = \frac{\tau}{4A}\partial_t {}^{(i)}\Psi_\ell^{\mathcal{P}} - \frac{Q}{2rA}\partial_t {}^{(i)}\Phi_\ell^{\mathcal{P}} + {}^{(i)}\kappa_\ell\,, \tag{114}
$$

where we have introduced $\tau = \sqrt{(\ell-1)(\ell+2)}$. $\hat{V}^{\mathcal{P}}_{G\,\ell}$ and $\hat{V}^{\mathcal{P}}_{MG\,\ell}$ are auxiliary potentials that are nonzero for $\ell = 1$ and will be useful later. The results for the axial sector read:

$$
V^{\mathcal{A}}_{G\,\ell} = \frac{r^2\left(A - 3rA'\right) + \left(\tau^2+1\right)r^2 - Q^2}{r^4}\,, \tag{115}
$$

$$
V^{\mathcal{A}}_{M\,\ell} = \frac{-A'r^3 + \ell(\ell+1)r^2 + 4Q^2}{r^4}\,, \tag{116}
$$

$$
V^{\mathcal{A}}_{MG\,\ell} = -\frac{2\tau Q}{r^3}\,, \tag{117}
$$

$$
{}^{(i)}h_{\ell t\phi} = A\partial_r(r\,{}^{(i)}\Phi_\ell^{\mathcal{A}}) + {}^{(i)}\sigma_\ell\,, \tag{118}
$$

$$
{}^{(i)}h_{\ell r\phi} = \frac{r}{A}\partial_t {}^{(i)}\Phi_\ell^{\mathcal{A}} + {}^{(i)}\chi_\ell\,, \tag{119}
$$

$$
{}^{(i)}f_{\ell\theta\phi} = \frac{1}{2}\ell(\ell+1)\tau\,{}^{(i)}\Psi_\ell^{\mathcal{A}} + {}^{(i)}\delta_\ell\,. \tag{120}
$$

Similarly to [1], in the limit $Q = 0$ wave equations decouple and ${}^{(i)}h_{\ell\mu\nu}$ is expressed by the gravitational master scalars only and ${}^{(i)}f_{\ell\mu\nu}$ is expressed by the electromagnetic master scalars only.

So far we have recovered Zerilli's linear results and now we want to go beyond the linear order. Firstly let's invert linear relations (111)-(115) to express master scalars in terms of metric and matter perturbations:

$$
{}^{(1)}\Phi_\ell^{\mathcal{P}} = \frac{4rA\left(r\partial_r {}^{(1)}h_{\ell+} - A^{(1)}h_{\ell rr}\right)}{\ell(\ell+1)\left(rA' - 2A + \ell(\ell+1)\right)} - \frac{2r^{(1)}h_{\ell+}}{\ell(\ell+1)}\,, \tag{121}
$$

$$
{}^{(1)}\Psi_\ell^{\mathcal{P}} = \frac{4r^2\left(\partial_r {}^{(1)}f_{\ell t\theta} - \partial_t {}^{(1)}f_{\ell r\theta}\right)}{\ell(\ell+1)\tau} + \frac{8QA\left(r\partial_r {}^{(i)}h_{\ell+} - A^{(1)}h_{\ell rr}\right)}{\ell(\ell+1)\tau\left(rA' - 2A + \ell(\ell+1)\right)}\,, \tag{122}
$$

$$
{}^{(1)}\Phi_\ell^{\mathcal{A}} = \frac{\left(r\left(\partial_r {}^{(1)}h_{\ell t\phi} - \partial_t {}^{(1)}h_{\ell r\phi}\right) - 2^{(i)}h_{\ell t\phi}\right)}{\ell(\ell+1)\tau^2 r} + \frac{4Q^{(1)}f_{\ell\theta\phi}}{\tau^2}\,, \tag{123}
$$

$$
{}^{(1)}\Psi_\ell^{\mathcal{A}} = \frac{2^{(1)}f_{\ell\theta\phi}}{\tau\ell(\ell+1)}\,. \tag{124}
$$

In higher orders the system of equations for scalar sources and for nonlinear functions ${}^{(i)}\alpha_\ell$, ${}^{(i)}\beta_\ell$, ${}^{(i)}\gamma_\ell$, ${}^{(i)}\lambda_\ell$, ${}^{(i)}\kappa_\ell$ is underdetermined and we can put one constraint in each sector on the form of the master functions. We choose to adapt the formulas (122)-(125) as the definitions of ${}^{(i)}\Phi_\ell^{\mathcal{P}}$, ${}^{(i)}\Psi_\ell^{\mathcal{P}}$, ${}^{(i)}\Phi_\ell^{\mathcal{A}}$ and ${}^{(i)}\Psi_\ell^{\mathcal{A}}$ in higher orders:

$$
{}^{(i)}\Phi_\ell^{\mathcal{P}} = \frac{4rA\left(r\partial_r {}^{(i)}h_{\ell+} - A^{(i)}h_{\ell rr}\right)}{\ell(\ell+1)\left(rA' - 2A + \ell(\ell+1)\right)} - \frac{2r^{(i)}h_{\ell+}}{\ell(\ell+1)}\,, \tag{125}
$$

$$
{}^{(i)}\Psi_\ell^{\mathcal{P}} = \frac{4r^2\left(\partial_r {}^{(i)}f_{\ell t\theta} - \partial_t {}^{(i)}f_{\ell r\theta}\right)}{\ell(\ell+1)\tau} + \frac{8QA\left(r\partial_r {}^{(i)}h_{\ell+} - A^{(i)}h_{\ell rr}\right)}{\ell(\ell+1)\tau\left(rA' - 2A + \ell(\ell+1)\right)}\,, \tag{126}
$$

$$
{}^{(i)}\Phi_\ell^{\mathcal{A}} = \frac{\left(r\left(\partial_r {}^{(i)}h_{\ell t\phi} - \partial_t {}^{(i)}h_{\ell r\phi}\right) - 2{}^{(i)}h_{\ell t\phi}\right)}{\ell(\ell+1)\tau^2 r} + \frac{4Q^{(i)}f_{\ell\theta\phi}}{\tau^2}\,, \tag{127}
$$

$$
{}^{(i)}\Psi_\ell^{\mathcal{A}} = \frac{2^{(i)}f_{\ell\theta\phi}}{\tau\ell(\ell+1)}\,. \tag{128}
$$

From these definitions we have constraints on the inhomogeneous functions:

$$
\frac{2rA\partial_r {}^{(i)}\gamma_\ell}{(rA' - 2A + \ell(\ell+1))} - \frac{2A^{2(i)}\beta_\ell}{(rA' - 2A + \ell(\ell+1))} - {}^{(i)}\gamma_\ell = 0\,, \tag{129}
$$

$$
{}^{(i)}\delta_\ell = 0\,. \tag{130}
$$

Now we plug (111)-(115) and (119)-(121) into Einstein equations. The homogeneous part containing master scalars vanishes from the equations and we are left with a system of equations containing functions ${}^{(i)}\alpha_\ell$, ${}^{(i)}\beta_\ell$, ${}^{(i)}\gamma_\ell$, ${}^{(i)}\lambda_\ell$, ${}^{(i)}\kappa_\ell$, ${}^{(i)}\sigma_\ell$, ${}^{(i)}\chi_\ell$ and ${}^{(i)}\delta_\ell$ and scalar sources ${}^{(i)}\tilde{S}^{\mathcal{P}}_{G\,\ell}$, ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\,\ell}$, ${}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell}$, ${}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell}$ on the left hand side and tensor sources ${}^{(i)}S_{\ell\,\mu\nu}$ on the right hand side. We can solve such a system and obtain the expressions for the sources and inhomogeneous functions.[1]. The sources for the polar sector are:

$$
\begin{aligned}
&{}^{(i)}\tilde{S}^{\mathcal{P}}_{G\,\ell} = \\
&-\frac{4A^2\left(\tau^2r^2+4Q^2\right){}^{(i)}S^G_{\ell\;rr}}{\ell(\ell+1)r\left(rA'-2A+\ell(\ell+1)\right)^2} + \frac{4{}^{(i)}S^G_{\ell\;tt}\left(2r^3A'-4r^2A+\left(\ell(\ell+1)+2\right)r^2-4Q^2\right)}{\ell(\ell+1)r\left(rA'-2A+\ell(\ell+1)\right)^2} + \\
&+\frac{8A\partial_r {}^{(i)}S^G_{\ell\;+}}{\ell(\ell+1)\left(rA'-2A+\ell(\ell+1)\right)} + \frac{8A{}^{(i)}S^G_{\ell\;r\theta}}{rA'-2A+\ell(\ell+1)} - \frac{4rV^{\mathcal{P}}_{G\,\ell}{}^{(i)}S^G_{\ell\;+}}{\ell(\ell+1)\tau^2} + \\
&+\frac{4{}^{(i)}S^G_{\ell\;-}\left(\frac{Q^2(8A)}{r^3(rA'-2A+\ell(\ell+1))} - A' - rV^{\mathcal{P}}_{M\,\ell}\right)}{\ell(\ell+1)} - \frac{16Q^{(i)}S^M_{\ell\;t}}{\ell(\ell+1)\left(rA'-2A+\ell(\ell+1)\right)}\,,
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
&\frac{\ell(\ell+1)}{4}\tau{}^{(i)}\tilde{S}^{\mathcal{P}}_{M\,\ell} = \frac{\ell(\ell+1)}{4}{}^{(i)}\hat{S}^{\mathcal{P}}_{M\,\ell} = \\
&r^2\partial_r {}^{(i)}S^M_{\ell\;t} - r^2\partial_t {}^{(i)}S^M_{\ell\;r} - {}^{(i)}S^M_{\ell\;t}\left(2r - \frac{8Q^2}{r\left(rA'-2A+\ell(\ell+1)\right)}\right) + \\
&+\frac{8Q\left(\frac{r^2(rA'-2A+2)}{4} - Q^2\right){}^{(i)}S^G_{\ell\;tt}}{r^2\left(rA'-2A+\ell(\ell+1)\right)^2} + \frac{8QA^{2(i)}S^G_{\ell\;rr}\left(\frac{r^2(rA'-2A+2(\ell(\ell+1)-1))}{4} + Q^2\right)}{r^2\left(rA'-2A+\ell(\ell+1)\right)^2} + \\
&+\frac{4\ell(\ell+1)QA{}^{(i)}S^G_{\ell\;r\theta}}{r\left(rA'-2A+\ell(\ell+1)\right)} + \frac{4QA\partial_r {}^{(i)}S^G_{\ell\;+}}{r\left(rA'-2A+\ell(\ell+1)\right)} + 2QA'\partial_r {}^{(i)}S^G_{\ell\;-} + \\
&-\frac{2Q^{(i)}S^G_{\ell\;-}\left(A'+rV^{\mathcal{P}}_{M\,\ell}\right)}{r} + 2QA\partial_r^{2(i)}S^G_{\ell\;-} - \frac{2Q\partial_t^{2(i)}S^G_{\ell\;-}}{A} - \frac{rV^{\mathcal{P}}_{MG\,\ell}{}^{(i)}S^G_{\ell\;+}}{\tau}\,,
\end{aligned} \tag{132}
$$

[1]The scalar sources can also be constructed in another way. Namely, once we have master scalars expressed by the perturbations of g and F, we can express the master wave equations in terms of $h_{\mu\nu}$ and $f_{\mu\nu}$. Then, we can find a linear combination of the homogeneous parts of Einstein–Maxwell equations and their derivatives that equals this wave equation. These linear combinations are exactly the sources for the wave equations (see equations (43), (44) in [4]). However, in the thesis I choose the more straightforward method.

where the ${}^{(i)}\hat{S}^{\mathcal{P}}_{M\,\ell}$ is an auxiliary source regular for $l = 1$. The sources for the axial sector are:

$$
{}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell} = \frac{2r\left(\partial_r {}^{(i)}S^G_{\ell\ t\phi} - \partial_t {}^{(i)}S^G_{\ell\ r\phi}\right)}{\tau^2}\,, \tag{133}
$$

$$
{}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell} = \frac{2{}^{(i)}S^M_{\ell\ \phi}}{\tau}\,. \tag{134}
$$

The inhomogeneous functions for the polar sector read:

$$
\begin{aligned}
{}^{(i)}\alpha_\ell = &-\frac{2r^2\left(r^2A^2{}^{(i)}S^G_{\ell\ rr} + r^2{}^{(i)}S^G_{\ell\ tt} + 2A{}^{(i)}S^G_{\ell\ +}\right)}{\ell(\ell+1)r^2\left(rA' - 2A + \ell(\ell+1)\right)} + \\
&- \frac{16Q^2A{}^{(i)}S^G_{\ell\ -}}{\ell(\ell+1)r^2\left(rA' - 2A + \ell(\ell+1)\right)}\,,
\end{aligned} \tag{135}
$$

$$
{}^{(i)}\beta_\ell = r\left(\frac{2r{}^{(i)}S^G_{\ell\ tr}}{\ell(\ell+1)} + \frac{\partial_t {}^{(i)}\alpha_\ell}{A}\right)\,, \tag{136}
$$

$$
{}^{(i)}\gamma_\ell = \frac{r\partial_r {}^{(i)}\alpha_\ell + {}^{(i)}\alpha_\ell}{A} - \frac{{}^{(i)}\alpha_\ell\left(rA' + \ell(\ell+1)\right)}{2A^2}\,, \tag{137}
$$

$$
{}^{(i)}\kappa_\ell = \frac{r^2{}^{(i)}S^M_{\ell\ r}}{\ell(\ell+1)} + \frac{2Q\partial_t {}^{(i)}S^G_{\ell\ -}}{A\ell(\ell+1)}\,, \tag{138}
$$

$$
{}^{(i)}\lambda_\ell = \frac{r^2{}^{(i)}S^M_{\ell\ t}}{\ell(\ell+1)} + \frac{2QA\partial_r {}^{(i)}S^G_{\ell\ -}}{\ell(\ell+1)}\,, \tag{139}
$$

and for the axial sector:

$$
{}^{(i)}\sigma_\ell = \frac{2r^2}{\tau^2}{}^{(i)}S^G_{\ell\ t\phi}\,, \tag{140}
$$

$$
{}^{(i)}\chi_\ell = \frac{2r^2}{\tau^2}{}^{(i)}S^G_{\ell\ r\phi}\,, \tag{141}
$$

$$
{}^{(i)}\delta_\ell = 0\,. \tag{142}
$$

3.4 Special cases: $\ell = 1$ and $\ell = 0$

Cases with $l = 0$ and $l = 1$ need special treatment - this can be immediately seen from the polar expansion of tensors, because some of them do not admit $\ell = 1$ or $\ell = 0$ expansion. Below we briefly describe how to deal with these cases.

Polar $\ell = 1$

For $\ell = 1$ we don't have $\ell = 1$ coefficient in the polar expansion of ${}^{(i)}h_-$. It means that we also loose Einstein equation ${}^{(i)}E_{\ell\,-} = 0$. However, since ${}^{(i)}h_{\ell\,-} = 0$ was our gauge condition, we can set ${}^{(i)}E_{\ell\,-} = 0$ as a new gauge condition in this case. It means that the results derived for $\ell = 2$ are applicable to the case $\ell = 1$ with one change: due to

the singularities in the source ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\,\ell}$ and potentials $V^{\mathcal{P}}_{G\ell}$, $V^{\mathcal{P}}_{MG\,\ell}$ we redefine a "matter" master scalar: ${}^{(i)}\hat{\Psi}^{\mathcal{P}}_{\ell} = \tau {}^{(i)}\Psi^{\mathcal{P}}_{\ell}$. Due to this redefinition, the set of wave equations now reads:

$$r(-\bar{\Box} + \tau^2 \hat{V}^{\mathcal{P}}_{G\,\ell})\frac{{}^{(i)}\Phi^{\mathcal{P}}_{\ell}}{r} + \hat{V}^{\mathcal{P}}_{MG\,\ell}\,{}^{(i)}\hat{\Psi}^{\mathcal{P}}_{\ell} = {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\,\ell}\,, \tag{143}$$

$$r(-\bar{\Box} + V^{\mathcal{P}}_{M\,\ell})\frac{{}^{(i)}\hat{\Psi}^{\mathcal{P}}_{\ell}}{r} + \tau^2 \hat{V}^{\mathcal{P}}_{MG\,\ell}\,{}^{(i)}\Phi^{\mathcal{P}}_{\ell} = {}^{(i)}\hat{S}^{\mathcal{P}}_{M\,\ell}\,, \tag{144}$$

where $\hat{V}^{\mathcal{P}}_{G\,\ell}$, $\hat{V}^{\mathcal{P}}_{MG\,\ell}$ and ${}^{(i)}\hat{S}^{\mathcal{P}}_{M\,\ell}$ are defined in (108), (110) and (133). What's more the gravitational master scalar ${}^{(i)}\Phi^{\mathcal{P}}_{1}$ is a pure gauge and one can get rid of it acting with a gauge transformation $\zeta_\mu = ({}^{(i)}\zeta_{1\,t}, {}^{(i)}\zeta_{1\,r}, {}^{(i)}\zeta_{1\,\theta}, 0)$, where:

$${}^{(i)}\zeta_{1\,t} = -\,\partial_t {}^{(i)}\zeta_{1\,\theta}\,, \tag{145}$$

$${}^{(i)}\zeta_{1\,r} = \frac{2\,{}^{(i)}\zeta_{1\,\theta}}{r} - \partial_r {}^{(i)}\zeta_{1\,\theta}\,, \tag{146}$$

$${}^{(i)}\zeta_{1\,\theta} = -\,\frac{r}{2}\,{}^{(i)}\Phi^{\mathcal{P}}_{1}\,. \tag{147}$$

Polar $\ell = 0$

For $\ell = 0$ in the polar sector there are no ${}^{(i)}h_{-}$, ${}^{(i)}h_{t\theta}$, ${}^{(i)}h_{r\theta}$, ${}^{(i)}f_{t\theta}$ and ${}^{(i)}f_{r\theta}$ components in the polar expansion. Using the gauge freedom we can set ${}^{(i)}h_{0\,+} = {}^{(i)}h_{0\,tr}$ and we are left with three variables only: ${}^{(i)}h_{0\,tt}$, ${}^{(i)}h_{0\,rr}$ and ${}^{(i)}f_{0\,tr}$. Equations for these variables can be integrated directly (see eq. (80)-(82) in [2]):

$$\frac{A}{r}\partial_t {}^{(i)}h_{0\,rr} = {}^{(i)}S^{G}_{0\;tr}\,, \tag{148}$$

$$\frac{A}{r}\partial_r\left(A\,{}^{(i)}h_{0\,rr} - \frac{{}^{(i)}h_{0\,tt}}{A}\right) = \frac{{}^{(i)}S^{G}_{0\;tt}}{A} + A\,{}^{(i)}S^{G}_{0\;rr}\,, \tag{149}$$

$$\partial_t\left({}^{(i)}f_{0\,tr} + \frac{Q}{2r^2}\left(\frac{{}^{(i)}h_{0\,tt}}{A} - A\,{}^{(i)}h_{0\,rr}\right)\right) = -A\,{}^{(i)}S^{M}_{0\;r}\,. \tag{150}$$

Axial $\ell = 1$

For $\ell = 1$ case there is no ${}^{(i)}h_{\ell\,\theta\varphi}$ coefficient in the polar expansion and we can use the remaining gauge freedom to set ${}^{(i)}h_{1\,r\varphi} = 0$. We are left with ${}^{(i)}h_{1\,t\varphi}$, ${}^{(i)}f_{1\,t\varphi}$, ${}^{(i)}f_{1\,r\varphi}$ and ${}^{(i)}f_{1\,\theta\varphi}$. The solution to perturbation Einstein–Maxwell equations is given by:

$${}^{(i)}f_{1\,t\phi} = -\frac{\partial_t {}^{(i)}f_{1\,\theta\phi}}{2}\,, \tag{151}$$

$${}^{(i)}f_{1\,r\phi} = -\frac{\partial_r {}^{(i)}f_{1\,\theta\phi}}{2}\,, \tag{152}$$

$${}^{(i)}f_{1\,\theta\phi} = {}^{(i)}\Psi^{\mathcal{A}}_{1} + \frac{4C_1 Q}{3r^2\,(rA' + 2A - 2)} \tag{153}$$

$$-\frac{r^2}{2A}\partial_r\left(\frac{{}^{(i)}h_{1\,t\phi}}{r^2}\right) - \frac{Q\,{}^{(i)}f_{1\,\theta\phi}}{Ar^2} + \frac{C_1}{Ar^2} = \int^{t} {}^{(i)}S^{G}_{1\,r\phi}\,dt'\,, \tag{154}$$

where C_1 is an arbitrary constant corresponding to the linearised Kerr–Newman solution. Master function ${}^{(i)}\Psi^{\mathcal{A}}_{1}$ fulfils an inhomogeneous wave equation:

$$r(-\bar{\Box} + V^{\mathcal{A}}_{M\,1})\frac{{}^{(i)}\Psi^{\mathcal{A}}_{1}}{r} = {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,1}\,, \tag{155}$$

where

$${}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,1} = 2{}^{(i)}S^{M}_{1\,\phi} - \frac{4AQ\int^{t}{}^{(i)}S^{G}_{1\,r\phi}\,dt'}{r^2}\,. \tag{156}$$

Function ${}^{(i)}\Psi^{\mathcal{A}}_{1}$ corresponds to a shifted function $\Phi^{(0)}_{0}$ from (45).

3.5 Summary

To sum up, the algorithm of solving perturbation Einstein–Maxwell equations in our setting is the following:

1. Solve wave equations for the master scalar functions (95) (in the first order master equations are homogeneous) and reconstruct metric and matter fields according to (111)-(115) and (119)-(121).

2. Move to a desired gauge, if necessary.

3. Construct sources for Einstein equations using (9) and go to the higher order.

So far, there was no general scheme of treating nonlinear perturbations in Einstein–Maxwell systems. The article [2] provides such a tool by simplifying the procedure of solving perturbation Einstein equations a lot: at each perturbation order, there are only wave equations that need to be integrated. Such a scheme might be useful e.g. for the nonlinear studies of the cosmic censorship conjecture. Unfortunately, due to a limited time of my PhD and due to other projects described in this thesis, I have not applied these results yet.

4 The problem of ultracompact rotating gravastars

4.1 Introduction

In the two previous papers, the focus was put on the formalism of perturbation methods and on providing tools that could be used to solve some problems. In the article [3] we apply the nonlinear perturbation scheme to answer a physical question: can the spacetime around the rotating gravastar be the same as around the Kerr black hole? This hope of matching the gravastar with the Kerr metric was sown by the results of Uchikata and Yoshida [38], Pani [39], Uchikata et al. [40], Posada [41]. From the first three papers [38, 39, 40] it follows that the I, Love and Q numbers of the rotating gravastar solution tend to those of Kerr as we compress the gravastar and in the limiting case they are equal. The last from this list, Posada [41], goes even further and hypothesises that his results provide a solution to the problem of the source of a slowly rotating Kerr black hole. There are also other second order perturbative constructions that were matched with the Kerr metric [42, 43, 44]. In the paper [3] we want to go beyond the second order approximation. We utilise the nonlinear perturbation scheme similar to [4, 2] and construct a rotating version of the ultracompact gravastar up to the third perturbation order and then we try to match it with a Kerr black hole.

Let's start with the description of the background metric. The original gravastar solution was proposed by Mazur and Mottola [45] as a static, spherically symmetric, two layer object. The interior layer of the gravastar has an equation of state $p = -\rho$ and the outer layer has an equation of state $p = \rho$, where p is the pressure and ρ is the density of the perfect fluid. The exterior is a Schwarzschild solution. All three patches of this solution are smoothly matched via Israel junction conditions [46]. The limiting version of this solution, later referred by other authors as the ultracompact gravastar, was introduced by Visser and Wiltshire [47], Mazur and Mottola [48]. The ultracompact gravastar is the de Sitter solution matched to the Schwarzschild spacetime via singular shell located at $r = 2M$. The metric of this solution is given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right) , \tag{157}$$

where instead of usual angle θ we use $u = \cos\theta$ and:

$$f(r) = \begin{cases} \frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) & r \leq 2M , \\ 1 - \frac{2M}{r} & r > 2M , \end{cases} \tag{158}$$

$$h(r) = \begin{cases} 1 - \frac{r^2}{4M^2} & r \leq 2M , \\ 1 - \frac{2M}{r} & r > 2M . \end{cases} \tag{159}$$

This solution can be obtained as the limiting case of the Schwarzschild constant density

star [9], given by:

$$f(r) = \begin{cases} \frac{1}{4}\left(\sqrt{1-\frac{2Mr^2}{R^3}} - 3\sqrt{1-\frac{2M}{R}}\right)^2 & r \leq R\,, \\ 1-\frac{2M}{r} & r > R\,, \end{cases} \tag{160}$$

$$h(r) = \begin{cases} 1-\frac{2Mr^2}{R^3} & r \leq R\,, \\ 1-\frac{2M}{r} & r > R\,, \end{cases} \tag{161}$$

$$\rho(r) = \begin{cases} \frac{M}{\frac{4}{3}\pi R^3} & r \leq R\,, \\ 0 & r > R\,, \end{cases} \tag{162}$$

$$p(r) = \begin{cases} \frac{3MR^3\left(1-\sqrt{(1-\frac{2M}{R})(1-\frac{2Mr^2}{R^3})}\right)+3M^2\left(r^2-3R^2\right)}{4\pi R^3(Mr^2+R^2(4R-9M))} & r \leq R\,, \\ 0 & r > R\,, \end{cases} \tag{163}$$

where R is the junction radius. Israel junction conditions for the static, spherically symmetric perfect fluid configurations reduce to the continuity of the metric (1st junction condition) and vanishing pressure on the matching surface (2nd junction condition). When $R > \frac{9}{4}M$, the pressure of the solution is regular, but for $R = \frac{9}{4}M$ the pressure in the centre becomes infinite (the limiting value of $R = \frac{9}{4}M$ is called the Buchdahl limit [49, 50]). For solutions with $2M < R < \frac{9}{4}M$, as we squeeze the star, the pressure singularity moves away from the center of the solution towards the matching radius R and the singularity radius is given by $r_{sing} = \frac{R\sqrt{9M-4R}}{\sqrt{M}}$. Finally, for $R = 2M$, the pressure becomes finite again and the interior equation of state becomes $p + \rho = 0$, therefore it becomes a de Sitter metric with a cosmological constant $\Lambda = \frac{3}{4M^2}$. However, in this limiting case, the pressure is no longer vanishing on the boundary and the matching surface located at $R = 2M$ does not fulfil second junction condition. It means that it produces a nonzero contribution to the energy–momentum tensor. Such a shell possess a δ-like singularity in the transverse pressure, but this singularity produces a finite contribution to the Komar mass–energy integral. What's more, for $R = 2M$ the matching surface is null and one needs to use junction conditions for the null surfaces to determine it's energy–momentum tensor (see [48] for the details).

When I speak about the matching of the rotating gravastar with the Kerr metric I always mean matching using the first junction condition (continuity of the metric), since already in the zeroth order the second junction condition is not fulfilled. If the first junction condition would succeed, then calculating the energy–momentum tensor of the shell would make sense. However, since, as I will show below, even the first junction condition cannot be fulfilled for matching gravastar with Kerr, there is no point in using the second junction condition at all.

Due to the fact that the standard Schwarzschild coordinates are singular at $r = 2M$, we prefer to use the ingoing Eddington–Finkelstein (EF) coordinates (v, r, u, φ), where $v = t + r + 2M\ln(\frac{r-2M}{2M})$. The interior metric in EF coordinates reads:

$$\bar{g} = \frac{1}{4}\left(1-\frac{r^2}{4M^2}\right)dv^2 + drdv + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right)\,, \tag{164}$$

and the exterior metric in EF coordinates reads:

$$\bar{g} = \left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 \left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right) . \tag{165}$$

4.2 Setup

Perturbation expansion and gauge choice

In the paper [3] we follow the same scheme as in [2, 4], therefore I do not introduce the same formalism again here. However, we use the convention compatible with Bruni et al. [6] and we expand the metric in series with the $i!$ coefficient in the denominator:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{\infty} \frac{a^i}{i!} {}^{(i)}h_{\mu\nu} , \tag{166}$$

where a stands for the perturbation expansion parameter (it will turn out to be the a parameter of the Kerr metric). As well as in [2] we use the Regge–Wheeler gauge. Due to the use of u rather than θ, the polar expansion needs to be transformed (see Eq. (6)-(11) in [3]).

General gauge transformations

Gauge transformations were already discussed in the previous papers [1, 2], but always in a context of transformations within a given perturbation order. Strictly speaking, we were so far considering the impact of acting with a gauge vector of order i ${}^{(i)}\xi$ on the metric of order i ${}^{(i)}h$. Such a discussion was sufficient, because we were assuming to solve perturbation Einstein equations order by order, fixing a gauge at every order and not moving back. However, if one has a solution up to order $i > j$, then changing a gauge in order j would also change the solution up to the order i. The question how to deal with an impact of lower order gauge transformations on higher order metric perturbations was answered by Bruni et al. [6] and we have used their result for our needs.

Why would we care about such gauge transformations? Imagine that we solve Einstein equations for the interior metric up to the third order and we want to match it with the exterior metric. Matching two spacetimes is in fact searching for the gauge in which the spacetimes fulfil junction conditions. If the matching conditions were not fulfilled in the first order, we would gauge–transform the first order metric to perform the matching, but such a gauge transformation would affect the solutions of order 2 and 3. This is exactly the case in our study: to be absolutely sure that we use the whole gauge freedom to match the interior with the exterior, we consider general first, second and third order gauge transformations and their impact on the metric.

Let's assume that we act with a gauge transformation $\xi = \sum_{i=0}^{\infty} \frac{a^i}{i!} {}^{(i)}\xi$ on the metric

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{\infty} \frac{a^i}{i!} {}^{(i)}h_{\mu\nu}$. According to [6], the metric transforms as:

$$\sum_{i=1}^{\infty} \frac{a^i}{i!} {}^{(i)}h_{\mu\nu} \to \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \cdots$$
$$\cdots \frac{a^{l_1+2l_2+\ldots+kl_k+\ldots}}{(2!)^{l_2} \cdots (k!)^{l_k} \cdots l_1! l_2! \cdots l_k! \cdots} \pounds_{{}^{(1)}\xi}^{l_1} \pounds_{{}^{(2)}\xi}^{l_2} \cdots \pounds_{{}^{(k)}\xi}^{l_k} \cdots \left(\sum_{i=1}^{\infty} \frac{a^i}{i!} {}^{(i)}h_{\mu\nu} \right) , \tag{167}$$

where $\pounds_{{}^{(i)}\xi}$ denotes a Lie derivative with respect to the vector ${}^{(i)}\xi$. Explicit form of these transformations up to the third order is:

$${}^{(1)}h_{\mu\nu} \to {}^{(1)}h_{\mu\nu} + \pounds_{{}^{(1)}\xi} \bar{g}_{\mu\nu} , \tag{168}$$
$${}^{(2)}h_{\mu\nu} \to {}^{(2)}h_{\mu\nu} + (\pounds_{{}^{(2)}\xi} + \pounds_{{}^{(1)}\xi}^2) \bar{g}_{\mu\nu} + 2\pounds_{{}^{(1)}\xi} {}^{(1)}h_{\mu\nu} , \tag{169}$$
$${}^{(3)}h_{\mu\nu} \to {}^{(3)}h_{\mu\nu} + (\pounds_{{}^{(1)}\xi}^3 + 3\pounds_{{}^{(1)}\xi} \pounds_{{}^{(2)}\xi} + \pounds_{{}^{(3)}\xi}) \bar{g}_{\mu\nu} + 3(\pounds_{{}^{(1)}\xi}^2 + \pounds_{{}^{(2)}\xi}) {}^{(1)}h_{\mu\nu} + $$
$$+ 3\pounds_{{}^{(1)}\xi} {}^{(2)}h_{\mu\nu} . \tag{170}$$

Although these formulas seem obscure, they are extremely useful - they allow to have full control over the gauge even after solving Einstein equations. The linear version of these transformations (what means acting with a gauge vector of order i on a metric perturbation of order i) is given by:

$${}^{(i)}h_{\ell v\varphi} \to {}^{(i)}h_{\ell v\varphi} - {}^{(i)}\dot{\xi}_{\varphi} , \tag{171}$$
$${}^{(i)}h_{\ell r\varphi} \to {}^{(i)}h_{\ell r\varphi} + \frac{2{}^{(i)}\xi_{\varphi}}{r} - {}^{(i)}\xi'_{\varphi} , \tag{172}$$
$${}^{(i)}h_{\ell u\varphi} \to {}^{(i)}h_{\ell u\varphi} + {}^{(i)}\xi_{\varphi} , \tag{173}$$
$${}^{(i)}h_{\ell vv} \to {}^{(i)}h_{\ell vv} - \frac{1}{4} \left(f {}^{(i)}\xi_r + 2\xi_v \right) f' + 2{}^{(i)}\dot{\xi}_v , \tag{174}$$
$${}^{(i)}h_{\ell vr} \to {}^{(i)}h_{\ell vr} + \frac{1}{2} f' {}^{(i)}\xi_r + \xi'_v + {}^{(i)}\dot{\xi}_r , \tag{175}$$
$${}^{(i)}h_{\ell rr} \to {}^{(i)}h_{\ell rr} + 2{}^{(i)}\xi'_r , \tag{176}$$
$${}^{(i)}h_{\ell +} \to {}^{(i)}h_{\ell +} + 2r f {}^{(i)}\xi_r - \ell(\ell+1) {}^{(i)}\xi_u + 4r {}^{(i)}\xi_v , \tag{177}$$
$${}^{(i)}h_{\ell -} \to {}^{(i)}h_{\ell -} - {}^{(i)}\xi_u , \tag{178}$$
$${}^{(i)}h_{\ell vu} \to {}^{(i)}h_{\ell vu} - {}^{(i)}\xi_v - {}^{(i)}\dot{\xi}_u , \tag{179}$$
$${}^{(i)}h_{\ell ru} \to {}^{(i)}h_{\ell ru} - {}^{(i)}\xi_r + \frac{2}{r} {}^{(i)}\xi_u - {}^{(i)}\xi'_u , \tag{180}$$

dots and primes correspond to derivatives with respect to v and r, respectively.

Matching

In the previous paragraphs I was referring to the Israel junction conditions and I argued that in this case it is sufficient to focus only on the first junction condition,

which I will now introduce. Let's assume that we have two spacetimes, which we call interior spacetime with coordinates $x^{-\mu} = (v^-, r^-, u^-, \varphi^-)$ and exterior spacetime with coordinates $x^{+\mu} = (v^+, r^+, u^+, \varphi^+)$. We want to match them via 3-dimensional hypersurface Σ given by a condition $r^\pm = r_b^\pm(u^\pm)$. The first Israel junction condition [46, 51] is given by:

$$[[\mathfrak{g}_{ab}^\pm]] = 0\,, \tag{181}$$

where $[[E]] = E^+(r_b^+) - E^-(r_b^-)$ and $\mathfrak{g}$ is a metric induced on the matching hypersurface. Now, since the hypersurface Σ is common for the two spacetimes, we introduce intrinsic coordinates on Σ: $y^a = (V, U, \Phi)$. Then we express interior and exterior coordinates on the hypersurface Σ by y^a (Eq. (19) in [3]):

$$x^{-\mu}\big|_{r_b^-} = \left(A^- V, r_b^-(U), F^-(U), \Phi\right)\,, \tag{182}$$

$$x^{+\mu}\big|_{r_b^+} = \left(A^+ V, r_b^+(U), F^+(U), \Phi\right)\,, \tag{183}$$

where $r_b^\pm(U) = 2M + \frac{a^2}{M^2}\eta^\pm(U) + \mathcal{O}(a^4)$, $F^\pm(U) = U + \frac{a^2}{M^2}\lambda^\pm(U) + \mathcal{O}(a^4)$ and we polar–expand $\eta^\pm$ into $\eta^\pm(U) = \eta_0^\pm + \eta_2^\pm P_2(U)$.

This allows us to write the metric induced on the matching hypersurface for both interior and exterior and have it expressed in coordinates y^a:

$$\mathfrak{g}_{VV} = \left(A^\pm\right)^2 g_{vv}^\pm\,, \tag{184}$$

$$\mathfrak{g}_{VU} = A^\pm g_{vr}^\pm r_b^{\pm\prime}(U) + A^\pm g_{vu}^\pm F^{\pm\prime}(U)\,, \tag{185}$$

$$\mathfrak{g}_{V\Phi} = A^\pm g_{v\varphi}^\pm\,, \tag{186}$$

$$\mathfrak{g}_{UU} = \left(F^{\pm\prime}(U)\right)^2 g_{uu}^\pm + \left(r_b^{\pm\prime}(U)\right)^2 g_{rr}^\pm + 2F^{\pm\prime}(U) r_b^{\pm\prime}(U) g_{ru}^\pm, \tag{187}$$

$$\mathfrak{g}_{U\Phi} = F^{\pm\prime}(U) g_{u\varphi}^\pm + r_b^{\pm\prime}(U) g_{r\varphi}^\pm\,, \tag{188}$$

$$\mathfrak{g}_{\Phi\Phi} = g_{\varphi\varphi}^\pm\,. \tag{189}$$

We are free to choose $F^+(U) = U$ and $A^+ = 1$ (see e.g. [38]), we also denote $A^- = A$.

The matching procedure described in section V of [3] consists of three steps:

1. Solving perturbation Einstein equations for the interior and choosing a solution which is regular at $r = 0$ and at $r = 2M$.

2. Trying to match the interior with the exterior. To do this, we act with a general gauge transformation (168) up to the third order and try solve matching conditions (182).

3. Going to the higher perturbation order, if the matching is successful.

Kerr metric expansion

Since we try to match the rotating gravastar with the Kerr metric, we have to expand the latter. We take the Kerr metric in EF coordinates, so that in the zeroth order it

is exactly (166). What's more, we move with the expanded Kerr to the RW gauge. Although this step is not necessary, it simplifies formulas. The nonzero coefficients of the Kerr metric in RW gauge up to the third perturbation order read:

$$
\begin{aligned}
{}^{(1)}h^{+}_{1,\,v\varphi} &= -\frac{2M}{r}\,, & {}^{(2)}h^{+}_{2,\,+} &= -\frac{4M(2M+r)}{r^2}\,,\\
{}^{(2)}h^{+}_{0,\,vv} &= \frac{4M^2}{3r^4}\,, & {}^{(3)}h^{+}_{1,\,v\varphi} &= \frac{24M^3}{5r^5}\,,\\
{}^{(2)}h^{+}_{2,\,vv} &= \frac{4M\left(6M^2-Mr-3r^2\right)}{3r^5}\,, & {}^{(3)}h^{+}_{3,\,v\varphi} &= \frac{4M\left(-6M^2+5Mr+5r^2\right)}{5r^5}\,,\\
{}^{(2)}h^{+}_{2,\,vr} &= \frac{4M(M+r)}{r^4}\,, & {}^{(3)}h^{+}_{3,\,r\varphi} &= -\frac{4M(9M+5r)}{5r^4}\,.\\
{}^{(2)}h^{+}_{2,\,rr} &= -\frac{8M}{r^3}\,,
\end{aligned}
\tag{190}
$$

4.3 Seeking for the regular interior solution

Einstein equations

We assume that with the rotation the equation of state of the gravastar does not change and we take:

$$p+\rho=0\,. \tag{191}$$

In the article [3], we provided the results for the metric perturbations without the perturbations of p and ρ and in the summary we only mentioned that allowing for the perturbations of p and ρ does not change anything for the results. Here we provide the full calculation including perturbations of p and ρ as well (within the equation of state (192)). Einstein equations take the Λ-vacuum form:

$$G_{\mu\nu}=8\pi\rho g_{\mu\nu}\,, \tag{192}$$

and the density is also expanded in series of a:

$$\rho=\rho_0+\sum_{i=1}^{\infty}\frac{a^{2i}}{i!}{}^{(i)}\delta\rho\,. \tag{193}$$

Then we follow the standard procedure described in the introduction to this thesis (equations (1)-(9)). For the purpose of examining the problem of matching gravastar with Kerr, it is sufficient to solve Einstein equations up to the third order. They can be directly integrated and there is no need for the master scalar formalism described for the Einstein–Maxwell case. However, to show that the solution we take is regular, we need to solve Einstein equations up to the 6th perturbation order. Direct integration of Einstein equations up to the 6th perturbation order is too complex and the master scalar formalism is really helpful. We provide the solution up to the 6th order and the Kreschmann scalar expansion in a Mathematica file [52]. We examine the Kretschmann

scalar in Appendix A and we disregard the parts of the solution that produce singularities in the expansion coefficients of the Kretschmann scalar.

Regular interior solution

According to the Appendix A, a regular interior solution up to the 3rd perturbation order reads:

$$g_{vv} = -\frac{1}{4}\left(1 - \frac{r^2}{M^2}\right) + a^2\left(\frac{c_{20}}{32M^2} - \frac{2}{3}r^2\Omega_{11}^2 P_2(u) + \frac{2}{3}r^2\left(\pi^{(2)}\delta\rho + \Omega_{11}^2\right)\right), \quad (194)$$

$$g_{vr} = \frac{1}{2}, \quad (195)$$

$$g_{+} = \frac{1}{8}r^2\left(\frac{a^2 c_{20}}{M^2} + 8\right), \quad (196)$$

$$g_{v\varphi} = -\frac{1}{6}a^3 r^2\left(1 - u^2\right)\Omega_{31} - ar^2\left(1 - u^2\right)\Omega_{11}, \quad (197)$$

where I have listed only nonzero terms. P_2 is the Legendre polynomial of order 2 and c_{20}, ${}^{(2)}\delta\rho$, Ω_{11} and Ω_{31} are arbitrary constants.

4.4 Matching ultracompact gravastar with Kerr

Before trying to match the interior solution with the exterior solution, we act on the interior with a gauge transformation up to the 3rd order:

$$\xi = a\,{}^{(1)}\xi + \frac{a^2}{2}{}^{(2)}\xi + \frac{a^3}{3!}{}^{(3)}\xi, \quad (198)$$

and we polar-expand gauge components:

$${}^{(1)}\xi_{\varphi}(v,r) = (1-u^2)\,{}^{(1)}\xi_{1\,\varphi}(v,r), \quad (199)$$

$${}^{(2)}\xi_{v}(v,r) = {}^{(2)}\xi_{0\,t}(v,r) + {}^{(2)}\xi_{2\,t}(v,r)P_2(u), \quad (200)$$

$${}^{(2)}\xi_{r}(v,r) = {}^{(2)}\xi_{0\,r}(v,r) + {}^{(2)}\xi_{2\,r}(v,r)P_2(u), \quad (201)$$

$${}^{(2)}\xi_{u}(v,r) = {}^{(2)}\xi_{2\,u}(v,r)P_2'(u), \quad (202)$$

$${}^{(3)}\xi_{\varphi}(v,r) = (1-u^2)\,{}^{(1)}\xi_{1\,\varphi}(v,r) + (1-u^2)\,{}^{(1)}\xi_{1\,\varphi}(v,r)P_3'(u). \quad (203)$$

Where $P_\ell(u)$ denotes the ℓth Legendre polynomial of u. I emphasise the dependence of gauge on v, because using v-dependent gauge transformations can still result in v-independent metric after the transformation.

Matching procedure reduces to solving the junction conditions (182) for:

1. $\eta^{\pm}$, λ^{-} and A responsible for the relation between coordinates on the matching hypersurface.

2. Constants arising from Einstein equations - the only free constants we have are Ω_{11}, c_{20}, Ω_{31} and ${}^{(2)}\delta\rho$.

3. Gauge functions ${}^{(i)}\xi_{\ell\,\mu}(v,r)$. If the gauge was not earlier settled to RW, these functions would be crucial for the matching. However, since both interior and exterior are expressed in the RW gauge, these functions won't play a crucial role in matching. Nevertheless, to be sure that we do not miss any possibility to match, we keep them in our equations.

The next step is expanding metric induced on the matching hypersurface $\mathfrak{g}^{\pm}_{ab}$ into series in a and solving equations for the above unknowns order by order.

1st order

We want to keep interior metric to be independent of v. A gauge transformation ensuring such independence is given by (see (172)-(181)):

$$ {}^{(1)}\xi_{1\,\varphi} = q_{11} v r^2 + {}^{(1)}\gamma_{1\varphi}(r)\,, \tag{204} $$

where q_{11} is an arbitrary constant and ${}^{(1)}\gamma_{1\varphi}(r)$ is an arbitrary function of r. From the expansion of (182) in the first order in a we obtain:

$$ \frac{{}^{(1)}h^{+}_{1\,v\varphi}(2M)}{A} - {}^{(1)}h^{-}_{1\,v\varphi}(2M) = -\partial_v {}^{(1)}\xi_{1\,\varphi}(v,2M)\,, \tag{205} $$

what, together with (205) yields:

$$ \Omega_{11} = -\frac{1}{4AM^2} + q_{11}\,. \tag{206} $$

2nd order

In the second order, again, we use a gauge transformation that guarantees the v-independence of metric:

$$ {}^{(2)}\xi_{0\,v} = -4M^2 f q_{20} v + {}^{(2)}\gamma_{0\,v}(r)\,, \tag{207} $$

$$ {}^{(2)}\xi_{0\,r} = 8M^2 q_{20} v + {}^{(2)}\gamma_{0\,r}(r)\,, \tag{208} $$

$$ {}^{(2)}\xi_{2\,v} = {}^{(2)}\gamma_{2\,v}(r)\,, \tag{209} $$

$$ {}^{(2)}\xi_{2\,r} = {}^{(2)}\gamma_{2\,r}(r)\,, \tag{210} $$

$$ {}^{(2)}\xi_{2\,u} = {}^{(2)}\gamma_{2\,u}(r)\,, \tag{211} $$

where q_{20} is an arbitrary constant and ${}^{(2)}\gamma_{\ell\,\mu}(r)$ are arbitrary functions of r. From the expansion of (182) in the second order in a we obtain:

$$ \begin{aligned} {}^{(2)}h^{+}_{0\,vv}(2M) - A^{2\,(2)}h^{-}_{0\,vv}(2M) &= \frac{A^2\eta_0^- + 2\eta_0^+}{2M^3} + \frac{16}{3}A^2M^2 q_{11}\left(q_{11} - 2\Omega_{11}\right) + \\ &+ \frac{A^2}{2M}{}^{(2)}\gamma_{0\,v}(2M)\,, \end{aligned} \tag{212} $$

$$ \begin{aligned} {}^{(2)}h^{+}_{2\,vv}(2M) - A^{2\,(2)}h^{-}_{2\,vv}(2M) &= \frac{A^2\eta_2^- + 2\eta_2^+}{2M^3} - \frac{16}{3}A^2M^2 q_{11}\left(q_{11} - 2\Omega_{11}\right) + \\ &+ \frac{A^2}{2M}{}^{(2)}\gamma_{2\,v}(2M)\,, \end{aligned} \tag{213} $$

$$2\eta_2^+ - \eta_2^- A = AM^{2\,(2)}\gamma_{2\,v}(2M)\,, \tag{214}$$

$$[[{}^{(2)}h_{0\,+}(2M)]] = -\frac{8(\eta_0^+ - \eta_0^-)}{M} + 8\lambda'(U) + 8M^{(2)}\gamma_{0\,v}(2M)\,, \tag{215}$$

$$[[{}^{(2)}h_{2\,+}(2M)]] = -\frac{8(\eta_2^+ - \eta_2^-)}{M} + 8M^{(2)}\gamma_{2\,v}(2M) - 6^{(2)}\gamma_{2\,u}(2M)\,, \tag{216}$$

$$[[{}^{(2)}h_{2\,-}(2M)]] = {}^{(2)}\gamma_{2\,u}(2M) + \frac{16U\lambda(U) + 8\left(1-U^2\right)\lambda'(U)}{3\left(U^2-1\right)^2}\,, \tag{217}$$

what leads to:

$$\eta_0^- = -M^{2\,(2)}\gamma_{0\,v}(2M) - \frac{M}{8}c_{20} - \frac{M}{6} - \frac{32}{9}\pi M^{5\,(2)}\delta\rho\,, \tag{218}$$

$$\eta_2^- = -\frac{M}{3} - M^{2\,(2)}\gamma_{2\,v}(2M)\,, \tag{219}$$

$$\eta_0^+ = -\frac{M}{6} - \frac{32}{9}\pi M^{5\,(2)}\delta\rho\,, \tag{220}$$

$$\eta_2^+ = \frac{M}{6}\,, \tag{221}$$

$$A = -1\,, \tag{222}$$

$$\lambda(U) = 0\,, \tag{223}$$

$${}^{(2)}\gamma_{2\,u}(2M) = 0\,. \tag{224}$$

3rd order

In the third order, the conditions for the v-independence of the metric are:

$${}^{(3)}\xi_{1\,\varphi} = q_{31}r^2 v + {}^{(3)}\gamma_{1\,\varphi}(r)\,, \tag{225}$$

$${}^{(3)}\xi_{3\,\varphi} = {}^{(3)}\gamma_{3\,\varphi}(r)\,, \tag{226}$$

where q_{31} is an arbitrary constant and ${}^{(3)}\gamma_{\ell\,\mu}(r)$ are arbitrary functions of r. From the expansion of (182) in the third order in a we obtain the conditions:

$$\begin{aligned}{}^{(3)}h^+_{1\,v\varphi}(2M) - A\,{}^{(3)}h^-_{1\,v\varphi}(2M) =& 3q_{11}\left(c_{20} - 64M^4 q_{20}\right) + \frac{3\left(5c_{20}+8\right)}{20M^2} + \\ &+ 4M^2\left(q_{31} - 12q_{20}\right) + 32\pi M^{2\,(2)}\delta\rho\,,\end{aligned} \tag{227}$$

$${}^{(3)}h^+_{3\,v\varphi}(2M) - A\,{}^{(3)}h^-_{3\,v\varphi}(2M) = \frac{3}{10M^2}\,, \tag{228}$$

$$\begin{aligned}5M^{2\,(3)}\xi_{3,\varphi}(2M) =& 6^{(2)}\gamma_{2\,r}(2M)\left(4M^2 q_{11} + 1\right) + \\ &+ 2\left(3M^{(2)}\gamma_{2\,v}(2M) + 1\right)\left(M^{(1)}\gamma'_{1\,\varphi}(2M) - {}^{(1)}\gamma_{1\,\varphi}(2M)\right)\,.\end{aligned} \tag{229}$$

The condition (228) can be fulfilled by setting

$$\Omega_{31} = \frac{3\left(4M^2 q_{11} + 1\right)\left(c_{20} - 64M^4 q_{20}\right)}{16M^4} + q_{31} + 8\pi^{(2)}\delta\rho\,, \tag{230}$$

and (230) can be fulfilled by the appropriate choice of ${}^{(i)}\gamma_{\ell\mu}$ (e.g. setting them to zero). Condition (229) leads to a contradiction and cannot be fulfilled - we do not have any free parameters in ${}^{(3)}h_{3\,v\varphi}$ and ${}^{(3)}h_{3\,v\varphi}$. This fact leads to the conclusion that within our assumptions gravastar cannot be matched with Kerr spacetime.

4.5 Discussion of the interior solution and matching surface

The solution (195)-(198) is an exact solution to Einstein equations. It turns out that it is just a de Sitter spacetime in rotating coordinates. Let's act on the solution (195)-(198) with the following gauge transformation:

$$
\begin{aligned}
{}^{(1)}\xi_1 &= \left(0,0,0,r^2\Omega_{11}v\right)\,, && (231)\\
{}^{(2)}\xi_0 &= \left(-\frac{c_{20}r}{16M^2}+\frac{c_{20}\left(r^2-4M^2\right)}{128M^4}v,\frac{c_{20}v}{16M^2},0,0\right)\,, && (232)\\
{}^{(2)}\xi_2 &= (0,0,0,0)\,, && (233)\\
{}^{(3)}\xi_1 &= \left(0,0,0,(r^2\Omega_{13}-\frac{3c_{20}r^2\Omega_{11}}{8M^2})v\right)\,, && (234)\\
{}^{(3)}\xi_3 &= (0,0,0,0)\,. && (235)
\end{aligned}
$$

Using formulas (169)-(171), we obtain the transformed metric:

$$
\begin{aligned}
g'_{vv} &= -\frac{1}{4}\left(1-\frac{r^2}{4M^2}-\frac{8}{3}\pi a^2 r^{2\,(2)}\delta\rho\right)\,, && (236)\\
g'_{vr} &= \frac{1}{2}\,, && (237)\\
g'_{+} &= r^2\,, && (238)\\
g'_{v\varphi} &= 0\,, && (239)
\end{aligned}
$$

what is just the de Sitter metric with cosmological constant term $\Lambda = \frac{3}{4M^2} + 8\pi^{(2)}\delta\rho a^2$.

We can also ask where is the surface that we match across up to the second order. Due to the transformations of the Kerr metric in EF coordinates to RW form, we have changed the r coordinate. Because of that the coordinate radius of the outer Kerr horizon changed from $r_H \simeq 2M - \frac{a^2}{2M}$ to $r_H^{RW} \simeq 2M - \frac{a^2}{4M}$. The location of the matching hypersurface in the exterior coordinates is given by $r_b^+(U) = 2M + \frac{a^2}{M^2}(\eta_0^+ + \eta_2^+ P_2(U))$. The coordinate distance between r_b^+ and r_H^{RW} is given by ${}^{(2)}\delta r = r_b^+ - r_H^{RW} = \frac{a^2\left(9u^2-128\pi^{(2)}\delta\rho M^4\right)}{36M}$. It means that for ${}^{(2)}\delta\rho = 0$ the matching is performed above the horizon radius apart from $u = 0$ (corresponding to the equator $\theta = \pi/2$) where it touches the horizon. For ${}^{(2)}\delta\rho > \frac{9}{128\pi M^4}$ the matching is performed under the horizon of a Kerr black hole. Due to the dependence of r_b^+ on u there is no possibility to match exactly on the Kerr horizon.

4.6 Summary and discussion

Searching for the material source of the Kerr metric is a long standing problem in General Relativity. Articles [38, 39, 40, 41] brought a hope that a rotating ultracompact

gravastar can be an example of such a source. One may argue that such a source would not be a legitimate example for at least three reasons:

1. The interior solution is an exotic negative pressure fluid.

2. Only the first junction condition would be fulfilled, the second would only provide a singular energy–momentum tensor of the matching surface.

3. The matching is performed extremely close to the horizon and in the zeroth order on the horizon.

On the other hand, static gravastars were proposed as an alternative to a Schwarzschild black hole [45, 48]. Thus, one can treat our study as an attempt to find the gravastar–like alternative to a Kerr black hole rather than as an attempt to construct a physically plausible star–like object that would serve as a source for the Kerr black hole. Nevertheless, we show that such a matching between rotating ultracompact gravastar and Kerr metric is not likely to be possible. Now I want to provide some remarks on our results and compare it to the other authors' results:

1. We have assumed a specific equation of state of the gravastar - one may argue, that with other equation of state matching would be possible. Some authors, e.g. [53, 54] claim that to match gravastar with Kerr, one should include a vortex solution inside.

2. The interior solution was chosen to be regular at $r = 0$ and $r = 2M$. To ensure this regularity, we calculated the Kretschmann scalar at $r = 0$ and $r = 2M$ and put to zero terms that produced singularities in it's Taylor expansion. However, it may happen that the singularities in the expansion are artefacts of the perturbative expansion ([53]). As an example we can take a function $f(r) = \frac{a^2}{r^2+a^2}$ that is regular at $r = 0$ for all real values of a, but it's Taylor expansion around $a = 0$ produces terms singular at $r = 0$. Unfortunately, we do not know a way to deal with this obstacle and it seems to be an inherent flaw of the perturbative expansion (all authors that we are aware of dismiss terms which are singular - with one exception of [53], but we discuss this paper later). This possibility leads to the following statements: if there exist a rotating gravastar solution that is continuously matched to the Kerr black hole, perturbative approach is not a right way to seek for this solution.

3. We would like to emphasise the need of going to higher orders with the known perturbative sources of the Kerr black hole, e.g. [43, 42] to verify if the matching survives. As we have found out, the system of equations arising from the first matching condition up to the second perturbation order is well-determined, what means that one could match almost anything to anything up to the second order. Only in the third order one has real conditions for the metric and if they are not compatible, the matching cannot be performed.

4. Author of [41] base the conclusion that rotating gravastar may be a source for the Kerr black hole on the limiting values of the multipole moments of the solution

he obtains. It seems that this claim is not fully justified. There is no proof that the equality of certain multipole moments between two spacetimes means that these are the same spacetimes. In a recent paper [55] authors provide examples of "mimickers" of the Kerr black hole - solutions which have the same value of some multipole moments as the Kerr black hole, but they are different spacetimes.

5. Recently there appeared an interesting article [53] that tackles exactly the same problem as our work [3]. The conclusion of [53] is that it's possible to match the rotating gravastar with the Kerr black hole on the horizon of the Kerr black hole up to the second perturbation order. However, from our calculations it follows that such a matching is not possible. This discrepancy comes from the fact that authors of [53] allow for the perturbations which are singular at $r = 0$, what introduces one more free constant, which can be used to perform the matching exactly on the horizon. The justification for such a scenario is the possibility that the singularities they have in perturbations are not real (see comment 2 above). I see downturns of such an approach. First of all, authors of [53] do not provide convincing arguments that the singularity they produce is not a real curvature singularity. What's more, by allowing for singular terms one would be able to match anything with almost anything, since a lot of freedom in the solutions would appear (but probably this freedom would be associated with curvature singularities in the solution).

5 Final remarks on the thesis

To sum up, in this thesis I presented the results of my studies on both linear and nonlinear perturbations of exact solutions to Einstein equations. In the two papers [1, 2] the focus was put on the formalism - in [1], together with A. Jansen and A. Rostworowski, we provided an extension of the Kodama–Ishibashi equations to the Eisnstein–Maxwell–scalar case and in [2] I provided the nonlinear extension of the Zerilli master equations describing the perturbations of the Reissner–Nordström black hole. I would like to emphasise the fact that without an "ansatz" approach introduced by A. Rostworowski in [4], obtaining results from [1] and [2] would be much more difficult. The third paper, [3], was more practical - it addressed the physical question: if the vacuum spacetime outside the compact object is the Kerr spacetime, is it possible that the source of this solution is the rotating gravastar? Although the perturbation methods are not one–hundred percent conclusive, they provide strong arguments that the answer to such a question is *no*.

Appendices

A Kretschmann scalar for a rotating gravastar solution

In this appendix I a present a perturbation expansion of the Kretschmann scalar. The Kretschmann scalar is calculated for the metric which is the solution to Einstein equations for the perturbations of the interior gravastar (de Sitter) solution up to the sixth order. The full solution itself is extremely lengthy and can be found in a Mathematica Notebook [52]. Method of solving Einstein equations is based on the formalism [4], which was already described and extended in this thesis.

The aim of this calculation is to find the regular part of the metric up to the third order. However, in this case the singularities which appear in metric at order i show up in the Kretschmann scalar at order $2i$, that's why we need the solution up to the sixth order. The full solution contains the following constants: Π_{11}, Ω_{11}, c_{20}, d_{20}, c_{22}, d_{22}, Π_{31}, Ω_{31}, Π_{33}, Ω_{33}, c_{40}, d_{40}, c_{42}, d_{42}, c_{44}, d_{44}, Π_{51}, Ω_{51}, Π_{53}, Ω_{53}, Π_{55}, Ω_{55}, c_{60}, d_{60}, c_{62}, d_{62}, c_{64}, d_{64}, c_{66}, d_{66}, where lower indices of a given contant $C_{i\ell}$ refer to the perturbation order and the multipole expansion number, respectively. The solution contains the perturbations of the density as well: ${}^{(2)}\delta\rho$, ${}^{(4)}\delta\rho$ and ${}^{(6)}\delta\rho$. Once the Kretschmann scalar expansion blows up, I set the appropriate constants to zero to make the solution regular. The Kretschmann scalar expansion is given by:

$$K = \bar{K} + \frac{a^2}{2}{}^{(2)}K + \frac{a^4}{4!}{}^{(4)}K + \frac{a^6}{6!}{}^{(6)}K + \mathcal{O}(a^8)\,. \tag{240}$$

A.1 Second order

Second order expansion of the Kretschmann scalar reads:

$${}^{(2)}K = -\frac{192(10M^2 - r^2)}{M^2 r^8}\Pi_{11} + \frac{64\pi^{(2)}\delta\rho}{M^2} - 192\left(\frac{r^2 + 8M^2}{M^2 r^8}\right)\Pi_{11}P_2(u)\,, \tag{241}$$

where lower index refers to the the polar expansion and P_ℓ is the Legendre polynomial of order ℓ. To make ${}^{(2)}K$ regular at $r = 0$, we set $\Pi_{11} = 0$.

A.2 Fourth order

$${}^{(4)}K = {}^{(4)}K_0 + {}^{(4)}K_2 P_2(u) + {}^{(4)}K_4 P_4(u)\,, \tag{242}$$

where

$$
\begin{aligned}
4!^{(4)}K_0 =& \frac{81d_{22}^2\left(560M^4-120M^2r^2+3r^4\right)\coth^{-1}\left(\frac{2M}{r}\right)^2}{20M^2r^{10}}+\\
&-\frac{27d_{22}\coth^{-1}\left(\frac{2M}{r}\right)\left(12c_{22}\left(560M^4-120M^2r^2+3r^4\right)+5d_{22}r\left(84M^2-11r^2\right)\right)}{5Mr^{10}}+\\
&+\frac{16\left(81c_{22}^2\left(560M^6-120M^4r^2+3M^2r^4\right)+80\pi r^{10}\left(3\delta\rho+3^{(4)}\delta\rho+16\pi\delta\rho^2M^2\right)\right)}{5M^2r^{10}}+\\
&+\frac{72\left(3c_{22}d_{22}\left(84M^2-11r^2\right)+16d_{20}^2r^3\right)}{r^9}+\\
&+\frac{9d_{22}^2\left(5040M^6-2760M^4r^2+443M^2r^4-16r^6\right)}{5r^8\left(Mr^2-4M^3\right)^2}\,,
\end{aligned}
\tag{243}
$$

$$
\begin{aligned}
4!^{(4)}K_2 =& \frac{81d_{22}^2\left(400M^4-72M^2r^2+r^4\right)\coth^{-1}\left(\frac{2M}{r}\right)^2}{14M^2r^{10}}+\\
&+\frac{216\left(14d_{20}r^2\left(r^2-12M^2\right)-3c_{22}\left(400M^4-72M^2r^2+r^4\right)\right)\coth^{-1}\left(\frac{2M}{r}\right)}{7Mr^{10}}+\\
&+\frac{54d_{22}^2\left(29r^2-300M^2\right)\coth^{-1}\left(\frac{2M}{r}\right)}{7Mr^9}+\\
&+\frac{18d_{22}^2\left(3600M^6-1848M^4r^2+265M^2r^4-8r^6\right)}{7r^8\left(Mr^2-4M^3\right)^2}+\\
&+\frac{432c_{22}\left(6c_{22}\left(400M^4-72M^2r^2+r^4\right)+d_{22}r\left(300M^2-29r^2\right)\right)}{7r^{10}}+\\
&+\frac{864d_{20}\left(c_{22}\left(48M^2-4r^2\right)+3d_{22}r\right)}{r^8}\,,
\end{aligned}
\tag{244}
$$

$$
\begin{aligned}
4!^{(4)}K_4 =& -\frac{243d_{22}^2\left(720M^4-40M^2r^2+13r^4\right)\coth^{-1}\left(\frac{2M}{r}\right)^2}{140M^2r^{10}}+\\
&-\frac{243d_{22}\coth^{-1}\left(\frac{2M}{r}\right)\left(4c_{22}\left(720M^4-40M^2r^2+13r^4\right)+5d_{22}r\left(36M^2+r^2\right)\right)}{35Mr^{10}}+\\
&+\frac{1944c_{22}\left(2c_{22}\left(720M^4-40M^2r^2+13r^4\right)+5d_{22}r\left(36M^2+r^2\right)\right)}{35r^{10}}+\\
&+\frac{27d_{22}^2\left(6480M^6-2520M^4r^2-27M^2r^4+64r^6\right)}{35r^8\left(Mr^2-4M^3\right)^2}\,.
\end{aligned}
\tag{245}
$$

To make ${}^{(4)}K$ regular at $r=0$ and $r=2M$, we set $c_{22}=d_{20}=d_{22}=0$.

A.3 Sixth order

$$
{}^{(6)}K = {}^{(6)}K_0 + {}^{(6)}K_2P_2(u) + {}^{(6)}K_4P_4(u) + {}^{(6)}K_6P_6(u)\,,
\tag{246}
$$

where

$$
\begin{aligned}
6!^{(6)}K_0 =& \frac{32}{3}\pi\left(32\pi^{(2)}\delta\rho^{(4)}\delta\rho + \frac{3^{(6)}\delta\rho}{M^2}\right) + \\
&+\frac{4800M^4\left(-3360M^6+1120M^4r^2-90M^2r^4+r^6\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)^2}{7r^{12}}+ \\
&+\frac{480M\left(3360M^6-1120M^4r^2+90M^2r^4-r^6\right)\Pi_{33}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{7r^{12}}+ \\
&+\frac{6400M^5\left(2520M^4-630M^2r^2+29r^4\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)}{7r^{11}}+ \\
&+\frac{2(\frac{r^2}{M^2}-10)\Pi_{31}^2}{3r^8}-\frac{320M^2\left(2520M^4-630M^2r^2+29r^4\right)\Pi_{33}\Omega_{33}}{7r^{11}}+ \\
&+-\frac{12(3360M^6-1120M^4r^2+90M^2r^4-r^6)\Pi_{33}^2}{7M^2r^{12}}+ \\
&-\frac{1280M^4\left(151200M^8-100800M^6r^2+22530M^4r^4-1805M^2r^6+28r^8\right)\Omega_{33}^2}{21r^{10}\left(r^2-4M^2\right)^2},
\end{aligned}
\tag{247}
$$

$$
\begin{aligned}
6!^{(6)}K_2 =& \frac{1600\left(-11200M^{10}+3600M^8r^2-276M^6r^4+3M^4r^6\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)^2}{7r^{12}}+ \\
&-\frac{480M\left(200M^4-42M^2r^2+r^4\right)\Pi_{31}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{7r^{10}}+ \\
&+\frac{160M\left(11200M^6-3600M^4r^2+276M^2r^4-3r^6\right)\Pi_{33}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{7r^{12}}+ \\
&+\frac{6400M^5\left(8400M^4-2000M^2r^2+87r^4\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)}{21r^{11}}+ \\
&+\frac{2\left(7\Pi_{31}r\left(8M^2+r^2\right)+480M^4\Omega_{33}\left(19r^2-150M^2\right)\right)\Pi_{31}}{21M^2r^9}+ \\
&-\frac{6400M^4\left(25200M^6-10200M^4r^2+1051M^2r^4-16r^6\right)\Omega_{33}^2}{63r^{10}\left(r^2-4M^2\right)}+ \\
&+\frac{4(11200M^6-3600M^4r^2+276M^2r^4-3r^6)\Pi_{33}^2}{7M^2r^{12}}+ \\
&-\frac{24\left(200M^4-42M^2r^2+r^4\right)\Pi_{31}\Pi_{33}}{7M^2r^{10}}+ \\
&+\frac{320M^2\left(8400M^4-2000M^2r^2+87r^4\right)\Pi_{33}\Omega_{33}}{21r^{11}},
\end{aligned}
\tag{248}
$$

$$
\begin{aligned}
6!^{(6)}K_4 =& \frac{14400M^4\left(-10080M^6+2880M^4r^2-190M^2r^4+r^6\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)^2}{77r^{12}}+\\
&-\frac{480M\left(80M^4-r^4\right)\Pi_{31}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{7r^{10}}+\\
&+\frac{1440M\left(10080M^6-2880M^4r^2+190M^2r^4-r^6\right)\Pi_{33}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{77r^{12}}+\\
&+\frac{57600M^5\left(2520M^4-510M^2r^2+19r^4\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)}{77r^{11}}+\\
&+\frac{36(-10080M^6+2880M^4r^2-190M^2r^4+r^6)\Pi_{33}^2}{77M^2r^{12}}+\\
&-\frac{2880M^2\left(2520M^4-510M^2r^2+19r^4\right)\Pi_{33}\Omega_{33}}{77r^{11}}+\\
&-\frac{1280M^4\left(453600M^8-280800M^6r^2+56790M^4r^4-4005M^2r^6+64r^8\right)\Omega_{33}^2}{77r^{10}\left(r^2-4M^2\right)^2}+\\
&+\frac{8\left(3\Pi_{33}\left(80M^4-r^4\right)+200M^4r\Omega_{33}\left(12M^2+r^2\right)\right)\Pi_{31}}{7M^2r^{10}}\,,
\end{aligned}
\tag{249}
$$

$$
\begin{aligned}
6!^{(6)}K_6 =& -\frac{40000M^4\left(1792M^6-336M^4r^2+24M^2r^4+3r^6\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)^2}{77r^{12}}+\\
&+\frac{4000M\left(1792M^6-336M^4r^2+24M^2r^4+3r^6\right)\Pi_{33}\Omega_{33}\coth^{-1}\left(\frac{2M}{r}\right)}{77r^{12}}+\\
&+\frac{32000M^5\left(6720M^4-700M^2r^2+69r^4\right)\Omega_{33}^2\coth^{-1}\left(\frac{2M}{r}\right)}{231r^{11}}+\\
&-\frac{100(1792M^6-336M^4r^2+24M^2r^4+3r^6)\Pi_{33}^2}{77M^2r^{12}}+\\
&-\frac{1600M^2\left(6720M^4-700M^2r^2+69r^4\right)\Pi_{33}\Omega_{33}}{231r^{11}}+\\
&-\frac{6400M^4\left(403200M^8-210000M^6r^2+35080M^4r^4-3725M^2r^6+448r^8\right)\Omega_{33}^2}{693r^{10}\left(r^2-4M^2\right)^2}\,.
\end{aligned}
\tag{250}
$$

To make ${}^{(6)}K$ regular at $r=0$ and $r=2M$, we set $\Pi_{31}=\Pi_{33}=\Omega_{33}=0$.

A.4 Regular part of the Kretschmann scalar

Up to the third order, we are left with the solution containing four nonzero constants: Ω_{11}, c_{20}, Ω_{31} and ${}^{(2)}\delta\rho$. The Kretschmann scalar corresponding to this solution (up to the sixth order) reads:

$$
\begin{aligned}
K =& \frac{3}{2M^4}+\frac{32\pi}{M^2}{}^{(2)}\delta\rho a^2+\frac{32}{3}\pi\left(\frac{3}{M^2}{}^{(4)}\delta\rho+\frac{16\pi^{(2)}\delta\rho^2}{M^2}\right)a^4+\\
&+\frac{32}{3}\pi\left(\frac{3}{M^2}{}^{(6)}\delta\rho+32\pi^{(2)}\delta\rho^{(4)}\delta\rho\right)a^6+\mathcal{O}(a^8)\,.
\end{aligned}
\tag{251}
$$

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Master equations and stability of Einstein-Maxwell-scalar black holes

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Abstract: We derive master equations for linear perturbations in Einstein-Maxwell scalar theory, for any spacetime dimension D and any background with a maximally symmetric $n = (D-2)$-dimensional spatial component. This is done by expressing all fluctuations analytically in terms of several master scalars. The resulting master equations are Klein-Gordon equations, with non-derivative couplings given by a potential matrix of size 3, 2 and 1 for the scalar, vector and tensor sectors respectively. Furthermore, these potential matrices turn out to be symmetric, and positivity of the eigenvalues is sufficient (though not necessary) for linear stability of the background under consideration. In general these equations cannot be fully decoupled, only in specific cases such as Reissner-Nordström, where we reproduce the Kodama-Ishibashi master equations. Finally we use this to prove stability in the vector sector of the GMGHS black hole and of Einstein-scalar theories in general.

Contents

1 Introduction

General Relativity admits a wide variety of Black Hole solutions, especially when coupled to matter. For any such black hole, or black brane, one of the central questions is how it behaves when perturbed, and in particular whether or not it is a (linearly) stable solution.

To answer this question one has to solve the linearized perturbation equations. This can be a messy task, because the metric, and the matter fields, have many components which all need to be fluctuated, and which may all couple.

The field of black holes perturbations was born with the seminal Regge-Wheeler [1] paper, where the stability of the Schwarzschild solution under *axial* linear perturbations was analysed. This study was extended to *polar* perturbations by Zerilli [2, 3] and given a new perspective by Moncrief [4]. Later, it was extended to the Schwarzschild-de Sitter [5] and Schwarzschild-anti-de Sitter [6] backgrounds. Finally, the problem of linear fluctuations was treated in full generality by Kodama&Ishibashi (KI). In the outstanding work [7] the problem was generalized for perturbations of general maximally-symmetric black holes i.e. to arbitrary spacetime dimension, cosmological constant and to any of the three maximally symmetric horizon topologies (spherical, planar and hyperbolic), and then extended to include Maxwell field (electro-vacuum) in [8]. These master equations have also been generalised to Gauss-Bonnet gravity in [9, 10] and to Lovelock gravity in [11].
The key result of black hole perturbation theory is that the general perturbation can be given in terms of only few scalar functions satisfying scalar wave equation with some potentials. In particular, in the case of maximally symmetric black holes with an electric charge [8] it was shown that the full problem of linear fluctuations can be reduced to solving 5 fully decoupled scalar wave equations with potentials, the so called *master scalar wave equations*. These are equations on the *master scalars*, in which all the fluctuations are expressed in a fully analytic way. [1]. Moreover, this structure can be extended beyond linear approximation and we find it really remarkable that in metric perturbation approach to Einstein equations, solving a full set of perturbation Einstein equations (at any perturbation order) can be reduced to the problem of solving a couple of scalar wave equations [12, 13].

In this work we generalize Kodama&Ishibashi (KI) results [8] to theories which in addition to a charge have a scalar field, with an arbitrary potential and an arbitrary potential coupling to the gauge field. This covers a broad class of actions, which have been heavily studied in the context of holography [14]. The simplest examples are Einstein-scalar theories in anti de Sitter spacetimes, with the physics depending heavily on the choice of potential and the background solutions typically being numerical, see e.g. [15–17]. Including the gauge field some analytic examples are the GMGHS (Gibbons, Maeda, Garfinkle, Horowitz, Strominger) black hole [18, 19] of which we will analyze the stability in section 4.4, and the asymptotically Lifshitz black brane of [20].

Although our results are an extension of KI, our derivation (initiated in [12, 21]) is slightly different. Instead of making manipulation with linearized Einstein equations (con-

[1] More precisely, in higher dimensions, $D > 4$, the master scalars in vector and tensor sectors come in a number of copies corresponding to different polarizations of gravitational waves in these sectors.

sisting mainly in taking different linear combination of these equations and their derivatives to arrive at master scalar equations), we take the structure of the outcome of previous work as the initial input for our procedure: we make an *ansatz* that all gauge invariant characteristics of fluctuation (see below for their definition) are given in terms of linear combinations of master scalars and their derivatives, where the master scalar themselves satisfy scalar wave equations coupled with interaction potentials. As the final results, we express all the perturbations analytically in terms of the three sets of master scalars, one for each helicity h. We find such an *ansatz approach* to solve for the fluctuations to be a very robust technique: interestingly it works also for time-dependent backgrounds, for example in the cosmological perturbations context [12, 21]). The main advantage is that once we decide on the correct form of the ansatz (i.e. the highest order of derivatives of master scalars in the linear combinations for gauge invariants and/or the form of the couplings between master scalars in master scalar wave equations), finding the function coefficients of these linear combinations and the actual form of the coupling potentials is an purely algorithmic task (although rather unthinkable to achieve in the pre-computer algebra packages era).

We express all the perturbations analytically into three sets of master scalars, one for each helicity h.
Each set of master scalars satisfies a coupled master equation of the form,

$$\Box\Phi_s^{(h)} - W_{s,s'}^{(h)}(r)\Phi_{s'}^{(h)} = 0\,, \tag{1.1}$$

where $\Box$ stands for the wave operator on the background metric (see eq.(2.2) below) and the potential matrix W couples the different master scalars with the same helicity, which are labeled by their spin. The components of the perturbations are first expressed into gauge invariant combinations (Eq. 3.4), which are then expressed in terms of these master scalars (Eqs. 3.5, 3.7 and appendix C). The potentials are given in Eqs. 3.6, 3.9, 3.12 and 3.14. In section 3.4 we discuss when these still coupled master equations can be further decoupled into single equations. Then in section 4 we discuss a sufficient criterion for linear stability and apply it to several specific cases.

The master equations we derive here are made available in a Mathematica notebook, along with a check of their correctness, at [22].

2 Setup

We consider the class of Einstein-Maxwell-scalar theories described by the following action,

$$S = \int d^{n+2}x\sqrt{-g}\left(R - 2\Lambda - \eta(\partial\phi)^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi)\right)\,, \tag{2.1}$$

where R is the Ricci scalar, Λ is a cosmological constant, η is an arbitrary normalization factor for the scalar field ϕ[2], $F = dA$ is the field strength and V and Z are two arbitrary functions of the scalar field, with $V(0) = 0$.

[2]This can be absorbed into ϕ but we keep it explicit to make it easier to substitute a particular model.

Any time-independent $n+2$ dimensional solution with a maximally symmetric n-dimensional spatial part can be written as[3]

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{\zeta(r)^2}{f(r)}dr^2 + S(r)^2 dX^2_{(n,K)}\,, \\ A &= a(r)dt\,. \end{aligned} \tag{2.2}$$

Here $dX^2_{(n,K)}$ is one of the three maximally symmetric n-dimensional spaces,

$$dX^2_{(n,K)} = \begin{cases} dx_1^2 + ... + dx_n^2, & K = \ \ 0\,, \quad \text{planar} \\ d\Omega_{(n)}, & K = +1\,, \ \text{spherical} \\ dH_{(n)}, & K = -1\,, \ \text{hyperbolic} \end{cases} . \tag{2.3}$$

Here we note that while Kodama&Ishibashi approach [7, 8] is coordinate independent we prefer to work with the fixed Fefferman-Graham, or Schwarzschild-like, coordinate system (2.2). However, since our final result, eq. (1.1) is a scalar equation it can be easily expressed in any coordinate system; similarly the rules that express gauge invariant quantities in terms of master scalars can be easily transformed (see Appendix D).

In order to avoid cluttering the presentation, we leave any complications relating to spherical or hyperbolic symmetry to Appendix A, focussing here on the planar case. In the results presented we do show the most general expressions, where the dependence on the topology shows up only through the parameter K defined above and the eigenvalues $-k^2$ of the corresponding Laplace operator.

The equations of motion following from Eq. (2.1) lead to the following equations for the background[4]:

$$\begin{aligned} \phi'' &= \phi'\left(\frac{\zeta'}{\zeta} - n\frac{S'}{S}\right) - \frac{a'^2 Z' + 4\eta f'\phi' - 2\zeta^2 V'}{4\eta f}\,, \\ a'' &= a'\left(\frac{\zeta'}{\zeta} - n\frac{S'}{S} - \frac{Z'\phi'}{Z}\right)\,, \\ S'' &= \frac{\zeta' S'}{\zeta} - \frac{\eta}{n} S\phi'^2\,, \\ 0 &= S^2\left(2\eta f\phi'^2 - Za'^2\right) - 2nSf'S' - 2n(n-1)fS'^2 + 2\zeta^2\left(n(n-1)K - S^2(V+\Lambda)\right)\,, \\ f'' &= Za'^2 + \frac{f'\zeta'}{\zeta} - (n-2)\frac{f'S'}{S} - \frac{2(n-1)}{S^2}\left(\zeta^2 K - fS'^2\right) - \frac{2\eta}{n} f\phi'^2\,, \end{aligned} \tag{2.4}$$

where in a slight abuse of notation, primes indicate radial derivatives except when acting on V or Z, where they indicate a derivative with respect to ϕ.

[3]Note that we could set either $\zeta = 1$ or $S = r$ by a gauge transformation, but we choose to keep it in this more general form.

[4]Here and in everything that follows, we extrapolate the dimensional dependence from our calculations at $n = 2, ..., 9$.

We further note that although we will work with the Fefferman-Graham coordinates Eq. (2.2) here, the final potentials in the master equations Eq. (1.1) will be exactly equal in the Eddington-Finkelstein coordinates parametrized as,

$$ds^2 = -f(r)dt^2 + 2\zeta(r)dtdr + S(r)^2 dX^2_{(n,K)}\,, \tag{2.5}$$

Any differences between the two will be shown in Appendix D.

We will not specialize to any specific background, but consider any background that satisfies these equations.

In the following section we will perturb this general solution and derive the master equations that describe these perturbations.

3 Master Equations

Perturbing the background solution to first order, we have to perturb all the fields: the metric, the gauge field and the scalar. Because of the maximal symmetry of the spatial part of the background, we can express the spatial dependence of these fluctuations using the eigenfunctions of the Laplacian of the n-dimensional maximally symmetric space, which in the present planar case are just plane waves, giving the following perturbations:

$$\begin{aligned} \delta g_{\mu\nu} &= h_{\mu\nu}(t,r)e^{ikx}\,, \\ \delta A_\mu &= a_\mu(t,r)e^{ikx}\,, \\ \delta\phi &= \varphi(t,r)e^{ikx}\,, \end{aligned} \tag{3.1}$$

where we've chosen the plane waves to propagate along the first spatial coordinate in $X = \big(x \equiv x_{(1)}, y \equiv x_{(2)}, ..., z \equiv x_{(n)}\big)$.

The derivation of the master equations takes the following steps:

1. Organise the fluctuations into three different sectors or channels, according to their transformations under the little group.

2. Rewrite the the fluctuations into gauge-invariant combinations (or equivalently choose a gauge that is fixed uniquely).

3. Rewrite those gauge-invariant combinations as linear combinations of master scalars and their derivatives, where the master scalars themselves satisfy Eq. (1.1).

We shall now discuss each step in turn.

3.1 Sectors

The perturbations Eq. (3.1) naturally decouple into three sets of equations, as summarized in Table (3.1). The sectors are classified by their helicity, 0, 1 or 2, or whether the fluctuation transforms as a scalar, vector or tensor once the momentum is fixed. Various

different names are used in the literature for these sectors, we will stick to scalar, vector and tensor since these seem to be the most natural and context-independent.

In Table (3.1) we summarize how the different components fall into the three sectors. We adopt a convention where indices i, j take values from (t, r, x) and indices $\alpha \neq \beta$ take values from $(x_2 \equiv y, ..., x_n \equiv z)$.

h \ s	$h_{\mu\nu}$	a_μ	φ	copies $\times N_{\text{coupled}}$	names
0	h_{ij} h 6 1	a_i 3	φ 1	1×11	<u>scalar</u> parity-even, polar sound
1	$h_{i\alpha}$ $(n-1) \times 3$	a_α $n-1$	–	$(n-1) \times 4$	<u>vector</u> parity-odd, axial shear
2	$h_{\alpha,\beta}$ $h_{\alpha\alpha} - h_{\beta\beta}$ $\frac{1}{2}(n-1)(n-2)$ $n-2$	–	–	$\frac{1}{2}(n+1)(n-2) \times 1$	<u>tensor</u> scalar
total	$\frac{1}{2}(n+2)(n+3)$	$n+2$	1	$\frac{1}{2}(n+3)(n+4)$	

Table 1. Decoupling of perturbations into sectors. Under each perturbation we note the number of components involved. In the rightmost column we list some names that are common in the literature for these sectors, here we use the underlined ones.

Each of the $1 + (n-1) + (1/2)(n+1)(n-2) = (1/2)n(n+1) - 1$ copies in the fourth column will include one gravitational master scalar. This number is equal to the number of graviton polarisations, which can be counted as a symmetric n by n matrix, subtracting the trace.

The scalar sector is the most complicated one, since it receives contributions from every field. In particular the scalar field itself of course falls into this sector. So do the components of the gauge field and metric with i indices, and finally the trace of the spatial metric perturbations, h. In the table we also list the number of components each of these has. For the scalar sector there are 11 fluctuations in total, which all couple to each other. In the next subsection we will show that the 11 coupled PDEs for fluctuations in this sector are in fact 11 equations for 7 gauge invariant characteristics of scalar perturbations.

The vector sector consists of those fluctuations with one α index. This can be the gauge field, or the metric where the other index is an i. Together these give 4 components, times $(n-1)$ for the number of values that α can take. These are not all coupled though, this sector further decouples into $(n-1)$ identical copies of sets of 4 coupled equations, one for each value of α. In the next subsection we will show that each copy of the 4 coupled PDEs for fluctuations in this sector is in fact a copy of 4 equations for 3 gauge invariant characteristics of vector perturbations.

Finally the tensor sector can only have contributions from the metric, and consists of those metric fluctuations with two distinct indices α, β, of which there are $\frac{1}{2}(n-1)(n-2)$, and differences of diagonal components, of which there are $n-2$ linearly independent ones. This sector is particularly simple, since none of these couple to each other. Thus it falls

into $(1/2)(n+1)(n-2)$ decoupled equations. All components in this sector are in fact gauge invariant - see the next subsection.

Since the tensor sector equations are all identical we can consider only one tensor perturbation, which we will take to be h_{yz}. Furthermore since the vector sector consists of identical sets of coupled equations for each value of α we can also consider only one copy of those, which we shall take along the z direction, so we perturb h_{tz}, h_{rz}, h_{xz} and a_z. In the scalar sector we need all 11 perturbations, but instead of h_{xx} and h we use different linear combinations.

The perturbations we take are:

$$\begin{aligned}
\delta g_{\mu\nu} &= \begin{pmatrix} h_{tt} & 1/2h_{tr} & ik\,h_{tx} & 0 & ... & 0 & h_{tz} \\ 1/2h_{tr} & h_{rr} & ik\,h_{rx} & 0 & ... & 0 & h_{rz} \\ ik\,h_{tx} & ik\,h_{rx} & h_{xx} & 0 & ... & 0 & ikh_{xz} \\ 0 & 0 & 0 & h_{yy} & 0 & 0 & h_{yz} \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ h_{tz} & h_{rz} & ikh_{xz} & h_{yz} & 0 & 0 & h_{zz} \end{pmatrix} e^{ikx}\,, \\
\delta A_\mu &= \Big(a_t\,,\, a_r\,,\, ika_x\,,\, 0\,,\, ...\,,\, 0\,,\, a_z\Big)\, e^{ikx}\,, \\
\delta\phi &= \varphi e^{ikx}\,,
\end{aligned} \tag{3.2}$$

where each function now depends on (t, r), and we further rewrite:

$$\begin{aligned}
h_{xx} &= \frac{1}{n}\left(h_+ - (n-1)k^2 h_-\right)\,, \\
h_{yy} = \cdots = h_{zz} &= \frac{1}{n}\left(h_+ + k^2 h_-\right)\,.
\end{aligned}$$

This particular convention comes from the decomposition into scalar, vector and tensor components for general maximally symmetric topologies that we do in Appendix A, taking the planar case.

3.2 Gauge Invariant Fluctuations

We can now use gauge transformations to simplify this further.

We do an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ and an infinitesimal gauge transformation $A_\mu \to A_\mu + \nabla_\mu \lambda$, where ξ^μ and λ are arbitrary functions of (t, r). If we keep the background fields invariant, the perturbations have to transform as,

$$\begin{aligned}
\delta g_{\mu\nu} &\to \delta g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu\,, \\
\delta A_\mu &\to \delta A_\mu + \nabla_\mu \lambda - \xi^\nu \nabla_\nu A_\mu - A_\nu \nabla_\mu \xi^\nu\,, \\
\delta\phi &\to \delta\phi - \xi^\nu \nabla_\nu \phi\,.
\end{aligned} \tag{3.3}$$

Now we further decompose the fluctuations into gauge-independent and gauge-dependent ones. We find the following set of gauge-independent combinations:

helicity 2:

$$\mathfrak{h}_{yz} \equiv h_{yz}\,,$$

helicity 1:

$$\begin{aligned}
\mathfrak{h}_{tz} &\equiv h_{tz} - \partial_t h_{xz}\,,\\
\mathfrak{h}_{rz} &\equiv h_{rz} - \partial_r h_{xz} + 2\frac{S'}{S} h_{xz}\,,\\
\mathfrak{a}_z &\equiv a_z\,,
\end{aligned}$$

helicity 0:

$$\begin{aligned}
\mathfrak{h}_{tt} &\equiv h_{tt} - 2\partial_t h_{tx} + \partial_t^2 h_- + \frac{f'}{2nSS'}(h_+ + k^2 h_-)\,,\\
\mathfrak{h}_{tr} &\equiv h_{tr} - 2\partial_r h_{tx} + \partial_t\partial_r h_- + 2\frac{f'}{f} h_{tx} - \frac{f'}{f}\partial_t h_- - \frac{\zeta^2}{nfSS'}\partial_t(h_+ + k^2 h_-)\,,\\
\mathfrak{h}_{rr} &\equiv h_{rr} - \frac{\zeta^2}{nfS'S}\partial_r(h_+ + k^2 h_-) + \left(\frac{\zeta^2}{2nf^2S^2S'}(Sf' + 2fS') - \eta\frac{\zeta^2}{n^2fS'^2}\phi'^2\right)(h_+ + k^2 h_-)\,,\\
\mathfrak{h}_{rx} &\equiv h_{rx} - \frac{1}{2}\partial_r h_- + \frac{S'}{S} h_- - \frac{\zeta^2}{2nfSS'}(h_+ + k^2 h_-)\,,\\
\mathfrak{a}_t &\equiv a_t - \partial_t a_x - \frac{a'}{2nSS'}(h_+ + k^2 h_-)\,,\\
\mathfrak{a}_r &\equiv a_r - \partial_r a_x + \frac{a'}{2f}\partial_t h_- - \frac{a'}{f} h_{tx}\,,\\
\boldsymbol{\varphi} &\equiv \varphi - \frac{\phi'}{2nSS'}(h_+ + k^2 h_-)\,,
\end{aligned} \tag{3.4}$$

Note here that although K does not appear in these expressions, these *are* the correct expressions for any K. Although these expressions are independent of K, the definition of the components through Eq. (3.2) does need to be modified, see Appendix A.

Note the structure of these definitions. The gauge invariants are formed by some subset of the components, "dressed" with the other components and their derivatives to make them gauge invariant. In the tensor sector this is trivial. If we demand this structure and that the coefficients are known algebraically in terms of the background, this choice of gauge invariants is unique in the vector sector. In the scalar sector there is however another choice. The one we have chosen is the Detweiler gauge [23, 24]. Instead one could have chosen the Regge-Wheeler gauge, where h_+ is taken as the basis for a gauge invariant instead of h_{rx}. We find the Detweiler gauge simpler to work with, however we note that since the master equations are gauge invariant, it actually does not matter for the final potentials. Intermediate results in the Regge-Wheeler gauge are discussed in Appendix D.

When these gauge invariants are substituted into the perturbation equations, the remaining non-gauge-invariant "dressing" components (h_{xz} in the vector sector and h_{tx}, $h_\pm$ and a_x in the scalar sector) automatically drop out. Instead of using the gauge invariant

components one may use the gauge freedom to set these components to zero, fixing the gauge. We stress that fixing the gauge, to Regge-Wheeler or Detweiler gauge, is completely equivalent to working with gauge-invariant variables.

This step, and the next, are summarized in Table 3.2.

h \ s	$h_{\mu\nu}$	a_μ	φ
0	$\mathfrak{h}_{tt},\ \mathfrak{h}_{tr},\ \mathfrak{h}_{rr},\ \mathfrak{h}_{rx}$ $h_{tx}(\xi_t),\ h_+(\xi_r),\ h_-(\xi_x)$ $\Phi_2^{(0)}$	$\mathfrak{a}_t,\ \mathfrak{a}_r$ $a_x(\lambda)$ $\Phi_1^{(0)}$	φ - $\Phi_0^{(0)}$
1	$\mathfrak{h}_{tz},\ \mathfrak{h}_{rz}$ $h_{xz}(\xi_z)$ $\Phi_2^{(1)}$	$\mathfrak{a}_z$ - $\Phi_1^{(1)}$	-
2	$\mathfrak{h}_{yz}$ - $\Phi_2^{(2)}$	-	-

Table 2. Decoupling of sectors into gauge-independent components. For each sector and each field we list first the gauge-invariant components, then in the line below the gauge-dependent ones and behind them in brackets the gauge parameter that can be used to set it to zero. The bottom line is the master scalar of that field in that sector.

3.3 Master Equations

The former two steps, the decoupling of independent sectors and the decoupling of gauge-dependent modes, are quite standard and technically simple, and it is no surprise that this can be done. The final step from the gauge invariant fluctuations to the master scalars is technically more difficult, and here it is also not clear why the equations can be written in the simple form that we will see.

Conceptually however this step is also very simple. We assume that we can express all the fluctuations at a given spin and helicity into a single so-called "master scalar", which satisfies a Klein-Gordon equation with a certain potential. We make an ansatz for the coefficients relating the gauge-invariant components to the master scalars, and for the potentials. Then we insert this ansatz into the perturbation equations and try to find a solution.

Around a vacuum solution, where the gauge field and scalar are zero in the background and only consist of the fluctuations, the different spins also decouple. So for a given helicity and spin (h, s) we can express all the fluctuations in terms of a single master scalar $\Phi_s^{(h)}$ that satisfies a Klein-Gordon equation with a potential $W_{s,s}^{(h)}$.

The coefficients and potentials in the ansatz are found by plugging the ansatz into the perturbation equations, and using the background equations (2.4) and the master equations (1.1) (which involve the as yet unknown potentials) to simplify. In each equation every coefficient of the master scalar and its derivatives must individually vanish, provided the master equations have been imposed. This system of equations is such that for a

given helicity and spin one has to solve a single simple first order ODE, that can be solved analytically. This gives one integration constant for each spin and helicity, corresponding to an arbitrary normalization of the master scalar. The rest of the equations are algebraic and can easily be solved, although in practice it can be rather difficult even with Mathematica.

What changes when there is a gauge field and/or scalar field in the background, is that now the master scalars in a given sector couple through the non-derivative interaction potentials $W^{(h)}_{s,s'}$. Remarkably, and non-trivially, these interaction potentials can be made symmetric through a choice of the aforementioned integration constants. This leaves one free integration constant per sector, that does not affect the potentials but just scales all the master equations by the same constant. In order to get the master equations in this form, the gauge-invariants also have to receive contributions from the other master scalars at different spins.

We will now look at the results of this procedure sector by sector.

3.3.1 Tensor sector

In the tensor sector, only present for $n > 2$, nothing changes with respect to the vacuum case. The fluctuation is proportional to the master scalar as,

$$\mathfrak{h}_{yz} \equiv S^2 \Phi_2^{(2)} \,, \tag{3.5}$$

and the master scalar satisfies a free Klein-Gordon equation,

$$W^{(2)}(r) = \frac{k^2}{S^2} \,. \tag{3.6}$$

Note that this single term comes simply from the Laplacian acting on a scalar eigenfunction, meaning that tensor modes satisfy a free, massless scalar field equation.

3.3.2 Vector sector

The vector sector, which is present only for $n > 1$, is still quite simple and similar to the vacuum case, in that there is no mixing between different fields at the level of the master scalars,

$$\begin{aligned} \mathfrak{h}_{tz} &\equiv \frac{n}{k\tilde{k}} \frac{fSS'}{\zeta} \Phi_2^{(1)} + \frac{1}{k\tilde{k}} \frac{fS^2}{\zeta} \partial_r \Phi_2^{(1)} \,, \\ \mathfrak{h}_{rz} &\equiv \frac{1}{k\tilde{k}} \frac{\zeta S^2}{f} \partial_t \Phi_2^{(1)} \,, \\ \mathfrak{a}_z &\equiv \frac{1}{k} \frac{S}{\sqrt{Z}} \Phi_1^{(1)} \,, \end{aligned} \tag{3.7}$$

where we have defined $\tilde{k} \equiv \sqrt{k^2 - nK}$ and we have chosen an overall normalization of the master scalars to reflect the singularity of the zero momentum, or in spherical setting $l = 1$, limits, see appendix B.1.

However as mentioned we get two equations which are coupled through the potential matrix,

$$W^{(1)} = \begin{pmatrix} W^{(1)}_{1,1} & W^{(1)}_{1,2} \\ W^{(1)}_{1,2} & W^{(1)}_{2,2} \end{pmatrix} \,, \tag{3.8}$$

where

$$
\begin{aligned}
W^{(1)}_{1,1}(r) &= \frac{k^2}{S^2} - \frac{f'S'}{\zeta^2 S} + (n-2)\left(\frac{K}{S^2} - \frac{fS'^2}{\zeta^2 S^2}\right) + \frac{Za'^2}{\zeta^2} + \frac{f\eta\phi'^2}{n\zeta^2} - \frac{1}{8\eta\zeta^2}\frac{Z'}{Z}\mathcal{V} \\
&\quad - \frac{Z'^2}{Z^2}\frac{f\phi'^2}{4\zeta^2} - \frac{Z'}{Z}\frac{fS'\phi'}{\zeta^2 S} + \frac{f\phi'^2 Z''}{2\zeta^2 Z}\,, \\
W^{(1)}_{1,2}(r) &= -\sqrt{k^2 - nK}\frac{\sqrt{Z}a'}{\zeta S}\,, \\
W^{(1)}_{2,2}(r) &= \frac{k^2}{S^2} - n\left(\frac{f'S'}{\zeta^2 S} - \frac{fS'^2}{\zeta^2 S^2} + \frac{K}{S^2}\right) + \eta\frac{f\phi'^2}{\zeta^2}\,,
\end{aligned}
\tag{3.9}
$$

and

$$
\mathcal{V}(r) = -2\zeta^2 V' + a'^2 Z'\,. \tag{3.10}
$$

Note that as expected, when the background gauge field vanishes, the equations decouple. Also note that the factor $\sqrt{k^2 - nK}$ in the interaction potential. In the planar case this becomes simply k, and the equations decouple in the zero momentum limit. In the spherical case this factor becomes equal to $\sqrt{(l-1)(l+n)}$, so that the equations again decouple if $l = 1$. This makes sense too, because at $l = 1$ there are no dynamical degrees of freedom in the metric, only in the gauge field.

3.3.3 Scalar sector

Here it gets significantly more complicated, and we present the gauge invariants in Appendix C. These are expressed in terms of three master scalars $\Phi^{(0)}_2, \Phi^{(0)}_1$ and $\Phi^{(0)}_0$, which in vacuum would correspond to the gravitational, gauge field and scalar fluctuations respectively. In the general case however, both the metric and the gauge field fluctuations receive contributions from all three master scalars, while the scalar field gets a contribution from the gravitational master scalar in addition to its own.

These three master scalars again satisfy the coupled Klein-Gordon equations 1.1 with the potential matrix,

$$
W^{(0)} = \begin{pmatrix} W^{(0)}_{0,0} & W^{(0)}_{0,1} & W^{(0)}_{0,2} \\ W^{(0)}_{0,1} & W^{(0)}_{1,1} & W^{(0)}_{1,2} \\ W^{(0)}_{0,2} & W^{(0)}_{1,2} & W^{(0)}_{2,2} \end{pmatrix}\,. \tag{3.11}
$$

The three diagonal potentials are,

$$
\begin{aligned}
W^{(0)}_{0,0}(r) &= \frac{k^2}{S^2} + \frac{\phi'}{\mathcal{D}^2\zeta^2}\Bigg(\frac{\zeta^2 k^2}{nS'}\mathcal{A} + \mathcal{F}\mathcal{D}S\mathcal{V} + 2\eta f\phi'\left(\mathcal{F}\mathcal{P} + 4\zeta^4 k^2\left(k^2-(n-1)K\right)\right) + \\
&\quad 2\eta\mathcal{F}\mathcal{D}\phi'\left(Sf' + (n-2)fS'\right)\Bigg) - \frac{1}{4\eta\zeta^2}\left(\mathcal{V}' - 2a'^2 Z'^2/Z\right)\,, \\
W^{(0)}_{1,1}(r) &= \frac{k^2}{S^2} + \frac{Za'^2}{\mathcal{D}^2\zeta^2}\Bigg(n^2 S'^2\mathcal{F}\left(Sf' - 2(n-1)fS'\right) + 2fn^2S^2Za'^2S'^2 + \\
&\quad 4f\zeta^2\left(nS'^2\left((2n-3)k^2 - n(n-1)K\right) + k^2\eta S^2\phi'^2\right) + 4\zeta^4k^4\Bigg) + \\
&\quad \frac{1}{Z}\left(\frac{Z'}{8\eta\zeta^2}\mathcal{V} + \frac{f(n-1)S'\phi'}{\zeta^2 S}Z' - \frac{fZ''\phi'^2}{2\zeta^2}\right) + \frac{2nfSS'Z'\phi'a'^2}{\mathcal{D}\zeta^2} - \\
&\quad \frac{(n-1)\left(nf'S' - f\eta S\phi'^2\right)}{\zeta^2 nS} + \frac{3fZ'^2\phi'^2}{4\zeta^2Z^2}\,, \\
W^{(0)}_{2,2}(r) &= \frac{k^2}{S^2} + \frac{n-1}{nS^2\mathcal{D}^2}\Bigg(4n^2\left(k^2-nK\right)fS^2a'^2ZS'^2 - 8n\zeta^4k^4K + 8\eta\zeta^2fS^2\phi'^2k^2\left(k^2-nK\right) + \\
&\quad 2n^2S'^2\mathcal{F}\left(Sf'\left(2k^2-nK\right) + 2fS'\left((n-2)k^2 - n(n-1)K\right)\right) + \\
&\quad 8\zeta^2\left(nS'\left(fS'\left(k^4 - n(n-2)k^2K + n^2(n-1)K^2\right) - k^4Sf'\right)\right)\Bigg)\,,
\end{aligned}
\tag{3.12}
$$

where we have further defined,

$$
\begin{aligned}
\mathcal{F}(r) &= 2fS' - Sf'\,, \\
\mathcal{D}(r) &= 2\zeta^2k^2 - nS'\mathcal{F}\,, \\
\mathcal{P}(r) &= \left(\eta S^2\phi'^2 - nS'^2\right)\mathcal{F}\,, \\
\mathcal{A}(r) &= 4n\eta fS'S^2Za'^2\phi'\,,
\end{aligned}
\tag{3.13}
$$

and where again primes indicate radial derivatives except when acting on V, Z or $\mathcal{V}$, where they indicate a derivative with respect to ϕ.

The three interaction potentials are,

$$
\begin{aligned}
W^{(0)}_{0,1}(r) &= -\frac{k\sqrt{Z}a'}{\sqrt{2}\mathcal{D}^2\zeta\sqrt{\eta}}\Bigg(\mathcal{A}+\mathcal{D}S\mathcal{V}+\frac{\mathcal{D}^2Z'}{SZ}+2\eta\mathcal{D}Sf\phi'^2Z'/Z+ \\
&\qquad 4\eta f\phi'\left(\mathcal{P}-n(n-1)S'^2\mathcal{F}-2\zeta^2S'\left((1-2n)k^2+n(n-1)K\right)\right)\Bigg), \\
W^{(0)}_{0,2}(r) &= k\sqrt{k^2-nK}\frac{\sqrt{n-1}}{\sqrt{n}\sqrt{\eta}S\mathcal{D}^2}\left(\mathcal{A}+\mathcal{D}S\mathcal{V}+4\eta f\phi'\left(\mathcal{P}+2\zeta^2nS'\left(k^2-(n-1)K\right)\right)\right), \\
W^{(0)}_{1,2}(r) &= -\sqrt{k^2-nK}\frac{\sqrt{2}\sqrt{n-1}\sqrt{Z}a'}{\zeta\sqrt{n}S\mathcal{D}^2}\Bigg(2fn^2S^2Za'^2S'^2+n^2Sf'S'^2\mathcal{F}+ \\
&\qquad 4f\zeta^2\left(nS'^2\left(k^2(n-2)-K(n-1)n\right)+k^2\eta S^2\phi'^2\right)+\mathcal{D}fnSS'\phi'\frac{Z'}{Z}+4\zeta^4k^4\Bigg),
\end{aligned}
\tag{3.14}
$$

We note again that if either the scalar or the gauge field vanish on the background, their respective master scalars decouple from the rest. Furthermore at zero momentum in the planar case, all equations decouple.

3.4 Full decoupling

The final master equations are a single decoupled equation for the tensor sector, two coupled equations for the vector sector and three coupled equations for the scalar sector.

We would like to be able to decouple the vector and scalar sector further to fully decoupled equations. If this is possible, the decoupled potentials would be the eigenvalues of the potential matrix,

$$
W^{(1)}_{\pm} = \frac{1}{2}\left(W^{(1)}_{1,1}+W^{(1)}_{2,2}\pm\sqrt{\left(W^{(1)}_{1,1}-W^{(1)}_{2,2}\right)^2+4\left(W^{(1)}_{1,2}\right)^2}\right). \tag{3.15}
$$

However, it is only possible to decouple the equations in this way if the eigenvectors of the potential matrix do not depend on r. Computing the eigenvalues and taking the r-derivative, one finds that the equations can be decoupled under the condition that:

$$
\partial_r \log\left(W^{(1)}_{1,2}\right) = \partial_r \log\left(W^{(1)}_{1,1}-W^{(1)}_{2,2}\right) \tag{3.16}
$$

More simply, they can be decoupled if

$$
\frac{W^{(1)}_{1,1}-W^{(1)}_{2,2}}{W^{(1)}_{1,2}} = \text{const.}, \tag{3.17}
$$

and in that case, the r-dependence in the square root in the eigenvalues factors out, leaving a square root only of constants.

In the sound channel the algebra is a bit more complicated, with the decoupled potentials being the following eigenvalues of the potential matrix (*if* it can be decoupled):

$$
W^{(0)}_{\sigma} = \frac{1}{3}\left(\mathcal{T}+\sigma\frac{\mathcal{T}^2+3\mathcal{U}}{\mathcal{C}}+\frac{\mathcal{C}}{\sigma}\right), \tag{3.18}
$$

where σ are the three roots of $\sigma^3 = 2$, and

$$\begin{aligned}
\mathcal{T} &= \mathfrak{tr}\left(W^{(0)}\right) , \\
\mathcal{U} &= \frac{1}{2}\left(\mathfrak{tr}\left(\left(W^{(0)}\right)^2\right) - \mathcal{T}^2\right) , \\
\mathcal{D} &= \det\left(W^{(0)}\right) , \\
\mathcal{C} &= \left(27\mathcal{D} + 2\mathcal{T}^3 + 9\mathcal{T}\mathcal{U} + \mathcal{R}\right)^{1/3} , \\
\mathcal{R} &= \left(\left(27\mathcal{D} + 2\mathcal{T}^3 + 9\mathcal{T}\mathcal{U}\right)^2 - 4\left(\mathcal{T}^2 + 3\mathcal{U}^2\right)^3\right)^{1/2} .
\end{aligned} \tag{3.19}$$

Again this decoupling can only be done when the eigenvectors are constant, we have however not been able to derive a simple criterion such as Eq. (3.17) in this case.

3.5 Comparison to Kodama-Ishibashi

To compare with the results of Kodama and Ishibashi [8] for Reissner-Nordström we turn off the scalar field,

$$\phi(r) = 0 , \quad V(\phi) = 0 , \quad Z(\phi) = 1 , \tag{3.20}$$

and we insert the Reissner-Nordström solution,

$$\begin{aligned}
\zeta(r) &= 1 , \\
S(r) &= r , \\
a(r) &= \sqrt{\frac{2n}{n-1}} Q r^{1-n} , \\
f(r) &= K - \lambda r^2 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}} .
\end{aligned} \tag{3.21}$$

From the decoupling condition Eq. (3.17) we see by inserting this background that the equations can be decoupled:

$$\frac{W^{(1)}_{1,1} - W^{(1)}_{2,2}}{W^{(1)}_{1,2}} = \frac{1}{\sqrt{k^2 - Kn}} \frac{\left(n^2 - 1\right)}{\sqrt{2n(n-1)}} \frac{\left(K - \lambda + Q^2\right)}{Q} , \tag{3.22}$$

and since we've set the scalar to zero the scalar sector also has only two equations, so we can apply the same criterion and find:

$$\frac{W^{(0)}_{1,1} - W^{(0)}_{2,2}}{W^{(0)}_{1,2}} = \frac{1}{\sqrt{k^2 - Kn}} \frac{n+1}{2} \frac{\left(K - \lambda + Q^2\right)}{Q} . \tag{3.23}$$

So the equations can indeed be fully decoupled and the resulting potentials are the eigenvalues of our potential matrices.

In order to compare these with KI we first have to transform them to the Schrödinger form,

$$V_S(r) = W(r) + \left(\frac{n}{4}\frac{S'}{S^2\zeta^2}\left(2Sf' + (n-2)fS'\right) - \frac{n}{2}\frac{f\zeta'}{\zeta^3}\frac{S'}{S} + \frac{n}{2}\frac{f}{\zeta^2}\frac{S''}{S}\right)\mathbb{1} . \tag{3.24}$$

We will see how this arises in the next section, and note that this redefinition drops out in the condition of Eq. (3.17).

Computing the eigenvalues, which we shall call $\tilde{W}^{(h)}$, from these potential matrices in Schrödinger form, we obtain for the tensor sector:

$$\tilde{W}^{(2)} = \frac{1}{4r^{2(n+1)}}\left(r^{2n}\left(4k^2 + K(n-2)n - \lambda n(n+2)r^2\right) + 2Mn^2r^{n+1} + (2-3n)nQ^2r^2\right) , \tag{3.25}$$

which agrees with Eq. (3.7) in [8] (with $\lambda_L = k^2 + 2(n-1)K$).

In the vector sector we obtain the two eigenvalues:

$$\begin{aligned}\tilde{W}^{(1)}_{\pm} &= \frac{1}{4}r^{-2(n+1)}\left(r^{2n}\left(4k^2 + (n-2)n\left(K - \lambda r^2\right)\right) - 2M\left(n^2+2\right)r^{n+1} + n(5n-2)Q^2r^2\right) \\ &\quad \pm \frac{\Delta^{(1)}}{r^{(n+1)}} , \\ \Delta^{(1)} &= \sqrt{(n^2-1)^2M^2 + 2n(n-1)(k^2-nK)Q^2} ,\end{aligned} \tag{3.26}$$

which agrees with Eq. (4.38) in [8] (with $k_V = k^2 - K$).

Finally in the scalar sector we obtain two significantly more complicated eigenvalues:

$$\begin{aligned}\tilde{W}^{(0)}_{\pm} &= \tilde{W}^{(0)}_1(r) \pm \Delta^{(0)}\tilde{W}^{(0)}_2(r) , \\ \Delta^{(0)} &= \sqrt{(n^2-1)^2M^2 + 4(n-1)^2(k^2-nK)Q^2} ,\end{aligned} \tag{3.27}$$

with $\tilde{W}^{(0)}_{1,2}$ functions too long to reproduce here.

This again agrees with KI, Eq. (5.61 - 5.63), although superficially they appear very different, in particular the structure of Eq. (3.27) is not visible in [8].

4 Stability

If we define $\Phi^{(h)}_s(t,r) = e^{-i\omega t}S(r)^{-n/2}\Psi^{(h)}_s(r)$, and evaluate Eq. (1.1) in Eddington-Finkelstein coordinates (2.5), we obtain the following Schrödinger-like equation,

$$X \equiv \partial_r\left(\frac{f}{\zeta}\partial_r\Psi(r)\right) - 2i\omega\partial_r\Psi(r) - \zeta V_S(r)\Psi(r) = 0 , \tag{4.1}$$

where for simplicity we drop s indices and h labels, but Ψ is still a vector with 1, 2 or 3 components for the tensor, vector and scalar channel respectively, and V_S is the corresponding Schrödinger potential matrix, that is related to the original potential W in Eq. (1.1) as in Eq. (3.24).

From the Schrödinger-like equation (4.1) it is possible to derive a sufficient, but not necessary, condition for linear stability of the corresponding fluctuation [25]. We review the argument here.

Start by defining the vanishing integral,

$$I \equiv -\int_{r_h}^{\infty}\bar{\Psi}X = \int_{r_h}^{\infty}\left(-\bar{\Psi}\partial_r\left(\frac{f}{\zeta}\partial_r\Psi\right) + 2i\omega\bar{\Psi}\partial_r\Psi + \zeta\bar{\Psi}V_S\Psi\right) . \tag{4.2}$$

By partial integration, this can be written as

$$I = \int_{r_h}^{\infty} \left(\frac{f}{\zeta} |\partial_r \Psi|^2 + 2i\omega \bar{\Psi} \partial_r \Psi + \zeta \bar{\Psi} V_S \Psi \right) - \frac{f}{\zeta} \bar{\Psi} \partial_r \Psi |_{r_h}^{\infty} \,. \tag{4.3}$$

Provided Ψ is regular at the horizon and dies off sufficiently fast at infinity, which are exactly the conditions for quasinormal modes, the boundary term vanishes.

From the above we obtain,

$$\mathrm{Im}(I) = 0 = \int_{r_h}^{\infty} \left(\omega \bar{\Psi} \partial_r \Psi + \bar{\omega} \Psi \partial_r \bar{\Psi} \right) \,, \tag{4.4}$$

where we have used that W, and thus V, is a real and symmetric matrix.

Now integrating the last term by parts we get

$$(\omega - \bar{\omega}) \int_{r_h}^{\infty} \bar{\Psi} \partial_r \Psi = \bar{\omega} |\Psi(r_h)|^2 - \bar{\omega} |\Psi(\infty)|^2 \,, \tag{4.5}$$

where the last term vanishes again assuming that Ψ dies off sufficiently fast.

Inserting this into I we finally obtain:

$$J \equiv \int_{r_h}^{\infty} \left(\frac{f}{\zeta} |\partial_r \Psi|^2 + \zeta \bar{\Psi} V_S \Psi \right) = -\frac{|\omega|^2}{\omega_I} |\Psi(r_h)|^2 \,. \tag{4.6}$$

From this we see that ω_I is negative, meaning the perturbation is stable, if and only if the integral J is positive. Since we do not know $\Psi(r)$, this is not directly useful. However a sufficient condition is that the eigenvalues of V are positive everywhere outside of the black hole.

We stress that it is not required that although in practice it helps if the equations can be further decoupled, it is not necessary for this argument. The only requirement is that the potential matrix be symmetric (or more generally Hermitian), which it explicitly is for any theory within our setup.

4.1 $\mathcal{S}$-deformation

We can get something more by transforming the integral with what is called an $\mathcal{S}$-deformation [8].

For some arbitrary, possibly matrix valued, function $\mathcal{S}$, define

$$\begin{aligned} \tilde{D} &\equiv \partial_r + \frac{\zeta}{f} \mathcal{S} \,, \\ \tilde{V}_S &\equiv V_S + \frac{1}{\zeta} \left(\mathcal{S}' - \frac{\zeta}{f} \mathcal{S}^2 \right) \,, \\ \tilde{J} &\equiv \int_{r_h}^{\infty} dr \left(\frac{f}{\zeta} |\tilde{D} \Psi|^2 + \zeta \bar{\Psi} \tilde{V}_S \Psi \right) \,. \end{aligned} \tag{4.7}$$

Provided the boundary term $S|\Psi|^2|_{r_h}^{\infty}$ vanishes and $\mathcal{S}$ is real and symmetric, or more generally a Hermitian matrix, $\tilde{J} = J$.

So if we can find any $\mathcal{S}$-deformation that makes the deformed potential positive everywhere, the corresponding perturbation is stable.

In practice, if the system can be decoupled it is usually easier to first decouple and then find an $\mathcal{S}$-deformation for the decoupled potentials. However, it is also possible, and indeed if they do not decouple the only way, to deform the potential matrix with a Hermitian matrix $\mathcal{S}$, and then try to show positivity of the eigenvalues of the deformed potential matrix.

This was used in [26] to prove stability of Reissner-Nordström black holes. If an analytic $\mathcal{S}$-deformation cannot be found one can also look for a numerical $\mathcal{S}$-deformation that is regular and makes the transformed potential vanish [27].

In the following sections we find analytic analytic $\mathcal{S}$-deformations to prove stability for various specific cases.

4.2 Stability of tensor perturbations

The tensor perturbations are the simplest of all, having a potential $W^{(2)}$ that comes only from the eigenvalue of the Laplacian, with the additional contribution of Eq. (3.24).

We can deform this with the $\mathcal{S}$-deformation

$$\mathcal{S}^{(2)} = -\frac{n}{2}\frac{fS'}{\zeta S}\,, \tag{4.8}$$

to obtain the manifestly positive deformed potential:

$$\tilde{V}_S^{(2)} = \frac{k^2}{S^2}\,. \tag{4.9}$$

Hence the tensor modes are always stable.

4.3 Stability of vector perturbations in Einstein-scalar theory

The vector sector is already significantly more complicated and we cannot prove stability in general.

However for specific cases we can, and curiously in the cases where it can be done, it can be done with the same simple $\mathcal{S}$-deformation:

$$\mathcal{S}^{(1)} = +\frac{n}{2}\frac{fS'}{\zeta S}\,, \tag{4.10}$$

the negative of the tensor one.

In particular if we turn off the gauge field, leaving just Einstein-scalar theory with an arbitrary potential, we are left with a single decoupled equation and the rather simple potential $W_{2,2}^{(1)}$. The modified potential becomes,

$$\tilde{V}_S^{(1)} = \frac{k^2 - nK}{S^2}\,. \tag{4.11}$$

This is manifestly positive for $K = 0$ and -1, and in the spherical case $k^2 = l(l+n-1)$, so the numerator becomes $(l-1)(l+n)$, also manifestly positive since in the vector sector $l \geq 1$.

4.4 Stability of vector perturbations of the GMGHS black hole

We now turn to a more involved application that has all the fields in our ansatz, an asymptotically flat charged black hole with a dilaton in 3+1 dimensions, which minimizes the action given by our ansatz (2.1), with

$$\begin{aligned} \eta &= 2\,, \\ V(\phi) &= 0\,, \\ \Lambda &= 0\,, \\ Z(\phi) &= 4e^{-2\alpha\phi}\,, \end{aligned} \tag{4.12}$$

where α is a free parameter corresponding to the dilaton coupling. For $\alpha = 0$ this reduces to the usual Reissner-Nordström action, while for $\alpha = 1$ this is the low energy effective action obtained from heterotic string theory.

The solution is given by [18, 19],

$$\begin{aligned} f(r) &= \left(1 - \frac{R_+}{r}\right)\left(1 - \frac{R_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}\,, \\ \zeta(r) &= 1\,, \\ S(r) &= r\left(1 - \frac{R_-}{r}\right)^{\frac{\alpha^2}{1+\alpha^2}}\,, \\ a(r) &= \sqrt{\frac{R_+R_-}{1+\alpha^2}}\frac{1}{r}\,, \\ e^{\alpha\phi(r)} &= \left(1 - \frac{R_-}{r}\right)^{\frac{\alpha^2}{1+\alpha^2}}\,, \end{aligned} \tag{4.13}$$

where $R_+ \geq R_-$ are the horizon and singularity respectively, in which the charge and mass can be expressed as: $2M = R_+ + \frac{1-\alpha^2}{1+\alpha^2}R_-$ and $Q^2 = \frac{R_+R_-}{1+\alpha^2}$.

For this solution, the master equations can be completely decoupled. Indeed this was already noted in [28], where stability was also argued for by numerical inspection of the decoupled potentials for several parameter values, although without analytical proof. Perturbations of this background were also analysed in the small charge approximation in [29].

Here we can prove stability analytically in the vector sector with the same $\mathcal{S}$-deformation of Eq. (4.10), and the process is only slightly more involved. The deformed potential takes the form,

$$\begin{aligned} \tilde{V}^{(1)}_{S\pm} = \frac{(1 - R_-/r)^{\frac{2}{1+\alpha^2}}}{2r(r-R_-)^2}\Bigg(&2R_-\left(1 + \delta l\left(3 + \delta l\right)\right) + 3\delta R + 2\delta l\left(3 + \delta l\right)\left(\delta r + \delta R\right) \pm \Delta \\ &+ \frac{4R_-}{1+\alpha^2}\Bigg)\,, \end{aligned} \tag{4.14}$$

where

$$\Delta^2 = \frac{16\delta l(\delta l + 3)\left(\alpha^2+1\right)R_-(\delta R + R_-) + \left(3\delta R\left(\alpha^2+1\right) + 2\left(\alpha^2+3\right)R_-\right)^2}{\left(\alpha^2+1\right)^2}\,, \tag{4.15}$$

and to be able to more easily show positivity we have defined the manifestly positive, or at least non-negative, quantities:

$$\begin{aligned} \delta R &= R_+ - R_- \,, \\ \delta r &= r - R_+ \,, \\ \delta l &= l - 1 \,. \end{aligned} \tag{4.16}$$

Now in $\tilde{V}^{(1)}_{S+}$ every symbol is positive, and there are no minus signs in the expression, so this is manifestly positive.

To show the same for $\tilde{V}^{(1)}_{S-}$, we have to show that Δ is smaller than the sum of the other terms in the expression. Or equivalently, we can show that Δ^2 is smaller than the square of the sum of the remaining terms. Simply writing this out using the same definitions as above, this is immediately seen to be true.

In the scalar channel the equations can also be decoupled. However the resulting potentials are very complicated and we have not been able to do a similar analytic stability proof in this case.

5 Discussion

We have reduced the problem of linear fluctuations in Einstein-Maxwell-scalar theories with maximally symmetric horizons to a small set of master scalars, one for each graviton polarization, in which everything can be expressed analytically. The equations fall into three sectors, tensor, vector and scalar, consisting of respectively a single decoupled equations and 2 and 3 coupled equations.

Although the potentials in the resulting master equations are rather complicated, the form is conceptually very simple. In fact it is not clear to us why we could obtain such a simple form, in particular with symmetric potential matrices in the coupled equations.

Furthermore, in several cases, such as Reissner-Nordström and the GMGHS black hole of [18, 19], the coupled equations can be decoupled further into fully decoupled equations. However there also exist analytic solutions, such as the asymptotically Lifshitz black brane of [20], where this cannot be done. It is not clear to us on a general level what distinguishes these theories.

The symmetry of the potential matrices allows one to derive a sufficient condition for stability, and the full decoupling makes it simpler to apply.

Furthermore our way of deriving the master equations is conceptually very simple and we believe would be rather simple to generalize to for instance other matter content. One simply has to find an ansatz which is sufficiently, but not too, general and ask Mathematica nicely to solve it for you.

We expect that our results can be straightforwardly generalized to include also time-dependence in the background, as was done in [8, 12, 21]

Finally we wish to comment on the differences with the Kovtun-Starinets (KS) approach [30] to solve fluctuation equations for black-branes, a widely used approach in holography. In appendix E we go into more detail. Instead of expressing all gauge-invariants

in terms of a master scalar, KS single out one gauge-invariant, $\mathfrak{h}_{tt}$, and decouple its equation from the others. We believe that the Kodama-Ishibashi approach that we follow has several advantages. On a more conceptual level, one explicitly solves all the equations of motion by solving the master scalar equations, and one can reconstruct analytically all of the components of the metric and the matter fields. Furthermore the master equations one obtains are covariant. On a practical, numerical, level, we find that the master equations are more accurate in finding the quasinormal modes in several ways. We discuss this in appendix E, along with a quantitative comparison.

In holographic studies a full quasinormal mode analysis is often lacking, especially in the most complicated sound channel. We hope that this work simplifies this sufficiently to make such a complete analysis more accessible and thus more common.

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A Spherical Case

A convenient way to parametrize all three n-dimensional maximally symmetric spaces at once is as $X = (x_1 \equiv x, x_2 \equiv y, x_3, ..., x_{n-1}, x_n \equiv z)$ and

$$\begin{aligned} dX^2_{(1)} &= dx_n^2 \,, \\ dX^2_{(i)} &= \frac{1}{\left(1 - Kx^2_{n-i+1}\right)} dx^2_{n-i+1} + \left(1 - Kx^2_{n-i+1}\right) dX^2_{(i-1)} \,, \end{aligned} \tag{A.1}$$

where $i = 2, ..., n$ and

$$K = \begin{cases} +1 & \text{spherical} \\ 0 & \text{planar} \\ -1 & \text{hyperbolic} \end{cases} . \tag{A.2}$$

The difficult part of the spherical case with respect to the planar is the decomposition into the three sectors. That is, to find an ansatz for the fluctuations analogous to Eq. (3.2), in such a way that the decomposition given in Table 3.1 still applies. Note that the very

simple ansatz in the planar case no longer suffices because under a rotation components get mixed.

Once such an ansatz has been found, everything else goes through in exactly the same manner, so here we discuss this ansatz.

To find this ansatz we follow the treatment of the harmonics on maximally symmetric spaces in Appendix B of [31].

The basic ingredient for constructing the different components is of course the scalar eigenfunction S of the Laplacian on the maximally symmetric space D^2, with eigenvalue k^2,

$$D^2 S + k^2 S = 0\,. \tag{A.3}$$

We can choose S that depends only on x, and the equation becomes

$$\left(1 - Kx^2\right) S''(x) = nKxS'(x) - k^2 S(x)\,, \tag{A.4}$$

with $k^2 = l(l+n-1)$.

With this we can immediately write the (t,r) part of the fluctuations as $(a,b \in \{t,r\})$

$$\delta g_{ab} = \begin{pmatrix} h_{tt} & 1/2h_{tr} \\ 1/2h_{tr} & h_{rr} \end{pmatrix} S(x)\,, \tag{A.5}$$

this piece can remain unchanged apart from the change of the scalar eigenfunction $S(x)$.

Any vector can be decomposed into a longitudinal and transverse part,

$$\begin{aligned} V &= V_L + V_T\,, \\ D^i V_{T,i} &= 0\,, \\ V_{L,i} &= \partial_i S\,. \end{aligned} \tag{A.6}$$

For the longitudinal part we already have an explicit expression in terms of S, and furthermore since S depends only on x it reduces to a single component. This will also contribute to the scalar sector and can be used to express h_{tx} and h_{rx}.

The transverse part will contribute to the vector sector and is not readily expressible in terms of the scalar S. It must however satisfy the equation (as does the longitudinal part),

$$D^2 V_T + \left(k^2 - K\right) V_T = 0\,, \tag{A.7}$$

with a shifted eigenvalue.

As in the planar case there are $n-1$ solutions, but it suffices to find a single one, since by symmetry all should satisfy the same equations. The simplest solution to this equation is,

$$\begin{aligned} V_{T,n} = S_V(x)\,, \quad V_{T,i} = 0 (i \neq n)\,, \\ \left(1 - Kx^2\right) S_V''(x) = (n-2)KxS_V'(x) - \left(k^2 + (n-2)K\right) S_V(x)\,. \end{aligned} \tag{A.8}$$

Note that we had to choose the last component to be nonzero in order to avoid explicit dependence on the other coordinates on the sphere.

With these we can express the ai part of the metric fluctuations, where $a \in \{t, r\}$ and i goes over the remaining coordinates, as,

$$\delta g_{ai} = \begin{pmatrix} h_{tx} S'(x) & 0 & \dots & 0 & h_{tz} S_V(x) \\ h_{rx} S'(x) & 0 & \dots & 0 & h_{rz} S_V(x) \end{pmatrix} , \tag{A.9}$$

and any symmetric tensor can be decomposed into a transverse, traceless and symmetric part, a longitudinal part and a trace part,

$$\begin{aligned} T &= T_{TT} + T_L + T_T \,, \\ T_{T,ij} &= S\Omega_{ij} \,, \\ T_{L,ij} &= D_i V_j + D_j V_i \,, \\ D^i T_{TT,ij} &= 0 \,, \\ T^i_{TT,i} &= 0 \,. \end{aligned} \tag{A.10}$$

So we already have explicit expressions for the trace and longitudinal part. The former contributes to the scalar sector through h_+, and the latter comes in two parts since for the vector it's built on we can take the longitudinal or transverse vector. It is more convenient to redefine these as:

$$\begin{aligned} T_{LT,ij} &= D_i V_{T,j} + D_j V_{T,i} \,, \\ T_{LL,ij} &= D_i V_{L,j} + D_j V_{L,i} - \frac{2}{n} D^k V_{L,k} \Omega_{ij} \,. \end{aligned} \tag{A.11}$$

Here T_{LT} will contribute to the vector sector through h_{xz}, and T_{LL} will contribute to the scalar sector through h_-.

The transverse traceless part must satisfy the equation (as do the other components),

$$D^2 T_{TT} + \left(k^2 - 2K\right) T_{TT} = 0 \,, \tag{A.12}$$

Again it suffices to find a single solution to this equation, which we have found to be,

$$\begin{aligned} T_{TT,yz} = T_{TT,zy} &= \left(1 - Ky^2\right)^{-(n-1)/2} \left(\prod_{i=3}^{n-1} (1 - Kx_i^2) \right) S_T(x) \,, \quad (T_{T,ij} = 0 \text{ otherwise})) \,, \\ \left(1 - Kx^2\right) S_T''(x) &= (n-4) K x S_T'(x) - \left(k^2 + 2K \left(1 - (n-3) \frac{Kx^2}{1 - Kx^2} \right) \right) S_T(x) \,. \end{aligned} \tag{A.13}$$

This component is in the tensor sector.

Summarizing, the full metric perturbation we do is,

$$\begin{aligned} \delta g_{\mu\nu} = &\left(h_{tt} dt^2 + h_{tr} dt dr + h_{rr} dr^2 + \frac{1}{n} h_+ \Omega_{ij} dx^i dx^j \right) S + 2 \left(h_{tx} dt dx + h_{rx} dr dx \right) S'(x) \\ &+ 2 \left(h_{tz} dt dz + h_{rz} dr dz \right) S_V + \left(\frac{1}{2} h_- T_{LL,ij} + h_{xz} T_{LT,ij} + h_{yz} T_{TT,ij} \right) dx^i dx^j \,, \end{aligned} \tag{A.14}$$

and this reduces exactly to Eq. (3.2) when $K = 0$.

B Special cases

In this appendix we discuss the special cases, which are $l = 1$ and $l = 0$ in the spherical case and $k = 0$ in the planar case. Although these cases are special, at the end of the day if one is interested in the quasinormal mode spectrum, one can use the potentials derived for the general case here as well.

B.1 Spherical $l = 1$ in vector sector

In the following derivation, we follow [32]. For $l = 1$, there is no h_{xz} in the harmonic decomposition, therefore $\mathfrak{h}_{tz}$ and $\mathfrak{h}_{rz}$ defined in (3.4) are not gauge invariant anymore. We can fix the gauge to set $h_{rz} = 0$. Then, from equations $E_{tz} = 0$ and $E_{rz} = 0$, we find:

$$\begin{aligned} \partial_r \left(\frac{h_{tz}}{S^2} \right) &= -\frac{\sqrt{Z} a'}{S} \Phi_1^{(1)} - c_1 \frac{\zeta}{S^{n+2}} \,, \\ a_z &= \frac{S}{\sqrt{Z}} \Phi_1^{(1)} \,, \end{aligned} \tag{B.1}$$

c_1 being an arbitrary constant. $\Phi_1^{(1)}$ fulfils an *inhomogeneous* wave equation:

$$\Box \Phi_1^{(1)} - W_{1,1}^{(1)} \Phi_1^{(1)} = c_1 \frac{\sqrt{Z} a'}{\zeta S^{n+1}} \,, \tag{B.2}$$

where potential $W_{1,1}^{(1)}$ is given by (3.12) with $K = 1$ and $k = \sqrt{n}$ ($l = 1$). The interpretation of (B.2) is the following: particular (stationary) solutions contribute to the angular momentum of the black hole (e.g. to linearised Kerr–Newman black hole in 3+1 dimensions), whereas the homogeneous solution is the dynamical degree of freedom of an electromagnetic field.

B.2 Spherical $l = 0$ and $l = 1$ in scalar sector

At $l = 0$ (with $K = 1$, so $k = 0$) the only dynamical degree of freedom is in the scalar field. Naively plugging this in into our potentials, one sees that now the interaction terms between the scalar master scalar and the others, $W_{0,1}^{(0)}$ and $W_{0,2}^{(0)}$ vanish. So one is left with a decoupled master equation for the only physical degree of freedom which is in the scalar field, and the potential is simply the one we found, $W_{0,0}^{(0)}$, which is perfectly regular for $k = 0$.

To obtain this result, we firstly use the fact that for $l = 0$ there are no h_{tx}, h_{rx}, h_- and a_x components of perturbations and no ξ_x gauge vector component. Gauge invariants defined for $k^2 > n$ do not make sense anymore. Instead, we can use ξ_t, ξ_r and λ to set e.g. h_{tr}, h_+ and a_t to zero. Making such a choice, we are left with four variables: h_{tt}, h_{tr}, a_r, φ. As expected, scalar field is the only dynamical variable in the system:

$$\varphi = \Phi_0^{(0)} \,. \tag{B.3}$$

This master scalar satisfies the wave equation with the potential $W_{0,0}^{(0)}$ of the general case, with $l = 0$ plugged in, but, as well as in the vector $l = 1$ case, this wave equation is

inhomogeneous:

$$\Box\Phi_0^{(0)} - W_{0,0}^{(0)}\Phi_0^{(0)} = \frac{c_0(a'^2 Z' - 2\zeta^2 V')}{4f\zeta\eta S^{n-1}S'} + \frac{c_0\left(Sf' + f(n-1)S'\right)\phi'}{f\zeta S^n S'} - \frac{c_0\eta\phi'^3}{\zeta n S^{n-2}S'^2}\,, \tag{B.4}$$

c_0 being an arbitrary constant. The other fluctuations can be found from the Einstein equations directly. In contrary to $l \geq 2$, however, not all of them can be directly expressed by a master scalar and it's derivatives, but integration for h_{tt} and a_r will be necessary:

$$\begin{aligned}
h_{rr} &= \frac{c_0\zeta^3}{f^2 S^{n-1}S'} + \frac{2\zeta^2\eta S\phi'}{fnS'}\Phi_0^{(0)}\,,\\
f\partial_r\left(\frac{h_{tt}}{f}\right) &= \left(\frac{S\left(2\zeta^2 V' - Z'a'^2 - 4\eta f'\phi'\right)}{2nS'} + \frac{2f\eta^2 S^2\phi'^3}{n^2 S'^2} - \frac{2(n-1)\eta f\phi'}{n}\right)\Phi_0^{(0)} + \\
&\quad - \frac{2f\eta S\phi'}{nS'}\partial_r\Phi_0^{(0)} + \frac{c_0\zeta\left(f\eta S^2\phi'^2 - nSf'S' - fn(n-1)S'^2\right)}{fnS^n S'^2},\\
\partial_t a_r &= \frac{a'h_{tt}}{2f} + \frac{1}{n}a'\left(\frac{nZ'}{Z} - \frac{\eta S\phi'}{S'}\right)\Phi_0^{(0)} - \frac{c_0\zeta a'}{2fS^{n-1}S'}\,.
\end{aligned} \tag{B.5}$$

The constant c_0 corresponds to a static perturbation of the zeroth order solution (e.g. to a shift of mass in the Reissner-Nordström background).

The case $l = 1$ (with $K = 1$, so $k^2 = n$) is special because since the metric has spin 2, it does not have any dynamical degrees of freedom with $l = 1$. This is the reason that we see factors of $\sqrt{k^2 - nK}$ in the potentials. In particular this factor occurs as a prefactor in all interaction potentials of the gravitational master scalar with the others, so $W_{1,2}^{(1)}$, $W_{0,2}^{(0)}$ and $W_{1,2}^{(0)}$. So simply plugging in $l = 1$ in the potentials we have the master equations for the scalar and gauge field fluctuations, which do have physical degrees of freedom with $l = 1$, and they decouple from the unphysical gravitational degrees of freedom.

To see this more concretely, note that for $l = 1$ there is no h_- component (the spherical harmonic can be explicitly solved in this case to be $S(x) = x$, and the T_{LL} tensor that defines h_- is given by second derivatives of this, hence vanishing). This means that the gauge-invariants of (3.4) are no longer gauge invariant. In particular they transform under

the infinitesimal coordinate transformation component ξ_x as,

$$
\begin{aligned}
\mathfrak{h}_{tt} &\to \mathfrak{h}_{tt} + \frac{f'}{SS'}\xi_x + 2\partial_t^2\xi_x\,,\\
\mathfrak{h}_{tr} &\to \mathfrak{h}_{tr} + 2\partial_t\partial_r\xi_x - \frac{2\left(Sf'S' + \zeta^2\right)}{fSS'}\partial_t\xi_x\,,\\
\mathfrak{h}_{rr} &\to \mathfrak{h}_{rr} + \frac{\zeta^2\left(Sf'S' + 2fS'^2 - \frac{2f\eta S^2\phi'^2}{n}\right)}{f^2S^2S'^2}\xi_x - \frac{2\zeta^2}{fSS'}\partial_r\xi_x\,,\\
\mathfrak{h}_{rx} &\to \mathfrak{h}_{rx} + \frac{\left(2S'^2 - \frac{\zeta^2}{f}\right)}{SS'}\xi_x - \partial_r\xi_x\,,\\
\mathfrak{a}_t &\to \mathfrak{a}_t - \frac{a'}{SS'}\xi_x\,,\\
\mathfrak{a}_r &\to \mathfrak{a}_r + \frac{a'}{f}\partial_t\xi_x\,,\\
\varphi &\to \varphi - \frac{\phi'}{SS'}\xi_x\,,
\end{aligned}
\tag{B.6}
$$

but are still invariant under the other components.

So we have an extra gauge choice that we are free to make. If we take the simple $\xi_x = -S^2\Phi_2^{(0)}$ and plug this into equations (C.1) to (C.5) expressing the gauge-invariants in terms of the master scalars, we see that $\Phi_2^{(0)}$ completely drops out from all expressions. This confirms the fact that there are no dynamical degrees of freedom in the metric with $l = 1$.

B.3 Planar $k = 0$

Without momentum there is no way to distinguish between what at finite momentum were different channels, and so by symmetry one expects the physics to be the same in all channels. This is seen explicitly in the approach of Kovtun and Starinets [30], where the equations for metric fluctuations become identical in each channel, and those for vector perturbations become identical to each other as well.

In our results, setting the momentum to zero makes all the equations decouple. However, the potentials for a given field are not all identical. In particular, the potential for the metric master scalar in the scalar channel vanishes, but in the vector channel it does not. The gauge field potentials do not vanish and are not equal to each other either.

This is not what we expect by symmetry, but there is an elegant solution. Defining Schrödinger-like potentials in Fefferman-Graham or Schwarzschild-like coordinates as[5],

$$
\frac{d^2\Psi}{dr_\star^2} + \left(\omega^2 - \tilde{V}_S\right)\Psi = 0\,, \tag{B.7}
$$

with $\partial_{r_\star} = \frac{f}{\zeta}\partial_r$, then if two of these potentials can be written in terms of a single super potential W_S as

$$
\tilde{V}_{S,\pm} = W_S^2 \mp \frac{dW_S}{dr_\star} + \beta\,, \tag{B.8}
$$

[5]This makes $\tilde{V}_S = f(r)V_S$ with V_S defined in Eq. (3.24).

then these two potentials are isospectral, having the same set of quasinormal modes [33, 34] (see also appendix A of [35]).

As expected by symmetry, we can write the zero-momentum potentials in this way:

$$
\begin{aligned}
(\tilde{V}_S)^{(0)}_{2,2}(r) &= W^2_{S,2} + \frac{dW_{S,2}}{dr_\star}\,, \\
(\tilde{V}_S)^{(1)}_{2,2}(r) &= W^2_{S,2} - \frac{dW_{S,2}}{dr_\star}\,, \\
W_{S,2} &= \frac{n}{2}\frac{fS'}{\zeta S}\,.
\end{aligned}
\tag{B.9}
$$

Curiously, this super potential is exactly equal to the $\mathcal{S}$-deformation we had to do to show stability in the vector sector.

Similarly for the gauge field we can write the potentials as,

$$
\begin{aligned}
(\tilde{V}_S)^{(0)}_{1,1}(r) &= W^2_{S,1} + \frac{dW_{S,1}}{dr_\star}\,, \\
(\tilde{V}_S)^{(1)}_{1,1}(r) &= W^2_{S,1} - \frac{dW_{S,1}}{dr_\star}\,, \\
W_{S,1} &= -(n-2)\frac{fS'}{2\zeta S} - \frac{f\phi' Z'}{2\zeta Z} - \frac{fSa'^2 Z}{\zeta S f' - 2f\zeta S'}\,.
\end{aligned}
\tag{B.10}
$$

C Scalar sector master scalars

The relation between the gauge invariant fluctuations in the scalar sector and the master scalars is as follows. For the scalar field and gauge field they are, using $\tilde{k} \equiv \sqrt{k^2 - nK}$ and $\tilde{n} = \sqrt{(n-1)/n}$,

$$
\begin{aligned}
\boldsymbol{\varphi} &= -\frac{1}{\sqrt{\eta}}\Phi_0^{(0)} + \frac{k}{n\tilde{n}\tilde{k}}\frac{S\phi'}{S'}\Phi_2^{(0)}\,, \\
\mathfrak{a}_r &= -\frac{\sqrt{2}}{k}\frac{S\zeta}{f\sqrt{Z}}\partial_t\Phi_1^{(0)} - \frac{1}{\tilde{n}k\tilde{k}}\frac{S^2 a'}{f}\partial_t\Phi_2^{(0)}\,, \\
\mathfrak{a}_t &= \frac{1}{\tilde{n}k\tilde{k}}\frac{Sa'}{\mathcal{D}nS'}\left(nS'\left(k^2 Sf' + 2fS'\left((n-2)k^2 - n(n-1)K\right)\right) + 2\zeta^2 k^4\right)\Phi_2^{(0)} \\
&-\frac{f}{2\mathcal{D}\zeta Z^{3/2}}\Bigg[-4\zeta\sqrt{\eta}S^2 a' Z^{3/2}\phi'\Phi_0^{(0)} + \frac{2\sqrt{2}}{k}\mathcal{D}SZ\partial_r\Phi_1^{(0)} \\
&+\frac{\sqrt{2}}{k}\left(2S'Z\left(nS^2 a'^2 Z + \mathcal{D}(n-1)\right) + \mathcal{D}S\phi' Z'\right)\Phi_1^{(0)}\Bigg]\,.
\end{aligned}
\tag{C.1}
$$

Note the factors of $1/\tilde{n}$ which, here and below, always occur in front of the gravitational master scalar, indicating the fact that in 3 dimensions there are no dynamical degrees of freedom in the metric. For $n = 1$ there is no h_- component and with the same procedure as in appendix B.2 above we can show that the same potentials apply to this case.

For the metric fluctuations, we can first express $\mathfrak{h}_{tr}$ directly in terms of the others,

$$
\mathfrak{h}_{tr} = 2\partial_t\mathfrak{h}_{rx} + \frac{4\eta}{k^2}S^2\phi'\partial_t\boldsymbol{\varphi} - \frac{2n}{k^2}\frac{SS'f}{\zeta^2}\partial_t\mathfrak{h}_{rr}\,. \tag{C.2}
$$

The remaining three are as follows,

$$\mathfrak{h}_{rr} = -2\frac{\zeta^2 S}{\mathcal{D}f}\left(-\sqrt{\eta}\phi'\mathcal{F}\Phi_0^{(0)} + \sqrt{2}\zeta k a'\sqrt{Z(\phi)}\Phi_1^{(0)}\right) + \frac{k}{n\tilde{n}\tilde{k}}\frac{2\zeta^2 S}{fS'}\partial_r\Phi_2^{(0)}$$
$$+\frac{k}{\tilde{n}\tilde{k}}\frac{\zeta^2}{\mathcal{D}f^2S'^2}\Bigg(2Sf'S'\left(2fS'^2-\frac{1}{n}\zeta^2k^2\right)-S^2f'^2S'^2+4f\big(-fS'^4+\zeta^2S'^2\big(k^2-(n-1)K\big)+\frac{\eta}{n^2}k^2\zeta^2S^2\phi'^2$$
$$-\frac{\eta}{2n}\mathcal{F}S^2S'\phi'^2\big)\Bigg)\Phi_2^{(0)}\,, \tag{C.3}$$

$$\mathfrak{h}_{rx} = \frac{\zeta S^2}{\mathcal{D}}\left(2\sqrt{\eta}\zeta\phi'\Phi_0^{(0)} - \frac{\sqrt{2}}{k}nS'a'\sqrt{Z}\Phi_1^{(0)}\right) + \frac{1}{\tilde{n}k\tilde{k}}S^2\partial_r\Phi_2^{(0)}$$
$$+\frac{1}{n\tilde{n}k\tilde{k}}\frac{\zeta^2 S}{S'f\mathcal{D}}\big(nS'\big(k^2Sf'+2fS'\left((n-2)k^2-n(n-1)K\right)\big)+2k^4\zeta^2\big)\,\Phi_2^{(0)} \tag{C.4}$$

$$\mathfrak{h}_{tt} = \frac{fS}{\sqrt{\eta}\mathcal{D}^2\zeta\sqrt{Z}}\Bigg(-4\eta\zeta Sf\mathcal{D}\sqrt{Z}\phi'\partial_r\Phi_0^{(0)} + \frac{2\sqrt{2\eta}}{k}nSS'f\mathcal{D}Za'\partial_r\Phi_1^{(0)}$$
$$+\zeta\sqrt{Z}\Bigg\{\Bigg[2\eta\big(4fS'\big(n(n-2)S'^2f+\zeta^2\big(k^2-n(n-1)K\big)\big)-2Sf'\big(n(n-3)S'^2f+k^2\zeta^2\big)-nS^2f'^2S'\big)\Bigg]\phi'$$
$$+Sa'^2\mathcal{D}Z'-2\zeta^2S\left(-nS'\mathcal{F}+2\zeta^2k^2\right)V'+4\eta^2S^2f\mathcal{F}\phi'^3+4\eta nS^2S'fa'^2\phi'Z\Bigg\}\Phi_0^{(0)}$$
$$-\frac{\sqrt{2\eta}}{k}a'\Bigg\{2fn^2S^2Z^2a'^2S'^2+fnSS'Z'\left(-nS'\mathcal{F}+2\zeta^2k^2\right)\phi'+4f\zeta^2\eta k^2S^2Z\phi'^2$$
$$+2Z\Bigg[2f^2(n-1)n^2S'^4-nSf'S'\left(f(n-3)nS'^2+\zeta^2k^2\right)-n^2S^2f'^2S'^2-2f\zeta^2nS'^2\left(k^2+n(n-1)K\right)+2\zeta^4k^4\Bigg]$$
$$\Bigg\}\Phi_1^{(0)}\Bigg) - \frac{2}{\tilde{n}k\tilde{k}}\frac{S^2f^2}{\zeta^2}\partial_r^2\Phi_2^{(0)} - \frac{2}{\tilde{n}k\tilde{k}}\frac{fS}{\mathcal{D}\zeta^3}\Bigg[Sf'\left(-fnS\zeta'S'+f\zeta(n-2)nS'^2+2\zeta^3k^2\right)+\zeta nS^2f'^2S'$$
$$-2f\left(f\zeta n^2S'^3-fnS\zeta'S'^2+\zeta^3S'\left(k^2(1-2n)+K(n-1)n\right)+\zeta^2k^2S\zeta'\right)\Bigg]\partial_r\Phi_2^{(0)}$$
$$+\frac{1}{n\tilde{n}k\tilde{k}}\frac{1}{S'\mathcal{D}^2}\Bigg\{-k^2n^2S^3f'^3S'^2+2nS^2f'^2S'\left(fnS'^2\left(2K(n-1)n-3k^2(n-2)\right)-2\zeta^2k^4\right)$$
$$+4Sf'\Bigg[fnS'^2\left(fnS'^2\left(-(2n(n-4)+9)k^2+2n(n-1)(n-2)K\right)+\zeta^2k^2\left((7-3n)k^2+n(n-1)K\right)\right)-\zeta^4k^6\Bigg]$$
$$+8fS'\Bigg[fnS'^2\left(fnS'^2\left((n(2n-5)+4)k^2-2n(n-1)^2K\right)+\zeta^2\big(-((n-3)n+4))k^4+n(n-1)Kk^2+n^2(n-1)^2K^2\big)\right)$$
$$+\zeta^4k^4\left(k^2-n(n-1)K\right)+\frac{n-1}{2}\tilde{k}^2fS^2(n^2Za'^2S'^2+2k^2\eta\zeta^2\phi'^2)\Bigg]\Bigg\}\Phi_2^{(0)} \tag{C.5}$$

D Transformations

D.1 Regge-Wheeler gauge invariants for the scalar sector

Firstly, let's remind slight difference between using Detweiler gauge and using Detweiler gauge invariants: Detweiler gauge is a certain choice of gauge, where we put all the scalar sector metric coefficients apart from $h_{tt}, h_{tr}, h_{rr}, h_{rx}$ to zero, whereas in Detweiler gauge invariant formulation, we work with $\mathfrak{h}^D_{tt}, \mathfrak{h}^D_{tr}, \mathfrak{h}^D_{rr}, \mathfrak{h}^D_{rx}, \mathfrak{a}^D_t, \mathfrak{a}^D_r, \boldsymbol{\varphi}^D$, which do not feel gauge transformations at all (from now superscripts D and RW correspond to Detweiler and Regge–Wheeler respectively). Importantly, in Detweiler gauge non-zero quantities correspond exactly to Detweiler gauge invariants: $h^D_{tt} = \mathfrak{h}^D_{tt}, h^D_{tr} = \mathfrak{h}^D_{tr}, h^D_{rr} = \mathfrak{h}^D_{rr}, h^D_{rx} = \mathfrak{h}^D_{rx}, a^D_t = \mathfrak{a}^D_t, a^D_r = \mathfrak{a}^D_r, \varphi^D = \boldsymbol{\varphi}^D$ (compare with 3.4), analogously for Regge–Wheeler.

Regge-Wheeler (RW) gauge invariants are another set of gauge invariants which can be build in a way described in 3.2. In principle, we could build them from the beginning, by "dressing" $h_{tt}, h_{tr}, h_{rr}, h_+, a_t, a_r, \varphi$ with the remaining metric components to make them gauge invariant. However, RW gauge invariants, as well as any other set of independent gauge invariants in this sector, must be function of Detweiler gauge invariants. To find this relation, we use the previous observation, that in the Detweiler gauge non-zero quantities correspond exactly to Detweiler gauge invariants, the same for RW. It means that it's sufficient to find the transformation between Detweiler and RW gauge and translate it into relation between gauge invariants, which reduces to moving from $h^D_+ = 0$ and $h^D_{rx} \neq 0$ to $h^{RW}_+ \neq 0$ and $h^{RW}_{rx} = 0$. It can be done by acting with a gauge vector $\zeta^\mu = (0, h^D_{rx}, 0, .., 0)e^{ikx}$. Finally, the relations between Detweiler and RW gauge invariants read:

$$
\begin{aligned}
\mathfrak{h}^{RW}_{tt} =& \mathfrak{h}^D_{tt} + \frac{ff'}{\zeta^2}\mathfrak{h}^D_{rx}\,, \\
\mathfrak{h}^{RW}_{tr} =& \mathfrak{h}^D_{tr} - 2\partial_t \mathfrak{h}^D_{rx}\,, \\
\mathfrak{h}^{RW}_{rr} =& \mathfrak{h}^D_{rr} + \left(\frac{2\zeta'}{\zeta} - \frac{f'}{f}\right)\mathfrak{h}^D_{rx} - 2\partial_r \mathfrak{h}^D_{rx}\,, \\
\mathfrak{h}^{RW}_{+} =& \frac{-2nfSS'}{\zeta^2}\mathfrak{h}^D_{rx}\,, \\
\mathfrak{a}^{RW}_{t} =& \mathfrak{a}^D_t - \frac{fa'}{\zeta^2}\mathfrak{h}^D_{rx}\,, \\
\mathfrak{a}^{RW}_{r} =& \mathfrak{a}^D_r\,, \\
\boldsymbol{\varphi}^{RW} =& \boldsymbol{\varphi}^D - \frac{f\phi'}{\zeta^2}\mathfrak{h}^D_{rx}\,.
\end{aligned}
\tag{D.1}
$$

D.2 Transformation between Fefferman-Graham and Eddington-Finkelstein coordinates

Potentials in master equations (1.1) have the same form for both Fefferman-Graham (FG), or Schwarzschild-like, and Eddington-Finkelstein (EF) coordinates. The difference in equations appears only in the form of laplacian - for numerical purposes it's probably more useful to use EF coordinates, since master equations involve only first time derivatives then.

How to express Detweiler gauge invariants in EF coordinates in terms of Detweiler gauge invariants in FG coordinates? (Again, these are distinct quantities related by some functions). Let's start with linear metric and gauge vector perturbations, which transform as:

$$\begin{aligned} h_{\mu\nu}^{EF} &= L_\mu^\alpha L_\nu^\beta h_{\alpha\beta}^{FG}\,, \\ a_\mu^{EF} &= L_\mu^\alpha a_\alpha^{FG}\,, \end{aligned} \tag{D.2}$$

where

$$(L_\nu^\mu) = \begin{pmatrix} 1 & -\frac{\zeta}{f} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{1}_n \end{pmatrix}\,. \tag{D.3}$$

Since we already know how to express $h_{\mu\nu}$ and a_μ by Detweiler gauge invariants, we can use the transformation rule (D.2) to express $\mathfrak{h}_{\mu\nu}^{D,EF}$ and $\mathfrak{a}_\mu^{D,EF}$ by $\mathfrak{h}_{\mu\nu}^{D,FG}$ and $\mathfrak{a}_\mu^{D,FG}$. For vector sector it reads (we need to add another superscripts: FG or EF and in vector sector we can omit D and RW, since they are the same):

$$\begin{aligned} \mathfrak{h}_{tz}^{EF} &= \mathfrak{h}_{tz}^{FG}\,, \\ \mathfrak{h}_{rz}^{EF} &= \mathfrak{h}_{rz}^{FG} - \frac{\zeta}{f}\mathfrak{h}_{tz}^{FG}\,, \\ \mathfrak{a}_z^{EF} &= \mathfrak{a}_z^{FG}\,. \end{aligned} \tag{D.4}$$

and for scalar sector:

$$\begin{aligned} \mathfrak{h}_{tt}^{D,EF} &= \mathfrak{h}_{tt}^{D,FG}\,, \\ \mathfrak{h}_{tr}^{D,EF} &= \mathfrak{h}_{tr}^{D,FG} - 2\frac{\zeta}{f}\mathfrak{h}_{tt}^{D,FG}\,, \\ \mathfrak{h}_{rr}^{D,EF} &= \mathfrak{h}_{rr}^{D,FG} - \frac{\zeta}{f}\mathfrak{h}_{tr}^{D,FG} + \left(\frac{\zeta}{f}\right)^2 \mathfrak{h}_{tt}^{D,FG}\,, \\ \mathfrak{h}_{rx}^{D,EF} &= \mathfrak{h}_{rx}^{D,FG}\,, \\ \mathfrak{a}_t^{D,EF} &= \mathfrak{a}_t^{D,FG}\,, \\ \mathfrak{a}_r^{D,EF} &= \mathfrak{a}_r^{D,FG} - \frac{\zeta}{f}\mathfrak{a}_t^{D,FG}\,. \end{aligned} \tag{D.5}$$

RW gauge invariants transform analogously as (D.5), with one difference: $\mathfrak{h}_+^{RW,EF} = \mathfrak{h}_+^{RW,FG}$.

To fully move to EF coordinates, namely to express $\mathfrak{h}_{\mu\nu}^{D,EF}$ in terms of master scalars like we did for FG coordinates ((C.1)-(C.5)), one should transform derivatives as well: $\partial_t \to \partial_t$, $\partial_r \to \partial_r + \frac{\zeta}{f}\partial_t$. For example, gauge invariants in EF coordinates in vector sector,

express by master scalars in the following way:

$$
\begin{aligned}
\mathfrak{h}_{tz}^{EF} &\equiv n\frac{fSS'}{\zeta}\Phi_2^{(1)} + \frac{fS^2}{\zeta}\partial_r\Phi_2^{(1)}\,,\\
\mathfrak{h}_{tz}^{EF} &\equiv \frac{nfSS'\Phi_2^{(1)}}{\zeta} + \frac{fS^2\partial_r\Phi_2^{(1)}}{\zeta} + S^2\partial_t\Phi_2^{(1)}\,,\\
\mathfrak{h}_{rz}^{EF} &\equiv -S^2\partial_r\Phi_2^{(1)} - nSS'\Phi_2^{(1)}\\
\mathfrak{a}_z^{EF} &\equiv \sqrt{k^2-nK}\frac{S}{\sqrt{Z}}\Phi_1^{(1)}\,.
\end{aligned}
\tag{D.6}
$$

Having transformation rules from this paragraph and from (D.1) one can move from (3.7, (C.1)-(C.5)) to the desired gauge and coordinate system without performing calculations from the beginning.

E Quasinormal modes of AdS planar black holes

In this appendix we derive master scalar wave equations for gravitational black brane perturbations in our approach. To ease the comparison with the Kovtun-Starinets (KS) approach [30], widely used in holography, we stick in this section to KS notation and discuss in detail the scalar (sound in KS terminology) sector of perturbations. The background line element, eq. (KS-4.2), reads

$$
ds^2 = a(u)\left(-f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + b(u)du^2\,, \tag{E.1}
$$

with

$$
a(u) = \frac{(r_0/R)^2}{u} \qquad \text{and} \qquad b(u) = \frac{R^2}{4u^2 f(u)}\,, \tag{E.2}
$$

where u is the AdS bulk variable (with the planar black hole horizon located at $u = 1$ and AdS boundary at $u = 0$), $f(u) = 1 - u^2$, R is the AdS radius and r_0 is related to black hole temperature: $T = r_0/(\pi R^2)$. We take gravitational fluctuations in the form $h_{\mu\nu}(t, u, z) = h_{\mu\nu}(t, u)e^{iqz}$. Under a gauge transformation induced by a gauge vector $\xi_\mu(t, u, z) = \xi_\mu(t, u)e^{iqz}$ these fluctuations transform as

$$
h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu\,. \tag{E.3}
$$

There are seven components of $h_{\mu\nu}$ that enter linearized Einstein equations in the KS sound sector, namely h_{tt}, h_{tu}, h_{uu}, h_{uz}, h_{tz}, h_{zz}, and $h = h_{xx} + h_{yy}$ and out of them four gauge invariant characteristics of fluctuations can be constructed. The Detweiler gauge invariants (i.e. gauge invariants obtained by dressing h_{tt}, h_{tu}, h_{uu}, and h_{uz} with linear combinations

of h_{tz}, h_{zz}, and h and their derivatives) read:

$$\mathfrak{h}_{tt} = h_{tt} + \frac{1}{2}\left(f(u) + \frac{a(u)}{a'(u)}f'(u)\right)h + \frac{2i}{q}\dot{h}_{tz} + \frac{\ddot{h}_{tz}}{2q^2} - \frac{\ddot{h}_{zz}}{q^2}\,, \tag{E.4}$$

$$\begin{aligned}\mathfrak{h}_{tu} = h_{tu} &- \frac{i}{q}\left(\frac{a'(u)}{a(u)} + \frac{f'(u)}{f(u)}\right)h_{tz} + \frac{i}{q}h'_{tz} - \left(\frac{b(u)}{2a'(u)} + \frac{1}{4q^2}\left(\frac{a'(u)}{a(u)} + \frac{f'(u)}{f(u)}\right)\right)\dot{h} \\ &+ \frac{1}{2q^2}\left(\frac{a'(u)}{a(u)} + \frac{f'(u)}{f(u)}\right)\dot{h}_{zz} + \frac{1}{4q^2}\dot{h}' - \frac{1}{2q^2}\dot{h}'_{zz}\,,\end{aligned} \tag{E.5}$$

$$\mathfrak{h}_{uu} = h_{uu} - \frac{b(u)}{a'(u)}h' - \frac{a'(u)b'(u) - 2b(u)a''(u)}{2\left(a'(u)\right)^2}h\,, \tag{E.6}$$

$$\mathfrak{h}_{uz} = h_{uz} - \frac{ia'(u)}{2qa(u)}h_{zz} + \left(\frac{q^2b(u)}{2a'(u)} - \frac{a'(u)}{4a(u)}\right)h + \frac{h'}{4} - \frac{h'_{zz}}{2}\,, \tag{E.7}$$

(and $\mathfrak{h}_{tt}$ corresponds to KS Z_2, cf. eq. (KS-3.12)). Indeed, it can be easily checked that the above expressions are gauge invariant, and moreover when (E.4-E.7) are solved for h_{tt}, h_{uu}, h_{tu}, and h_{uz} and these solutions are substituted into Einstein equations $E_{\mu\nu} := R_{\mu\nu} + \frac{4}{R}g_{\mu\nu} = 0$, all gauge dependent terms drop out at linear order and the linearized equations read:

$$E_{tu} = \frac{3u\left(u^2-1\right)}{R^2}\dot{\mathfrak{h}}_{uu} + \frac{q^2R^2u}{2r_0^2}\mathfrak{h}_{tu} + \frac{iqR^2u}{2r_0^2}\dot{\mathfrak{h}}_{uz} + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,, \tag{E.8}$$

$$\begin{aligned}E_{tz} &= 2iq\left(u^2-1\right)u^2\dot{\mathfrak{h}}_{uu} - 2iq\left(u^2-1\right)u^2\mathfrak{h}'_{tu} - 4iqu^3\mathfrak{h}_{tu} - 2\left(u^2-1\right)u^2\dot{\mathfrak{h}}'_{uz} \\ &\quad + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,,\end{aligned} \tag{E.9}$$

$$\begin{aligned}E_{uu} &= -\frac{R^2u}{2r_0^2\left(u^2-1\right)}\ddot{\mathfrak{h}}_{uu} + \frac{2u\left(u^2-2\right)}{R^2}\mathfrak{h}'_{uu} + \left(\frac{q^2R^2u}{2r_0^2} + \frac{4\left(2u^4-4u^2+1\right)}{R^2\left(u^2-1\right)}\right)\mathfrak{h}_{uu} \\ &\quad + \frac{R^2\left(2u^2-1\right)}{r_0^2\left(u^2-1\right)^2}\dot{\mathfrak{h}}_{tu} + \frac{R^2u}{r_0^2\left(u^2-1\right)}\dot{\mathfrak{h}}'_{tu} - \frac{R^2u}{2r_0^2\left(u^2-1\right)}\mathfrak{h}''_{tt} - \frac{R^2\left(u^2-2\right)}{2r_0^2\left(u^2-1\right)^2}\mathfrak{h}'_{tt} \\ &\quad + \frac{R^2u\left(u^2-3\right)}{2r_0^2\left(u^2-1\right)^3}\mathfrak{h}_{tt} + \frac{iqR^2u}{r_0^2}\mathfrak{h}'_{uz} + \frac{iqR^2\left(2u^2-1\right)}{r_0^2\left(u^2-1\right)}\mathfrak{h}_{uz} + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,,\end{aligned} \tag{E.10}$$

$$\begin{aligned}E_{uz} &= \frac{iqu\left(u^2-3\right)}{R^2}\mathfrak{h}_{uu} + \frac{iqR^2u}{2r_0^2\left(u^2-1\right)}\dot{\mathfrak{h}}_{tu} - \frac{iqR^2u}{2r_0^2\left(u^2-1\right)}\mathfrak{h}'_{tt} + \frac{iqR^2}{2r_0^2\left(u^2-1\right)^2}\mathfrak{h}_{tt} \\ &\quad - \frac{R^2u}{2r_0^2\left(u^2-1\right)}\ddot{\mathfrak{h}}_{uz} + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,,\end{aligned} \tag{E.11}$$

$$
\begin{aligned}
R^2 E_{tt} = & \frac{4r_0^2 u^2 \left(u^2-1\right)^2 \left(u^2+1\right)}{R^4} \mathfrak{h}'_{uu} + 2u^2 \left(u^2-1\right) \ddot{\mathfrak{h}}_{uu} + \frac{8r_0^2 u \left(u^2-1\right)\left(2u^4-u^2+1\right)}{R^4} \mathfrak{h}_{uu} \\
& - u\left(u^2+1\right) \mathfrak{h}'_{tt} + 2u^2\left(u^2-1\right) \mathfrak{h}''_{tt} + \left(\frac{q^2 R^4 u}{2r_0^2} + \frac{u^4+6u^2-3}{u^2-1}\right) \mathfrak{h}_{tt} \\
& - 2u\left(u^2+1\right) \dot{\mathfrak{h}}_{tu} - 4u^2\left(u^2-1\right) \dot{\mathfrak{h}}'_{tu} - 2iqu\left(u^4-1\right) \mathfrak{h}_{zu} + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,, \qquad \text{(E.12)} \\
R^2 E_{zz} = & -\frac{4r_0^2\left(u^2-1\right)^2 u^2}{R^4} \mathfrak{h}'_{uu} - \frac{2\left(u^2-1\right) u \left(q^2 R^4 u + r_0^2\left(8u^2+4\right)\right)}{R^4} \mathfrak{h}_{uu} + 2u\dot{\mathfrak{h}}_{tu} \\
& - u\mathfrak{h}'_{tt} + \frac{\left(q^2 R^4 u + 2r_0^2\left(u^2+1\right)\right)}{2r_0^2\left(u^2-1\right)} \mathfrak{h}_{tt} - 4iq\left(u^2-1\right) u^2 \mathfrak{h}'_{uz} - 2iq\left(3u^2+1\right) u\mathfrak{h}_{uz} \\
& + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,, \qquad \text{(E.13)}
\end{aligned}
$$

$$
\begin{aligned}
R^2\left(E_{xx}+E_{yy}\right) = & -\frac{4r_0^2 u^2\left(u^2-1\right)^2 \mathfrak{h}'_{uu}}{R^4} + \frac{8r_0^2 u\left(-2u^4+u^2+1\right)}{R^4} \mathfrak{h}_{uu} + 2u\dot{\mathfrak{h}}_{tu} \\
& - u\mathfrak{h}'_{tt} + \frac{\left(u^2+1\right)}{u^2-1} \mathfrak{h}_{tt} + 2iqu\left(u^2-1\right) \mathfrak{h}_{uz} + \mathcal{O}\left(h_{\mu\nu}^2\right) = 0\,. \qquad \text{(E.14)}
\end{aligned}
$$

Now, we make our ansatz that the gauge invariant characteristics of perturbations are given in terms of linear combinations of a single master scalar $\Phi(t,u)$ satisfying scalar wave equation on the background solution (E.1), namely

$$
(-\Box + V)\Phi(t,u)e^{iqz} = 0\,, \qquad \text{(E.15)}
$$

where $\Box$ is the scalar wave operator corresponding to line element (E.1). Plugging such ansatz into linearized Einstein equations (E.8-E.14) one gets uniquely defined formulas, namely

$$
\begin{aligned}
\mathfrak{h}_{tt} = & \frac{8\left(r_0/R\right)^2}{3u^2\left(2\mathfrak{q}^2+3u\right)^2} \left[2\mathfrak{q}^2\left(6\mathfrak{q}^2 f(u) + 6\mathfrak{q}^2 + 9u^5 + 6\mathfrak{q}^2 u^4 + \left(4\mathfrak{q}^4-27\right)u^3 + 27u\right)\Phi(t,u)\right. \\
& \left. -3uf(u)\left(2\mathfrak{q}^2+3u\right)\left(\left(4\mathfrak{q}^2+3u^3+3u\right)\partial_u\Phi(t,u) - uf(u)\left(2\mathfrak{q}^2+3u\right)\partial_{uu}\Phi(t,u)\right)\right]
\end{aligned} \qquad \text{(E.16)}
$$

$$
\mathfrak{h}_{tu} = R^2 \frac{3uf(u)\left(2\mathfrak{q}^2+3u\right)\partial_{tu}\Phi(t,u) + 2\left(-12\mathfrak{q}^2 f(u) + 9\mathfrak{q}^2 + 9u^3 + 2\mathfrak{q}^4 u\right)\partial_t\Phi(t,u)}{3u^2 f(u)\left(2\mathfrak{q}^2+3u\right)} \qquad \text{(E.17)}
$$

$$
\mathfrak{h}_{zu} = 2ir_0\mathfrak{q}\left(\frac{2\mathfrak{q}^2\left(2\mathfrak{q}^2 u+3\right)}{3uf(u)\left(2\mathfrak{q}^2+3u\right)}\Phi(t,u) - \frac{1}{u}\partial_u\Phi(t,u)\right) \qquad \text{(E.18)}
$$

$$
\mathfrak{h}_{uu} = R^2 \frac{4\mathfrak{q}^2\left[-\left(4\mathfrak{q}^2 f(u) - 2\mathfrak{q}^2 - 3u^3\right)\Phi(t,u) + uf(u)\left(2\mathfrak{q}^2+3u\right)\partial_u\Phi(t,u)\right]}{3u^2 f^2(u)\left(2\mathfrak{q}^2+3u\right)} \qquad \text{(E.19)}
$$

and

$$
V = -\frac{8\mathfrak{q}^2\left(2\mathfrak{q}^2+3u^3+6\mathfrak{q}^2u^2+9u\right)}{R^2\left(2\mathfrak{q}^2+3u\right)^2}\,, \qquad \text{(E.20)}
$$

where $\mathfrak{q} = q/(2\pi T) = qR^2/(2r_0)$, cf. (KS-4.6). Now, from the master scalar wave equation (E.15) quasinormal modes of the planar black hole can be effectively computed

with [36] algorithm. The key point to be noted is that the scalar wave equation (E.15) is by definition *covariant* i.e. once the form of the potential is found in one coordinate system, for example (E.1) used here, it can be easily transformed (as a scalar) to any other coordinate system.

Now we compare the KS and KI approaches for doing numerics, where we focus on the approach of discretizing the QNM equation on a pseudospectral grid and computing the generalized eigenvalues, an approach first used in gravity in [37], and which is used in the publicly available package *QNMspectral*[36].

First, the only time derivatives come from the Laplacian, which in Eddington-Finkelstein coordinates gives a linear dependence of the equations on the quasinormal mode frequency. This is of great practical convenience, since it naturally has the form of a generalized eigenvalue equation when discretized. In contrast, the KS approach introduces higher orders of the frequency into the equation through the decoupling process, up to fourth order in the sound channel. In order to turn this equation into a generalized eigenvalue equation one has to linearize it in the frequency by introducing extra functions $\omega^p\Phi$ and writing it as a coupled system of equations, effectively increasing the size of the matrix by a factor of four.

Second, from the KS equation one obtains a lot of numerical artifacts near $\omega = \pm k$, which are approximate solutions of the discretized equation but not physical solutions. In the KI form these are completely absent.

Finally, at the same level of numerical accuracy the KI equations give much more accurate results for the physical frequencies, even when disregarding the factor 4 increase in matrix size due to the higher powers of the frequency.

As a quantitative illustration we compute the QNMs at $\mathfrak{q} = 1$ using both equations, with the package [36]. From the KS approach we use (KS-4.35) with the replacement $Z_2(u) = u^2(1-u)^{-i\omega/2}\tilde{Z}_2(u)$, so that the normalizable and ingoing mode is regular at both endpoints. From the KI approach, we use Eq. (E.15) but in Eddington-Finkelstein coordinates $ds^2 = -f(u)dt^2 - 2u^{-2}dudt + u^{-1}(dx^2 + dy^2 + dz^2)$, with $f(u) = u^{-2}(1-u^4)$. The potential is simply Eq. (E.20) with the replacement $u \to u^2$. We also have to rescale the master scalar as $\Phi = u^2\tilde{\Phi}$, where now the non-normalizable term goes as $\tilde{\Phi} \sim \log(u)$ and the normalizable term goes to a constant.

For both we use $N = 40$ grid points and check for convergence by comparing with the same computation at $N = 50$ grid points. so we are comparing the eigenvalues of a 40×40 matrix to a $40^4 \times 40^4$ matrix. Illustrating the points made above, the KS computation takes about 150 ms versus about 10 ms for the KI computation. The KS computation has many unphysical poles around $\mathfrak{w} = \pm 1$ while the KI has none. Finally we show the physical modes that are visible with this accuracy in Table 3.

A final curiosity is that as one takes the zero-momentum limit in the planar case, one obtains different potentials for different helicities, even though without any momentum these can no longer be distinguished. On the other hand in the Kovtun-Starinets formalism one obtains identical equations independent of the helicity, only of the type of field (metric, gauge field or scalar). It turns out however that these different potentials actually give rise

j	$\mathfrak{w}_j$
0	$\pm\underline{0.7414299655} - \underline{0.2862800072}i$
1	$\pm\underline{1.733511}095 - \underline{1.343007}549i$
2	$\pm\underline{2.70}5540 - \underline{2.35}7062i$
3	$\pm 3.689 - 3.364i$
4	$\pm 4.7 - 4.4i$

Table 3. QNMs in sound channel of AdS_5-Schwarzschild black brane at momentum $\mathfrak{q} = 1$. Full results are computed using the KI approach, only showing converged digits. Underlined digits are what is visible at the same numerical accuracy using the KS equation.

to the same spectrum of QNMs (as tested in Reissner-Nordström backgrounds in various dimensions), so the potentials are iso-spectral.

Nonlinear perturbations of Reissner-Nordström black holes

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We develop a nonlinear perturbation theory of Reissner-Nordström black holes. We show that, at each perturbation level, Einstein-Maxwell equations can be reduced to four inhomogeneous wave equations, two for the polar and two for the axial sector. The gravitational part of these equations is similar to Regge-Wheeler and Zerilli equations with source and additional coupling to the electromagnetic sector. We construct solutions to the inhomogeneous part of wave equations in terms of sources for Einstein-Maxwell equations. We discuss the $\ell = 0$ and $\ell = 1$ cases separately.

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I. INTRODUCTION

Perturbative methods play an important role in General Relativity. They find application to stability analysis, gravitational radiation, cosmology, rotating stars, the accretion disk, self-force, etc. Sometimes linear analysis gives sufficient insight into physical phenomena, but sometimes going beyond linear order can change qualitatively linear predictions (e.g., the Bizoń-Rostworowski conjecture of instability of anti-de Sitter spacetime [1]). In this paper, we study nonlinear perturbations of spherically symmetric solutions to Maxwell-Einstein equations. Linear perturbation theory of the Schwarzschild solution was formulated by Regge and Wheeler [2] and Zerilli [3] and then generalized to a Reissner-Nordström black hole by Zerilli [4] (see also Refs. [5–8], and [9]). Perturbations of Reissner-Nordström have also been recently discussed in the context of stability of the Cauchy horizon (issue crucial for the *strong cosmic censorship conjecture*; see Refs. [10,11], and [12]). All of these calculations are, however, only linear (or numerical only), and there was no robust procedure to move beyond linearity. Master equations from the present article provide a straightforward procedure to move beyond first-order estimates; at higher orders, there are still only wave equations (now inhomogeneous) to solve.

Taking into account higher-order perturbation terms makes the computations significantly more difficult; equations at each order beyond linear include all the previous-order terms. This issue has been treated by some authors—e.g., second-order perturbations of Schwarzschild were studied by Tomita and Tajima [13], Garat and Price [14], Gleiser *et al.* [15], Nakano and Ioka [16], and Brizuela *et al.* [17]. Recently, Rostworowski [18] provided a robust framework to deal with nonlinear (in principle of any order) gravitational perturbations of spherically symmetric spacetimes. The present article is an extension of Ref. [18] to both gravitational and electromagnetic nonlinear perturbations of Reissner-Nordström black holes. It also generalizes Zerilli's master equations [4] to any perturbation order.

Our approach is based on assumptions similar to those from Ref. [18]. We rewrite them explicitly here, since there are some differences:

(1) At each perturbation level, there are four master scalar variables, two in the polar and two in the axial sector. In each sector, they fulfill a system of two linearly coupled inhomogeneous (homogeneous at the linear order) wave equations with potentials.
(2) At each perturbation level, Regge-Wheeler variables and electromagnetic tensor components are linear combinations of master scalar variables from the suitable sector and their derivatives up to the second order. At the nonlinear orders, one needs to include additional functions to fulfill Maxwell-Einstein equations.
(3) At the linear level, relations from the previous point can be inverted to express master scalars as combinations of Regge-Wheeler variables and electromagnetic tensor components. At the nonlinear level, we take the same expressions for the master scalar functions.

In our considerations, we restrict ourselves to axially symmetric perturbations only (going beyond axial symmetry is a straightforward procedure, that conceptually adds little to this paper). During calculations, we stick to the Regge-Wheeler (RW) gauge. For practical implementations, after finding a solution in the RW gauge, one should move to an asymptotically flat gauge to ensure regularity of higher-order source functions (see Brizuela *et al.* [17]).

The paper is organized as follows. In Sec. II, we briefly introduce the Reissner-Nordström metric, and in Sec. III, we discuss the general form of perturbation expansion of

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Einstein-Maxwell equations. In Secs. IV, V, and VI, we remind the reader of polar expansion in axial symmetry, discuss gauge choice, and present source identities. The main result of this paper, namely providing inhomogeneous wave equations for Einstein-Maxwell equations of any perturbation order, is contained in Sec. VII.

II. BACKGROUND METRIC

The Reissner-Nordström solution describes a static, spherically symmetric black hole with an electric charge. In static coordinates $(t \in (-\infty, \infty), r \in (r_+, \infty), \theta \in (0, \pi), \phi \in [0, 2\pi))$, its metric is given by (we use $G = c = 4\pi\epsilon_0 = 1$)

$$\bar{g} = -A dt^2 + \frac{1}{A} dr^2 + r^2 d\Omega^2, \tag{1}$$

where $A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $r_+ = M + \sqrt{M^2 - Q^2}$, and M and Q are mass and charge of a black hole, respectively (we assume $|Q| < M$). Together with an electromagnetic tensor $\bar{F}$ with only nonzero terms $\bar{F}_{tr} = -\bar{F}_{rt} = \frac{Q}{r}$, metric (1) solves Einstein-Maxwell equations,

$$\bar{R}_{\mu\nu} = 8\pi \bar{T}_{\mu\nu}, \tag{2}$$

$$\bar{\nabla}_\mu \bar{F}^{\mu\nu} = 0, \tag{3}$$

$$\bar{F}_{[\mu\nu,\lambda]} = 0, \tag{4}$$

where $\bar{\nabla}$ and $\bar{R}_{\mu\nu}$ are, respectively, the covariant derivative and Ricci tensor with respect to the metric $\bar{g}$ and the comma denotes a partial derivative. $\bar{T}_{\mu\nu}$ is given by

$$\bar{T}_{\mu\nu} = \frac{1}{4\pi}\left(\bar{F}_{\mu\alpha}\bar{F}^{\alpha}_{\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}\right) \tag{5}$$

III. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS OF EINSTEIN-MAXWELL SYSTEMS

Let us assume that metric $\bar{g}$ and electromagnetic tensor $\bar{F}$ solve Einstein-Maxwell equations (2)–(4). Now, we seek for new solutions g and F that we expand around $\bar{g}$ and $\bar{F}$ with respect to the perturbation parameter ϵ:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i>0} {}^{(i)}h_{\mu\nu}\epsilon^i, \tag{6}$$

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + \sum_{i>0} {}^{(i)}f_{\mu\nu}\epsilon^i. \tag{7}$$

We plug (6) and (7) into Einstein-Maxwell equations, to obtain their perturbative form of order i,

$$\Delta_L {}^{(i)}h_{\mu\nu} - 8\pi {}^{(i)}t_{\mu\nu} = {}^{(i)}S^G_{\mu\nu}, \tag{8}$$

$$\bar{\nabla}^\mu {}^{(i)}f_{\mu\nu} - {}^{(i)}\Theta_\nu = {}^{(i)}S^M_\nu, \tag{9}$$

$${}^{(i)}f_{[\mu\nu,\lambda]} = 0, \tag{10}$$

where

$$\Delta_L {}^{(i)}h_{\mu\nu} = \frac{1}{2}(-\bar{\nabla}^\alpha\bar{\nabla}_\alpha {}^{(i)}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu {}^{(i)}h^\alpha_\alpha - 2\bar{R}_{\mu\alpha\nu\beta}{}^{(i)}h^{\alpha\beta} + \bar{\nabla}_\mu\bar{\nabla}^{\alpha(i)}h_{\nu\alpha} + \bar{\nabla}_\nu\bar{\nabla}^{\alpha(i)}h_{\mu\alpha}), \tag{11}$$

$$\begin{aligned} {}^{(i)}t_{\mu\nu} &= 2{}^{(i)}f_{\alpha(\mu}\bar{F}^\alpha{}_{\nu)} - \frac{1}{2}{}^{(i)}f_{\alpha\beta}\bar{F}^{\alpha\beta}\bar{g}_{\mu\nu} \\ &\quad + \left(\frac{1}{2}\bar{F}_{\alpha\sigma}\bar{F}_\beta{}^\sigma\bar{g}_{\mu\nu} - \bar{F}_{\mu\alpha}\bar{F}_{\nu\beta}\right){}^{(i)}h^{\alpha\beta} \\ &\quad - \frac{1}{4}\bar{F}^2{}^{(i)}h_{\mu\nu} - {}^{(i)}h_{\alpha(\mu}\bar{T}^\alpha{}_{\nu)}, \end{aligned} \tag{12}$$

$${}^{(i)}\Theta_\nu = \bar{g}^{\alpha\beta}(\bar{F}_{\sigma\nu}{}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} + \bar{F}_{\beta\sigma}{}^{(i)}\delta\Gamma^\sigma_{\alpha\nu}), \tag{13}$$

$${}^{(i)}\delta\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}\bar{g}^{\sigma\delta}(-\bar{\nabla}_\delta{}^{(i)}h_{\alpha\beta} + \bar{\nabla}_\alpha{}^{(i)}h_{\beta\delta} + \bar{\nabla}_\beta{}^{(i)}h_{\delta\alpha}). \tag{14}$$

Tensor sources ${}^{(i)}S^G_{\mu\nu}$ and vector sources ${}^{(i)}S^M_\nu$ are expressed by ${}^{(j<i)}h_{\mu\nu}$ and ${}^{(j<i)}f_{\mu\nu}$; therefore, perturbative Einstein equations should be solved order by order (see Appendix A for the construction of sources). For $i = 1$, both sources vanish.

IV. POLAR EXPANSION

In a spherically symmetric background, in $3 + 1$ dimensions, vector and tensor components split into two sectors that transform differently under rotations: polar and axial (for the details, see, e.g., Refs. [2,3,19,20]). Symmetric tensors have seven polar and three axial components, and antisymmetric tensors have three polar and three axial components. Below, we list expansions of all the components of both symmetric and antisymmetric tensors and of vectors in axial symmetry (P_ℓ denotes ℓth Legendre polynomial).

The symmetric tensor, polar sector is

$$S_{ab}(t,r,\theta) = \sum_{0\le\ell} S_{\ell ab}(t,r)P_\ell(\cos\theta), \quad a,b = t,r, \tag{15}$$

$$S_{a\theta}(t,r,\theta) = \sum_{1\le\ell} S_{\ell a\theta}(t,r)\partial_\theta P_\ell(\cos\theta), \quad a = t,r, \tag{16}$$

$$\frac{1}{2}\left(S_{\theta\theta}(t,r,\theta) + \frac{S_{\phi\phi}(t,r,\theta)}{\sin^2\theta}\right) = \sum_{0\le\ell} S_{\ell+}(t,r)P_\ell(\cos\theta), \tag{17}$$

$$\frac{1}{2}\left(S_{\theta\theta}(t,r,\theta)-\frac{S_{\phi\phi}(t,r,\theta)}{\sin^2\theta}\right) = \sum_{2\le\ell} S_{\ell-}(t,r)(-\ell(\ell+1)P_\ell(\cos\theta)-2\cot\theta\partial_\theta P_\ell(\cos\theta)). \tag{18}$$

The symmetric tensor, axial sector is

$$S_{a\phi}(t,r,\theta)=\sum_{1\le\ell} S_{\ell a\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta), \qquad a=t,r, \tag{19}$$

$$S_{\theta\phi}(t,r,\theta) = \sum_{2\le\ell} S_{\ell\theta\phi}(t,r)(-\ell(\ell+1)\sin\theta P_\ell(\cos\theta) - 2\cos\theta\partial_\theta P_\ell(\cos\theta)). \tag{20}$$

The antisymmetric tensor, polar sector is

$$A_{tr}(t,r,\theta) = \sum_{0\le\ell} A_{\ell tr}(t,r)P_\ell(\cos\theta), \tag{21}$$

$$A_{a\theta}(t,r,\theta)=\sum_{1\le\ell} A_{\ell a\theta}(t,r)\partial_\theta P_\ell(\cos\theta), \qquad a=t,r. \tag{22}$$

The antisymmetric tensor, axial sector is

$$A_{a\phi}(t,r,\theta) = \sum_{1\le\ell} A_{\ell a\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta), \qquad a=t,r, \tag{23}$$

$$A_{\theta\phi}(t,r,\theta) = \sum_{0\le\ell} A_{\ell\theta\phi}(t,r)\sin\theta P_\ell(\cos\theta). \tag{24}$$

The vector, polar sector is

$$V_a(t,r,\theta) = \sum_{0\le\ell} V_{\ell a}(t,r)P_\ell(\cos\theta), \qquad a=t,r, \tag{25}$$

$$V_\theta(t,r,\theta) = \sum_{1\le\ell} V_{\ell\theta}(t,r)\partial_\theta P_\ell(\cos\theta). \tag{26}$$

The vector, axial sector is

$$V_\phi(t,r,\theta) = \sum_{1\le\ell} V_{\ell\phi}(t,r)\sin\theta\partial_\theta P_\ell(\cos\theta). \tag{27}$$

Since the background is spherically symmetric, differential operators acting on ${}^{(i)}h_{\mu\nu}$ and ${}^{(i)}f_{\mu\nu}$ do not mix axial and polar sectors; therefore, Einstein-Maxwell equations split into two sectors as well: there are seven Einstein and three Maxwell equations in the polar sector and three Einstein and one Maxwell equation in the axial sector. In our paper, we consider separately $\ell \ge 2$, $\ell = 1$, and $\ell = 0$.

V. GAUGE CHOICE

Under a gauge transformation induced by a vector X^μ, tensors transform as $t_{\mu\nu} \to t_{\mu\nu} + \mathcal{L}_X t_{\mu\nu}$ (see Appendix B for the explicit form of transformations). Throughout the paper, we use Regge-Wheeler gauge [2]; namely, we set ${}^{(i)}h_{\ell tr}$, ${}^{(i)}h_{\ell r\theta}$, and ${}^{(i)}h_{\ell -}$ to zero in the polar sector and ${}^{(i)}h_{l\theta\phi} = 0$ in the axial sector. It turns out that variables we use correspond exactly to RW gauge invariants; therefore, a result for $\ell \ge 2$ can be read as expressions for RW gauge invariants. However, throughout the paper, we stick to fixed RW gauge because the discussion of $\ell = 0$ and $\ell = 1$ cases is more straightforward then. When the background quantities $\bar{g}$ and $\bar{F}$ fulfill Einstein equations, the left-hand sides of perturbation equations (8)–(10) of order i do not feel gauge transformations of order i, but source functions ${}^{(i)}S^G_{\mu\nu}$ and ${}^{(i)}S^G_{\mu\nu}$ depend on the gauge transformations of order $j < i$ explicitly, so such a formulation is not fully gauge invariant. This, however, is not a problem, since equations are solved order by order and for the practical implementations one goes to the asymptotically flat gauge before moving to the next order anyway.

VI. SOURCES FOR EINSTEIN-MAXWELL EQUATIONS

Sources ${}^{(i)}S^G_{\ell\,\mu\nu}$ and ${}^{(i)}S^M_{\ell\,\nu}$ are built of ${}^{(j)}h_{\ell\mu\nu}$ and ${}^{(j)}f_{\ell_{\mu\nu}}$ with $j < i$. These sources are not independent but fulfill five identities:

$$\bar{\nabla}^{\mu\,(i)}S^G{}_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\nu{}^{(i)}S^{G^\mu}{}_\mu - 2\bar{F}^\mu{}_\nu{}^{(i)}S^M{}_\mu = 0, \tag{28}$$

$$\bar{\nabla}^{\mu\,(i)}S^M{}_\mu = 0, \tag{29}$$

which come from the Bianchi identity and contracted Jacobi identity for tensor $F_{\mu\nu}$. One can check that they hold using (8)–(10) directly. The explicit form of identities (28) and (29) for polar-expanded sources in the polar sector reads [we introduce $\tau = \sqrt{(\ell-1)(\ell+2)}$]

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\,tr} + \frac{2QA}{r^2}{}^{(i)}S^M_{\ell\,r} + A\partial_r{}^{(i)}S^G_{\ell\,tr} - \frac{1}{2A}\partial_t{}^{(i)}S^G_{\ell\,tt} - \frac{1}{2}A\partial_t{}^{(i)}S^G_{\ell\,rr} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^G_{\ell\,t\theta} - \frac{1}{r^2}\partial_t{}^{(i)}S^G_{\ell\,+} = 0, \tag{30}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\, rr} + \frac{2Q}{r^2 A}{}^{(i)}S^M_{\ell\, t} + \frac{1}{2A}\partial_r{}^{(i)}S^G_{\ell\, tt} + \frac{1}{2}A\partial_r{}^{(i)}S^G_{\ell\, rr} - \frac{1}{A}\partial_t{}^{(i)}S^G_{\ell\, tr} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^G_{\ell\, r\theta} - \frac{\partial_r}{r^2}{}^{(i)}S^G_{\ell\, +} = 0, \tag{31}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\, r\theta} + \frac{1}{2A}{}^{(i)}S^G_{\ell\, tt} - \frac{1}{2}A{}^{(i)}S^G_{\ell\, rr} + A\partial_r S_{r\theta} - \frac{1}{A}\partial_t{}^{(i)}S^G_{\ell\, t\theta} - \frac{\tau^2}{r^2}{}^{(i)}S^G_{\ell\, -} = 0, \tag{32}$$

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^M_{\ell\, r} + A\partial_r{}^{(i)}S^M_{\ell\, r} - \frac{1}{A}\partial_t{}^{(i)}S^M_{\ell\, t} - \frac{\ell(\ell+1)}{r^2}{}^{(i)}S^M_{\ell\, \theta} = 0, \tag{33}$$

and in the axial sector reads

$$\left(A' + \frac{2A}{r}\right){}^{(i)}S^G_{\ell\, r\phi} + A\partial_r{}^{(i)}S^G_{\ell\, r\phi} - \frac{\partial_t{}^{(i)}S^G_{\ell\, t\phi}}{A} + - \frac{\tau^2{}^{(i)}S^G_{\ell\, \theta\phi}}{r^2} = 0. \tag{34}$$

VII. GRAVITATIONAL AND ELECTROMAGNETIC PERTURBATIONS

Now, we polar expand Eqs. (8)–(10):

$${}^{(i)}E_{\ell\mu\nu} = \Delta_L{}^{(i)}h_{\ell\mu\nu} - 8\pi{}^{(i)}t_{\ell\mu\nu} = {}^{(i)}S^G_{\ell\, \mu\nu}, \tag{35}$$

$${}^{(i)}J_{\ell\nu} = \bar{\nabla}^{\mu}{}^{(i)}f_{\ell\mu\nu} - {}^{(i)}\Theta_{\ell\nu} = {}^{(i)}S^M_{\ell\, \nu}, \tag{36}$$

$${}^{(i)}f_{\ell(\mu\nu,\alpha)} = 0. \tag{37}$$

A. Polar sector, $\ell \geq 2$

First, from (37), we have

$${}^{(i)}f_{\ell\, tr} = \partial_r{}^{(i)}f_{\ell\, t\theta} - \partial_t{}^{(i)}f_{\ell\, r\theta}, \tag{38}$$

and from ${}^{(i)}E_{\ell-}$,

$$\frac{1}{4}\left(\frac{1}{A}{}^{(i)}h_{\ell\, tt} - A{}^{(i)}h_{\ell\, rr}\right) - {}^{(i)}S_{\ell-} = 0. \tag{39}$$

We use relations (38) and (39) to eliminate ${}^{(i)}f_{\ell_{tr}}$ and ${}^{(i)}h_{\ell\, tt}$ from Eqs. (35)–(37). Then, we are left with five variables: ${}^{(i)}h_{\ell\, tt}$, ${}^{(i)}h_{\ell\, tr}$, ${}^{(i)}h_{\ell\, +}$, ${}^{(i)}f_{\ell_{t\theta}}$, and ${}^{(i)}f_{\ell_{r\theta}}$.

Remaining equations can all be fulfilled by introducing two master scalar variables ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ which solve a system of two coupled inhomogeneous (homogeneous at the linear order) wave equations [21]:

$$r(-\bar{\Box} + V^{\mathcal{P}}_{G\ell})\frac{{}^{(i)}\Phi^{\mathcal{P}}_{\ell}}{r} + V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} = {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}, \tag{40}$$

$$r(-\bar{\Box} + V^{\mathcal{P}}_{M\ell})\frac{{}^{(i)}\Psi^{\mathcal{P}}_{\ell}}{r} + V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} = {}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}. \tag{41}$$

Following the idea of Ref. [18], we express leftover variables by linear combinations of master scalar functions, their partial derivatives up to the second order (to solve homogeneous part of Einstein-Maxwell equations), and additional source functions (to solve the inhomogeneous part of equations). These combinations and potentials $V^{\mathcal{P}}_{G\ell}$, $V^{\mathcal{P}}_{M\ell}$, and $V^{\mathcal{P}}_{MG\ell}$ are defined uniquely:

$$V^{\mathcal{P}}_{G\ell} = \tau^2\hat{V}^{\mathcal{P}}_{G\ell} = \frac{\tau^2(-r^2A'^2 - 2A(-2A + \ell(\ell+1) + 2) + \ell^2(\ell+1)^2)}{r^2(rA' - 2A + \ell(\ell+1))^2} + \frac{8Q^2\tau^2A}{r^4(rA' - 2A + \ell(\ell+1))^2}, \tag{42}$$

$$V^{\mathcal{P}}_{M\ell} = \frac{-rA' + \ell(\ell+1)}{r^2} + \frac{4Q^2(2A(2r^3A' + \tau^2r^2 + 4Q^2) - r^4A'^2 - 4r^2A^2 + (\ell(\ell+1))^2r^2)}{r^6(rA' - 2A + \ell(\ell+1))^2}, \tag{43}$$

$$V^{\mathcal{P}}_{MG\ell} = \tau\hat{V}^{\mathcal{P}}_{MG\ell} = \frac{2\tau Q(2A(r^3A' + 4Q^2 - 2r^2) - r^4A'^2 + (\ell(\ell+1))^2r^2)}{r^5(rA' - 2A + \ell(\ell+1))^2}, \tag{44}$$

$${}^{(i)}h_{\ell\, tr} = -r\partial_{tr}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \left(\frac{rA'}{2A} - \frac{\tau^2}{rA' - 2A + \ell(\ell+1)}\right)\partial_t{}^{(i)}\Phi^{\mathcal{P}}_{\ell} - \frac{2\tau Q\partial_t}{r(rA' - 2A + \ell(\ell+1))}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\alpha_{\ell}, \tag{45}$$

$${}^{(i)}h_{\ell\, tt} = -r\partial_{rr}{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \left(-\frac{\tau^2}{rA' - 2A + \ell(\ell+1)} - \frac{rA'}{2A}\right)\partial_r{}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{r}{2A}\left(\frac{A'}{r} + V^{\mathcal{P}}_{G\ell}\right){}^{(i)}\Phi^{\mathcal{P}}_{\ell} + -\frac{2\tau Q}{r(rA' - 2A + \ell(\ell+1))}\partial_r{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + \frac{r}{2A}V^{\mathcal{P}}_{MG\ell}{}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\beta_{\ell}, \tag{46}$$

$$ {}^{(i)}h_{\ell+} = -A\partial_r {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{\left(\frac{2A(rA'-2A+2)}{rA'-2A+\ell(\ell+1)} - \ell(\ell+1)\right)}{2r} {}^{(i)}\Phi^{\mathcal{P}}_{\ell} - \frac{2\tau QA}{r^2(rA'-2A+\ell(\ell+1))} {}^{(i)}\Psi^{\mathcal{P}}_{\ell} + {}^{(i)}\gamma_{\ell}, \tag{47} $$

$$ {}^{(i)}f_{\ell t\theta} = \frac{A\tau}{4}\partial_r {}^{(i)}\Psi^{\mathcal{P}}_{\ell} - \frac{QA}{2r}\partial_r {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + \frac{QA}{2r^2} {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\lambda_{\ell}, \tag{48} $$

$$ {}^{(i)}f_{\ell r\theta} = \frac{\tau}{4A}\partial_t {}^{(i)}\Psi^{\mathcal{P}}_{\ell} - \frac{Q}{2rA}\partial_t {}^{(i)}\Phi^{\mathcal{P}}_{\ell} + {}^{(i)}\kappa_{\ell}. \tag{49} $$

At the linear level, ${}^{(1)}\alpha_{\ell} = {}^{(1)}\beta_{\ell} = {}^{(1)}\gamma_{\ell} = {}^{(1)}\lambda_{\ell} = {}^{(1)}\kappa_{\ell} = {}^{(1)}\tilde{S}^{\mathcal{P}}_{G\ell} = {}^{(1)}\tilde{S}^{\mathcal{P}}_{M\ell} = 0$ and relations (45)–(46) can be inverted to express ${}^{(1)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(1)}\Psi^{\mathcal{P}}_{\ell}$ as functions of ${}^{(i)}h_{\ell\mu\nu}$ and ${}^{(i)}f_{\ell\mu\nu}$. At higher orders, we treat linear level expressions for ${}^{(1)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(1)}\Psi^{\mathcal{P}}_{\ell}$ as definitions of ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$:

$$ {}^{(i)}\Phi^{\mathcal{P}}_{\ell} = \frac{4rA(r\partial_r {}^{(i)}h_{\ell+} - A {}^{(i)}h_{\ell rr})}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))} - \frac{2r {}^{(i)}h_{\ell+}}{\ell(\ell+1)}, \tag{50} $$

$$ {}^{(i)}\Psi^{\mathcal{P}}_{\ell} = \frac{4r^2(\partial_r {}^{(i)}f_{\ell t\theta} - \partial_t {}^{(i)}f_{\ell r\theta})}{\ell(\ell+1)\tau} + \frac{8QA(r\partial_r {}^{(i)}h_{\ell+} - A {}^{(i)}h_{\ell rr})}{\ell(\ell+1)\tau(rA'-2A+\ell(\ell+1))}. \tag{51} $$

Having these definitions, we may express the left-hand side of (40) and (41) as combinations of ${}^{(i)}E_{\ell\mu\nu}$, ${}^{(i)}J_{\ell\nu}$, and their derivatives. Finding these combinations, we use (35) and (36) to build sources for wave equations:

$$ \begin{aligned} {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell} = &-\frac{4A^2(\tau^2r^2+4Q^2) {}^{(i)}S^G_{\ell rr}}{\ell(\ell+1)r(rA'-2A+\ell(\ell+1))^2} + \frac{4 {}^{(i)}S^G_{\ell tt}(2r^3A'-4r^2A+(\ell(\ell+1)+2)r^2-4Q^2)}{\ell(\ell+1)r(rA'-2A+\ell(\ell+1))^2} \\ &+ \frac{8A\partial_r {}^{(i)}S^G_{\ell+}}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))} + \frac{8A {}^{(i)}S^G_{\ell r\theta}}{rA'-2A+\ell(\ell+1)} - \frac{4rV^{\mathcal{P}}_{G\ell} {}^{(i)}S^G_{\ell+}}{\ell(\ell+1)\tau^2} \\ &+ \frac{4 {}^{(i)}S^G_{\ell-}\left(\frac{Q^2(8A)}{r^3(rA'-2A+\ell(\ell+1))} - A' - rV^{\mathcal{P}}_{M\ell}\right)}{\ell(\ell+1)} - \frac{16Q {}^{(i)}S^M_{\ell t}}{\ell(\ell+1)(rA'-2A+\ell(\ell+1))}, \end{aligned} \tag{52} $$

$$ \begin{aligned} \frac{\ell(\ell+1)}{4}\tau {}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell} &= \frac{\ell(\ell+1)}{4} {}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell} \\ &= r^2\partial_r {}^{(i)}S^M_{\ell t} - r^2\partial_t {}^{(i)}S^M_{\ell r} - {}^{(i)}S^M_{\ell t}\left(2r - \frac{8Q^2}{r(rA'-2A+\ell(\ell+1))}\right) + \frac{8Q\left(\frac{r^2(rA'-2A+2)}{4} - Q^2\right) {}^{(i)}S^G_{\ell tt}}{r^2(rA'-2A+\ell(\ell+1))^2} \\ &\quad + \frac{8QA^2 {}^{(i)}S^G_{\ell rr}\left(\frac{r^2(rA'-2A+2(\ell(\ell+1)-1))}{4} + Q^2\right)}{r^2(rA'-2A+\ell(\ell+1))^2} + \frac{4\ell(\ell+1)QA {}^{(i)}S^G_{\ell r\theta}}{r(rA'-2A+\ell(\ell+1))} + \frac{4QA\partial_r {}^{(i)}S^G_{\ell+}}{r(rA'-2A+\ell(\ell+1))} \\ &\quad + 2QA'\partial_r {}^{(i)}S^G_{\ell-} + -\frac{2Q {}^{(i)}S^G_{\ell-}(A'+rV^{\mathcal{P}}_{M\ell})}{r} + 2QA\partial_r^2 {}^{(i)}S^G_{\ell-} - \frac{2Q\partial_t^2 {}^{(i)}S^G_{\ell-}}{A} - \frac{rV^{\mathcal{P}}_{MG\ell} {}^{(i)}S^G_{\ell+}}{\tau}. \end{aligned} \tag{53} $$

${}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}$ and ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}$ are given uniquely up to the source identities (30)–(33). We introduced auxiliary quantities $\hat{V}^{\mathcal{P}}_{G\ell}$, $\hat{V}^{\mathcal{P}}_{MG\ell}$, and ${}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}$, which are nonzero (or nonsingular) for $\ell = 1$.

At the nonlinear level, part of the solution (45)–(49) consisting of master scalars ${}^{(i)}\Phi^{\mathcal{P}}_{\ell}$ and ${}^{(i)}\Psi^{\mathcal{P}}_{\ell}$ and their derivatives fulfills the homogeneous part of Einstein-Maxwell equations (35), (36), and (37), whereas part of (45)–(49) consisting of functions ${}^{(i)}\alpha_{\ell}$, ${}^{(i)}\beta_{\ell}$, ${}^{(i)}\gamma_{\ell}$, ${}^{(i)}\lambda_{\ell}$, and ${}^{(i)}\kappa_{\ell}$ is responsible for the inhomogeneous part of the Einstein-Maxwell equations. To find these functions, we plug (45)–(49) into Eqs. (35) and (36) and into definitions (50) and (51) to ensure consistency. Then, we solve these equations for ${}^{(i)}\alpha_{\ell}$, ${}^{(i)}\beta_{\ell}$, ${}^{(i)}\gamma_{\ell}$, ${}^{(i)}\lambda_{\ell}$, and ${}^{(i)}\kappa_{\ell}$. These functions, as well as scalar sources for wave equations ${}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}$ and ${}^{(i)}\tilde{S}^{\mathcal{P}}_{M\ell}$, are defined uniquely [up to the source identities (30)–(33)]:

$$ \begin{aligned} {}^{(i)}\alpha_{\ell} = &-\frac{2r^2(r^2A^2 {}^{(i)}S^G_{\ell rr} + r^2 {}^{(i)}S^G_{\ell tt} + 2A {}^{(i)}S^G_{\ell+})}{\ell(\ell+1)r^2(rA'-2A+\ell(\ell+1))} + \\ &- \frac{16Q^2A {}^{(i)}S^G_{\ell-}}{\ell(\ell+1)r^2(rA'-2A+\ell(\ell+1))}, \end{aligned} \tag{54} $$

$$^{(i)}\beta_\ell = r\left(\frac{2r^{(i)}S^G_{\ell\, tr}}{\ell(\ell+1)} + \frac{\partial_t {}^{(i)}\alpha_\ell}{A}\right), \tag{55}$$

$$^{(i)}\gamma_\ell = \frac{r\partial_r {}^{(i)}\alpha_\ell + {}^{(i)}\alpha_\ell}{A} - \frac{{}^{(i)}\alpha_\ell(rA' + \ell(\ell+1))}{2A^2}, \tag{56}$$

$$^{(i)}\kappa_\ell = \frac{r^{2(i)}S^M_{\ell\, r}}{\ell(\ell+1)} + \frac{2Q\partial_t {}^{(i)}S^G_{\ell\, -}}{A\ell(\ell+1)}, \tag{57}$$

$$^{(i)}\lambda_\ell = \frac{r^{2(i)}S^M_{\ell\, t}}{\ell(\ell+1)} + \frac{2QA\partial_r {}^{(i)}S^G_{\ell\, -}}{\ell(\ell+1)}. \tag{58}$$

B. Polar sector, $\ell = 1$

For $\ell = 1$, there is no $S_{\ell -}$ coefficient in a symmetric tensor decomposition; therefore, we do not have a $^{(i)}h_{\ell -}$ metric coefficient, and we lose the algebraic Einstein equation (39). However, since one of the gauge conditions was $^{(i)}h_{\ell -} = 0$, we gain additional gauge freedom, which we can use to keep algebraic relation (39). That means our $\ell \geq 2$ results are directly applicable to $\ell = 1$ as well. The only obstacle is that for the $\ell = 1$ coefficient $\tau = 0$ and singular terms appear in the source for wave equation (41) and in the definition (51). We can deal with it by introducing $^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell = \tau {}^{(i)}\Psi^{\mathcal{P}}_\ell$, which, together with $^{(i)}\Phi^{\mathcal{P}}_\ell$, fulfills a set of wave equations,

$$r(-\bar{\Box} + \tau^2 \hat{V}^{\mathcal{P}}_{G\ell})\frac{{}^{(i)}\Phi^{\mathcal{P}}_\ell}{r} + \hat{V}^{\mathcal{P}}_{MG\ell}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell}, \tag{59}$$

$$r(-\bar{\Box} + V^{\mathcal{P}}_{M\ell})\frac{{}^{(i)}\hat{\Psi}^{\mathcal{P}}_\ell}{r} + \tau^2 \hat{V}^{\mathcal{P}}_{MG\ell}\, {}^{(i)}\Phi^{\mathcal{P}}_\ell = {}^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}, \tag{60}$$

where $\hat{V}^{\mathcal{P}}_{G\ell}$, $\hat{V}^{\mathcal{P}}_{MG\ell}$, and $^{(i)}\hat{S}^{\mathcal{P}}_{M\ell}$ are defined in (42), (44), and (53). For $\ell = 1$, the system is simpler—there is no coupling to the gravitational master scalar in (60). Now, scalar sources for both equations are regular for $\ell = 1$. Metric and electromagnetic tensor perturbations are then given by

$$\begin{aligned} {}^{(i)}h_{1\, tr} = &-r\partial_{tr} {}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{rA'}{2A}\partial_t {}^{(i)}\Phi^{\mathcal{P}}_1 \\ &- \frac{2Q\partial_t}{r(rA' - 2A + 2)}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\alpha_1, \end{aligned} \tag{61}$$

$$\begin{aligned} {}^{(i)}h_{1\, rr} = &-r\partial_{rr} {}^{(i)}\Phi^{\mathcal{P}}_1 - \frac{rA'}{2A}\partial_r {}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{A'}{2A}\, {}^{(i)}\Phi^{\mathcal{P}}_1 + \\ &- \frac{2Q}{r(rA' - 2A + 2)}\partial_r {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + \frac{r}{2A}\hat{V}^{\mathcal{P}}_{MG1}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\beta_1, \end{aligned} \tag{62}$$

$$\begin{aligned} {}^{(i)}h_{1+} = &-A\partial_r {}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{A-1}{r}\, {}^{(i)}\Phi^{\mathcal{P}}_1 \\ &- \frac{2QA}{r^2(rA' - 2A + 2)}\, {}^{(i)}\Psi^{\mathcal{P}}_1 + {}^{(i)}\gamma_1, \end{aligned} \tag{63}$$

$$^{(i)}f_{1\, t\theta} = \frac{A}{4}\partial_r {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{QA}{2r}\partial_r {}^{(i)}\Phi^{\mathcal{P}}_1 + \frac{QA}{2r^2}\, {}^{(i)}\Phi^{\mathcal{P}}_1 + {}^{(i)}\lambda_1, \tag{64}$$

$$^{(i)}f_{1\, r\theta} = \frac{1}{4A}\partial_t {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{Q}{2rA}\partial_t {}^{(i)}\Phi^{\mathcal{P}}_1 + {}^{(i)}\kappa_1. \tag{65}$$

Since there is no $^{(i)}S^G_{\ell -}$ source term, $^{(i)}\alpha_1$, $^{(i)}\beta_1$, $^{(i)}\gamma_1$, $^{(i)}\lambda_1$, and $^{(i)}\kappa_1$ for $\ell = 1$ are given by

$$^{(i)}\alpha_1 = -\frac{r^2A^{2(i)}S^G_{1\, rr} + r^{2(i)}S^G_{1\, tt} + 2A^{(i)}S^G_{1\, +}}{(rA' - 2A + 2)}, \tag{66}$$

$$^{(i)}\beta_1 = r^{2(i)}S^G_{1\, tr} + \frac{r\partial_t {}^{(i)}\alpha_1}{A}, \tag{67}$$

$$^{(i)}\gamma_1 = \frac{r\partial_r {}^{(i)}\alpha_1 + {}^{(i)}\alpha_1}{A} - \frac{{}^{(i)}\alpha_1(rA' + 2)}{2A^2}, \tag{68}$$

$$^{(i)}\kappa_1 = \frac{r^2}{2}\, {}^{(i)}S^M_{1\, r}, \tag{69}$$

$$^{(i)}\lambda_1 = \frac{r^2}{2}\, {}^{(i)}S^M_{1\, t}. \tag{70}$$

Although direct implementation of previous results provides a general solution to $\ell = 1$ equations, it can be misleading; it looks like there are two dynamical variables, whereas there should be only one [7] (for the Schwarzschild case $\ell = 1$, gravitational modes are pure gauge [22]). However, by the following gauge transformation, one can get rid of $^{(i)}\Phi^{\mathcal{P}}_1$ from (75)–(79),

$$^{(i)}\zeta_{1\, t} = -\partial_t {}^{(i)}\zeta_{1\, \theta}, \tag{71}$$

$$^{(i)}\zeta_{1\, r} = \frac{2\, {}^{(i)}\zeta_{1\, \theta}}{r} - \partial_r {}^{(i)}\zeta_{1\, \theta}, \tag{72}$$

$$^{(i)}\zeta_{1\, \theta} = -\frac{r}{2}\, {}^{(i)}\Phi^{\mathcal{P}}_1, \tag{73}$$

and the solution reads

$$\begin{aligned} {}^{(i)}h_{1\, tt} = &-\frac{2A^2Q}{r(rA' - 2A + 2)}\partial_r {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 - \frac{rA}{2}\hat{V}^{\mathcal{P}}_{MG1}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 \\ &+ A^{2(i)}\beta_1 + rA^{(i)}\tilde{S}^{\mathcal{P}}_{G\ell} \end{aligned} \tag{74}$$

$$^{(i)}h_{1\, tr} = -\frac{2Q\partial_t}{r(rA' - 2A + 2)}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\alpha_1, \tag{75}$$

$$^{(i)}h_{1\, rr} = -\frac{2Q}{r(rA' - 2A + 2)}\partial_r {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + \frac{r}{2A}\hat{V}^{\mathcal{P}}_{MG\, 1}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + {}^{(i)}\beta_1, \tag{76}$$

$$^{(i)}h_{1\, +} = -\frac{2QA}{(rA' - 2A + 2)}\, {}^{(i)}\hat{\Psi}^{\mathcal{P}}_1 + r^{2(i)}\gamma_1, \tag{77}$$

$$^{(i)}f_{1\,t\theta} = \frac{A}{4}\partial_r {}^{(i)}\hat{\Psi}_1^{\mathcal{P}} + {}^{(i)}\lambda_1, \tag{78}$$

$$^{(i)}f_{1\,r\theta} = \frac{1}{4A}\partial_t {}^{(i)}\hat{\Psi}_1^{\mathcal{P}} + {}^{(i)}\kappa_1. \tag{79}$$

The cost of performing this transformation is the loss of algebraic relation (39). From our results, one can also move to a gauge used by some authors (Refs. [6,7]) in which $^{(i)}h_{1+} = 0$.

C. Polar sector, $\ell = 0$

In this case, we follow Rostworowski [18]. Using gauge freedom, we set $^{(i)}h_{0\,+} = 0$ and $^{(i)}h_{0\,tr=0}$, and leftover nonzero variables are $^{(i)}h_{0\,tt}$, $^{(i)}h_{0\,rr}$, and $^{(i)}f_{0\,tr}$. From $^{(i)}E_{0\,01}$, $^{(i)}E_{0\,00} + A^{2\,(i)}E_{0\,11}$, and $^{(i)}J_{0\,1}$ (the only independent equations), we have, respectively,

$$\frac{A}{r}\partial_t {}^{(i)}h_{0\,rr} = {}^{(i)}S^G_{0\,tr}, \tag{80}$$

$$\frac{A}{r}\partial_r\left(A\,{}^{(i)}h_{0\,rr} - \frac{{}^{(i)}h_{0\,tt}}{A}\right) = \frac{{}^{(i)}S^G_{0\,tt}}{A} + A\,{}^{(i)}S^G_{0\,rr}, \tag{81}$$

$$\partial_t\left({}^{(i)}f_{0\,tr} + \frac{Q}{2r^2}\left(\frac{{}^{(i)}h_{0\,tt}}{A} - A\,{}^{(i)}h_{0\,rr}\right)\right) = -A\,{}^{(i)}S^M_{0\,r}. \tag{82}$$

These equations can be therefore integrated directly, starting from (80).

D. Axial sector, $\ell \geq 2$

First, we use (37) to obtain

$$^{(i)}f_{\ell\,t\phi} = -\frac{\partial_t f_{\theta\phi}}{\ell(\ell+1)}, \tag{83}$$

$$^{(i)}f_{\ell\,r\phi} = -\frac{\partial_r f_{\theta\phi}}{\ell(\ell+1)}. \tag{84}$$

We are left with three variables $^{(i)}h_{\ell\,t\phi}$, $^{(i)}h_{\ell\,r\phi}$, and $^{(i)}f_{\ell\theta\phi}$. In the same manner as before, we can fulfill equations (35) and (36) by introducing two master scalar variables $^{(i)}\Phi^{\mathcal{A}}_\ell$ and $^{(i)}\Psi^{\mathcal{A}}_\ell$, which solve a system of two coupled wave equations:

$$r(-\bar{\Box} + V^{\mathcal{A}}_{G\,\ell})\frac{{}^{(i)}\Phi^{\mathcal{A}}_\ell}{r} + V^{\mathcal{A}}_{MG\,\ell}\,{}^{(i)}\Psi^{\mathcal{A}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell}, \tag{85}$$

$$r(-\bar{\Box} + V^{\mathcal{A}}_{M\,\ell})\frac{{}^{(i)}\Psi^{\mathcal{A}}_\ell}{r} + V^{\mathcal{A}}_{MG\,\ell}\,{}^{(i)}\Phi^{\mathcal{A}}_\ell = {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell}. \tag{86}$$

Following the procedure described in the previous section, we find three potentials and express $h_{t\phi}$, $h_{r\phi}$, and $f_{\theta\phi}$ by master scalars and their derivatives:

$$V^{\mathcal{A}}_{G\,\ell} = \frac{r^2(A - 3rA') + (\tau^2+1)r^2 - Q^2}{r^4}, \tag{87}$$

$$V^{\mathcal{A}}_{M\,\ell} = \frac{-A'r^3 + \ell(\ell+1)r^2 + 4Q^2}{r^4}, \tag{88}$$

$$V^{\mathcal{A}}_{MG\,\ell} = -\frac{2\tau Q}{r^3}, \tag{89}$$

$$^{(i)}h_{\ell\,t\phi} = A\partial_r(r\,{}^{(i)}\Phi^{\mathcal{A}}_\ell) + {}^{(i)}\sigma_\ell, \tag{90}$$

$$^{(i)}h_{\ell\,r\phi} = \frac{r}{A}\partial_t {}^{(i)}\Phi^{\mathcal{A}}_\ell + {}^{(i)}\chi_\ell, \tag{91}$$

$$^{(i)}f_{\ell\theta\phi} = \frac{1}{2}\ell(\ell+1)\tau\,{}^{(i)}\Psi^{\mathcal{A}}_\ell + {}^{(i)}\delta_\ell. \tag{92}$$

Now, we invert the above relations for linear order and treat the following expressions as definitions of $^{(i)}\Phi^{\mathcal{A}}_\ell$ and $^{(i)}\Psi^{\mathcal{A}}_\ell$ at the nonlinear order:

$$^{(i)}\Phi^{\mathcal{A}}_\ell = \frac{(r(\partial_r {}^{(i)}h_{\ell\,t\phi} - \partial_t {}^{(i)}h_{\ell\,r\phi}) - 2\,{}^{(i)}h_{\ell\,t\phi})}{\ell(\ell+1)\tau^2 r} + \frac{4Q\,{}^{(i)}f_{\ell\theta\phi}}{\tau^2}, \tag{93}$$

$$^{(i)}\Psi^{\mathcal{A}}_\ell = \frac{2\,{}^{(i)}f_{\ell\theta\phi}}{\tau\ell(\ell+1)}. \tag{94}$$

Finally, we find inhomogeneous functions $^{(i)}\sigma_\ell$, $^{(i)}\chi_\ell$, and $^{(i)}\delta_\ell$,

$$^{(i)}\sigma_\ell = \frac{2r^2}{\tau^2}\,{}^{(i)}S^G_{\ell\,t\phi}, \tag{95}$$

$$^{(i)}\chi_\ell = \frac{2r^2}{\tau^2}\,{}^{(i)}S^G_{\ell\,r\phi}, \tag{96}$$

$$^{(i)}\delta_\ell = 0, \tag{97}$$

and scalar sources $^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell}$ and $^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell}$,

$$^{(i)}\tilde{S}^{\mathcal{A}}_{G\,\ell} = \frac{2r(\partial_r {}^{(i)}S^G_{\ell\,t\phi} - \partial_t {}^{(i)}S^G_{\ell\,r\phi})}{\tau^2}, \tag{98}$$

$$^{(i)}\tilde{S}^{\mathcal{A}}_{M\,\ell} = \frac{2\,{}^{(i)}S^M_{\ell\,\phi}}{\tau}. \tag{99}$$

E. Axial sector, $\ell = 1$

Since $^{(i)}h_{\ell\theta\phi}$ does not appear for $\ell = 1$, we can use gauge freedom to set $^{(i)}h_{1\,r\phi} = 0$. From (37), we have

$$^{(i)}f_{1\,t\phi} = -\frac{\partial_t {}^{(i)}f_{1\,\theta\phi}}{2}, \tag{100}$$

$$ {}^{(i)}f_{1\,r\phi} = -\frac{\partial_r {}^{(i)}f_{1\,\theta\phi}}{2}. \tag{101} $$

Remaining equations contain ${}^{(i)}h_{1\,t\phi}$ and ${}^{(i)}f_{1\,\theta\phi}$ only. From ${}^{(i)}E_{1\,r\phi} = {}^{(i)}S^G_{1\,r\phi}$, we find

$$ -\frac{r^2}{2A}\partial_r\left(\frac{{}^{(i)}h_{1\,t\phi}}{r^2}\right) - \frac{Q{}^{(i)}f_{1\,\theta\phi}}{Ar^2} + \eta(r) = \int^t {}^{(i)}S^G_{1\,r\phi}dt', \tag{102} $$

where $\eta(r)$ is some function of r. It is not arbitrary—from ${}^{(i)}E_{1\,t\phi} = {}^{(i)}S^G_{1\,t\phi}$ and source identity (34), we find $\eta = \frac{C_1}{Ar^2}$, C_1 being an arbitrary constant.

Let us introduce ${}^{(i)}\Psi^{\mathcal{A}}_1$ such that ${}^{(i)}f_{1\,\theta\phi} = {}^{(i)}\Psi^{\mathcal{A}}_1 + \frac{4C_1Q}{3r^2(rA'+2A-2)}$. From (36), we find that ${}^{(i)}\Psi^{\mathcal{A}}_1$ fulfills an inhomogeneous (homogeneous at the linear level) wave equation,

$$ r(-\bar{\Box} + V^{\mathcal{A}}_{M\,1})\frac{{}^{(i)}\Psi^{\mathcal{A}}_1}{r} = {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,1}, \tag{103} $$

where

$$ V^{\mathcal{A}}_{M\,1} = \frac{4Q^2 - r^3A' + 2r^2}{r^4}, \tag{104} $$

$$ {}^{(i)}\tilde{S}^{\mathcal{A}}_{M\,1} = 2{}^{(i)}S^M_{1\,\phi} - \frac{4AQ\int^t {}^{(i)}S^G_{1\,r\phi}dt'}{r^2}. \tag{105} $$

We note that at the linear level setting ${}^{(i)}\Psi^{\mathcal{A}}_1 = 0$ corresponds to the linearized Kerr-Newman metric.

VIII. SUMMARY

Nonlinear perturbation theory of the Reissner-Nordström solution has not been present in the literature so far, and the present article fills this gap. Basing on a systematic approach to gravitational perturbations by Rostworowski [18], we have shown that one can fulfill perturbative Einstein-Maxwell equations at any perturbation order by solving two inhomogeneous master wave equations at each sector (cases $\ell = 0$, 1 needed special treatment). This makes treatment of higher-order perturbations of Reissner-Nordström clear and would be especially useful for numerical purposes. To summarize, a complete order by order algorithm of solving Einstein-Maxwell equations within our formalism would be:

(1) Solve wave equations (40), (41), (85), and (86), and calculate RW variables and electromagnetic tensor components according to (45)–(49) and (90)–(92),
(2) Move to asymptotically flat gauge, and calculate sources to Einstein-Maxwell equations (Appendix A),
(3) Construct sources to wave equations [Eqs. (52), (53), (98), and (99)], and move to the next order.

Applications of presented calculations possibly include nonlinear studies on the strong cosmic censorship conjecture and on astrophysical systems, where electromagnetic field is taken into account.

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APPENDIX A: SOURCES FOR EINSTEIN-MAXWELL EQUATIONS

Let us fix index i and assume that we already know the solution to Einstein-Maxwell equations (35)–(37) up to ith order:

$$ \tilde{g}_{\mu\nu} = \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}h_{\ell\mu\nu}, \tag{A1} $$

$$ \tilde{F}_{\mu\nu} = \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}f_{\ell\mu\nu}. \tag{A2} $$

Using this solution, we can calculate the Einstein tensor $G_{\mu\nu}(\tilde{g})$ and the energy-momentum tensor $T_{\mu\nu}(\tilde{g}, \tilde{F})$. Although these tensors fulfill Einstein-Maxwell equations up to order i, they contribute to the $i+1$ (and higher) perturbation equations. Finally, tensor and vector sources of order $i+1$ are given by

$$ {}^{(i+1)}S^G{}_{\mu\nu} = [i+1](-G_{\mu\nu}(\tilde{g}) + 8\pi T_{\mu\nu}(\tilde{g}, \tilde{F})), \tag{A3} $$

$$ {}^{(i+1)}S^E{}_{\nu} = [i+1](-\nabla^\mu(\tilde{g}_{\alpha\beta})\tilde{F}_{\mu\nu}), \tag{A4} $$

where $[k](\ldots)$ denotes the kth-order expansion in ϵ of a given quantity.

Although in most cases expressions for the sources ${}^{(i+1)}S^G{}_{\mu\nu}$ and ${}^{(i+1)}S^G{}_{\mu\nu}$ are complicated, their construction is a purely algebraic task and can be easily performed using computer algebra.

APPENDIX B: GAUGE TRANSFORMATIONS

Under a gauge transformation $x^\mu \to x^\mu + X^\mu$, tensors transform as $t_{\mu\nu} \to t_{\mu\nu} + \mathcal{L}_X t_{\mu\nu}$. For $X^\mu = {}^{(i)}\zeta^\mu\epsilon^i$, perturbation functions of order i transform in the following way:

$$ {}^{(i)}h_{\ell\mu\nu} \to {}^{(i)}h_{\ell\mu\nu} + \mathcal{L}_{{}^{(i)}\zeta_\ell}\bar{g}_{\mu\nu}, \tag{B1} $$

$$ {}^{(i)}f_{\ell\mu\nu} \to {}^{(i)}f_{\ell\mu\nu} + \mathcal{L}_{{}^{(i)}\zeta_\ell}\bar{F}_{\mu\nu}. \tag{B2} $$

The explicit form of these transformations in a polar sector is

$$^{(i)}h_{\ell tt} \to {}^{(i)}h_{\ell tt} + 2\partial_t {}^{(i)}\zeta_{\ell t} - AA'^{(i)}\zeta_{\ell r}, \tag{B3}$$

$$^{(i)}h_{\ell tr} \to {}^{(i)}h_{\ell tr} + \partial_r {}^{(i)}\zeta_{\ell t} + \partial_t {}^{(i)}\zeta_{\ell r} - \frac{A'}{A}{}^{(i)}\zeta_{\ell t}, \tag{B4}$$

$$^{(i)}h_{\ell t\theta} \to {}^{(i)}h_{\ell t\theta} + \partial_t {}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell t}, \tag{B5}$$

$$^{(i)}h_{\ell rr} \to {}^{(i)}h_{\ell rr} + 2\partial_r {}^{(i)}\zeta_{\ell\theta} + \frac{A'}{A}{}^{(i)}\zeta_{\ell r}, \tag{B6}$$

$$^{(i)}h_{\ell r\theta} \to {}^{(i)}h_{\ell r\theta} + \partial_r {}^{(i)}\zeta_{\ell\theta} - \frac{2}{r}{}^{(i)}\zeta_{\ell\theta} + {}^{(i)}\zeta_{\ell r}, \tag{B7}$$

$$^{(i)}h_{\ell +} \to {}^{(i)}h_{\ell +} + 2A\frac{{}^{(i)}\zeta_{\ell r}}{r} - \ell(\ell+1)\frac{{}^{(i)}\zeta_{\ell\theta}}{r^2}, \tag{B8}$$

$$^{(i)}h_{\ell -} \to {}^{(i)}h_{\ell -} + {}^{(i)}\zeta_{\ell\theta}, \tag{B9}$$

$$^{(i)}f_{\ell t\theta} \to {}^{(i)}f_{\ell t\theta} + \frac{AQ}{r^2}{}^{(i)}\zeta_{\ell r}, \tag{B10}$$

$$^{(i)}f_{\ell r\theta} \to {}^{(i)}f_{\ell r\theta} + \frac{Q}{Ar^2}{}^{(i)}\zeta_{\ell t}, \tag{B11}$$

$$^{(i)}f_{\ell tr} \to {}^{(i)}f_{\ell tr} + \frac{Q}{Ar^2}{}^{(i)}\zeta_{\ell t}, \tag{B12}$$

$$^{(i)}f_{\ell tr} \to {}^{(i)}f_{\ell tr} + Q\partial_r\left(\frac{A}{r^2}{}^{(i)}\zeta_{\ell r}\right) - \frac{Q}{r^2A}\partial_t {}^{(i)}\zeta_{\ell t} \tag{B13}$$

and in the axial sector is

$$^{(i)}h_{\ell t\phi} \to {}^{(i)}h_{\ell t\phi} + \partial_t {}^{(i)}\zeta_{\ell\phi}, \tag{B14}$$

$$^{(i)}h_{\ell r\phi} \to {}^{(i)}h_{\ell r\phi} + \partial_r {}^{(i)}\zeta_{\ell\phi} - 2\frac{{}^{(i)}\zeta_{\ell\phi}}{r}, \tag{B15}$$

$$^{(i)}h_{\ell\theta\phi} \to {}^{(i)}h_{\ell\theta\phi} + {}^{(i)}\zeta_{\ell\phi}. \tag{B16}$$

$$^{(i)}f_{\ell t\phi} \to {}^{(i)}f_{\ell t\phi}, \tag{B17}$$

$$^{(i)}f_{\ell r\phi} \to {}^{(i)}f_{\ell r\phi}, \tag{B18}$$

$$^{(i)}f_{\ell\theta\phi} \to {}^{(i)}f_{\ell\theta\phi}. \tag{B19}$$

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Ultracompact rotating gravastars and the problem of matching with Kerr spacetime

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A number of authors provided arguments that a rotating gravastar is a good candidate for a source of the Kerr metric. These arguments were based on the second order perturbation analysis. In the following paper, we construct a perturbative solution of the rotating gravastar up to the third perturbation order and show that once we demand finiteness of the Kretschmann scalar expansion, it cannot be continuously matched with the Kerr spacetime.

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I. INTRODUCTION

Gravastars, proposed by Mazur and Mottola [1] as an alternative to black holes, have been studied extensively in the recent years [2–9]. One of the issues concerning gravastars is to find a rotating gravastar solution. So far only perturbative versions of such a solution exist [10–12]. These studies indicate that in the ultracompact limit [13] the rotating gravastar can be a source of the Kerr metric (i.e., I, Love, Q numbers tend to those of Kerr in this limit). Similar perturbation-type sources (thin shells) of the Kerr metric were studied earlier by, e.g., [14–16]. On the other hand, constructing perturbation sources of the Kerr metric have been criticized by Krasiński [17].

In this work, we take perturbation approach to check if the matching of the gravastar with the Kerr spacetime survives at higher orders. It means that we want to construct a rotating analogue of [13] with the Kerr spacetime outside. We use slightly different framework to [10–12] and instead of solving Einstein equations both for interior and exterior, we *a priori* assume that an exterior solution is the Kerr metric. Then we seek for an interior solution and try to match it with the Kerr metric.

Most of the work on rotating gravastars was based on Hartle's structure equations [18] (see also [19–21]). Hartle's framework allows to study slowly rotating perfect fluid objects up to the second order in the angular momentum. To go beyond the second order, we find it easier to follow Rostworowski [22], who provided a nonlinear extension of Regge-Wheeler and Zerilli formalisms. Formalism given by [22] is dedicated to (Λ-) vacuum spacetimes and can be easily adapted to our needs. The difference between Hartle's framework and our approach is only on the level of ansatz on metric perturbation form and they are physically equivalent within the range of applicability of Hartle's framework. We find that [22] provides a very powerful tool for dealing with nonlinear perturbations. Although in the present article we describe perturbation analysis only up to the third order, we solved Einstein equations up to the sixth order to calculate the Kretschmann scalar and we think it's possible to go further if needed.

The paper is organized as follows: in Secs. II–IV we provide preliminaries, in Sec. V we discuss the matching, in Sec. VI we expand the Kerr metric, in Secs. VII and VIII we solve interior Einstein equations and try to match interior and exterior metrics and in Sec. IX we summarize and discuss our calculations.

II. BACKGROUND SOLUTION

As a background, we take the ultracompact gravastar model [13]. In static coordinates (t, r, u, φ), where $u = \cos\theta$, its metric is given by:

$$\bar{g} = f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right), \tag{1}$$

where

$$f(r) = \begin{cases} \frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) & r \le R, \\ 1 - \frac{2M}{r} & r > R, \end{cases} \tag{2}$$

$$h(r) = \begin{cases} 1 - \frac{r^2}{4M^2} & r \le R, \\ 1 - \frac{2M}{r} & r > R. \end{cases} \tag{3}$$

An induced metric is continuous across the (null) matching surface $r = 2M$. There is a nonzero stress-energy tensor induced on this shell, see [13] for the details. The exterior metric is a solution to vacuum Einstein equations and the interior metric is a solution to Einstein equations with a

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cosmological constant $\Lambda = \frac{3}{4M^2}$. Both interior and exterior metrics are singular at $r = 2M$. To keep them regular, also in higher perturbation orders, we use Eddington–Finkelstein (EF) coordinates (v, r, u, φ). Interior metric in EF coordinates reads:

$$\bar{g} = \frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right)dv^2 + drdv + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right). \tag{4}$$

and exterior metric in EF coordinates reads:

$$\bar{g} = \left(1 - \frac{2M}{r}\right)dv^2 + 2drdv + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right). \tag{5}$$

III. POLAR EXPANSION

In a spherically symmetric background, in $3+1$ dimensions, vector and tensor components split into two sectors: polar and axial (for the details see e.g., [23–27]). Symmetric tensors have 7 polar and 3 axial components. Below we list the expansion of the components of symmetric tensors in axial symmetry (P_ℓ denotes the ℓth Legendre polynomial). In the polar sector we have:

$$S_{ab}(r,u) = \sum_{0\le\ell} S_{\ell ab}(r)P_\ell(u), \qquad a,b = v,r, \tag{6}$$

$$S_{au}(r,u) = -\sum_{1\le\ell} S_{\ell au}(r)\partial_u P_\ell(u), \qquad a = v,r, \tag{7}$$

$$\frac{1}{2}\left((1-u^2)S_{uu}(r,u) + \frac{S_{\varphi\varphi}(r,u)}{(1-u^2)}\right) = \sum_{0\le\ell} S_{\ell+}(r)P_\ell(u), \tag{8}$$

$$\frac{1}{2}\left((1-u^2)S_{uu}(r,u) - \frac{S_{\varphi\varphi}(r,u)}{(1-u^2)}\right) = \sum_{2\le\ell} S_{\ell-}(r)(-\ell(\ell+1)P_\ell(u) + 2u\partial_u P_\ell(u)). \tag{9}$$

In the axial sector we have:

$$S_{a\varphi}(r,u) = \sum_{1\le\ell} S_{\ell a\varphi}(r)(-1+u^2)\partial_u P_\ell(u), \qquad a = v,r, \tag{10}$$

$$S_{u\varphi}(r,u) = \sum_{2\le\ell} S_{\ell u\varphi}(r)(\ell(\ell+1)P_\ell(u) - 2u\partial_u P_\ell(u)). \tag{11}$$

IV. METRIC PERTURBATIONS

We assume that there exists an exact, stationary and axially symmetric solution to Einstein equations, which we expand into series in a parameter a (which will be an angular momentum per unit mass of a an exterior metric) around the static metric (2):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{\infty} \frac{a^i}{i!}\,{}^{(i)}h_{\mu\nu} \tag{12}$$

After perturbation expansion we polar–expand metric perturbations according to (6)–(11). Thus, apart from the perturbation index i, all perturbations gain an index ℓ corresponding to the ℓth Legendre polynomial.

For axial perturbations we take:

$$ {}^{(i)}h_\ell = \begin{pmatrix} 0 & 0 & 0 & {}^{(i)}h_{\ell v\varphi}(r)(-1+u^2)\partial_u P_\ell(u) \\ 0 & 0 & 0 & {}^{(i)}h_{\ell r\varphi}(r)(-1+u^2)\partial_u P_\ell(u) \\ 0 & 0 & 0 & 0 \\ {}^{(i)}h_{\ell v\varphi}(r)(-1+u^2)\partial_u P_\ell(u) & {}^{(i)}h_{\ell r\varphi}(r)(-1+u^2)\partial_u P_\ell(u) & 0 & 0 \end{pmatrix}. \tag{13}$$

Using the gauge freedom, we set ${}^{(i)}h_{\ell u\varphi}(r) = 0$, what corresponds to the Regge-Wheeler (RW) gauge.

For the polar perturbations we take:

$$ {}^{(i)}h_\ell = \begin{pmatrix} {}^{(i)}h_{\ell vv}(r)P_\ell(u) & {}^{(i)}h_{\ell vr}(r)P_\ell(u) & 0 & 0 \\ {}^{(i)}h_{\ell vr}(r)P_\ell(u) & {}^{(i)}h_{\ell rr}(r)P_\ell(u) & 0 & 0 \\ 0 & 0 & {}^{(i)}h_{\ell+}(r)\frac{P_\ell(u)}{1-u^2} & 0 \\ 0 & 0 & 0 & {}^{(i)}h_{\ell+}(r)(1-u^2)P_\ell(u) \end{pmatrix}. \tag{14}$$

Using the gauge freedom, we set ${}^{(i)}h_{\ell ru} = {}^{(i)}h_{\ell vu} = {}^{(i)}h_{\ell -} = 0$, what also corresponds to the RW gauge. Note that in [18] there are no ${}^{(i)}h_{\ell vr}$ and ${}^{(i)}h_{\ell r\varphi}$ coefficients in the metric ansatz. This fact arises from the fact that Hartle uses static coordinates. For EF coordinates in the background both ${}^{(i)}h_{\ell vr}$ and ${}^{(i)}h_{\ell r\varphi}$ turn out to be nonzero in most cases.

In the interior, we solve perturbation Einstein equations with a cosmological constant $\Lambda = \frac{3}{4M^2}$. For a given order i and a given multipole ℓ, they have the following form:

$$\delta^{(i)}G_{\ell\mu\nu} + \frac{3}{4M^2}{}^{(i)}h_{\ell\mu\nu} = {}^{(i)}S_{\ell\mu\nu}, \tag{15}$$

where $\delta^{(i)}G_{\ell\mu\nu}$ denotes the components of the Einstein tensor expansion built of metric perturbations of order i. ${}^{(i)}S_{\mu\nu}$ denotes a source for the ith order Einstein equations consisting of metric perturbations of orders lower that i. We provide an explicit form of Eqs. (15) in the Appendix A.

V. MATCHING INTERIOR WITH EXTERIOR

We match the exterior metric with the interior metric on a three-dimensional hypersurface located at $r^{\pm} = r_b^{\pm}$, where "+" and "−" stand for exterior and interior, respectively. From the first Israel junction condition ([28,29]) we demand the continuity of the induced metric at the matching hypersurface:

$$[[\mathfrak{g}_{ab}]] = 0, \tag{16}$$

where $[[E]] = E^+(r_b^+) - E^-(r_b^-)$. Following [11], we introduce intrinsic coordinates on the three-dimensional hypersurface: $y^a = (V, U, \Phi)$. Then we express interior and exterior coordinates $x^{\pm\mu}$ on a hypersurface in terms of y^a:

$$x^{-\mu}|_{r_b^-} = (A^- V, r_b^-(U), F^-(U), \Phi), \tag{17}$$

$$x^{+\mu}|_{r_b^+} = (A^+ V, r_b^+(U), F^+(U), \Phi), \tag{18}$$

where $r_b^{\pm}(U) = 2M + \frac{a^2}{M^2}\eta^{\pm}(U) + \mathcal{O}(a^4)$, $F^{\pm}(U) = U + \frac{a^2}{M^2}\lambda^{\pm}(U) + \mathcal{O}(a^4)$. We expand $\eta^{\pm}$ into $\eta^{\pm}(U) = \eta_0^{\pm} + \eta_2^{\pm}P_2(U)$.

The metric induced on this hypersurface is given by:

$$\mathfrak{g}_{ab}^{\pm} = \begin{pmatrix} (A^{\pm})^2 g_{vv}^{\pm} & A^{\pm}g_{vr}^{\pm}r_b^{\pm\prime}(U) + A^{\pm}g_{vu}^{\pm}F^{\pm\prime}(U) & A^{\pm}g_{v\varphi}^{\pm} \\ A^{\pm}g_{vr}^{\pm}r_b^{\pm\prime}(U) + A^{\pm}g_{vu}^{\pm}F^{\pm\prime}(U) & (F^{\pm\prime}(U))^2 g_{uu}^{\pm} + (r_b^{\pm\prime}(U))^2 g_{rr}^{\pm} + 2F^{\pm\prime}(U)r_b^{\pm\prime}(U)g_{ru}^{\pm} & F^{\pm\prime}(U)g_{u\varphi}^{\pm} + r_b^{\pm\prime}(U)g_{r\varphi}^{\pm} \\ A^{\pm}g_{v\varphi}^{\pm} & F^{\pm\prime}(U)g_{u\varphi}^{\pm} + r_b^{\pm\prime}(U)g_{r\varphi}^{\pm} & g_{\varphi\varphi}^{\pm} \end{pmatrix}. \tag{19}$$

Using the freedom of choice of coordinates V, U, Φ, we set $F^+(U) = U$ and $A^+ = 1$ (see, e.g., [10]). For simplicity, we denote $A^- = A$.

The location of the matching hypersurface is not known *a priori* and $\eta^{\pm}(U)$ and $\lambda^-(U)$ are unknown functions that need to be found. Our procedure of matching interior and exterior metrics for a given perturbation order is the following:

(1) We solve perturbation Einstein equations for the interior. These solutions contain two constants per ℓ in every perturbation order, but most of these constants need to be set to zero to keep the Kretschmann scalar expansion regular at $r = 0$ and $r = 2M$. However, this is not straightforward to apply, because in our case the singularities in the expansion of the Kretschmann scalar occur in higher perturbation orders than the singularities of the metric itself (in the opposition to the exterior case, e.g., Raposo *et al.* [30]). Therefore, to settle constants in the third order, we solved Einstein equations up to the sixth perturbation order to study behavior of the Kretschmann scalar. Since these expressions are too long to be listed in this paper, we make them available in the *Mathematica* notebook [31].

(2) We act with the general gauge transformation on the interior metric, and then we solve matching conditions (16) for constants arising from Einstein equations, for $\eta^{\pm}(U)$, $\lambda(U)$, and for gauge components. Finding a proper gauge is a part of the matching problem and using the result of Bruni *et al.* [32], we are able to control the impact of the gauge from the lower perturbation order on the metric functions in the higher perturbation order.

(3) If the matching is successful, we go to the higher perturbation order.

The second junction condition tells about the energy content of the matching hypersurface—already in the background solution there is a thin shell located at $r = 2M$ (since this is a null hypersurface, second junction condition needs to be modified, see [29,13] for the details). However, in the next sections we show that even the first

junction condition is not possible to fulfill, therefore we do not find it necessary to discuss second junction condition at all.

VI. KERR METRIC EXPANSION

As an exterior metric, we take the Kerr solution. In the advanced EF coordinates it reads:

$$ds^2 = -\left(1 - \frac{2Mr}{a^2u^2 + r^2}\right)dv^2 + 2dvdr + \frac{a^2u^2 + r^2}{1 - u^2}du^2 + (1 - u^2)\left(\frac{2a^2Mr(1 - u^2)}{a^2u^2 + r^2} + a^2 + r^2\right)d\varphi^2 + \frac{4aMr(1 - u^2)}{a^2u^2 + r^2}dvd\varphi + 2a(1 - u^2)drd\varphi. \tag{20}$$

Since we solve the interior equations in RW gauge, we prefer to use the Kerr metric in RW gauge as well. To do this, we expand (20) into series in a up to the 3rd order, and then act with the gauge transformations (B1)–(B3) to move to the RW gauge. Finally, we obtain:

$$\begin{aligned} ds^2 = &-\left(\left(1 - \frac{2M}{r}\right) - \frac{a^2M(u^2(6M^2 - Mr - 3r^2) - 2M^2 + Mr + r^2)}{r^5}\right)dv^2 \\ &+ \left(\frac{2a^2M(1 - 3u^2)}{r^3}\right)dr^2 + \left(\frac{r^2}{1 - u^2} + \frac{a^2M(3u^2 - 1)(2M + r)}{r^2(u^2 - 1)}\right)du^2 \\ &+ \left(r^2(1 - u^2) + \frac{a^2M(u^2 - 1)(3u^2 - 1)(2M + r)}{r^2}\right)d\varphi^2 \\ &+ 2\left(1 + \frac{a^2M(3u^2 - 1)(M + r)}{r^4}\right)dvdr + 2\left(\frac{a^3M(1 - u^2)(5u^2 - 1)(9M + 5r)}{5r^4}\right)drd\varphi \\ &+ 2\left(\frac{2aM(1 - u^2)}{r} - \frac{a^3M(u^2 - 1)(M^2(6u^2 - 2) + M(r - 5ru^2) + r^2(1 - 5u^2))}{r^5}\right)dvd\varphi + \mathcal{O}(a^4), \end{aligned} \tag{21}$$

For simplicity, we omit "+" and "−" coordinate superscripts and use them only when it is necessary to differentiate the interior from the exterior. We expand (21) into series in a. Below we list nonzero components of this expansion after the polar decomposition.

$$\begin{aligned} &{}^{(1)}h^+_{1v\varphi} = -\tfrac{2M}{r}, && {}^{(2)}h^+_{2+} = -\tfrac{4M(2M+r)}{r^2}, \\ &{}^{(2)}h^+_{0vv} = \tfrac{4M^2}{3r^4}, && {}^{(3)}h^+_{1v\varphi} = \tfrac{24M^3}{5r^5}, \\ &{}^{(2)}h^+_{2vv} = \tfrac{4M(6M^2 - Mr - 3r^2)}{3r^5}, && {}^{(3)}h^+_{3v\varphi} = \tfrac{4M(-6M^2 + 5Mr + 5r^2)}{5r^5}, \\ &{}^{(2)}h^+_{2vr} = \tfrac{4M(M+r)}{r^4}, && {}^{(3)}h^+_{3r\varphi} = -\tfrac{4M(9M+5r)}{5r^4}. \\ &{}^{(2)}h^+_{2rr} = -\tfrac{8M}{r^3}, \end{aligned} \tag{22}$$

VII. INTERIOR SOLUTION

A. The first order

1. Axial $\ell = 1$

For $\ell = 1$ there is no $h_{u\varphi}$ component and we can use the remaining gauge freedom to set ${}^{(1)}h^-_{1r\varphi} = 0$. Linearized Einstein equation are homogeneous (A1)–(A3) and yield:

$${}^{(1)}h^-_{1v\varphi} = \Omega_{11}r^2 + \frac{\Pi_{11}}{r} \tag{23}$$

where Ω_{11} and Π_{11} are arbitrary constants. We set $\Pi_{11} = 0$ to make the Kretschmann scalar expansion regular at r = 0, therefore we are left with ${}^{(1)}h^-_{1v\varphi} = \Omega_{11}r^2$. It turns out that this solution is a pure gauge, but we will discuss it later.

B. The second order

1. Polar $\ell = 0$

For $\ell = 0$ there are no h_-, h_{vu}, h_{ru} components in the polar decomposition and we have an additional gauge freedom, which we use to set ${}^{(2)}h^-_{0vr}$, ${}^{(2)}h^-_{0+}$ to zero.

The only nonzero variables left are ${}^{(2)}h^{-}_{0vv}$ and ${}^{(2)}h^{-}_{0rr}$. Solution to Einstein equations (A4)–(A10) with $\ell = 0$ and with sources (A13)–(A15) reads:

$$ {}^{(2)}h^{-}_{0vv} = \frac{4r^2\Omega_{11}^2}{3} - \frac{c_{20}(r^2-4M^2)}{64M^4} + \frac{d_{20}}{r}, \tag{24} $$

$$ {}^{(2)}h^{-}_{0rr} = \frac{c_{20}}{r^2-4M^2}. \tag{25} $$

where c_{20} and d_{20} are arbitrary constants. This solution is singular at $r=0$ and $r=2M$. To avoid singularity in the Kretschmann scalar expansion at $\mathrm{r}=0$, we set $d_{20}=0$. Singularity at $r=2M$ can be removed using a transformation generated by a gauge vector ${}^{(2)}\xi_0$ (${}^{(2)}\xi_{0v} = \frac{c_{20}((r^2-4M^2)\tanh^{-1}(\frac{r}{2M})+2Mr)}{64M^3}$, ${}^{(2)}\xi_{0r} = \frac{c_{20}\tanh^{-1}(\frac{r}{2M})}{8M}$, ${}^{(2)}\xi_{0u}=0$, ${}^{(2)}\xi_{0\varphi}=0$), what yields:

$$ {}^{(2)}h^{-}_{0vv} = \frac{4r^2\Omega_{11}^2}{3} + \frac{c_{20}}{16M^2}, \tag{26} $$

$$ {}^{(2)}h^{-}_{0vr} = 0, \tag{27} $$

$$ {}^{(2)}h^{-}_{0rr} = 0, \tag{28} $$

$$ {}^{(2)}h^{-}_{0+} = \frac{c_{20}r^2}{4M^2}. \tag{29} $$

2. Polar $\ell = 2$

Solution to Einstein equations (A4)–(A10) with $\ell = 2$ and with sources (A16)–(A19) reads:

$$ {}^{(2)}h^{-}_{2vv} = \frac{(r^2-4M^2)^2}{128M^4}\,{}^{(2)}h^{-}_{2rr} - \frac{4}{3}r^2\Omega_{11}^2, \tag{30} $$

$$ {}^{(2)}h^{-}_{2vr} = -\frac{1}{4}\left(1-\frac{r^2}{4M^2}\right){}^{(2)}h^{-}_{2rr}, \tag{31} $$

$$ {}^{(2)}h^{-}_{2rr} = \frac{c_{22}}{16M^4r^3} + \frac{d_{22}(3(r^2-4M^2)^2\coth^{-1}(\frac{2M}{r}) + 2Mr(5r^2-12M^2))}{32M^3r^3(r^2-4M^2)^2}, \tag{32} $$

$$ {}^{(2)}h^{-}_{2+} = \frac{c_{22}(4M^2+r^2)}{128M^6r} + \frac{d_{22}(3M(4M^2+r^2)\coth^{-1}(\frac{2M}{r}) - 2r(3M^2+r^2))}{256M^6r}, \tag{33} $$

where c_{22} and d_{22} are arbitrary constants. To avoid singularity in the Kretschmann scalar expansion at $r=0$ and $r=2M$ we need to set $c_{22}=0$, $d_{22}=0$, what yields:

$$ {}^{(2)}h^{-}_{2vv} = -\frac{4}{3}r^2\Omega_{11}^2, \tag{34} $$

$$ {}^{(2)}h^{-}_{2vr} = 0, \tag{35} $$

$$ {}^{(2)}h^{-}_{2rr} = 0, \tag{36} $$

$$ {}^{(2)}h^{-}_{2+} = 0. \tag{37} $$

C. The third order

1. Axial $\ell = 1$

The solution to Einstein equations (A1)–(A3) with $\ell = 1$ reads:

$$ {}^{(3)}h^{-}_{1v\varphi} = \Omega_{31}r^2 + \frac{\Pi_{31}}{r}. \tag{38} $$

To avoid singularity in the Kretschmann scalar expansion at $\mathrm{r}=0$, we set $\Pi_{31}=0$.

2. Axial $\ell = 3$

Solution to Einstein equations (A1)–(A3) with $\ell = 3$ reads:

$$ {}^{(3)}h^{-}_{3v\varphi} = \frac{(r^2-4M^2)}{r^3}\Pi_{33} + \frac{(-120M^4r + 20M^2r^3 + 60(4M^5-M^3r^2)\coth^{-1}(\frac{2M}{r}) + r^5)}{3r^3}\Omega_{33}, \tag{39} $$

$$ {}^{(3)}h^{-}_{3r\varphi} = \frac{8M^2}{r^3}\Pi_{33} + \frac{8M^2(\frac{r(-120M^4+20M^2r^2+r^4)}{r^2-4M^2} - 60M^3\coth^{-1}(\frac{2M}{r}))}{3r^3}\Omega_{33}, \tag{40} $$

where Ω_{33} and Π_{33} are arbitrary constants. Singularities at $r = 0$ and $r = 2M$ lead to the singularity in the Kretschmann scalar expansion, therefore $\Omega_{33} = 0$, $\Pi_{33} = 0$.

VIII. MATCHING

A. First order

Before matching, we act with the general gauge transformation on the interior metric. Although we consider stationary metrics, we take gauge vectors that depend on v coordinate. It might happen that acting with gauge vectors depending on v explicitly, we obtain metric independent of v (we discuss such a case in Sec. IX). From the matching conditions (16) we have:

$$\frac{{}^{(1)}h^{+}_{1v\varphi}(2M)}{A} - {}^{(1)}h^{-}_{1v\varphi}(2M) = -\partial_v {}^{(1)}\xi_{1\varphi}(v, 2M), \tag{41}$$

To keep transformed metric v-independent, we use (B4) and (B5) and obtain a condition:

$${}^{(1)}\xi_{1\varphi} = q_{11} v r^2 + {}^{(1)}\gamma_{1\varphi}(r), \tag{42}$$

where q_{11} is an arbitrary constant and γ_1 is an arbitrary function of r. From (41) we obtain:

$$\Omega_{11} = -\frac{1}{4AM^2} + q_{11}. \tag{43}$$

B. Second order

We act with the most general second order gauge transformation (B1)–(B2) on the interior metric. To keep transformed metric v-indepedent, we use (B7)–(B13) and obtain conditions:

$${}^{(2)}\xi_{0v} = -4M^2 f q_{20} v + {}^{(2)}\gamma_{0v}(r), \tag{44}$$

$${}^{(2)}\xi_{0r} = 8M^2 q_{20} v + {}^{(2)}\gamma_{0r}(r), \tag{45}$$

$${}^{(2)}\xi_{2v} = {}^{(2)}\gamma_{2v}(r), \tag{46}$$

$${}^{(2)}\xi_{2r} = {}^{(2)}\gamma_{2r}(r), \tag{47}$$

$${}^{(2)}\xi_{2u} = {}^{(2)}\gamma_{2u}(r), \tag{48}$$

where q_{20} is an arbitrary constant and ${}^{(i)}\gamma_{\ell\mu}$ are functions of r.

Matching conditions (16) yield:

$${}^{(2)}h^{+}_{0vv}(2M) - A^2 {}^{(2)}h^{-}_{0vv}(2M) = \frac{A^2\eta_0^- + 2\eta_0^+}{2M^3} + \frac{16}{3}A^2M^2 q_{11}(q_{11} - 2\Omega_{11}) + \frac{A^2}{2M}{}^{(2)}\gamma_{0v}(2M), \tag{49}$$

$${}^{(2)}h^{+}_{2vv}(2M) - A^2 {}^{(2)}h^{-}_{2vv}(2M) = \frac{A^2\eta_2^- + 2\eta_2^+}{2M^3} - \frac{16}{3}A^2M^2 q_{11}(q_{11} - 2\Omega_{11}) + \frac{A^2}{2M}{}^{(2)}\gamma_{2v}(2M), \tag{50}$$

$$2\eta_2^+ - \eta_2^- A = AM^2 {}^{(2)}\gamma_{2v}(2M), \tag{51}$$

$$[[{}^{(2)}h_{0+}(2M)]] = -\frac{8(\eta_0^+ - \eta_0^-)}{M} + 8\lambda'(U) + 8M{}^{(2)}\gamma_{0v}(2M), \tag{52}$$

$$[[{}^{(2)}h_{2+}(2M)]] = -\frac{8(\eta_2^+ - \eta_2^-)}{M} + 8M{}^{(2)}\gamma_{2v}(2M) - 6{}^{(2)}\gamma_{2u}(2M), \tag{53}$$

$$[[{}^{(2)}h_{2-}(2M)]] = {}^{(2)}\gamma_{2u}(2M) + \frac{16U\lambda(U) + 8(1-U^2)\lambda'(U)}{3(U^2-1)^2}. \tag{54}$$

After plugging solutions to perturbation equations into (49)–(54), we obtain:

$$\eta_0^- - = -M^2 {}^{(2)}\gamma_{0v}(2M) - \frac{4MU}{3}\lambda_1 - \frac{M}{8}c_{20} - \frac{M}{6} - \frac{1}{4}M(3U^2-1){}^{(2)}\gamma_{2u}(2M), \tag{55}$$

$$\eta_2^- = -\frac{M}{3} - M^2 {}^{(2)}\gamma_{2v}(2M) + \frac{1}{2}M{}^{(2)}\gamma_{2u}(2M), \tag{56}$$

$$\eta_0^+ = -\frac{M}{6} + \frac{2\lambda_1 MU}{3} + \frac{1}{8}M(3U^2-1){}^{(2)}\gamma_{2u}(2M), \tag{57}$$

$$\eta_2^+ = \frac{M}{6} - \frac{1}{4}M{}^{(2)}\gamma_{2u}(2M), \tag{58}$$

$$A = -1, \tag{59}$$

$$\lambda(U) = \lambda_1(U^2-1) + \frac{3}{8}U(U^2-1){}^{(2)}\gamma_{2u}(2M). \tag{60}$$

where λ_1 is an arbitrary constant. To keep η_0^- independent of U, we have to set $\lambda_1 = 0$ and ${}^{(2)}\gamma_{2u}(2M) = 0$, what leads to:

$$\eta_0^- - = -M^2 {}^{(2)}\gamma_{0v}(2M) - \frac{M}{8}c_{20} - \frac{M}{6}, \tag{61}$$

$$\eta_2^- = -\frac{M}{3} - M^2 {}^{(2)}\gamma_{2v}(2M), \tag{62}$$

$$\eta_0^+ = -\frac{M}{6}, \tag{63}$$

$$\eta_2^+ = \frac{M}{6}, \tag{64}$$

$$A = -1, \tag{65}$$

$$\lambda(U) = 0, \tag{66}$$

$${}^{(2)}\gamma_{2u}(2M) = 0. \tag{67}$$

C. Third order

Again, we act with the most general third order gauge transformation (B1)–(B3) on the interior metric. To keep transformed metric v-indepedent, we use (B4)–(B6) and obtain conditions:

$${}^{(3)}\xi_{1\varphi} = q_{31} r^2 v + {}^{(3)}\gamma_{1\varphi}(r), \tag{68}$$

$${}^{(3)}\xi_{3\varphi} = {}^{(3)}\gamma_{3\varphi}(r), \tag{69}$$

where q_{31} is an arbitrary constant and ${}^{(i)}\gamma_{\ell\mu}$ are functions of r. Using (43) and (61)–(67), third order matching conditions (16) yield:

$${}^{(3)}h^+_{1v\varphi}(2M) - A\,{}^{(3)}h^-_{1v\varphi}(2M) = \frac{3(5c_{20}+8)}{20M^2} + 3c_{20}q_{11} - 192M^4 q_{11}q_{20} + M^2 4(q_{31} - 12q_{20}), \tag{70}$$

$${}^{(3)}h^+_{3v\varphi}(2M) - A\,{}^{(3)}h^-_{3v\varphi}(2M) = \frac{3}{10M^2}, \tag{71}$$

$$5M^2 {}^{(3)}\xi_{3,\varphi}(2M) = 6\,{}^{(2)}\gamma_{2r}(2M)(4M^2 q_{11} + 1) + 2(3M\,{}^{(2)}\gamma_{2v}(2M) + 1)(M\,{}^{(1)}\gamma'_{1\varphi}(2M) - {}^{(1)}\gamma_{1\varphi}(2M)). \tag{72}$$

Condition (72) can be fulfilled just by setting all the gauge components to zero. Setting $\xi_{2u} = 0$ and plugging (38)–(40) into (70), we obtain:

$$\Omega_{31} = \frac{9}{80M^4} + q_{31} + \frac{3(4M^2 q_{11} + 1)(c_{20} - 64M^4 q_{20})}{16M^4}. \tag{73}$$

However, (71) does not have any free parameters and it cannot be fulfilled (we obtain contradiction $-\frac{3}{10M^2} = 0$). That makes impossible to match interior with exterior in the third order.

IX. DISCUSSION AND SUMMARY

Although we found the matching impossible, it is interesting to know what is the interior solution we obtained. The regular interior solution up to the third order reads:

$$ds^2 = \begin{pmatrix} -\frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) + a^2\left(\frac{c_{20}}{32M^2} + r^2(1-u^2)\Omega_{11}^2\right) & \frac{1}{2} & 0 & \frac{1}{6}ar^2(u^2-1)(6\Omega_{11} + a^2\Omega_{31}) \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-u^2} + \frac{a^2 c_{20} r^2}{8M^2(1-u^2)} & 0 \\ \frac{1}{6}ar^2(u^2-1)(6\Omega_{11} + a^2\Omega_{31}) & 0 & 0 & r^2(1-u^2) + \frac{a^2 c_{20} r^2 (1-u^2)}{8M^2} \end{pmatrix}. \tag{74}$$

It turns out that this is an exact solution to Einstein equations—a gauge–transformed de Sitter space. To see this, let us take the gauge vector with components:

$${}^{(1)}\xi_1 = (0, 0, 0, r^2\Omega_{11} v), \tag{75}$$

$${}^{(2)}\xi_0 = \left(-\frac{c_{20} r}{16M^2} + \frac{c_{20}(r^2 - 4M^2)}{128M^4} v, \frac{c_{20} v}{16M^2}, 0, 0\right), \tag{76}$$

$${}^{(2)}\xi_2 = (0, 0, 0, 0), \tag{77}$$

$$^{(3)}\xi_1 = \left(0, 0, 0, \left(r^2\Omega_{13} - \frac{3c_{20}r^2\Omega_{11}}{8M^2}\right)v\right), \quad (78)$$

$$^{(3)}\xi_3 = (0, 0, 0, 0). \quad (79)$$

Acting with those vectors on (74) (using formulas (B1)–(B3), we obtain

$$ds^2 = \begin{pmatrix} -\frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-u^2} & 0 \\ 0 & 0 & 0 & -r^2(u^2-1) \end{pmatrix}, \quad (80)$$

what is exactly the background de Sitter metric, so all perturbations we obtained are a pure gauge. One can ask, if allowing for a change in the background density does not affect this result, but the answer is no. We repeated the calculation allowing for the perturbations of density and pressure (within the equation of state $p = -\rho$), but they do not change the conclusions.

We would also like to comment on a recent article [33] that concerns the same problem as our work. Authors of [33] use Hartle formalism to match the rotating gravastar with the Kerr black hole up to the second perturbation order. They succeed to do that (as we do in the second order), but there are two main differences between our approaches. The first difference is the choice of the matching surface. We do not fix the matching surface and we treat it as a variable to be found. Authors of [33] fix the matching surface to be the horizon of the Kerr black hole. It seems to be contradictory to our results, because we do not have a freedom to perform matching on the horizon, but there comes the second difference between our papers. We dismiss solutions which produce singularities both at $r = 0$ and at $r = 2M$, whereas authors of [33] allow for the solutions which have a singularity at $r = 0$ in the second perturbation order. Because of that, they have additional freedom in the interior solution and they are able to match it with Kerr on the horizon. The justification they make for allowing such a singular solution is the possibility that the singularity is not real, but it appears as an artefact of the perturbative expansion. This argument touches the sensitive point of the perturbative expansion. It may happen that a function which is not singular at some point, in this case at $r = 0$, has singular expansion coefficients when expanded in the perturbation parameter (see Summary and Discussion in [33]). Authors of [33] do not determine whether such a scenario is the origin of the singularity they allow for. On the other hand, we cannot exclude that the singular terms in the Kretschmann scalar that we put to zero are such artificial singularities. If this is the case and if we did not set $c_{22} = 0$, we would be able to match solutions in the third order. Unfortunately, this ambiguity seems to be an inherent limitation of the perturbation theory.

To sum up, we made an attempt to match the ultracompact rotating gravastar with the Kerr metric using the nonlinear perturbation theory. The solution we choose is a general solution to the perturbation equations around a static gravastar that does not produce the singularities in the Kretschmann scalar expansion. Although the matching can be performed up to the second order, in the third order it is is no longer possible. What is more, the interior of the ultracompact rotating gravastar is just the de Sitter metric. Since some of the proposed sources of the Kerr metric are based on the second perturbation order calculations, we find it necessary to check if these results survive at the higher perturbation orders.

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APPENDIX A: EINSTEIN EQUATIONS

Einstein equations (15) of order i divide into two parts: the homogeneous part $\delta G_{\ell\mu\nu}$ consisting of metric perturbations of order i and sources $^{(i)}S_{\ell\mu\nu}$ consisting of metric perturbations of orders j ($j < i$). These equations need to be solved order by order: after solving Einstein equations up to order i one can construct explicit form of $i + 1$ order source.

1. Homogeneous part

In the axial sector in the RW gauge, there are two nonzero variables: $^{(i)}h_{\ell v\varphi}$ and $^{(i)}h_{\ell r\varphi}$ (for simplicity, we omit i and ℓ indices in formulas (A1)–(A10)). Homogeneous part of Einstein equations reads (where we introduce $E_{\mu\nu} = \delta G_{\mu\nu} + \frac{3}{4M^2}h_{\mu\nu}$):

$$2i!r^2E_{v\varphi} = (2f + \ell(\ell+1) - 2)h_{v\varphi} - r^2fh''_{v\varphi}, \quad (A1)$$

$$2i!r^2E_{r\varphi} = 2r^2h''_{v\varphi} - 4h_{v\varphi} + (\ell(\ell+1) - 2)h_{r\varphi}, \quad (A2)$$

$$2i!E_{u\varphi} = fh'_{r\varphi} + 2h'_{v\varphi} + f'h_{r\varphi}. \quad (A3)$$

In the polar sector in the RW gauge, there are four nonzero variables: $^{(i)}h_{\ell vv}$, $^{(i)}h_{\ell vr}$, $^{(i)}h_{\ell rr}$, $^{(i)}h_{\ell +}$. Homogeneous part of Einstein equations reads:

$$8i!r^4E_{vv} = 2f^3r^3h'_{rr} + 8f^2r^3h'_{vr} - 2f^2r^2h''_+ + 4fr^2(2rf' + 2f + \ell(\ell+1))h_{vr} + f(2rf' + \ell(\ell+1) - 2)h_+ + fr(2f - rf')h'_+ + f^2r^2(4rf' + 2f + \ell(\ell+1))h_{rr} + 4r^2(2f + \ell(\ell+1))h_{vv} + 8fr^3h'_{vv}, \tag{A4}$$

$$4i!r^4E_{vr} = -2f^2r^3h'_{rr} + (-2rf' - \ell(\ell+1) + 2)h_+ - fr^2(4rf' + 2f + \ell(\ell+1))h_{rr} - 2r^2(4rf' + 4f + \ell(\ell+1))h_{vr} + r(rf' - 2f)h'_+ - 8fr^3h'_{vr} + 2fr^2h''_+ - 8r^3h'_{vv} - 8r^2h_{vv}, \tag{A5}$$

$$2i!r^4E_{rr} = r^2(2rf' + \ell(\ell+1))h_{rr} + 2fr^3h'_{rr} + 8r^3h'_{vr} - 2r^2h''_+ + 4rh'_+ - 4h_+, \tag{A6}$$

$$2i!E_{vu} = h_{vr}f' + fh'_{vr} + 2h'_{vv}, \tag{A7}$$

$$4i!r^3E_{ru} = r^2(rf' + 2f)h_{rr} - 4r^3h'_{vr} + 8r^2h_{vr} - 2rh'_+ + 4h_+, \tag{A8}$$

$$4i!r^2E_+ = -4r^2(4rf' + 4f + \ell(\ell+1) - 4)h_{vr} - fr^3(rf' + 2f)h'_{rr} - 4r^3(rf' + 2f)h'_{vr} + 2r(rf' - 2f)h'_+ + 4(f - rf')h_+ - r^2(4f^2 + f(6rf' + \ell(\ell+1) - 4) + r^2f'^2)h_{rr} + 2fr^2h''_+ - 8r^4h''_{vv} - 16r^3h'_{vv}, \tag{A9}$$

$$4i!E_- = fh_{rr} + 4h_{vr}. \tag{A10}$$

2. Sources

Below we list the nonzero components of sources for Einstein equations. Sources for the ith order perturbation equations can be found in the following way (see, e.g., appendix A of [34]). Let us assume that we already know the solution to perturbation Einstein equations up to the ith order (it consists of metric perturbations ${}^{(j)}h_{\mu\nu}$ with $j \le i$):

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}h_{\ell\mu\nu}\frac{a^j}{j!}. \tag{A11}$$

Using this solution we can calculate the Einstein tensor $G_{\mu\nu}(\tilde{g})$, which satisfies the ith order perturbation equations and contributes to the $i+1$th (and higher) order perturbation equations. Finally, the source of the order $i+1$ is given by:

$${}^{(i+1)}S_{\mu\nu} = [i+1](-G_{\mu\nu}(\tilde{g})), \tag{A12}$$

where $[k]\,(\cdots)$ denotes the kth order expansion of a given quantity. Although in most cases expressions for the sources ${}^{(i+1)}S_{\mu\nu}$ are complicated, their construction is a purely algebraic task and can be easily performed using computer algebra. Below we list nonzero components of ith order sources in terms of explicit solutions ${}^{(j)}h_{\mu\nu}$ found for lower orders.

The source for the second order:

$${}^{(2)}S_{0vv} = 4\left(1 - \frac{r^2}{4M^2}\right)\Omega_{11}^2, \tag{A13}$$

$${}^{(2)}S_{0vr} = -8\Omega_{11}^2, \tag{A14}$$

$${}^{(2)}S_{0+} = -16\Omega_{11}^2, \tag{A15}$$

$${}^{(2)}S_{2vv} = \left(\frac{r^2}{M^2} - 8\right)\Omega_{11}^2, \tag{A16}$$

$${}^{(2)}S_{2vr} = 8\Omega_{11}^2, \tag{A17}$$

$${}^{(2)}S_{2vu} = \frac{8}{3}r\Omega_{11}^2, \tag{A18}$$

$${}^{(2)}S_{2+} = 16r^2\Omega_{11}^2. \tag{A19}$$

The sources for the third order are zero.

APPENDIX B: GAUGE TRANSFORMATIONS

Consider a gauge transformation induced by a gauge vector $\xi = \sum_{i=0}^{\infty}\frac{a^i}{i!}{}^{(i)}\xi$. According to [32], metric perturbations transform in the following way:

$${}^{(1)}h_{\mu\nu} \to {}^{(1)}h_{\mu\nu} + \pounds_{{}^{(1)}\xi}\bar{g}_{\mu\nu}, \tag{B1}$$

$$^{(2)}h_{\mu\nu} \to {}^{(2)}h_{\mu\nu} + (\pounds_{^{(2)}\xi} + \pounds^2_{^{(1)}\xi})\bar{g}_{\mu\nu} + 2\pounds_{^{(1)}\xi}{}^{(1)}h_{\mu\nu}, \tag{B2}$$

$$^{(3)}h_{\mu\nu} \to {}^{(3)}h_{\mu\nu} + (\pounds^3_{^{(1)}\xi} + 3\pounds_{^{(1)}\xi}\pounds_{^{(2)}\xi} + \pounds_{^{(3)}\xi})\bar{g}_{\mu\nu} + 3(\pounds^2_{^{(1)}\xi} + \pounds_{^{(2)}\xi})^{(1)}h_{\mu\nu} + 3\pounds_{^{(1)}\xi}{}^{(2)}h_{\mu\nu}, \tag{B3}$$

where $\pounds_{^{(i)}\xi}$ denotes a Lie derivative with respect to $^{(i)}\xi$.

An explicit form of (B1)–(B3) for a gauge vector of order i acting on a metric components of order i reads (for clarity, we omit i indices, dots and primes correspond to derivatives with respect to v and r, respectively):

$$h_{\ell v\varphi} \to h_{\ell v\varphi} - \dot{\xi}_\varphi, \tag{B4}$$

$$h_{\ell r\varphi} \to h_{\ell r\varphi} + \frac{2\xi_\varphi}{r} - \xi'_\varphi, \tag{B5}$$

$$h_{\ell u\varphi} \to h_{\ell u\varphi} + \xi_\varphi, \tag{B6}$$

$$h_{\ell vv} \to h_{\ell vv} - \frac{1}{4}(f\xi_r + 2\xi_v)f' + 2\dot{\xi}_v, \tag{B7}$$

$$h_{\ell vr} \to h_{\ell vr} + \frac{1}{2}f'\xi_r + \xi'_v + \dot{\xi}_r, \tag{B8}$$

$$h_{\ell rr} \to h_{\ell rr} + 2\xi'_r, \tag{B9}$$

$$h_{\ell +} \to h_{\ell +} + 2rf\xi_r - \ell(\ell+1)\xi_u + 4r\xi_v, \tag{B10}$$

$$h_{\ell -} \to h_{\ell -} - \xi_u, \tag{B11}$$

$$h_{\ell vu} \to h_{\ell vu} - \xi_v - \dot{\xi}_u, \tag{B12}$$

$$h_{\ell ru} \to h_{\ell ru} - \xi_r + \frac{2}{r}\xi_u - \xi'_u. \tag{B13}$$

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